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Binomial sums resulting to the ratio, G/π and $1/\pi$

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Abstract

In this brief article, we present few series involving binomial coefficients where important mathematical constants, namely Catalan's constant G and inverse of π appear. We provide the explicit evaluation of series via the construction of generating functions and applications of Wallis' well-know integral formula.

1 Introduction

In mathematical literature several constants are introduced and among such constants, we define Catalan's constant by

$$G = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(2n-1)^2},$$

which is named after Eugène Charles Catalan, who found fast converging series and used it to calculate their sums. He published a memoir about it in 1865 [2]. This fascinating constant possesses several integral and series representations (see [1]). The elementary integral forms (see [[1], p. 2]) of G are as follows:

$$G = \int_0^1 \frac{\tan^{-1} x}{x} dx = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{x}{\sin x} dx,$$

which will be helpful in course of analysis of our mains result. Next, we state that binomial sums involving the coefficients are central binomial coefficients, namely $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$ for all $n \geq 0$. These coefficients play a very crucial roles in many fields like number theory, combinatorics, particle physics and analysis. Several interesting and intriguing identities are given by Lehmer [4] and many facts regarding the coefficients can be found in [3].

Central binomial coefficients are simply generated by the infinite binomial series expansion of the function $\frac{1}{\sqrt{1-4x}}$ for $x \in [-1/4, 1/4)$, which in other words, it can be expressed as:

$$\frac{1}{\sqrt{1-4x}} = \sum_{n=0}^{\infty} \binom{2n}{n} x^n = 1 + 2x + 6x^2 + 20x^3 + \cdots. \tag{1}$$

Furthermore, it can be easily obtained that

$$\int_0^{\frac{\pi}{2}} \sin^{2n} x dx = \frac{\pi}{2 \cdot 4^n} \binom{2n}{n}, \ n \geq 0, \tag{2}$$

which is famously known as Wallis' integral formula. Our main results solely rely on the aforementioned facts (1) and (2), in the process of both construction of power series and evaluation of main portions that will be resulting to ratio G/π and Ramanujan-types formula for $1/\pi$. We proceed rest portions of the work in following sections. In Section 2, we just record the mains aims of recent paper. In Section 3, we will develop the building blocks as lemmas that will assist in course of completion of main gist of paper. In Section 5, we will have glances of more series yielding ratios G/π and $1/\pi$.

Now we highlight our main goals as follows:

2 Main results

Hereafter, we present our main results.

Theorem 1. *If n be a positive integer, then we have*

$$\sum_{n=1}^{\infty}\binom{2n}{n}\binom{2n-2}{n-1}\frac{1}{16^n(2n-1)^2}=\frac{G}{\pi}-\frac{1}{2\pi}.$$

Theorem 2. *If n be a positive integer, then the followings hold:*

$$\sum_{n=1}^{\infty}\binom{2n}{n}\binom{2n-2}{n-1}\frac{1-n}{2^{4n-1}(2n-1)^2}=\frac{G}{\pi}-\frac{1}{\pi}.$$

and

$$\sum_{n=1}^{\infty}\binom{2n}{n}\binom{2n-2}{n-1}\frac{2n}{16^n(2n-1)^2}=\frac{G}{\pi}.$$

Theorem 3. *The following equality holds:*

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2(2n+1)16^n}\binom{2n}{n}^2=\frac{3}{4\pi}-\frac{G}{2\pi}.$$

All the aforementioned theorems will be proved in compact and explicit manner without any sorts of different approaches to each of them. We will prove them together by establish the elementary relation among them. To do so, we built the following lemmas.

3 Lemmas and their proofs

This section holds some preliminaries that are going to be building blocks for series stated in previous section. We proceed as follows:

Lemma 1. *For all $x\in[-1,1]$, the following relation holds:*

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^{2n-1}}{4^n(2n-1)^2}=\frac{\sqrt{1-x^2}+x\sin^{-1}x-1}{x}.$$

Proof. We have the generating function of central binomial coefficients (1), which can be further written as

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^{2n-2}}{4^n}=\frac{1}{x^2}\left(\frac{1}{\sqrt{1-x^2}}-1\right),$$

which on integration implies

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^{2n-1}}{4^n(2n-1)}=\frac{1-\sqrt{1-x^2}}{x}.$$

The latter expression is easy to get since

$$\int\frac{1}{x^2}\left(\frac{1}{\sqrt{1-x^2}}-1\right)dx=\frac{1-\sqrt{1-x^2}}{x}+C_1$$

and $C_1=0$ which is obtained if $x\to 0$. Further, we divide by x and on integrating, we have

$$\begin{aligned}\sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^{2n-1}}{4^n(2n-1)^2} &= \int\frac{1-\sqrt{1-x^2}}{x^2}dx\\ &= \frac{\sqrt{1-x^2}+x\sin^{-1}(x)-1}{x}+C_2,\end{aligned}$$

where C_2 is constant of integration. As $x\to 0, C_2=0$ and therefore,

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{x^{2n-1}}{4^n(2n-1)^2}=\frac{\sqrt{1-x^2}+x\sin^{-1}x-1}{x} \tag{3}$$

is proved.

Lemma 2. *If $x\in[-1,1]$, then*

$$\sum_{n=1}^{\infty}\binom{2n}{n}\frac{nx^{n-1}}{4^n(2n-1)^2}=\frac{\sin^{-1}\left(\sqrt{x}\right)}{2\sqrt{x}}.$$

Proof. The proof is based on Lemma 1 which can be obtained by multiplying x on both sides of (3) and followed by differentiation with respect to x. Details are left to the readers.

Lemma 3. *For all $x\in[-1,1]$, we have*

$$\sum_{n=1}^{\infty}\frac{nx^{2n}\binom{2n}{n}}{4^n(2n-1)^2(2n+1)}=\frac{1}{8}\left(\sqrt{1-x^2}+2x\sin^{-1}x-\frac{\sin^{-1}x}{x}\right).$$

Proof. Enforcing substitution $x \to x^2$ and multiplying both sides of Lemma 2 by x^2, integrating, and performing some algebraic simplification leads to the desired result. We encourage curious readers to pursue the results.

Now we are ready to prove the closed forms of the theorems from the previous section.

4 Proofs of the Theorems

Here we display the proofs of the formulas mentioned in Section 2 in respective manner.

Proof. To prove Theorem 1, we divide both sides of Lemma 1 by x, replace x by $\sin y$ and hence on integrating from 0 to $\pi/2$, we get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{4^n(2n-1)^2} \int_0^{\frac{\pi}{2}} \left(\sin^2 y\right)^{n-1} dy &= \sum_{n=1}^{\infty} \binom{2n}{n}\binom{2n-2}{n-1} \frac{2\pi}{16^n(2n-1)^2} \\ &= \int_0^{\frac{\pi}{2}} \frac{\cos y - 1 + y \sin y}{\sin^2 y} dy \\ &= \int_0^{\frac{\pi}{2}} \frac{y}{\sin y} dy - \int_0^{\frac{\pi}{2}} \frac{1-\cos y}{\sin^2 y} dy \\ &= 2G - 1.\end{aligned}$$

In the first line of the latter expression, we invoke Wallis' integral formula (2) and an elementary integral result

$$\int_0^{\frac{\pi}{2}} \frac{1-\cos y}{\sin^2 y} dy = \int_0^{\frac{\pi}{2}} \frac{dy}{1+\cos y} = \tan\left(\frac{y}{2}\right)\Big|_0^{\frac{\pi}{2}} = 1.$$

Therefore, a simple routine simplification leads

$$\sum_{n=1}^{\infty} \binom{2n}{n}\binom{2n-2}{n-1} \frac{2}{16^n(2n-1)^2} = \frac{2G}{\pi} - \frac{1}{\pi}, \tag{4}$$

and multiplying by $1/2$ thus, completes the proof of Theorem 1.

Proof. On similar fashion, we prove Theorem 2, in order to achieve we replace x by x^2 in Lemma 2, then put $x = \sin y$ and integrating from 0 to $\pi/2$, we have

$$\sum_{n=1}^{\infty} \binom{2n}{n} \frac{n}{4^n(2n-1)^2} \int_0^{\frac{\pi}{2}} \left(\sin^2 y\right)^{n-1} dy$$

$$= \sum_{n=1}^{\infty} \binom{2n}{n} \binom{2n-2}{n-1} \frac{2n\pi}{16^n(2n-1)^2} = \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{y}{\sin y} dy = G.$$

Dividing both sides of last equality by π, we produce

$$\sum_{n=1}^{\infty} \binom{2n}{n} \binom{2n-2}{n-1} \frac{2n}{16^n(2n-1)^2} = \frac{G}{\pi} \tag{5}$$

which is the required second conclusion as stated in Theorem 2. Subtracting (4) and (5) and simplifying we meet the announced closed form

$$\sum_{n=1}^{\infty} \binom{2n}{n} \binom{2n-2}{n-1} \frac{1-n}{2^{4n-1}(2n-1)^2} = \frac{G}{\pi} - \frac{1}{\pi},$$

which completes the first part of Theorem 2.

Next, we present the proof of the Theorem 3.

Proof. We begin with Lemma 3 where we set $x = \sin y$ and integrating from 0 to $\pi/2$ gives us

$$\sum_{n=1}^{\infty} \frac{n}{16^n(2n-1)^2(2n+1)} \binom{2n}{n}^2 = \frac{1}{4\pi} + \frac{1}{2\pi} \int_0^{\frac{\pi}{2}} y \sin y dy - \frac{G}{2\pi}.$$

We manipulate the results; $\int_0^{\pi/2} \cos y dy = 1$ and integral relation of G to yield right hand quantity. Applying integration by parts, we easily show that

$$\int_0^{\frac{\pi}{2}} y \sin y dy = \int_0^{\frac{\pi}{2}} \cos y dy = 1.$$

On plugging the result back to the previous equation and performing simplification, proves the conclusion.

5 Some additional series

Based on the key notion (introduced lemmas), we have simple Ramanujan-like formula for $1/\pi$ and ratio of G/π:

$$\sum_{n=1}^{\infty}\frac{n^2}{(2n-1)^2(2n+1)16^n}\binom{2n}{n}^2=\frac{G}{4\pi}+\frac{1}{8\pi},$$
$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2(2n+1)16^n}\binom{2n}{n}\binom{2n+1}{n+1}=\frac{4}{9\pi},$$
$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2(2n+1)(2n+3)16^n}\binom{2n}{n}^2=\frac{31}{128\pi}-\frac{13G}{64\pi}$$
$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2(2n+1)(2n+4)16^n}\binom{2n}{n}^2=\frac{539}{2700\pi}-\frac{G}{6\pi},$$
$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2(2n+1)(2n+5)16^n}\binom{2n}{n}^2=\frac{1345}{9216\pi}-\frac{59G}{512\pi}.$$

We welcome interested readers to pursue the result. In view of above conclusions, we can observe a definite pattern being followed by the last three series, thus, if $j\geq 3$ is a positive integer, then the following conjecture holds:

$$\sum_{n=1}^{\infty}\frac{n}{(2n-1)^2(2n+1)(2n+j)16^n}\binom{2n}{n}^2=\frac{a_j}{\pi}-\frac{b_jG}{\pi},$$

where determining the general expressions of a_j and b_j remain open. Likewise, based on the lemmas we introduced, we claim that the following holds:

$$\sum_{n=1}^{\infty}\frac{{}_2F_1\left(\frac{1}{2},n;n+1;y^2\right)}{(2n-1)^24^n}\binom{2n}{n}y^{2n}=\frac{\arcsin^2 y}{2} \tag{6}$$

where $y\in[-1,1]$ and ${}_2F_1(\cdot)$ denotes Gauss hypergeometric function. At $y=1/2$, we have

$$12\sum_{n=1}^{\infty}\frac{{}_2F_1\left(\frac{1}{2},n;n+1;\frac{1}{4}\right)}{16^n(2n-1)^2}\binom{2n}{n}=\zeta(2).$$

If $\alpha = \sqrt{8 - \sqrt{10 - 2\sqrt{5}} - \sqrt{3} - \sqrt{15}}$, then setting $y = \alpha/4$ in [6], we have a curious series

$$1200 \sum_{n=1}^{\infty} \frac{{}_2F_1\left(\frac{1}{2}, n; n+1; \frac{\alpha^2}{16}\right)}{4^n(2n-1)^2} \binom{2n}{n} \left(\frac{\alpha}{4}\right)^{2n} = \zeta(2).$$

Here notation $\zeta(2) = \sum_{n=1}^{\infty} 1/n^2$ is Riemann zeta function. Utilizing the lemmas, we can investigate more other sorts of series resulting the ratio, G/π and $1/\pi$.

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New Inequalities involving Fibonacci and Lucas numbers

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Abstract

In this paper, some discrete inequalities are presented. Using them, new inequalities for Lucas and Fibonacci numbers are obtained.

1 Introduction

The sequences $(F_n)_{n\geq 0}$ of Fibonacci numbers and $(L_n)_{n\geq 0}$ Lucas numbers are defined by $F_0 = 0, F_1 = 1$ and for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$, and $L_0 = 2, L_1 = 1$ and for all $n \geq 2$, $L_n = L_{n-1} + L_{n-2}$, respectively. In the last years, a lot of inequalities involving the elements of these sequences have appeared in the literature ([1],[4],[2], [3], [5],[6]). In this paper, and identity and some discrete inequalities are used to derive new inequalities involving Fibonacci and Lucas numbers.

2 Basic results

Hereafter, we present three results that will be used later on. We begin with the following identity.

Lemma 1. *Let $n \geq 3$ be a positive integer. If a_i are positive reals for $1 \leq i \leq n$, then it holds*

$$\sum_{i=2}^{n} a_i \left(\sum_{j=1}^{i-1} a_j \right)^{-1} \left(\sum_{j=1}^{i} a_j \right)^{-1} = a_1^{-1} \sum_{i=2}^{n} a_i \left(\sum_{i=1}^{n} a_i \right)^{-1}.$$

Proof. We argue by mathematical induction on $n \geq 3$. For $n = 3$, we have

$$\frac{a_2}{a_1(a_1 + a_2)} + \frac{a_3}{(a_1 + a_2)(a_1 + a_2 + a_3)} = \frac{a_2 + a_3}{a_1(a_1 + a_2 + a_3)}$$

that trivially holds after clearing denominators. Suppose that the identity holds for $n \geq 3$. That is,

$$\sum_{i=2}^{n} a_i \left(\sum_{j=1}^{i-1} a_j \right)^{-1} \left(\sum_{j=1}^{i} a_j \right)^{-1} = a_1^{-1} \sum_{i=2}^{n} a_i \left(\sum_{i=1}^{n} a_i \right)^{-1},$$

and we have to see that it also holds for $n + 1$. Indeed,

$$\begin{aligned}
\sum_{i=2}^{n+1} a_i \left(\sum_{j=1}^{i-1} a_j \right)^{-1} \left(\sum_{j=1}^{i} a_j \right)^{-1} &= \sum_{i=2}^{n} a_i \left(\sum_{j=1}^{i-1} a_j \right)^{-1} \left(\sum_{j=1}^{i} a_j \right)^{-1} \\
&\quad + a_{n+1} \left(\sum_{i=1}^{n} a_i \right)^{-1} \left(\sum_{i=1}^{n+1} a_i \right)^{-1} \\
&= \left(\sum_{i=2}^{n} a_i \right) a_1^{-1} \left(\sum_{i=1}^{n} a_i \right)^{-1} + a_{n+1} \left(\sum_{i=1}^{n} a_i \right)^{-1} \left(\sum_{i=1}^{n+1} a_i \right)^{-1} \\
&= \left[\left(\sum_{i=2}^{n} a_i \right)^2 + a_1 \left(\sum_{i=2}^{n} a_i \right) \right] \left(\sum_{i=1}^{n} a_i \right)^{-1} a_1^{-1} \left(\sum_{i=1}^{n+1} a_i \right)^{-1} \\
&\quad + \left[a_{n+1} \left(\sum_{i=2}^{n} a_i \right) + a_1 a_{n+1} \right] \left(\sum_{i=1}^{n} a_i \right)^{-1} a_1^{-1} \left(\sum_{i=1}^{n+1} a_i \right)^{-1} \\
&= \left(\sum_{i=2}^{n+1} a_i \right) a_1^{-1} \left(\sum_{i=1}^{n+1} a_i \right)^{-1}
\end{aligned}$$

Lemma 2. *Let $n \geq 3$ be a positive integer. If a_i are positive reals for $1 \leq i \leq n$, then it holds*

$$\left(\sum_{i=2}^{n} a_i\right)\left(2\sum_{j=1}^{i-1} a_j + a_i\right)^{-2} \leq \left(\sum_{i=2}^{n} a_i\right)\left(4a_1\sum_{i=1}^{n} a_i\right)^{-1}.$$

Proof. Putting $x_i = a_1 + \ldots + a_i$ and $y_i = a_1 + \ldots + a_i + a_{i+1}$ for $1 \leq i \leq n-1$ in the well-known inequality $\dfrac{1}{xy} \geq \dfrac{4}{(x+y)^2}$, and after multiplication by a_{i+1} of both sides of the inequality, we obtain

$$\begin{aligned}
\frac{a_2}{a_1(a_1+a_2)} &\geq \frac{4a_2}{(2a_1+a_2)^2},\\
\frac{a_3}{(a_1+a_2)(a_1+a_2+a_3)} &\geq \frac{4a_3}{(2a_1+2a_2+a_3)^2},\\
&\vdots\\
\frac{a_n}{(a_1+\ldots+a_{n-1})(a_1+\ldots+a_n)} &\geq \frac{4a_n}{(2a_1+\ldots+2a_{n-1}+a_n)^2}.
\end{aligned}$$

Adding up the preceding, yields

$$\begin{aligned}
\left(\sum_{i=2}^{n} 4a_i\right)\left(2\sum_{j=1}^{i-1} a_j + a_i\right)^{-2} &\leq \sum_{i=2}^{n}\left[a_i\left(\sum_{j=1}^{i-1} a_j\right)^{-1}\left(\sum_{j=1}^{i} a_j\right)^{-1}\right]\\
&= \left(\sum_{i=2}^{n} a_i\right)a_1^{-1}\left(\sum_{i=1}^{n} a_i\right)^{-1}
\end{aligned}$$

from which the statement follows.

Lemma 3. *Let x, y, z be positive real numbers. Then, it holds*

$$\frac{x^4y + y^4z + z^4x}{xy^4 + yz^4 + zx^4} \leq 1.$$

Proof. The inequality claimed is equivalent to $x^4y + y^4z + z^4x \leq xy^4 + yz^4 + zx^4$. WLOG we may assume that $0 < x \leq y \leq z$.

Then, $\frac{1}{2}(y-x)(z-y)(z-x)((x+y)^2+(y+z)^2+(z+x)^2) \geq 0$ or $(x^2(z-y)+y^2(x-z)+z^2(y-x))(x^2+y^2+z^2+xy+yz+zx) \geq 0$ which is equivalent to

$$\big(x^2(z-y)+y^2(x-z)+z^2(y-x)\big)(x^2+y^2+z^2+xy+yz+zx) \geq 0.$$

Finally, after some computations, we obtain

$$x^4(z-y)+y^4(x-z)+z^4(y-x) \geq 0 \Leftrightarrow x^4y+y^4x+z^4x \geq x^4y+y^4z+z^4x.$$

Equality holds when $x=y=z$, and the proof is complete.

Lemma 4. *Let $n \geq 2$ be an integer number. If $a_i, b_i > 0, 1 \leq i \leq n$, then*

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\ldots+\frac{a_n^2}{b_n} \geq \frac{(a_1+a_2+\ldots+a_n)^2}{b_1+b_2+\ldots+b_n} \text{ (Bergstrom)}$$

Proof. For $n=2$ we have

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2} \geq \frac{(a_1+a_2)^2}{b_1+b_2} \Leftrightarrow (b_1+b_2)(a_1^2b_2+a_2^2b_1) \geq b_1b_2(a_1+a_2)^2$$

$(a_1b_2-a_2b_1)^2 \geq 0$ (true!). Equality holds for$\frac{a_1}{b_1}=\frac{a_2}{b_2}$.

Using the preceding, yields

$$\frac{a_1^2}{b_1}+\frac{a_2^2}{b_2}+\ldots+\frac{a_n^2}{b_n} \geq \frac{(a_1+a_2)^2}{b_1+b_2}+\frac{a_3^2}{b_3}+\ldots+\frac{a_n^2}{b_n} \geq \ldots \geq \frac{(a_1+a_2+\ldots+a_n)^2}{b_1+b_2+\ldots+b_n}.$$

This completes the proof.

3 Applications

In the sequel, some inequalities will be given. We begin with

Problem 1. *Let $n \geq 2$ be an integer number. Prove that*

$$\frac{F_2}{(F_5-2)^2}+\frac{F_3}{(F_6-2)^2}+\ldots+\frac{F_n}{(F_{n+3}-2)^2} \leq \frac{L_{n+3}+L_{n+1}-5L_0}{5F_3^2(F_{n+2}-F_2)}.$$

Solution. Let $S_n = F_1 + F_2 + \ldots + F_n = F_{n+2} - F_2$. Then, for $1 \leq i \leq n$ we have

$$F_{i+3} - 2F_2 = F_{i+2} - F_{i+1} - 2F_2 = S_i + S_{i-1}.$$

In particular,

$$\begin{aligned} F_5 - 2F_2 &= S_2 + S_1 = F_1 + F_2 + F_3 = 2F_1 + F_2, \\ F_6 - 2F_2 &= S_3 + S_2 = 2F_1 + 2F_2 + F_3, \\ &\vdots \\ F_{n+3} - 2F_2 &= S_n + S_{n-1} = 2F_1 + 2F_2 + \ldots + 2F_{n-1} + F_n. \end{aligned}$$

For $1 \leq i \leq n$ we set $a_i = F_i$ in Lemma 2, and we obtain

$$\frac{F_2}{(F_5-2)^2} + \frac{F_3}{(F_6-2)^2} + \ldots + \frac{F_n}{(F_{n+3}-2)^2}$$

$$= \frac{F_2}{(2F_1+F_2)^2} + \frac{F_3}{(2F_1+2F_2+F_3)^2} + \ldots + \frac{F_n}{(2F_1+2F_2+\ldots+2F_{n-1}+F_n)^2}$$

$$\leq \frac{F_2+F_3+\ldots+F_n}{4F_1(F_1+F_2+\ldots+F_n)} = \frac{F_{n+2}-F_3}{F_3^2(F_{n+2}-F_2)}.$$

Since $5F_n = L_{n+1} + L_{n-1}$, as can be easily proven, we have $F_{n+2} - F_3 = \frac{1}{5}(L_{n+3} - L_{n+1} - 5L_0)$ and

$$\frac{F_{n+2}-F_3}{F_3^2(F_{n+2}-F_2)} = \frac{L_{n+3}+L_{n+1}-5L_0}{5F_3^2(F_{n+2}-F_2)}.$$

Hence,

$$\frac{F_2}{(F_5-2)^2} + \frac{F_3}{(F_6-2)^2} + \ldots + \frac{F_n}{(F_{n+3}-2)^2} \leq \frac{L_{n+3}+L_{n+1}-5L_0}{5F_3^2(F_{n+2}-F_2)}.$$

Likewise, putting $a_i = Li$ for $1 \leq i \leq n$, it holds

$$\frac{L_2}{(L_5-2)^2} + \frac{L_3}{(L_6-2)^2} + \ldots + \frac{L_n}{(L_{n+3}-2)^2} \leq \frac{L_{n+2}-L_3}{L_1L_3(L_{n+2}-L_2)}$$

on account that $L_1 + L_2 + \ldots + L_n = L_{n+2} - L_2$.

Problem 2. *Let $\{a_n\}_{n\geq 1}$ be an arithmetic progression with $a_1 > 0$ and $d > 0$. Then it holds*

$$\frac{1}{\sqrt{a_1F_1}}+\frac{1}{\sqrt{a_2F_2}}+\ldots+\frac{1}{\sqrt{a_nF_n}} \geq \frac{2n^2}{n+a_nF_{n+2}-d(F_{n+3}-3)-a_1}$$

Solution. On account of AM-GM inequality, we have for all $x > 0$, $\sqrt{x} \leq \frac{x+1}{2}$ and $\frac{1}{\sqrt{x}} \geq \frac{2}{x+1}$. Using it and Bergstrom's inequality, yields

$$\frac{1}{\sqrt{a_1F_1}}+\frac{1}{\sqrt{a_2F_2}}+\ldots+\frac{1}{\sqrt{a_nF_n}} \geq \frac{2}{1+a_1F_1}+\frac{2}{1+a_2F_2}+\ldots+\frac{2}{1+a_nF_n}$$

$$\geq \frac{2n^2}{n+a_1F_1+a_2F_2+\ldots+a_nF_n}$$

Now, using the definition of Fibonacci numbers, we have $a_k \cdot F_k = (a_1+(d-1)r)F_k = (a_1-d)F_k + kdF_k$. But,

$$k\cdot F_k = k(F_{k+2}-F_{k+1}) = (k+2)F_{k+2}-(k+1)F_{k+1}-2(F_{k+2}-F_{k+1})-F_{k+1}$$

$$= (k+2)F_{k+2}-(k+1)F_{k+1}-2(F_{k+2}-F_{k+1})-(F_{k+3}-F_{k+2}).$$

Therefore,

$$\sum_{k=1}^{n} k\cdot F_k = \sum_{k=1}^{n}[(k+2)F_{k+2}-(k+1)F_{k+1}]-2\sum_{k=1}^{n}(F_{k+2}-F_{k+1})-\sum_{k=1}^{n}(F_{k+3}-F_{k+2})$$

$$= (n+2)F_{n+2}-2F_2-2(F_{n+2}-F_2)-(F_{n+3}-F_3) = nF_{n+2}-F_{n+3}+F_3$$

and hence,

$$\begin{aligned}\sum_{k=1}^{n} a_k\cdot F_k &= (a_1-d)\sum_{k=1}^{n}F_k + d\sum_{k=1}^{n}k\cdot F_k\\ &= (a_1-d)(F_{n+2}-F_2)+d\,(n\cdot F_{n+2}-F_{n+3}+F_3)\\ &= a_nF_{n+2}-d\,(F_{n+3}-F_3-F_2)-a_1F_2\end{aligned}$$

from which the desired inequality follows.

Problem 3. *Let $n \geq 2$ be an integer number. Then, it holds*

$$\frac{1}{\sqrt{3F_1}}+\frac{1}{\sqrt{4F_2}}+\ldots+\frac{1}{\sqrt{(n+2)F_n}} \geq \frac{2n^2}{(1+F_{n+1})n+F_n}.$$

Solution. Putting $a_1 = 3, d = 1$ in last problem, and taking into account that $3F_1 + 4F_2 + \ldots + (n+2)F_n = (1 + F_{n+1})n + F_n$, we have

$$\frac{1}{\sqrt{3F_1}} + \frac{1}{\sqrt{4F_2}} + \ldots + \frac{1}{\sqrt{(n+2)F_n}} \geq \frac{2}{1+3F_1} + \frac{2}{1+4F_2} + \ldots + \frac{2}{1+(n+2)F_n}$$

$$\geq \frac{2n^2}{n + 3F_1 + 4F_2 + \ldots + (n+2)F_n}$$

and then the desired inequality follows.

Putting $a_1 = 2, d = 1$ in problem 2, and taking into account that

$$2L_1 + 3L_2 + \ldots + nL_n = \sum_{k=1}^{n} (k+1)L_k = (n+1)L_{n+2} - (L_{n+3} - L_3 - L_2) - 2L_2$$

$$= nL_{n+2} - L_{n+1} + 1,$$

we have that it holds

$$\frac{1}{\sqrt{2L_1}} + \frac{1}{\sqrt{3L_2}} + \ldots + \frac{1}{\sqrt{nL_n}} \geq \frac{2n^2}{(1 + L_{n+2})n - L_n + 1}.$$

Problem 4. *Let $n \geq 2$ be an integer number. Prove that*

$$\sqrt{\frac{F_1^2 + F_2^2}{F_1}} + \sqrt{\frac{F_1^2 + F_1^2 + F_3^2}{F_2}} + \ldots + \sqrt{\frac{F_1^2 + F_2^2 + \ldots + F_n^2}{F_n}}$$

$$\geq \frac{2n^2 F_{n+2}}{(n+1)F_{n+2} - F_1}.$$

Solution. Since for all $i \geq 0$ is $F_{i+2} = F_{i+1} + F_i$, then $F_{i+2} - F_i = F_{i+1}$ and $F_{i+2} \cdot F_{i+1} - F_{i+1} \cdot F_i = F_{i+1}^2$. Adding up the preceding identities, yields $F_2^2 + F_3^2 + \ldots + F_{k+1}^2 = F_{k+2} \cdot F_{k+1} - F_1 \cdot F_2$ or $F_1^2 + F_2^2 + F_3^2 + \ldots + F_{k+1}^2 = F_{k+2} \cdot F_{k+1}$ from which it follows

$$\frac{1}{F_1^2 + F_2^2 + F_3^2 + \ldots + F_{k+1}^2} = \frac{1}{F_{k+2} \cdot F_{k+1}}$$

and, after multiplication by F_k, we get

$$\frac{F_k}{F_1^2+F_2^2+F_3^2+\ldots+F_{k+1}^2}=\frac{F_{k+2}-F_{k+1}}{F_{k+2}\cdot F_{k+1}}.$$

Adding up the preceding identities, we obtain

$$\sum_{k=1}^{n}\frac{F_k}{F_1^2+F_2^2+F_3^2+\ldots+F_{k+1}^2}=1-\frac{1}{F_{n+2}}.$$

On account of AM-GM inequality, we have for all $x>0$, $\sqrt{x}\leq\frac{x+1}{2}$ and $\frac{1}{\sqrt{x}}\geq\frac{2}{x+1}$. Using it, yields

$$\begin{aligned}
&\sqrt{\frac{F_1^2+F_2^2}{F_1}}+\sqrt{\frac{F_1^2+F_1^2+F_3^2}{F_2}}+\ldots+\sqrt{\frac{F_1^2+F_2^2+\ldots+F_n^2}{F_n}}\\
&\geq\frac{2}{1+\frac{F_1}{F_1^2+F_2^2}}+\frac{2}{1+\frac{F_2}{F_1^2+F_2^2+F_3^2}}+\ldots+\frac{2}{1+\frac{F_n}{F_1^2+F_2^2+\ldots+F_{n+1}^2}}\\
&\overset{Bergstrom}{\geq}\frac{2n^2}{n+\frac{F_1}{F_1^2+F_2^2}+\frac{F_2}{F_1^2+F_2^2+F_3^2}+\ldots+\frac{F_n}{F_1^2+F_2^2+\ldots+F_{n+1}^2}}\\
&=\frac{2n^2}{n+1-\frac{1}{F_{n+2}}}=\frac{2n^2F_{n+2}}{(n+1)F_{n+2}-1}.
\end{aligned}$$

Problem 5. *Let m,n be positive integers with $n\geq 2$. Then, it holds*

$$2\sum_{n=1}^{m}\sum_{i=1}^{n}\frac{F_1^4F_{i+1}+F_{i+1}^4F_{i+2}+F_{i+2}^4F_i}{F_iF_{i+1}^4+F_{i+1}F_{i+2}^4+F_{i+2}F_i^4}\leq m(m+1).$$

Proof. Setting $x=F_i, y=F_{i+1}, z=F_{i+2}$, for $1\leq i\leq n$ in Lemma 3, we have

$$\sum_{i=1}^{n}\frac{F_1^4F_{i+1}+F_{i+1}^4F_{i+2}+F_{i+2}^4F_i}{F_iF_{i+1}^4+F_{i+1}F_{i+2}^4+F_{i+2}F_i^4}\leq\sum_{k=1}^{n}1=n.$$

Now, we have

$$\sum_{n=1}^{m}\sum_{i=1}^{n}\frac{F_1^4F_{i+1}+F_{i+1}^4F_{i+2}+F_{i+2}^4F_i}{F_iF_{i+1}^4+F_{i+1}F_{i+2}^4+F_{i+2}F_i^4}\leq\sum_{n=1}^{m}n=\frac{m(m+1)}{2}$$

from which the statement follows. □

Likewise, putting $x = L_i, y = L_{i+1}, z = L_{i+2}$, for $1 \leq i \leq n$, we have

$$2\sum_{n=1}^{m}\sum_{i=1}^{n}\frac{L_1^4L_{i+1}+L_{i+1}^4L_{i+2}+L_{i+2}^4L_i}{L_iL_{i+1}^4+L_{i+1}L_{i+2}^4+L_{i+2}L_i^4} \leq m(m+1), \forall n, m \in \mathbb{N}, n \geq 2.$$

Problem 6. *Let $n \geq 2$ be an integer number. Prove that*

$$1+\frac{1}{n^2}\sum_{1\leq i<j\leq n}\frac{\left(\sqrt{L_iL_{j+1}}-\sqrt{L_{i+1}L_j}\right)^2}{L_iL_j} < \frac{1}{n}\sum_{k=1}^{n}\frac{L_{k+1}}{L_k}$$

Here the subscripts are taken modulo n.

Solution. The inequality claimed may be written in the most convenient form

$$1+\frac{1}{n^2}\sum_{1\leq i<j\leq n}\frac{\left(\sqrt{\frac{L_i}{L_{i+1}}}-\sqrt{\frac{L_j}{L_{j+1}}}\right)^2}{\frac{L_iL_j}{L_{i+1}L_{j+1}}} < \frac{1}{n}\sum_{k=1}^{n}\frac{L_{k+1}}{L_k}$$

Putting $a_i = \dfrac{L_i}{L_{i+1}}$, $(1 \leq i \leq n)$ and taking all the subscripts modulo n, we have

$$a_1\,a_2\,\ldots\,a_n = \frac{L_1}{L_2}\frac{L_2}{L_3}\ldots\frac{L_n}{L_1} = 1.$$

We claim that if $a_1, a_2, \ldots, a_n$ are $n \geq 2$ positive real numbers which product is one, then it holds

$$1+\frac{1}{n^2}\sum_{1\leq i<j\leq n}\frac{(\sqrt{a_i}-\sqrt{a_j})^2}{a_ia_j} \leq \frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}$$

Indeed, for $x_k \geq 0$, $(1 \leq k \leq n)$ we have

$$\frac{1}{n}\sum_{1\leq i<j\leq n}(x_i-x_j)^2 = \sum_{k=1}^{n}x_k^2-\frac{1}{n}\left(\sum_{k=1}^{n}x_k\right)^2 \leq \sum_{k=1}^{n}x_k^2-\frac{1}{n}\left(n\sqrt[n]{\prod_{k=1}^{n}x_k}\right)^2$$

on account that

$$\frac{1}{n}\sum_{k=1}^{n} x_k \geq \sqrt[n]{\prod_{k=1}^{n} x_k} \Leftrightarrow \sum_{k=1}^{n} x_k \geq n\sqrt[n]{\prod_{k=1}^{n} x_k}$$

Putting $x_k = \sqrt{\frac{1}{a_k}}$, $(1 \leq k \leq n)$ in the preceding and taking into account the constrain, we get

$$\sum_{k=1}^{n} \frac{1}{a_k} - n \geq \frac{1}{n} \sum_{1\leq i<j\leq n} \left(\sqrt{\frac{1}{a_i}} - \sqrt{\frac{1}{a_j}}\right)^2$$

Dividing both terms by n we obtain

$$\frac{1}{n}\sum_{k=1}^{n} \frac{1}{a_k} - 1 \geq \frac{1}{n^2} \sum_{1\leq i<j\leq n} \left(\sqrt{\frac{1}{a_i}} - \sqrt{\frac{1}{a_j}}\right)^2$$

or

$$\frac{1}{n}\sum_{k=1}^{n} \frac{1}{a_k} - 1 \geq \frac{1}{n^2} \sum_{1\leq i<j\leq n} \frac{(\sqrt{a_i} - \sqrt{a_j})^2}{a_i a_j}$$

and the claim follows. Equality holds for $a_1 = a_2 = \ldots = a_n = 1$.

As a consequence

$$1 + \frac{1}{n^2} \sum_{1\leq i<j\leq n} \frac{\left(\sqrt{\frac{L_i}{L_{i+1}}} - \sqrt{\frac{L_j}{L_{j+1}}}\right)^2}{\frac{L_i L_j}{L_{i+1} L_{j+1}}} \leq \frac{1}{n}\sum_{k=1}^{n} \frac{L_{k+1}}{L_k}$$

from which the statement follows. Since all the Lucas numbers are distinct, then the equality does not occur and this finish the proof.

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On Muirhead's Inequality

Antoine Mhanna

Abstract

We revisit Muirhead's inequality together with some related known inequalities. We give a short proof to the Power-Mean inequality and Minkowski's determinant inequality.

1 Introduction and preliminaries

The first section introduces some notations together with known results and inequalities starting by Muirhead's inequality. For $a = (a_1, \dots, a_n)$, $x = (x_1, \dots, x_n)$ two real n-tuples and σ a permutation in S_n, we denote by $[x^a_\sigma]$ the product $x^{a_1}_{\sigma(1)} x^{a_2}_{\sigma(2)} \cdots x^{a_n}_{\sigma(n)}$ whenever the product is well defined.

The vector $\tilde{a}_j$ is the vector a without the a_j entry and $[x^{\tilde{a}_j}_\sigma]$ is the product without $x^{a_j}_{\sigma(j)}$ that equals to $\dfrac{x^{a_1}_{\sigma(1)} x^{a_2}_{\sigma(2)} \cdots x^{a_n}_{\sigma(n)}}{x^{a_j}_{\sigma(j)}}$.

In general we only consider nonnegative entry vectors except mentioning otherwise.

Throughout the paper let $a = (a_1, \dots, a_n) \in \mathbb{R}^{n+}$ and $b = (b_1, \dots, b_n) \in \mathbb{R}^{n+}$ two vectors with nonnegative entries arranged in decreasing order: $a_1 \geq \cdots \geq a_n$ and $b_1 \geq \cdots \geq b_n$.

We say b majorizes a or $a \prec b$ if and only if

$$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i \quad \text{for all } k, 1 \leq k \leq n \text{ and } \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.$$

Muirhead's inequality: If $a \prec b$ then for any nonnegative n-tuple x, it holds:

$$\sum_{\sigma \in S_n} [x_\sigma^a] \leq \sum_{\sigma \in S_n} [x_\sigma^b]. \tag{1}$$

The inequality was first proved for integer exponents in [8], see also ([5],[7]). Mainly all proofs of (1) use induction and rely on the following general property.

Proposition 1. *Let $a \prec b$ with precisely m different entries, then there exist at most $m-2$ vectors $c^{(l)}$ $l \geq 1$, such that $a = c^{(0)} \prec c^{(1)} \prec c^{(2)} \prec \cdots \prec c^{(j)} \prec b = c^{(j+1)}$ where $j \leq m-1$ and for $l \geq 0$, $c^{(l)}$ differs from $c^{(l+1)}$ by only two entries.*

Proposition 1 is an important result that can be used to prove the next two classical statements.

Corollary 1 ([7], Chap. 5). *Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two nonnegative real vectors arranged in decreasing order,*

1. *If $a \prec b$ then $\prod_{i=k}^{n} a_i \geq \prod_{i=k}^{n} b_i$ for every k, $1 \leq k \leq n$.*
2. *If $\prod_{i=1}^{n} a_i \leq \prod_{i=1}^{n} b_i$ then $\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i$.*

Proof. For *(1)* the case $n = 2$ is elementary, applying Proposition 1 we get $\prod_{i=k}^{n} c_i^{(l)} \geq \prod_{i=k}^{n} c_i^{(l+1)}$ for every k and l, $1 \leq k \leq n$, $0 \leq l \leq j$. To prove *(2)* we go by induction, the result is true for $n = 1$ assuming it holds for $n-1$ variables and $\sum_{i=1}^{n} a_i > \sum_{i=1}^{n} b_i$ we have

$$\sum_{i=1}^{n-1} a_i \leq \sum_{i=1}^{n-1} b_i \leq \sum_{i=1}^{n} b_i < \sum_{i=1}^{n-1} a_i + a_n.$$

So there exist $h \leq a_n$ such that $\sum_{i=1}^{n} b_i = \sum_{i=1}^{n-1} a_i + a_n - h$ applying *(1)* for $k = 1$ to $a = (a_1, a_2, \ldots, a_n - h)$ and b yields to a contradiction.

Next we present a proof of Muirhead's inequality similar to other known proofs see ([5], [7]). Apparently the only difference is that here the sequence $c^{(l)}$ (from Proposition 1) is reversely constructed, see Example 1.

Proof. [Proof of (1)] The case $n = 2$ is proved by inspection: $(a_1, a_2) \prec (b_1, b_2)$ and $b_2 \leq a_2 \leq a_1 \leq b_1$ with $a_2 - b_2 = b_1 - a_1 = r$ we get:

$$x_1^{b_2} x_2^{b_2} (x_1^{a_1-b_2} x_2^{a_2-b_2} + x_2^{a_1-b_2} x_1^{a_2-b_2} - x_1^{b_1-b_2} - x_2^{b_1-b_2}) = x_1^{b_2} x_2^{b_2} \cdot T$$

and

$$T = (x_2^r - x_1^r)(x_1^{a_1-b_2} - x_2^{a_1-b_2}) \leq 0.$$

Since the sum is symmetric in x_i, the inequality is true if for some j, $a_j = b_j$ from the previous cases (lower dimension); this is because we can write (1) as

$$\sum_{i=1}^{n} x_i^{a_j} \Big[\sum_{\sigma, \sigma(j)=i} [\tilde{x_\sigma^{b_j}}] - \sum_{\sigma, \sigma(j)=i} [\tilde{x_\sigma^{a_j}}] \Big] \geq 0.$$

Thus we only need to give a vector $c = (c_1, \ldots, c_n)$ such that $a \prec c \prec b$ with $a_k = c_k$ for some k and $c_h = b_h$ for some index h. We assume $n \geq 3$ and $a_i \neq b_i$ for every i.

Take p to be minimal with $a_i - b_i = r_i > 0$ for all $p \leq i \leq n$ and take q to be minimal with $a_i - b_i = d_i < 0$ for all $q \leq i \leq p - 1$.

If $|r_n| \leq |d_q|$ set $c = \begin{cases} c_q = a_q + r_n \\ c_n = b_n \\ c_i = a_i \text{ otherwise,} \end{cases}$ if $|r_n| \geq |d_q|$ set

$$c = \begin{cases} c_n = a_n + d_q \\ c_q = b_q \\ c_i = a_i \text{ otherwise.} \end{cases}$$

Example 1. We give an example to illustrate the construction of sequence $c^{(l)}$, say $a = (18, 16, 6, 5, 3, 3)$ and $b = (20, 15, 8, 5, 2, 1)$. We have $a \prec b$ and omitting equal entries we may apply the previous proof to get successively $c^{(1)} = (18, 16, 8, 5, 3, 1)$, $c^{(2)} = (19, 16, 8, 5, 2, 1)$.

2 Some applications

We first state some applications of (1) that recently appeared in Amer. Math. Monthly [1] and College Mathematics Journal [2] respectively.

Problem 12066: Let n and k be two integers greater than 1. Let A be an $n \times n$ positive definite Hermitian matrix. Prove:

$$(\det(A))^{1/n} \leq \left(\frac{(\mathrm{Tr}(A))^k - \mathrm{Tr}(A^k)}{n^k - n} \right)^{1/k}.$$

We first rewrite the inequality as:

$$\left(\sum_{i=1}^{n} a_i^n \right)^k - \sum_{i=1}^{n} a_i^{kn} \geq (n^k - n) \prod_{i=1}^{n} a_i^k$$

where the eigenvalue of A are $\lambda_i = a_i^n$. Using the multinomial expansion:

$$\left(\sum_{i=1}^{n} a_i^n \right)^k - \sum_{i=1}^{n} a_i^{kn} = \sum_{\substack{k_1+k_2+\cdots+k_n=k \\ 0 \leq k_t < k}} \binom{k}{k_1, \ldots, k_n} \prod_{t=1}^{n} a_t^{nk_t}$$

and by symmetry we see that the sequence of powers of the vector $l = (nk_1, \ldots, nk_n)$ majorizes the n-tuple $r = (k, \ldots, k)$ for any partition with $k_1 + \cdots + k_n = k$, in fact if $n \geq k$ we have nothing to prove if $n < k$, assume without loss of generality (upon rearranging the entries of l) that $k_1 > \cdots > k_n$ and there is j less than n such that $jk > n(k_1 + \cdots + k_j)$ so $n(k_{j+1} + \cdots + k_n) > (n-j)k$ which means that for some i, $j+1 \leq i \leq n$, $nk_i > k$ a contradiction since we rearranged the vector l in decreasing order and supposed $jk > n(k_1 + \cdots + k_j)$ where then for some $m \in [1, j]$, $nk_m < k$. The inequality holds with equality if and only if all eigenvalues are equal.

Another application of this inequality is the following:

Problem 1176: Let A be an $n \times n$ positive semidefinite Hermitian matrix having eigenvalues $(\lambda_1, \ldots, \lambda_n)$, $\lambda_1 \geq \cdots \geq \lambda_n$ and $\lambda_1 > 0$. For p and q nonnegative integers prove:

$$\frac{\mathrm{Tr}(A^p)+\mathrm{Tr}(A^{p+1})+\ldots\mathrm{Tr}(A^{p+q})}{\mathrm{Tr}(A^{p+1})+\mathrm{Tr}(A^{p+2})+\ldots\mathrm{Tr}(A^{p+q+1})}\leq\frac{n}{\mathrm{Tr}(A)}.$$

We want to prove

$$\frac{\sum\limits_{i=1}^{n}\lambda_i^p}{\sum\limits_{i=1}^{n}\lambda_i^{p+q}}\leq\frac{n}{\sum\limits_{i=1}^{n}\lambda_i^q}$$

for any nonnegative reals p and q, $(pq\neq 0)$ which implies the result by using the next property. This is just true by Muirhead's inequality as

$$\left(\sum_{i=1}^{n}\lambda_i^p\right)\left(\sum_{i=1}^{n}\lambda_i^q\right)\leq n\sum_{i=1}^{n}\lambda_i^{p+q}.$$

Equality occurs only if all eigenvalues are equal.

Proposition 2 ([3]). *Let $p_1,\ldots,p_n$ respectively $q_1,\ldots,q_n$ be n real respectively positive real numbers. If $m=\min\limits_i\left(\frac{p_i}{q_i}\right)$ and $M=\max\limits_i\left(\frac{p_i}{q_i}\right)$ then $m\leq\frac{p_1+\cdots+p_n}{q_1+\cdots+q_n}\leq M$.*

For sake of completeness we present a short proof of the classical Power-Mean inequality see [6] and the references therein;

Power-Mean Inequality: Let $a_1\geq a_2\geq\cdots\geq a_n\geq 0$, $\lambda\geq 1$ and for $1\leq i\leq n$, $\alpha_i\geq 0$ with $\sum_{i=1}^n\alpha_i=1$. Then

$$\sum_{i=1}^{n}\alpha_i a_i\leq\left(\sum_{i=1}^{n}\alpha_i a_i^\lambda\right)^{\frac{1}{\lambda}}.$$

Proof. Clearly the result holds for $n=1$ suppose it is true for any n and we want to prove it for $n+1$ variables. If for example $a_1=a_2$ the inequality holds by assumption, otherwise $a_1>a_2$ taking the derivative in a_1 of the function defined by

$$f(a_1)=\left(\sum_{i=1}^{n}\alpha_i a_i^\lambda\right)^{\frac{1}{\lambda}}-\sum_{i=1}^{n}\alpha_i a_i$$

we obtain

$$f'(a_1) = \alpha_1 \left(\frac{a_1^{\lambda-1}}{(\sum_{i=1}^n \alpha_i a_i^\lambda)^{\frac{\lambda-1}{\lambda}}} - 1 \right) \geq 0$$

and the inequality follows.

Along the same lines we can prove the Weighted Arithmetic Geometric inequality.

Weighted AM-GM Inequality: Let $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and for $1 \leq i \leq n$, $\alpha_i \geq 0$ with $\sum_{i=1}^n \alpha_i = 1$. We have

$$\sum_{i=1}^n \alpha_i a_i \geq \prod_{i=1}^n a_i^{\alpha_i}. \tag{2}$$

Another interesting question is to ask when we can replace symmetric sums by cyclic ones. For this we define the cyclic permutation $\mathfrak{c}$ over the set of indexes $(1, \ldots, n)$ by $\mathfrak{c}(i) = i + 1$. Of course all indexes are to be assumed $\pmod n$ and in $[1; n]$. We denote by $\mathfrak{c}^n$ the iteration of n cyclic permutation with the convention that $\mathfrak{c}^0$ is the identity permutation. For example:

$$x_{\mathfrak{c}^3(1)}^{15} x_{\mathfrak{c}^3(2)}^{8} x_{\mathfrak{c}^3(3)}^{7} x_{\mathfrak{c}^3(4)} = x_4^{15} x_1^8 x_2^7 x_3.$$

In the particular case when $b = (b_1, 0, \ldots, 0)$ we have:

Lemma 1. *Suppose* $a = (a_1, \ldots, a_n) \prec b = (b_1, 0, \ldots, 0)$ *and* $x = (x_1, \ldots, x_n)$ *with nonnegative entries then:*

$$\sum_{i=1}^n x_i^{b_1} \geq \sum_{i=0}^{n-1} [x_{\mathfrak{c}^i}^a].$$

Proof. By (2) we have:

$$\frac{a_1 x_{\mathfrak{c}^i(1)}^{b_1} + a_2 x_{\mathfrak{c}^i(2)}^{b_1} + \cdots + a_n x_{\mathfrak{c}^i(n)}^{b_1}}{b_1} \geq x_{\mathfrak{c}^i(1)}^{a_1} \cdots x_{\mathfrak{c}^i(n)}^{a_n},$$

summing over i we obtain the inequality.

Remark. If $a = \frac{p}{q}$ is a nonnegative rational number and q is odd, then for $x_1 \geq x_2$ it is easy to see that $x_1^a - x_2^a \geq 0$ if $x_1 \geq -x_2$.

Lemma 2. *Let $a = (a_1, a_2) \prec b = (b_1, 0)$ where a_1, a_2, b_1 are positive rational numbers with odd denominators and let $x = (x_1, \ldots, x_n)$ with $x_i + x_j \geq 0$ for every $i \neq j$, (x has at most one negative entry), then*

$$\sum_{i=1}^{n} x_i^{b_1} \geq \sum_{i=1}^{n} x_i^{a_1} x_{i+1}^{a_2},$$

where $x_{n+1} \equiv x_1$.

Proof. For $n = 2$, $x_1^{b_1} + x_2^{b_1} - x_1^{a_1}x_2^{a_2} - x_2^{a_1}x_1^{a_2} = (x_1^{a_2} - x_2^{a_2})(x_1^{a_1} - x_2^{a_1}) \geq 0$, suppose the result holds for $x = (x_1, \ldots, x_{n-1})$ we shall prove it for n variables. Assume without loss of generality x_1 to be the maximal variable (positive) and take the derivative of the function $f(x_1) = x_1^{b_1} - x_1^{a_1}x_2^{a_2} - x_n^{a_1}x_1^{a_2}$ to get by previous remark

$$f'(x_1) = a_1 x_1^{a_1 - 1}(x_1^{a_2} - x_2^{a_2}) + a_2 x_1^{a_2 - 1}(x_1^{a_1} - x_n^{a_1}) \geq 0 \quad (1).$$

In particular we only need to consider the case $x_1 = x_2$ or $x_1 = x_n$ but in either case the inequality reduces to the case of $n - 1$ variables which is true by induction.

The well known Minkowski's determinant inequality -see ([4], page 510)- states that for any A and B in $\mathbb{M}_n(\mathbb{C})$ positive semi-definite (Hermitian) we have

$$\mathrm{Det}(A + B) \geq \left(\mathrm{Det}(A)^{\frac{1}{n}} + \mathrm{Det}(B)^{\frac{1}{n}}\right)^n. \tag{3}$$

Denote by $\lambda_i(M)$, $1 \leq i \leq n$ the eigenvalues of the Hermitian positive semi-definite matrix M in decreasing order. For any k, $1 \leq k \leq n$, $\sum_{i=1}^{k} \lambda_i(M) = \|M\|_k$ is by definition the Ky-Fan k-norm of M see [4]. We can write for $1 \leq k \leq n$:

$$\sum_{i=1}^{k} \lambda_i(A + B) \leq \sum_{i=1}^{k} \lambda_i(A) + \sum_{i=1}^{k} \lambda_i(B) = \sum_{i=1}^{k} (\lambda_i(A) + \lambda_i(B))$$

and $\sum_{i=1}^{n} \lambda_i(A+B) = \sum_{i=1}^{n}(\lambda_i(A)+\lambda_i(B)) = \mathrm{Tr}(A+B)$. Setting $a_i = \lambda_i(A+B)$, $b_i = \lambda_i(A)+\lambda_i(B)$ and using Corollary 1 we have

$$\mathrm{Det}(A+B) \geq \prod_{i=1}^{n}(\lambda_i(A)+\lambda_i(B)) \geq \left(\mathrm{Det}(A)^{\frac{1}{n}} + \mathrm{Det}(B)^{\frac{1}{n}}\right)^n. \quad (4)$$

The last bound of (4) can be deduced from [9] and is proved as follows: Suppose $\mathrm{Det}(B) \neq 0$ (if not it is easy to conclude), dividing by $\mathrm{Det}(B)$ and letting $\dfrac{\lambda_i(A)}{\lambda_i(B)} = c_i$ we need to prove

$$\prod_{i=1}^{n}(c_i+1) \geq (\prod_{i=1}^{n} c_i^{\frac{1}{n}} + 1)^n,$$

equivalently

$$\sum_{k=0}^{n} e_k(c_1,\ldots,c_n) \geq \sum_{k=0}^{n} \binom{n}{k} \prod_{i=1}^{n} c_i^{\frac{k}{n}}$$

where $e_k(c_1,\ldots,c_n)$ is the elementary symmetric function of order k. Clearly $e_k(c_1,\ldots,c_n) \geq \binom{n}{k}\prod_{i=1}^{n} c_i^{\frac{k}{n}}$ with equality if and only if all c_i's are equal.

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The best constant in some geometric inequalities

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Abstract

The purpose of this article is to obtain the best constant that appear in some geometric inequalities.

1 Introduction

First recall the fundamental triangle theorem of Blundon (see [2], [3]):

Theorem 1. *For any triangle ABC, the inequality $s_1 \leq s \leq s_2$ holds, where s_1, s_2 represent the semiperimeter of two isosceles triangles $A_1B_1C_1$ and $A_2B_2C_2$ which have the same circumradius R and irradius r as the triangle ABC, where*

$$s_1^2 = 2R^2 + 10Rr - r^2 - 2\sqrt{R(R-2r)^3}$$
$$s_2^2 = 2R^2 + 10Rr - r^2 + 2\sqrt{R(R-2r)^3}.$$

Also recall the Blundon theorem for acute triangles (see [1])

Theorem 2. *For any acute triangle holds*

$s_1 \leq s \leq s_2$ *if* $2 \leq \dfrac{R}{r} \leq \sqrt{2}+1$ *and* $s_3 \leq s \leq s_2$ *if* $\dfrac{R}{r} \geq \sqrt{2}+1$, *where s_1, s_2 represent the semiperimeter of isosceles triangles from Theorem 1 and s_3 the semiperimeter of a right angle triangle, where $s_3 = 2R + r$.*

A geometric interpretation can be found in [5].

We will search the best constant k for the inequality of type

$$E(a,b,c,k) \geq 0 \text{ or } E(a,b,c,k) \leq 0 \tag{1}$$

where E is symmetric and homogeneous in a,b,c.
We reduce the inequality (1) at inequality of form

$$F(R,r,s) \geq k \text{ or } F(R,r,s) \leq k. \tag{2}$$

In (2), we consider R,r constant and s variable.
If we denote $x = R/r$ inequality (2) is equivalent to

$$F(x,1,s) \geq k \text{ or } F(x,1,s) \leq k \tag{3}$$

for all $x \geq 2$.
Let $f : [s_1, s_2] \to \mathbb{R}$, defined by $f(s) = F(x,1,s)$, $x \geq 2$.

For the inequality $F(x,1,s) \geq k$ or $f(s) \geq k$, we have:

- If f is increasing we obtain $f(s) \geq f(s_1) \geq k$, so $k \leq f(s_1) = h(x)$, for all $x \geq 2$. So the best constant is $k_1 = \inf\limits_{x \geq 2} h(x)$.
- If f is decreasing then $f(s) \geq f(s_2) \geq k$, hence $k \leq f(s_2) = g(x)$, for all $x \geq 2$, and the best constant is $k_2 = \inf\limits_{x \geq 2} g(x)$.

For inequality $F(x,1,s) \leq k$ or $f(s) \leq k$, we have:

- if f is increasing we have $f(s) \leq f(s_2) \leq k$ or $k \geq f(s_2) = u(x)$, for all $x \geq 2$.
  So the best constant is $k_3 = \sup\limits_{x \geq 2} u(x)$.
- If f is decreasing we have $f(s) \leq f(s_1) \leq k$ or $k \geq f(s_1) = v(x)$, for all $x \geq 2$, so the best constant is $k_4 = \sup\limits_{x \geq 2} v(x)$.

In the same way we proceed in the case of acute triangle using Theorem 2.

2 Applications

In the following we present some applications.

I. Find the best constant α such that the inequality

$$\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}+\frac{c^2}{(a+b)^2}\leq \alpha+\frac{3-4\alpha}{2}\,\frac{r}{R},$$

holds in every ABC.

We will use the following well-known identities:

$$\sum_{cyc}\frac{a}{b+c}=\frac{2s^2-2r^2-2Rr}{s^2+r^2+2Rr}$$

and

$$\sum_{cyc}\frac{bc}{(a+b)(a+c)}=\frac{s^2-2Rr+r^2}{s^2+2Rr+r^2}$$

So

$$\begin{aligned}\sum_{cyc}\left(\frac{a}{b+c}\right)^2&=\left(\sum_{cyc}\frac{a}{b+c}\right)^2-2\sum_{cyc}\frac{bc}{(a+b)(a+c)}\\&=\frac{2[s^4-(6r^2+4Rr)s^2+6R^2r^2+4Rr^3+r^4]}{(s^2+r^2+2Rr)^2}.\end{aligned}$$

We consider the function $f:(0,+\infty)\to\mathbb{R}$ defined by

$$\begin{aligned}f(t)&=\frac{2[t^2-(6r^2+4Rr)t+6R^2r^2+4Rr^3+r^4]}{(t+r^2+2Rr)^2}\quad\text{with}\\f'(t)&=\frac{4R[(2R+2r)t-5R^2r-6Rr^2-2r^3]}{(t+2Rr+r^2)^3}.\end{aligned}$$

From Gerretsen inequality we have $t=s^2\geq 16Rr-5r^2$. We will prove $16Rr-5r^2\geq\dfrac{5R^2r+6Rr^2+2r^2}{2R+2r}$ which is equivalent to

$$(2x+2)(16x-5)\geq 5x^2+6x+2,\ \forall\, x\geq 2.$$

This inequality is true if $x > \dfrac{2\sqrt{97}-8}{27} \simeq 0,43$, which is true since $x \geq 2$. We deduce that $f'(t) > 0$ and ; f is increasing. Then $f(t) \leq f(s_2^2)$, therefore

$$\sum_{cyc} \frac{a^2}{(b+c)^2} \leq \frac{29R^3 - 5R^2r - 8Rr^2 - 4r^3 + (7R+2r)\sqrt{R(R-2r)^3}}{2R(3R+2r)^2}.$$

The inequality from the statement is equivalent to

$$\frac{29x^3 - 5x^2 - 8x - 4 + (7x+2)\sqrt{x(x-2)^3}}{2x(3x+2)^2} \leq \alpha + \left(\frac{3}{2} - 2\alpha\right)\frac{1}{x},$$

for all $x \geq 2$, or

$$\begin{aligned}\alpha &\geq \frac{x}{x-2}\left[\frac{29x^3 - 5x^2 - 8x - 4 + (7x+2)\sqrt{x(x-2)^3}}{2x(3x+2)^2} - \frac{3}{2x}\right] \\ &= \frac{29x^2 + 26x + 8 + (7x+2)\sqrt{x(x-2)}}{2(3x+2)^2}, \quad \forall\, x \geq 2.\end{aligned}$$

So the best constant is

$$\alpha_0 = \sup_{x \geq 2} \frac{29x^2 + 26x + 8 + (7x+2)\sqrt{x(x-2)}}{2(3x+2)^2} = 2.$$

Remark 1. In every ABC it holds:

$$\frac{a^2}{(b+c)^2} + \frac{b^2}{(c+a)^2} + \frac{c^2}{(a+b)^2} + \frac{5r}{2R} \leq 2.$$

II. Find the best constant α such that the inequality

$$\frac{a^2}{(b+c)^2} + \frac{b^2}{(c+a)^2} + \frac{c^2}{(a+b)^2} \geq \alpha + \frac{3-4\alpha}{2}\,\frac{r}{R}$$

is true for every triangle ABC.

We saw in **I** that f is increasing. So

$$f(t) \geq f(s_1^2) = \frac{29R^3 - 5R^2r - 8Rr^2 - 4r^3 - (7R + 2r)\sqrt{R(R-2r)^3}}{2R(3R+2r)^2}$$

therefore

$$\frac{29x^3 - 5x^2 - 8x - 4 - (7x+2)\sqrt{x(x-2)^3}}{2x(3x+2)^2} \geq \alpha + \left(\frac{3}{2} - 2\alpha\right)\frac{1}{x},$$

for all $x \geq 2$, or

$$\alpha \leq \frac{29x^2 + 26x + 8 - (7x+2)\sqrt{x(x-2)}}{2(3x+2)^2}, \forall\, x \geq 2.$$

So the best constant is

$$\alpha_1 = \min_{x \geq 2} \frac{29x^2 + 26x + 8 - (7x+2)\sqrt{x(x-2)}}{2(3x+2)^2} = \frac{11}{9}.$$

Remark 2. In every triangle ABC it holds:

$$\frac{a^2}{(b+c)^2} + \frac{b^2}{(c+a)^2} + \frac{c^2}{(a+b)^2} + \frac{17r}{18R} \geq \frac{11}{9}.$$

III. Find the best positive constant α such that the inequality

$$\alpha\left(\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2}\right) + 8(1-\alpha)\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \geq 1,$$

holds in every ABC .

We use the know identities: $\sum_{cyc} \tan\frac{A}{2} = \frac{4R+r}{s}$, $\prod_{cyc} \sin\frac{A}{2} = \frac{r}{4R}$.
The inequality from statement may be written as

$$\alpha\left[\frac{(4R+r)^2}{s^2} - 2\right] + (8 - 8\alpha)\frac{r}{4R} \geq 1. \tag{4}$$

We consider the function $f:[s_1,s_2]\to\mathbb{R}$ defined by

$$f(s)=\alpha\left[\frac{(4R+r)^2}{s^2}-2\right]+(8-8\alpha)\frac{r}{4R},$$

which is decreasing in s with R,r fixed.
So from Blundon's inequality we obtain

$$f(s)\geq f(s_2) \tag{5}$$

From (4) and (5) it follows that $f(s_2)\geq 1$ or

$$\alpha\left[\frac{(4x+1)^2}{s_2^2(x)}-2\right]+\frac{2}{x}-\frac{2\alpha}{x}\geq 1,\ \forall\, x\geq 2,$$

where $s_2(x)=2x^2+10x-1-2\sqrt{x(x-2)^3}$.
We obtain after carrying out some computations

$$\alpha\geq u(x)=\frac{2x^2+10x-1+2\sqrt{x(x-2)^3}}{12x^2+8x-1-(4x+4)\sqrt{x(x-2)}},\ \forall\, x\geq 2$$

So $\alpha\geq\sup\limits_{x\geq 2}u(x)=\dfrac{1}{2}$. Therefore the best constant is $\alpha=\dfrac{1}{2}$.

Remark 3. If we take $\alpha=1/2$, we obtain

$$\sum_{cyc}\tan\frac{A}{2}+8\prod_{cyc}\sin\frac{A}{2}\geq 2,$$

which is the statement of a problem of Leon Banoff published in Crux Mathematicorum no 5 (1984).

IV. Find the best constants $\alpha,\beta,\gamma\geq -2$ such that the inequality

$$\left(\frac{r}{r_a}\right)^3+\left(\frac{r}{r_b}\right)^3+\left(\frac{r}{r_c}\right)^3\leq\frac{\alpha R+\beta r}{R+\gamma r},$$

is true in ever ytriangle ABC .

First we will prove that

$$\left(\frac{r}{r_a}\right)^3+\left(\frac{r}{r_b}\right)^3+\left(\frac{r}{r_c}\right)^3\le\frac{8R-13r}{8R+11r}. \tag{6}$$

We have

$$\left(\frac{r}{r_a}\right)^3+\left(\frac{r}{r_b}\right)^3+\left(\frac{r}{r_c}\right)^3=\frac{(s-a)^3+(s-b)^3+(s-c)^3}{s^3} \tag{7}$$
$$=\frac{9s^3-3s^2\sum a+3s\sum a^2-\sum a^3}{s^3}$$
$$=\frac{3s^3+6s(s^2-r^2-4Rr)-2s(s^2-6Rr-3r)}{s^3}=1-\frac{12Rr}{s^2}$$

From (6) and (7) it follows that we need to prove that

$$1-\frac{12Rr}{s^2}\le\frac{8R-13r}{8R+11r}\quad\text{or}\quad s^2\le\frac{8R^2+11Rr}{2}. \tag{8}$$

By Gerretsen's inequality $s^2\le 4R^2+4Rr+3r$. So to prove (8), it remains to show that

$$8R^2+8Rr+6r\le 8R^2+11Rr\quad\text{or}\quad 3r(R-2r)\ge 0,$$

which clearly holds on account of Euler's inequality.

In the following we will prove that the best inequality of this type is the inequality (6). Indeed, suppose that it exists other constants $\alpha,\beta,\gamma\in\mathbb{R}$ and $\gamma\ge -2$ such that

$$\left(\frac{r}{r_a}\right)^3+\left(\frac{r}{r_b}\right)^3+\left(\frac{r}{r_c}\right)^3\le\frac{\alpha R+\beta r}{R+\gamma r}\le\frac{8R-13r}{8R+11r} \tag{9}$$

is true in every $\triangle ABC$ or

$$1-\frac{12Rr}{s^2}\le\frac{\alpha R+\beta r}{R+\gamma r}\le\frac{8R-13r}{8R+11r}. \tag{10}$$

According to Blundon's inequality and (10), we obtain

$$1-\frac{12Rr}{2R^2+10Rr-r^2+2\sqrt{R(R-2r)^3}}\le\frac{\alpha R+\beta r}{R+\gamma r}\le\frac{8R-13r}{8R+11r}. \tag{11}$$

In the case of isosceles triangle with sides $b = c = 1$, $a = 0$, from (11) since $R = 1/2$, $r = 0$ it follows that $1 \leq \alpha \leq 1$ or $\alpha = 1$. In the case of equilateral triangle $R = 2r$, from (11)

$$\frac{1}{9} \leq \frac{2+\beta}{2+\gamma} \leq \frac{1}{9} \quad \text{or} \quad \gamma = 9\beta + 16 \geq -2 \quad \text{or} \quad \beta \geq -2$$

So inequality (11) may be written as

$$1 - \frac{12x}{2x^2 + 10x - 1 + 2\sqrt{x(x-2)^3}} \leq \frac{x+\beta}{x+9\beta+16} \leq \frac{8x-13}{8x+11}. \tag{12}$$

The second part of inequality (12) $(x \geq 2 \geq -9\beta - 16)$, may be written as:

$8x^2 + 8\beta x + 11x + 11\beta \leq 8x^2 + (72\beta + 128)x - 13x - 13(9\beta + 16)$
or

$(64\beta + 104)x - 13(9\beta + 16) - 11\beta \geq 0, \ \forall\, x \geq 2$. So

$$64\beta \geq -104 \quad \text{or} \quad \beta \geq -\frac{13}{8}. \tag{13}$$

The first side of (12) may be written as

$$\frac{2\beta + 4}{x + 9\beta + 16} \leq \frac{3x}{2x^2 + 10x - 1 + 2\sqrt{x(x-2)^3}}, \ \forall\, x \geq 2,$$

or

$$(x-2)[(4\beta+5)x + \beta + 2] + (4\beta + 8)\sqrt{x(x-2)^3} \leq 0, \ \forall\, x \geq 2,$$

or

$$(4\beta+5)x + \beta + 2 + (4\beta + 8)\sqrt{x(x-2)} \leq 0, \ \forall\, x \geq 2,$$

or

$$4\beta + 5 + \frac{\beta + 2}{x} + (4\beta + 8)\sqrt{1 - \frac{2}{x}} \leq 0, \ \forall\, x \geq 2.$$

Taking $x \to \infty$ we obtain

$$\beta \leq -\frac{13}{8}. \tag{14}$$

From (13) and (14) we get that (6) is the best inequality of this type in the "statement".

V. Find the best constant α such that the inequality

$$\frac{bc}{(s-a)^2}+\frac{ca}{(s-b)^2}+\frac{ab}{(s-c)^2}\leq \alpha\frac{R}{r}+12-2\alpha,$$

is true in every acute triangle.

We have

$$\sum_{cyc}\frac{bc}{(s-a)^2}=\frac{r-8R}{r}+\frac{(4R+r)^2}{rs^2}.$$

So the inequality from statement is equivalent to

$$F(s,R,r)=\left[\frac{(4R+r)^3}{rs^2}-11-\frac{8R}{r}\right]\frac{r}{R-2r}\leq\alpha.$$

Since F is decreasing we obtain

$$F(s,R,r)\leq F(s_1,R,r)\leq\alpha_1 \quad \text{if} \quad 2\leq\frac{R}{r}\leq\sqrt{2}+1$$
$$F(s,R,r)\leq F(s_3,R,r)\leq\alpha_2 \quad \text{if} \quad \frac{R}{r}\geq\sqrt{2}+1.$$

Consider the functions $f_1:\left[2,\sqrt{2}+1\right]\to\mathbb{R}$ defined by $f_1(x)=F(s_1,R,r)$ and $f_2:\left[\sqrt{2}+1,+\infty\right)\to\mathbb{R}$ defined by $f_2(x)=F(s_3,R,r)$. One can show that

$$\alpha_1=\sup_{x\in[2,\sqrt{2}+1]}f_1(x)$$

and

$$\alpha_2=\sup_{x\in[\sqrt{2}+1,+\infty)}f_2(x)$$

are the best constant for inequalities (3)

$$f_1(x)=\left(\frac{1}{x-2}\right)\left(\frac{(4x+1)^3}{2x^2+10x-1-2\sqrt{x(x-2)^3}}-11-8x\right)$$
$$f_2(x)=\left(\frac{1}{x-2}\right)\left(\frac{(4x+1)^3}{(2x+1)^2}-11-8x\right).$$

Using WA we obtain

$$\alpha_1 = \sup_{x\in[2,\sqrt{2}+1]} f_1(x) = f_1\left(\sqrt{2}+1\right) = 10+2\sqrt{2}$$
$$\alpha_2 = \sup_{x\in[\sqrt{2}+1,+\infty)} f_2(x) = 10+2\sqrt{2}.$$

So the best constant is $10+2\sqrt{2}$.

Remark 4. In every acute triangle ABC

$$\frac{bc}{(s-a)^2}+\frac{ca}{(s-b)^2}+\frac{ab}{(s-c)^2} \le \left(10+2\sqrt{2}\right)\frac{R}{r}-8-4\sqrt{2}.$$

VI. Find the best constant k such that the inequality

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2} \ge k\frac{R-2r}{2R}.$$

it is true in every acute triangle.

From identity $\sum_{cyc}\frac{a}{b+c}=\frac{2s^2-2r^2-2Rr}{s^2+r^2+2Rr}$, the inequality from the statement is equivalent to

$$k \le \frac{R}{R-2r}\,\frac{s^2-10Rr-7r^2}{s^2+r^2+2Rr} \quad \text{or} \quad k \le F(R,r,s)=f(s),$$

where $f:(0,+\infty)\to\mathbb{R}$ given by $f(s)=\frac{R}{R-2r}\,\frac{s^2-10Rr-7r^2}{s^2+r^2+2Rr}$ satisfies $f'(s)=\frac{2s^2(12Rr+8r^2)}{s^2+r^2+2Rr}>0$.

Hence f is increasing and $k\le f(s_1)\le f(s)$ if $2\le\frac{R}{r}\le\sqrt{2}+1$ and $k\le f(s_3)\le f(s)$ if $\frac{R}{r}\ge\sqrt{2}+1$.

So $k\le u(x)=\frac{x}{x-2}\,\frac{x^2-4-\sqrt{x(x-2)}}{x^2+6x-\sqrt{x(x-2)^3}}$ if $x\in\left[2,\sqrt{2}+1\right]$.

Therefore the best constant is $k_1 = \inf\limits_{x\in[2,\sqrt{2}+1]} u(x) = \sqrt{2}-1$.

Also if $\dfrac{R}{r} = x \geq \sqrt{2}+1$, $k \leq f(s_3) = v(x) = \dfrac{x}{x-2}\dfrac{2x^2-3x-3}{2x^2+3x+1}$.

So the best constant is $k_2 = \inf\limits_{x\geq\sqrt{2}+1} \dfrac{x}{x-2}\dfrac{2x^2-3x-3}{2x^2+3x+1} = \sqrt{2}-1$.

Therefore, in general, the best constant is $\sqrt{2}-1$.

Remark 5. In every acute triangle holds

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\left(\sqrt{2}-1\right)\frac{r}{R} \geq \frac{2+\sqrt{2}}{2}.$$

VII. Find the best constant α such that

$$\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}+\frac{c^2}{(a+b)^2} \geq \alpha + \frac{3-4\alpha}{2}\,\frac{r}{R}$$

is true in every acute triangle.

We see in **I** that the function f is increasing. So according Blundon inequality in acute triangle, we have since $\alpha \leq f(s)$ that $\alpha \leq f(s_1^2) \leq f(s^2)$ if $2 \leq \dfrac{R}{r} \leq \sqrt{2}+1$ or

$$\alpha \leq u(x) = \frac{29x^2+26x+8-(7x+2)\sqrt{x(x-2)}}{2(3x+2)^2}.$$

So the best constant is $\alpha_2 = \inf\limits_{2\leq x\leq\sqrt{2}+1} u(x) = \dfrac{4-\sqrt{2}}{2}$.

Also if $\dfrac{R}{r} \geq \sqrt{2}+1$, we have $\alpha \leq f(s_3^2) \leq f(s^2)$ or

$$\alpha \leq v(x) = \frac{16x^5+4x^4-46x^3-55x^2-22x-3}{2(x+1)^2(2x+1)^2(x-2)}.$$

So the best constant is $\alpha_3 = \inf\limits_{x\geq\sqrt{2}+1} v(x) = \dfrac{4-\sqrt{2}}{2}$. Therefore the best constant is $\dfrac{4-\sqrt{2}}{2}$.

Remark 6. In every acute triangle it is true the inequality

$$\frac{a^2}{(b+c)^2}+\frac{b^2}{(c+a)^2}+\frac{c^2}{(a+b)^2}\geq\frac{4-\sqrt{2}}{2}\,\frac{5-2\sqrt{2}}{2}\,\frac{r}{R}.$$

VIII. Find the best constant k such that the inequality

$$\cos^4 A+\cos^4 B+\cos^4 C\geq k$$

is true in every acute triangle.

Let $x=a^2,\ y=b^2,\ z=c^2$. Since a,b,c are the sides of an acute triangle, then x,y,z is the sides of an triangle with R,r the circumradius and inradius and s semiperimeter. From the cosine law, for all $s\in(0,+\infty)$ we have

$$\sum_{cyc}\cos^4 A=\sum_{cyc}\frac{(y+z-x)^4}{16y^2z^2}=\frac{1}{16x^2y^2z^2}\sum_{cyc}x^2(y+z-x)^4$$
$$=\frac{1}{16\sigma_3^2}\left[48\sigma_3^2+\sigma_1^6-10\sigma_1^4+8\sigma_3\sigma_1^3+32\sigma_2^2\sigma_1^2-32\sigma_1\sigma_2\sigma_3-32\sigma_2^2\right]$$
$$=\frac{(256R^2r^2+32r^4)s^2-(2048R^3r^3+1536R^2r^2+384Rr^5+32r^6)}{256R^2r^2s^2}$$
$$=\frac{256R^2r^2+32r^4}{256R^2r^2}-\frac{2084R^3r^3+1536R^2r^4+384Rr^5+32r^6}{256R^2r^2s^2}=f(s).$$

So f is increasing (we use $\sigma_1=2s,\ \sigma_2=s^2+r^2+4Rr,\ \sigma_3=4Rrs$). We obtain $f(s)=F(R,r,s)\geq k$ or $f(s)\geq f(s_1)\geq k$ or

$$k\leq f(s_1)=u(x)=\frac{256x^2+32}{256x^2}-\frac{2048x^3+1536x^2+384x+32}{256x^2(2x^2+10x-1)+2\sqrt{x(x-2)^3}}.$$

Therefore the best constant is

$$k_0=\inf_{x\geq 2}u(x)=\frac{73}{384}.$$

Remark 7. In every acute triangle the following inequality holds

$$\cos^4 A+\cos^4 B+\cos^4 C\geq\frac{73}{384}.$$

IX. Find the best constant k such that the inequality

$$\cos^3 A + \cos^3 B + \cos^3 C \leq k.$$

holds in every acute triangle. It is known the identity

$$\sum_{cyc} \cos^3 A = \frac{4R^3 + 12R^2r + 6Rr^2 + r^3 - 3rs^2}{4R^3} = F(s, R, r) = f(s),$$

f is decreasing. So we search k such that $f(s) \leq k$.
In the case $2 \leq \dfrac{R}{r} \leq \sqrt{2} + 1$, according Blundon theorem in acute triangle we have $f(s) \leq f(s_1) \leq k$ or

$$u(x) = f(s_1) \leq k, \ \forall\, x \in \left[2, \sqrt{2} + 1\right)$$
$$u(x) = \frac{4x^3 + 6x^2 - 24x + 4 + 6\sqrt{x(x-2)^3}}{4x^3}, \ \forall\, x \in \left[2, \sqrt{2} + 1\right]$$

So the best constant in this case is $k = \sup\limits_{x \in [2, \sqrt{2}+1]} u(x) = \dfrac{1}{\sqrt{2}}$.

If $R/r \geq \sqrt{2} + 1$ we have $f(s) \leq f(s_3) \leq k$ or $v(s) = f(s_3) \leq k$, $\forall\, x \in \left[\sqrt{2} + 1, +\infty\right)$, where $v(x) = \dfrac{4x^3 - 6x - 2}{4x^3}$ $\forall\, x \geq \sqrt{2} + 1$.
So the best constant is $k_2 = \sup\limits_{x \in [\sqrt{2}+1, +\infty)} v(x) = \dfrac{1}{\sqrt{2}}$.
Therefore the best constant is $\dfrac{1}{\sqrt{2}}$.

Remark 8. 1) In every acute triangle one has

$$\cos^3 A + \cos^3 B + \cos^3 C \leq \frac{1}{\sqrt{2}} \simeq 0,707.$$

2) In [6] Yu-Dong Wu and Nu-Chun Hu find that $k_0 \simeq 1,225\ldots$ is the best constant for which $\sum_{cyc} \cos^3 A \leq k$ holds in every triangle ABC. In [4] we give an easier proof of this inequality.

X. Find the best constant k such that the inequality

$$\cos^3 A + \cos^3 B + \cos^3 C \geq k$$

holds in every triangle ABC .

By **IX**, function f is decreasing or $f(s) \geq f(s_2) \geq k$.
So $v(x) = f(s_2) \geq k$, $\forall\, x \geq 2$, where

$$v(x) = \frac{4x^3 + 6x^2 - 24x + 4 - 6\sqrt{x(x-2)^3}}{4x^3}\,,\ \forall\, x \geq 2.$$

So the best constant is

$$k_2 = \inf_{x \geq 2} v(x) = \frac{3}{8} \simeq 0,375.$$

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Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.
2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

jose.luis.diaz@upc.edu

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

October 30, 2022

Elementary Problems

E–101. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Determine two positive integers which sum is 3948 and their lest common multiple is 82861.

E–102. *Proposed by Michel Bataille, Rouen, France.* Let ABC be a triangle with sides a, b, c and opposite angles

$$A = \frac{6\pi}{11},\ B = \frac{3\pi}{11},\ C = \frac{2\pi}{11}.$$

Prove that $a^2(a-b) + b^2(b-c) + c^2(c-a) = bc^2$.

E–103. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Find the smallest integer value of $n \geq 1$, such that

$$2022 = {a_1}^3 + {a_2}^3 + \cdots + {a_n}^3$$

for integers $a_1, a_2, \ldots, a_n$.

E–104. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $a_1, a_2, \ldots, a_{20}$ be distinct integer numbers. Show that among them, there are two a_i and a_j such that $a_i - a_j$ is a multiple of 19.

E–105. *Proposed by Toyesh Prakash Sharma, Agra College, India.* If $1 < a, b, c < 6$ and $a + b + c = 6$ then show that

$$\frac{\ln a}{b+c} + \frac{\ln b}{c+a} + \frac{\ln c}{a+b} \geq \frac{3}{4}\ln 2.$$

E–106. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let ABC be a triangle with the circumcenter O and the orthocenter H. Points A_1, B_1, C_1 are the circumcenters of triangles HBC, HAC and HAB, respectively. Show that if $\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = \overrightarrow{0}$, then ABC is an equilateral triangle.

Easy–Medium Problems

EM–101. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.* Let ℓ be an exterior common tangent to congruent external circles Ω and Ω' of radius r. The circle Ω_1 touches ℓ and also touches Ω and Ω' externally. The circle Ω_2 touches Ω_1, Ω and Ω' externally. The circle Ω_3 touches Ω_2, Ω and Ω' externally. The circle Ω_4 touches Ω_3, Ω and Ω' externally and also touches the line of centers of Ω and Ω'. Prove that the radius of Ω_2 is twice the length of the radius of Ω_4.

EM–102. *Proposed by Michel Bataille, Rouen, France.* Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) = f(y) \cdot f(xf(y))$$

for all real numbers x, y.

EM–103. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a_0, a_1, a_2 be real numbers such that $3a_0 \geq a_1a_2$. If the polynomial with real coefficients $A(z) = a_0 + a_1z + a_2z^2 + z^3$ has all its zeros real and nonnegative, then prove that

$$9a_0^2 + a_1^2a_2^2 \geq \frac{4}{3}a_1^3 + 6a_0a_1a_2.$$

EM–104. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let ABC be a right triangle with $\angle BAC = 90°$. If M is the middle point of BC, D is the foot of the altitude from A, Γ is the circumcircle of $\triangle ABC$ and exists the circle Ω tangent to BC at M, tangent to AD at P, and tangent to circle Γ at R, then calculate $\angle ABC$ and $\angle ACB$, respectively.

EM–105. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* The union of nineteen planar surfaces, each of area equal to 1, has a total area equal to 10. Prove that the overlap of some two of these surfaces has an area greater than or equal to $1/19$.

EM–106. *Proposed by Nicolae Papacu, Slobozia, Romania.* Find all positive integers x, y, z such that $2^{x+1} + 45^y = z^2$.

Medium–Hard Problems

MH–101. *Proposed by Michel Bataille, Rouen, France.* Find the extreme of

$$\frac{ab(a+b-1)}{(a^2+b^2+1)^3}$$

when a, b are real numbers satisfying $a^2+b^2 = a+b$.

MH–102. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be three positive numbers. Prove that

$$\sqrt[3]{(a^a+b^a+c^a)(a^b+b^b+c^b)(a^c+b^c+c^c)} \geq a^{\frac{a+b+c}{3}}+b^{\frac{a+b+c}{3}}+c^{\frac{a+b+c}{3}}.$$

MH–103. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a scalene triangle with centroid G and symmedian point K, such that AK is the angle bisector of the angle BAG. D, E and F are the feet of perpendiculars from B to the line AG, from C to the line AG and from G to AC respectively. Prove that the area of triangle ABC is $[ABC] = 6[DEF]$.

MH–104. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* On a 2023×2023 board symmetrical with respect to the main diagonal, the integer numbers from 1 to 2023 are placed so that in each row and column each number appears once and only once. Show that all the numbers appear on the main diagonal.

MH–105. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* If $[1, 2, \ldots, 2n]$ represents the least common multiple of the integers $1, 2, \ldots, 2n$, then show that

$$\frac{\left[1, 2, \ldots, 2n\right]}{\binom{n}{0}^2+\binom{n}{1}^2+\ldots+\binom{n}{n}^2}$$

is an integer number.

MH–106. *Proposed by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA.* If n is a multiple of 8, prove that the number of subsets of $\{1, 2, \ldots, n\}$ having cardinality divisible by 4 is $2^{n-2} + 2^{(n-2)/2}$.

Advanced Problems

A–101. *Proposed by Michel Bataille, Rouen, France.* For each positive integer n, let A_n and G_n be the arithmetic mean and the geometric mean of the n integers $n^2+1, n^2+2, \ldots, n^2+n$. Find

$$\lim_{n\to\infty}(A_n - G_n).$$

A–102. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In how many ways can the rational $2022/2021$ be written as the product of two rational numbers of the form $(n+1)/n$, where n is a positive integer?

A–103. *Proposed by Joseph Santmyer, US Federal Government (retired), Las Crues, New Mexico, USA.* Let us denote by A and B the following integrals

$$A = \int_0^{2\pi} \cos(t)\sin[e^{\cos(t)}\cos(\sin(t)-t)]\cosh[e^{\cos(t)}\cos(\sin(t)-t)]dt,$$
$$B = \int_0^{2\pi} \sin(t)\cos[e^{\cos(t)}\cos(\sin(t)-t)]\sinh[e^{\cos(t)}\cos(\sin(t)-t)]dt.$$

Prove that

$$2\pi\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^{2n}}{(2n+1)!(2n)!} = A - B.$$

A–104. *Proposed by Gabriel T. Prăjitură, SUNY Brockport, Brockport, NY, USA.* Let f be a real function with the intermediate value property. If there is p a point at which f is not continuous, prove that there is an interval (α,β) such that for every $y \in (\alpha,\beta)$, $f^{-1}\{y\}$ has infinite cardinality.

A–105. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $A_n \in \mathcal{M}_n(\mathbb{R})$ be the matrix with entries $a_{ij} = \dfrac{1}{i+j-1}$, $1 \le i,j \le n$. Compute $\det(A_n)$ and the sum of the entries of A_n^{-1}.

A–106. *Proposed by Vasile Mircea Popa, "Lucian Blaga" University of Sibiu, Romania.* Calculate the following integral:

$$\int_1^{\infty} \frac{\sqrt{x}\ln^2 x}{x^3 + x\sqrt{x} + 1} dx.$$

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

jose.luis.diaz@upc.edu

The Pigeonhole Principle

José Luis Díaz-Barrero

1 Introduction

The pigeonhole principle, for short PHP, is one of the simplest statements in Mathematics and one of the most applied in a large number of topics such as combinatorial geometry, graph theory, enumerative combinatorics, number theory, algebra, analysis, and so on. It is also a very powerful tool for solving problems in Olympiads and other mathematical contests (see [1]). Although the pigeonhole principle appears as early as 1624 in a book attributed to Jean Leurechon, it is commonly called Dirichlet's box principle or Dirichlet's drawer principle after an 1834 treatment of the principle by Peter Gustav Lejeune Dirichlet under the name "drawer principle" or "shelf principle".

In this mathlesson, we present the principle and some of its applications. These problems have been proposed, discussed and solved in the training sessions that are given every year by the Barcelona Math Circle (BMC) to students interested in mathematics in general and, in particular, in mathematical contests.

2 The Pigeonhole Principle

In mathematics, the pigeonhole principle claims that if $n+1$ objects are put into n boxes, then at least one box must contain more than one object. The proof of this claim is almost immediate. If

in every box there would be at most one object, we would have at most n objects, which contradicts the fact that we arrange $n+1$ objects into n boxes.

The same reasoning may be applied to prove the following

Theorem 1 (General Pigeonhole Principle). *Let m and n be any two positive integers. If m objects are distributed among n boxes, then one of the boxes must contain at least $\left\lceil \frac{m}{n} \right\rceil$ objects, where $\lceil x \rceil$ represents the ceiling function of x.*

Proof. We argue by contradiction. Suppose none of the boxes contain at least $\left\lceil \frac{m}{n} \right\rceil$ objects. Then all of the boxes contain at most $\left\lceil \frac{m}{n} \right\rceil - 1$ objects. Adding up the total number of objects in the n boxes, we have that there are not more than

$$n\left(\left\lceil \frac{m}{n} \right\rceil - 1\right)$$

objects in total. Let $m = kn + r$, where $0 \leq r < n$. Then,

$$\left\lceil \frac{m}{n} \right\rceil = \begin{cases} k, & \text{if } r = 0; \\ k+1, & \text{if } r > 0. \end{cases}$$

We now have

$$n \cdot \left\lceil \frac{m}{n} \right\rceil = \begin{cases} nk, & \text{if } r = 0; \\ nk + n, & \text{if } r > 0, \end{cases}$$

and so the total number of objects is

$$n \cdot \left(\left\lceil \frac{m}{n} \right\rceil - 1\right) = n \cdot \left\lceil \frac{m}{n} \right\rceil - n = \begin{cases} nk - n, & \text{if } r = 0; \\ nk, & \text{if } r > 0. \end{cases}$$

$$< nk + r = m.$$

Hence, there are not more than m objects in total, which is a contradiction. Thus, one of the boxes must contain at least $\left\lceil \frac{m}{n} \right\rceil$ objects.

Remark. Note that $\left\lceil \frac{m}{n} \right\rceil = \left\lfloor \frac{m-1}{n} \right\rfloor + 1$.

3 Applications

The following are some applications of PHP. We start with

Problem 1. *Each* 1×1 *square of a* 5×41 *chessboard is painted either red or blue. Prove that it is possible to choose three rows and three columns of the chessboard so that the nine* 1×1 *squares into which they intersect are painted with the same color.*

Solution. Since we have used only two colors, the second version of the pigeonhole principle implies that, in each column (of five 1×1 squares), one of the two colors must occur at least $\lfloor \frac{5-1}{2} \rfloor + 1 = 3$ times. Call such a color **dominant** and, for each of the 41 columns, take note of its dominant color. Since one out of the two colors dominates in each of the 41 columns, by invoking once more the second version of the pigeonhole principle we conclude that one of the two colors is the dominant one in at least $\lfloor \frac{41-1}{2} \rfloor + 1 = 21$ columns. Assume, without loss of generality, that such a color is red, and call a column as **red** if red is its dominant color. Then, choose 21 red columns and, in each of them, choose three 1×1 red squares.

Notice that there are exactly $\binom{5}{3} = 10$ possible ways of choosing three of the five 1×1 squares of each column. Therefore, out of the 21 red columns (and once again from the second version of the pigeonhole principle), the three 1×1 chosen red squares occupy exactly the same positions in at least $\lfloor \frac{21-1}{10} \rfloor + 1 = 3$ columns. In other words, we have at least three columns having 1×1 red squares in the same set of three rows, and this is exactly what we were asked to prove.

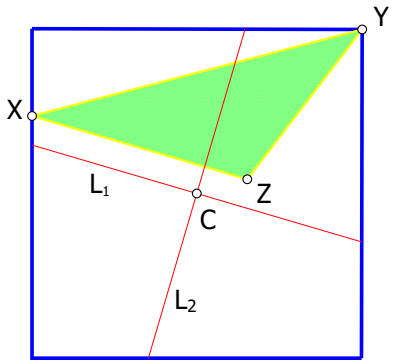
Problem 2. *Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum.*

Solution. Note that there are $2^{10} - 2 = 1022$ distinct nonempty proper subsets of our set of 10 two-digit numbers. Also note that

the sum of the elements of any subset of our set of 10 two-digit numbers must be between 10 and $90 + 91 + 92 + 93 + 94 + 95 + 96 + 97 + 98 + 99 = 945 < 1022$. The Pigeonhole Principle then implies that there are two distinct subsets whose members have the same sum. Let these sets be A and B. Note that $A - (A \cap B)$ and $B - (A \cap B)$ are two distinct sets whose members have the same sum. These two sets are subsets of our set of 10 distinct two-digit numbers, so this proves the claim.

Problem 3. *Suppose a triangle can be placed inside a square of area* 1 *in such a way that the center of the square is not inside the triangle. Show that one side of the triangle has length less than* 1.

Solution. Through the center C of the square, draw a line L_1 parallel to the closet side of the triangle and a second line L_2 perpendicular to L_1 at C. The lines L_1 and L_2 divide the square into four congruent quadrilaterals.



Scheme for solving problem 3.

Since C is not inside the triangle, the triangle can lie in at most two adjacent quadrilaterals. By the Pigeonhole Principle, two of the vertices of the triangle must belong to the same quadrilateral. Now the furthest distance between two points in the quadrilateral is the distance between two of its opposite vertices, which is at most 1. So the side of the triangle with two vertices lying in the same quadrilateral must have length less than 1.

Problem 4. *Let α be any irrational real number. Prove that there exists a positive integer n so that the distance of $n\alpha$ from the closest integer is less than 10^{-10}.*

Solution. Let us represent the set of all positive real numbers by a circle as shown in the figure below. That is we think of the circle having circumference 1, and two real numbers are represented by the same point on the circle if their difference is an integer. Now

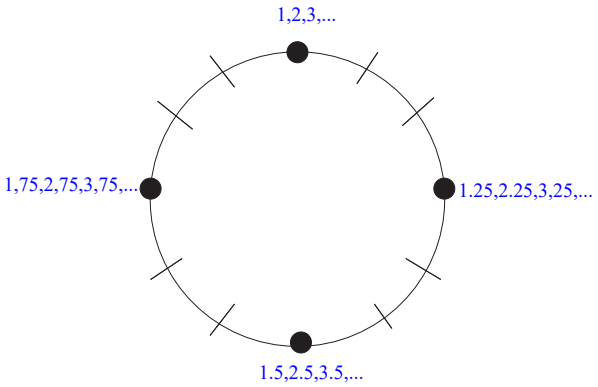


Figure 1: Scheme for solving problem 4.

let us divide the perimeter of the circle into 10^{10} equal parts. (The figure shows a subdivision to 12 parts for better picture quality.) Let us take the first $10^{10}+1$ elements of the sequence $\alpha, 2\alpha, 3\alpha, \ldots$. As there are 10^{10} arcs, by the PHP there has to be one arc T that contains at least two elements of the sequence. Let $i\alpha$ and $j\alpha$ be two such elements. Let $i < j$, then this implies that the distance from $(j-i)\alpha$ to the nearest integer is less than the length of the arc T, that is, 10^{-10}.

Problem 5. *Let $a_1, a_2, \ldots, a_{13}$ be distinct real numbers. Prove that there exists two of them, say a and b, such that*

$$0 < \frac{a-b}{1+ab} \leq 2 - \sqrt{3}.$$

Solution. For each $i \in \{1, 2, \ldots, 13\}$ there exists a real number $x_i \in (-\pi/2, \pi/2)$ such that $\tan(x_i) = a_i$. If we divide the interval

$(-\pi/2, \pi/2)$ into 12 parts of length $\pi/12$, then by the Pigeonhole Principle, at least two of the numbers x_i and x_j lie in the same subinterval. Then, we have

$$0 < x_i - x_j \leq \frac{\pi}{12} \Rightarrow \tan(0) < \tan(x_i - x_j) \leq \tan\Big(\frac{\pi}{12}\Big)$$

Calling $a = \tan(x_i)$ and $b = \tan(x_j)$ from the preceding we have

$$0 < \frac{a-b}{1+ab} \leq \tan\Big(\frac{\pi}{12}\Big) = 2 - \sqrt{3}.$$

Problem 6. *Suppose that each positive integer is colored either red or blue. Show that there exist three positive integers $a < b < c$ having the same color and such that $c + a = 2b$.*

Solution. We will show, in fact, that the triple of positive integers $a < b < c$ can be chosen from the set $\{1, 2, \ldots, 13\}$. To start with, look at the numbers $5, 7$ and 9. At least two of these must have the same color by the PHP, which we can assume, without loss of generality, to be red. There are three cases to consider, according to which pair of numbers in the set $\{5, 7, 9\}$ is known to be red.

- Suppose first that 5 and 7 are red. If either $3, 6$ or 9 is red, then one of the triples $3 < 5 < 7, 5 < 6 < 7$ or $5 < 7 < 9$ is entirely red. Thus we can assume that $3, 6$ and 9 are blue, and then $3 < 6 < 9$ is a blue triple.
- Next, suppose that 7 and 9 are red. Then the same argument applies, but shifted up by two numbers. Namely, if either $5, 8$ or 11 is red, then at least one of the triples $5 < 7 < 9, 7 < 8 < 9$ or $7 < 9 < 11$ is entirely red. Otherwise, $5 < 8 < 11$ is a blue triple.
- Finally, if 5 and 9 are red members of $\{5, 7, 9\}$, then we again apply the same argument, this time using skips twice as large. Specifically, if either $1, 7$ or 13 is red, then at least one of the triples $1 < 5 < 9, 5 < 7 < 9$ or $5 < 9 < 13$ is entirely red. Otherwise, $1 < 7 < 13$ is a blue triple.

Note that the colors of the numbers $2, 4, 10$ and 12 are irrelevant to this proof.

References

[1] Soberón, P. *Problem Solving in Combinatorics*. Birkhaüser, 2013, pp. x+174. ISBN: 978-3-0348-0596-4. DOI: 10.1007/978-3-0348-0597-1.

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Contests

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to find interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Problems and solutions from the 36th edition of the Iberoamerican Mathematical Olympiad

O. Rivero Salgado

1 Introduction

The 36th edition of the Iberoamerican Mathematical Olympiad took place in October 2021 in Costa Rica. Due to the Covid pandemic, the 91 participants cannot gather together and the competition followed an online format. It was developed in two consecutive days, and contestants had to solve 3 problems each day in a maximum time of four hours and a half. Each problem was graded with an integer mark between 0 and 7 points, so the maximum possible score was 42 points, achieved this year by three students. According to the usual standards, at most half of the participants can get a medal, and then these are awarded in the proportion 1:2:3 for gold, silver and bronze, respectively.

The Spanish team made a very good performance, achieving two silver medals (Leonardo Costa and Àlex Rodríguez) and two bronze medals (Bernat Pagès and Miguel Valdivieso). The chief of the delegation was Óscar Rivero and the deputy leader was Marc Felipe.

We present now the problems of the competition, and include the solutions given to them by our team. In all the cases, the solutions follow the ideas presented by the contestants, but we have done some little modifications to ease the exposition.

2 Problems and solutions

We now present the problems that appeared in the paper and their solutions.

Problem 1. Let $P = \{p_1, p_2, \ldots, p_{10}\}$ be a set of 10 different prime numbers and let A be the set of all integers greater than 1 such that their prime factorizations contain only primes in P. Each element in A is colored in the following way:

(a) each element in P has a distinct color,
(b) if m, $n \in A$, then mn has the same color as either m or n,
(c) for each pair of distinct colors $\mathcal{R}$ and $\mathcal{S}$, there are no j, k, m, $n \in A$ (not necessarily distinct), with j, k colored $\mathcal{R}$ and m, n colored $\mathcal{S}$, such that both j divides m and n divides k.

Show that there is some prime in P such that all of its multiples in A have the same color.

Solution by Leonardo Costa. First of all, by mathematical induction, all the powers of $p_1, p_2, \ldots, p_{10}$ must be painted using one of the colors of $p_1, p_2, \ldots, p_{10}$. We call those colors $1, 2, \ldots, 10$. Since any number of A is the product of powers of p_i, for $i \in \{1, \ldots, 10\}$, every number in A is of one of the colors $1, 2, \ldots, 9, 10$.

Consider the directed graph with vertices $1, \ldots, 10$ where i is connected with j (in this order) if there exists a vertex a of color i and a vertex b of color j such that $a \mid b$. Note that by condition (c), if i is connected with j, then j is not connected with i. Suppose that for any $i \in \{1, \ldots, 10\}$ there exists j such that i is connected to j. Then we may consider for any vertex a another node b such that a is connected with b, and this process can be successively iterated. Since the graph is directed, in some moment we will come back to a vertex we have already visited and we will have a cycle.

Let $x_1, x_2, \ldots, x_k$ be the vertices of the cycle. By simplicity, let $C_i = \{x \in A \mid x \text{ is of color } i\}$. Since x_1 is connected to x_2, $p_{x_1}p_{x_2} \in C_{x_2}$ (if it were in C_{x_1}, $p_{x_2} | p_{x_1}p_{x_2}$ and hence x_2 would be connected to x_1). Analogously, $p_{x_i}p_{x_{i+1}} \in C_{x_{i+1}}$. But then $p_{x_1}p_{x_2}\cdots p_{x_k} \in C_{x_1} \cup C_{x_2} \cup \ldots \cup C_{x_k}$.

Hence, if $p_{x_1}p_{x_2}\cdots p_{x_k} \in C_{x_j}$, then x_{j+1} is connected to x_j, contradicting condition (c). Note that in the previous sentence indices are taken modulo k. Finally, since we had assumed that for all $i \in \{1,\ldots,10\}$ there exists j such that i is connected with j, there exists $i \in \{1,\ldots,10\}$ such that for all j, i is not connected with j. Hence, for that i, since $p_i \in C_i$, every multiple of p_i will be also of color i, as desired.

Problem 2. Consider an acute triangle ABC, with $AC > AB$, and let Γ be its circumcircle. Let E and F be the midpoints of the sides AC and AB, respectively. The circumcircle of triangle CEF intersects Γ at X and C, with $X \neq C$. The line BX and the line tangent to Γ at A intersect at Y. Let P be the point on segment AB such that $YP = YA$, with $P \neq A$, and let Q be the point where AB intersects the line parallel to BC passing through Y. Show that F is the midpoint of PQ.

Solution by Bernat Pagès. Let $R = AC \cap QY$. Moving angles,

$$\gamma = \angle BCA = \angle FEA = \angle QRA,$$

and similarly

$$\gamma = \angle BCA = \angle QAY = \angle QPY,$$

where we have used that $AY = PY$. Further,

$$\beta = \angle CBA = \angle RQA = \angle PQY,$$

while

$$\angle QYP = 180^\circ - \angle PQY - \angle QPY = 180^\circ - \beta - \gamma = \alpha.$$

Then, YPQ is similar to ABC. Since $\angle RAP = \angle RYP = \alpha$, $RAYP$ is cyclic. Further, YAR is similar to YQA (comparing angles), and this means $YA^2 = YQ \cdot YR$. Using powers of Y in ABC, $YA^2 = YX \cdot YB$. These two equations imply that $QRXB$ is cyclic.

Note now that $\angle RXY = 180^\circ - \angle RXB = 180^\circ - \angle RQB = \angle ABR = \beta$. Since $\angle RAY + \angle RXY = \alpha + \beta + \gamma = 180^\circ$, so $XRAY$ is cyclic. All this together implies that $RAYXP$ is cyclic.

We now claim that R, F and X are collinear. Indeed, we already know that $\angle RXB = 180^\circ - \beta$. Regarding $\angle FXB$, observe that

$$\angle FXB = \angle FXC + \angle DXB = \angle FEA + \angle CAB = \alpha + \gamma.$$

Then, $\angle RXB = \angle FXB$, and then R, F and X are collinear.

Finally, using power of a point in the circles $RQXB$ and $AYXPR$, we obtain that

$$FX \cdot FR = FQ \cdot FB = FP \cdot FA.$$

Hence, $FQ \cdot FB = FP \cdot FA$. Since $FB = FA$, we conclude that $FQ = FP$, as desired.

Problem 3. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive integers and let $b_1, b_2, b_3, \ldots$ be the sequence of real numbers given by

$$b_n = \frac{a_1 \cdot a_2 \ldots a_n}{a_1 + a_2 + \ldots + a_n}, \quad \text{for } n \geq 1.$$

Show that, if for every one million consecutive terms of the sequence $b_1, b_2, b_3, \ldots$ there is at least one integer, then there is some k such that $b_k > 2021^{2021}$.

Solution by Àlex Rodríguez. We proceed by contradiction, so assume that for any k, $b_k \leq 2021^{2021}$. Then, there exist infinitely many indexes i such that b_i is an integer between 1 and 2021^{2021}. By the pigeonhole principle, and since the distance between the integers is at most 10^6 there exist infinitely many pairs with $n < m$ such that $b_n = b_m$, $m - n < A := 10^6 \cdot 2021^{2021} + 1$. Then:

$$\frac{\prod_{i=1}^{n} a_i}{\sum_{i=1}^{n}} = \frac{\prod_{i=1}^{m} a_i}{\sum_{i=1}^{m}},$$

and from here we see that

$$\frac{\sum_{i=n+1}^{m} a_i}{\sum_{i=1}^{n} a_i} + 1 = \prod_{i=n+1}^{m} a_i.$$

In particular, the fraction in the left is an integer (and therefore, greater or equal than one). This implies that

$$\sum_{i=n+1}^{m} a_i \geq \sum_{i=1}^{m} a_i \geq \sum_{i=1}^{n} n.$$

In particular, we have seen that there exists a_i, with $n+1 \leq i \leq m$, and such that $a_i(m-n) \geq n$. By hypothesis,

$$a_i \cdot A > a_i(m-n) \geq n,$$

which implies that $a_i > \frac{n}{A}$, where n is arbitrarily large, but A is a constant. This means that a_n is not bounded (for any T there exists i such that $a_i > T$).

For our further use, note that $\frac{ab}{a+b} \geq 1$ if $a, b \geq 2$. Let us study now the growth of b_n:

$$b_{n+1} = \frac{\prod_{i=1}^{n} a_i \cdot a_{n+1}}{\sum_{i=1}^{n} a_i + a_{n+1}} = b_n\Big(\frac{a_{n+1} \cdot \sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} a_i + a_{n+1}}\Big).$$

We may assume that $\sum_{i=1}^{n} a_i \geq 2$ for all $n \geq 2$, so $b_{n+1} \geq b_n$ if and only if $a_{n+1} \geq 1$ for all $n \geq 2$. We study now the minimum of b_n after one point. Let $t \in [3, 10^6 + 3]$ be an integer such that b_t is an integer (then $b_t \geq 1$). The unique option for the sequence (b_n) to decrease at that point is that $a_n = 1$, which can only happen 10^6 times in a row from t until b_n is an integer. The worst case scenario occurs when $a_n = 1$ for all $n = t+1, t+2, \ldots, t+10^6$. Then, since $b_t \geq 1$,

$$\prod_{i=1}^{t} a_i \geq \sum_{i=1}^{t} a_i \geq 3,$$

because $t \geq 3$. Therefore,

$$b_{t+10^6} = \frac{\prod_{i=1}^{t} a_i \prod_{i=t+1}^{t+10^6} a_i}{\sum_{i=1}^{t} a_i + \sum_{i=t+1}^{t+10^6} a_i} = \frac{\prod_{i=1}^{t} a_i}{\sum_{i=1}^{t} a_i + 10^6} \tag{1}$$

$$\geq 1 - \frac{10^6}{\sum_{i=1}^{t} a_i + 10^6} \geq 1 - \frac{10^6}{3 + 10^6} = B > 0. \tag{2}$$

This means we have found a lower, $b_n \geq B$, where B is a positive constant. Finally, consider the result which asserts that a_n is unbounded, and we add it to the relation between b_{n+1} and b_n. Let k be an integer such that $a_k > T$, and sufficiently large such that $\sum_{i=1}^{k-1} a_i \geq \frac{2021^{2021}}{B}$.

Let $S = \sum_{i=1}^{k-1} a_i$. Then, $\prod_{i=1}^{k-1} a_i \geq SB$,

$$2021^{2021} \geq b_k = \frac{\prod_{i=1}^{k-1} a_i \cdot a_k}{S + a_k} \geq \frac{SBa_k}{S + a_k} \geq \frac{2021^{2022} a_k}{\frac{2021^{2022}}{B} + a_k},$$

where in the last inequality we have used that $\frac{x}{x+\alpha}$ is an increasing function for any $\alpha \geq 1$. Hence, if $C = \frac{2021^{2021} \cdot 2021^{2022}}{B}$,

$$C + 2021^{2021} a_k \geq 2021^{2022} a_k,$$

or what it is the same,

$$C \geq 2021^{2021} \cdot (2021 - 1) \cdot a_k,$$

contradicting the fact that (a_k) is unbounded.

Problem 4. Let a, b, c, x, y, z be real numbers such that

$$a^2+x^2 = b^2+y^2 = c^2+z^2 = (a+b)^2+(x+y)^2 = (b+c)^2+(y+z)^2$$
$$= (c+a)^2 + (z+x)^2.$$

Show that $a^2 + b^2 + c^2 = x^2 + y^2 + z^2$.

Solution by Àlex Rodrǵuez. Write k for the common value of $a^2 + x^2$, $b^2 + y^2$, and so on. Then, adding the first three equations,

$$3k = a^2 + x^2 + b^2 + y^2 + c^2 + z^2.$$

Similarly, multiplying by two and subtracting the last three equations,

$$\begin{aligned} 3k &= -2(ab + bc + ca + xy + yz + zx) \\ &= -(a + b + c)^2 - (x + y + z)^2 + a^2 + b^2 + c^2 + x^2 + y^2 + z^2 \\ &= -(a + b + c)^2 - (x + y + z)^2 + 3k. \end{aligned}$$

Then,

$$a + b + c = x + y + z = 0.$$

By homogeneity, we may assume that $a^2+x^2 = b^2+y^2 = c^2+z^2 = 1$ (the case where they are zero is trivial). Then,

$$a = \cos\alpha,\ x = \sin\alpha,\ b = \cos\beta,\ y = \sin\beta,\ c = \cos\gamma,\ z = \sin\gamma,$$

for some α, β, $\gamma \in \mathbb{R}$. Define complex numbers

$$z_1 = a + ix = e^{i\alpha}, \quad z_2 = b + iy = e^{i\beta}, \quad z_3 = c + iz = e^{i\gamma}.$$

Observe that $z_1 + z_2 + z_3 = 0$. Multiplying by $e^{-i\alpha}$, we obtain

$$1 + e^{i(\beta-\alpha)} + e^{i(\gamma-\alpha)}.$$

In particular, this means that $e^{i(\beta-\alpha)}$ and $e^{i(\gamma-\alpha)}$ are conjugate numbers with real part $-1/2$. In particular, z_1, z_2, z_3 give rise to an equilateral triangle over the unit circle. Then, we may assume without loss of generality that $\beta = \alpha + 2\pi/3$ and $\gamma = \alpha + 4\pi/3$. Further,

$$z_1 \cdot z_2 = e^{i\alpha} z_2, \quad z_1 \cdot z_3 = e^{i\alpha} z_3, \quad z_2 \cdot z_3 = e^{i\alpha} \cdot z_1,$$

and therefore

$$z_1 z_2 + z_1 z_3 + z_2 z_3 = e^{i\alpha}(z_1 + z_2 + z_3) = 0.$$

This means that

$$z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1 z_2 + z_2 z_3 + z_3 z_1) = 0.$$

In particular, if $\Re(w)$ stands for the real part of w,

$$0 = \Re(z_1^2+z_2^2+z_3^2) = \cos^2\alpha - \sin^2\alpha + \cos^2\beta - \sin^2\beta + \cos^2\gamma - \sin^2\gamma.$$

We conclude that

$$a^2 + b^2 + c^2 = x^2 + y^2 + z^2,$$

as desired.

Problem 5. For a finite set C of integers, we define $S(C)$ to be the sum of the elements of C. Find two nonempty sets A and B, whose intersection is empty and whose union is the set $\{1, 2, \ldots, 2021\}$, such that the product $S(A)S(B)$ is a perfect square.

Solution by Miguel Valdivieso. Let $D = \{n \in \mathbb{N} \mid 1 \leq n \leq 2021\}$ and consider

$$A := \{n \in \mathbb{N} \mid 1 \leq n \leq 242\} \cup \{n \in \mathbb{N} \mid 1779 \leq n \leq 2021\}, \quad B = D \backslash A.$$

We will show that A and B satisfy the conditions of the statement. Indeed, using the definition of B, it is immediate showing that the intersection is the empty set and the union is D. Further, we can check that

$$S(A) = \sum_{i=1}^{242} i + \sum_{i=1779}^{2021} i = 243 \cdot 2021.$$

Moreover, since $S(D) = 2021 \cdot 2011$, we have that

$$S(B) = S(D) - S(A) = 2021 \cdot (1011 - 243) 2021 \cdot 768.$$

We deduce that

$$S(A) \cdot S(B) = 2021^2 \cdot (243 \cdot 768) = (2^4 \cdot 3^3 \cdot 2021)^2,$$

as desired.

Remark. The key point is noting that $2021 \cdot 1011 = 43 \cdot 47 \cdot 3 \cdot 337$ must have a divisor of the form $4k + 1$, that is, that may be written as a sum of two squares. The only divisor of that form is $337 = 81 + 256$. From here, it is straightforward checking that any decomposition must be of the form $S(A) = 81 \cdot 3 \cdot 2021$, $S(B) = 256 \cdot 3 \cdot 2021$.

Problem 6. Consider a regular polygon with n sides, $n \geq 4$, and let V be a subset of r vertices of the polygon. Show that if $r(r-3) \geq n$ then there exist at least two congruent triangles whose vertices are in V.

Solution by Gustavo Ochoa, proposer of the problem (it follows the ideas of the solution written by Leonardo Costa during the contest). The distance between two vertices of V, A and B, will be measured as the number of sizes needed to go from A to B using the shortest path. We will denote it as $d(A, B)$.

If among the r vertices of V there are four of them, A, B, C and D such that $d(A, B) = d(C, D)$, then a trivial check shows that there are two congruent triangles. Conversely, it is clear that if there are two congruent triangles, then there are four vertices belonging to those two triangles, A, B, C and D, such that $d(A, B) = d(C, D)$. Hence, it is enough with showing that if $r(r-3) \geq n$, then in V there are four distinct vertices, A, B, C and D, such that $d(A, B) = d(C, D)$.

Fixing a vertex $A \in V$, the possible distances to the other vertices of the set belong to $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$. Further, fixing a distance d, there are at most two vertices at a distance d of A. Note that if there are four vertices of V, B_1, C_1, B_2 and C_2, such that

$$d(A, B_1) = d(A, B_2), \quad d(A, C_1) = d(A, C_2),$$

then the triangles AB_1C_1 and AB_2C_2 are congruent (exchanging C_1 and C_2 if needed).

From now on, let us suppose that there are no two congruent triangles with vertices in V, and let us conclude that $r(r-3) < n$. When considering the distances from a vertex $A \in V$ to the remaining $r-1$ vertices in V, only one of those $r-1$ distances may be repeated. Hence, the cardinal of the set of distances from A must be greater or equal than $r-2$.

The number of possible distances among vertices of V (allowing repetition of distances which share a vertex, but counting just once $d(A, B) = d(B, A)$) is $\frac{r(r-1)}{2}$. As we have already discussed, among those sharing a vertex only one distance may be repeated. Further, if d is the repeated distance for vertex A and B we can derive a contradiction: in particular, assume that there exist C_1, C_2, D_1 and D_2 such that $d(A, C_1) = d(A, C_2) = d(B, D_1) = d(B, D_2)$, then the triangles AC_1C_2 and BD_1D_2 are congruent.

The number of possible distances is then $\lfloor n/2 \rfloor$. Since at most r of them are repeated, the number of possible distances (allowing repetitions) is $\lfloor n/2 \rfloor + r$. Therefore,

$$\frac{r(r-1)}{2} \leq \lfloor \frac{n}{2} \rfloor + r.$$

If n is odd, this directly implies that $r(r-3) < n$ and we are done. If n is even, it could happen that $r(r-3) = n$ without two congruent triangles. However, this is not possible. To achieve the equality, one of the distances must be $n/2$ (a diameter AB). In this case, to have equality, there must be two vertices of V, say C and D, at the same distance from A. Hence, C and D are also at the same distance from B, and so the triangles ACB and ADB are congruent.

Comment. There is a limit case where n is a multiple of three and there are three vertices A, B and C with $d(A,B) = d(B,C) = d(C,A) = n/3$. One can easily reach a contradiction in this situation.

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Problems and Solutions from the Regional Olympiad of Honduras, 2022

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The XX edition of the Regional Mathematics Olympiad in Honduras (Olimpiada Departamental de Matemáticas) was held on May 20, 2022 at the national level. As usual in this type of events, one of the main objectives is to promote the study of science in general and, in particular, mathematics. Another objective is to support talent and initiative among young people in the various departments of the country. Usually, 1000 to 5000 students across the country participate in this regional olympiad.

1 Problems and solutions

Hereafter, we present the problems that appeared in the paper of the competition and their solutions.

Problem 1. Tom writes all the numbers from 1 to 20 in a row and gets a 31-digit number:

$$12345678910111213141516171819 20.$$

Then he erases 24 of the 31 digits in such a way that the remaining number is the largest possible. What number did he get?

Solution. Let $N = \overline{a_1a_2a_3a_4a_5a_6a_7}$ the maximum number that can be obtained by deleting 24 digits of the 31 digits written, then N is greater than or equal to the other 7 numbers obtained so

$$N \geq 9101112 > 9000000.$$

Note that the number 9101112 is formed by removing the first 8 digits and the last 16 digits. Digit $a_1 = 9$ must be the first 9 of the sequence, since if it is the last one, N at most will have 3 digits. Then the first 8 digits are necessarily deleted and it remains

$$9101112131415161718192 0.$$

Eliminating the digits between 7 and 9 and deleting the last digit, we get that $N \geq 9718192 > 9700000$. Let $a_2 \geq 7$, suppose that $a_2 \geq 8$ then in the hypothetical case where $a_2 = 8$ then N at most would be 6 digits. Likewise, in the case where $a_2 = 9$ we obtain that N would be at most 4 digits. Therefore it is proved that $a_2 = 7$ so we necessarily delete the digits between 7 and 9. The number we have so far is 97181920. If only one digit must be deleted so that 7 remains in a_3 then the choice must be made between 1 and 8.

Case 1. Let $a_3 = 1$ then $N = 97181920$.

Case 2. Let $a_3 = 8$ then $N_1 = 97811920$.

Note that $N_1 > N$ and therefore this is the largest number that must satisfy the property mentioned in the statement.

Problem 2. Several positive integers are written on the blackboard. The product of the two smallest is 16 and the product of the two largest is 225. Also, all of the numbers on the board are distinct. What is the sum of all the numbers written on the board?

Solution. Let $a < b$ be the smallest numbers and $c < d$ be the largest numbers then we will ask ourselves which numbers when multiplied give as result the number 16 and 225 respectively,

then we have $ab = 16$ which is satisfied by the pairs $(2, 8)$ and $(1, 16)$. Analogously for 225 we have $cd = 225$ with solutions $(9, 25)$, $(5, 45)$, $(3, 75)$, $(1, 225)$. At this point we have the following information $a < b < c < d$ where $8 \leq b$ as a consequence $9 \leq c$ and $c \leq 9$. So, $c = 9$ and it is satisfied that $d = 25$. For $b = 8$ and $c = 9$ results $a = 2$ so that the multiplication is fulfilled. Finally $a = 2$, $b = 8$, $c = 9$ and $d = 25$. The reason why it is not necessary to justify why $b = 8$ and $c = 9$ is because there are no integers between b and c. The final sum is $2 + 8 + 9 + 9 + 25 = 44$.

Problem 3. I am a 4-digit number located between 5000 and 6000. My ones digit is half as many as my tens digit. My hundreds digit is 3 times more than my tens digit. What number am I?

Solution. Let N be the 4-digit number. It is of the form $N = \overline{abcd}$. Then

$$5000 < N = \overline{abcd} < 6000.$$

Here we realize that $a = 5$. Interpreting the data provided by the problem we have $d = \frac{1}{2}c$ so $2d = c$ and $b = 3c$. Substituting c in $b = 3c$ we have $b = 3c = 6d$. As we know that any digit number is less than 10, then we have that $6d = b < 10$ and

$$d < \frac{10}{6} = 1.\overline{66}.$$

Since d is an integer, then $d \leq 1$. Since d is a digit, then $d \geq 0$. Then, $d = 1$ or $d = 0$. We analyze these two cases

- If $d = 0$, then from $2d = c$ we get $c = 0$ and from $b = 3c$ we obtain $b = 0$. In this case, we conclude that $N = 5000$. But from the statement, we know that $N > 5000$. Contradiction.
- If $d = 1$, then $b = 6$ and $c = 2$and the resulting number is 5621.

Problem 4. Four players scored goals in a soccer tournament, all of them scored different amounts of goals. Considering all four players, Miguel scored the least amount of goals. The other three have scored 20 goals in total. What is the highest number of goals Miguel could have scored?

Solution. Let us name the players A, B, C and Miguel. Moreover, let us name the names scored as a, b, c, d where d correspond to Miguel's goals, here we know that all goals are distinct and d is the least number of goals as follows $a \neq b \neq c \neq d$. Without loss of generality, we may assume that $a > b > c > d$. From here, we have $c \geq d + 1$ and $b > c \geq d + 1$. Then $b \geq d + 2$ and $a > b \geq d + 2$. Thus, $a \geq d + 3$. Adding a, b and c in terms of d we have $a + b + c \geq 3d + 6$. Since $a + b + c = 20$, then $20 \geq 3d + 6$ and $4.\overline{66} \geq d$. Since d is integer $4 \geq d$ and $d_{max} = 4$. Just to check that $d_{max} = 4$ we will find an example for a, b, c, d. Let $a = 9, b = 6, c = 5, d = 4$ then $a + b + c = 20$, $a > b > c > d$ with all of them distinct being d the smallest.

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Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Elementary Problems

E–95. *Proposed by Michel Bataille, Rouen, France.* Let α, β, γ be the angles of a triangle. Prove that

$$\frac{\left(1+\tan\frac{\alpha}{4}\right)\left(1+\tan\frac{\beta}{4}\right)\left(1+\tan\frac{\gamma}{4}\right)}{1+\tan\frac{\alpha}{4}\tan\frac{\beta}{4}\tan\frac{\gamma}{4}} = 2.$$

Solution 1 by the proposer. Expressing that $\tan\left(\frac{\alpha+\beta}{4}\right) = \tan\left(\frac{\pi}{4}-\frac{\gamma}{4}\right)$, we obtain

$$\frac{\tan\frac{\alpha}{4}+\tan\frac{\beta}{4}}{1-\tan\frac{\alpha}{4}\tan\frac{\beta}{4}} = \frac{1-\tan\frac{\gamma}{4}}{1+\tan\frac{\gamma}{4}}.$$

Setting $u = \tan\frac{\alpha}{4},\ v = \tan\frac{\beta}{4},\ w = \tan\frac{\gamma}{4}$, we deduce that

$$u+v+w+uv+vw+wu = 1+uvw.$$

Now, the required result follows from

$$\begin{aligned}(1+u)(1+v)(1+w) &= 1+(u+v+w)+(uv+vw+wu)+uvw\\ &= 1+(1+uvw)+uvw = 2(1+uvw).\end{aligned}$$

Solution 2 by Marian Ursărescu, "Roman-Vodă" National College, Roman, Romania. We must show that:

$$\left(1+\tan\frac{\alpha}{4}+\tan\frac{\beta}{4}+\tan\frac{\alpha}{4}\tan\frac{\beta}{4}\right)\left(1+\tan\frac{\gamma}{4}\right)$$

$$=2+2\tan\frac{\alpha}{4}\tan\frac{\beta}{4}\tan\frac{\gamma}{4}$$

$$\sum_{cyc}\tan\frac{\alpha}{4}+\sum_{cyc}\tan\frac{\alpha}{4}\tan\frac{\beta}{4}=1+\prod_{cyc}\tan\frac{\alpha}{4}\quad(1)$$

Now, $\alpha+\beta+\gamma=\pi$, then $\frac{\alpha}{4}+\frac{\beta}{4}+\frac{\gamma}{4}=\frac{\pi}{4}$ and hence,

$$\tan\left(\frac{\alpha}{4}+\frac{\beta}{4}+\frac{\gamma}{4}\right)=\tan\frac{\pi}{4}\ \Leftrightarrow\frac{\tan\left(\frac{\alpha}{4}+\frac{\beta}{4}\right)+\tan\frac{\gamma}{4}}{1-\tan\left(\frac{\alpha}{4}+\frac{\beta}{4}\right)\tan\frac{\gamma}{4}}=1$$

or

$$\frac{\tan\frac{\alpha}{4}+\tan\frac{\beta}{4}}{1-\tan\frac{\alpha}{4}\tan\frac{\beta}{4}}+\tan\frac{\gamma}{4}=1-\frac{\tan\frac{\alpha}{4}+\tan\frac{\beta}{4}}{1-\tan\frac{\alpha}{4}\tan\frac{\beta}{4}}\cdot\tan\frac{\gamma}{4}$$

$$\frac{\tan\frac{\alpha}{4}+\tan\frac{\beta}{4}+\tan\frac{\gamma}{4}-\tan\frac{\alpha}{4}\cdot\tan\frac{\beta}{4}\cdot\tan\frac{\gamma}{4}}{1-\tan\frac{\alpha}{4}\tan\frac{\beta}{4}-\tan\frac{\alpha}{4}\tan\frac{\gamma}{4}-\tan\frac{\beta}{4}\tan\frac{\gamma}{4}}=1$$

$$\Leftrightarrow\sum_{cyc}\tan\frac{\alpha}{4}+\sum_{cyc}\tan\frac{\alpha}{4}\tan\frac{\beta}{4}=1+\prod_{cyc}\tan\frac{\alpha}{4}$$

hence (1) its true.

Solution 3 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Since the angles of a triangle add up to $180°$, we have $\tan\left(\frac{\alpha}{4}+\frac{\beta}{4}+\frac{\gamma}{4}\right)=\tan 45°=1$. Therefore, applying the identity

$$\tan(x+y+z)=\frac{\tan x+\tan y+\tan z-\tan x\tan y\tan z}{1-\tan x\tan y-\tan y\tan z-\tan z\tan x}$$

with $x=\frac{\alpha}{4}$, $y=\frac{\beta}{4}$, $z=\frac{\gamma}{4}$, we obtain

$$1-\tan\frac{\alpha}{4}\tan\frac{\beta}{4}-\tan\frac{\beta}{4}\tan\frac{\gamma}{4}-\tan\frac{\gamma}{4}\tan\frac{\alpha}{4}$$

$$=\tan\frac{\alpha}{4}+\tan\frac{\beta}{4}+\tan\frac{\gamma}{4}-\tan\frac{\alpha}{4}\tan\frac{\beta}{4}\tan\frac{\gamma}{4}$$

or, equivalently,

$$1+\tan\frac{\alpha}{4}\tan\frac{\beta}{4}\tan\frac{\gamma}{4}$$
$$=\tan\frac{\alpha}{4}+\tan\frac{\beta}{4}+\tan\frac{\gamma}{4}+\tan\frac{\alpha}{4}\tan\frac{\beta}{4}+\tan\frac{\beta}{4}\tan\frac{\gamma}{4}+\tan\frac{\gamma}{4}\tan\frac{\alpha}{4}.$$

This may be written in the form

$$1+\tan\frac{\alpha}{4}\tan\frac{\beta}{4}\tan\frac{\gamma}{4}$$
$$=\Big(1+\tan\frac{\alpha}{4}\Big)\Big(1+\tan\frac{\beta}{4}\Big)\Big(1+\tan\frac{\gamma}{4}\Big)-\Big(1+\tan\frac{\alpha}{4}\tan\frac{\beta}{4}\tan\frac{\gamma}{4}\Big).$$

Transposing the last term of the RHS to the left, we get

$$2\Big(1+\tan\frac{\alpha}{4}\tan\frac{\beta}{4}\tan\frac{\gamma}{4}\Big)=\Big(1+\tan\frac{\alpha}{4}\Big)\Big(1+\tan\frac{\beta}{4}\Big)\Big(1+\tan\frac{\gamma}{4}\Big)$$

and

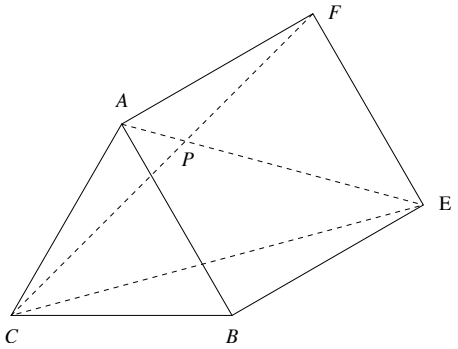
$$\frac{\big(1+\tan\frac{\alpha}{4}\big)\big(1+\tan\frac{\beta}{4}\big)\big(1+\tan\frac{\gamma}{4}\big)}{1+\tan\frac{\alpha}{4}\tan\frac{\beta}{4}\tan\frac{\gamma}{4}}=2,$$

which was to be proved.

Also solved by *Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA; and José Luis Díaz-Barrero and Josep Gibergans Báguena, BarcelonaTech, Barcelona, Spain.*

E–96. *Proposed by José Luis Díaz-Barrero and Josep Gibergans Báguena, BarcelonaTech, Barcelona, Spain.* Let ABC be an equilateral triangle. On the side AB, and not overlapping $\triangle ABC$, build a square $ABEF$. Let P be the intersection point of AE and CF. Show that $PC = PE$.

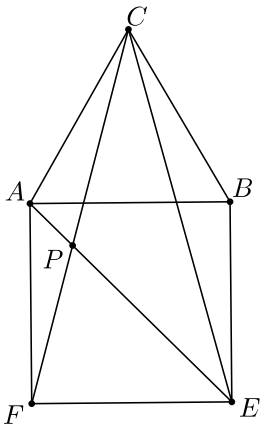
Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. See the diagram below. Triangle ACF is isosceles with $\angle ACF = \angle AFC = 15°$. Similarly, triangle BCE is isosceles with $\angle BCE = \angle BEC = 15°$.



Scheme for solving problem E-96

With $\angle ACB = 60°$, $\angle BEF = 90°$, and $\angle AEF = 45°$, it follows that triangle CEP is isosceles, with $\angle PCE = \angle PEC = 30°$. Thus, $PC = PE$.

Solution 2 by Michel Bataille, Rouen, France. Since $CA = AF$ and $\angle CAF = 60° + 90° = 150°$, we have $\angle ACF = 15°$. Similarly, from $BC = BE$ and $\angle CBE = 150°$, we obtain $\angle BCE = \angle BEC = 15°$. We deduce that $\angle PCE = 60° - 15° - 15° = 30°$ and $\angle CEP = \angle BEA - \angle BEC = 45° - 15° = 30°$. Thus, $\angle PCE = \angle CEP$ and $PC = PE$ follows.



Scheme for solving problem E-96

Also solved by *Miguel Amengual Covas, Cala Figuera, Mallorca, Spain, and the proposers.*

E–97. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* How many ways are there to represent 100000000 as the product of three factors? Factorizations which differ in the order of the factors are considered to be distinct.

Solution by the proposer. Suppose that $100000000 = abc$. Let us factorize each factor a, b or c into prime factors. Since $10^8 = 2^8\,5^8$, then clearly

$$a = 2^{a_1}\,5^{b_1}, \quad b = 2^{a_2}\,5^{b_2}, \quad c = 2^{a_3}\,5^{b_3}.$$

All the exponents above are nonnegative but can vanish and they satisfy

$$a_1 + a_2 + a_3 = b_1 + b_2 + b_3 = 8.$$

There is a one-to-one correspondence between the triples of factors (a, b, c) and sextuples of exponents $(a_1, a_2, a_3, b_1, b_2, b_3)$. Now, we have to count how many solutions in nonnegative integers has the equation $a_1 + a_2 + a_3 = 8$. The exponent a_1 can take any value from 0 through 8. If $a_1 = k$ fixed, then we have $8 - k + 1$ solutions. Hence, we have in total $9+8+7+6+5+4+3+2+1 = 45$ solutions. Likewise, the solutions of $b_1 + b_2 + b_3 = 8$ are 45. Combining all the triples of exponents for 2 and 5 we get that $45 \times 45 = 2025$ is the number of ways in which we can factorize 10^8 as the product of three factors, and we are done.

E–98. *Proposed by Ghițescu Cosmina, Bucharest, Romania.* Find all positive integers such that

$$\lfloor \sqrt{2n+1} + \sqrt{2n+2} \rfloor + \lfloor \sqrt{8n+6} \rfloor = 2\sqrt{n+2015}.$$

Solution by Michel Bataille, Rouen, France. It is easily checked that 289 is a solution. We show that there is no other solution. First, we prove that if n is any positive integer, then

$$\lfloor \sqrt{2n+1} + \sqrt{2n+2} \rfloor = \lfloor \sqrt{8n+6} \rfloor. \tag{1}$$

Let $m = \lfloor \sqrt{8n+6} \rfloor$. Since $8n+6 \equiv 2 \pmod 4$, $8n+6$ is not a perfect square, hence $m < \sqrt{8n+6} < m+1$. Also we remark that

$$8n+6 = (\sqrt{2n+1} + \sqrt{2n+2})^2 + (\sqrt{2n+1} - \sqrt{2n+2})^2$$

hence $m+1 > \sqrt{2n+1} + \sqrt{2n+2}$ and

$$\begin{aligned} 8n+6 &= (\sqrt{2n+1} + \sqrt{2n+2})^2 + \frac{1}{(\sqrt{2n+1} + \sqrt{2n+2})^2} \\ &\leq (\sqrt{2n+1} + \sqrt{2n+2})^2 + \frac{1}{(2+\sqrt{3})^2} \\ &< (\sqrt{2n+1} + \sqrt{2n+2})^2 + 1, \end{aligned}$$

so that $(\sqrt{2n+1} + \sqrt{2n+2})^2 > 8n+6-1 \geq m^2+1-1 = m^2$. The equality (1) follows.
Now, let n be a solution. From (1) we see that $m = \lfloor \sqrt{8n+6} \rfloor = \sqrt{n+2015}$ so that $\sqrt{n+2015} = m < \sqrt{8n+6}$. It follows that $n+2015 < 8n+6$, hence $n \geq 288$ and $m \geq 48$.
In addition, from $8n+6 = 8(m^2-2015)+6 \leq m^2+2m$, we deduce that $7m^2 - 2m - 16114 \leq 0$. But, $p(x) = 7x^2 - 2x - 16114$ is increasing on $[48, +\infty)$ with $p(48) < 0$ and $p(49) > 0$. Thus, m must equal 48 and $n = 48^2 - 2015 = 289$. This completes the proof.

Also solved by *the proposer*

E–99. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* All the positive integers from 19 to 101 are written consecutively to form the number $19202122\ldots99100101$. Find the largest power of 3 that divides it.

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA. The sum of the digits in the number $19202122\cdots99100101$ is

Digits	Sum
19	10
$2021 \cdots 29$	65
$3031 \cdots 39$	75
$4041 \cdots 49$	85
$5051 \cdots 59$	95
$6061 \cdots 69$	105
$7071 \cdots 79$	115
$8081 \cdots 89$	125
$9091 \cdots 99$	135
100101	3
	813

Because 813 is divisible by 3 but not divisible by 9, the number $19202122 \cdots 99100101$ is divisible by 3 but not divisible by 9. Thus, the largest power of 3 that divides $19202122 \cdots 99100101$ is the first power: $3^1 = 3$.

Solution 2 by the proposer. To test the divisibility of the number $19202122 \ldots 99100101$ by 3 or 9 we should find the sum of the digits of it. Note that 1 occurs 12 times in the digits from 19 to 101. That is, in $19, 21, 31, 41, 51, 61, 71, 81, 91, 100, 101$. Likewise, we have that $2, 3, 4$, and 5 occur 18 times; $6, 7, 8$ occur 18 times; and 9 occurs 19 times. Then, we find the sum of the digits of the given number is 813. This number is divisible by 3 because $8 + 1 + 3 = 12$, but it is not divisible by 9. Thus the highest power of 3 dividing $19202122 \ldots 99100101$ is 3.

Solution 3 by Alberto Espuny Díaz, Technische Universität Ilmenau, Germany. In order to answer the question, we want to apply the divisibility criteria for 3 and 9 (recall: we can add the digits of the number and check if the result is divisible by 3; if so, the original number was divisible by 3 too; the same is true for 9). In order to do this, we need to add the digits of the proposed number. We begin by considering the list of all non-negative integers less than 100,

$$\{00, 01, 02, \ldots, 09, 10, 11, 12, \ldots, 98, 99\}.$$

In total, there are 200 digits in this list, and clearly each digit appears the same number of times, that is, each appears exactly 20 times. Now, in the number we are considering, we start with 19; this means that each of the numbers $2, 3, 4, 5, 6, 7, 8$ is not counted twice, 9 is not counted once, and 1 is not counted eleven times (we do not care about the number of times that 0 appears). Finally, since we also add the number 100 and 101 to the list, the digit 1 appears three more times. In total, the number of times that each digit appears in our number is given in the following table:

Digit	1	2	3	4	5	6	7	8	9
Appearances	12	18	18	18	18	18	18	18	19

Therefore, the sum of the digits is

$$12 + 18(2 + 3 + 4 + 5 + 6 + 7 + 8) + 19 \cdot 9 = 813.$$

Adding the digits of the resulting number again, we reach 12. This is clearly divisible by 3, but not by 9. Thus, the highest power of 3 which divides our number is 3 itself.

E–100. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a and b be real numbers such that $\pi/4 \leq a, b \leq \pi/2$. Prove that

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \leq \frac{9}{2}.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA. The inequality is valid for real numbers a and b in any interval $[r, s]$, where $r > 0$ and $s = 2r$.

Without loss of generality, we may assume that $a \leq b$. Thus we may write $b = ta$, $1 \leq t \leq 2$. Consequently,

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{(a+b)^2}{ab} = \frac{(t+1)^2a^2}{ta^2} = \frac{(t+1)^2}{t} \leq \frac{9}{2}$$
$$\iff (2t-1)(t-2) \leq 0,$$

which is true since $t \in [1, 2]$.

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. We will establish the following generalization: Let $c > 0$, $\alpha > 1$, and a and b be real numbers such that $c \le a, b \le \alpha c$. Then

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \le \left(\sqrt{\alpha}+\frac{1}{\sqrt{\alpha}}\right)^2.$$

Taking $c = \pi/4$ and $\alpha = 2$, the result

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \le \left(\sqrt{2}+\frac{1}{\sqrt{2}}\right)^2 = \frac{9}{2}$$

follows.

Note

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{a}{b}+2+\frac{b}{a} = \left(\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right)^2.$$

Let $x = a/b$. With $c \le a, b \le \alpha c$, it follows that $1/\alpha \le x \le \alpha$. Now, consider the function

$$f(x) = \left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2$$

for $1/\alpha \le x \le \alpha$. Then

$$f'(x) = 2\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)\left(\frac{1}{2\sqrt{x}}-\frac{1}{2x^{3/2}}\right),$$

so $x = 1$ is the only critical point of f. Moreover,

$$f\left(\frac{1}{\alpha}\right) = f(\alpha) = \left(\sqrt{\alpha}+\frac{1}{\sqrt{\alpha}}\right)^2,$$

which is greater than 4 by the arithmetic mean - geometric mean inequality, and $f(1) = 4$. Thus, for $1/\alpha \le x \le \alpha$,

$$4 \le f(x) \le \left(\sqrt{\alpha}+\frac{1}{\sqrt{\alpha}}\right)^2.$$

Solution 3 by Daniel Văcaru, Pitești, Romania. We have

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \leq \frac{9}{2} \Leftrightarrow \frac{(a+b)^2}{ab} \leq \frac{9}{2} \Leftrightarrow 2(a+b)^2 \leq 9ab$$

$$\Leftrightarrow 2a^2-5ab+2b^2 \leq 0 \Leftrightarrow b^2\left[2\left(\frac{a}{b}\right)^2-5\frac{a}{b}+2\right] \leq 0$$

$$\Leftrightarrow 2\left(\frac{a}{b}\right)^2-5\left(\frac{a}{b}\right)+2=0$$

Putting $x=a/b$, we get $2x^2-5x+2 \leq 0$. Since $a,b \in \left[\frac{\pi}{4},\frac{\pi}{2}\right] \Rightarrow \frac{a}{b} \in \left[\frac{1}{2},2\right]$ and the parabola of equation $y=2x^2-5x+2$ takes negative values in $(1/2,2)$ and zero in $x=1/2$ and $x=2$, we conclude that the statement holds. Equality occurs when $a=\pi/4$ and $b=\pi/2$ or $a=\pi/2$ and $b=\pi/4$.

Solution 4 by the proposer. We have that if $0<\alpha \leq x \leq \beta$ then $x-\alpha \geq 0$ and $x-\beta \leq 0$. Multiplying up both expressions, we get

$$(x-\alpha)(x-\beta) \leq 0 \Leftrightarrow x^2-(\alpha+\beta)x+\alpha\beta \leq 0 \Leftrightarrow x^2+\alpha\beta \leq (\alpha+\beta)x$$

from which $x+\alpha\beta/x \leq \alpha+\beta$ follows. After division of both terms by $\sqrt{\alpha\beta}$, we obtain

$$\frac{x}{\sqrt{\alpha\beta}}+\frac{\sqrt{\alpha\beta}}{x} \leq \frac{\alpha+\beta}{\sqrt{\alpha\beta}}.$$

Using the preceding result and the well-known inequality $4xy \leq (x+y)^2$, we have

$$\begin{aligned}
4\,(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) &= 4\left(\frac{1}{\sqrt{\alpha\beta}}\cdot\sqrt{\alpha\beta}\right)(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \\
&= 4\left(\frac{a}{\sqrt{\alpha\beta}}+\frac{b}{\sqrt{\alpha\beta}}\right)\left(\frac{\sqrt{\alpha\beta}}{a}+\frac{\sqrt{\alpha\beta}}{b}\right) \\
&\leq 4\left(\frac{a}{\sqrt{\alpha\beta}}+\frac{b}{\sqrt{\alpha\beta}}+\frac{\sqrt{\alpha\beta}}{a}+\frac{\sqrt{\alpha\beta}}{b}\right)^2 \\
&\leq \left(\frac{\alpha+\beta}{\sqrt{\alpha+\beta}}+\frac{\alpha+\beta}{\sqrt{\alpha\beta}}\right)^2 = 4\,\frac{(\alpha+\beta)^2}{\alpha+\beta}
\end{aligned}$$

from which

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \leq \frac{(\alpha+\beta)^2}{\alpha+\beta}$$

follows. Putting $\alpha = \pi/4$ and $\beta = \pi/2$ in the last expression, we get

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \leq \frac{9}{2}.$$

Equality holds when $a = \pi/4$ and $b = \pi/2$ or $a = \pi/2$ and $b = \pi/4$, and the proof is complete.

Solution 5 by Alberto Espuny Díaz, Technische Universität Ilmenau, Germany. By operating, the desired inequality can be rewritten as

$$2 + \frac{a}{b} + \frac{b}{a} \leq \frac{9}{2}.$$

Now let $x = a/b$, so the inequality becomes

$$2 + x + \frac{1}{x} \leq \frac{9}{2}.$$

The bounds on a and b in the statement imply that we must have $1/2 \leq x \leq 2$. Furthermore, the function $f(x) = 2 + x + 1/x$ is convex in the positive reals. Therefore, it suffices to check that the inequality holds in the extremes of the interval that we are interested in, $x = 1/2$ and $x = 2$. And in both cases the inequality holds with equality.

Also solved by *Michel Bataille, Rouen, France.*

Easy–Medium Problems

EM–95. *Proposed by Michel Bataille, Rouen, France.* Let a, b be real numbers. Prove that $|\sin(a-b)|$ belongs to the closed interval with endpoints $|\sin a - \sin b|$ and $|\sin a + \sin b|$.

Solution by Michel Bataille, Rouen, France. We have to prove that

$$(|\sin(a-b)| - |\sin a - \sin b|)(|\sin(a-b)| - |\sin a + \sin b|) \leq 0. \tag{1}$$

If $\sin(a-b) = 0$, (1) writes as $|\sin^2 a - \sin^2 b| \leq 0$, which holds since $\sin^2 a = \sin^2 b$ (because $b \equiv a \pmod{\pi}$). Otherwise, we have

$$|\sin(a-b)| + |\sin a - \sin b| > 0, \qquad |\sin(a-b)| + |\sin a + \sin b| > 0$$

and (1) is successively equivalent to

$$(\sin^2(a-b) - (\sin a - \sin b)^2)(\sin^2(a-b) - (\sin a + \sin b)^2) \leq 0$$

$$(\sin^2 a + \sin^2 b - \sin^2(a-b))^2 \leq 4 \sin^2 a \sin^2 b$$

$$|\sin^2 a + \sin^2 b - \sin^2(a-b)| \leq 2|\sin a \sin b|. \tag{2}$$

Now, we have

$$\sin^2 a + \sin^2 b - \sin^2(a-b) = \sin^2 a + \sin^2 b - (\sin a \cos b - \sin b \cos a)^2$$

$$= 2 \sin^2 a \sin^2 b + 2 \sin a \cos b \sin b \cos a = 2 \sin a \sin b \cos(a-b)$$

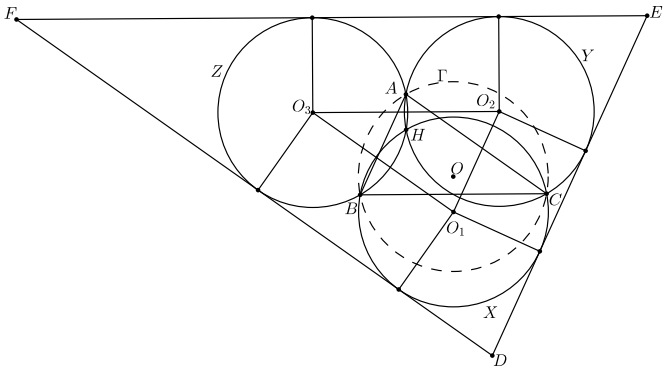
hence

$$|\sin^2 a + \sin^2 b - \sin^2(a-b)| = 2|\sin a \sin b||\cos(a-b)|$$

and (2) follows since $|\cos(a-b)| \leq 1$.

EM–96. *Proposed by Alexandru Benescu, Romania.* Let ABC be a triangle, H its orthocenter and X, Y, Z the circumscribed circles of $\triangle BHC$, $\triangle AHC$, $\triangle AHB$, respectively. Let DE be the common tangent to X and Y, EF to Y and Z and FD to X and Z, such that all 3 circles X, Y and Z are inside $\triangle DEF$. Prove that AD, BE and CF are concurrent.

Solution 1 by Michel Bataille, Rouen, France. Let Γ be the circumcircle of ΔABC and let O be its center. It is well-known that the reflections of H in BC, CA, AB are on Γ. It follows that X, Y, Z are the reflections of Γ in BC, CA, AB and that their respective centers O_1, O_2, O_3 are the reflections of O in BC, CA, AB. The line O_1O_2 through the centers of X and Y is perpendicular



Scheme for solving problem EM-96

to the line CH through their common points, hence is parallel to AB. Since X, Y have the same radius (the circumradius of ΔABC), the line DE is parallel to O_1O_2. Finally, DE is parallel to AB. In the same way, we obtain that EF is paralle to BC and FD is paralle to CA. Thus, the triangles ABC and DEF are homothetic and the lines AD, BE, CF intersect at the center of homothety.

Solution 2 by Alexandru Benescu, Romania modified by Marc Felipe i Alsina, BarcelonaTech, Barcelona, Spain. The circles Γ_A, Γ_B and Γ_C are have the same radius. Indeed, since the reflection H' of H on BC lies on the circumscribed circle Γ of ABC,

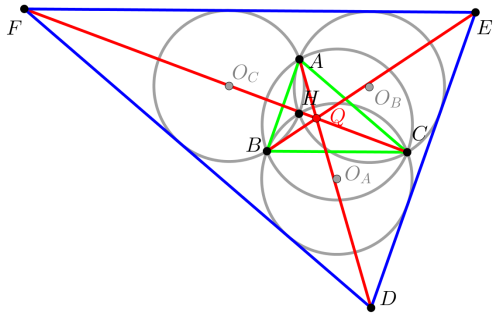


Figure 1: Scheme for solving problem ME-96

we have that the circumcircles Γ_A of $\triangle BHC$ and Γ of $\triangle BH'C$ are congruent because there is a symmetry that sends one to the other. Repeating the argument with the other sides of the triangle we see that Γ_A, Γ_B and Γ_C all have the same radius R as Γ.

Let O_B be the center of Γ_B and O_C be the center of Γ_C. The four sides of quadrilateral O_BAO_CH are radii, so it is a rhombus. This implies $AH \perp O_BO_C$, but O_BO_C is parallel to EF because the common tangent between two circles of the same radii are parallel to the line joining their centers. So, EF is perpendicular to AH, which means it is parallel to BC. Similarly, $AB \parallel DE$ and $AC \parallel DF$, which implies that $\triangle ABC$ and $\triangle DEF$ have parallel sides. This implies that both triangles are either congruent or homothetic. We can discard the congruent case since the first one fits inside a circle Γ of radius R and the second one contains Γ_A, also of radius R. The lines AD, BE and CF concur at the center Q of the homothety of the two triangles.

EM–97. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Consider the polynomial $A(x) = x^5 + x^4 - 4x^3 + x^2 + x + 3$. Prove that for every positive integer n, the polynomial $A(x)^n$ has at least one negative coefficient.

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY. We know that every polynomial of odd degree has at least one real zero. Furthermore, Descartes' rule of signs tells us that $A(x)$

has at most two real zeros. It follows that $A(x)$ has exactly one real zero r and two complex conjugate pairs of zeros. If r were positive, the AGM inequality would give us

$$A(r) = (r^5+r^4+r^2+r)-4r^3+3 \geq 4\sqrt[4]{r^{12}}-4r^3+3 = 3 > 0,$$

a contradiction. Thus $A(x)$ has one negative real root r, and we can write $A(x) = (x-r)Q(x)$ and $A(x)^n = (x-r)^n Q(x)^n$. If the polynomial $A(x)^n$ were to have all nonnegative coefficients, then $A(x)^n$ would have no real roots, but we see that $A(x)^n$ has at least the real root r of multiplicity n. This contradiction tells us that $A(x)^n$ must have at least one negative coefficient.

Solution 2 by the proposer. Observe that for any polynomial $A(x)$ its value at the point $x = 1$ is equal to the sum of all its coefficients. Consequently, the sum of the coefficients of the polynomial $A(x)^n$ is equal to $A(1)^n = (1+1-4+1+1+3)^n = 3^n$. But the free term of $A(x)^n$ is equal to $A(0)^n = 3^n$, while the coefficient of x^{5n} is equal to 1, and their sum is already 3^n+1. Hence one of the remaining coefficients of $A(x)^n$ is negative.

EM–98. *Proposed by Mihaela Berindeanu, Bucharest, Romania.*

Find all $x, y, z \in (0,3)$ such that $\begin{cases} (x^2+y^2)\sqrt{9-z^2} \geq 9z, \\ (y^2+z^2)\sqrt{9-x^2} \geq 9x, \\ (z^2+x^2)\sqrt{9-y^2} \geq 9y. \end{cases}$

Solution 1 by Michel Bataille, Rouen, France. We show that the only solution for (x,y,z) is $\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$.
It is readily checked that this triple is a solution. Conversely, let (x,y,z) be an arbitrary solution. From the first inequality, we obtain $(x^2+y^2)^2(9-z^2) \geq 81z^2$, that is,

$$9(x^2+y^2)^2 \geq z^2(81+(x^2+y^2)^2). \tag{1}$$

But, $81+(x^2+y^2)^2 \geq 18(x^2+y^2)$ (since $(9-(x^2+y^2))^2 \geq 0$) so that (1) gives $x^2+y^2-2z^2 \geq 0$. Similarly, we obtain $y^2+z^2-2x^2 \geq 0$ and $z^2+x^2-2y^2 \geq 0$.
Now, since $0 = (x^2+y^2-2z^2)+(y^2+z^2-2x^2)+(z^2+x^2-2y^2)$, the sum of three nonnegative terms, we must have $x^2+y^2 =$

$2z^2,\ y^2+z^2=2x^2,\ z^2+x^2=2y^2$. It follows that $x^2=y^2=z^2(=(x^2+y^2+z^2)/3)$, hence $x=y=z$ (since $x,y,z>0$). From (1), we now have $4x^4+81-36x^2\le 0$, that is, $(2x^2-9)^2\le 0$. Thus, $x^2=\frac{9}{2}$ and consequently $x=y=z=\frac{3\sqrt{2}}{2}$.

Solution 2 by the proposer. Dividing the first inequality by $z^2\sqrt{9-z^2}$, yields $\dfrac{x^2+y^2}{z^2}\ge\dfrac{9}{z\sqrt{9-z^2}}$. Applying AM-GM inequality, we get $z\sqrt{9-z^2}\le\dfrac{z^2+9-z^2}{2}\Rightarrow z\sqrt{9-z^2}\le\dfrac{9}{2}$. From the above, we obtain $\dfrac{x^2+y^2}{z^2}\ge 2\Rightarrow x^2+y^2\ge 2z^2$. Analogously, $y^2+z^2\ge 2x^2$ and $z^2+x^2\ge 2y^2$. Adding up the preceding, yields $2x^2+2y^2+2z^2\ge 2x^2+2y^2+2z^2$. The case of equality occurs for $z=\sqrt{9-z^2}$, $y=\sqrt{9-x^2}$, and $z=\sqrt{9-y^2}$. Solving the preceding, we get that the only solution (x,y,z) is

$$\left(\frac{3\sqrt{2}}{2},\frac{3\sqrt{2}}{2},\frac{3\sqrt{2}}{2}\right).$$

Also solved by *José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.*

EM–99. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Find all prime numbers $r<q<p$ such that $\sqrt{p^2-13q^2-r^2}$ is rational and $\sqrt{p+3r-q}$ is a positive integer.

Solution 1 by Michel Bataille, Rouen, France. It is readily checked $(2,3,61)$ is a solution for (r,q,p) (if $p=61,q=3$ and $r=2$, then $\sqrt{p^2-13q^2-r^2}=60$ and $\sqrt{p+3r-q}=8$). We show that there is no other solution.
First, if m is a nonnegative integer and $\sqrt{m}$ is a rational number, then $\sqrt{m}$ is an integer: indeed, we may suppose that $m\ge 1$; if $\sqrt{m}=u/v$ for some positive, coprime integers u,v, then $mv^2=u^2$ and if we had $v>1$, a prime factor of v would divide u, contradicting that u,v are coprime. Thus $v=1$ and the result follows.
Now, let (r,q,p) be an arbitrary solution. From the above property,

there exists a nonnegative integer s such that

$$p^2 = s^2 + 13q^2 + r^2. \tag{1}$$

Assume that r is odd; then p, q are odd and so is s. Since the square of an odd integer is congruent to 1 modulo 4, the relation (1) leads to a contradiction modulo 4. Thus, we must have $r = 2$. Assume that $q > 3$. Then p and q are congruent to 1 or 5 modulo 6, hence p^2 and q^2 are congruent to 1 modulo 6 and (1) yields $s^2 + 4 \equiv 0 \pmod 6$, a contradiction since the squares modulo 6 are $0, 1, 3, 4$). Thus, we must have $q = 3$.
The relation (1) now gives $p^2 - s^2 = 121$, that is, $(p - s)(p + s) = 11^2$. We must have $s \neq 0$ since otherwise $p = 11$, in which case $\sqrt{p + 3r - q}$ is not a positive integer. Thus $0 < p - s < p + s$ and we have $p - s = 1$ and $p + s = 11^2$, hence $2p = 122$ and $p = 61$. Finally $p = 61, q = 3, r = 2$ is the only possible solution and we are done.

Solution 2 by the proposer. $\sqrt{p^2 - 13q^2 - r^2} \in \mathbb{Q}^*$ if and only if $p^2 - 13q^2 - r^2$ it is a perfect square.
Observation: If all three prime numbers are odd, the value below the radical $p^2 - 13q^2 - r^2$ would be an odd number and the square of an odd number is of the form $4k$ or $4k + 1$.
But $r > 2 \Rightarrow p^2 \equiv q^2 \equiv r^2 \equiv 1$ modulo 4 and $p^2 - 13q^2 - r^2 \equiv 3$ modulo 4, so $p^2 - 13q^2 - r^2$ is of the form $4k + 3$. Taking into account the above observation, it is necessary to conclude that the smallest number in the suite $r < q < p$ is even, so $r = 2$. With $r = 2$, the expression under the radical becomes $p^2 - 13q^2 - 4$. Because q is a prime number and $q > r \Rightarrow q \geq 3$
I will prove by contradiction that $q = 3$
Suppose that $q > 3 \Rightarrow p > 3 \Rightarrow p^2 \equiv q^2 \equiv 1$ (modulo 6), so $p^2 - 13q^2 - 4 \equiv 2$ modulo 6. So, for $q > 3$, $p^2 - 13q^2 - 4$ is not a perfect square $\Rightarrow q = 3$. Now, we must determine the prime number for which $p^2 - 13 \cdot 9 - 4$ is a perfect square, so I'm looking for $a \in \mathbb{N}$, $a \neq 0$ so that $p^2 - 121 = a^2 \Rightarrow (p - a)(p + a) = 121$.
From which we have

$$\begin{cases} p - a = 1 \\ p + a = 121 \end{cases} \Rightarrow \begin{cases} p = 61 \\ a = 60 \end{cases}$$

$\begin{cases} p-a=11 \\ p+a=11 \end{cases} \Rightarrow \begin{cases} p=11 \\ a=0 \end{cases}$ Contradiction with $a \neq 0$.
With $r=2,\ q=3,\ p=61 \Rightarrow \sqrt{p+3r-q}=\sqrt{61+6-3}=8$.

EM–100. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Assume you have a dodecahedron, so that when it is thrown, each of the faces (which are numbered 1 to 12) occurs with equal probability. Determine the probability that after 24 throws of the dodecahedron, the product of all the numbers thrown will be divisible by 14.

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Because $14 = 2 \cdot 7$,

$$P(14 \nmid \text{product}) = P(\text{no sevens}) + P(\text{no evens}) - P(\text{no sevens} \cap \text{no evens}).$$

In n throws of the dodecahedron,

$$\begin{aligned} P(\text{no sevens}) &= \left(\frac{11}{12}\right)^n, \\ P(\text{no evens}) &= \left(\frac{1}{2}\right)^n, \text{ and} \\ P(\text{no sevens} \cap \text{no evens}) &= \left(\frac{5}{12}\right)^n, \end{aligned}$$

where the intersection between throwing no sevens and throwing no evens is the set $\{1, 3, 5, 9, 11\}$. Thus,

$$\begin{aligned} P(14 \mid \text{product}) &= 1 - P(14 \nmid \text{product}) \\ &= 1 - \left(\frac{11}{12}\right)^n - \left(\frac{1}{2}\right)^n + \left(\frac{5}{12}\right)^n. \end{aligned}$$

For 24 throws of the dodecahedron, this becomes

$$P(14 \mid \text{product}) = 1 - \left(\frac{11}{12}\right)^{24} - \left(\frac{1}{2}\right)^{24} + \left(\frac{5}{12}\right)^{24}.$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Let $P_{14}(n)$ denote the probability that the product of the numbers thrown for n throws of the dodecahedron is divisible by 14. Clearly, $P_{14}(1) = 0$, and $P_{14}(2) = \frac{1}{12}$, as either the first throw needs to produce a 7 and the second throw an even or the first throw needs to produce an even and the second throw a 7. Now, the product of the numbers thrown for n throws of the dodecahedron will be divisible by 14 if

- the first throw produces a 1, 3, 5, 9, or 11 and the remaining $n-1$ throws produce a product that is divisible by 14; or
- the first throw produces an even and the remaining $n-1$ throws produce at least on 7; or
- the first throw produces a 7 and the remaining $n-1$ throws produce at least one even.

Thus,

$$P_{14}(n) = \frac{5}{12}P_{14}(n-1) + \frac{1}{2}\left(1 - \left(\frac{11}{12}\right)^{n-1}\right) + \frac{1}{12}\left(1 - \frac{1}{2^{n-1}}\right).$$

Working backwards through this recurrence relations,

$$\begin{aligned} P_{14}(n) &= \left(\frac{5}{12}\right)^{n-2} P_{14}(2) + \frac{7}{12}\sum_{k=0}^{n-3}\left(\frac{5}{12}\right)^{k} - \frac{1}{2}\sum_{k=2}^{n-1}\left(\frac{5}{12}\right)^{n-1-k}\left(\frac{11}{12}\right)^{k} \\ &\quad -\frac{1}{12}\sum_{k=2}^{n-1}\left(\frac{5}{12}\right)^{n-1-k}\frac{1}{2^k} \\ &= 1 - \left(\frac{11}{12}\right)^{n} - \left(\frac{1}{2}\right)^{n} + \left(\frac{5}{12}\right)^{n}. \end{aligned}$$

For 24 throws of the dodecahedron, this becomes

$$P_{14}(24) = 1 - \left(\frac{11}{12}\right)^{24} - \left(\frac{1}{2}\right)^{24} + \left(\frac{5}{12}\right)^{24}.$$

Solution 3 by the proposer. Let A be the set of 24 tosses that do not contain an even number and B the set of 24 tosses that do not contain 7. The product of numbers shown in 24 tosses is a multiple of 14 if and only if both an even number and 7 appear

among the 24 tosses. Thus we need to count $|\overline{A} \cap \overline{B}|$. On account of Morgan's Law, we have to compute

$$|\overline{A} \cap \overline{B}| = |\overline{A \cup B}| = 12^{24} - |A \cup B|.$$

Now, $|A| = 6^{24}$ (since each toss can be any of the six outcomes, $1, 3, 5, 7, 9, 11$), $|B| = 11^{24}$ and $|A \cap B| = 5^{24}$. Thus, on account of PIE, we have

$$|A \cup B| = |A| + |B| - |A \cap B| = 6^{24} + 11^{24} - 5^{24}$$

and

$$|\overline{A} \cap \overline{B}| = 12^{24} - (6^{24} + 11^{24} - 5^{24}) = 12^{24} - 6^{24} - 11^{24} + 5^{24}.$$

Finally, the probability required

$$p = \frac{12^{24} - 6^{24} - 11^{24} + 5^{24}}{12^{24}} \simeq 0.8760.$$

Also solved by *Alberto Espuny Díaz, Technische Universität Ilmenau, Germany.*

Medium–Hard Problems

MH–95. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let a, b, c be nonzero real numbers such that $a^3 + b^3 + c^3 = 0$. Find the maximum value of

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\left(\frac{a}{c} + \frac{c}{b} + \frac{b}{a}\right)$$

and determine where the maximum holds.

Solution 1 by the proposer. Let

$$A = \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\left(\frac{a}{c} + \frac{c}{b} + \frac{b}{a}\right)$$

The conditions $a^3 + b^3 + c^3 = 0, a \neq 0, b \neq 0, c \neq 0$ imply that at least one of the three variables a, b, and c have to be negative, and at least one — positive. By the Dirichlet's pigeonhole principle implies that two of the variables have the same sign. Without loss of generality, we may assume that $sign(a) = sign(b) \neq sign(c)$. It follows that $ab > 0, sign(a + b) = -sign(c)$.

It is clear that $A(-a, -b, -c) = A(a, b, c)$, and without loss of generality, we may assume that $c > 0, a < 0, b < 0$.

Let $u > 0, v > 0$ be real numbers such that $a + b = -uc, ab = vc^2$.

$$\begin{aligned}(-uc)^3 = (a+b)^3 &= a^3 + b^3 + 3ab(a+b)\\ &= -c^3 + 3vc^2(-uc) = c^3(-1 - 3uv)\\ &\Leftrightarrow -u^3 = -1 - 3uv\end{aligned}$$

$$v = \frac{u^3 - 1}{3u} \tag{1}$$

Now

$$0 \leq (a-b)^2 = a^2 + b^2 - 2ab = (a+b)^2 - 4ab =$$

$$= u^2c^2 - 4vc^2 = \left(u^2 - 4\frac{u^3 - 1}{3u}\right)c^2 \Leftrightarrow u^3 \leq 4$$

$$u \leq \sqrt[3]{4} \tag{2}$$

with equality if and only if $a = b$.

From (1) and (2) the following inequality holds

$$v = \frac{u^3 - 1}{3u} = \frac{1}{3}\left(u^2 - \frac{1}{u}\right) \leq \frac{1}{3}\left(\left(\sqrt[3]{4}\right)^2 - \frac{1}{(\sqrt[3]{4})}\right)$$

$$v \leq \frac{1}{\sqrt[3]{4}} \tag{3}$$

$$\begin{aligned} A &= \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\left(\frac{a}{c} + \frac{c}{b} + \frac{b}{a}\right) = \\ &= \frac{a^2c + c^2b + b^2a}{abc} \cdot \frac{a^2b + b^2c + c^2a}{abc} = \\ &= \frac{abc^4 + (a^3 + b^3)c^3 + 3a^2b^2c^2 + (a^3 + b^3)abc + a^3b^3}{a^2b^2c^2} = \\ &= \frac{vc^6 + (-c^3)c^3 + 3v^2c^6 + (-c^3)vc^3 + v^3c^6}{v^2c^6} = \\ &= \frac{v^3 + 3v^2 - 1}{v^2} = 3 + v - \frac{1}{v^2} \end{aligned}$$

By (3), it follows that

$$A \leq 3 + \frac{1}{\sqrt[3]{4}} - \frac{1}{\left(\frac{1}{\sqrt[3]{4}}\right)^2} = \frac{1 + 3\sqrt[3]{4} - 4}{\sqrt[3]{4}} = \frac{3}{2}\left(2 - \sqrt[3]{2}\right)$$

The maximum value $A_{max} = \frac{3}{2}\left(2 - \sqrt[3]{2}\right)$ occurs if $a = b$, and since $0 = a^3 + b^3 + c^3 = 2a^3 + c^3$, it follows that $c = -\sqrt[3]{2}a$.

Finally the maximum value holds for any permutation of $(a, b, c) = (t, t, -t\sqrt[3]{2}), t \in \mathbb{R}, t \neq 0$.

Solution 2 by Michel Bataille, Rouen, France. Let P denote the product $\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\left(\frac{a}{c} + \frac{c}{b} + \frac{b}{a}\right)$. We claim that the required maximum value of P is $3 - \frac{3\sqrt[3]{2}}{2}$, attained when

$$(a, b, c) = (-m\sqrt[3]{2}, m, m)$$

(or a permutation) for some non-zero real number m.
If m is a non-zero real number and $(a,b,c) = (-m\sqrt[3]{2}, m, m)$ (or a permutation), we have $a^3+b^3+c^3 = 0$ and $P = 3 - \frac{3\sqrt[3]{2}}{2}$.
Now let $a,b,c \neq 0$ be such that $a^3+b^3+c^3 = 0$. We prove that

$$P \leq 3 - \frac{3\sqrt[3]{2}}{2}.$$

Without loss of generality, we suppose that $a \leq b \leq c$. Since $a^3+b^3+c^3 = 0$, we then have $a < 0$ and $c > 0$. Also, an easy calculation shows that

$$P = \frac{a^3b^3+b^3c^3+c^3a^3+3a^2b^2c^2}{a^2b^2c^2} = 3 + \frac{ab}{c^2} + \frac{bc}{a^2} + \frac{ca}{b^2},$$

so that

$$P - 3 = \frac{b(c^3+a^3)}{c^2a^2} + \frac{ca}{b^2} = \frac{ca}{b^2} - \frac{b^4}{c^2a^2} = f(b^2)$$

where f is the function defined by $f(x) = \frac{ca}{x} - \frac{x^2}{c^2a^2}$. Then $f'(x) = \frac{-2}{c^2a^2x^2}\left(x^3 + \frac{c^3a^3}{2}\right)$ and since $0 < -\frac{ca}{\sqrt[3]{2}} < -ca \leq \max(a^2,c^2)$, we see that $f\left(-\frac{ca}{\sqrt[3]{2}}\right) = -\frac{3\sqrt[3]{2}}{2}$ is the maximum value of $f(x)$ for $x \in (0, \max(a^2,c^2))$. Thus, $f(b^2) \leq -\frac{3\sqrt[3]{2}}{2}$ and $P \leq 3 - \frac{3\sqrt[3]{2}}{2}$.
The equality $f(b^2) = -\frac{3\sqrt[3]{2}}{2}$ holds if and only if $b^2 = \frac{-ca}{\sqrt[3]{2}}$. In the case $b = \frac{\sqrt{c}\sqrt{-a}}{2^{1/6}}$, setting $u = \sqrt{c^3}, v = \sqrt{-a^3}$, the condition $a^3+b^3+c^3 = 0$ leads to $(u/v)^2 + (1/\sqrt{2})(u/v) - 1 = 0$, hence $v = u\sqrt{2}$ and therefore $-a^3 = 2c^3$. Thus $a = -c\sqrt[3]{2}$ and $b = \frac{2^{1/6}\sqrt{c}\sqrt{c}}{2^{1/6}} = c$, meaning that $(a,b,c) = (-c\sqrt[3]{2}, c, c)$.
Similarly, $b = -\frac{\sqrt{c}\sqrt{-a}}{2^{1/6}}$ yields $(a,b,c) = (a, a, -a\sqrt[3]{2})$. The claim follows.

Solution 3 by Moti Levy, Rehovot, Israel.

$$\begin{aligned} S &:= \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right) \\ &= \frac{(a^2b+ac^2+b^2c)(a^2c+ab^2+bc^2)}{(abc)^2} \\ &= \frac{abc(a^3+b^3+c^3)+a^3b^3+a^3c^3+b^3c^3+3a^2b^2c^2}{(abc)^2} \\ &= 3+\frac{a^3b^3+a^3c^3+b^3c^3}{(abc)^2}. \end{aligned} \tag{4}$$

It is evident from (4) that S is bounded by 3 (since $a^3b^3+a^3c^3+b^3c^3<0$).

Let $t:=\frac{b}{a}$, then $a^3+b^3+c^3=0$ implies

$$c=-\sqrt[3]{1+t^3}a, \tag{5}$$

and

$$S(t)=3-\frac{t^6+t^3+1}{t^2(t^3+1)^{\frac{2}{3}}}. \tag{6}$$

The constraint $a^3+b^3+c^3=0$ implies that the three numbers cannot have the same sign, hence without loss of generality, we may assume that $t>0$.
The function $S(t)$ is twice differentiable for $t>0$.

$$\begin{aligned} \frac{dS}{dt} &= \frac{(1-t)(1+t+t^2)(2+t^3)(1+2t^3)}{t^3(t^3+1)^{\frac{5}{3}}}, \\ \frac{d^2S}{dt^2} &= -2\frac{(3+8t^3+12t^6+3t^9+t^{12})}{t^4(t^3+1)^{\frac{8}{3}}}. \end{aligned}$$

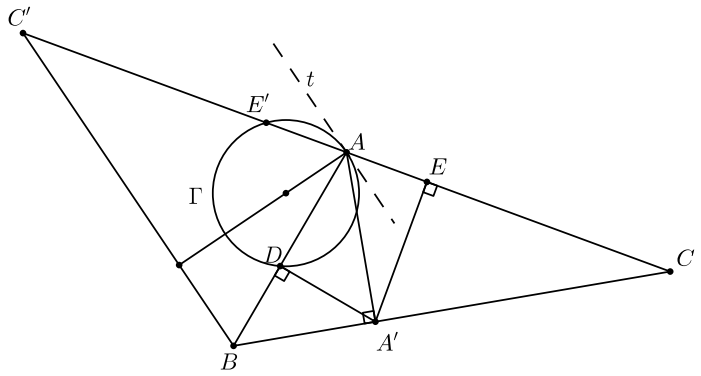
$S'(1)=0$ while $S''(1)<0$, hence $S(t)$ has maximum at $t=1$ and the maximum value is $S(1)=3-\frac{3}{2}\sqrt[3]{2}\cong 1.11012$.

The maximum holds when

$a=b=1,\ c=-\sqrt[3]{2}$, or $b=c=1,\ a=-\sqrt[3]{2}$, or $c=a=1,\ b=-\sqrt[3]{2}$, or $a=b=-1,\ c=\sqrt[3]{2}$, or $b=c=-1,\ a=\sqrt[3]{2}$, or $c=a=-1,\ b=\sqrt[3]{2}$.

MH–96. *Proposed by Michel Bataille, Rouen, France.* Let A' be the foot of the altitude from A in ΔABC and let D, E be the orthogonal projections of A' onto AB and AC. If E' is the reflection of E about A, show that the circumcircle of $\Delta ADE'$ is tangent to the median from A in ΔABC.

Solution 1 by the proposer. The inversion in the circle with centre A, radius AA' inverts the line BC into the circle with diameter AA', hence exhanges B and D as well as C and E. It

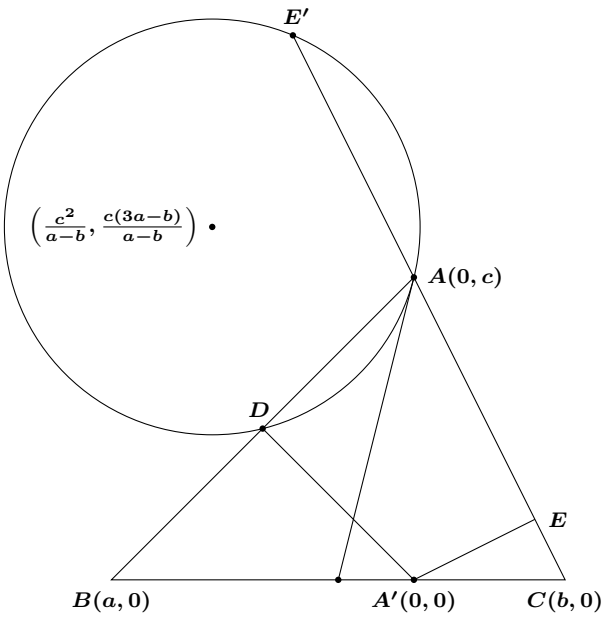


Scheme for solving problem MH-96

follows that the inverse of E' is the reflection C' of C about A. Now, let Γ be the circumcircle of $\Delta ADE'$. The line BC', as the inverse of Γ, is perpendicular to the diameter through A, hence is parallel to the tangent t at A. Thus, t, which is parallel to $C'B$ and passes through the midpoint A of CC', intersects BC at its midpoint. The result follows.

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Place the origin in A', and the x- axis along BC. Let $A = (0, c)$, $B = (a, 0)$, and $C = (b, 0)$.

It is easy to show that $D = \left(\frac{ac^2}{c^2+a^2}, \frac{a^2c}{c^2+a^2}\right)$, $E = \left(\frac{bc^2}{b^2+c^2}, \frac{b^2c}{b^2+c^2}\right)$, and $E' = \left(\frac{-bc^2}{b^2+c^2}, \frac{c\left(b^2+2c^2\right)}{b^2+c^2}\right)$.



The equations of the perpendiculars bisectors of AD and AE' are quickly found to be

$$y = \frac{b}{c}x + \frac{3c}{2}, \qquad y = \frac{a}{c}x + \frac{c}{2}.$$

Solving these simultaneous equations for x and y, the coordinates of the circumcenter of $\triangle ADE'$ are

$$\left(\frac{c^2}{a-b}, \frac{c(3a-b)}{a-b}\right).$$

The radius through A, then, has slope $\frac{a+b}{2c}$. The median of $\triangle ABC$ drawn from A has slope $\frac{-2c}{a+b}$.

The slopes are the negative reciprocals of each other and the median from A in $\triangle ABC$ is perpendicular to the radius at A.

We conclude that the circumcircle of $\triangle ADE'$ is tangent to the median from A in $\triangle ABC$ at A.

MH–97. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all positive real numbers for which

$$126x^7 - 127x^6 + 1 = 0.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. By inspection we see that $x = 1$ and $x = 1/2$ are solutions of the given equation. We can factor the polynomial:

$$126x^7-127x^6+1 = (x-1)(2x-1)(63x^5+31x^4+15x^3+7x^2+3x+1)$$

$$= (x-1)(2x-1)P(x).$$

Therefore, we have the solutions $x = 1/2$ and $x = 1$. These are the only positive real solutions since $P(x) > 1$ for $x > 0$.

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Let $f(x) = 126x^7 - 127x^6 + 1$. By Descartes' Rule of Signs, f has either zero or two positive real roots. By inspection,

$$f(1) = 126(1) - 127(1) + 1 = 0,$$

so f must have two positive real roots. To find the second positive real root, using the Rational Root Theorem, we find

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^6\left(126 \cdot \frac{1}{2} - 127\right) + 1 = \frac{1}{64}(63 - 127) + 1 = 0.$$

The two positive real numbers for which $126x^7 - 127x^6 + 1 = 0$ are

$$x = 1 \quad \text{and} \quad x = \frac{1}{2}.$$

Solution 3 by Michel Bataille, Rouen, France. It is readily checked that 1 and $\frac{1}{2}$ are solutions. We show that there are no other solutions.
Let $p(x) = 126x^7 - 127x^6 + 1$. We have $p(x) = (x - 1)q(x)$ where

$$q(x) = 126x^6 - (x^5 + x^4 + x^3 + x^2 + x + 1) =$$

$$120x^6+x^5(x-1)+x^4(x^2-1)+x^3(x^3-1)+x^2(x^4-1)+x(x^5-1)+x^6-1.$$

Clearly, $q(x) > 0$ when $x > 1$, hence $p(x) \neq 0$ if $x > 1$.
We have $q(x) = x^6 r(1/x)$ where

$$r(x) = 127-(x^6+x^5+x^4+x^3+x^2+x+1) = 127-\frac{1-x^7}{1-x} = \frac{s(x)}{1-x}.$$

The polynomial $s(x)$ satisfies

$$s(x) = x^7 - 127x + 126 = x^7 - 128 - 127(x-2) =$$

$$(x-2)(x^6+2x^5+4x^4+8x^3+16x^2+32x-63) =$$

$$(x-2)(x^6-1+2(x^5-1)+4(x^4-1)+8(x^3-1)+16(x^2-1)+32(x-1)),$$

hence $s(x)$ vanishes only once on $(1,\infty)$, namely for $x = 2$. It follows that if $0 < x < 1$, then $q(x) = 0$ only if $x = \frac{1}{2}$. This completes the proof.

Solution 4 by the proposer. It is easy to see that $x = 1$ and $x = 1/2$ are solutions of the given equation. We claim that they are the only real solutions. To prove our claim we will write the equation in the most convenient form

$$\frac{1}{x^6} = 127 - 126x$$

Since the function $f(x) = \dfrac{1}{x^6}$ is strictly convex in $(0,+\infty)$, as it is well known, and $g(x) = 127 - 126x$ is a straight line, then the graphs of f and g have at most two points in common. So, the only real solutions of $f(x) = g(x)$ are $x = 1/2$ and $x = 1$ as claimed.

Finally, we will proof that if $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a strictly convex function on I and $a, b \in \mathbb{R}$, then the equation $f(x) = ax + b$ has at most two real roots. We will argue by contradiction. Suppose that the equation $f(x) - ax - b = 0$ has three distinct roots. Namely, $x_1 < x_2 < x_3$. Then exists $t \in (0,1)$ such that $x_2 = (1-t)x_1 + tx_3$. Since f is strictly convex, then

$$f(x_2) = f[(1-t)x_1 + tx_3] < (1-t)f(x_1) + tf(x_3)$$

$$= (1-t)(ax_1+b) + t(ax_3+b) = ax_2 + b = f(x_2)$$

which is impossible and the proof is complete.

Also solved by *José Gibergnas-Báguena, BarcelonaTech, Barcelona, Spain.*

MH–98. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a scalene triangle. Erect a square outwardly from each side of triangle ABC. Let A_1, B_1, C_1 be the respective centers of the squares. The lines AA_1, BB_1, CC_1 concur in V_1 – the Vecten point. We denote by C_2, A_2, B_2 the symmetric points of C_1, A_1, B_1 with respect to AB, BC, CA, respectively. The lines AA_2, BB_2, CC_2 concur in V_2 – the Inner Vecten point. Knowing that V_1, V_2, A are collinear, find angle $\angle BAC$.

Solution by the proposer. It is clear that the triangles ABC_1, BCA_1, CAB_1 are isosceles and right-angled, with

$$\angle BA_1C = \angle CB_1A = \angle AC_1B = 90^\circ$$

$$\angle A_1BC = \angle A_1CB = \angle B_1AC = \angle B_1CA = \angle C_1BA = \angle C_1AB = 45^\circ.$$

The point V_1 lies on the line AA_1 and V_2 lies on the line AA_2. But A_1 and A_2 are symmetric with respect to BC, and they lie on the perpendicular bisector of BC, i.e. $A_1 \not\equiv A_2$ and A, A_1, A_2 collinear if and only if A be on the perpendicular bisector of BC, i.e. $AB = AC$, a contradiction with hypothesis that ABC is a scalene triangle.

It follows that A, V_1, V_2 are collinear if and only if $V_1 \equiv A$ or $V_2 \equiv A$.

Case 1. $V_1 \equiv A$, (Figure 3)

The point V_1 lies on BB_1 hence B, A, B_1 are collinear and A is between B and B_1.

$$\angle BV_1B_1 = 180^\circ = \angle BAB_1 = \angle BAC + \angle CAB_1 = \angle BAC + 45^\circ$$

$$\angle BAC = 135^\circ$$

Case 2. $V_2 \equiv A$, (Figure 4)

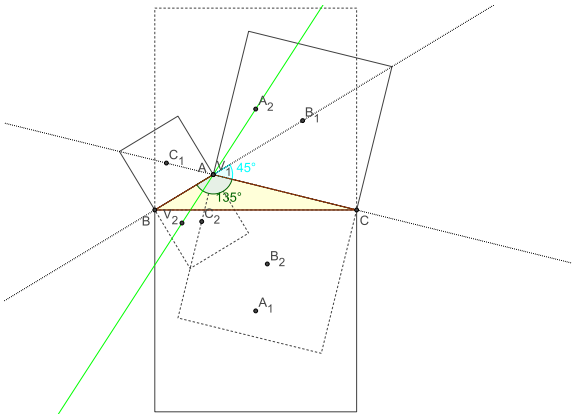


Figure 2: Scheme for solving problem MH-98

The point V_2 lies on BB_2 hence B, A, B_2 are collinear and A is not between B and B_2.

$$\angle BAC = \angle BAC_2 = \angle BAC_1 = 45^\circ$$

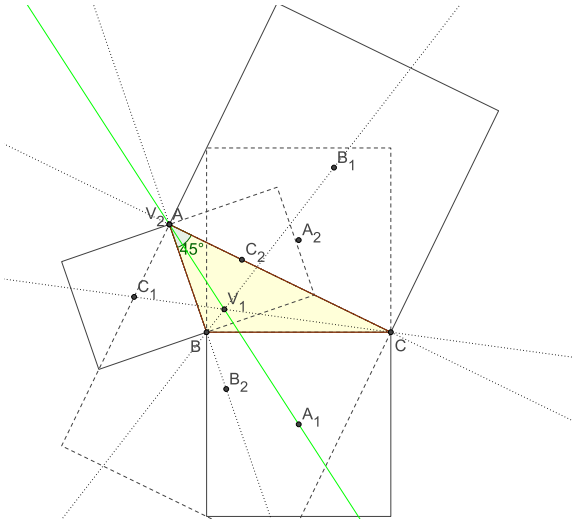


Figure 3: Scheme for solving problem MH-98

In conclusion, $\angle BAC = 45^\circ$ or $\angle BAC = 135^\circ$.

MH–99. *Proposed by Ander Lamaison Vidarte, Brno, Czech Republic.* Let $n \geq 3$ be an integer. We have n balls, labelled from 1 to n, and n holes forming a circle. An arrangement is a way of placing one ball on each hole. We say that an arrangement is bushy if, for each ball b, both neighbouring balls have labels smaller than b or both larger than b. We make n moves as follows: on the k-th move, we locate the ball labelled k and swap the position of its neighbours. We say that an arrangement is jumpy if, for each k, the two balls swapped on the k-th move both have labels smaller than k, or both larger than k. Prove that there exist as many jumpy arrangements as bushy arrangements.

Solution by Ander Lamaison Vidarte, Brno, Czech Republic. We can assume that, in a bushy arrangement, a number is assigned to each *hole*, rather than each ball. This is obviously equivalent, since each hole contains one numbered ball. We will define a bijection between the set B of bushy numberings of holes and the set J of jumpy arrangements of balls.

Consider a jumpy arrangement. Suppose that each ball is red at the beginning, and, on the k-th step, the ball labelled k is colored blue. Because the arrangement is jumpy, the two balls swapped on each step have the same color, and so the only time at which the color of the ball contained in a given hole h changes is when it contains the ball labelled k at step k. Because this must happen once at each hole, if we write on each hole the corresponding number k, each number from 1 to n appears in exactly one hole.

Now consider a hole with the number k. The ball contained in this hole is red until the k-th step, and blue after that. Because at step k the two balls that are swapped both have the same color, the two neighboring holes both have numbers smaller than k, or both larger than k. Thus the numbering on the holes is bushy. This defines our function $f : J \to B$.

We can also define its inverse function. Consider a bushy numbering of the holes. Start with n blank balls, one in each hole. At the k-th step, locate the hole with the number k, write the number k on the ball in that hole, and swap the balls in neighboring holes. The process maintains the property that the ball on the hole la-

belled k is unmarked until the k-th step, and marked afterwards. This is because the numbering of the holes is bushy. As such, each ball is assigned exactly one number.

Return each ball to the original position. We claim that the arrangement is jumpy. Indeed, the series of swaps in the procedure that we used is the same as in the statement, and because the hole numbering is bushy, at step k the two balls swapped are both marked (which means that their number is smaller than k) or both unmarked (which means their final number is greater than k). The arrangement is thus jumpy. This defines a function $g : B \to J$, which can easily be checked to be the inverse of f, meaning that $|J| = |B|$.

MH–100. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let x_1, x_2, x_3, x_4, x_5 be nonnegative real numbers. If $x_1 \le 4$, $x_1 + x_2 \le 13$, $x_1 + x_2 + x_3 \le 29$, $x_1 + x_2 + x_3 + x_4 \le 54$ and $x_1 + x_2 + x_3 + x_4 + x_5 \le 90$, then prove that $\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \sqrt{x_4} + \sqrt{x_5} \le 20$.

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. By Jensen's inequality,

$$\begin{aligned}
&\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \sqrt{x_4} + \sqrt{x_5} \\
&\qquad = 2\sqrt{\frac{x_1}{4}} + 3\sqrt{\frac{x_2}{9}} + 4\sqrt{\frac{x_3}{16}} + 5\sqrt{\frac{x_4}{25}} + 6\sqrt{\frac{x_5}{36}} \\
&\qquad \le 20\sqrt{\frac{x_1}{40} + \frac{x_2}{60} + \frac{x_3}{80} + \frac{x_4}{100} + \frac{x_5}{120}} \\
&\qquad = 20\sqrt{\frac{30x_1 + 20x_2 + 15x_3 + 12x_4 + 10x_5}{1200}}.
\end{aligned}$$

Now,

$$\begin{aligned}
&30x_1 + 20x_2 + 15x_3 + 12x_4 + 10x_5 \\
&\quad = 10x_1 + 5(x_1 + x_2) + 3(x_1 + x_2 + x_3) + 2(x_1 + x_2 + x_3 + x_4) \\
&\qquad\quad +10(x_1 + x_2 + x_3 + x_4 + x_5) \\
&\quad \le 10(4) + 5(13) + 3(29) + 2(54) + 10(90) \\
&\quad = 1200,
\end{aligned}$$

so

$$\sqrt{x_1}+\sqrt{x_2}+\sqrt{x_3}+\sqrt{x_4}+\sqrt{x_5}\le 20.$$

Solution 2 by Michel Bataille, Rouen, France. Let $S=\sqrt{x_1}+\sqrt{x_2}+\sqrt{x_3}+\sqrt{x_4}+\sqrt{x_5}$. We notice that

$$S=2\sqrt{\frac{x_1}{4}}+3\sqrt{\frac{x_2}{9}}+4\sqrt{\frac{x_3}{16}}+5\sqrt{\frac{x_4}{25}}+6\sqrt{\frac{x_5}{36}}$$

and that $2+3+4+5+6=20$.
Jensen's inequality applied to the concave function $x\mapsto\sqrt{x}$ gives

$$S\le 20\sqrt{\frac{2}{20}\cdot\frac{x_1}{4}+\frac{3}{20}\cdot\frac{x_2}{9}+\frac{4}{20}\cdot\frac{x_3}{16}+\frac{5}{20}\cdot\frac{x_4}{25}+\frac{6}{20}\cdot\frac{x_5}{36}},$$

that is,

$$S\le 20\sqrt{\frac{1}{20}\Big(\frac{x_1}{2}+\frac{x_2}{3}+\frac{x_3}{4}+\frac{x_4}{5}+\frac{x_5}{6}\Big)}.$$

Therefore, it suffices to prove that

$$\frac{x_1}{2}+\frac{x_2}{3}+\frac{x_3}{4}+\frac{x_4}{5}+\frac{x_5}{6}\le 20. \tag{1}$$

Now, since the left-hand side of (1) can be written as

$$\frac{1}{6}\cdot(x_5+x_4+x_3+x_2+x_1)+\left(\frac{1}{5}-\frac{1}{6}\right)(x_4+x_3+x_2+x_1)$$
$$+\left(\frac{1}{4}-\frac{1}{5}\right)(x_3+x_2+x_1)+\left(\frac{1}{3}-\frac{1}{4}\right)(x_2+x_1)+\left(\frac{1}{2}-\frac{1}{3}\right)x_1,$$

the hypotheses provide

$$\frac{x_1}{2}+\frac{x_2}{3}+\frac{x_3}{4}+\frac{x_4}{5}+\frac{x_5}{6}\le\frac{1}{6}\cdot 90+\frac{1}{30}\cdot 54+\frac{1}{20}\cdot 29+\frac{1}{12}\cdot 13+\frac{1}{6}\cdot 4=20$$

and (1) follows.

Solution 3 by the proposer. We prove a more general result. We claim that if

$$\begin{aligned} x_1 &\le y_1,\\ x_1+x_2 &\le y_1+y_2,\\ &\cdots\\ x_1+x_2+x_3+x_4+x_5 &\le y_1+y_2+y_3+y_4+y_5, \end{aligned}$$

then $\sqrt{x_1}+\sqrt{x_2}+\sqrt{x_3}+\sqrt{x_4}+\sqrt{x_5} \leq \sqrt{y_1}+\sqrt{y_2}+\sqrt{y_3}+\sqrt{y_4}+\sqrt{y_5}$.
Indeed, expanding the squares in the trivial inequality

$$\left(\frac{\sqrt{x_1}}{\sqrt[4]{y_1}} - \sqrt[4]{y_1}\right)^2 + \left(\frac{\sqrt{x_2}}{\sqrt[4]{y_2}} - \sqrt[4]{y_2}\right)^2 + \ldots + \left(\frac{\sqrt{x_5}}{\sqrt[4]{y_5}} - \sqrt[4]{y_5}\right)^2 \geq 0,$$

we get

$$\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \sqrt{x_4} + \sqrt{x_5}$$
$$\leq \frac{1}{2}\left[\left(\frac{x_1}{\sqrt{y_1}} + \frac{x_2}{\sqrt{y_2}} + \ldots + \frac{x_5}{\sqrt{y_5}}\right) + (\sqrt{y_1} + \sqrt{y_2} + \sqrt{y_3} + \sqrt{y_4} + \sqrt{y_5})\right].$$

On the other hand, we have

$$\begin{aligned}
\frac{x_1}{\sqrt{y_1}} + \frac{x_2}{\sqrt{y_2}} + \ldots + \frac{x_5}{\sqrt{y_5}} &= \frac{1}{\sqrt{y_5}}(x_5 + x_4 + x_3 + x_2 + x_1) \\
&+ \left(\frac{1}{\sqrt{y_4}} - \frac{1}{\sqrt{y_5}}\right)(x_4 + x_3 + x_2 + x_1) \\
&+ \left(\frac{1}{\sqrt{y_3}} - \frac{1}{\sqrt{y_4}}\right)(x_3 + x_2 + x_1) \\
&+ \left(\frac{1}{\sqrt{y_2}} - \frac{1}{\sqrt{y_3}}\right)(x_2 + x_1) \\
&+ \left(\frac{1}{\sqrt{y_1}} - \frac{1}{\sqrt{y_2}}\right)x_1.
\end{aligned}$$

Combining the preceding with constrains, we obtain that

$$\begin{aligned}
\frac{x_1}{\sqrt{y_1}} + \frac{x_2}{\sqrt{y_2}} + \ldots + \frac{x_5}{\sqrt{y_5}} &\leq \frac{1}{\sqrt{y_5}}(y_5 + y_4 + y_3 + y_2 + y_1) \\
&+ \left(\frac{1}{\sqrt{y_4}} - \frac{1}{\sqrt{y_5}}\right)(y_4 + y_3 + y_2 + y_1) \\
&+ \left(\frac{1}{\sqrt{y_3}} - \frac{1}{\sqrt{y_4}}\right)(y_3 + y_2 + y_1) \\
&+ \left(\frac{1}{\sqrt{y_2}} - \frac{1}{\sqrt{y_3}}\right)(y_2 + y_1) \\
&+ \left(\frac{1}{\sqrt{y_1}} - \frac{1}{\sqrt{y_2}}\right)y_1 \\
&= \sqrt{y_1} + \sqrt{y_2} + \sqrt{y_3} + \sqrt{y_4} + \sqrt{y_5}
\end{aligned}$$

and

$$\begin{aligned}&\sqrt{x_1}+\sqrt{x_2}+\sqrt{x_3}+\sqrt{x_4}+\sqrt{x_5}\\ &\le \frac{1}{2}\Big[(\sqrt{y_1}+\sqrt{y_2}+\ldots+\sqrt{y_5})+(\sqrt{y_1}+\sqrt{y_2}+\ldots+\sqrt{y_5})\Big]\\ &=\sqrt{y_1}+\sqrt{y_2}+\sqrt{y_3}+\sqrt{y_4}+\sqrt{y_5}.\end{aligned}$$

Equality holds, when $x_i = y_i$, $1 \le i \le 5$, and the claim is proven.

Putting $y_1 = 4$, $y_2 = 9$, $y_3 = 16$, $y_4 = 25$, $y_5 = 36$, we get

$$\sqrt{x_1}+\sqrt{x_2}+\sqrt{x_3}+\sqrt{x_4}+\sqrt{x_5} \le 20.$$

Equality holds when $x_1 = 4$, $x_2 = 9$, $x_3 = 16$, $x_4 = 25$ and $x_5 = 36$.

Also solved by *Ioan Viorel Codreanu, Satulung, Maramures, Romania.*

Advanced Problems

A–95. *Proposed by Michel Bataille, Rouen, France.* Let n be a nonnegative integer. Evaluate in closed form

$$\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2n+2k}{n+k}.$$

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Recall that the generating function for the central binomial coefficients is

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}.$$

It follows that

$$\sum_{k=-n}^{n} \binom{2n-2k}{n-k}\binom{2n+2k}{n+k}$$

is the coefficient of x^{2n} in

$$\left(\frac{1}{\sqrt{1-4x}}\right)^2 = \frac{1}{1-4x} = \sum_{n=0}^{\infty} (4x)^n,$$

which is 4^{2n}. Then

$$\begin{aligned}\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2n+2k}{n+k} &= \frac{1}{2}\left(\sum_{k=-n}^{n} \binom{2n-2k}{n-k}\binom{2n+2k}{n+k} + \binom{2n}{n}^2\right) \\ &= \frac{1}{2}\left(4^{2n} + \binom{2n}{n}^2\right).\end{aligned}$$

Solution 2 by the proposer. Let S_n be the sum to be calculated. The change of index $k = n - j$ yields

$$S_n = \sum_{j=0}^{n} \binom{2j}{j}\binom{4n-2j}{2n-j}.$$

We observe that

$$\left(\sum_{k=0}^{\infty}\binom{2k}{k}x^k\right)^2 = \left(\sum_{k=0}^{\infty}\binom{-1/2}{k}(-4x)^k\right)^2$$
$$= \left((1-4x)^{-1/2}\right)^2 = (1-4x)^{-1} = \sum_{k=0}^{\infty} 4^k x^k,$$

which implies

$$\sum_{i=0}^{k}\binom{2i}{i}\binom{2(k-i)}{k-i} = 4^k \qquad (k=0,1,\ldots).$$

It follows that

$$4^{2n} = \sum_{i=0}^{2n}\binom{2i}{i}\binom{2(2n-i)}{2n-i}$$
$$= \binom{2n}{n}^2 + \sum_{i=0}^{n-1}\binom{2i}{i}\binom{2(2n-i)}{2n-i} + \sum_{i=n+1}^{2n}\binom{2i}{i}\binom{2(2n-i)}{2n-i}.$$

The change of index $j = 2n-i$ in the last sum yields

$$4^{2n} - \binom{2n}{n}^2 = 2\sum_{j=0}^{n-1}\binom{2j}{j}\binom{2(2n-j)}{2n-j} = 2\left(S_n - \binom{2n}{n}^2\right)$$

and therefore

$$S_n = \frac{1}{2}\left(4^{2n} + \binom{2n}{n}^2\right).$$

Solution 3 by Morgan Orr, Ashley Herbig, Eli Lutz (students), Department of Mathematics, Christopher Newport University, Newport News, VA. We start by writing the series out like so:

$$\binom{2n}{n}\binom{2n}{n}+\binom{2n-2}{n-1}\binom{2n+2}{n+1}+\binom{2n-4}{n-2}\binom{2n+4}{n+2}+\ldots+\binom{0}{0}\binom{4n}{2n},$$

and recognize this is a portion of the coefficient of x^{2n} in the Cauchy Product of

$$\left(\sum_{n=0}^{\infty}\binom{2n}{n}x^n\right)\left(\sum_{n=0}^{\infty}\binom{2n}{n}x^n\right).$$

The entire coefficient of x^{2n} is given by

$$\sum_{k=-n}^{n} \binom{2n-2k}{n-k}\binom{2n+2k}{n+k},$$

which is related to the desired sum by

$$\binom{2n}{2n}\binom{2n}{2n}+\sum_{k=-n}^{n} \binom{2n-2k}{n-k}\binom{2n+2k}{n+k} = 2\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2n+2k}{n+k}.$$

Now, the generating function for the central binomial coefficients is

$$\sum_{n=0}^{\infty} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}},$$

so

$$\left(\sum_{n=0}^{\infty} \binom{2n}{n} x^n\right)^2 = \frac{1}{1-4x} = \sum_{n=0}^{\infty} (4x)^n$$

by the expansion for the geometric series. It follows that

$$\sum_{k=-n}^{n} \binom{2n-2k}{n-k}\binom{2n+2k}{n+k} = 4^{2n}.$$

Thus:

$$\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2n+2k}{n+k} = \frac{1}{2}\binom{2n}{2n}\binom{2n}{2n} + \frac{1}{2}4^{2n}.$$

Solution 4 by Moti Levy, Rehovot, Israel. We begin by changing the binomial coefficients:

$$\sum_{k=0}^{n} \binom{2(n-k)}{n-k}\binom{2(n+k)}{n+k} = \sum_{k=0}^{n} \binom{2k}{k}\binom{4n-2k}{2n-k}$$

The following combinatorial identity is well known. It is called the "remarkable property of the middle elements of Pascal's triangle" in [1], equation (5.39) on page 187.

$$\sum_{k=0}^{n} \binom{2k}{k}\binom{2n-2k}{n-k} = 4^n. \tag{1}$$

Replacing n by $2n$ in (1), we get

$$\sum_{k=0}^{2n} \binom{2k}{k}\binom{4n-2k}{2n-k} = 16^n. \tag{2}$$

$$\begin{aligned}\sum_{k=0}^{2n} \binom{2k}{k}\binom{4n-2k}{2n-k} &= \sum_{k=0}^{n} \binom{2k}{k}\binom{4n-2k}{2n-k} + \sum_{k=n+1}^{2n} \binom{2k}{k}\binom{4n-2k}{2n-k} \\ &= \sum_{k=0}^{n} \binom{2k}{k}\binom{4n-2k}{2n-k} + \sum_{k=n}^{2n} \binom{2k}{k}\binom{4n-2k}{2n-k} - \binom{2n}{n}^2 \\ &= \sum_{k=0}^{n} \binom{2k}{k}\binom{4n-2k}{2n-k} + \sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k+2n}{k+n} - \binom{2n}{n}^2 \\ &= 2\sum_{k=0}^{n} \binom{2k}{k}\binom{4n-2k}{2n-k} - \binom{2n}{n}^2. \end{aligned} \tag{3}$$

By (2) and (3) the required sum is

$$\sum_{k=0}^{n} \binom{2(n-k)}{n-k}\binom{2(n+k)}{n+k} = \frac{1}{2}\left(16^n + \binom{2n}{n}^2\right).$$

Remark: The required sum may obtained using generalized hypergeometric functions.

It can be shown that

$$\sum_{k=0}^{\infty} \binom{2(n-k)}{n-k}\binom{2(n+k)}{n+k} = \binom{2n}{n}^2 {}_3F_2\left[\begin{matrix} 1 & -n & n+\frac{1}{2} \\ & n+1 & -n+\frac{1}{2} \end{matrix}\middle| 1\right] \tag{4}$$

and then (4) can be evaluated by contiguous expression of Dixon's theorem given in [2].

References:

[1] Graham, Knuth and Patashnik, "*Concrete Mathematics*" 2nd edition, Addison-Wesley 1994.

[2] J. L. Lavoie, F. Grondin, A. K. Rathie and K. Arora, "*Generalizations of Dixon's Theorem on the Sum of* $3F2$", Mathematics of Computation, Vol. 62, No. 205 (Jan., 1994), pp. 267-276

A–96. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a scalene, non-right-angled triangle with altitudes AD, BE, CF and midpoints M_a, M_b, M_c of BC, CA, AB, respectively. Let J be the intersection point of EF and M_bM_c. Point K is the symmedian point of the triangle ABC. Knowing that J lies on the median AM_a, prove that the circle with center J and radius JK is tangent to the Euler line of $\triangle ABC$.

Solution 1 by Michel Bataille, Rouen, France. We use barycentric coordiantes relatively to (A, B, C) and Conway's notations

$$S_A = \frac{b^2 + c^2 - a^2}{2},\ S_B = \frac{c^2 + a^2 - b^2}{2},\ S_C = \frac{a^2 + b^2 - c^2}{2}$$

where $a = BC, b = CA, c = AB$.
Since the orthocenter H is the point $(S_BS_C : S_CS_A : S_AS_B)$, we have $E = (S_C : 0 : S_A)$ and $F = (S_B : S_A : 0)$ and therefore the equation of EF is $xS_A - yS_B - zS_C = 0$. Since the equation of M_bM_c is $x = y + z$, we obtain

$$J = (S_C - S_B : S_C - S_A : S_A - S_B) = (b^2 - c^2 : a^2 - c^2 : b^2 - a^2).$$

This point J being on the line AM_a $(y = z)$, we see that the condition $2a^2 = b^2 + c^2$ holds, that is, ΔABC is a root-mean-square triangle (see reference [1]). In such a triangle the centroid G satisfies $AG \perp GH$ ([1] p. 63). It follows that the circle with center J and radius JG is tangent to the Euler line HG. Thus, it suffices to show that $JK = JG$.
From the condition $2a^2 = b^2 + c^2$, we deduce that $J = (2 : 1 : 1)$, hence J is the midpoint of AM_a and since $4AM_a^2 = 2b^2 + 2c^2 - a^2 = 3a^2$, we readily obtain that $JG^2 = \frac{a^2}{48}$.
Since $K = (a^2 : b^2 : c^2)$, we have

$$\begin{aligned} 12a^2\overrightarrow{JK} &= 12a^2K - 12a^2J \\ &= (4a^2A + 4b^2B + 4c^2C) - (6a^2A + 3a^2B + 3a^2C) \\ &= -2a^2\overrightarrow{CA} + (4b^2 - 3a^2)\overrightarrow{CB} \end{aligned}$$

so that

$$\begin{aligned} 144a^4JK^2 &= 4a^4b^2 + a^2(4b^2 - 3a^2)^2 - 2a^2(4b^2 - 3a^2)(a^2 + b^2 - c^2) \\ &= a^2(15a^4 + 8b^4 - 22a^2b^2 + 8b^2c^2 - 6a^2c^2), \end{aligned}$$

or, allowing for $2a^2 = b^2 + c^2$,

$$\begin{aligned} 144a^4JK^2 &= a^2[15a^4 + 8b^4 - 16a^2b^2 + 8b^2c^2 - 6a^2(2a^2)] \\ &= a^2[3a^4 + 8b^2(b^2 + c^2 - 2a^2)] = 3a^6 \end{aligned}$$

and $JK^2 = \frac{a^2}{48} = JG^2$ follows.

[1] Bataille, M., On the Centres of Root-Mean-Square Triangles, *Crux Mathematicorum*, Vol. 44(2), February 2018, p. 63-8.

Solution 2 by the proposer. The conditions ABC be a scalene, non-right triangle means that the triangle DEF is non-degenerate and the lines EF and M_bM_c are non-parallel, i.e. J exist.

We first establish a lemma.

Lemma 1. *In a scalene, oblique triangle ABC, the line AJ is perpendicular to the Euler line.*

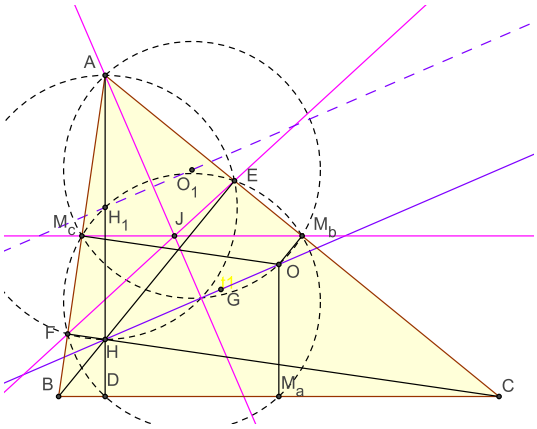


Figure 4: Scheme for solving problem A-96

Proof. It is well known that the perpendicular bisectors of the three sides of a triangle are concurrent at the circumcenter O of the triangle. So $\angle AM_bO = \angle AM_cO = 90°$ or A, M_b, M_c, O lie on the circle (O_1) with diameter AO and center O_1.

Let H be the orthocenter of the triangle ABC. Then $\angle AEH = \angle AFH = 90°$ or A, E, H, F lie on the circle (H_1) with diameter AH and center H_1.

EF, M_bM_c are chords of the nine-point circle, and by the power of point J, $JE.JF = JM_b.JM_c$. But EF is a chord of (H_1) and M_bM_c is a chord of (O_1), hence J lies on the radical axis of (H_1) and (O_1). The radical axis is a straight line perpendicular to the line joining centers H_1, O_1. The point A is a intersect point of (H_1) and (O_1), so A lies on its radical axis too.

In the triangle AHO, the line H_1O_1 is the midline, so $H_1O_1 \| HO$ and the radical axis $AJ \perp HO$. □

The centroid G lies on the median AM_a and on the Euler line HO too. But from Lemma 1, $AJ \perp OH$ hence $JG \perp OH$, thus OH is tangent to the circle ω with center J and radius JG.

It remains to show that K lies on ω.

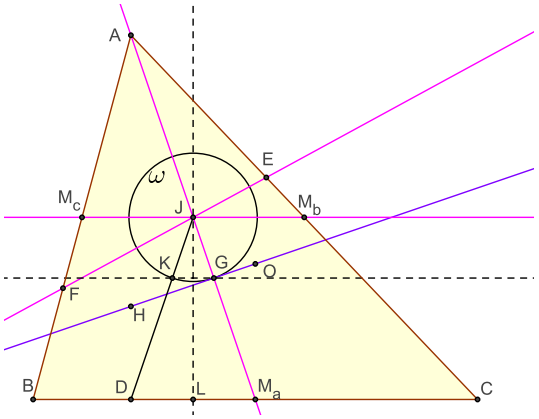


Figure 5: Scheme for solving problem A-96

Let $\angle CAB = \alpha, \angle ABC = \beta, \angle BCA = \gamma$. The points E, F are feet of the perpendiculars from B to AC and from C to AB respectively.

$$AF = AC \cos\alpha = b \cos\alpha$$

By the cosine formula,

$$a^2 = b^2 + c^2 - 2bc\cos\alpha; \quad \cos\alpha = \frac{-a^2 + b^2 + c^2}{2bc}$$

Now,

$$AF = b\frac{-a^2+b^2+c^2}{2bc} = \frac{-a^2+b^2+c^2}{2c}$$
$$FB = BC\cos\beta = a\frac{a^2-b^2+c^2}{2ac} = \frac{a^2-b^2+c^2}{2c}$$
$$F = \frac{FB}{AB}A + \frac{AF}{AB}B = \frac{a^2-b^2+c^2}{2c^2}A + \frac{-a^2+b^2+c^2}{2c^2}B$$

$$AE = AB\cos\alpha = c\frac{-a^2+b^2+c^2}{2bc} = \frac{-a^2+b^2+c^2}{2b}$$
$$EC = BC\cos\gamma = a\frac{a^2+b^2-c^2}{2ab} = \frac{a^2+b^2-c^2}{2b}$$
$$E = \frac{EC}{AC}A + \frac{AE}{AC}C = \frac{a^2+b^2-c^2}{2b^2}A + \frac{-a^2+b^2+c^2}{2b^2}C$$

$$BD = AB\cos\beta = c\frac{a^2-b^2+c^2}{2ac} = \frac{a^2-b^2+c^2}{2a}$$
$$DC = AC\cos\gamma = b\frac{a^2+b^2-c^2}{2ab} = \frac{a^2+b^2-c^2}{2a}$$
$$D = \frac{DC}{BC}B + \frac{BD}{BC}C = \frac{a^2+b^2-c^2}{2a^2}B + \frac{a^2-b^2+c^2}{2a^2}C$$

The point J lies on the EF, so

$$\begin{aligned} J &= xE + (1-x)F = \\ &= \frac{b^2(a^2-b^2+c^2) + (b^2-c^2)(-a^2+b^2+c^2)x}{2b^2c^2}A + \\ &+ \frac{(-a^2+b^2+c^2)(1-x)}{2c^2}B + \frac{(-a^2+b^2+c^2)x}{2b^2}C \end{aligned} \tag{1}$$

But J is the common point for AM_a, EF, M_bM_c, and M_bM_c is a midline of $\triangle ABC$, so J is the midpoint of AM_a.

$$J = \frac{1}{2}(A + M_a) = \frac{1}{2}A + \frac{1}{4}B + \frac{1}{4}C \tag{2}$$

From (1) and (2) we obtaining:

$$\begin{cases} \dfrac{(-a^2+b^2+c^2)x}{2b^2} = \dfrac{1}{4}; \quad (C) \\ \dfrac{(-a^2+b^2+c^2)(1-x)}{2c^2} = \dfrac{1}{4}; \quad (B) \end{cases}$$

$$\begin{cases} x = \dfrac{b^2}{2(-a^2+b^2+c^2)} \\ \dfrac{(-a^2+b^2+c^2)(1-\frac{b^2}{2(-a^2+b^2+c^2)})}{2c^2} - \dfrac{1}{4} = 0 \end{cases}$$

$$a^2 = \frac{b^2+c^2}{2};\ a = \frac{\pm\sqrt{b^2+c^2}}{\sqrt{2}} \tag{3}$$

It is well known that centroid G and symmedian point K are

$$G = \frac{1}{3}A + \frac{1}{3}B + \frac{1}{3}C$$

$$\begin{aligned} K &= \frac{a^2}{a^2+b^2+c^2}A + \frac{b^2}{a^2+b^2+c^2}B + \frac{c^2}{a^2+b^2+c^2}C \\ &= \frac{1}{3}A + \frac{2b^2}{3(b^2+c^2)}B + \frac{2c^2}{3(b^2+c^2)}C \end{aligned}$$

The vector

$$\begin{aligned} \overrightarrow{KG} = G - K &= \frac{1}{3}B + \frac{1}{3}C - \frac{2b^2}{3(b^2+c^2)}B - \frac{2c^2}{3(b^2+c^2)}C = \\ &= \frac{b^2-c^2}{3(b^2+c^2)}(C-B) = \frac{b^2-c^2}{3(b^2+c^2)}\overrightarrow{BC} \end{aligned}$$

Therefore $KG \| BC$.

Let L be the midpoint of DM_a. J is the midpoint of AM_a, hence J lies oh the perpendicular bisector of DM_a. Thus $\triangle DLJ \cong \triangle M_aLJ$, hence DJ is symmetric of M_aJ with respect to LJ.

The centroid G divided M_aA in ratio $1:2$ or $M_aG:GJ=2:1$

Let $K'\in DJ$ be point such that $DK':K'J=2:1$.

$$\begin{aligned}K' &= \frac{1}{3}D+\frac{2}{3}J\\ &= \frac{1}{3}\left(\frac{a^2+b^2-c^2}{2a^2}B+\frac{a^2-b^2+c^2}{2a^2}C\right)+\frac{2}{3}\left(\frac{1}{2}A+\frac{1}{4}B+\frac{1}{4}C\right)\\ &= \frac{1}{3}A+\frac{2b^2}{3(b^2+c^2)}B+\frac{2c^2}{3(b^2+c^2)}C\\ &= K\end{aligned}$$

It follows that K lies on DJ and therefore K is symmetric of G with respect to the perpendicular bisector LJ of DM_a. Hence $JK=JG$ and K lies on ω.

A–97. *Proposed by Nicolae Papacu, Slobozia, Romania.* Show that there exist three unique real numbers $a,b,c\in(0,\pi/2)$ such that

$$\begin{cases}a=\cot(a),\\ b=\tan(\cot(b)),\\ c=\cot(\tan(c)),\end{cases}$$

and $c<a<b$.

Solution by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. For $x\in(0,\pi/2)$, x is increasing from 0 to $\pi/2$ and $\cot x$ is decreasing from $+\infty$ to 0; hence, there is a unique $a\in(0,\pi/2)$ such that $a=\cot a$. Moreover, because

$$\cot\frac{\pi}{4}=1>\frac{\pi}{4}\quad\text{and}\quad\cot\frac{\pi}{3}=\frac{1}{\sqrt{3}}<\frac{\pi}{3},$$

it follows that $a\in(\pi/4,\pi/3)$.

Now, for $n=0,1,2,\ldots$, $\cot(\tan x)$ is decreasing for $x\in(\tan^{-1}n\pi,\tan^{-1}(n+1)\pi)$ with

$$\lim_{x\to\tan^{-1}n\pi^+}\cot(\tan x)=+\infty\quad\text{and}\quad\lim_{x\to\tan^{-1}(n+1)\pi^-}\cot(\tan x)=-\infty,$$

so there exists a unique $c_n \in (\tan^{-1} n\pi, \tan^{-1}(n+1)\pi)$ such that $c_n = \cot(\tan c_n)$. For $n \geq 1$,

$$c_n > \tan^{-1} \pi > \tan^{-1} \sqrt{3} = \frac{\pi}{3} > a.$$

However, because $\tan x > x$ for $x \in (0, \pi/2)$ and $\cot x$ is decreasing for $x \in (0, \pi)$, it follows that $\cot(\tan x) < \cot x$ for $x \in (0, \tan^{-1} \pi)$. This, in turn, implies $c_0 < a$.

Next, for $x \in (\cot^{-1} \pi/2, \pi/2)$, $\tan(\cot x)$ is decreasing with

$$\lim_{x \to \cot^{-1} \pi/2^+} \tan(\cot x) = +\infty \quad \text{and} \quad \lim_{x \to \pi/2^-} \tan(\cot x) = 0,$$

so there exists a unique $b \in (\cot^{-1} \pi/2, \pi/2)$ such that $b = \tan(\cot b)$. Moreover, for $x \in (\cot^{-1} \pi/2, \pi/2)$, $\tan(\cot x) > \cot x$, which implies $b > a$. For $n = 1, 2, 3, \ldots$, $\tan(\cot x)$ is decreasing for $x \in \left(\cot^{-1} \frac{2n+1}{2}\pi, \cot^{-1} \frac{2n-1}{2}\pi\right)$ with

$$\lim_{x \to \cot^{-1}(2n+1)\pi/2^+} \tan(\cot x) = +\infty$$

and

$$\lim_{x \to \cot^{-1}(2n-1)\pi/2^-} \tan(\cot x) = -\infty,$$

so there exists a unique $b_n \in \left(\cot^{-1} \frac{2n+1}{2}\pi, \cot^{-1} \frac{2n-1}{2}\pi\right)$ such that $b_n = \tan(\cot b_n)$. However,

$$b_n < \cot^{-1} \frac{\pi}{2} < \cot^{-1} 1 = \frac{\pi}{4} < a.$$

Finally, there exist three unique real numbers $a, b, c \in (0, \pi/2)$ such that

$$\begin{cases} a = \cot a, \\ b = \tan(\cot b), \\ c = \cot(\tan c), \end{cases}$$

and $c < a < b$.

Also solved by *the proposer.*

A–98. *Proposed by Toyesh Prakash Sharma, C. F. Andrews School, Agra, India.* Find the value of

$$\int_1^{e^3} \sqrt{\ln x}\,\mathrm{d}x + \int_{e^3}^{e^9} \frac{\mathrm{d}x}{\sqrt{\ln x}} + \int_0^{\sqrt{3}} e^{x^2}\,\mathrm{d}x + \int_{\sqrt{3}}^{3} e^{-x^2}\,\mathrm{d}x.$$

Solution 1 by Moti Levy, Rehovot, Israel. The *error function* $\operatorname{erf}(x)$ is defined by

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

The *imaginary error function* $\operatorname{erfi}(x)$ is defined by

$$\operatorname{erfi}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{t^2} dt.$$

Applying integration by parts

$$\begin{aligned}\int_1^{e^3} \sqrt{\ln(x)}dx &= x\sqrt{\ln(x)}\Big]_1^{e^3} - \frac{1}{2}\int_1^{e^3} \frac{1}{\sqrt{\ln(x)}}dx \\ &= \sqrt{3}e^3 - \frac{1}{2}\int_1^{e^3} \frac{1}{\sqrt{\ln(x)}}dx = \sqrt{3}e^3 - \int_0^{\sqrt{3}} e^{t^2}dt \\ &= \sqrt{3}e^3 - \frac{\sqrt{\pi}}{2}\operatorname{erfi}\left(\sqrt{3}\right).\end{aligned}$$

By change of integration variable $\sqrt{\ln(x)} = t$,

$$\int_{e^3}^{e^9} \frac{1}{\sqrt{\ln(x)}}dx = 2\int_{\sqrt{3}}^{3} e^{t^2}dt = \sqrt{\pi}\operatorname{erfi}(3) - \sqrt{\pi}\operatorname{erfi}\left(\sqrt{3}\right).$$

$$\int_0^{\sqrt{3}} e^{x^2}dx = \frac{\sqrt{\pi}}{2}\operatorname{erfi}\left(\sqrt{3}\right),$$

$$\int_{\sqrt{3}}^{3} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}\operatorname{erf}(3) - \frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\sqrt{3}\right)$$

$$\begin{aligned}&\int_1^{e^3} \sqrt{\ln(x)}dx + \int_{e^3}^{e^9} \frac{1}{\sqrt{\ln(x)}}dx + \int_0^{\sqrt{3}} e^{x^2}dx + \int_{\sqrt{3}}^{3} e^{-x^2}dx \\ &= \sqrt{3}e^3 + \sqrt{\pi}\operatorname{erfi}(3) - \sqrt{\pi}\operatorname{erfi}\left(\sqrt{3}\right) + \frac{\sqrt{\pi}}{2}\operatorname{erf}(3) - \frac{\sqrt{\pi}}{2}\operatorname{erf}\left(\sqrt{3}\right) \\ &\cong 2909.27\end{aligned}$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. With the substitution $w = \sqrt{\ln x}$, followed by integration by parts,

$$\begin{aligned}\int_1^{e^3} \sqrt{\ln x}\, dx &= \int_0^{\sqrt{3}} 2w^2 e^{w^2}\, dw = we^{w^2}\Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} e^{w^2}\, dw \\ &= \sqrt{3}e^3 - \int_0^{\sqrt{3}} e^{w^2}\, dw,\end{aligned}$$

so

$$\int_1^{e^3} \sqrt{\ln x}\, dx + \int_0^{\sqrt{3}} e^{x^2}\, dx = \sqrt{3}e^3.$$

Next, with the substitution $w = \sqrt{\ln x}$,

$$\int_{e^3}^{e^9} \frac{dx}{\sqrt{\ln x}} = \int_{\sqrt{3}}^{3} 2e^{w^2}\, dw = \sqrt{\pi}\Big(\operatorname{erfi}(3) - \operatorname{erfi}\Big(\sqrt{3}\Big)\Big),$$

where $\operatorname{erfi}(x)$ denotes the imaginary error function, and

$$\int_{\sqrt{3}}^{3} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}\Big(\operatorname{erf}(3) - \operatorname{erf}\Big(\sqrt{3}\Big)\Big),$$

where $\operatorname{erf}(x)$ is the error function. Finally,

$$\begin{aligned}&\int_1^{e^3} \sqrt{\ln x}\, dx + \int_{e^3}^{e^9} \frac{dx}{\sqrt{\ln x}} + \int_0^{\sqrt{3}} e^{x^2}\, dx + \int_{\sqrt{3}}^{3} e^{-x^2}\, dx \\ &\quad = \sqrt{3}e^3 + \sqrt{\pi}\Big(\operatorname{erfi}(3) - \operatorname{erfi}\Big(\sqrt{3}\Big)\Big) + \frac{\sqrt{\pi}}{2}\Big(\operatorname{erf}(3) - \operatorname{erf}\Big(\sqrt{3}\Big)\Big).\end{aligned}$$

Solution 3 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. From left to right, denote the integrals by A, B, C and D, respectively. Recall the definitions of the error function and the imaginary error function:

$$\operatorname{erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-t^2}\, dt, \ \operatorname{erfi}(x) = (2/\sqrt{\pi}) \int_0^x e^{t^2}\, dt.$$

Then we have

$$\begin{aligned}
A: \quad x = e^{t^2} \Longrightarrow A &= \int_0^{\sqrt{3}} 2t^2 e^{t^2}\, dt = \sqrt{3}e^3 - \frac{\sqrt{\pi}}{2}\operatorname{erfi}(\sqrt{3}).\\
B: \quad x = e^{t^2} \Longrightarrow B &= 2\int_{\sqrt{3}}^{3} e^{t^2}\, dt = 2\left(\int_0^3 e^{t^2}\, dt - \int_0^{\sqrt{3}} e^{t^2}\, dt\right).\\
&= \sqrt{\pi}(\operatorname{erfi}(3) - \operatorname{erfi}(\sqrt{3})).\\
C: \quad \int_0^{\sqrt{3}} e^{x^2}\, dx &= \frac{\sqrt{\pi}}{2}\operatorname{erfi}(\sqrt{3}) \text{ by definition.}\\
D: \quad \int_{\sqrt{3}}^{3} e^{-x^2}\, dx &= \int_0^3 e^{-x^2}\, dx - \int_0^{\sqrt{3}} e^{-x^2}\, dx\\
&= \frac{\sqrt{\pi}}{2}(\operatorname{erf}(3) - \operatorname{erf}(\sqrt{3})).
\end{aligned}$$

Therefore,

$$\begin{aligned}
A + B + C + D &= \sqrt{3}e^3 + \frac{\sqrt{\pi}}{2}\big(\operatorname{erf}(3) - \operatorname{erf}(\sqrt{3}) + 2\operatorname{erfi}(3) - 2\operatorname{erfi}(\sqrt{3})\big)\\
&\approx 2909.2659,
\end{aligned}$$

where we used a computer algebra system to determine numerical values.

A–99. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $A(x)$ be a polynomial of degree n with integer coefficients, and let $a_0 < a_1 < \ldots < a_n$ be integer numbers. Prove that $|A(a_i)| \geq n!\, 2^{-n}$ for some index $i \in \{0, 1, 2, \ldots, n\}$.

Solution 1 by the proposer. Consider the polynomial

$$B(x) = \prod_{k=0}^{n}(x - a_k) \sum_{i=0}^{n} \frac{A(a_i)}{x - a_i} \prod_{j \neq i} \frac{1}{a_i - a_j} \quad \text{(Lagrange)}.$$

We have, for $0 \leq i \leq n$ that it holds:

$$\begin{aligned}
B(a_i) &= A(a_i)\, \frac{(a_i - a_0)\ldots(a_i - a_{i-1})(a_i - a_{i+1})\ldots(a_i - a_n)}{(a_i - a_0)\ldots(a_i - a_{i-1})(a_i - a_{i+1})\ldots(a_i - a_n)}\\
&= A(a_i).
\end{aligned}$$

Since $A(x)$ and $B(x)$ take the same value in $n+1$ points and $\deg(B(x) \leq n$, then $A(x) = B(x)$. The leading coefficient of $B(x)$ is a nonzero integer of the form

$$\sum_{i=0}^{n} A(a_i) \prod_{j \neq i} \frac{1}{a_i - a_j}$$

and absolute value bigger than one. That is,

$$1 \leq \left| \sum_{i=0}^{n} A(a_i) \prod_{j \neq i} \frac{1}{a_i - a_j} \right|$$

$$\leq |A(a_0)| \prod_{j \neq 0} \frac{1}{|a_0 - a_j|} + \ldots + |A(a_i)| \prod_{j \neq i} \frac{1}{|a_i - a_j|} + |A(a_n)| \prod_{j \neq n} \frac{1}{|a_n - a_j|}.$$

Let $|A(a_i)| = \max\{|A(a_0)|, |A(a_1)|, \ldots, |A(a_n)|\}$, then we have

$$\begin{aligned} |A(a_i)| &\geq \left| \sum_{i=0}^{n} \prod_{j \neq i} \frac{1}{|a_i - a_j|} \right|^{-1} \geq \left| \sum_{i=0}^{n} \prod_{j \neq i} \frac{1}{|i - j|} \right|^{-1} \\ &= \left| \sum_{i=0}^{n} \frac{1}{i!(n-i)!} \right|^{-1} = n!\, 2^{-n} \end{aligned}$$

Solution 2 by Michel Bataille, Rouen, France. Classically we have

$$\frac{A(x)}{(x - a_0)(x - a_1) \cdots (x - a_n)} = \sum_{k=0}^{n} \frac{r_k}{x - a_k} \tag{1}$$

with

$$r_k = \frac{A(a_k)}{\prod_{j=0, j \neq k}^{n} (a_k - a_j)}.$$

From the hypothesis on the integers $a_0, \ldots, a_n$, the inequality $|a_i - a_j| \geq |i - j|$ holds and we deduce that $\prod_{j=0, j \neq k}^{n} (a_k - a_j) \geq \prod_{j=0, j \neq k}^{n} |k - j| = k!(n-k)!$. It follows that for $k = 0, 1, \ldots, n$ we have

$$|r_k| \leq \frac{|A(a_k)|}{k!(n-k)!} = \frac{|A(a_k)|}{n!} \binom{n}{k}.$$

From now on, we assume that $A(x)$ is monic.
Multiplying both sides of (1) by x and letting $x \to \infty$, we see that

$1 = \sum_{k=0}^{n} r_k$ and therefore

$$1 = \left|\sum_{k=0}^{n} r_k\right| \le \sum_{k=0}^{n} |r_k| \le \sum_{k=0}^{n} \frac{|A(a_k)|}{n!}\binom{n}{k} \le \frac{M}{n!}\sum_{k=0}^{n}\binom{n}{k} = \frac{M\cdot 2^n}{n!}$$

where M denotes $\max(|A(a_0)|, |A(a_1)|, \ldots, |A(a_n)|)$.
Thus, $M \ge \frac{n!}{2^n}$, which implies that $|A(a_i)| \ge \frac{n!}{2^n}$ for some index i.

A–100. *Proposed by Vasile Mircea Popa, "Lucian Blaga" University of Sibiu, Romania.* Calculate the integral

$$\int_{-\infty}^{\infty} \frac{\operatorname{arc\,cot}(x)}{\sqrt{x^4+x^2+1}}\,\mathrm{d}x.$$

In this problem we will consider that definition of the function $\operatorname{arc\,cot}(x)$ whose image is the interval $(0,\pi)$.

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Consider the more general problem: let $\alpha > 0$, and evaluate

$$I = \int_{-\infty}^{\infty} \frac{\operatorname{arc\,cot} x}{\sqrt{\alpha x^4+x^2+\alpha}}\,dx.$$

Write

$$I = \int_0^{\infty} \frac{\operatorname{arc\,cot} x}{\sqrt{\alpha x^4+x^2+\alpha}}\,dx + \int_{-\infty}^{0} \frac{\operatorname{arc\,cot} x}{\sqrt{\alpha x^4+x^2+\alpha}}\,dx;$$

with the change of variables $x \to -x$ in the second integral on the right side, this becomes

$$\begin{aligned} I &= \int_0^{\infty} \frac{\operatorname{arc\,cot} x}{\sqrt{\alpha x^4+x^2+\alpha}}\,dx + \int_0^{\infty} \frac{\operatorname{arc\,cot}(-x)}{\sqrt{\alpha x^4+x^2+\alpha}}\,dx \\ &= \pi\int_0^{\infty} \frac{1}{\sqrt{\alpha x^4+x^2+\alpha}}\,dx. \end{aligned}$$

Next, write

$$I = \pi\left(\int_0^1 \frac{1}{\sqrt{\alpha x^4+x^2+\alpha}}\,dx + \int_1^{\infty} \frac{1}{\sqrt{\alpha x^4+x^2+\alpha}}\,dx\right)$$

and make the change of variables $x \to \frac{1}{x}$ in the second integral on the right side to obtain

$$I = 2\pi \int_0^1 \frac{1}{\sqrt{\alpha x^4 + x^2 + \alpha}}\, dx.$$

Now, let $x = \tan\theta$. Then

$$\begin{aligned}
I &= 2\pi \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{\alpha \tan^4\theta + \tan^2\theta + \alpha}}\, d\theta \\
&= 2\pi \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{\alpha(1+\tan^2\theta)^2 - (2\alpha - 1)\tan^2\theta}}\, d\theta \\
&= \frac{2\pi}{\sqrt{\alpha}} \int_0^{\pi/4} \frac{1}{\sqrt{1 - (2 - \frac{1}{\alpha})\sin^2\theta\cos^2\theta}}\, d\theta \\
&= \frac{2\pi}{\sqrt{\alpha}} \int_0^{\pi/4} \frac{1}{\sqrt{1 - \frac{2\alpha-1}{4\alpha}\sin^2 2\theta}}\, d\theta \\
&= \frac{\pi}{\sqrt{\alpha}} \int_0^{\pi/2} \frac{1}{\sqrt{1 - \frac{2\alpha-1}{4\alpha}\sin^2\theta}}\, d\theta \\
&= \frac{\pi}{\sqrt{\alpha}} K\left(\frac{\sqrt{\alpha(2\alpha-1)}}{2\alpha}\right),
\end{aligned}$$

where K denotes the complete elliptic integral of the first kind. Thus, with $\alpha = 1$,

$$\int_{-\infty}^{\infty} \frac{\operatorname{arc\,cot} x}{\sqrt{x^4 + x^2 + 1}}\, dx = \pi K\left(\frac{1}{2}\right).$$

Solution 2 by Michel Bataille, Rouen, France. Let I be the integral to be evaluated. Since $\operatorname{arc\,cot}(-x) = \pi - \operatorname{arc\,cot}(x)$, we have

$$\int_{-\infty}^{0} \frac{\operatorname{arc\,cot}(x)}{\sqrt{x^4 + x^2 + 1}}\, dx = \int_0^{\infty} \frac{\pi - \operatorname{arc\,cot}(x)}{\sqrt{x^4 + x^2 + 1}}\, dx$$

and therefore

$$\begin{aligned}
I &= \int_{-\infty}^{\infty} \frac{\pi}{\sqrt{x^4 + x^2 + 1}}\, dx = \pi \int_0^{\infty} \frac{dx}{x\sqrt{(x + \frac{1}{x})^2 - 1}} \\
&= \pi \int_{-\infty}^{\infty} \frac{e^t dt}{e^t\sqrt{(e^t + e^{-t})^2 - 1}} = 2\pi \int_0^{\infty} \frac{dt}{\sqrt{4\cosh^2 t - 1}}.
\end{aligned}$$

Now, consider the substitution $t = \ln(\cot(u/2))$. Then $dt = -\frac{du}{\sin u}$ and $\cosh t = \frac{1}{\sin u}$, hence

$$I = 2\pi \int_{\frac{\pi}{2}}^{0} \frac{-du}{\sin u} \cdot \frac{1}{\frac{2}{\sin u}\sqrt{1 - \frac{\sin^2 u}{4}}} = \pi \int_{0}^{\frac{\pi}{2}} \frac{du}{\sqrt{1 - \frac{\sin^2 u}{4}}}$$

and finally $I = \pi K(1/2)$ where K denotes the elliptic integral of the first kind defined by $K(k) = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-k^2 \sin^2 x}}$ for $0 < k < 1$.

Solution 3 by the proposer. Let us denote:

$$A = \int_{-\infty}^{\infty} \frac{\operatorname{arc\,cot}(x)}{\sqrt{x^4 + x^2 + 1}} dx; \quad B = \int_{-\infty}^{\infty} \frac{\arctan(x)}{\sqrt{x^4 + x^2 + 1}} dx.$$

We have:

$$A + B = \int_{-\infty}^{\infty} \frac{\operatorname{arc\,cot}(x) + \arctan(x)}{\sqrt{x^4 + x^2 + 1}} dx = \frac{\pi}{2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{x^4 + x^2 + 1}} dx.$$

Because the integrand is even, we have:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{x^4 + x^2 + 1}} dx = 2 \int_{0}^{\infty} \frac{1}{\sqrt{x^4 + x^2 + 1}} dx.$$

We can write:

$$\begin{aligned} C &= \int_{0}^{\infty} \frac{1}{\sqrt{x^4 + x^2 + 1}} dx \\ &= \int_{0}^{1} \frac{1}{\sqrt{x^4 + x^2 + 1}} dx + \int_{1}^{\infty} \frac{1}{\sqrt{x^4 + x^2 + 1}} dx. \end{aligned}$$

The substitution $x = \frac{1}{t}$ gives for the second integral:

$$\int_{1}^{\infty} \frac{1}{\sqrt{x^4 + x^2 + 1}} dx = \int_{0}^{1} \frac{1}{\sqrt{t^4 + t^2 + 1}} dt.$$

So, we have:

$$C = 2 \int_{0}^{1} \frac{1}{\sqrt{t^4 + t^2 + 1}} dt.$$

Next, we will show that the C integral can be expressed using the complete elliptic integral of the first kind, which is defined as:

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta, \; k \in (-1, 1).$$

More exactly we will show that $C = K\left(\frac{1}{2}\right)$.
Beginning with the right-hand side, we can write:

$$K\left(\frac{1}{2}\right) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - \frac{1}{4} \sin^2 \theta}} d\theta.$$

Also, via the substitution $t = \tan\left(\frac{\theta}{2}\right) \Rightarrow d\theta = \frac{2}{1+t^2} dt$, we obtain:

$$K\left(\frac{1}{2}\right) = \int_0^1 \frac{1}{\sqrt{1 - \frac{1}{4}\left(\frac{2t}{1+t^2}\right)^2}} \frac{2}{1+t^2} dt = 2 \int_0^1 \frac{1}{\sqrt{t^4 + t^2 + 1}} dt = C.$$

The B integral is equal to zero since the integrand is odd.

So, we have: $A = \dfrac{\pi}{2} \cdot 2C$

We obtained the value of the integral required in the problem statement:

$$A = \pi K\left(\frac{1}{2}\right).$$

Solution 4 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. Let I denote the given integral. Then the substitution $x \mapsto -x$ followed by an application of the relation $\operatorname{arc\,cot}(-x) = \pi - \operatorname{arc\,cot} x$ yields

$$\begin{aligned} I = \int_{-\infty}^{\infty} \frac{\operatorname{arc\,cot}(-x)}{\sqrt{x^4 + x^2 + 1}} dx &= \pi \int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4 + x^2 + 1}} - \int_{-\infty}^{\infty} \frac{\operatorname{arc\,cot}(x)}{\sqrt{x^4 + x^2 + 1}} dx \\ &= \pi \int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4 + x^2 + 1}} - I, \end{aligned}$$

so that

$$\begin{aligned} I = \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^4+x^2+1}} &= \pi \int_0^{\infty} \frac{dx}{\sqrt{x^4+x^2+1}} \\ &= \pi\left(\int_0^1 \frac{dx}{\sqrt{x^4+x^2+1}} + \int_1^{\infty} \frac{dx}{\sqrt{x^4+x^2+1}}\right) \\ &= 2\pi \int_0^1 \frac{dt}{\sqrt{t^4+t^2+1}}, \end{aligned} \tag{1}$$

where we have substituted $x = 1/t$ in the integral over $[1, \infty)$.

Letting $t = \tan\theta$ in the last integral, we have

$$\begin{aligned} \int_0^1 \frac{dt}{\sqrt{t^4+t^2+1}} &= \int_0^{\pi/4} \frac{\sec^2\theta}{\sqrt{\tan^4\theta+\tan^2\theta+1}} d\theta \\ &= \int_0^{\pi/4} \frac{d\theta}{\sqrt{1-\sin^2\theta\cos^2\theta}} \\ &= \int_0^{\pi/4} \frac{d\theta}{\sqrt{1-\frac{1}{4}\sin^2 2\theta}} \\ &\overset{\theta\mapsto\theta/2}{=} \frac{1}{2}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-\frac{1}{4}\sin^2\theta}} = \frac{1}{2}K\left(\frac{1}{2}\right), \end{aligned} \tag{2}$$

where K denotes the complete elliptic integral of the first kind.

Combining (1) and (2), we see that

$$I = \pi K\left(\frac{1}{2}\right) \approx 5.2959.$$

Also solved by *by Moti Levy, Rehovot, Israel.*

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