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Triangles with one angle twice another

Miguel Amengual Covas

In this short note, we present some conditions which characterize those triangles for which one angle is two times the size of another, as well some properties about this family of triangles.

Let r, r_a, s, S represent, respectively, the inradius, the A-exradius, the semiperimeter and the area of a triangle ABC with side lengths a, b, c opposite angles A, B, C, respectively.

Let M, D, E, I, I_a be, respectively, the midpoint of side BC, the foot of the altitude from C, the foot of the bisector of $\angle CAB$, the incenter and the A-excenter.

We will prove that the eight conditions are equivalent[1]:

$$\angle A = 2 \cdot (\angle B); \tag{1}$$

$$a^2 = b(b + c); \tag{2}$$

$BE = PQ$, where P is on CA and Q is on BC such that PIQ is parallel to BC; (3)

$$CA + AI = BC; \tag{4}$$

EM is perpendicular to AB; (5)

$$CA = CP, \text{ where } \{P\} = BC \cap I_aM; \tag{6}$$

$$CA = 2 \cdot DM; \tag{7}$$

[1]For (3), see Problem 2424, Crux Mathematicorum [1999:112]. For (4), see Problem 2559, Crux Mathematicorum [2000:305]. For (6), see Virgil Nicula and Cosmin Pohoaţă, *Diviziune armonică*, Editura Gil (2007), Problema 37°, 51, 57. For (8), see Problem 2151, Crux Mathematicorum [1996:217].

$BC = 2 \cdot DT$, where T is the point on the side BC where the excircle touches BC. (8)

In fact, we will show the following:

$$(5) \Leftrightarrow \begin{matrix}(8)\\ \Updownarrow \\ (1)\\ \Updownarrow \\ (7)\end{matrix} \Leftrightarrow \begin{matrix}(6)\\ \Updownarrow \\ (2)\\ \Updownarrow \\ (4)\end{matrix} \Leftrightarrow (3).$$

Proof that (1) $\Longleftrightarrow$ (2). Let D be the point on CA so that A is between C and D, and such that $AD = AB$ (figure 1). By the external angle theorem applied to $\triangle ADB$ at A, $\triangle ADB$ is isosceles with

$$\angle DBA = \angle ADB = \frac{1}{2}(\angle CAB). \tag{9}$$

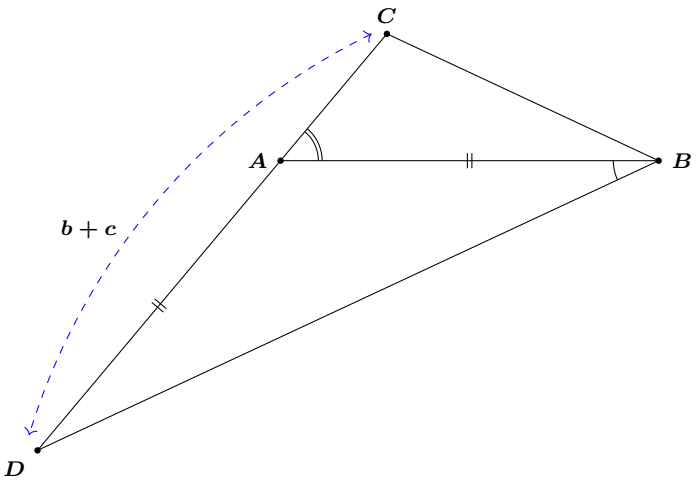

Figure 1: Construction for the proof that (1) $\Longleftrightarrow$ (2).

1. Suppose $\angle A = 2\angle B$. Then, triangles ABC and BDC (with a common angle at C and, using (9), $\angle CDB = \angle ADB = \frac{1}{2}(\angle CAB) = \angle ABC$) are similar.
   From the proportional sides, then, we have
   $$\frac{CA}{BC} = \frac{BC}{CD}.$$

This is just

$$\frac{b}{a} = \frac{a}{b+c} \quad \text{or, equivalently,} \quad a^2 = b(b+c).$$

2. Suppose $a^2 = b(b+c)$ or, equivalently, $\frac{b}{a} = \frac{a}{b+c}$. Then, triangles ABC and DBC are similar (S-A-S) and the corresponding angles CAB and DBC are equal. Thus,

$$\begin{aligned} \angle CAB &= \angle DBC \\ &= \angle DBA + \angle ABC \\ &= \frac{1}{2}(\angle CAB) + \angle ABC \qquad \text{(from (9))} \end{aligned}$$

and

$$\angle CAB = 2(\angle ABC). \qquad \square$$

Proof that (2) $\Longleftrightarrow$ (3). Because PQ is parallel to AB, triangle PQC is similar to $\triangle ABC$ and PQ has the same ratio to AB as the altitude from C of $\triangle PQC$, which differs from the *corresponding* altitude h by r, has to h. Thus,

$$\frac{PQ}{AB} = \frac{h-r}{h} = 1 - \frac{r}{h}.$$

The area of ABC is $\frac{1}{2}ch$ and also $r \cdot \frac{a+b+c}{2}$. Hence,

$$\frac{r}{h} = \frac{c}{a+b+c},$$

giving

$$\frac{PQ}{AB} = \frac{a+b}{a+b+c}$$

and

$$PQ = \frac{(a+b)c}{a+b+c}.$$

Since the square of the length of the bisector AE is

$$\frac{bc(a+b+c)(-a+b+c)}{(b+c)^2},$$

we obtain

$$\begin{aligned}&\frac{(a+b+c)^2(b+c)^2}{c}\big(BE^2-PQ^2\big)\\&=b(a+b+c)^3(-a+b+c)-c(a+b)^2(b+c)^2.\end{aligned}$$

We rewrite the RHS as a quartic polynomial in c,

$$\begin{aligned}&bc^4+\big(3b^2-a^2\big)c^3+2b(2b-a)(a+b)c^2\\&\quad+b(3b-2a)(a+b)^2c+b(b-a)(a+b)^3,\end{aligned}$$

and factor it (see Table 1), obtaining that

$$(bc+b^2-a^2)(c^3+2bc^2+2(a+b)bc+b(a+b)^2),$$

from which it follows that

$$BE=PQ \quad \text{if and only if} \quad bc+b^2-a^2=0, \text{ i.e., } a^2=b(b+c). \ \square$$

An alternative solution is given in *Crux Mathematicorum* [2000:126,127].

	b	$3b^2-a^2$	$2b(2b-a)(a+b)$	$b(3b-2a)(a+b)^2$	$b(b-a)(a+b)^3$
$\frac{a^2-b^2}{b}$		a^2-b^2	$2a^2b-2b^3$	$b^2(a+b)^2$	$-b(b-a)(a+b)^3$
	b	$2b^2$	$2b^2(a+b)$	$b^2(a+b)^2$	0

Table 1: Factorization of the quartic polynomial.

Proof that (2) $\Longleftrightarrow$ (4). Since $AI^2=\frac{bc(s-a)}{s}$, we have

$$\begin{aligned}&s(AI^2-(BC-CA)^2)\\&=bc(s-a)-s(a-b)^2\\&=\frac{1}{2}((-a+b+c)bc-(a+b+c)(a^2-2ab+b^2))\\&=\frac{1}{2}(-a^3+(b-c)a^2+b(b-c)-b(b^2-c^2))\\&=\frac{1}{2}(a-b+c)(bc+b^2-a^2)\\&=(s-b)(bc+b^2-a^2),\end{aligned}$$

from which it follows that $AI = BC - CA$ if and only if $bc + b^2 - a^2 = 0$, i.e., $a^2 = b(b+c)$. □

An alternative solution is given in *Crux Mathematicorum* [2001:466].

Proof that (1) $\iff$ (5). We have that

$$\begin{aligned}
\angle A = 2 \cdot \angle B &\iff \frac{1}{2}(\angle A) = \angle B \\
&\iff \triangle EAB \text{ is isosceles with } \angle EAB = \angle ABE \\
&\iff EA = EB \\
&\iff E \text{ lies on the perpendicular bisector of side } AB \\
&\iff EM \perp AB.
\end{aligned}$$

□

Proof that (2) $\iff$ (6). We begin with the following lemma.

Lemma 1. *We have*

$$P \in I_aM \iff PC = \frac{a(a+b)}{a+b+c}.$$

Proof. Applying Menelaus's theorem to the triad of points MPI_a on the sides of triangle ABC, we obtain (see figure 2)

$$\begin{aligned}
P \in I_aM &\iff \frac{AM}{MB} \cdot \frac{BP}{PE} \cdot \frac{EI_a}{I_aA} = 1 \\
&\iff \frac{PE}{BP} = \frac{EI_a}{I_aA} = \frac{r_a}{h_a + r_a} = \frac{S}{(s-a)h_a + S} \\
&\qquad = \frac{ah_a}{2(s-a)h_a + ah_a} = \frac{a}{b+c} \\
&\iff \frac{PE}{a} = \frac{PB}{b+c} = \frac{PE + PC}{a + (b+c)} \\
&\qquad = \frac{BE}{a+b+c} = \frac{ac}{(b+c)(a+b+c)} \\
&\iff PB = \frac{ac}{a+b+c} \\
&\iff PC = \frac{a(a+b)}{a+b+c}.
\end{aligned}$$

□

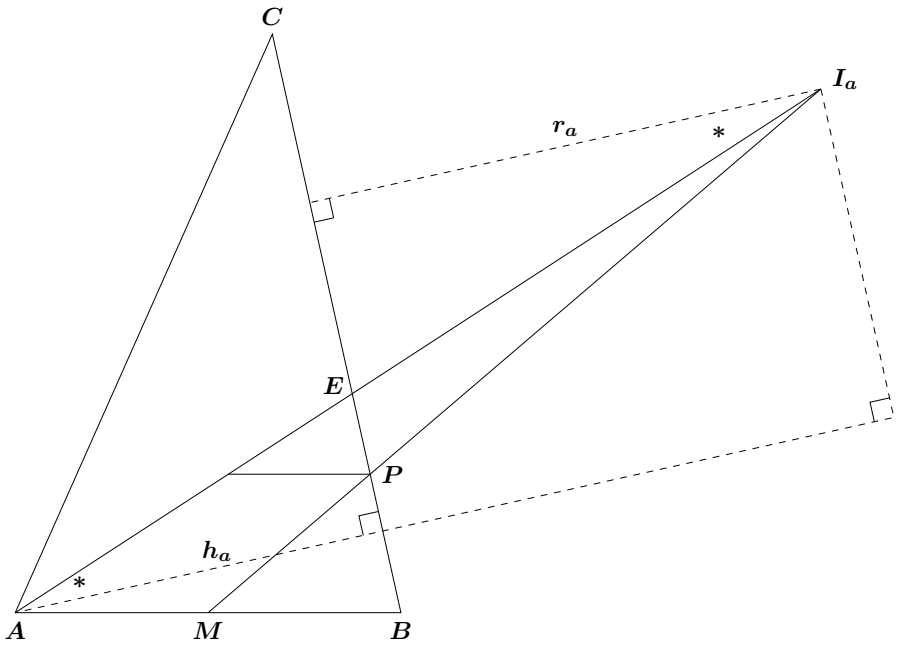

Figure 2: Construction for the proof of Lemma 1.

Thus,

$$PC = CA \iff \frac{a(a+b)}{a+b+c} = b \iff a^2 = b^2 + bc. \qquad \square$$

Proof that (2) $\iff$ (7). Let C' be the point on BA extended such that $C'A = AC$. Then, $\triangle ACC'$ is isosceles with $\angle CC'A = \frac{1}{2}(\angle A)$. Thus,

$$\begin{aligned}
\angle A = 2 \cdot (\angle B) &\iff \angle CC'A = \left(\frac{1}{2}(\angle A)\right) = \angle B \\
&\iff \triangle CC'B \text{ is isosceles with} \\
&\qquad D \text{ the midpoint of the basis } C'B \\
&\iff \overline{DB} = \frac{1}{2}\overline{C'B} = \frac{1}{2}\left(\overline{C'A} + \overline{AB}\right) = \frac{b+c}{2} \\
&\iff DM = DB - MB = \frac{b+c}{2} - \frac{c}{2} = \frac{b}{2} \\
&\iff CA = 2 \cdot DM. \qquad \square
\end{aligned}$$

Proof that (1) $\Longleftrightarrow$ (8). Let C' be the point on BA extended such that $C'A = AC$. Then, $\triangle ACC'$ is isosceles with $\angle CC'A = \frac{1}{2}(\angle A)$. Thus,

$$\begin{aligned}
\angle A = 2\cdot(\angle B) &\iff \angle CC'A = \left(\frac{1}{2}(\angle A)\right) = \angle B \\
&\iff \triangle CC'B \text{ is isosceles with} \\
&\qquad D \text{ the midpoint of the basis } C'B \\
&\iff \overline{DB} = \frac{1}{2}\overline{C'B} = \frac{1}{2}(\overline{C'A} + \overline{AB}) = \frac{b+c}{2} \\
&\iff DT = DB - TB = \frac{b+c}{2} - (s-a) = \frac{a}{2} \\
&\iff BC = 2\cdot DT. \qquad \square
\end{aligned}$$

An alternative solution is given in *Crux Mathematicorum* [1997:310].

Several engaging properties [1] for a triangle ABC with $\angle A = 2\cdot(\angle B)$ are listed and proved below. Notice that, for such a triangle, $\angle B$ is not necessarily the smallest angle in the triangle, nor is $\angle A$ necessarily the largest.

1. Let CI (extended) intersect AB at U (figure 3).

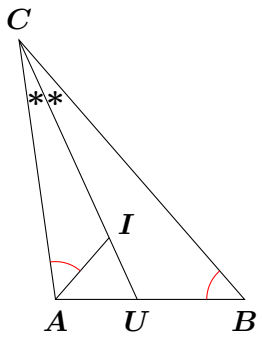

Figure 3: Construction for the first property.

Similar triangles CAI and CBU yield

$$\frac{CA}{AI} = \frac{BC}{UB},$$

and since $UB = \frac{ca}{a+b}$, then

$$AI = \frac{bc}{a+b} = a - b. \tag{10}$$

2. Triangles ABC and EAC are similar (see figure 4).

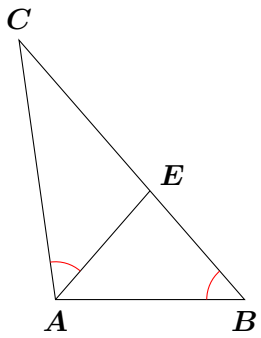

Figure 4: Construction for the second property.

From the proportional sides we get

$$\frac{AB}{BC} = \frac{AE}{CA}$$

and

$$AE = \frac{bc}{a}.$$

3. Let the circle through A, I, B intersect BC at P and CA at Q. From inspection of figure 5, quadrilateral $AQBI$ is cyclic and on chord AI

$$\angle AQI = \angle ABI = \frac{1}{2}(\angle ABC).$$

Making analogous use of the quadrilaterals $IQPB$, $PABQ$, $IAPB$ and $QAIP$, we have marked several angles with the same symbol φ, meaning $\frac{1}{2}(\angle ABC)$.

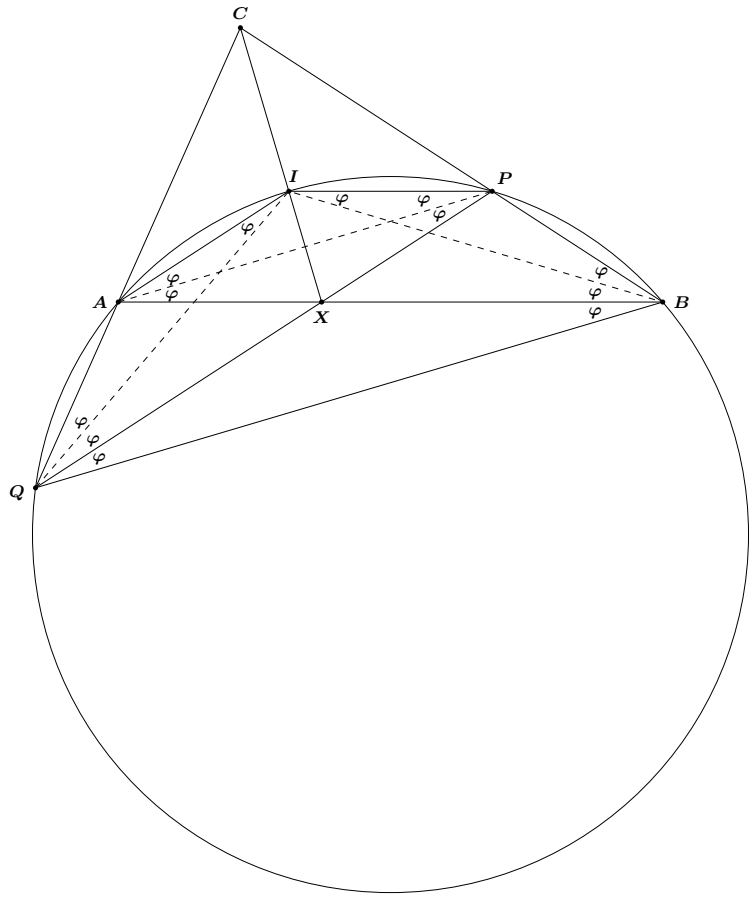

Figure 5: Construction for the third property.

It follows that

(a) Triangles AQI, IAP, PID are isosceles with

$$QA = AI = IP = PB,$$

which is equal to $a - b$ by (10).

(b) The pairs of equal alternate angles IPA, PAB and PAB, ABQ make

$$IP \parallel AB \quad \text{and} \quad AP \parallel QB.$$

(c) We have

$$PC = BC - PB = a - (a - b) = b = CA$$

and

$$CQ = CA + AQ = b + (a - b) = a = BC,$$

making $\triangle ABC$ and $\triangle PQC$ congruent (S-A-S) with

$$\overline{PQ} = c.$$

(d) The foot of the bisector of $\angle BCA$ lies on the line PQ.
Proof. Let X be the point where the side AB is crossed by the line PQ. Menelaus's theorem, applied to $\triangle ABC$ and transversal PXQ asserts that

$$\frac{AX}{XB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = 1,$$

and therefore

$$\frac{AX}{XB} \cdot \frac{1}{CA} \cdot \frac{BC}{1} = 1,$$

yielding $\frac{AX}{XB} = \frac{CA}{BC}$, and the conclusion follows by the converse of the internal angle bisector theorem. □

4. Let Y be the point where the line AB is crossed by the line OC. Then, the distance between the points A and Y equals CA.
Proof. If $\angle CAB = 90°$, then point Y coincides with vertex B and

$$\text{distance}(Y, A) = \text{distance}(B, A) = AB = CA,$$

where the last equality holds since ABC is a 90-45-45 degrees triangle.
If $\angle CAB \neq 90°$, let C' be the point on the circumcircle of $\triangle ABC$ diametrically opposite C. Then (see figure 6),

$$\angle ACY = \angle ACC' = 90° - \angle AC'C = 90° - \angle ABC,$$

where the second inequality holds in the right-angled triangle CAC' and the third holds on chord CA.
Since the angles of $\triangle CAY$ add up to 180 degrees and $\angle A = 2 \cdot (\angle B)$, we have $\angle AYC = 90° - \angle ABC$. Thus, $\angle AYC = \angle ACY$, making $\triangle CAY$ isosceles with $CA = AY$. □

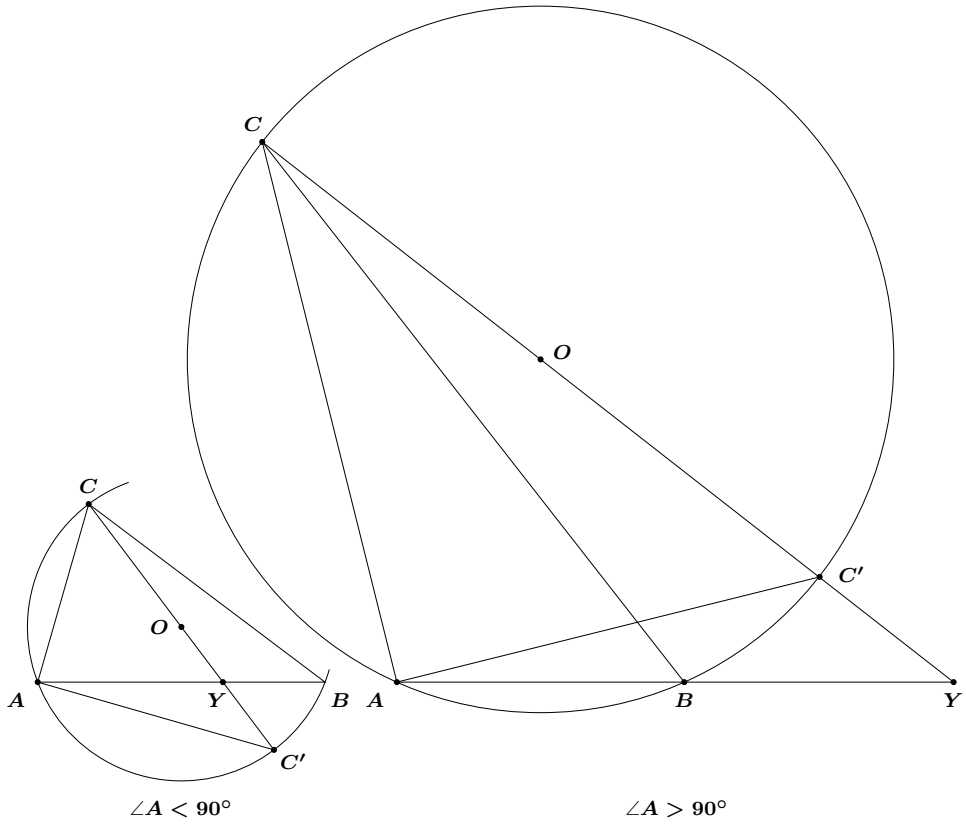

Figure 6: Construction for the fourth property.

5. Since the length of the tangent from vertex A of $\triangle ABC$ to the incircle is $s - a$ and the tangent from any vertex to the excircle beyond the opposite side is s (see figure 7),

$$\frac{AI_a}{AI} = \frac{s}{s-a}.$$

Subtracting 1 from each side gives

$$\frac{II_a}{AI} = \frac{a}{s-a}$$

and

$$II_a = \frac{a}{s-a} \cdot AI.$$

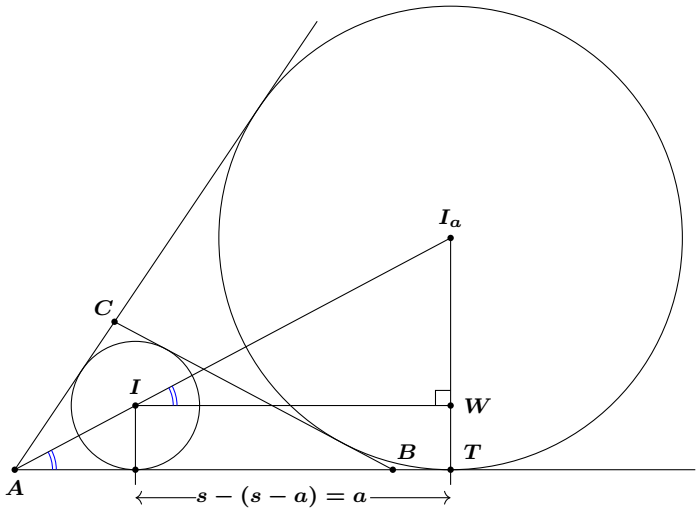

Figure 7: Construction for the fifth property.

Substituting for AI from (10),

$$II_a = \frac{a}{s-a}(a-b) = \frac{a^2-ab}{s-a} = \frac{b(-a+b+c)}{s-a},$$

where the last equality follows by substituting $b(b+c)$ for a^2), yielding

$$II_a = 2b.$$

Let the point of contact of the A-excircle with AB be T and let W the point on I_aT such that IW is parallel to AB. From right-angled triangle IWI_a where $IW = a$,

$$\cos\frac{A}{2} = \cos(\angle I_aIW) = \frac{IW}{II_a} = \frac{a}{2b},$$

and then $\cos A = 2\cos^2\frac{A}{2} - 1$ gives

$$\cos A = \frac{a^2}{2b^2} - 1 = \frac{b(b+c)}{2b^2} - 1 = \frac{c-b}{2b}.$$

Now, from the simple triangle relation $c = a\cos B + b\cos A$, it follows that

$$\cos B = \frac{c+b}{2a}.$$

6. Let d_a, d_b be the distances from the circumcenter of $\triangle ABC$ to the sides BC, CA, respectively. Then (see figure 8),

$$d_a = OM = R|\cos A|, \qquad d_b = ON = R\cos B,$$

where R denotes the circumcenter of $\triangle ABC$ and M, N the midpoints of sides BC, CA. Substitution of the expressions above for $\cos A$ and $\cos B$ in these formulae yields

$$\frac{d_a}{d_b} = \frac{a|c-b|}{b(c+b)} = \frac{a|c-b|}{a^2} = \frac{|c-b|}{a}.$$

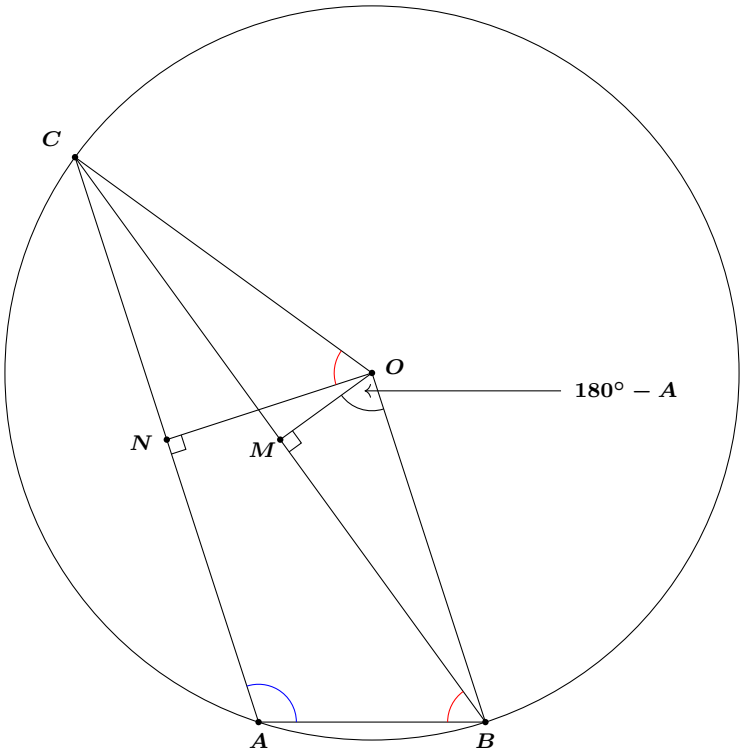

Figure 8: Construction for the sixth property.

7. The area of $\triangle ABC$ is rs and also $r_a(s-a)$, hence $rs = r_a(s-a)$. Thus,

$$\frac{r}{r_a} = \frac{s-a}{s} == \frac{-ab+b(b+c)}{ab+b(b+c)} = \frac{a-b}{a+b},$$

where the second equality follows by multiplying numerator and denominator by $2b$ and the third, by substituting a^2 for $b(b+c)$ and simplifying.

References

[1] Bellot Rosado, F. "Triángulos especiales (2)". *Revista Escolar de la Olimpiada Iberoamericana de Matemáticas* 18 (2005).

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A new proof of the Blundon-Eddy inequality and some applications

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Abstract

The purpose of this article is to give a new proof of the Blundon-Eddy inequality in a bicentric quadrilateral, writing the sides according to the radius R, inradius r and semiperimeter S. The Blundon theorem is proved in [1]. Also we give a Blundon-type inequality for the sum $\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}$.

1 Introduction

Let $ABCD$ be a bicentric quadrilateral with R the radius of circumcenter, r the radius of incenter, S the semiperimeter, F the area, d_1, d_2 the diagonals. We denote $x_3 = d_1d_2$.

The following lemma is known.

Lemma 1. *In every bicentric quadrilateral we have*

i) $F = Sr$,
ii) $F = \sqrt{abcd}$,
iii) $d_1d_2 = ac + bd$ *(Ptolomey's theorem) and*
iv) $d_1d_2 = 2r\left(\sqrt{4R^2 + r^2} + r\right)$.

Theorem 1. *In every bicentric quadrilateral we have*

$$\{2a, 2b, 2c, sd\} = \left\{S \pm \sqrt{S^2 - 2x_3 \pm 2\sqrt{x_3^2 - 4S^2r^2}}\right\},$$

where $x_3 = ac + bd = 2r(\sqrt{4R^2 + r^2} + r)$.

Proof. We have $ac+bd = x_3$, $ac\cdot bd = S^2r^2$, from which we obtain

$$\{ac, bd\} = \left\{\frac{x_3 \pm \sqrt{x_3^2 - 4r^2S^2}}{2}\right\}.$$

Without loss of generality, we may take $ac = \dfrac{x_3 + \sqrt{x_3^2 - 4r^2S^2}}{2}$. Since $S = a + c$, we obtain

$$\{a, c\} = \left\{\frac{1}{2}\Big(S \pm \sqrt{S^2 - 2x_3 - 2\sqrt{x_3^2 - 4r^2S^2}}\Big)\right\}.$$

In a similar way we obtain

$$\{b, d\} = \left\{\frac{1}{2}\Big(S \pm \sqrt{S^2 - 2x_3 + 2\sqrt{x_3^2 - 4r^2S^2}}\Big)\right\}. \qquad \square$$

Theorem 2. *In every bicentric quadrilateral we have the identities*

i) $(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2$
$= 16r^2S^2\Big[S^2 - \Big(\sqrt{4R^2 + r^2} + r\Big)^2\Big]^2\Big[S^2 - 8r\Big(\sqrt{4R^2 + r^2} - r\Big)\Big]$;

ii) $(a-c)^2(b-d)^2 = S^2\Big(S^2 - 8r\Big(\sqrt{4R^2 + r^2} - r\Big)\Big)$;

iii) $(a-b)(b-c)(c-d)(d-a) = 4r^2\Big[\Big(\sqrt{4R^2 + r^2} + r\Big)^2 - S^2\Big]$.

Proof. We denote

$$u = S^2 - 2x_3 + 2\sqrt{x_3^2 - 4r^2S^2},$$
$$v = S^2 - 2x_3 - 2\sqrt{x_3^2 - 4r^2S^2}.$$

By Theorem 1, we may assume that $2a = S + \sqrt{u}$, $2b = S + \sqrt{v}$, $2c = S - \sqrt{u}$, $2d = S - \sqrt{v}$.

i) We have

$$\begin{aligned}
&4^6\Pi(a-b)^2 \\
&= \big(\sqrt{u} - \sqrt{v}\big)^2\big(2\sqrt{u}\big)^2\big(\sqrt{u} + \sqrt{v}\big)^2\big(\sqrt{u} - \sqrt{v}\big)^2\big(2\sqrt{v}\big)^2\big(\sqrt{u} - \sqrt{v}\big)^2 \\
&= 4^2(u-v)^4uv,
\end{aligned}$$

or

$$\begin{aligned}
&\Pi(a-b)^2\\
&=\frac{1}{4^4}4^4(x_3^2-4S^2r^2)^2\left[(S^2-2x_3^2)^2-4x_3+8S^2r^2\right]\\
&=16r^4\Big(S^2-\Big(\sqrt{4R^2+r^2}+r\Big)^2\Big)^2(S^4-4x_3S^2+16r^2S^2)\\
&=16r^4S^2\Big(S^2-\Big(\sqrt{4R^3+r^2}+r\Big)^2\Big)\Big(S^2-8r\Big(\sqrt{4R^2+r^2}-r\Big)\Big).
\end{aligned}$$

ii) We have

$$\begin{aligned}
(a-c)^2(b-d)^2&=\frac{1}{4^2}\big(2\sqrt{u}\big)^2\big(2\sqrt{v}\big)^2=4uv\\
&=S^2\Big(S^2-8r\Big(\sqrt{4R^2+r^2}-r\Big)\Big).
\end{aligned}$$

iii) We have

$$\begin{aligned}
&(a-b)(b-c)(c-d)(d-a)\\
&=\frac{1}{16}(\sqrt{u}-\sqrt{v})(\sqrt{v}+\sqrt{u})(\sqrt{v}-\sqrt{u})(-\sqrt{v}-\sqrt{u})\\
&=\frac{1}{16}(u-v)^2=x_3^2-4S^2r^2\\
&=4r^2\Big[\Big(\sqrt{4R^2+r^2}+r\Big)^2-S^2\Big).
\end{aligned}$$

□

Theorem 3 (Blundon-Eddy for bicentric quadrilateral). *In every bicentric quadrilateral the inequality $S_1\le S\le S_2$ holds, where S_1 and S_2 represent the semiperimeter of two bicentric quadrilaterals $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ with the sides*

$$\begin{aligned}
&\{a_1,b_1,c_1,d_1\}\\
&=\left\{2r\sqrt{R^2-(r\pm d)^2},\sqrt{2r\Big(\sqrt{4R^2+r^2}-r\Big)},\sqrt{2r\Big(\sqrt{4R^2+r^2}-r\Big)}\right\}
\end{aligned}$$

and

$$\{a_2,b_2,c_2,d_2\}=\left\{\frac{2Rr}{R-\overline{d}},\frac{2Rr}{R-\overline{d}+\overline{d}},\frac{2Rr}{R+\overline{d}},\frac{2Rr}{R+\overline{d}}\right\},$$

with the semiperimeters

$$S_1 = \sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)} \quad \textit{and} \quad S_2 = \sqrt{4R^2+r^2}+r,$$

where $\overline{d} = OI$.

Proof. By Theorem 2 ii) and iii), since $\Pi(a-b) = \frac{1}{16}(u-v)^2 \geq 0$, it follows that $S^2 \geq S_1^2$ and $S^2 \leq S_2^2$ or $S_1 \leq S \leq S_2$.

We denote $AB = a$, $BC = b$, $CD = c$, $DA = d$.

In the following we solve the equality case from iii).

Case 1. According to iii), $S_2 = \sqrt{4R^2+r^2}+r$ if $a = b$, or $b = c$, or $d = c$, or $d = a$.

We consider $a = d$; the cases $b = c$, $d = c$ or $a = b$ are similar.

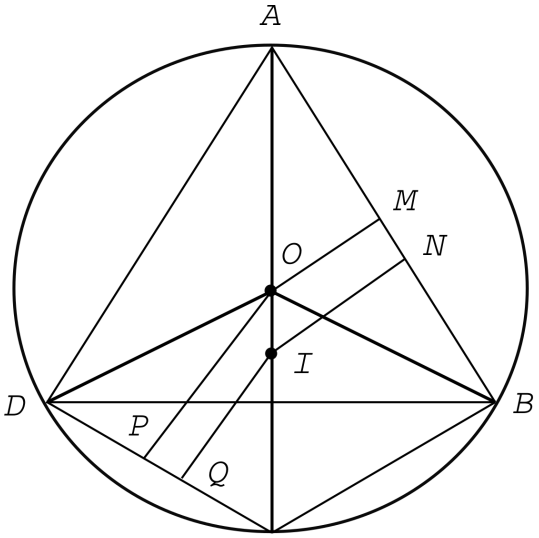

Figure 1: Scheme for the proof of Theorem 3.

Since triangle ABD is isosceles and $ABCD$ is inscribed in $\mathcal{C}(O,R)$, we have $\mu(\widehat{BCA}) = \mu(\widehat{ACB})$. So $I \in AC$ and $\mu(\widehat{DAC}) = \mu(\widehat{CAB})$ or $O \in AC$. Hence, $AO \equiv OB \equiv OC \equiv OD$.

Let $ON \perp AB$, $IN \perp AB$, $M, N \in (AB)$. Suppose $R > d$ (similar if $R < d$). We have $AB \equiv AD = a$, $BC \equiv DC = c$. Since $\triangle AOM \sim \triangle AIN$ and $\triangle QIC \sim \triangle COP$ and by Pythagoras in $\triangle AIN$ and $\triangle QIC$, denoting $AN = x$, $QC = y$, we obtain

$$\frac{a}{2x} = \frac{R}{R+\overline{d}}, \qquad \frac{2y}{b} = \frac{R-\overline{d}}{R},$$
$$x^2 + r^2 = (R+\overline{d})^2, \qquad y^2 + r^2 = (R-\overline{d})^2.$$

From this, we obtain

$$a = \frac{2R}{R+\overline{d}}\sqrt{(R+\overline{d})^2 - r^2} \quad \text{and} \quad b = \frac{2R}{R-+\overline{d}}\sqrt{(R-\overline{d})^2 - r^2},$$

with $a = d$, $b = c$.

We verify if indeed $S_2 = \sqrt{4R^2 + r^2} + r$. We have

$$S_2 = 2a + 2b = 2R\left(\frac{\sqrt{(R+\overline{d})^2 - r^2}}{R+\overline{d}} + \frac{\sqrt{(R-\overline{d})^2 - r^2}}{R-\overline{d}}\right)$$
$$= 2R\left(\frac{1}{R+\overline{d}} + \frac{1}{R-\overline{d}}\right) = \frac{4R^2 r}{R^2 - \overline{d}^2} = \sqrt{4R^2 + r^2} + r$$

(where we applied the Fuss theorem $\frac{1}{(R+\overline{d})^2} + \frac{1}{(R-\overline{d})^2} = \frac{1}{r^2}$).

So we obtain a bicentric quadrilateral $A_2B_2C_2D_2$ with the sides

$$a_2 = d_2 = \frac{2R}{R+\overline{d}}\sqrt{(R+\overline{d})^2 - r^2},$$
$$b_2 = c_2 = \frac{2R}{R-\overline{d}}\sqrt{(R-\overline{d})^2 - r^2}$$

and semiperimeter $S_2 = \sqrt{4R^2 + r^2} + r$.

Case 2. By ii), $S_1 = \sqrt{8r\left(\sqrt{4R^2 + r^2} - r\right)}$ if $a = c$ or $b = d$.

WLOG we consider $a = c$. We denote $\beta = 8r\left(\sqrt{4R^2 + r^2} - r\right)$.

So $a+c=b+d=S$ or $a=c=\dfrac{\sqrt{\beta}}{2}$. But $abcd=r^2S^2$ or $\dfrac{\beta}{4}bd=\beta r$, or $bd=4r^2$ and

$$b+d=\sqrt{\beta}. \tag{1}$$

From (1) it follows that

$$b,d\in\left\{\frac{\sqrt{\beta}\pm\sqrt{\beta-16r^2}}{2}\right\}. \tag{2}$$

Denoting

$$x=\sqrt{R^2-(r+\overline{d})^2}+\sqrt{R^2-(r-\overline{d})^2},$$
$$y=\sqrt{R^2-(r-\overline{d})^2}-\sqrt{R^2-(r+\overline{d})^2},$$

we have

$$x^2=2r\sqrt{4R^2+r^2}-2r^2, \tag{3}$$

or $x=\dfrac{\sqrt{\beta}}{2}$, $y^2=2r\sqrt{4R^2+r^2}-6r^2$ or $y=\sqrt{2r\left(\sqrt{4R^2+r^2}-3r\right)}$ or

$$\sqrt{\beta-16r^2}=2y. \tag{4}$$

From (2), (3) and (4) we obtain

$$a_1=c_1=\sqrt{R^2-(r-\overline{d})}+\sqrt{R^2-(r+\overline{d})^2}=\sqrt{2\left(\sqrt{4R^2+r^2}-r\right)}$$

and

$$\{b_1,d_1\}=\left\{2\sqrt{R^2-(r\pm\overline{d})^2}\right\},$$

the sides of bicentric quadrilateral $A_1B_1C_1D_1$ with semiperimeter $S_1=\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}$. □

Lemma 2. *In every bicentric quadrilateral we have*

i) $x_1+x_2=S^2$,
ii) $x_1x_2=2r\left(\sqrt{4R^2+r^2}-r\right)S^2$, *where* $x_1=ab+cd$, $x_2=ad+bc$.

Proof. i) $x_1 + x_2 = (a+c)(b+d) = S^2$.

ii) We have $\sigma[ABC]+\sigma[ADC] = \sigma[ABD]+\sigma[BCD] = \sigma[ABCD]$ or $ab\sin B + cd\sin B = 2F$ and $ad\sin A + bc\sin C = 2Sr$, or $x_1\dfrac{d_1}{2R} = 2Sr$ and $x_2\dfrac{d_2}{2R} = 2Sr$, or $x_1x_2d_1d_2 = 16R^2r^2S^2$ or, according to Lemma 1,

$$x_1x_2 = \frac{16R^2r^2S^2}{2r\left(\sqrt{4R^2+r^2}+r\right)} = 2r\left(\sqrt{4R^2+r^2}-r\right). \qquad \square$$

Theorem 4. *In every bicentric quadrilateral, the following inequality is true:*

$$\begin{aligned}&\sqrt{2r\left(\sqrt{4R^2+r^2}-r\right)}\\ &+2\sqrt{2}\sqrt[4]{2r\left(\sqrt{4R^2+r^2}-r\right)}\sqrt{\sqrt{2r\left(\sqrt{4R^2+r^2}-r\right)}+2r}+2r\\ &\le \sum\sqrt{ab} \le 4\sqrt{r\left(\sqrt{4R^2+r^2}+r\right)}+\sqrt{4R^2+r^2}+r.\end{aligned}$$

Proof. We have

$$\sum\sqrt{ab} = \sqrt{ab}+\sqrt{cd}+\sqrt{ad}+\sqrt{bc}+\sqrt{ac}+\sqrt{bd}.$$

We denote $y_1 = \sqrt{ab}+\sqrt{cd}$, $y_2 = \sqrt{ad}+\sqrt{bc}$, $y_3 = \sqrt{ac}+\sqrt{bd}$.

We have $\sum\sqrt{ab} = y_1+y_2+y_3$, $y_1^2 = x_1+2F$ or $y_1 = \sqrt{x_1+2Sr}$ and $y_2 = \sqrt{x_2+2Sr}$.

$$\begin{aligned}&y_1+y_2 = \sqrt{x_1+2Sr}+\sqrt{x_2+2Sr} \qquad \text{or}\\ &y_1+y_2 = \sqrt{x_1+x_2+4Sr+2\sqrt{(x_1+2Sr)(x_2+2Sr)}}.\end{aligned}$$

So we consider the function $f\colon [S_1, S_2] \to \mathbb{R}$

$$\begin{aligned}f(S) &= \sum\sqrt{ab} = y_1+y_2+y_3\\ &= \sqrt{S^2+4Sr+2\sqrt{2r\left(\sqrt{4R^2+r^2}-r\right)S^2+4r^2S^2+2rS^3}},\end{aligned}$$

which is an increasing function.

So $f(S_1) \leq f(S) \leq f(S_2)$ according Theorem 3 or

$$\sum \sqrt{a_1 b_1} \leq \sum \sqrt{ab} \leq \sum \sqrt{a_2 b_2}.$$

We have $a_1 = c_1 = \sqrt{R^2 - (r - +\overline{d})^2} + \sqrt{R^2 - (r + \overline{d})^2}$, $b_1 = 2\sqrt{R^2 - (r - +\overline{d})^2}$, $d_1 = 2\sqrt{R^2 - (r + \overline{d})^2}$ and

$$\sum \sqrt{a_1 b_1} = a_1 + 2\sqrt{a_1 b_1} + 2\sqrt{a_1 d_1} + \sqrt{b_1 d_1}. \tag{5}$$

But $b_1 d_1 = 4r^2$. So from (5) we obtain

$$\sum \sqrt{a_1 b_1} = \sqrt{R^2 - (r - \overline{d})^2} + \sqrt{R^2 - (r + \overline{d})^2} + 2r$$
$$+2\sqrt{2}\sqrt[4]{2r\left(\sqrt{4R^2 + r^2} - r\right)}\left(\sqrt[4]{R^2 - (r + \overline{d})^2} + \sqrt[4]{R^2 + (r + \overline{d})^2}\right). \tag{6}$$

Also if we denote $\sqrt[4]{R^2 - (r - \overline{d})^2} + \sqrt[4]{R^2 - (r + \overline{d})^2} = z$ we have

$$z = \sqrt{\sqrt{R^2 - (r - \overline{d})^2} + \sqrt{R^2 - (r + \overline{d})^2} + 2\sqrt[4]{[R^2 - (r + \overline{d})^2][R^2 + (r - \overline{d})^2]}}$$
$$= \sqrt{\sqrt{2r\left(\sqrt{4R^2 + r^2} - r\right)} + 2r}.$$

So

$$\sum \sqrt{a_1 b_1} = \sqrt{2r\left(\sqrt{4R^2 + r^2} - r\right)}$$
$$+2\sqrt{2}\sqrt[4]{2r\left(\sqrt{4R^2 + r^2} - r\right)}\sqrt{\sqrt{2r(4R^2 + r^2} - r) + 2r} + 2r,$$

$$\sum \sqrt{a_2 b_2} = 4\sqrt{a_2 b_2} + a_2 + b_2$$
$$= 4\sqrt{\frac{4R^2}{R^2 - \overline{d}^2}\sqrt{[(R - \overline{d})^2 - r^2][(R + \overline{\overline{d}})^2 - r^2]}}$$
$$+ \frac{2R}{R - \overline{d}}\sqrt{(R - \overline{d})^2 - r^2} + \frac{2R}{R + \overline{d}}\sqrt{(R + \overline{d})^2 - r^2}. \tag{7}$$

By the Fuss equality we have $R^2-\overline{d}^2=r(\sqrt{4R^2+r^2}-r)$ and

$$\begin{aligned}
&[(R-\overline{d})^2-r^2][(R+\overline{d}d)^2-r^2]\\
&=\left[2R^2-r\sqrt{4R^2+r^2}-2Rd\right]\left[2R^2-r\sqrt{4R^2+r^2}+2Rd\right]\\
&=\left(2R^2-r\sqrt{4R^2+r^2}\right)^2-4R^2\left(R^2+r^2-r\sqrt{4R^2+r^2}\right)\\
&=4R^4+4R^2r^2+r^4-4R^2r\sqrt{4R^2+r^2}-4R^4-4R^2r^2+4R^2r\sqrt{4R^2+r^2}\\
&=r^4.
\end{aligned}$$

Replacing in (7) we obtain

$$\begin{aligned}
\sum\sqrt{a_2b_2}&=\frac{8Rr}{\sqrt{r\left(\sqrt{4R^2+r^2}-r\right)}}+\sqrt{4R^2+r^2}+r\\
&=4\sqrt{r\left(\sqrt{4R^2+r^2}+r\right)}+\sqrt{4R^2+r^2}+r.
\end{aligned}$$

□

Corollary 1. *In every bicentric quadrilateral, the following inequality holds:*

$$\sqrt{ab}+\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}+\sqrt{cd}\le\frac{8\sqrt{2}}{3}R+\frac{20}{3}r.$$

Proof. Using the notations from Theorem 4, to prove the inequality from the statement, it will be sufficient to show that

$$4\sqrt{\sqrt{4x^2+1}+1}+\sqrt{4x^2+1}+1\le\frac{8\sqrt{2}}{3}x+\frac{20}{3},\ \forall\, x\ge\sqrt{2}.$$

We denote $t=\sqrt{\sqrt{4x^2+1}+1}$. We prove that

$$4t+t^2\le\frac{4\sqrt{2}}{3}\sqrt{t^4-2t^2}+\frac{20}{3},\ \forall\, t\ge 2,$$

or

$$3t^2+12t-20\le 4\sqrt{2}\sqrt{t^4-2t^2},\ \forall\, t\ge 2,$$

or

$$32t^4-64t^2-(3t^2+12t-20)^2\ge 0,\ \forall\, t\ge 2,$$

or

$$(t-2)^2(23t^2+20t-100)\ge 0,\ \forall\, t\ge 2,$$

which is true. □

Corollary 2. *Find the best real constants α, β such that the inequality*

$$\sqrt{ab}+\sqrt{ac}+\sqrt{ad}+\sqrt{bc}+\sqrt{bd}+\sqrt{cd} \leq \alpha R+\beta r$$

is true in every bicentric quadrilateral.

Proof. We prove that the inequality from Corollary 1 is the best of type $\sum \sqrt{ab} \leq \alpha R+\beta r$. Assume there exist $\alpha_0, \beta_0 \in \mathbb{R}$ such that

$$\sum \sqrt{ab} \leq \alpha_0 R+\beta_0 r \leq \frac{8\sqrt{2}}{3} R+\frac{20}{3} r$$

is true in every bicentric quadrilateral.

From Theorem 4 we have that

$$\begin{aligned}\sum \sqrt{ab} &\leq 4\sqrt{r\left(\sqrt{4R^2+r^2}+r\right)}+\sqrt{4R^2+r^2}+r \\ &\leq \alpha_0 R+\beta_0 r \leq \frac{8\sqrt{2}}{3} R+\frac{20}{3} r\end{aligned}$$

holds in every bicentric quadrilateral or

$$4\sqrt{\sqrt{4x^2+1}+1}+\sqrt{4x^2+1}+1 \leq \alpha_0 x+\beta_0 \leq \frac{8\sqrt{2}}{3} x+\frac{20}{3}, \forall\, x \geq \sqrt{2}.$$

For $x=\sqrt{2}$ we obtain $12 \leq \sqrt{2}\alpha_0+\beta_0 \leq 12$ or $\beta_0+\sqrt{2}\alpha_0=12$.

So

$$\begin{aligned}&4\sqrt{\sqrt{4x^2+1}+1}+\sqrt{4x^2+1}+1 \\ &\leq \alpha_0 x+12-\sqrt{2}\alpha_0 \leq \frac{8\sqrt{2}}{3} x+\frac{20}{3}, \ \forall\, x \geq \sqrt{2},\end{aligned}$$

or

$$\alpha_0 \geq \frac{4\sqrt{\sqrt{4x^2+1}+1}+\sqrt{4x^2+1}-11}{x-\sqrt{2}}, \ \forall\, x \geq \sqrt{2}.$$

If we denote $t=\sqrt{\sqrt{4x^2+1}+1}$, $\forall\, t \geq 2$, we obtain

$$\alpha_0 \geq \frac{2(t^2+4t-12)}{\sqrt{t^4-2t^2-2\sqrt{2}}}, \ \forall\, t \geq 2$$

or

$$\alpha_0 \geq \frac{2(t+6)\left(\sqrt{t^4-2t^2}+2\sqrt{2}\right)}{t^3+2t^2+2t+4},\ \forall\, t \geq 2.$$

So

$$\alpha_0 \geq \max_{t\geq 2} \frac{2(t+6)\left(\sqrt{t^4-2t^2}+2\sqrt{2}\right)}{t^3+2t^2+2t+4} = \frac{8\sqrt{2}}{3}.$$

So $\alpha_0 \geq \dfrac{8\sqrt{2}}{3}$.

Also

$$\alpha_0 x + 12 - \sqrt{2}\alpha_0 \leq \frac{8\sqrt{2}}{3}\, x + \frac{20}{3},\ \forall\, x \geq \sqrt{2}$$

or

$$\alpha_0 + \frac{12}{x} - \frac{\sqrt{2}\alpha_0}{x} \leq \frac{8\sqrt{2}}{3} + \frac{20}{3x},\ \forall\, x \geq \sqrt{2}.$$

Taking $x \to \infty$ it follows that $\alpha_0 \leq \dfrac{8\sqrt{2}}{3}$. So $\alpha_0 = \dfrac{8\sqrt{2}}{3}$.

Since $\beta_0 + \sqrt{2}\alpha_0 = 12$, we obtain $\beta_0 = \dfrac{20}{3}$, a contradiction.

It results that the inequality from Corollary 1 is the best of type

$$\sum \sqrt{ab} \leq \alpha R + \beta r. \qquad \square$$

Corollary 3. *In every bicentric quadrilateral the following inequality is true:*

$$12r \leq \sqrt{ab} + \sqrt{ac} + \sqrt{ad} + \sqrt{bc} + \sqrt{bd} + \sqrt{cd}.$$

Proof. We denote $x = R/r$. According to Theorem 4, it will be sufficient to prove that

$$\sqrt{2\left(\sqrt{4x^2+1}-1\right)}+$$

$$+2\sqrt{2}\sqrt[4]{2\left(\sqrt{4x^2+1}-1\right)}\sqrt{\sqrt{2\left(\sqrt{4x^2+1}-1\right)}+2}+2 \geq 12,\ \forall\, x \geq \sqrt{2}.$$

We denote $t=\sqrt{2\left(\sqrt{4x^2+1}-1\right)}$ or $x=\dfrac{1}{4}\sqrt{t^4+4t^2}$.

We obtain $t+2\sqrt{2}\sqrt{t(t+2)}\geq 10,\ \forall\, t\geq 2$, which is true. □

Remark 1. *Similarly to Corollary 2, we may prove that the best constant α for which the inequality $\sum\sqrt{ab}\geq \alpha R+(12-\sqrt{2}\alpha)r$ is true in every bicentric quadrilateral is $\alpha=0$.*

Theorem 5. *In every bicentric quadrilateral, the following inequalities hold:*

$$\begin{aligned}2\sqrt[4]{2r\left(\sqrt{4R^2+r^2}-r\right)}+\sqrt{2\sqrt{2}\sqrt{r\left(\sqrt{4R^2+r^2}-r\right)}+4r}\\ \leq \sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\\ \leq 2\sqrt{\sqrt{4R^2+r}+r+2\sqrt{r\left(\sqrt{4R^2+r^2}+r\right)}}.\end{aligned}$$

Proof. We denote $u=\sqrt{ac}+\sqrt{bd}$. We have

$$\begin{aligned}\sum\sqrt{a}&=\sqrt{a}+\sqrt{c}+\sqrt{b}+\sqrt{d}\\ &=\sqrt{a+c+2\sqrt{ac}}+\sqrt{b+d+2\sqrt{bd}}=\sqrt{S+2\sqrt{ac}}+\sqrt{S+2\sqrt{bd}}\\ &=\sqrt{2S+2u+2\sqrt{\left(S+2\sqrt{ac}\right)\left(S+2\sqrt{bd}\right)}}\\ &=\sqrt{2S+2u+2\sqrt{S^2+4F+2Su}}\\ &=\sqrt{2S+2u+2\sqrt{S^2+4Sr+2Su}}\end{aligned}\tag{8}$$

with

$$u=\sqrt{ac+bd+2\sqrt{abcd}}=\sqrt{2r\left(\sqrt{4R^2+r^2}-r\right)+2Sr}.\tag{9}$$

We consider the function $f\colon [S_1,S_2]\to\mathbb{R},\ f(S)=\sum\sqrt{a}$. From (8) and (9) it follows that f is increasing. So

$$f(S_1)\leq f(S)\leq f(S_2)\tag{10}$$

or

$$\begin{aligned}\sqrt{a_1}+\sqrt{b_1}+\sqrt{c_1}+\sqrt{d_1}\leq \sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\\ \leq \sqrt{a_2}+\sqrt{b_2}+\sqrt{c_2}+\sqrt{d_2}\end{aligned}$$

with

$$\begin{aligned}\sum\sqrt{a_1} &= \sqrt{a_1}+\sqrt{b_1}+\sqrt{c_1}+\sqrt{d_1}\\ &= 2\sqrt{a_1}+\sqrt{b_1}+\sqrt{d_1}\\ &= 2\sqrt[4]{2r\left(\sqrt{4R^2+r^2}-r\right)}+\sqrt{2}\left(\sqrt[4]{R^2-(r+\overline{d})^2}+\sqrt[4]{R^2-(r-\overline{d})^2}\right).\end{aligned}$$

We have

$$\begin{aligned}&\sqrt[4]{R^2-(r-\overline{d})^2}+\sqrt[4]{R^2-(r+\overline{d})^2}\\ &=\sqrt{\sqrt{R^2-(r-\overline{d})^2}+\sqrt{R^2-(r+\overline{d})^2}+2\sqrt[4]{[R^2-(r-\overline{d})^2][R^2+(r+\overline{d})^2]}}\\ &=\sqrt{\sqrt{2r\left(\sqrt{4R^2+r^2}-r\right)}+2r}.\end{aligned}$$

So

$$\sum\sqrt{a_1}=2\sqrt[4]{2r\left(\sqrt{4R^2+r^2}-r\right)}+\sqrt{\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}+4r}. \quad (11)$$

Also

$$\sum\sqrt{a_2}=2\sqrt{a_2}+2\sqrt{b_2}=2\left(\sqrt{a_2+b_2+2\sqrt{a_2\,b_2}}\right). \quad (12)$$

We have $a_2+b_2=\sqrt{4R^2+r}+r=S_2$,

$$\begin{aligned}a_2\cdot b_2 &= \frac{4R^2}{R^2-d^2}\sqrt{[(R-\overline{d})^2-r^2][(R+\overline{d})^2-r^2]}\\ &= \frac{4R^2r^2}{R^2-\overline{d}^2}=\frac{4R^2r^2}{r\left(\sqrt{4R^2+r^2}-r\right)}=r\left(\sqrt{4R^2+r^2}+r\right).\end{aligned} \quad (13)$$

From (12) and (13) we obtain

$$\sum\sqrt{a_2}=\sqrt{\sqrt{4R^2+r^2}+r+2\sqrt{r\left(\sqrt{4R^2+r^2}+r\right)}}. \quad (14)$$

The statement follows from (10), (11) and (14). □

Corollary 4. *In every bicentric quadrilateral, the following inequality holds:*

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\le 2\sqrt{2}\sqrt{R}+\left(4\sqrt{2}-2\sqrt[4]{8}\right)\sqrt{r}. \quad (15)$$

Proof. From Theorem 5 we have

$$\sum \sqrt{a} \leq 2\sqrt{\sqrt{4R^2+r^2}+r}+2\sqrt{r\left(\sqrt{4R^2+r^2}+r\right)}.$$

If we denote $x=\dfrac{R}{r}$ to prove (15), it will be sufficient to prove that

$$2\sqrt{\sqrt{4x^2+1}+1}+2\sqrt{\sqrt{4x^2+1}+1} \leq 2\sqrt{2}\sqrt{x}+4\sqrt{2}-2\sqrt[4]{8},\ \forall\, x \geq \sqrt{2}. \tag{16}$$

If we denote $t=\sqrt{\sqrt{4x^2+1}+1} \geq 2$, the inequality (16) may be written as

$$\sqrt{t^2+2t} \leq \sqrt[4]{t^4-2t^2}+2\sqrt{2}-\sqrt[4]{8},\ \ \forall\, t \geq 2,$$

which is true according to Wolphram Alpha. □

Corollary 5. *If α, β are real numbers with the property that the inequality*

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d} \leq \alpha\sqrt{R}+\beta\sqrt{r} \tag{17}$$

is true in every triangle ABC, then we have the inequality

$$2\sqrt{2}\sqrt{R}+\left(4\sqrt{2}-2\sqrt[4]{8}\right)\sqrt{r} \leq \alpha\sqrt{R}+\beta\sqrt{r}$$

in every triangle ABC.

Proof. If the bicentric quadrilateral is a square, then by (17) we get

$$\sqrt[4]{2}\alpha+\beta \geq 4\sqrt{2}. \tag{18}$$

If $a=b,\ c=d=0,\ R=\dfrac{a}{2},\ r=0$, we obtain $\alpha \geq 2\sqrt{2}$.

Since $R \geq \sqrt{2}\,r$ we have that

$$\begin{aligned}
&\left(\alpha-2\sqrt{2}\right)\sqrt{R}+\left(\beta-4\sqrt{2}+2\sqrt[4]{8}\right)\sqrt{r}\\
&\geq \left(\sqrt[4]{2}\alpha-2\sqrt[4]{8}+\beta-4\sqrt{2}+2\sqrt[4]{8}\right)\sqrt{r}\\
&= \left(\sqrt[4]{2}\alpha+\beta-4\sqrt{2}\right)\sqrt{r} \geq 0.
\end{aligned}$$

So $\alpha\sqrt{R}+\beta\sqrt{r} \geq 2\sqrt{2}\sqrt{R}+\left(4\sqrt{2}-2\sqrt[4]{8}\right)\sqrt{r}$. □

Corollary 6. *In every bicentric quadrilateral the following inequality is true:*

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d}\geq 4\sqrt{2}\sqrt{r}. \tag{19}$$

Proof. We may prove the inequality similarly to Corollary 3. But we find an other easy proof using the AM-GM inequality. We have

$$\sum\sqrt{a}\geq 4\sqrt[4]{\sqrt{abcd}}=4\sqrt[4]{F}=4\sqrt[4]{Sr}.$$

So to prove (19) we need to prove that $\sqrt[4]{Sr}\geq\sqrt{2}\sqrt{r}$ or $S\geq 4r$, or from Blundon inequality $S\geq\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}\geq 4r$, or $\sqrt{4R^2+r^2}\geq 3r$, or $R\geq\sqrt{2}r$, which is true. □

Remark 2. *In the same way as in Corollary 3, we can prove that the inequality* (19) *is the best of type* $\sum\sqrt{a}\geq\alpha\sqrt{R}+\beta\sqrt{r}$.

Theorem 6. *In every bicentric quadrilateral, the following inequality holds:*

$$\begin{aligned}\sqrt{2r\left(\sqrt{4R^2+r^2}-r\right)}\sqrt{\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}+4r}+4r\sqrt[4]{2r\left(\sqrt{4R^2+r^2}-r\right)}\\ \leq\sqrt{abc}+\sqrt{abd}+\sqrt{acd}+\sqrt{bcd}\\ \leq 2\sqrt{r\left(\sqrt{4R^2+r^2}+r\right)}\sqrt{\sqrt{4R^2+r^2}+r+2\sqrt{r\left(\sqrt{4R^2+r^2}+r\right)}}.\end{aligned}$$

Proof. We consider the function $f\colon[S_1,S_2]\to\mathbb{R}$

$$\begin{aligned}f(S)&=\sum\sqrt{abc}=\sqrt{ac}\left(\sqrt{b}+\sqrt{d}\right)+\sqrt{bd}\left(\sqrt{a}+\sqrt{c}\right)\\ &=\sqrt{ac}\sqrt{b+d+2\sqrt{bd}}+\sqrt{bd}\sqrt{a+c+2\sqrt{ac}}\\ &=\sqrt{ac(b+d)+2\sqrt{abcd}\sqrt{ac}}+\sqrt{bd(a+c)+2\sqrt{abcd}\sqrt{bc}}\\ &=\sqrt{acS+2Sr\sqrt{ac}}+\sqrt{bdS+2Sr\sqrt{bd}}\\ &=\sqrt{S}\left(\sqrt{ac+2r\sqrt{ac}}+\sqrt{bd+2r\sqrt{bd}}\right)\\ &=\sqrt{S}\sqrt{ac+bd+2r\sqrt{ac+bd+2\sqrt{abcd}}+2\sqrt{ac+2r\sqrt{ac}\left(bd+2r\sqrt{bd}\right)}}\\ &=\sqrt{S}\sqrt{x_3+2r\sqrt{x_3+2Sr}+2\sqrt{S^2r^2+4r^3S+2r^2S\left(\sqrt{x_3+2Sr}\right)}},\end{aligned}$$

where $x_3 = 2r\big(r + \sqrt{4R^2 + r^2}\big)$.

It follows that f is increasing in S or $f(S_1) \le f(S) \le f(S_2)$, or

$$\begin{aligned}\sqrt{a_1b_1c_1} + \sqrt{a_1b_1d_1} + \sqrt{a_1c_1d_1} + \sqrt{b_1c_1d_1} &\le \sum \sqrt{abc} \\ &\le \sqrt{a_2b_2c_2} + \sqrt{a_2b_2d_2} + \sqrt{a_2c_2d_2} + \sqrt{b_2c_2d_2}. \qquad (20)\end{aligned}$$

We have

$$\begin{aligned}\sum \sqrt{a_1b_1c_1} &= \sqrt{a_1^2b_1} + \sqrt{a_1b_1d_1} + \sqrt{a_1^2d_1} + \sqrt{a_1b_1d_1} \\ &= a_1\Big(\sqrt{b_1} + \sqrt{d_1}\Big) + 2\sqrt{a_1b_1d_1} \\ &= a_1\sqrt{S_1 + 2\sqrt{b_1d_1}} + 2\sqrt{a_1 \cdot 4r^2} \\ &= a_1\sqrt{S_1 + 4r} + 4r\sqrt{a_1} \\ &= \sqrt{2r\Big(\sqrt{4R^2+r^2}-r\Big)}\sqrt{\sqrt{8r\Big(\sqrt{4R^2+r^2}-r\Big)}+4r} \\ &\quad + 4r\sqrt[4]{2r\Big(\sqrt{4R^2 + r^2} - r\Big)}. \qquad (21)\end{aligned}$$

Also

$$\sum \sqrt{a_2b_2c_2} = 2\sqrt{a_2b_2}\Big(\sqrt{a_2} + \sqrt{b_2}\Big). \qquad (22)$$

But

$$a_2b_2 = \frac{4R^2r^2}{R^2 - \overline{d}^2} = \frac{4R^2r^2}{r\big(\sqrt{4R^2 + r^2} - r\big)} = r\Big(\sqrt{4R^2 + r^2} + r\Big)$$

and $a_2 + b_2 = S_2 = \sqrt{4R^2 + r^2} + r$. Replacing in (22) we obtain

$$\begin{aligned}&\sum \sqrt{a_2b_2c_2} \\ &= 2\sqrt{r\Big(\sqrt{4R^2+r^2}+r\Big)}\sqrt{\sqrt{4R^2+r^2}+r+2\sqrt{r\Big(\sqrt{4R^2+r^2}+r\Big)}}. \quad \square\end{aligned}$$

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Some New Infinite Sums

Ankush Kumar and Toyesh Prakash

Abstract

Usually when calculating an indefinite integral by parts it is necessary to apply this algorithm several times until the final result is obtained. Using this idea it is possible to derive identities that allow us to calculate the sum of some numerical series.

Main result and applications

To compute some infinite sums we will use the following theorem.

Theorem 1. *Let $f(x)$ be a function whose anti derivative exists. Then,*

$$f(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{(n+1)!}\left(x\,\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}f(x) + (n+1)\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right).$$

Proof. By applying integration by parts, we have

$$\int f(x)\,\mathrm{d}x = xf(x) - \int xf'(x)\,\mathrm{d}x.$$

Integrating $\int xf'(x)\,\mathrm{d}x$ by parts again, we get

$$\begin{aligned}\int f(x)\,\mathrm{d}x &= xf(x) - \left(\frac{x^2}{2}f'(x) - \int \frac{x^2}{2}f''(x)\,\mathrm{d}x\right)\\ &= xf(x) - \frac{x^2}{2}f'(x) + \frac{1}{2}\int x^2 f''(x)\,\mathrm{d}x.\end{aligned}$$

Carrying out the same procedure with the last integral that appears in the preceding expression, we obtain

$$\begin{aligned}\int f(x)\,\mathrm{d}x &= xf(x) - \frac{1}{2}x^2 f'(x) + \frac{1}{3\cdot 2}x^3 f''(x) \\ &\quad - \frac{1}{4\cdot 3\cdot 2}x^4 f'''(x) + \frac{1}{5\cdot 4\cdot 3\cdot 2}x^5 f''''(x) + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)!}\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x).\end{aligned}$$

Differentiating both sides of the above expression yields

$$f(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{(n+1)!}\left(x\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}f(x) + (n+1)\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}\right).$$

This completes the proof. □

Next, we will compute in two ways the following sum:

$$\sum_{n=0}^{\infty} \frac{(-x)^n}{(n+1)!}(x+n+1),$$

where x is a nonzero real number. Using traditional techniques, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-x)^n}{(n+1)!}(x+n+1) &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)!} + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)!} + 1 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{n+1}}{(n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)!} + 1 - \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)!} \\ &= 1.\end{aligned}$$

On the other hand, applying Theorem 1 to the function $f(x) = e^x$, we get

$$e^x = \sum_{n=0}^{\infty} \frac{(-x)^n}{(n+1)!}(xe^x + (n+1)e^x),$$

from which we obtain

$$\sum_{n=0}^{\infty} \frac{(-x)^n}{(n+1)!}(x+n+1) = 1.$$

Applying Theorem 1 to the function $f(x) = xe^x$, we obtain

$$xe^x = \sum_{n=0}^{\infty} \frac{(-x)^n}{(n+1)!}(x((n+1)e^x + xe^x) + (n+1)(ne^x + xe^x)),$$

from which it follows that

$$\begin{aligned} x &= \sum_{n=0}^{\infty} \frac{(-x)^n}{(n+1)!}\big(x(n+1) + x^2 + n(n+1) + x(n+1)\big) \\ &= \sum_{n=0}^{\infty} \frac{(-x)^n}{(n+1)!}(x^2 + 2x(n+1) + n(n+1)). \end{aligned}$$

Likewise, for $f(x) = \sin x$ we get

$$\sin x = \sum_{n=0}^{\infty} \frac{(-x)^n}{(n+1)!}\Big(x\cos\Big(x + \frac{n\pi}{2}\Big) + (n+1)\sin\Big(x + \frac{n\pi}{2}\Big)\Big),$$

and for $f(x) = \cos x$ we get

$$\cos x = \sum_{n=0}^{\infty} \frac{(-x)^n}{(n+1)!}\Big(-x\sin\Big(x + \frac{n\pi}{2}\Big) + (n+1)\cos\Big(x + \frac{n\pi}{2}\Big)\Big).$$

Putting $x = 1, 2, 3, 4, 5$ in the preceding first identity, we obtain the following sums:

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}\big(n^2 + 3n + 3\big) &= 1, \\ \sum_{n=0}^{\infty} \frac{(-2)^n}{(n+1)!}\big(n^2 + 5n + 8\big) &= 2, \\ \sum_{n=0}^{\infty} \frac{(-3)^n}{(n+1)!}\big(n^2 + 7n + 15\big) &= 3, \\ \sum_{n=0}^{\infty} \frac{(-4)^n}{(n+1)!}\big(n^2 + 9n + 24\big) &= 4, \\ \sum_{n=0}^{\infty} \frac{(-5)^n}{(n+1)!}\big(n^2 + 11n + 35\big) &= 5, \end{aligned}$$

and for $x = \sqrt{2}$, we get

$$\sum_{n=0}^{\infty} \frac{\left(2n^2 + 2\left(\sqrt{2}+1\right)n + \left(1 + 2\sqrt{2}\right)\right)}{\left(-\sqrt{2}\right)^n (n+1)!} = \sqrt{2}.$$

Likewise, using the expressions for $\sin x$ and $\cos x$ new sums can be derived.

Acknowledgement

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Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.
2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

May 15, 2022

Elementary Problems

E–95. *Proposed by Michel Bataille, Rouen, France.* Let α, β, γ be the angles of a triangle. Prove that

$$\frac{\left(1+\tan\frac{\alpha}{4}\right)\left(1+\tan\frac{\beta}{4}\right)\left(1+\tan\frac{\gamma}{4}\right)}{1+\tan\frac{\alpha}{4}\tan\frac{\beta}{4}\tan\frac{\gamma}{4}} = 2.$$

E–96. *Proposed by José Luis Díaz-Barrero and Josep Gibergans Báguena, BarcelonaTech, Barcelona, Spain.* Let ABC be an equilateral triangle. On the side AB, and not overlapping $\triangle ABC$, build a square $ABEF$. Let P be the intersection point of AE and CF. Show that $PC = PE$.

E–97. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* How many ways are there to represent 100000000 as the product of three factors? Factorizations which differ in the order of the factors are considered to be distinct.

E–98. *Proposed by Ghiţescu Cosmina, Bucharest, Romania.* Find all positive integers such that

$$\lfloor\sqrt{2n+1}+\sqrt{2n+2}\rfloor + \lfloor\sqrt{8n+6}\rfloor = 2\sqrt{n+2015}.$$

E–99. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* All the positive integers from 19 to 101 are written consecutively to form the number $19202122\ldots99100101$. Find the largest power of 3 that divides it.

E–100. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a and b be real numbers such that $\pi/4 \leq a, b \leq \pi/2$. Prove that

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \leq \frac{9}{2}.$$

Easy–Medium Problems

EM–95. *Proposed by Michel Bataille, Rouen, France.* Let a, b be real numbers. Prove that $|\sin(a-b)|$ belongs to the closed interval with endpoints $|\sin a - \sin b|$ and $|\sin a + \sin b|$.

EM–96. *Proposed by Alexandru Benescu, Romania.* Let ABC be a triangle, H its orthocenter and X, Y, Z the circumscribed circles of $\triangle BHC$, $\triangle AHC$, $\triangle AHB$, respectively. Let DE be the common tangent to X and Y, EF to Y and Z and FD to X and Z, such that all 3 circles X, Y and Z are inside $\triangle DEF$. Prove that AD, BE and CF are concurrent.

EM–97. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Consider the polynomial $A(x) = x^5 + x^4 - 4x^3 + x^2 + x + 3$. Prove that for every positive integer n, the polynomial $A(x)^n$ has at least one negative coefficient.

EM–98. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Find all $x, y, z \in (0, 3)$ such that
$$\begin{cases} (x^2+y^2)\sqrt{9-z^2} \geq 9z, \\ (y^2+z^2)\sqrt{9-x^2} \geq 9x, \\ (z^2+x^2)\sqrt{9-y^2} \geq 9y. \end{cases}$$

EM–99. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Find all prime numbers $r < q < p$ such that $\sqrt{p^2 - 13q^2 - r^2}$ is rational and $\sqrt{p + 3r - q}$ is a positive integer.

EM–100. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Assume you have a dodecahedron, so that when it is thrown, each of the faces (which are numbered 1 to 12) occurs with equal probability. Determine the probability that after 24 throws of the dodecahedron, the product of all the numbers thrown will be divisible by 14.

Medium–Hard Problems

MH–95. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let a, b, c be nonzero real numbers such that $a^3 + b^3 + c^3 = 0$. Find the maximum value of

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\left(\frac{a}{c} + \frac{c}{b} + \frac{b}{a}\right)$$

and determine where the maximum holds.

MH–96. *Proposed by Michel Bataille, Rouen, France.* Let A' be the foot of the altitude from A in ΔABC and let D, E be the orthogonal projections of A' onto AB and AC. If E' is the reflection of E about A, show that the circumcircle of $\Delta ADE'$ is tangent to the median from A in ΔABC.

MH–97. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all positive real numbers for which

$$126x^7 - 127x^6 + 1 = 0.$$

MH–98. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a scalene triangle. Erect a square outwardly from each side of triangle ABC. Let A_1, B_1, C_1 be the respective centers of the squares. The lines AA_1, BB_1, CC_1 concur in V_1 – the Vecten point. We denote by C_2, A_2, B_2 the symmetric points of C_1, A_1, B_1 with respect to AB, BC, CA, respectively. The lines AA_2, BB_2, CC_2 concur in V_2 – the Inner Vecten point. Knowing that V_1, V_2, A are collinear, find angle $\angle BAC$.

MH–99. *Proposed by Ander Lamaison Vidarte, Brno, Czech Republic.* Let $n \geq 3$ be an integer. We have n balls, labelled from 1 to n, and n holes forming a circle. An arrangement is a way of placing one ball on each hole. We say that an arrangement is bushy if, for each ball b, both neighboring balls have labels smaller than b or both larger than b.

We make n moves as follows: on the k-th move, we locate the ball labelled k and swap the position of its neighbors. We say that an arrangement is jumpy if, for each k, the two balls swapped on the k-th move both have labels smaller than k, or both larger than k. Prove that there exist as many jumpy arrangements as bushy arrangements.

MH–100. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let x_1, x_2, x_3, x_4, x_5 be nonnegative real numbers. If $x_1 \leq 4$, $x_1 + x_2 \leq 13$, $x_1 + x_2 + x_3 \leq 29$, $x_1 + x_2 + x_3 + x_4 \leq 54$ and $x_1 + x_2 + x_3 + x_4 + x_5 \leq 90$, then prove that $\sqrt{x_1} + \sqrt{x_2} + \sqrt{x_3} + \sqrt{x_4} + \sqrt{x_5} \leq 20$.

Advanced Problems

A–95. *Proposed by Michel Bataille, Rouen, France.* Let n be a nonnegative integer. Evaluate in closed form

$$\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2n+2k}{n+k}.$$

A–96. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a scalene, non-right-angled triangle with altitudes AD, BE, CF and midpoints M_a, M_b, M_c of BC, CA, AB, respectively. Let J be the intersection point of EF and M_bM_c. Point K is the symmedian point of the triangle ABC. Knowing that J lies on the median AM_a, prove that the circle with center J and radius JK is tangent to the Euler line of $\triangle ABC$.

A–97. *Proposed by Nicolae Papacu, Slobozia, Romania.* Show that there exist three unique real numbers $a, b, c \in (0, \pi/2)$ such that

$$\begin{cases} a = \cot(a), \\ b = \tan(\cot(b)), \\ c = \cot(\tan(c)), \end{cases}$$

and $c < a < b$.

A–98. *Proposed by Toyesh Prakash Sharma, C. F. Andrews School, Agra, India.* Find the value of

$$\int_1^{e^3} \sqrt{\ln x}\,\mathrm{d}x + \int_{e^3}^{e^9} \frac{\mathrm{d}x}{\sqrt{\ln x}} + \int_0^{\sqrt{3}} e^{x^2}\,\mathrm{d}x + \int_{\sqrt{3}}^{3} e^{-x^2}\,\mathrm{d}x.$$

A–99. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $A(x)$ be a polynomial of degree n with integer coefficients, and let $a_0 < a_1 < \ldots < a_n$ be integer numbers. Prove that $|A(a_i)| \geq n!\,2^{-n}$ for some index $i \in \{0, 1, 2, \ldots, n\}$.

A–100. *Proposed by Vasile Mircea Popa, "Lucian Blaga" University of Sibiu, Romania.* Calculate the integral

$$\int_{-\infty}^{\infty} \frac{\operatorname{arc\,cot}(x)}{\sqrt{x^4+x^2+1}}\,\mathrm{d}x.$$

In this problem we will consider that definition of the function $\operatorname{arc\,cot}(x)$ whose image is the interval $(0,\pi)$.

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Classical inequalities and applications

José Luis Díaz-Barrero

1 Introduction

Inequalities, elementary or not, play an important role in many branches of mathematics and appear frequently in mathematical olympiads and contests. Many elementary inequalities have been extensively documented in the well-known works of Hardy, Littlewood, and Pólya [9] and Mitrinovic [10, 11]. Our goal in this Mathlesson is to list some classical inequalities and use them and properties of elementary functions to obtain new inequalities. Furthermore, we work out some examples completely and also give several applications.

2 Classical discrete inequalities

In the sequel we state a list of results that include, among others, the well-known *Standard Dozen Inequalities*. It is a collection of twelve famous inequalities that are used and applied to prove elementary inequalities such as the ones that appear in Mathematical Olympiads and other contests around the world. We begin with the following.

Theorem 1 (Means). *Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Then, $HM \leq GM \leq AM \leq QM$, where*

$$HM = \frac{n}{1/a_1 + 1/a_2 + \ldots + 1/a_n}, \quad GM = \sqrt[n]{a_1 a_2 \ldots a_n},$$
$$AM = \frac{a_1 + a_2 + \ldots + a_n}{n}, \qquad QM = \sqrt{\frac{a_1^2 + a_2^2 + \ldots + a_n^2}{n}}.$$

Equality holds if and only if $a_1 = a_2 = \ldots = a_n$.

Theorem 2 (Weighted Power Mean). *Let $a_1, a_2, \ldots, a_n$ be a set of positive real numbers and let $w_1, w_2, \ldots, w_n$ be nonnegative real numbers with a positive sum. Then,*

$$f(r) = \left(\frac{w_1 a_1^r + w_2 a_2^r + \ldots + w_n a_n^r}{w_1 + w_2 + \ldots + w_n} \right)^{1/r}$$

is a non-decreasing function of r, with the convention that, when $r = 0$, $f(0)$ is the weighted geometric mean. Namely,

$$f(0) = \left(\prod_{k=1}^{n} a_k^{w_k} \right)^{\frac{1}{w_1 + \ldots + w_n}}.$$

In particular, $f(1) \geq f(0) \geq f(-1)$ gives the well-known AM-GM-HM inequality.

Theorem 3 (Cauchy-Buniakowski-Schwarz). *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Then,*

$$\left(\sum_{k=1}^{n} a_k b_k \right)^2 \leq \left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} b_k^2 \right).$$

A generalization of Cauchy's inequality [1] is the following.

Theorem 4 (Hölder). *Let*

$$a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n; \ldots; z_1, z_2, \ldots, z_n$$

be nonnegative real numbers, and let $\lambda_a, \lambda_b, \ldots, \lambda_z$ be positive real numbers which sum is one. Then,

$$\sum_{k=1}^{n} a_k^{\lambda_a} b_k^{\lambda_b} \cdots z_k^{\lambda_z} \leq \left(\sum_{k=1}^{n} a_k \right)^{\lambda_a} \left(\sum_{k=1}^{n} b_k \right)^{\lambda_b} \cdots \left(\sum_{k=1}^{n} z_k \right)^{\lambda_z}.$$

Corollary 1. *Let $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ be real numbers and let p, q be positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then,*

$$\sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q \right)^{1/q},$$

with equality when $|b_k| = c|a_k|^{p-1}$, $1 \leq k \leq n$.

Theorem 5 (Minkowski). *Let $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ be real numbers and let $p > 1$. Then,*

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p \right)^{1/p} \leq \left(\sum_{k=1}^{n} |a_k|^p \right)^{1/p} + \left(\sum_{k=1}^{n} |b_k|^p \right)^{1/p}.$$

Equality holds when the sequences $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are proportional.

Theorem 6 (Rearrangement). *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be sequences of positive real numbers, and let $c_1, \ldots, c_n$ be a permutation of $b_1, \ldots, b_n$. The sum $S = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$ is maximal if the two sequences $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are sorted in the same way and minimal if the two sequences are sorted oppositely, one increasing and the other decreasing.*

Theorem 7 (Chebyshev). *If $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$ are two nondecreasing sequences of real numbers, then*

$$\sum_{k=1}^{n} a_k b_{n+1-k} \leq \frac{1}{n} \left(\sum_{k=1}^{n} a_k \right) \left(\sum_{k=1}^{n} b_k \right) \leq \sum_{k=1}^{n} a_k b_k.$$

A function $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ is convex on I if and only if $f(px + (1-p)y) \leq pf(x) + (1-p)f(y)$ for all $x, y \in I$ and $0 \leq p \leq 1$. If the inequality holds in the opposite direction, then f is said to be concave.

Theorem 8 (Jensen). *Suppose that $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function and suppose that the nonnegative real numbers p_k, $1 \leq k \leq n$, satisfy $p_1 + p_2 + \cdots + p_n = P_n$. Then, for all $a_k \in I$, $1 \leq k \leq n$, one has*

$$f\left(\frac{1}{P_n} \sum_{k=1}^{n} p_k a_k \right) \leq \frac{1}{P_n} \sum_{k=1}^{n} p_k f(a_k).$$

If f is concave, the preceding inequality is reversed.

Notice that when $n = 2$ and $P_n = 1$ Jensen's inequality is nothing more than the definition of convexity previously given.

Theorem 9 (Ky Fan). *Let $0 < a_k \leq 1/2$ for $k = 1, 2, \ldots, n$. Then,*

$$\left(\prod_{k=1}^{n} a_k\right) \Big/ \left(\sum_{k=1}^{n} a_k\right)^n \leq \left(\prod_{k=1}^{n} (1 - a_k)\right) \Big/ \left(\sum_{k=1}^{n} (1 - a_k)\right)^n.$$

Let $x_1 \geq x_2 \geq \ldots \geq x_n; y_1 \geq y_2 \geq \ldots \geq y_n$ be real numbers. If $x_1 + x_2 + \ldots + x_k \geq y_1 + y_2 + \ldots + y_k$ for all $k = 1, 2, \ldots, n-1$ and for $k = n$ we have

$$x_1 + x_2 + \ldots + x_n = y_1 + y_2 + \ldots + y_n,$$

then it is said that the sequence $\{x_k\}_{k=1}^n$ majorizes the sequence $\{y_k\}_{k=1}^n$. We write this in the form

$$(x_1; x_2; \ldots; x_n) \succ (y_1; y_2; \ldots; y_n).$$

Theorem 10 (Karamata or Majorization). *Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function on I and suppose that $(x_1; x_2; \ldots; x_n) \succ (y_1; y_2; \ldots; y_n)$, where $x_k, y_k \in I$. Then,*

$$f(x_1) + f(x_2) + \ldots + f(x_n) \geq f(y_1) + f(y_2) + \ldots + f(y_n).$$

Notice that when $m = (x_1 + x_2 + \ldots + x_n)/n$ is the arithmetic mean of the numbers $x_1, \ldots, x_n$, then $(x_1; x_2; \ldots; x_n) \succ (m; m; \ldots; m)$ and Karamata's inequality becomes Jensen's inequality.

Theorem 11 (Popoviciu). *Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be convex on I and let $a, b, c \in I$. Then, for any positive reals p, q, r,*

$$\begin{aligned} &f(a) + qf(b) + rf(c) + (p+q+r) f\left(\frac{ap + bq + cr}{p+q+r}\right) \\ &\geq (p+q) f\left(\frac{ap+bq}{p+q}\right) + (q+r) f\left(\frac{bq+cr}{q+r}\right) + (r+p) f\left(\frac{cr+ap}{r+p}\right). \end{aligned}$$

Corollary 2. *Let $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ be convex on I and let $a, b, c \in I$. Then,*

$$f(a) + f(b) + f(c) + 3f\left(\frac{a+b+c}{3}\right) \geq 2\left[f\left(\frac{a+b}{2}\right) + f\left(\frac{b+c}{2}\right) + f\left(\frac{c+a}{2}\right)\right].$$

Theorem 12 (Popoviciu Generalized). *Let $n \geq 3$ be a positive integer. Then,*

$$(n-2)\sum_{k=1}^{n} f(x_k) + nf\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) \geq 2\sum_{1\leq i<j\leq n} f\left(\frac{x_i + x_j}{2}\right),$$

where $f\colon I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function and $x_1, x_2, \ldots, x_n \in I$.

Let $x_1, \ldots, x_n$ be positive real numbers and $\mathrm{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n$. The p-mean of $x_1, \ldots, x_n$ is defined as

$$[\mathrm{p}] = \frac{1}{n!}\sum_{\sigma\in S_n} x_{\sigma(1)}^{p_1} x_{\sigma(2)}^{p_2} \cdots x_{\sigma(n)}^{p_n},$$

where S_n is the set of all permutations of $\{1, 2, \ldots, n\}$.

Theorem 13 (Muirhead). *Let $a_1, a_2, \ldots, a_n; b_1, b_2, \ldots, b_n$ be positive numbers. Suppose that the sequence $\mathrm{a} = (a_1, a_2, \ldots, a_n)$ majorizes the sequence $\mathrm{b} = (b_1, b_2, \ldots, b_n)$. Then, for any positive real numbers $x_1, x_2, \ldots, x_n$ we have $[\mathrm{a}] \geq [\mathrm{b}]$, or, equivalently,*

$$\sum_{\sigma\in S_n} x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} \cdots x_{\sigma(n)}^{a_n} \geq \sum_{\sigma\in S_n} x_{\sigma(1)}^{b_1} x_{\sigma(2)}^{b_2} \cdots x_{\sigma(n)}^{b_n}.$$

Since $(1, 0, \ldots, 0) \succ (1/n, 1/n, \ldots, 1/n)$, the AM-GM inequality is a consequence.

Theorem 14 (Shur). *Let a, b, c be nonnegative real numbers. For any $r > 0$, we have*

$$a^r(a-b)(a-c) + b^r(b-a)(b-c) + c^r(c-a)(c-b) \geq 0.$$

Theorem 15 (Bernoulli). *The following statements hold.*

1. *If $n \geq 1$ is an integer and $x \geq -1$ is a real number, then $(1+x)^n \geq 1+nx$.*
2. *If $\alpha > 1$ or $\alpha < 0$, then, for $x \geq -1$, we have $(1+x)^\alpha \geq 1+\alpha x$.*
3. *If $0 < \alpha < 1$, then, for $x \geq -1$, we have $(1+x)^\alpha \leq 1+\alpha x$.*

Theorem 16 (Newton-MacLaurin). *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers. Define the symmetric polynomials $s_0, s_1, \ldots, s_n$ by $(x+a_1)(x+a_2)\ldots(x+a_n) = s_0 + s_1 x + \ldots + s_n x^n$ and define the symmetric averages $d_k = s_k / \binom{n}{k}$. Then,*

$$(a)\quad d_k^2 \geq d_{k+1} d_{k-1} \quad \text{and} \quad (b)\quad d_1 \geq \sqrt{d_2} \geq \sqrt[3]{d_3} \geq \ldots \geq \sqrt[n]{d_n}.$$

Theorem 17 (Chrystal-Huygens). *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be positive real numbers. Then, for all $n \geq 2$,*

$$\sqrt[n]{(a_1+b_1)(a_2+b_2)\ldots(a_n+b_n)} \geq \sqrt[n]{a_1 a_2 \ldots a_n} + \sqrt[n]{b_1 b_2 \ldots b_n}.$$

Theorem 18 (Schweitzer, Pólya and Szegö). *Let $0 < a < A$, $0 < b < B$ and let $a_1, a_2, \ldots, a_n \in [a, A]$ and $b_1, b_2, \ldots, b_n \in [b, B]$. Then, for all $n \geq 2$,*

$$1 \leq \frac{(a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2)}{(a_1 b_1 + a_2 b_2 + \ldots + a_n b_n)^2} \leq \frac{1}{4}\left(\sqrt{\frac{AB}{ab}} + \sqrt{\frac{ab}{AB}}\right)^2.$$

Theorem 19 (Radon). *If $p > 0$, $x_k \geq 0$, $a_k > 0$, $1 \leq k \leq n$, then*

$$\sum_{k=1}^{n} \frac{x_k^{p+1}}{a_k^p} \geq \left(\sum_{k=1}^{n} x_k\right)^{p+1} \Big/ \left(\sum_{k=1}^{n} a_k\right)^p.$$

Surely, the particular case when $p = 1$ is the most well known, called in the literature Bergström's inequality.

Corollary 3 (Bergström). *If $x_k \in \mathbb{R}$, $a_k > 0$, $1 \leq k \leq n$, then*

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \ldots + \frac{x_n^2}{a_n} \geq \frac{(x_1 + x_2 + \ldots + x_n)^2}{a_1 + a_2 + \ldots + a_n}.$$

Theorem 20 (Euler). *Let r and R be the inradius and circumradius of a triangle ABC. Then, $R \geq 2r$ and equality holds when $\triangle ABC$ is equilateral.*

Theorem 21 (Erdős-Mordell). *Let M be an interior point to the triangle ABC and let A', B', C' the projections of M on the sides BC, CA and AB, respectively. Then,*

$$MA + MB + MC \geq 2(MA' + MB' + MC').$$

Equality holds when $\triangle ABC$ is equilateral and M is its center of gravity.

Theorem 22 (Ptolomey). *Let $ABCD$ be a convex quadrilateral. Then,*

$$AB \cdot CD + AD \cdot BC \geq AC \cdot BD.$$

Equality holds when $ABCD$ is a cyclic quadrilateral.

3 Applications

In what follows, using elementary techniques and the results listed in the previous section, some inequalities are obtained. We begin with an inequality that can be proved using mean inequalities.

Problem 1. *Let a, b, c be nonnegative real numbers. Prove that*

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} \geq \frac{\sqrt{ab}}{1+a+b} + \frac{\sqrt{bc}}{1+b+c} + \frac{\sqrt{ca}}{1+c+a}.$$

Solution. We claim that $\dfrac{a}{1+a} + \dfrac{b}{1+b} \geq \dfrac{a+b}{1+a+b}$. Indeed,

$$\frac{a}{1+a} + \frac{b}{1+b} - \frac{a+b}{1+a+b} = \frac{ab(2+a+b)}{(1+a)(1+b)(1+a+b)} \geq 0.$$

Likewise, $\dfrac{b}{1+b} + \dfrac{c}{1+c} \geq \dfrac{b+c}{1+b+c}$ and $\dfrac{c}{1+c} + \dfrac{a}{1+a} \geq \dfrac{c+a}{1+c+a}$. Adding up the preceding inequalities yields

$$2\left(\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c}\right) \geq \frac{a+b}{1+a+b} + \frac{b+c}{1+b+c} + \frac{c+a}{1+c+a}.$$

Applying the AM-GM inequality yields

$$\frac{a+b}{1+a+b}+\frac{b+c}{1+b+c}+\frac{c+a}{1+c+a}\geq\frac{2\sqrt{ab}}{1+a+b}+\frac{2\sqrt{bc}}{1+b+c}+\frac{2\sqrt{ca}}{1+c+a}.$$

Therefore,

$$2\left(\frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}\right)\geq\frac{2\sqrt{ab}}{1+a+b}+\frac{2\sqrt{bc}}{1+b+c}+\frac{2\sqrt{ca}}{1+c+a}$$

and the statement follows. Equality holds when $a=b=c=0$, and we are done. □

Now we will insert a term as a key idea to prove an inequality. This technique is used in the following problem.

Problem 2. *Let a,b,c be positive numbers. Prove that*

$$(a^ab^bc^c)^2\left(a^{-(b+c)}+b^{-(c+a)}+c^{-(a+b)}\right)^3\geq 27.$$

Solution. The stated inequality can be written as

$$\left(a^ab^bc^c\right)^2\geq\left(\frac{3}{a^{-(b+c)}+b^{-(c+a)}+c^{-(a+b)}}\right)^3.$$

To prove it, we will prove the following chain of inequalities:

$$\left(a^ab^bc^c\right)^2\geq a^{b+c}b^{c+a}c^{a+b}\geq\left(\frac{3}{a^{-(b+c)}+b^{-(c+a)}+c^{-(a+b)}}\right)^3.$$

The right one immediately follows applying the GM-HM inequality. To prove the left one, we suppose WLOG that $a\geq b\geq c$. Then, $\ln a\geq\ln b\geq\ln c$ and, by rearrangement, we have

$$a\ln a+b\ln b+c\ln c\geq b\ln a+c\ln b+a\ln c$$

and

$$a\ln a+b\ln b+c\ln c\geq c\ln a+a\ln b+b\ln c.$$

Adding up the preceding inequalities yields

$$2(a\ln a+b\ln b+c\ln c)\geq(b+c)\ln a+(c+a)\ln b+(a+b)\ln c$$

or, equivalently,

$$2\big(\ln a^a + \ln b^b + \ln c^c\big) \geq \ln a^{b+c} + \ln b^{c+a} + \ln c^{a+b},$$

from which the statement immediately follows. Equality holds when $a = b = c$. □

A change of variables combined with the CBS inequality is the key idea to prove the following inequality [4].

Problem 3. *Let x, y, z be positive real numbers. Then,*

$$\left(\frac{x}{y} + \frac{z}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{y}{z} + \frac{x}{\sqrt[3]{xyz}}\right)^2 + \left(\frac{z}{x} + \frac{y}{\sqrt[3]{xyz}}\right)^2 \geq 12.$$

Solution. Setting $a = \dfrac{x}{\sqrt[3]{xyz}}$, $b = \dfrac{y}{\sqrt[3]{xyz}}$ and $c = \dfrac{z}{\sqrt[3]{xyz}}$ yields

$$\left(a + \frac{b}{c}\right)^2 + \left(b + \frac{c}{a}\right)^2 + \left(c + \frac{a}{b}\right)^2 \geq 12.$$

To prove the preceding inequality we set $\vec{u} = \left(a + \dfrac{b}{c}, b + \dfrac{c}{a}, c + \dfrac{a}{b}\right)$ and $\vec{v} = (1, 1, 1)$ into the CBS inequality, and we obtain

$$\begin{aligned}
&\left(a + \frac{b}{c}\right)^2 + \left(b + \frac{c}{a}\right)^2 + \left(c + \frac{a}{b}\right)^2 \\
&= \frac{1}{3}(1^2 + 1^2 + 1^2)\left[\left(a + \frac{b}{c}\right)^2 + \left(b + \frac{c}{a}\right)^2 + \left(c + \frac{a}{b}\right)^2\right] \\
&\geq \frac{1}{3}\left[\left(a + \frac{b}{c}\right) + \left(b + \frac{c}{a}\right) + \left(c + \frac{a}{b}\right)\right]^2. \qquad (1)
\end{aligned}$$

Now, taking into account that $abc = 1$ and applying the AM-GM inequality twice, we get

$$\begin{aligned}
\left(a + \frac{b}{c}\right) + \left(b + \frac{c}{a}\right) + \left(c + \frac{a}{b}\right) &\geq 3\sqrt[3]{\left(a + \frac{b}{c}\right)\left(b + \frac{c}{a}\right)\left(c + \frac{a}{b}\right)} \\
&\geq 3\sqrt[3]{2^3\sqrt{\frac{ab}{c}\,\frac{bc}{a}\,\frac{ca}{b}}} = 6.
\end{aligned}$$

Therefore, the RHS of (1) becomes

$$\frac{1}{3}\left[\left(a+\frac{b}{c}\right)+\left(b+\frac{c}{a}\right)+\left(c+\frac{a}{b}\right)\right]^2 \geq 12.$$

Equality holds when $x = y = z$. □

The following result is a source of many interesting inequalities.

Lemma. *Let $\alpha \in \mathbb{C}$ and $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \cdots, b_n$ be sequences of complex numbers. Then,*

$$\mathrm{Re}\left(\overline{\alpha}\sum_{k=0}^{n} a_k b_k\right) \leq \frac{1}{2}\left(\sum_{k=0}^{n}|a_k|^2 + |\alpha|^2\sum_{k=0}^{n}|b_k|^2\right). \tag{2}$$

Proof. Inequality (2) immediately follows from the identity

$$\begin{aligned}\sum_{k=0}^{n}|a_k-\alpha\overline{b}_k|^2 &= \sum_{k=0}^{n}(a_k-\alpha\overline{b}_k)(\overline{a}_k-\overline{\alpha}b_k)\\ &= \sum_{k=0}^{n}|a_k|^2+|\alpha|^2\sum_{k=0}^{n}|b_k|^2-2\mathrm{Re}\left(\overline{\alpha}\sum_{k=0}^{n}a_k b_k\right)\\ &\geq 0,\end{aligned}$$

valid for any complex number $\alpha \in \mathbb{C}$. □

From the preceding we get the following result [1, 10].

Corollary (Complex CBS inequality). *Let $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ be sequences of complex numbers. Then,*

$$\left|\sum_{k=0}^{n} a_k b_k\right|^2 \leq \sum_{k=0}^{n}|a_k|^2\sum_{k=0}^{n}|b_k|^2,$$

with equality if and only if there is a complex number $\alpha \in \mathbb{C}$ such that $a_k = \alpha\overline{b}_k$ for any $k \in \{0, 1, \ldots, n\}$.

Proof. Setting $\alpha = \left(\sum_{k=0}^{n} a_k b_k\right)\Big/\left(\sum_{k=0}^{n}|b_k|^2\right)$, $b_k \neq 0$ $(1 \leq k \leq n)$ into (2), we obtain

$$\sum_{k=0}^{n}|a_k-\alpha\overline{b}_k|^2 = \sum_{k=0}^{n}|a_k|^2 - \left|\sum_{k=0}^{n}a_k b_k\right|^2\Big/\left(\sum_{k=0}^{n}|b_k|^2\right) \geq 0,$$

which proves the statement. Finally, from the last expression it follows that equality holds if and only if $a_k = \alpha\bar{b}_k$ for $0 \le k \le n$, and the proof is complete. □

Problem 4. *Let $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \cdots, b_n$ be sequences of complex numbers. Prove that*

$$\mathrm{Re}\left(\sum_{k=0}^{n} a_k b_k\right) \le \frac{1}{n}\sum_{k=0}^{n} |a_k|^2 + \frac{2n+1}{6}\sum_{k=0}^{n} |b_k|^2.$$

Solution. Setting $\alpha = k$, $0 \le k \le n$, into (1) yields

$$k\mathrm{Re}\left(\sum_{k=0}^{n} a_k b_k\right) \le \frac{1}{2}\left(\sum_{k=0}^{n} |a_k|^2 + k^2\sum_{k=0}^{n} |b_k|^2\right). \tag{3}$$

Adding up the inequalities in (3) and taking into account the well known identities

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6},$$

we have

$$\begin{aligned}&\frac{n(n+1)}{2}\mathrm{Re}\left(\sum_{k=0}^{n} a_k b_k\right)\\ &\le \frac{1}{2}\left((n+1)\sum_{k=0}^{n} |a_k|^2 + \frac{n(n+1)(2n+1)}{6}\sum_{k=0}^{n} |b_k|^2\right).\end{aligned}$$

After simplifying terms in the preceding inequality the statement follows. □

A simple observation about nonnegative functions jointly with mean inequalities can be used to get new proposals [3]. An example is presented in the following problem.

Problem 5. *Let n be a positive integer. Prove that*

$$\left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{2^k C(n,k)}\right)^n > e^{(n+1)-3^n}.$$

Solution. The graph of $f(x) = e^x$ and the tangent line $y = 1 + x$ show that we have for all $x \in \mathbb{R}$ the well-known inequality $1 + x \leq e^x$. Setting $x = \dfrac{1}{n}\sum_{k=1}^{n} 2^k C(n,k) - 1$ into it, taking into account that $\sum_{k=1}^{n} 2^k C(n,k) = 3^n - 1$ and the HM-AM inequality, yields

$$\exp\left(\sum_{k=1}^{n} 2^k C(n,k) - n\right) \geq \left(\frac{1}{n}\sum_{k=1}^{n} 2^k C(n,k)\right)^n \geq \left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{2^k C(n,k)}\right)^{-n}.$$

That is,

$$\left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{2^k C(n,k)}\right)^n \exp\left(\sum_{k=1}^{n} 2^k C(n,k) - n\right) = \left(\frac{1}{n}\sum_{k=1}^{n} \frac{1}{2^k C(n,k)}\right)^n e^{3^n-(n+1)} \geq 1. \qquad \square$$

Now, we give an inequality involving sequences of nonnegative integers [6].

Problem 6. *Let F_n be the n-th Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$ and, for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$, and let L_n be the n-th Lucas number, defined by $L_0 = 2$, $L_1 = 1$ and, for all $n \geq 2$, $L_n = L_{n-1} + L_{n-2}$. Prove that, for all $n \geq 1$,*

$$\frac{1}{2}\left(F_n^{1/F_n} + L_n^{1/L_n}\right) \leq 2 - \frac{F_{n+1}}{F_{2n}}.$$

Solution. Let $f\colon (0,1] \to \mathbb{R}$ be the function defined by $f(x) = x\ln x + \ln(2-x)$. Then,

$$f'(x) = \ln x + \frac{1-x}{2-x} \quad \text{and} \quad f''(x) = \frac{(x-1)(x-4)}{x(2-x)^2} \geq 0.$$

Thus, f' is increasing and $f'(x) < f'(1) = 0$. Since $f' < 0$, then f is decreasing and $f(x) \geq f(1) = 0$. That is, $x\ln x + \ln(2-x) \geq 0$

and $\ln x^x \geq \ln(1/(2-x))$, from which it immediately follows that

$$x + x^{-x} \leq 2.$$

Equality holds when $x = 1$. Now, setting $x = 1/F_n$ and $x = 1/L_n$ into the preceding inequality yields

$$\frac{1}{F_n} + \left(\frac{1}{F_n}\right)^{-1/F_n} \leq 2 \quad \text{and} \quad \frac{1}{L_n} + \left(\frac{1}{L_n}\right)^{-1/L_n} \leq 2.$$

Adding up the preceding expressions and taking into account that $F_n + L_n = 2F_{n+1}$ and $F_n L_n = F_{2n}$, as can be easily proved by induction, we get

$$\frac{F_n + L_n}{F_n L_n} + \left(\frac{1}{F_n}\right)^{-1/F_n} + \left(\frac{1}{L_n}\right)^{-1/L_n} \leq 4$$

or, equivalently,

$$\frac{F_{n+1}}{F_{2n}} + \frac{1}{2}\left[\left(\frac{1}{F_n}\right)^{-1/F_n} + \left(\frac{1}{L_n}\right)^{-1/L_n}\right] \leq 2,$$

from which the statement immediately follows. Equality holds when $n = 1$, and we are done. □

Convex functions are very useful when dealing with inequalities. Next, we give an example [8].

Problem 7. *Let a, b, c be three positive numbers such that $a + b + c = 1$. Prove that*

$$(a^a + b^a + c^a)(a^b + b^b + c^b)(a^c + b^c + c^c) \geq \left(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c}\right)^3.$$

Solution. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = \ln(a^x + b^x + c^x)$. We have $f'(x) = \dfrac{a^x \ln a + b^x \ln b + c^x \ln c}{a^x + b^x + c^x}$ and

$$f''(x) = \frac{a^x \ln^2 a + b^x \ln^2 b + c^x \ln^2 c}{a^x + b^x + c^x} - \frac{(a^x \ln a + b^x \ln b + c^x \ln c)^2}{(a^x + b^x + c^x)^2} \geq 0.$$

The last inequality can be easily obtained applying the Cauchy-Buniakowski-Schwarz inequality to the vectors $\vec{u} = (\sqrt{a^x}, \sqrt{b^x}, \sqrt{c^x})$

and $\vec{v} = (\sqrt{a^x}\ln a, \sqrt{b^x}\ln b, \sqrt{c^x}\ln c)$. Therefore, f is convex and, applying Jensen's inequality, we get

$$f\left(\frac{a+b+c}{3}\right) \le \frac{1}{3}(f(a)+f(b)+f(c))$$

or

$$\begin{aligned}&\ln\left(a^{\frac{a+b+c}{3}} + b^{\frac{a+b+c}{3}} + c^{\frac{a+b+c}{3}}\right)\\ &\le \frac{1}{3}\left[\ln(a^a+b^a+c^a) + \ln\left(a^b+b^b+c^b\right) + \ln(a^c+b^c+c^c)\right].\end{aligned}$$

Now taking into account the properties of the logarithm function and the constrain, yields

$$(a^a+b^a+c^a)(a^b+b^b+c^b)(a^c+b^c+c^c) \ge \left(a^{\frac{a+b+c}{3}}+b^{\frac{a+b+c}{3}}+c^{\frac{a+b+c}{3}}\right)^3,$$

from which the statement follows. Equality holds when $a = b = c = 1/3$. □

There are a lot of efficient strategies for proving inequalities. Nevertheless, the use of probability techniques as a method for proving inequalities has seldom appeared in the literature. Next, we present an example connecting probability and inequalities.

Problem 8. *Let m, n be positive integers and let $x_{ij} \in [0,1]$ for all $i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$. Prove that*

$$\prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m} x_{ij}\right) + \prod_{i=1}^{m}\left(1 - \prod_{j=1}^{n}(1-x_{ij})\right) \ge 1.$$

Solution. Since $0 \le x_{ij} \le 1$, we can assume that $x_{ij} = p[A_{ij}]$ where A_{ij} are independent events in some probability space $(E, \mathcal{P}(E), p)$. It is well known that

$$\bigcup_{j=1}^{n}\left(\bigcap_{i=1}^{m} A_{ij}\right) \subseteq \bigcap_{i=1}^{m}\left(\bigcup_{j=1}^{n} A_{ij}\right).$$

Taking into account Morgan's Law yields

$$\bigcup_{j=1}^{n}\left(\bigcap_{i=1}^{m} A_{ij}\right) = \overline{\bigcap_{j=1}^{n}\overline{\left(\bigcap_{i=1}^{m} A_{ij}\right)}} \subseteq \bigcap_{i=1}^{m}\overline{\left(\bigcap_{j=1}^{n}\overline{A}_{ij}\right)} = \bigcap_{i=1}^{m}\left(\bigcup_{j=1}^{n} A_{ij}\right).$$

Thus,

$$p\left[\overline{\bigcap_{j=1}^{n}\overline{\left(\bigcap_{i=1}^{m}A_{ij}\right)}}\right] \leq p\left[\bigcap_{i=1}^{m}\overline{\left(\bigcap_{j=1}^{n}\overline{A}_{ij}\right)}\right]$$

or, equivalently,

$$1-\prod_{j=1}^{n}\left(1-\prod_{i=1}^{m}x_{ij}\right) \leq \prod_{i=1}^{m}\left(1-\prod_{j=1}^{n}(1-x_{ij})\right). \qquad \square$$

Problem 9. *Let a, b, c be three real numbers greater or equal than one. Prove that it is always possible to construct a triangle whose sides have lengths $a^{1/a}, b^{1/b}, c^{1/c}$.*

Solution. We have $[a] \leq a < [a]+1$. Applying Bernoulli's inequality yields

$$2^a \geq 2^{[a]} = (1+1)^{[a]} \geq 1+[a] > a \geq 1,$$

from which we get $2 > a^{1/a} \geq 1$. Likewise, $2 > b^{1/b} \geq 1$ and $2 > c^{1/c} \geq 1$. Now, we have

$$\begin{aligned} a^{1/a}+b^{1/b} &\geq 1+1 = 2 > c^{1/c}, \\ b^{1/b}+c^{1/c} &\geq 1+1 = 2 > a^{1/a}, \\ c^{1/c}+a^{1/a} &\geq 1+1 = 2 > b^{1/b}. \end{aligned}$$

Therefore, with segments of lengths $a^{1/a}, b^{1/b}, c^{1/c}$ it is always possible to build up a triangle. $\square$

Problem 10. *Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be real numbers. Show that $r^2 \leq 1$, where $r = \dfrac{S_{xy}}{S_x S_y}$.*

(Here r^2 is the determination coefficient, which is the square of the correlation coefficient r).

Solution. The covariance S_{xy} of the given data is defined by

$$S_{xy} = \frac{1}{n}\sum_{k=1}^{n}(x_k-\overline{x})(y_k-\overline{y})$$

and the standard deviations S_x and S_y are given by

$$S_x = \left(\frac{1}{n}\sum_{k=1}^{n}(x_k - \overline{x})^2\right)^{1/2} \quad \text{and} \quad S_y = \left(\frac{1}{n}\sum_{k=1}^{n}(y_k - \overline{y})^2\right)^{1/2},$$

respectively. Taking into account the CBS inequality, we have

$$r^2 = \frac{S_{xy}^2}{S_x^2 S_y^2} = \frac{\left(\sum_{k=1}^{n}(x_k - \overline{x})(y_k - \overline{y})\right)^2}{\left(\sum_{k=1}^{n}(x_k - \overline{x})^2\right)\left(\sum_{k=1}^{n}(y_k - \overline{y})^2\right)} \leq 1. \qquad \square$$

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Contests

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to find interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Problems and solutions from the 8th edition of the BarcelonaTech Mathcontest

O. Rivero Salgado and J. L. Díaz-Barrero

1 Problems and solutions

Hereafter, we present the four problems that appeared in the paper given to the contestants of the BarcelonaTech Mathcontest 2021, as well as their official solutions.

Problem 1. The numbers $\{1, 2, \ldots, 80\}$ are written in the blackboard. Gaudí plays the following *solitaire* game. Whenever there are at least two numbers written in the blackboard, he chooses two among them, a and b, erases them, and writes instead either the number $8a + b$ or the number $a - 6b$. He wins if in the moment when only one number is written in the blackboard, this is 2021. Does he have a winning strategy?

Solution. Consider the numbers modulo 7. Then, in each step, two numbers a and b are chosen and replaced either by $8a + b \equiv a + b$ modulo 7 or by $a - 6b \equiv a + b$ modulo 7. That is, working modulo 7, the operation consists on replacing two numbers a and b by $a + b$. Then, the resulting number modulo 7 once the game is finished will be $1 + 2 + \ldots + 80 \equiv 6$ modulo 7, and 2021 is congruent with 5. Therefore, he cannot win the game.

Problem 2. Let $ABCD$ be a parallelogram. A variable line drawn through A cuts BC and CD at F and G, respectively, and cuts the parallel to BD through C at E. Prove that

$$\frac{1}{AF}+\frac{1}{AG}=\frac{2}{AE}.$$

Solution. Let $\{O\}=AC\cap BD$ and $\{P\}=AE\cap BD$. Since OP is the mid parallel to $\triangle ACE$ then $AE=2AP$.

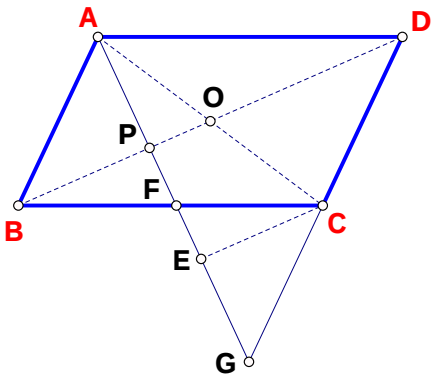

Figure 1: Scheme for solving problem 2.

We have the following triangle similarities: $\triangle APD\sim\triangle FPB$, $\triangle ABF\sim\triangle GCF$, $\triangle GCF\sim\triangle GCA$, from which we obtain

$$\begin{aligned}
&\frac{AP}{AD}=\frac{FP}{FB}\iff\frac{1}{AP}=\frac{FB}{FP\cdot AD}=\frac{FB}{FP\cdot BC},\\
&\frac{AF}{BF}=\frac{GF}{CF}\iff\frac{1}{AF}=\frac{CF}{BF\cdot GF},\\
&\frac{AG}{AD}=\frac{GF}{CF}\iff\frac{1}{AG}=\frac{CF}{AD\cdot GF}=\frac{CF}{BC\cdot GF}.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{1}{AF}+\frac{1}{AG}&=\frac{CF}{FG}\left(\frac{1}{BF}+\frac{1}{BC}\right)\\
&=\frac{CF}{FG}\cdot\frac{BF+BC}{BF\cdot BC}\cdot\frac{1}{AP}\cdot\frac{PF\cdot BC}{BF}\\
&=\frac{1}{AP}\cdot\frac{CF}{FG}\cdot\frac{BF+BC}{BF\cdot BC}\cdot\frac{PF}{BF}.
\end{aligned}$$

Since $\triangle DPA \sim \triangle BPF$, then we have

$$\frac{FB}{PF} = \frac{AD}{AP} = \frac{BC}{AP} \iff FB(AP + PF) = PF(BC + FB),$$

from which

$$\frac{PF}{AP + PF} = \frac{FB}{BC + FB} \iff \frac{BC + FB}{BF} = \frac{AF}{PF}$$

follows. Since $\triangle CFG \sim \triangle BFA$, then we have

$$\frac{CF}{FG} = \frac{BF}{AF} \iff \frac{CF}{FG} \cdot \frac{AF}{BF} = 1.$$

Finally, we have

$$\frac{1}{AF} + \frac{1}{AG} = \frac{1}{AP} \cdot \frac{CF}{FG} \cdot \frac{AF}{PF} \cdot \frac{PF}{BF} = \frac{1}{AP} = \frac{2}{AE},$$

and we are done.

Problem 3. Let a, b, n be integers such that $0 < a \leq b < n$. Prove that there exists a prime p that divides both $\binom{n}{a}$ and $\binom{n}{b}$.

Solution. We argue by contradiction. Suppose that $\binom{n}{a}$ and $\binom{n}{b}$ are relatively prime. We need the following identity:

$$\binom{n}{b}\binom{b}{a} = \binom{n}{a}\binom{n-a}{b-a},$$

valid for $0 \leq a \leq b \leq n$. Indeed, suppose we have a class with n students. Then we may choose $\binom{n}{b}$ committees of b students and for each b students we have $\binom{b}{a}$ subcommittees with a girls. Thus, the total number of students may be counted (committed) in two ways:

$$\binom{n}{b}\binom{b}{a} = \binom{n}{a}\binom{n-a}{b-a}.$$

From the preceding, we get that

$$\binom{n}{a} \,\Big|\, \binom{n}{b}\binom{b}{a}.$$

But since $\binom{n}{a}$ and $\binom{n}{b}$ are coprime, it follows that $\binom{n}{a}$ divides $\binom{b}{a}$ which is impossible because it is clear that $\binom{b}{a} < \binom{n}{a}$. Thus,

$$GCD\left(\binom{n}{a}, \binom{n}{b}\right) > 1,$$

and we are done.

Problem 4. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2 - bc}{2a^2 + ab + ac} + \frac{b^2 - ca}{2b^2 + bc + ba} + \frac{c^2 - ab}{2c^2 + ca + cb} \leq 0.$$

Solution. First, we write the inequality claimed as

$$\sum_{\text{cyclic}} \frac{a}{2a + b + c} - \sum_{\text{cyclic}} \frac{bc}{a(2a + b + c)} \leq 0.$$

Applying Cauchy's inequality to the vectors $\vec{u} = \left(\sqrt{a+b}, \sqrt{a+c}\right)$ and $\vec{v} = \left(\frac{1}{\sqrt{a+b}}, \frac{1}{\sqrt{a+c}}\right)$ yields $4 \leq \left(\frac{1}{a+b} + \frac{1}{a+c}\right)(2a+b+c)$, and we obtain

$$\frac{a}{2a + b + c} \leq \frac{1}{4}\left(\frac{a}{a+b} + \frac{a}{a+c}\right).$$

Likewise, we get

$$\frac{b}{a + 2b + c} \leq \frac{1}{4}\left(\frac{b}{b+c} + \frac{b}{b+a}\right)$$

and

$$\frac{c}{a + b + 2c} \leq \frac{1}{4}\left(\frac{c}{a+c} + \frac{c}{b+c}\right).$$

Adding up the preceding inequalities yields

$$\sum_{\text{cyclic}} \frac{a}{2a + b + c} \leq \frac{1}{4}(1 + 1 + 1) = \frac{3}{4}.$$

Applying again Cauchy's inequality to the vectors

$$\vec{u} = \left(\sqrt{\frac{bc}{a(2a + b + c)}}, \sqrt{\frac{ca}{b(a + 2b + c)}}, \sqrt{\frac{ab}{c(a + b + 2c)}}\right)$$

and

$$\vec{v} = \left(\sqrt{bca(2a+b+c)}, \sqrt{cab(a+2b+c)}, \sqrt{abc(a+b+2c)}\right)$$

yields

$$\begin{aligned}&(bc+ca+ab)^2\\ &\leq \left(\frac{bc}{a(2a+b+c)} + \frac{ca}{b(a+2b+c)} + \frac{ab}{c(a+b+2c)}\right) 4abc(a+b+c).\end{aligned}$$

Then,

$$\sum_{\text{cyclic}} \frac{bc}{a(2a+b+c)} \geq \frac{(ab+bc+ca)^2}{4abc(a+b+c)} \geq \frac{3}{4}.$$

Indeed,

$$\frac{(ab+bc+ca)^2}{4abc(a+b+c)} \geq \frac{3}{4} \Leftrightarrow a^2b^2+b^2c^2+c^2a^2 - abc(a+b+c) \geq 0,$$

which immediately follows from the well-known inequality $x^2+y^2+z^2 \geq xy+yz+zx$. Finally, we have

$$\sum_{\text{cyclic}} \frac{a}{2a+b+c} - \sum_{\text{cyclic}} \frac{bc}{a(2a+b+c)} \leq \frac{3}{4} - \frac{3}{4} = 0.$$

Equality holds when $a=b=c$, and we are done.

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Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

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Elementary Problems

E–89. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In an urn we have balls of 20 different colors. Balls are drawn at random from the urn and placed on a table. What is the minimum number of balls that must be drawn from the urn to ensure that, among them, there are at least 102 that share the same color?

Solution 1 by Michel Bataille, Rouen, France. Answer: 2021.

Suppose that we have drawn 101 balls of each of the 20 colors. Then we do not have 102 balls of the same color. It follows that the desired number of draws must be greater than 101×20. However, drawing one more ball ensure that 102 balls are of the same color. Thus the desired number is $101 \times 20 + 1 = 2021$.

Solution 2 by the proposer. Note that if we take 20 balls from the urn it could be possible that all of them were of different colors.

So, we can ensure that there are two of the same color if we take 21 balls from de urn (on account of the pigeonhole principle). Likewise, we need $41 = 20 \times 2 + 1$ balls to ensure that there are at least 3 of the same color, because with 40 balls it may be that each color appears exactly two times. Continuing this procedure we arrive the result and we need to draw $20 \times 101 + 1 = 2021$ balls from the urn to ensure that among them there are 102 of the same color.

Also solved by *Alberto Espuny Díaz, Technische Universität Ilmenau, Ilmenau, Germany.*

E–90. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Find all functions $f\colon \mathbb{N}^* \to \mathbb{N}^*$ that for all $x, y \in \mathbb{N}^*$ satisfy

$$\frac{x(f(x)+y)}{yf(x)+1} + \frac{y(f(y)+x)}{xf(y)+1} = \frac{[f(x+y)]^2}{xy+1}.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. Let $x = y = 1$. Then,

$$\frac{f(1)+1}{f(1)+1} + \frac{f(1)+1}{f(1)+1} = \frac{[f(2)]^2}{2}, \quad \text{or} \quad f(2) = 2.$$

Now, suppose $f(1) = a$ for some $a \in \mathbb{N}^*$, and let $x = 1$ and $y = 2$. Then,

$$\frac{[f(3)]^2}{3} = \frac{a+2}{2a+1} + \frac{6}{3}, \quad \text{or} \quad [f(3)]^2 = 7 + \frac{a+5}{2a+1}.$$

For $\frac{a+5}{2a+1}$ to be an integer requires $a + 5 \geq 2a + 1$, or $a \leq 4$.

- $a = 0$: $\frac{a+5}{2a+1} = 12$, but $[f(3)]^2 = 12$ does not have a solution in $\mathbb{N}^*$.
- $a = 1$: $\frac{a+5}{2a+1} = 2$, so $f(3) = 3$.
- $a = 2$: $\frac{a+5}{2a+1}$ is not an integer.
- $a = 3$: $\frac{a+5}{2a+1}$ is not an integer.
- $a = 4$: $\frac{a+5}{2a+1} = 1$, but $[f(3)]^2 = 8$ does not have a solution in $\mathbb{N}^*$.

Thus, $f(1) = 1$. Finally, take $y = 1$. Then,

$$\frac{x(f(x)+1)}{f(x)+1} + \frac{1+x}{1+x} = x+1 = \frac{[f(x+1)]^2}{1+x}, \quad \text{or} \quad f(x+1) = x+1.$$

The only function $f\colon \mathbb{N}^* \to \mathbb{N}^*$ that for all $x, y \in \mathbb{N}^*$ satisfies

$$\frac{x(f(x)+y)}{yf(x)+1} + \frac{y(f(y)+x)}{xf(y)+1} = \frac{[f(x+y)]^2}{xy+1}$$

is $f(x) = x$.

Solution 2 by Michel Bataille, Rouen, France. Let $E(x, y)$ denote the given relation and let f be a solution.

First, it is readily checked that $E(1, 1)$ and $E(2, 2)$ give $f(2) = 2$ and $f(4) = 4$, respectively.

Second, if $x \in \mathbb{N}^*$, $E(x, 1)$ and a little algebra lead to

$$[f(x+1)]^2 = (x+1)^2 - \frac{(x^2-1)(f(1)-1)}{xf(x)+1}. \tag{1}$$

It follows that the assumption $f(1) > 1$ implies that $f(x+1) < x+1$ for any integer $x > 1$. With $x = 3$, we would obtain $f(4) < 4$, a contradiction. Therefore we must have $f(1) = 1$. The relation (1) then yields $f(x+1) = x+1$ for any positive integer x and we can conclude that $f(n) = n$ for all positive integers n.

Conversely, the identity function $n \mapsto n$ is a solution (readily checked).

We conclude that the identity function of $\mathbb{N}^*$ is the unique solution.

Solution 3 by Víctor Martín Chabrera, Barcelona, Spain. Let us restrict ourselves to the case $y = 1$. We have the following:

$$x + \frac{f(1)+x}{xf(1)+1} = \frac{[f(x+1)]^2}{x+1}.$$

Multiplying both sides by $x+1$,

$$\begin{aligned}
[f(x+1)]^2 &= x(x+1) + \frac{(f(1)+x)(x+1)}{xf(1)+1} \\
&= x(x+1) + (x+1) - (x+1) + \frac{(f(1)+x)(x+1)}{xf(1)+1} \\
&= (x+1)^2 + \frac{(f(1)+x)(x+1) - (x+1)(xf(1)+1)}{xf(1)+1} \\
&= (x+1)^2 + \frac{(f(1)+x-xf(1)-1)(x+1)}{xf(1)+1} \\
&= (x+1)^2 - \frac{(f(1)-1)(x-1)(x+1)}{xf(1)+1} \\
&= (x+1)^2 - \frac{(f(1)-1)(x^2-1)}{xf(1)+1}.
\end{aligned}$$

Plugging $x=2$ into the equation, we get

$$[f(3)]^2 = 9 - \frac{3(f(1)-1)}{2f(1)+1}.$$

Note that, since $f(1) \geq 1$, $0 \leq \frac{3(f(1)-1)}{2f(1)+1} \leq \frac{3f(1)}{2f(1)} = \frac{3}{2}$.

So, since the fraction must be an integer number, this means that it is either 0 or 1. However, we can discard the latter case, since that would imply $[f(3)]^2 = 8$, which is not possible since 8 is not a perfect square. So, we are only left with the case that the fraction is equal to 0, which implies $f(1) = 1$.

Plugging this into $[f(x+1)]^2 = (x+1)^2 - \frac{(f(1)-1)(x^2-1)}{xf(1)+1}$, we immediately obtain that $f(x) = x$. It is trivial to check that this function satisfies the functional equation of the statement.

Also solved by *the proposer.*

E–91. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Each 1×1 square of a 7×211 rectangle is painted

either black or white. Prove that it is possible to choose four rows and four columns of the rectangle so that the sixteen 1×1 squares into which they intersect are painted with the same color.

Solution 1 by the proposer. Since we have used only two colors, the second version of the pigeonhole principle implies that, in each column (of seven 1×1 squares), one of the two colors must occur at least $\lceil \frac{7}{2} \rceil = 4$ times. Call such a color **dominant** and, for each of the 211 columns, take note of its dominant color. Since one out of the two colors dominates in each of the 211 columns, by invoking once more the second version of the pigeonhole principle we conclude that one of the two colors is the dominant one in at least $\lceil \frac{211}{2} \rceil = 106$ columns. Assume, without loss of generality, that such a color is black, and call a column as **black** if black is its dominant color. Then, choose 106 black columns and, in each of them, choose four 1×1 black squares.

Notice that there are exactly $\binom{7}{4} = 35$ possible ways of choosing four of the seven 1×1 squares of each column. Therefore, out of the 106 black columns (and once again from the second version of the pigeonhole principle), the four 1×1 chosen black squares occupy exactly the same positions in at least $\lceil \frac{106}{35} \rceil = 4$ columns. In other words, we have at least four columns having 1×1 black squares in the same set of four rows, and this is exactly what we were asked to prove.

Solution 2 by Víctor Martín Chabrera, Barcelona, Spain. Consider the 211 columns of the rectangle. Let us call a column white if it has more white squares than black squares, and black otherwise. By the pigeonhole principle, either at least 106 columns are white or at least the same amount are black.

Assume WLOG at least 106 are white. Each of these columns has at least 4 white squares. Let us choose exactly 4 of those white squares and paint them in green. Note that if two squares are green, they were originally white, so it is enough to see that we can find such a 4×4 "subsquare" painted in green.

Note that we could have only painted the green squares in one of the $\binom{7}{4} = 35$ possible combinations. As $106 > 3 \cdot 35$, this

means that, by the pigeonhole principle, there are 4 columns out of the at least 106 white ones that are painted with the same green combination. The four green squares of each of these four columns give the desired 4×4 "subsquare".

E–92. *Proposed by Michel Bataille, Rouen, France.* Let m, n be positive integers such that $n \geq m$. Prove that $m+1$ divides $(mn-1) \cdot \binom{n}{m}$.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. If $n = m$, then

$$(mn-1) \cdot \binom{n}{m} = (m^2-1) \cdot \binom{m}{m} = m^2 - 1 = (m+1)(m-1),$$

which is clearly divisible by $m+1$. Now, suppose $n = m+k$ for some positive integer k. Then,

$$\begin{aligned} mn - 1 = m(m+k) - 1 &= m(m+1) + mk - m - 1 \\ &= (m+1)(m+k-1) - k, \end{aligned}$$

and

$$(mn-1) \cdot \binom{n}{m} = (m+1)(m+k-1) \cdot \binom{m+k}{m} - k \cdot \binom{m+k}{m}.$$

Because the binomial coefficient $\binom{m+k}{m}$ is an integer, the term

$$(m+1)(m+k-1) \cdot \binom{m+k}{m}$$

is divisible by $m+1$. Additionally,

$$\begin{aligned} k \cdot \binom{m+k}{m} &= k\frac{(m+k)!}{m!k!} = \frac{(m+k)!}{m!(k-1)!} \\ &= (m+1)\frac{(m+k)!}{(m+1)!(k-1)!} = (m+1) \cdot \binom{m+k}{m+1}, \end{aligned}$$

so the term

$$k \cdot \binom{m+k}{m}$$

is also divisible by $m+1$. Thus, $m+1$ divides $(mn-1) \cdot \binom{n}{m}$ for all positive integers n, m such that $n \geq m$.

Solution 2 by the proposer. We have

$$\begin{aligned}(mn-1)\cdot\binom{n}{m} &= [n(m+1)-(n+1)]\cdot\binom{n}{m}\\ &= n(m+1)\cdot\binom{n}{m}-(n+1)\cdot\binom{n}{m}\\ &= n(m+1)\cdot\binom{n}{m}-(m+1)\cdot\binom{n+1}{m+1}\\ &= (m+1)\left(n\cdot\binom{n}{m}-\binom{n+1}{m+1}\right),\end{aligned}$$

and the result follows since $n\cdot\binom{n}{m}-\binom{n+1}{m+1}$ is an integer.

Also solved by *Víctor Martín Chabrera, Barcelona, Spain.*

E–93. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $A(x_1,y_1)$, $B(x_2,y_2)$, $C(x_3,y_3)$ be the vertices of a triangle where a, b, c are the lengths of its sides. Prove that its incenter lies at the point

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\ \frac{ay_1+by_2+cy_3}{a+b+c}\right).$$

Solution 1 by Michel Bataille, Rouen, France. Let $[XYZ]$ denote the area of triangle XYZ. If M is a point interior to the triangle ABC, we know that

$$[MBC]\overrightarrow{MA}+[MCA]\overrightarrow{MB}+[MAB]\overrightarrow{MC}=\overrightarrow{0}.$$

Taking $M=I$, the incenter of ΔABC, we have $[IBC]=\dfrac{ra}{2}$ where r is the inradius, and similarly, $[ICA]=\dfrac{rb}{2}$, $[IAB]=\dfrac{rc}{2}$. It follows that

$$a\overrightarrow{IA}+b\overrightarrow{IB}+c\overrightarrow{IC}=\overrightarrow{0}.$$

If O is the origin, we obtain

$$a(\overrightarrow{OA}-\overrightarrow{OI})+b(\overrightarrow{OB}-\overrightarrow{OI})+c(\overrightarrow{OC}-\overrightarrow{OI})=\overrightarrow{0},$$

that is,

$$a\overrightarrow{OA} + b\overrightarrow{OB} + c\overrightarrow{OC} = (a+b+c)\overrightarrow{OI}.$$

This means that the coordinates of I are

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right).$$

Solution 2 by the proposer. The equation of side AB is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = x(y_1 - y_2) - y(x_1 - x_2) + x_1y_2 - x_2y_1 = 0.$$

Likewise, the equations of sides BC and CA are $x(y_2-y_3)-y(x_2-x_3)+x_2y_3-x_3y_2 = 0$ and $x(y_3-y_1)-y(x_3-x_1)+x_3y_1-x_1y_3 = 0$, respectively.

The equation of bisector ℓ_a is

$$\frac{|x(y_1 - y_2) - y(x_1 - x_2) + x_1y_2 - x_2y_1|}{c} = \frac{|x(y_3 - y_1) - y(x_3 - x_1) + x_3y_1 - x_1y_3|}{b}$$

and the equation of bisector ℓ_b is

$$\frac{|x(y_1 - y_2) - y(x_1 - x_2) + x_1y_2 - x_2y_1|}{c} = \frac{|x(y_2 - y_3) - y(x_2 - x_3) + x_2y_3 - x_3y_2|}{a}.$$

Solving the preceding system yields

$$I = (x, y) = \left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c}\right).$$

Solution 3 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. The incenter of the triangle ABC lies at the intersection of the triangle's angle bisectors. Let A' denote the point at which the bisection of angle A intersects the side BC. By the angle bisector theorem,

$$A' = B + \frac{c}{b+c}(C - B) = \frac{b}{b+c}B + \frac{c}{b+c}C.$$

It then follows that the line segment from A to A' is given parametrically by

$$(1-t)A + t\left(\frac{b}{b+c}B + \frac{c}{b+c}C\right).$$

Similarly, let B' denote the point at which the bisection of angle B intersects the side AC. Then,

$$B' = C + \frac{a}{a+c}(A-C) = \frac{a}{a+c}A + \frac{c}{a+c}C,$$

and the line segment from B to B' is given parametrically by

$$(1-s)B + s\left(\frac{a}{a+c}A + \frac{c}{a+c}C\right).$$

The segments AA' and BB' intersect for $t = \frac{b+c}{a+b+c}$ and $s = \frac{a+c}{a+b+c}$ at the point

$$\frac{a}{a+b+c}A + \frac{b}{a+b+c}B + \frac{c}{a+b+c}C.$$

Thus, the incenter of the triangle ABC lies at the point

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right).$$

Also solved by *Víctor Martín Chabrera, Barcelona, Spain, and Daniel Văcaru, Pitești, Romania.*

E–94. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Find all nonnegative integers n and p such that $n^2+9n+1 = 11^p$.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. With $n = 0$, $n^2+9n+1 = 1 = 11^0$, and with $n = 1$, $n^2+9n+1 = 11 = 11^1$, so the ordered pairs $(n,p) = (0,0)$ and $(1,1)$ satisfy $n^2+9n+1 = 11^p$. To see that these are the only ordered pairs of nonnegative integers such that $n^2+9n+1 = 11^p$, note $n^2+9n+1 > 11$ for $n > 1$; therefore, only positive integer powers of 11 need be considered. Now,

- $n = 1$: $n^2 + 9n + 1 = 11 \equiv 0 \pmod{11}$;
- $n = 2$: $n^2 + 9n + 1 = 23 \equiv 1 \pmod{11}$;
- $n = 3$: $n^2 + 9n + 1 = 37 \equiv 4 \pmod{11}$;
- $n = 4$: $n^2 + 9n + 1 = 53 \equiv 9 \pmod{11}$;
- $n = 5$: $n^2 + 9n + 1 = 71 \equiv 5 \pmod{11}$;
- $n = 6$: $n^2 + 9n + 1 = 91 \equiv 3 \pmod{11}$;
- $n = 7$: $n^2 + 9n + 1 = 113 \equiv 3 \pmod{11}$;
- $n = 8$: $n^2 + 9n + 1 = 137 \equiv 5 \pmod{11}$;
- $n = 9$: $n^2 + 9n + 1 = 163 \equiv 9 \pmod{11}$;
- $n = 10$: $n^2 + 9n + 1 = 191 \equiv 4 \pmod{11}$;
- $n = 11$: $n^2 + 9n + 1 = 221 \equiv 1 \pmod{11}$,

and

$$\begin{aligned}(n+11)^2 + 9(n+11) + 1 &= n^2 + 9n + 1 + 22n + 220 \\ &= n^2 + 9n + 1 + 11(2n + 20) \\ &\equiv n^2 + 9n + 1 \pmod{11}.\end{aligned}$$

This implies $n^2 + 9n + 1$ is divisible by 11 (and can therefore possibly equal a positive integer power of 11) if and only if n is of the form $1 + 11m$ for some nonnegative integer m. Next,

$$(1+11m)^2+9(1+11m)+1=11+121m+121m^2=11(1+11m+11m^2).$$

If $m = 0$, we recover the solution $(n, p) = (1, 1)$ noted earlier. However, for any $m \geq 1$,

$$1 + 11m + 11m^2 \equiv 1 \pmod{11},$$

which means that $1 + 11m + 11m^2$ can never be a positive integer power of 11. Accordingly, no number of the form $n = 1 + 11m$ for any integer $m \geq 1$ yields $n^2 + 9n + 1$ that is equal to a positive integer power of 11.

In summary, the only nonnegative integer solutions of $n^2+9n+1 = 11^p$ are $(n, p) = (0, 0)$ and $(n, p) = (1, 1)$.

Solution 2 by Michel Bataille, Rouen, France. The pairs $(0, 0)$ and $(1, 1)$ are obvious solutions for (n, p). We show that there are no other solutions.

Let (n, p) be a solution. If $p = 0$ then $n(n + 9) = 0$, hence $n = 0$ and $(n, p) = (0, 0)$.
Now, suppose that $p \geq 1$. Then we have

$$(n - 1)^2 = 11^p - 11n, \tag{1}$$

so that 11 divides $(n - 1)^2$ (since $p \geq 1$). Since 11 is a prime, 11 also divides $n - 1$ and we have $n = 11k + 1$ for some nonnegative integer k. From (1), we obtain $11^{p-1} - 11k(k + 1) = 1$. If $p > 1$, 11 divides the left side but not the right side, which cannot occur. Therefore we must have $p = 1$ and so $11k(k + 1) = 0$. Thus $k = 0$, $n = 11k + 1 = 1$ and $(n, p) = (1, 1)$. This completes the proof.

Solution 3 by Víctor Martín Chabrera, Barcelona, Spain. It is trivial to see that the only solutions (n, p) with $p \leq 1$ are $(0, 0)$ and $(1, 1)$.

Now, let us assume (n, p) is a solution, with $p \geq 2$. Observe that $n^2 - 9n + 1 = (n - 1)^2 + 11n$. Reducing modulo 11 we get $(n - 1)^2 \equiv 0 \pmod{11}$. As 11 is a prime, this means $(n - 1)$ is a multiple of 11, which also means that $(n - 1)^2$ is a multiple of 121. So, reducing modulo 121, we get

$$11n \equiv (n - 1)^2 + 11n \equiv 11^p \equiv 0 \pmod{121}.$$

This means that both n and $n - 1$ are multiples of 11, which is absurd, and we conclude that the only solutions are $(0, 0)$ and $(1, 1)$.

Solution 4 by Todor Zaharinov, Sofia, Bulgaria. It is clear that $n = 0$, $p = 0$ is a solution.

Denote $P(n) = n^2 + 9n + 1$.

Let $p \geq 1$. Then $11 \mid P(n)$ or $P(n) \equiv 0 \pmod{11}$.

Directly calculate

n	0	1	2	3	4	5	6	7	8	9	10
$P(n) \pmod{11}$	1	0	1	4	9	5	3	3	5	9	4

$$P(11m + 1) \equiv P(1) \equiv 0 \pmod{11}.$$

Hence $11 \mid P(n)$ if and only if $n = 11m+1$ for nonnegative integer m.

$$\begin{aligned} P(n) &= n^2 + 9n + 1 \\ &= (11m+1)^2 + 9(11m+1) + 1 \\ &= 121m^2 + 121m + 11 \\ &= 11(11m(m+1)+1) \\ &= 11^p. \end{aligned}$$

$11 \nmid (11m(m+1)+1)$, hence $(11m(m+1)+1) = 1$ and either $m = 0, n = 1, p = 1$ or $m = -1, n = -10, p = 1$, but $n < 0$.

In conclusion, the solutions for nonnegative integers are $n = 0$, $p = 0$ and $n = 1$, $p = 1$.

Also solved by *the proposer.*

Easy–Medium Problems

EM–89. *Proposed by Michel Bataille, Rouen, France.* Let a, b, c be positive real numbers such that $abc \geq 1$. Prove that

$$2(a^2 + b^2 + c^2) \geq a + b + c + 3.$$

Solution 1 by Daniel Văcaru, Pitești, Romania. We have $a+b+c \geq 3\sqrt[3]{abc} \Rightarrow a + b + c \geq 3$. We have $a^2 + b^2 + c^2 \geq 3\left(\sqrt[3]{abc}\right)^2 \Rightarrow a^2+b^2+c^2 \geq 3$. Furthermore, we have $a^2+b^2+c^2 \geq \dfrac{(a+b+c)^2}{3} = \dfrac{(a+b+c)(a+b+c)}{3} \geq a + b + c$. Then,

$$2\left(a^2 + b^2 + c^2\right) = \left(a^2 + b^2 + c^2\right) + \left(a^2 + b^2 + c^2\right) \geq a + b + c + 3.$$

Solution 2 by the proposer. Since $a^2 + b^2 + c^2 \geq ab + bc + ca$, it is sufficient to show that

$$a^2 + b^2 + c^2 + ab + bc + ca \geq a + b + c + 3,$$

or even that

$$\left(a^2 + \frac{1}{a} - a - 1\right) + \left(b^2 + \frac{1}{b} - b - 1\right) + \left(c^2 + \frac{1}{c} - c - 1\right) \geq 0 \quad (1)$$

(since $ab \geq \frac{1}{c}, bc \geq \frac{1}{a}, ca \geq \frac{1}{b}$). Now, for positive x we have

$$x^2 + \frac{1}{x} - x - 1 = \frac{x^3 + 1 - x^2 - x}{x} = \frac{(x-1)^2(x+1)}{x} \geq 0,$$

hence (1) holds, the left-hand side being the sum of three nonnegative numbers.

Also solved by *Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA; Víctor Martín Chabrera, Barcelona, Spain, and Sarah B. Seales, Peoria, AZ, USA.*

EM–90. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $\{a_n\}_{n\geq 0}$ be the sequence defined by $a_n = 18^n + 20^n$. Find the remainder of a_{722} when divided by 722.

Solution 1 by Michel Bataille, Rouen, France. We show that the requested remainder is 2. We will apply the following lemma:

Let k be a positive integer and a, b be integers such that $b \equiv a \pmod{p^k}$, where p is a prime. Then, $b^p \equiv a^p \pmod{p^{k+1}}$.

Proof. From $b = a + cp^k$ for some integer c and the binomial theorem, we deduce that

$$b^p = (a + cp^k)^p = a^p + pa^{p-1}(cp^k) + dp^{k+1}$$

for some integer d. Thus $b^p \equiv a^p \pmod{p^{k+1}}$. □

Now, observing that $18^2 \equiv 1 \equiv 20^2 \pmod{19}$ and applying the lemma twice, we obtain

$$18^{2\times 19} \equiv 1 \pmod{19^2}, \quad 18^{2\times 19^2} \equiv 1 \pmod{19^3}$$

and similarly, $20^{2\times 19^2} \equiv 1 \pmod{19^3}$. It follows that

$$a_{722} = 18^{2\times 19^2} + 20^{1\times 19^2} \equiv 2 \pmod{19^3},$$

which means that $a_{722} = 2 + m\cdot 19^3$ for some integer m. Clearly, m is positive and since a_{722} is even, m must be even as well. Setting $m = 2\ell$, we obtain $a_{722} = 2 + (19\ell)\cdot 722$ and the result follows.

Solution 2 by the proposer. We have

$$\begin{aligned} a_n &= 18^n + 20^n = (19-1)^n + (19+1)^n \\ &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k 19^{n-k} + \sum_{k=0}^{n} \binom{n}{k} 19^{n-k} \\ &= 2\left(19^n + \binom{n}{2} 19^{n-2} + \ldots + \binom{n}{n-2} 19^2 + 1\right), \end{aligned}$$

when n is even. Thus,

$$\begin{aligned} a_{722} &= 2\left(19^{722} + \binom{722}{2}19^{720} + \ldots + \binom{722}{720}19^2 + 1\right) \\ &= 2 + 2\cdot 19^2\left(19^{720} + \binom{722}{2}19^{718} + \ldots + \binom{722}{720}\right) \\ &= 2 + 722\left(19^{722} + \binom{n}{2}19^{720} + \ldots + \binom{n}{n-2}19^2 + 1\right) \\ &\equiv 2 \pmod{722}. \end{aligned}$$

So, the remainder is 2, and we are done.

Solution 3 by Víctor Martín Chabrera, Barcelona, Spain. Consider the polynomial $P(x) = (19+x)^{722} = \sum_{k=0}^{722}\binom{722}{k}19^k x^{722-k}$. The polynomial $Q(x) = \frac{P(x)+P(-x)}{2}$ will be $P(x)$ after removing all its monomials of odd degree. By construction of Q, we have

$$Q(1) = \sum_{\substack{0\le k\le 722\\ k \text{ even}}}\binom{722}{k}19^k = \sum_{j=0}^{361}\binom{722}{2j}19^{2j} = \sum_{j=0}^{361}\binom{722}{2j}361^j.$$

Let $m = 18^{722} + 20^{722}$. Taking into account that $m = 2Q(1)$,

$$m = 2 + 2\sum_{j=1}^{361}\binom{722}{2j}361^j = 2 + 722\sum_{j=1}^{361}\binom{722}{2j}361^{j-1},$$

from where we immediately see that $m \equiv 2 \pmod{722}$.

Also solved by *Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA, and Todor Zaharinov, Sofia, Bulgaria.*

EM–91. *Proposed by Ghiţescu Cosmina, student 8^{th} grade, Bucureşti, Romania.* Let ABC be a triangle with the sides $a > b > c$. If R is the circumradius of $\triangle ABC$, r_a, r_b, r_c are the rays of the excircles, p is the semiperimeter and $a^2 + b^2 + c^2 = 8R^2$, then prove that

$$\frac{3\sqrt{3}}{\sqrt{2}+1}(\sin A + \sin B + \sin C) < \frac{p}{r_a + r_b + r_c - 4R}.$$

Solution 1 by Michel Bataille, Rouen, France. We shall use some known relations between the elements of the triangle:

$$r_a + r_b + r_c = r + 4R, \quad a^2 + b^2 + c^2 = 2p^2 - 2r^2 - 8rR,$$

$$\sin A + \sin B + \sin C = \frac{p}{R}, \quad \cos A \cos B \cos C = \frac{p^2 - (r+2R)^2}{4R^2},$$

where r denotes the inradius.

The inequality to be proved can be written as $\frac{3\sqrt{3}}{\sqrt{2}+1} \cdot \frac{p}{R} < \frac{p}{r}$, that is,

$$\frac{R}{r} > \frac{3\sqrt{3}}{\sqrt{2}+1}. \tag{1}$$

From the hypothesis $a^2+b^2+c^2 = 8R^2$, we deduce that $2p^2-2r^2-8rR = 8R^2$, hence $p^2 = (r+2R)^2$ so that $\cos A \cos B \cos C = 0$. As a result, the triangle is right-angled, and a, as its largest side, is its hypotenuse. But in such a triangle, we have $r = p - a = \frac{b+c-a}{2}$ (if I is the incenter and E and F are its projections on AC and AB, then $AEIF$ is a square with side $r = AE = \frac{b+c-a}{2}$).
Now, since $(b+c)^2 \le 2(b^2+c^2)$, we obtain that

$$r \le \frac{\sqrt{2(b^2+c^2)} - a}{2} = \frac{\sqrt{2a^2} - a}{2} = \frac{a(\sqrt{2}-1)}{2} = R(\sqrt{2}-1).$$

It follows that $\frac{R}{r} \ge \sqrt{2}+1$ and (1) holds because $\sqrt{2}+1 > \frac{3\sqrt{3}}{\sqrt{2}+1}$.

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. We start with two known necessary and sufficient conditions for a triangle with semiperimeter p and inradius r to be right-angled:

$$a^2 + b^2 + c^2 = 8R^2, \qquad p = 2R + r.$$

Since we are told that $a > b > c$, the given triangle ABC is right-angled with hypothenuse $a(= 2R)$.

We will use the following well-known relations, which are valid for every triangle:

$$\sin A + \sin B + \sin C = \frac{p}{R}, \qquad r_a + r_b + r_c = 4R + r,$$

from which it follows that

$$\frac{p}{(\sin A+\sin B+\sin C)(r_a+r_b+r_c-4R)} = \frac{p}{\frac{p}{R}\cdot r} = \frac{R}{r} = \frac{R}{p-2R} = \frac{\frac{a}{2}}{p-a} = \frac{a}{-a+b+c}. \tag{2}$$

But

$$(b+c)^2 \le 2\left(b^2+c^2\right) = \text{(by the Pythagorean theorem)} = 2a^2$$

and

$$b+c \le a\sqrt{2}, \tag{3}$$

with equality only when the triangle is isosceles right-angled.

Since we are assuming that b and c are not equal, using (2) and (3), we obtain

$$\frac{p}{(\sin A+\sin B+\sin C)(r_a+r_b+r_c-4R)} > \frac{1}{-1+\sqrt{2}}$$

or, equivalently,

$$\left(1+\sqrt{2}\right)(\sin A+\sin B+\sin C) < \frac{p}{r_a+r_b+r_c-4R},$$

which is a refinement of the proposed inequality.

Also solved by *Marian Ursărescu "Roman-Vodă" National College, Roman, Romania; Daniel Văcaru, Pitești, Romania, and the proposer.*

EM–92. *Proposed by Óscar Rivero Salgado, BarcelonaTech, Barcelona, Spain.* Find all the solutions in the set of positive integers of the equation

$$\frac{x^4+y^3}{x^2+y} = x+y.$$

Solution 1 by Michel Bataille, Rouen, France. As it is readily checked, the pairs $(1,2)$ and $(2,2)$ are solutions for (x,y). We show that there are no other solutions.

Let (x,y) be a solution. We first show that if $x \geq 2$, then $y \leq x$. Indeed, if $x \geq 2$, then

$$x^3 + y^3 < x^4 + y^3 = (x+y)(x^2+y),$$

hence $x^2 - xy + y^2 < x^2 + y$, from which we deduce that $y - x < 1$, that is, $y \leq x$.

Second, we show that we must have $x \leq 2$. To this end, we remark that the equation gives

$$xy(x+1) - x^3(x-1) = y^3 - y^2 \geq 0$$

and, therefore, $y(x+1) \geq x^2(x-1)$. Assuming $x \geq 3$, we would have $y \leq x$, hence $x^2(x-1) \leq x(x+1)$ or $(x-1)^2 \leq 2$, contradicting $x \geq 3$.

Now, if $x = 1$, we must have $y^2 - y - 2 = 0$, hence $y = 2$ and $(x,y) = (1,2)$. If $x = 2$, we obtain $y^3 - y^2 - 6y + 8 = 0$. Since $y = 1$ or $y = 2$ (because $y \leq x$), we must have $y = 2$ so that $(x,y) = (2,2)$. This completes the proof.

Solution 2 by the proposer. The proposed equation can be rewritten as

$$x^4 + y^3 = x^3 + x^2y + xy + y^2.$$

Suppose that $x, y > 2$. In that case, we will prove that we have the following inequalities,

$$\frac{x^4}{2} > x^3, \quad \frac{y^3}{2} > y^2, \quad \frac{x^4 + y^3}{2} \geq x^2 + xy,$$

which, when added together, make it impossible to achieve the initial equality. The first two inequalities are immediate. For the latter, by virtue of the inequality between the arithmetic and geometric means, we have

$$\frac{x^4 + y^3}{2} \geq x^2 y^{3/2} = \frac{x^2 y^{3/2}}{2} + \frac{x^2 y^{3/2}}{2} \geq x^2 + xy.$$

In the last step, we used that $\frac{y^{3/2}}{2} > \frac{y}{2} > 1$ y $\frac{xy^{1/2}}{2} > \frac{x}{2} > 1$.

We can only study the cases in which one of the variables is at most 2.

- Case $x = 1$. In this case we get
$$y^3 + 1 = (y+1)^2 = y^2 + 2y + 1,$$
or alternatively $y(y^2 - y - 2) = y(y-2)(y+1)$, whose only solution in the positive integers is $y = 2$.
- Case $x = 2$. In this case we get
$$y^3 + 16 = (y+4)(y+2) = y^2 + 6y + 8,$$
or alternatively $y^3 - y^2 - 6y + 8 = 0$. Any solution in the positive integers would be a divisor of 8, and it is immediately proven that the only possible solution is $y = 2$.
- Case $y = 1$. We obtain
$$x^4 + 1 = (x^2+1)(x+1) = x^3 + x^2 + x + 1,$$
or $x(x^3 - x^2 - x - 1) = 0$, that has no solutions in the set of positive integers.
- Case $y = 2$. Now, we have
$$x^4 + 8 = (x^2+2)(x+2) = x^3 + 2x^2 + 2x + 4,$$
or alternatively $x^4 - x^3 - 2x^2 - 2x + 4 = 0$. Any solution in the positive integers will divide 4. It is verified that $x = 1$ and $x = 2$ work (they were already obtained before).

Solution 3 by Víctor Martín Chabrera, Barcelona, Spain. We can see case by case that the only solutions (x, y) with $x, y \leq 2$ are $(1, 2)$ and $(2, 2)$. Let us show these are the only ones:

- If $y > x$ and $y > 2$ let us rewrite the equation as $\frac{x^4+y^3}{x+y} = x^2 + y$. We have the following:
$$\frac{x^4 + y^3}{x + y} \geq \frac{x^3 + y^3}{x + y} = x^2 - xy + y^2 = x^2 + y(y - x) \geq x^2 + y.$$
So, there will only be a solution when the two inequalities are equalities, which implies $x = 1$ and $y = x + 1 = 2$.

- If $x \geq y$ and $x > 2$, let us rewrite the expression as $x^4 + y^3 = (x+y)(x^2+y)$. We can bound the left hand side by $x^4+y^3 > x^4$, and the right hand side by $(x+y)(x^2+y) \leq 2x(x^2+x) < 4x^3$. Note that, for $x = 3$, we have $81 = x^4 > 2x(x^2+x) = 72$, and for $x \geq 4$, $x^4 \geq 4x^3$, making the equality impossible.

EM–93. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let ABC be an isosceles triangle with $AB = BC$. Let M be an interior point such that $\angle AMC = 2 \cdot \angle ABC$. Let K be a point on AM such that $\angle BKM = \angle ABC$. Prove that $4\,KM \cdot MC \leq BK^2$.

Solution by the proposer. Let P be the intersection point of CM and BK. On account of $\angle AMC = 2 \cdot \angle ABC$ and $\angle BKM = \angle ABC$, we have

$$\angle KPM = \angle AMC - \angle BKM = \angle ABC,$$

from which it follows that $PM = MK$.

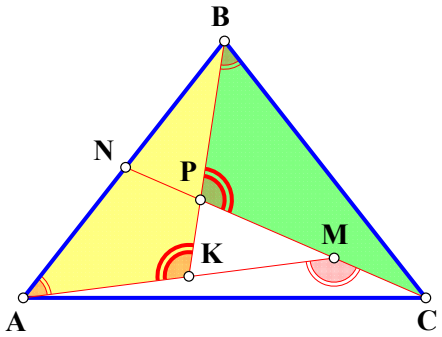

Figure 1: Scheme for solving problem EM–93.

On the other hand, triangles AKB and BPC are congruent because

$$\angle KAB = \angle BKM - \angle ABK = \angle ABC - \angle ABK = \angle PBC,$$

$$\angle AKB = 180^\circ - \angle KPM = \angle BPC,$$

and furthermore $AB = BC$. From the above, we have

$$BK = PC = PM + MC = MK + MC,$$

and

$$\sqrt{KM \cdot MC} \leq \frac{KM + MC}{2} = \frac{BK}{2},$$

from which the statement follows.

Also solved by *Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

EM–94. *Proposed by Toma–Ioan Dumitrescu, Bucharest, Romania.* Let a, b, c be positive reals such that $a + b + c = 5$. Prove that

$$\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} + \frac{a^2 + b^2 + c^2}{a + b + c} \geq 10.$$

Solution 1 by Michel Bataille, Rouen, France. It is well-known that for any real numbers x, y, z the inequality $xy + yz + zx \leq x^2 + y^2 + z^2$ holds. It follows that

$$25 = (a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) \leq 3(a^2 + b^2 + c^2)$$

and, therefore,

$$\frac{a^2 + b^2 + c^2}{a + b + c} = \frac{a^2 + b^2 + c^2}{5} \geq \frac{25/3}{5} = \frac{5}{3}. \tag{1}$$

Besides, the Cauchy-Schwarz inequality yields

$$\left(\frac{a^4}{ab} + \frac{b^4}{bc} + \frac{c^4}{ca}\right)(ab + bc + ca) \geq (a^2 + b^2 + c^2)^2$$

and, remarking that $\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} = \frac{a^4}{ab} + \frac{b^4}{bc} + \frac{c^4}{ca}$, we deduce that

$$\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a} \geq \frac{(a^2 + b^2 + c^2)^2}{ab + bc + ca} \geq a^2 + b^2 + c^2 \geq \frac{25}{3} \tag{2}$$

(using $ab + bc + ca \leq a^2 + b^2 + c^2$ again).

Now, by addition, (1) and (2) give

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+\frac{a^2+b^2+c^2}{a+b+c}\geq\frac{25}{3}+\frac{5}{3}=10.$$

Equality holds when $a=b=c=5/3$.

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA. By the Cauchy-Schwarz inequality,

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}=\frac{a^4}{ab}+\frac{b^4}{bc}+\frac{c^4}{ca}\geq\frac{(a^2+b^2+c^2)^2}{ab+bc+ca}.$$

Next,

$$a^2+b^2+c^2\geq ab+bc+ca$$

because it is equivalent to

$$\frac{1}{2}(a-b)^2+\frac{1}{2}(b-c)^2+\frac{1}{2}(c-a)^2\geq 0,$$

and, by the arithmetic mean–quadratic mean inequality,

$$a^2+b^2+c^2\geq\frac{(a+b+c)^2}{3}.$$

Thus,

$$\begin{aligned}\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+\frac{a^2+b^2+c^2}{a+b+c}&\geq(a^2+b^2+c^2)\left(1+\frac{1}{a+b+c}\right)\\&\geq\frac{1}{3}\big((a+b+c)^2+a+b+c\big)\\&=\frac{1}{3}(25+5)=10.\end{aligned}$$

Also solved by *Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA, and the proposer.*

Medium–Hard Problems

MH–89. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let F_n be the n-th Fibonacci number defined by $F_0 = 0$, $F_1 = 1$ and, for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Prove that

$$\sum_{k=1}^{n} \frac{F_k^3}{F_{k+1}} \geq \frac{1}{F_n F_{n+1}} \left(\sum_{k=1}^{n} \sqrt{\frac{F_k^5}{F_{k+1}}} \right)^2.$$

(Here, the subscripts are taken modulo n).

Solution 1 by the proposer. Consider the discrete distribution of probability (X, f_X) defined in the following table.

X	$\sqrt{F_1/F_2}$	$\sqrt{F_2/F_3}$	$\dots$	$\sqrt{F_n/F_1}$
f_X	$\dfrac{F_1^2}{F_n F_{n+1}}$	$\dfrac{F_2^2}{F_n F_{n+1}}$	$\dots$	$\dfrac{F_n^2}{F_n F_{n+1}}$

Then,

$$E(X) = \sum_{k=1}^{n} \sqrt{\frac{F_k}{F_{k+1}}} \cdot \frac{F_k^2}{F_n F_{n+1}} = \frac{1}{F_n F_{n+1}} \sum_{k=1}^{n} \sqrt{\frac{F_k^5}{F_{k+1}}}.$$

Likewise,

$$E(X^2) = \sum_{k=1}^{n} \frac{F_k}{F_{k+1}} \cdot \frac{F_k^2}{F_n F_{n+1}} = \frac{1}{F_n F_{n+1}} \sum_{k=1}^{n} \frac{F_k^3}{F_{k+1}}.$$

Since $\mathrm{Var}(X) = E(X^2) - E^2(X) \geq 0$, then $E(X^2) \geq E^2(X)$ and we have

$$\frac{1}{F_n F_{n+1}} \sum_{k=1}^{n} \frac{F_k^3}{F_{k+1}} \geq \left(\frac{1}{F_n F_{n+1}} \sum_{k=1}^{n} \sqrt{\frac{F_k^5}{F_{k+1}}} \right)^2,$$

from which the statement follows. Equality holds when $n = 1$, and we are done.

Solution 2 by Michel Bataille, Rouen, France. (We do not take subscripts modulo n.)

From the Cauchy-Schwarz inequality, we obtain

$$\left(\sum_{k=1}^{n}\frac{F_k^3}{F_{k+1}}\right)\left(\sum_{k=1}^{n}F_k^2\right)\geq\left(\sum_{k=1}^{n}\sqrt{\frac{F_k^3}{F_{k+1}}}\cdot F_k\right)^2=\left(\sum_{k=1}^{n}\sqrt{\frac{F_k^5}{F_{k+1}}}\right)^2. \quad (1)$$

But we have $\sum_{k=1}^{n}F_k^2=F_nF_{n+1}$: this is true for $n=1$ and, it if holds for some positive integer n, then $\sum_{k=1}^{n+1}F_k^2=F_nF_{n+1}+F_{n+1}^2=F_{n+1}(F_n+F_{n+1})=F_{n+1}F_{n+2}$, hence by induction the formula holds for all positive integer n. Dividing (1) by F_nF_{n+1}, we deduce that

$$\sum_{k=1}^{n}\frac{F_k^3}{F_{k+1}}\geq\frac{1}{F_nF_{n+1}}\left(\sum_{k=1}^{n}\sqrt{\frac{F_k^5}{F_{k+1}}}\right)^2.$$

Also solved by *Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA; Moti Levy, Rehovot, Israel; Víctor Martín Chabrera, Barcelona, Spain, and Marian Ursărescu, "Roman-Vodă" National College, Roman, Romania.*

MH–90. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a scalene triangle. Let N be the center of the nine-point circle of $\triangle ABC$ and $\angle\alpha=\angle BAC$. If AN is parallel to BC, prove that

$$\cos\alpha=\frac{-bc}{2R^2}.$$

Solution 1 by the proposer. Let O,H,G,M be the center of circumcircle, the orthocenter, the centroid of $\triangle ABC$ and the midpoint of BC, respectively.

It is well known that N lies on the Euler Line HGO and $HN=NO$; $HG=2GO$; $OG=2GN$; $AG=2GM$.

Without loss of generality, we may assume that $b>c$. Consider Cartesian coordinates with $M=(0,0)$, $B=(-a/2,0)$, $C=(a/2,0)$, $A=(-6,3t)$.

Let $A_H = AH \cap BC$ and $A' = AN \cap OM$. The line $HA_H \perp BC$, $OM \perp BC$ and N is the midpoint of HO, hence N lies on the perpendicular bisector of $A_H M$ and from $AN \| BC$, it follows that $N = (-3, 3t)$, $A' = (0, 3t)$.

$GM/AM = 1/3$ or $G = (-2, t)$. Let G' be the pedal of G on OM. $\triangle ONA' \sim \triangle OGG'$ and $OG/ON = 2/3$, hence $O - G = 2/3(O - N)$, $1/3 \cdot O = G - 2/3 \cdot N$, $O = 3 \cdot G - 2 \cdot N = (0, -3t)$.

$$\begin{aligned} b^2 = AC^2 &= AA_H^2 + A_H C^2 = 9t^2 + (a/2 + 6)^2, \\ c^2 = AB^2 &= AA_H^2 + A_H B^2 = 9t^2 + (a/2 - 6)^2, \\ R^2 = OA^2 &= OA'^2 + AA'^2 = 36t^2 + 36, \\ R^2 = OB^2 &= OM^2 + BM^2 = 9t^2 + a^2/4. \end{aligned}$$

Hence $a^2 = 36(3t^2 + 4)$. The area of $\triangle ABC$ is

$$S = \frac{1}{2} AA_H \cdot BC = \frac{3}{2} at, \quad S = \frac{abc}{4R}.$$

By the law of cosines we have

$$\begin{aligned} \cos\alpha &= \frac{a^2(b^2 + c^2 - a^2)}{2a^2bc} \\ &= \frac{18(-144 + a^2 - 180t^2)(4 + 3t^2)}{2a^2bc} \\ &= \frac{-1296t^2(4 + 3t^2)}{2a^2bc} \\ &= \frac{-16S^2}{2a^2bc} = \frac{-a^2b^2c^2}{2a^2bcR^2} \\ &= \frac{-bc}{2R^2}. \end{aligned}$$

Solution 2 by Michel Bataille, Rouen, France. We introduce

$$S_A = \frac{b^2 + c^2 - a^2}{2}, \quad S_B = \frac{c^2 + a^2 - b^2}{2}, \quad S_C = \frac{a^2 + b^2 - c^2}{2}$$

(Conway's notations) and recall the barycentric coordinates of the circumcenter O and orthocenter H of ΔABC relatively to (A, B, C),

$$O = (a^2 S_A : b^2 S_B : c^2 S_C), \qquad H = (S_B S_C : S_C S_A : S_A S_B),$$

and the equalities $2(S_BS_C+S_CS_A+S_AS_B) = a^2S_A+b^2S_B+c^2S_C = 8F^2$ where F is the area of ΔABC [see for example Focus On... No 39, *Crux Mathematicorum*, Vol 46(1), January 2020, p. 26-31].

Since N is the midpoint of OH, we easily obtain

$$N = (2S_BS_C + a^2S_A : 2S_CS_A + b^2S_B : 2S_AS_B + c^2S_C).$$

The equation of the parallel to BC through A being $y + z = 0$, the hypothesis $AN \parallel BC$ gives $2S_CS_A + b^2S_B + 2S_AS_B + c^2S_C = 0$, that is, $2a^2S_A + b^2S_B + c^2S_C = 0$ (since $S_B + S_C = a^2$) or $a^2S_A + 8F^2 = 0$.

Now, because $S_A = bc\cos\alpha$ and $F = \dfrac{abc}{4R}$, the latter readily becomes $\cos\alpha = \dfrac{-bc}{2R^2}$, as required.

Also solved by *Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

MH–91. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* A triplet $u, v, w \in \mathbb{R}[x, y, z]$ is called Barcelonian if there exist polynomials $A, B, C \in \mathbb{R}[x, y, z]$ such that

$$u^{2021}A + v^{2021}B + w^{2021}C = 2021.$$

(a) Is the triplet of polynomials $u = x + 2y + z$, $v = y + 2z + 3$, $w = x + y + z$ Barcelonian?
(b) Is the triplet of polynomials $u = x + 2y + z$, $v = y + 2z + 3$, $w = x + y - z + 4$ Barcelonian?

Solution 1 by Moti Levy, Rehovot, Israel. Claim: A triplet of polynomials u, v, w is Barcelonian if and only if the ideal $I := \langle u, v, w\rangle$, generated by the triplet u, v, w, is equal to the whole ring $\mathbb{R}[x, y, z]$.

Proof of claim:

($\Leftarrow$) Suppose a triplet of polynomials u, v, w is Barcelonian. Then, the unit 1 belongs to the ideal $\langle u, v, w\rangle$, hence the ideal is equal to the whole ring.

($\Rightarrow$) Suppose a triplet of polynomials u, v, w generates the whole ring. That is, $\langle u, v, w\rangle = \mathbb{R}[x, y, z]$. We will prove by mathematical induction that the triplet of polynomials $u^{2021}, v^{2021}, w^{2021}$ generates the whole ring.

Suppose $\langle u^m, v^m, w^m\rangle = \mathbb{R}[x, y, z]$. Then, there exist polynomials A, B and C such that

$$u^m A + v^m B + w^m C = 1.$$

Multiply both sides by u, so

$$u^{m+1} A + v^m (uB) + w^m (uC) = u.$$

Hence, $u \in \langle u^{m+1}, v^m, w^m\rangle$. It follows that $u^m \in \langle u^{m+1}, v^m, w^m\rangle$ and this implies that $\mathbb{R}[x, y, z] = \langle u^m, v^m, w^m\rangle \subseteq \langle u^{m+1}, v^m, w^m\rangle$. We conclude that $\langle u^{m+1}, v^m, w^m\rangle = \mathbb{R}[x, y, z]$.

Therefore, by symmetry of u, v, w and the induction, we have $\langle u^{2021}, v^{2021}, w^{2021}\rangle = \mathbb{R}[x, y, z]$.

a) The triplet $(x + 2y + z)$, $(y + 2z + 3)$, $(x + y + z)$ is not Barcelonian since there are no three real numbers α, β, γ such that

$$\alpha(x + 2y + z) + \beta(y + 2z + 3) + \gamma(x + y + z) = 1,$$
$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$$

has only the zero solution.

b) The triplet $(x + 2y + z)$, $(y + 2z + 3)$, $(x + y - z + 4)$ is Barcelonian since

$$-\frac{1}{7}(x + 2y + z) + \frac{1}{7}(y + 2z + 3) + \frac{1}{7}(x + y - z + 4) = 1.$$

Solution 2 by Víctor Martín Chabrera, Barcelona, Spain.
(a) It is easy to see that the application $(x, y, z) \mapsto (u, v, w)$ is a bijective affine map. In particular, this means $(u, v, w) = (0, 0, 0)$ is the image of the application for some input values. For that

case, however, the LHS of the equation is 0, and so such triplet of polynomials cannot be Barcelonian.

(b) We can easily see that $u = v + w - 7$. Since $(x, y, z) \mapsto (x, v, w)$ is a bilinear affine map, we can write the equation as

$$(v+w-7)^{2021}A(x,v,w)+v^{2021}B(x,v,w)+w^{2021}C(x,v,w) = 2021.$$

Let us try to make A, B and C independent from x. So, we want to find $A(v,w)$, $B(v,w)$ and $C(v,w)$ such that

$$(v + w - 7)^{2021}A(v,w) + v^{2021}B(v,w) + w^{2021}C(v,w) = 2021.$$

Let $p(v,w) = \left(\frac{v+w}{7} - 1\right)^{2021}$. We can see that its independent term is equal to -1, and so we can write it as $p(v,w) = q(v,w) - 1$, where $q(v,w)$ will have null independent term. We can also see that $(v + w - 7)^{2021} = 7^{2021}p(v,w) = 7^{2021}(q(v,w) - 1)$.

Now, let us set $A(v,w)$ in the following manner:

$$A(v,w) = -\frac{2021}{7^{2021}}\left[1 + q(v,w) + q^2(v,w) + \ldots + q^{4041}(v,w)\right].$$

And so, we will have the following:

$$\begin{aligned}
&(v + w - 7)^{2021}A(v,w)\\
&= 7^{2021}(q(v,w) - 1) \cdot \left(-\frac{2021}{7^{2021}}\left[1 + q(v,w) + \ldots + q^{4041}(v,w)\right]\right)\\
&= -2021(q^{4042}(v,w) - 1) = 2021 - 2021q^{4042}(v,w).
\end{aligned}$$

So, note that, apart from the independent term, this polynomial will be made out of monomials of degree at least 4042. Let $a_{i,j} = [v^i w^j](2021 - 2021q^{4042}(v,w))$. Now, we are left with giving values to $B(v,w)$ and $C(v,w)$. So, let $b_{i,j} = [v^i w^j]B(v,w)$, and $c_{i,j} = [v^i w^j]C(v,w)$. We will consider these coefficients to be 0 if one of its coefficients is negative.

Let $s_{i,j} = a_{i,j} + b_{i-2021,j} + c_{i,j-2021}$. We need $s_{0,0} = 2021$ and $s_{i,j} = 0$ if $i + j > 0$. The first condition is true by construction, regardless of the choices of $b_{i,j}$ and $c_{i,j}$. Also, by construction, $a_{i+j} = 0$ if $0 < i + j < 4042$.

Now, we are just left with choosing $b_{i,j}$ and $c_{i,j}$ when $i + j > 0$. We will consider the following four cases.

- If $i < 2021$ and $j < 2021$, we will have $i + j < 4042$, and as $a_{i,j}$, $b_{i-2021,j}$ and $c_{i,j-2021}$ will be 0, $s_{i,j}$ will automatically be 0.
- If $i < 2021$ and $j \geq 2021$, as we will have $b_{i-2021,j} = 0$, we will set $c_{i,j-2021} = -a_{i,j}$.
- Similarly, if $i \geq 2021$ and $j < 2021$, as we will have $c_{i,j-2021} = 0$, we will set $b_{i-2021,j} = -a_{i,j}$.
- The last case is when $i \geq 2021$ and $j \geq 2021$. Here, we will set $b_{i-2021,j} = c_{i,j-2021} = -\frac{a_{i,j}}{2}$.

As we have found polynomials A, B and C that satisfy the equation, we conclude that the initial triplet is Barcelonian.

Solution 3 by the proposer. (a) The answer is NO. Indeed, the polynomials u, v, w are equal to zero at $x = 3/2$, $y = 0$, $z = -3/2$. This leads to $0 = 2021$, which is impossible.

(b) The answer is YES. Indeed, we have that $u - v - w = 1$ and then the following identity holds:

$$(u - v - w)^n = \sum_{n_1+n_2+n_3=n} \frac{(-1)^{n_1+n_2} n!}{n_1! n_2! n_3!} u^{n_1} v^{n_2} w^{n_3} = 1.$$

For $n \geq 3 \cdot 2021 + 1 = 6064$ in each term the sum in the preceding identity at least one power n_i is greater or equal than 2021 (there can be more than one n_i with the same property). We may thus rewrite the above identity as

$$u^{2021} A + v^{2021} B + w^{2021} C = 1,$$

where A, B, C are not unique. Multiplying both terms of the preceding expression by 2021 yields

$$\begin{aligned} u^{2021}(2021 \cdot A) + v^{2021}(2021 \cdot B) + w^{2021}(2021 \cdot C) \\ = u^{2021} A' + v^{2021} B' + w^{2021} C' = 2021, \end{aligned}$$

and we may conclude that triple (u, v, w) is Barcelonian.

MH–92. *Proposed by Óscar Rivero Salgado, BarcelonaTech, Barcelona, Spain.* Let A and B be two symmetric $n \times n$ matrices with real entries and such that $A(A-B) = B$. Show that

$$\mathrm{Tr}(B^3(B-A)) \geq \mathrm{Tr}(A^3(B-A)),$$

where Tr stands for the trace.

Solution 1 by Michel Bataille, Rouen, France. We denote by M^T the transpose of the matrix M.

Since $A^T = A$ and $B^T = B$, we have

$$B = B^T = (A(A-B))^T = (A-B)^T A^T = (A^T - B^T)A = (A-B)A$$

so that $A(A-B) = (A-B)A$ and, therefore, $AB = BA$.

Being symmetric with real entries, A and B are diagonalizable in $\mathcal{M}_n(\mathbb{R})$ and all their eigenvalues are real. But for commuting diagonalizable matrices, there exists a common basis of eigenvectors (see at the end for a quick proof of this known result). Therefore, there exists an invertible matrix P such that $A = PDP^{-1}$ and $B = P\Delta P^{-1}$ where $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $\Delta = \mathrm{diag}(\mu_1, \ldots, \mu_n)$. Here $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and $\mu_1, \ldots, \mu_n$ are the eigenvalues of B.

This said, we have to prove that $\mathrm{Tr}[(B^3 - A^3)(B-A)] \geq 0$.

We calculate

$$(B^3 - A^3)(B-A) = P(\Delta^3 - D^3)P^{-1} \cdot P(\Delta - D)P^{-1} = PUP^{-1},$$

where $U = (\Delta^3 - D^3)(\Delta - D) = \mathrm{diag}(\nu_1, \ldots, \nu_n)$ with

$$\nu_k = (\mu_k^3 - \lambda_k^3)(\mu_k - \lambda_k) \quad (k = 1, \ldots, n).$$

Thus,

$$\mathrm{Tr}[(B^3 - A^3)(B-A)] = \mathrm{Tr}(U) = \nu_1 + \ldots + \nu_n$$

and the desired result follows since $\nu_k \geq 0$ for $k = 1, \ldots, n$ (for $x, y \in \mathbb{R}$, $x^3 - y^3$ and $x - y$ have the same sign).

Proof of the result about commuting diagonalizable matrices. Let A, B be diagonalizable with $AB = BA$ and let E_λ be an eigenspace of A, associated with the eigenvalue λ. If $X \in E_\lambda$, then $A(BX) = B(AX) = B(\lambda X) = \lambda BX$, hence $BX \in E_\lambda$. Let $B_1 \colon E_\lambda \to E_\lambda$ be defined by $B_1 X = BX$ for $X \in E_\lambda$ and let m and m_1 be the minimal polynomial of B and B_1, respectively. For $X \in E_\lambda$, we have $m(B_1)X = m(B)X = 0$, hence $m(B_1) = 0$ and m_1 divides m. Since m does not have multiple roots (B being diagonalizable), the same is true of m_1 and B_1 is diagonalizable. This means that we can find a basis of E_λ formed of eigenvectors of B_1, hence of B. Doing the same for each eigenspace of A, the union of the bases found is a basis formed of eigenvectors of both A and B.

Solution 2 by the proposer. From the given condition,

$$AB = A^2 - B.$$

Using that A and B are both symmetric we have that

$$BA = B^T A^T = (AB)^T = (A^2)^T - B^T = A^2 - B = AB.$$

Since both matrices commute and are symmetric, this means that there exists a common set of eigenvectors simultaneously diagonalizing both of them. Further, we may easily check that any matrix of the form $A^a B^b$, with $a, b \in \mathbb{N}$, is symmetric and diagonalizes in the same basis of eigenvectors used for both A and B. In particular, if a common eigenvector v satisfies $Av = \lambda v$ and $Bv = \mu v$, then $A^a B^b v = \lambda^a \mu^b v$.

Let $\{\lambda_i\}$ and $\{\mu_i\}$ stand for the ordered eigenvalues of A and B, respectively, with respect to the common eigenbasis. Then, the given inequality is equivalent to

$$\sum_{i=1}^{n} \mu_i^3(\mu_i - \lambda_i) \geq \sum_{i=1}^{n} \lambda_i^3(\mu_i - \lambda_i),$$

or alternatively to

$$\sum_{i=1}^{n} (\mu_i^3 - \lambda_i^3)(\mu_i - \lambda_i) \geq 0.$$

The latter clearly holds since

$$(\mu_i^3 - \lambda_i^3)(\mu_i - \lambda_i) = (\mu_i - \lambda_i)^2(\mu_i^2 + \lambda_i^2 + \mu_i\lambda_i) \geq 0.$$

Observe that in the last step we have used that for any real numbers a and b, $a^2 + b^2 + ab = \frac{3(a+b)^2+(a-b)^2}{4} \geq 0$.

Remark. Alternatively, we may deduce the commutativity of A and B by noting that

$$0 = A^2 - AB - B + A - A - I + I = A(A - B - I) + (A - B - I) + I,$$

or alternatively as

$$(A + I)(B - A + I) = I.$$

This means that the matrix $A + I$ is the inverse of $B - A + I$, so

$$I = (B - A + I)(A + I) = -A^2 + BA + B + I,$$

and consequently $(A - B)A = B$. Comparing this expression with the condition, we arrive to

$$AB = BA.$$

Solution 3 by Víctor Martín Chabrera, Barcelona, Spain. Let $X = B - A$, $Y = A + B$. We can write A and B as $\frac{Y-X}{2}$ and $\frac{Y+X}{2}$, respectively. So, we can rewrite the relation $A(A - B) = B$ as $\frac{Y-X}{2}(-X) = \frac{Y+X}{2}$ or, more conveniently, as $YX = X^2 - Y - X$.

Let us show that both matrices X and Y commute. Take into account that, by construction, both are symmetric:

$$\begin{aligned} XY &= X^\top Y^\top = (YX)^\top = \left(X^2 - Y - X\right)^\top \\ &= \left(X^2\right)^\top - Y^\top - X^\top = X^2 - Y - X = YX. \end{aligned}$$

Now, we can rewrite the desired inequality $\mathrm{Tr}(B^3(B - A)) \geq \mathrm{Tr}(A^3(B - A))$ as

$$\mathrm{Tr}\left(\left(\frac{Y + X}{2}\right)^3 X\right) \geq \mathrm{Tr}\left(\left(\frac{Y - X}{2}\right)^3 X\right)$$

or, equivalently, and more conveniently, as

$$\mathrm{Tr}\big(\big[(\boldsymbol{Y}+\boldsymbol{X})^3-(\boldsymbol{Y}-\boldsymbol{X})^3\big]\boldsymbol{X}\big)\geq 0.$$

Developing further, and taking into account that $\boldsymbol{XY}=\boldsymbol{YX}$, this can be equivalently rewritten as

$$\mathrm{Tr}\big(6\boldsymbol{Y}^2\boldsymbol{X}^2+2\boldsymbol{X}^4\big)\geq 0.$$

We will prove this last inequality. Since both $\boldsymbol{X}^2$ and $\boldsymbol{Y}^2$ are positive semidefinite symmetric matrices, and $\boldsymbol{X}$ commutes with $\boldsymbol{Y}$, then the products $\boldsymbol{Y}^2\boldsymbol{X}^2$ and $\boldsymbol{X}^4$ will also be positive semidefinite (since $\boldsymbol{Y}^2\boldsymbol{X}^2=(\boldsymbol{YX})(\boldsymbol{YX})^\top$ and $\boldsymbol{X}^4=(\boldsymbol{X}^2)(\boldsymbol{X}^2)^\top$). Since a positive semidefinite matrix has non-negative trace, we get

$$\mathrm{Tr}\big(6\boldsymbol{Y}^2\boldsymbol{X}^2+2\boldsymbol{X}^4\big)=6\mathrm{Tr}\big(\boldsymbol{Y}^2\boldsymbol{X}^2\big)+2\mathrm{Tr}\big(\boldsymbol{X}^4\big)\geq 0,$$

completing the proof.

MH–93. *Proposed by Daniel Vilardell Regué, CFIS, Barcelona-Tech, Barcelona, Spain.* Let $\boldsymbol{F_n}$ be the $\boldsymbol{n}$-th Fibbonacci number. Prove that for all odd $\boldsymbol{n}$ the following equality holds:

$$\boldsymbol{F_n}=\sum_{\boldsymbol{k}=\boldsymbol{0}}^{\frac{\boldsymbol{n-1}}{\boldsymbol{2}}}\binom{\frac{\boldsymbol{n-1}}{\boldsymbol{2}}+\boldsymbol{k}}{\boldsymbol{2k}}.$$

Solution 1 by Michel Bataille, Rouen, France. We shall use the well-known relation

$$\boldsymbol{F_{n+1}}=\sum_{\boldsymbol{k}=\boldsymbol{0}}^{\lfloor\frac{\boldsymbol{n}}{\boldsymbol{2}}\rfloor}\binom{\boldsymbol{n-k}}{\boldsymbol{k}},\tag{1}$$

which holds for any nonnegative integer $\boldsymbol{n}$ (and is easily proved by induction using $\binom{s+1}{r+1}=\binom{s}{r}+\binom{s}{r+1}$ or by application of the general formula

$$\sqrt{1+x}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\Big(\frac{x}{4}\Big)^k=\left(\frac{1+\sqrt{1+x}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{1+x}}{2}\right)^{n+1},$$

see R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1994, formula 5-74).

We have to prove that

$$F_{2m+1} = \sum_{k=0}^{m} \binom{m+k}{2k}. \tag{2}$$

From (1) with $n = 2m$, we obtain $F_{2m+1} = \sum_{k=0}^{m} \binom{2m-k}{k}$ so that

$$\begin{aligned} F_{2m+1} &= \sum_{k=0}^{m} \binom{2m-k}{(2m-k)-k} = \sum_{k=0}^{m} \binom{2m-k}{2(m-k)} \\ &= \sum_{j=0}^{m} \binom{2m-(m-j)}{2j} = \sum_{j=0}^{m} \binom{m+j}{2j} \end{aligned}$$

and (2) follows.

Solution 2 by Moti Levy, Rehovot, Israel. Actually, we will prove that the following two equalities hold:

$$\begin{aligned} F_{2m-1} &= \sum_{k=0}^{m-1} \binom{m-1+k}{2k}, \\ F_{2m} &= \sum_{k=1}^{m} \binom{m-1+k}{2k-1}. \end{aligned}$$

Let

$$S_{2m-1} = \sum_{k=0}^{m-1} \binom{m-1+k}{2k}, \qquad S_{2m} = \sum_{k=1}^{m} \binom{m-1+k}{2k-1}.$$

Then, for $m = 1$,

$$S_1 = \sum_{k=0}^{0} \binom{m-1+k}{2k} = F_1, \tag{3}$$

and

$$S_2 = \sum_{k=1}^{1} \binom{1-1+k}{2k-1} = 1 = F_2. \tag{4}$$

$$\begin{aligned} S_{2m-1} + S_{2m} &= \sum_{k=0}^{m-1} \binom{m-1+k}{2k} + \sum_{k=1}^{m} \binom{m-1+k}{2k-1} \\ &= 1 + \sum_{k=1}^{m-1} \binom{m-1+k}{2k} + \sum_{k=1}^{m-1} \binom{m-1+k}{2k-1} + 1 \\ &= 1 + \sum_{k=1}^{m-1} \left(\binom{m-1+k}{2k} + \binom{m-1+k}{2k-1} \right) + 1 \\ &= 1 + \sum_{k=1}^{m-1} \binom{m+k}{2k} + 1 \\ &= \sum_{k=0}^{m} \binom{m+k}{2k} = S_{2m+1}. \end{aligned} \tag{5}$$

We used above the combinatorial identity $\binom{m-1}{k} + \binom{m-1}{k-1} = \binom{m}{k}$.

Similarly, one can show that

$$S_{2m} + S_{2m+1} = S_{2m+2}. \tag{6}$$

By (5), (6), (3) and (4), it is evident that the sequence (S_n) obeys the recurrence rule and initial conditions of the Fibonacci sequence, hence it is identical to the Fibonacci sequence.

Solution 3 by Víctor Martín Chabrera, Barcelona, Spain. Using the change of variables $j = \frac{n-1}{2} - k$, we can rewrite the given expression as

$$\begin{aligned} \sum_{k=0}^{\frac{n-1}{2}} \binom{\frac{n-1}{2}+k}{2k} &= \sum_{k=0}^{\frac{n-1}{2}} \binom{\frac{n-1}{2}+k}{\frac{n-1}{2}-k} \\ &= \sum_{j=0}^{\frac{n-1}{2}} \binom{n-1-j}{j} = \sum_{j\in\mathbb{Z}} \binom{n-1-j}{j}. \end{aligned}$$

Here, we are using the more general definition of the binomial coefficients, stating that $\binom{N}{K} = 0$ if $K \notin \{0, \dots, N\}$, which is the case for values of j that lie outside the range $\left[0, \frac{n-1}{2}\right]$.

We will show that the formula

$$F_n = \sum_{j\in\mathbb{Z}} \binom{n-1-j}{j}$$

holds regardless of the parity of n using induction. For $n \in \{0,1\}$ it trivially holds. Given a value $n > 1$, assume the formula holds for each $0 \le k < n$. Then, using the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$,

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \binom{n-1-j}{j} &= \sum_{j\in\mathbb{Z}} \binom{n-2-j}{j-1} + \sum_{j\in\mathbb{Z}} \binom{n-2-j}{j} \\ &= \sum_{i\in\mathbb{Z}} \binom{n-3-i}{i} + \sum_{j\in\mathbb{Z}} \binom{n-2-j}{j} \\ &= F_{n-2} + F_{n-1} = F_n,\end{aligned}$$

completing the proof.

Solution 4 by the proposer. To prove this equality, we are going to use a double counting technique and we are going to count the number of compositions of n as the sum of odd positive numbers.

We first divide the set of compositions into two sets, the ones that have 1 as the last element and the ones that do not. Those sets are disjoint and each composition is in one of those two sets. Let c_t be the number of compositions of t as the sum of odd positive numbers. Then, we have that the first set is equal to c_{n-1} since we can add a 1 the each composition of $n-1$ at the end. We also have that the second set is equal to c_{n-2} since we can add 2 to the last number in each composition of $n-2$.

We know that the number of partitions of 1 and 2 in odd parts is both 1, therefore we have that

$$c_n = c_{n-1} + c_{n-2}, \qquad c_1 = c_2 = 1,$$

and we know that recurrence is the same as the Fibbonacci recurrence. Therefore $c_n = F_n$.

We will now count that in a different way. We will first find the number of ways to compose a number n in k odd parts and then

we will sum that for all k. The number of ways to compose n in t odd parts is the same as the ways to compose $n-t$ into even parts which is the same as the number of ways to compose $\frac{n-t}{2}$ (if n is odd and t even we know that the sum of t odd numbers is an even number so the number of compositions for t even will be 0).

We also know that the number of compositions of a number into t parts is $\binom{n+t-1}{t-1}$ Therefore, setting $t = 2k+1$, we find that the number of partitions

$$\sum_{k=0}^{\frac{n-1}{2}} \binom{\frac{n-(2k+1)}{2} + 2k + 1 - 1}{2k+1-1} = \sum_{k=0}^{\frac{n-1}{2}} \binom{\frac{n+1}{2} + k}{2k},$$

from where we can conclude that

$$F_n = \sum_{k=0}^{\frac{n-1}{2}} \binom{\frac{n-1}{2} + k}{2k}.$$

Also solved by *Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA, and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.*

MH–94. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let ABC be an acute triangle, with $\measuredangle(BAC) = 45^\circ$. Outside the triangle take the points X, Y, Z so that AB and AC are hypotenuses in the isosceles right triangles ABX and AYC and the quadrilateral $ABZC$ is a parallelogram. If $CX \cap BY = \{W\}$, then show that AW and AZ are isogonal lines.

Solution 1 by the proposer. Let us denote $CX \cap AB = \{X'\}$, $BY \cap AC = \{Y'\}$, $AW \cap BC = W'$.

Then,

$$\frac{AX'}{X'B} = \frac{\sigma(XAX')}{\sigma(XBX')} = \frac{XA \cdot XX' \cdot \sin(AXX') \cdot \frac{1}{2}}{XB \cdot XX' \cdot \sin(BXX') \cdot \frac{1}{2}},$$

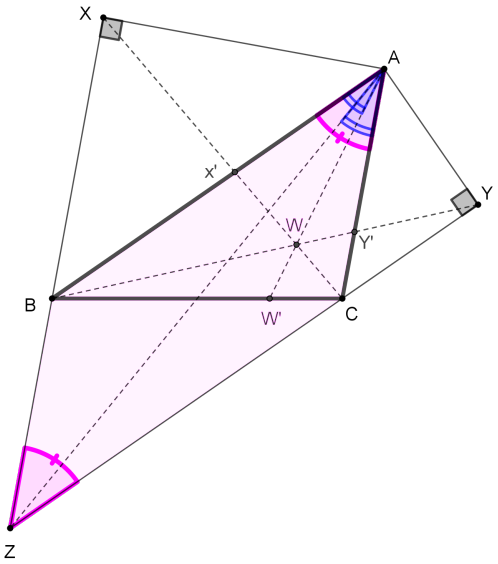

Figure 2: Scheme for solution 1 to problem MH–94.

where $XA = XB$ (ABX is a right isosceles triangle), so

$$\frac{AX'}{X'B} = \frac{\sin(AXX')}{\sin(BXX')}. \tag{1}$$

From the Sine And Cosine Rules, we have

$$\begin{cases} \dfrac{\sin(AXX')}{AC} = \dfrac{\sin(XAC)}{XC}, \\ \dfrac{\sin(BXX')}{BC} = \dfrac{\sin(XBC)}{XC}, \end{cases}$$

so

$$\begin{aligned} \frac{\sin(AXX')}{\sin(BXX')} &= \frac{AC \cdot \sin(XAC)}{BC \cdot \sin(XBC)} \\ &= \frac{AC \cdot \sin 90^\circ}{BC \cdot \sin(45^\circ + B)} = \frac{AC}{BC \cdot \sin(A + B)}. \end{aligned}$$

Relationship (1) becomes

$$\frac{AX'}{X'B} = \frac{AC}{BC \cdot \sin C}.$$

Analogously,

$$\frac{AY'}{Y'C} = \frac{AB}{BC \cdot \sin B}.$$

From Ceva's Theorem

$$\frac{BW'}{W'C} \cdot \frac{CY'}{Y'A} \cdot \frac{X'A}{X'B} = 1AB \cdot \frac{AC}{BC \cdot \sin C} = 1,$$

so

$$\frac{BW'}{W'C} = \frac{AB \cdot \sin C}{AC \cdot \sin B} = \frac{AB^2}{AC^2}.$$

So, in $\triangle ABC$, AW' is symmedian and AZ is the median from A (AZ is a diagonal in the parallelogram $ABZC$). The median and the symmedian are isogonal lines, so

$$\measuredangle(BAW) = \measuredangle(ZAC).$$

Solution 2 by Michel Bataille, Rouen, France. Since AC is parallel to BZ, we have that $\angle ZBC = \angle ACB$, hence $\measuredangle(ZBA) = \measuredangle(ABC) + \measuredangle(ACB) = 180^\circ - 45^\circ = 135^\circ$. Observing that $\measuredangle(ABX) = 45^\circ$, we deduce that Z, B, X are collinear. Similarly, Z, C, Y are collinear. Note also that the points A, X, Y, Z are on the circle with diameter AZ (since $\measuredangle(AXZ) = \measuredangle(AYZ) = 90^\circ$).

Let E be the midpoint of AB and let $\measuredangle(AXY) = \alpha$, $\measuredangle(AYX) = \beta$. Note that $\measuredangle(XAY) = 45^\circ + 45^\circ + 45^\circ = 135^\circ$, hence $\alpha + \beta = 45^\circ$. Since $\measuredangle(XAE) = 45^\circ$, we have $\measuredangle(EXY) = 45^\circ - \alpha = \beta$.

Now, suppose that we have already proved that AW and XY are perpendicular. Then, since $XE \perp AB$, we obtain that $\measuredangle(BAW) = \measuredangle(EXY) = \beta$; we also have $\measuredangle(ZAC) = \measuredangle(AZB) = \beta$ (since $AC \parallel ZB$), hence $\measuredangle(ZAC) = \measuredangle(BAW) = \beta$ and AW, AZ are isogonal.

It remains to prove that $AW \perp XY$. To this end, we show that the dot product $\overrightarrow{AW} \cdot \overrightarrow{XY}$ is 0.

Let $AB = c$, $AC = b$. Then, $BX = \frac{c}{\sqrt{2}}$, $BZ = b$, $CY = \frac{b}{\sqrt{2}}$, $CZ = c$, hence

$$(b\sqrt{2} + c)B = (b\sqrt{2})X + cZ, \qquad (c\sqrt{2} + b)C = (c\sqrt{2})Y + bZ,$$

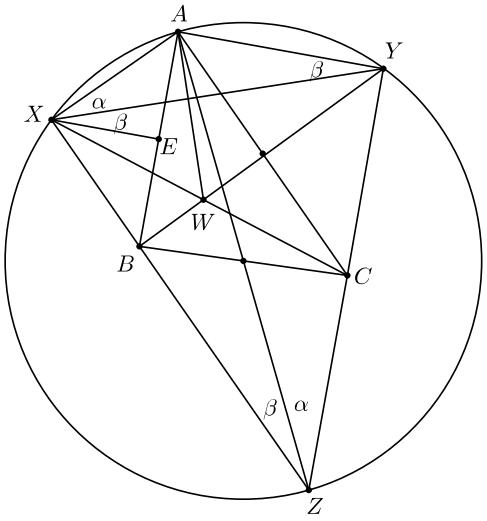

Figure 3: Scheme for solution 2 to problem MH–94.

from which we deduce that the point

$$(c^2\sqrt{2}+bc)C+(b^2\sqrt{2})X=(b^2\sqrt{2}+bc)B+(c^2\sqrt{2})Y$$

is the point of intersection W of CX and BY and, therefore, $mW=(b^2\sqrt{2})X+(c^2\sqrt{2})Y+bcZ$, where $m=b^2\sqrt{2}+c^2\sqrt{2}+bc$.

We have $\overrightarrow{AZ}\cdot\overrightarrow{AY}=AZ\cdot\frac{b}{\sqrt{2}}\sin\alpha=\frac{b}{\sqrt{2}}AY=\frac{b^2}{2}$ (since $\measuredangle(ZAY)=90^\circ-\alpha$) and, similarly, $\overrightarrow{AZ}\cdot\overrightarrow{AX}=\frac{c^2}{2}$; also $\overrightarrow{AX}\cdot\overrightarrow{AY}=\frac{b}{\sqrt{2}}\cdot\frac{c}{\sqrt{2}}\cdot\frac{-\sqrt{2}}{2}=-\frac{bc}{2\sqrt{2}}$. A simple calculation then gives

$$m\overrightarrow{AW}\cdot\overrightarrow{AY}=((b^2\sqrt{2})\overrightarrow{AX}+(c^2\sqrt{2})\overrightarrow{AY}+bc\overrightarrow{AZ})\cdot\overrightarrow{AY}=\frac{b^2c^2}{\sqrt{2}},$$

and similarly $m\overrightarrow{AW}\cdot\overrightarrow{AX}=\frac{b^2c^2}{\sqrt{2}}$. It follows that

$$m\overrightarrow{AW}\cdot\overrightarrow{XY}=m\overrightarrow{AW}\cdot\overrightarrow{AY}-m\overrightarrow{AW}\cdot\overrightarrow{AX}=0,$$

as desired.

Solution 3 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. On the one hand, since $\angle BAC=\angle ACY(=45^\circ)$, we have

$AB \parallel CY$. On the other hand, the point Z given by our construction is such that $AB \parallel ZC$. Thus, points Y, C, Z are collinear (Euclides).

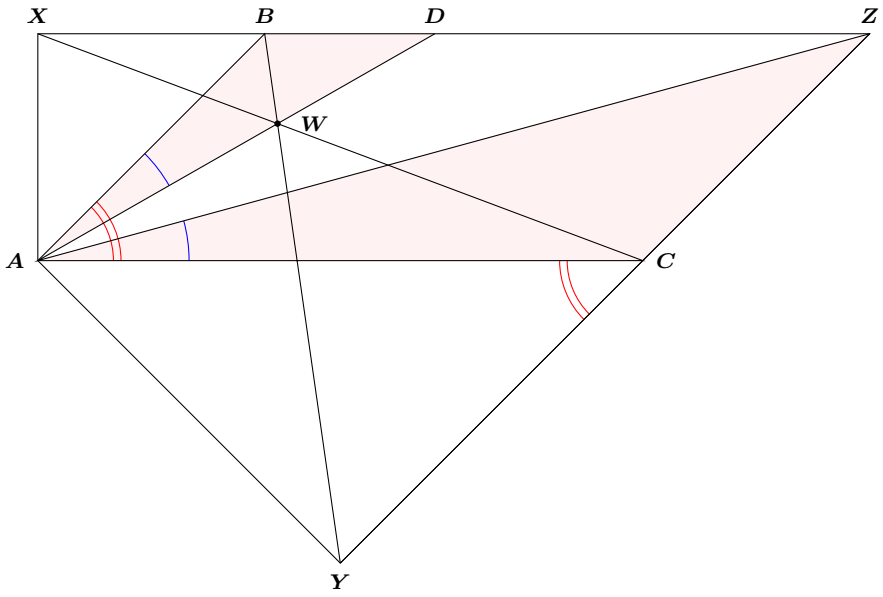

Figure 4: Scheme for solution 3 to problem MH–94.

Applying Menelaus's theorem to the triad of points B, W, Y on the sides of triangle XCZ, we obtain

$$\frac{XW}{WC} \cdot \frac{CY}{YZ} \cdot \frac{ZB}{BX} = 1,$$

where

$$\frac{XW}{WB} = \left(\begin{array}{c} \text{from similar triangles} \\ XWD \text{ and } CWA \end{array} \right) = \frac{XD}{CA} = \frac{XB + BD}{CA}$$
$$= \left(\begin{array}{c} \text{from isosceles} \\ \text{right-angled} \\ \triangle AXB \end{array} \right) = \frac{\frac{AB}{\sqrt{2}} + BD}{CA},$$

$$\frac{CY}{YZ} = \frac{CY}{YC+CZ} = \frac{CY}{YC+AB} = \left(\begin{array}{c} \text{from isosceles} \\ \text{right-angled} \\ \triangle AYC \end{array} \right) = \frac{\frac{AC}{\sqrt{2}}}{\frac{AC}{\sqrt{2}}+AB},$$

and

$$\frac{ZB}{BX} = \frac{CA}{BX} = (\text{from isosceles right-angled triangle } AXB) = \frac{CA}{\frac{AB}{\sqrt{2}}}.$$

Therefore,

$$\frac{\frac{AB}{\sqrt{2}} + BD}{CA} \cdot \frac{\frac{AC}{\sqrt{2}}}{\frac{AC}{\sqrt{2}} + AB} \cdot \frac{CA}{\frac{AB}{\sqrt{2}}} = 1,$$

yielding

$$\frac{BD}{AB} = \frac{AB}{AC}.$$

Since $AB = CZ$, then,

$$\frac{BD}{AB} = \frac{CZ}{AC},$$

making triangles ABD and ACZ similar (side-angle-side) with $\angle BAD = \angle ZAC$, that is,

$$\angle BAW = \angle ZAC$$

(being $ABZC$ a parallelogram, $\angle ABD = \angle ABZ = \angle ACZ$). Thus, AW and AZ are equally inclined to the arms of $\angle A$ and hence are isogonal, as claimed.

Advanced Problems

A–89. *Proposed by Vasile Mircea Popa, "Lucian Blaga" University of Sibiu, Romania.* Calculate the integral

$$\int_0^\infty \frac{\sqrt{x}\ln x}{x^2+x+1}\,\mathrm{d}x.$$

Solution 1 by Moti Levy, Rehovot, Israel. Let

$$I=\int_0^\infty \frac{\sqrt{x}}{x^2+x+1}\ln(x)\,\mathrm{d}x.$$

By change of variable $t=\sqrt{x}$,

$$I=2\int_0^\infty \frac{t^2\ln(t^2)}{t^4+t^2+1}\,\mathrm{d}t.$$

Then,

$$\begin{aligned} I &= 4\int_1^\infty \frac{t^2\ln(t)}{t^4+t^2+1}\,\mathrm{d}t+4\int_0^1 \frac{t^2\ln(t)}{t^4+t^2+1}\,\mathrm{d}t \\ &= -4\int_0^1 \frac{\ln(t)}{t^4+t^2+1}\,\mathrm{d}t+4\int_0^1 \frac{t^2\ln(t)}{t^4+t^2+1}\,\mathrm{d}t \\ &= 4\int_0^1 \frac{t^2-1}{t^4+t^2+1}\ln(t)\,\mathrm{d}t. \end{aligned}$$

After integration by parts,

$$I=2\int_0^1 \frac{1}{t}\ln\left(t^2+t+1\right)\mathrm{d}t-2\int_0^1 \frac{1}{t}\ln\left(t^2-t+1\right)\mathrm{d}t.$$

It follows that

$$\begin{aligned} \int_0^1 \frac{1}{t}\ln\left(t^2+t+1\right)\mathrm{d}t &= \int_0^1 \frac{1}{t}\ln\left(\left(1-e^{i\frac{2\pi}{3}}t\right)\left(1-e^{-i\frac{2\pi}{3}}t\right)\right)\mathrm{d}t \\ &= \int_0^1 \frac{1}{t}\ln\left(1-e^{i\frac{2\pi}{3}}t\right)\mathrm{d}t+\int_0^1 \frac{1}{t}\ln\left(1-e^{-i\frac{2\pi}{3}}t\right)\mathrm{d}t. \end{aligned}$$

An integral representation of the Dilogarithm function $\mathbf{Li_2}(z)$ is (see Wikipedia entry, "Polylogarithm")

$$\mathbf{Li_2}(z) = -\int_0^1 \frac{1}{t}\ln(1-zt)\,\mathrm{d}t, \quad |z|<1.$$

Hence,

$$\int_0^1 \frac{1}{t}\ln\left(t^2+t+1\right)\mathrm{d}t = -\mathbf{Li_2}\left(e^{i\frac{2\pi}{3}}\right) - \mathbf{Li_2}\left(e^{-i\frac{2\pi}{3}}\right).$$

Now we use a known property of the Dilogarithm function (see Wikipedia entry, "Polylogarithm"),

$$\mathbf{Li_2}\left(e^{2\pi i x}\right) + \mathbf{Li_2}\left(e^{-2\pi i x}\right) = 2\pi^2 B_2(x) = 2\pi^2\left(x^2 - x + \frac{1}{6}\right).$$

It follows that

$$-\mathbf{Li_2}\left(e^{i\frac{2\pi}{3}}\right) - \mathbf{Li_2}\left(e^{-i\frac{2\pi}{3}}\right) = -2\pi^2\left(\left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right) + \frac{1}{6}\right) = \frac{1}{9}\pi^2.$$

Similarly,

$$\begin{aligned}\int_0^1 \frac{1}{t}\ln\left(t^2-t+1\right)\mathrm{d}t &= \int_0^1 \frac{1}{t}\ln\left(1-e^{i\frac{\pi}{3}}t\right)\mathrm{d}t + \int_0^1 \frac{1}{t}\ln\left(1-e^{-i\frac{\pi}{3}}t\right)\mathrm{d}t \\ &= -\mathbf{Li_2}\left(e^{i\frac{\pi}{3}}\right) - \mathbf{Li_2}\left(e^{-i\frac{\pi}{3}}\right) \\ &= -2\pi^2\left(\left(\frac{1}{6}\right)^2 - \left(\frac{1}{6}\right) + \frac{1}{6}\right) = -\frac{1}{18}\pi^2.\end{aligned}$$

We conclude that the definite integral is equal to

$$I = 2\left(\frac{1}{9}\pi^2 + \frac{1}{18}\pi^2\right) = \frac{1}{3}\pi^2.$$

Solution 2 by the proposer. Let us denote

$$\begin{aligned}I &= \int_0^\infty \frac{\sqrt{x}\ln x}{x^2+x+1}\,\mathrm{d}x, \\ A &= \int_0^1 \frac{\sqrt{x}\ln x}{x^2+x+1}\,\mathrm{d}x, \\ B &= \int_1^\infty \frac{\sqrt{x}\ln x}{x^2+x+1}\,\mathrm{d}x.\end{aligned}$$

We consider the A integral and we have successively:

$$\begin{aligned}
A &= \int_0^1 \frac{(1-x)x^{\frac{1}{2}}\ln x}{1-x^3}\,\mathrm{d}x \\
&= \int_0^1 \frac{x^{\frac{1}{2}}\ln x}{1-x^3}\,\mathrm{d}x - \int_0^1 \frac{x^{\frac{3}{2}}\ln x}{1-x^3}\,\mathrm{d}x \\
&= \int_0^1 \sum_{n=0}^{\infty} x^{3n+\frac{1}{2}}\ln x\,\mathrm{d}x - \int_0^1 \sum_{n=0}^{\infty} x^{3n+\frac{3}{2}}\ln x\,\mathrm{d}x \\
&= \sum_{n=0}^{\infty}\left(\int_0^1 x^{3n+\frac{1}{2}}\ln x\,\mathrm{d}x - \int_0^1 x^{3n+\frac{3}{2}}\ln x\,\mathrm{d}x\right).
\end{aligned}$$

Next, using the following relationship:

$$\int_0^1 x^a \ln x\,\mathrm{d}x = -\frac{1}{(a+1)^2}, \quad a \in \mathbb{R},\ a \geq 0,$$

we obtain

$$A = \sum_{n=0}^{\infty}\left[\frac{1}{\left(3n+\frac{5}{2}\right)^2} - \frac{1}{\left(3n+\frac{3}{2}\right)^2}\right] = \sum_{n=0}^{\infty}\left[\frac{\frac{1}{9}}{\left(n+\frac{5}{6}\right)^2} - \frac{\frac{1}{9}}{\left(n+\frac{1}{2}\right)^2}\right].$$

We also have the following relationship of the trigamma function:

$$\psi_1(x) = \sum_{n=0}^{\infty}\frac{1}{(x+n)^2}.$$

And we obtained the value of the A integral:

$$A = \frac{1}{9}\left[\psi_1\left(\frac{5}{6}\right) - \psi_1\left(\frac{1}{2}\right)\right].$$

Now we consider the B integral and substitute $y = \frac{1}{x}$ to obtain

$$B = -\int_0^1 \frac{y^{-\frac{1}{2}}\ln y}{y^2+y+1}\,\mathrm{d}y.$$

Proceeding similarly to the A integral, we obtain

$$B = \frac{1}{9}\left[\psi_1\left(\frac{1}{6}\right) - \psi_1\left(\frac{1}{2}\right)\right].$$

This results in

$$I = A + B = \frac{1}{9}\left[\psi_1\left(\frac{1}{6}\right) + \psi_1\left(\frac{5}{6}\right) - 2\psi_1\left(\frac{1}{2}\right)\right].$$

Finally, using trigamma's reflection formula

$$\psi_1(x) + \psi_1(1-x) = \frac{\pi^2}{\sin^2(\pi x)},$$

we obtain

$$\psi_1\left(\frac{1}{6}\right) + \psi_1\left(\frac{5}{6}\right) = \frac{\pi^2}{\sin^2\left(\frac{\pi}{6}\right)} = 4\pi^2.$$

The special value of the trigamma function, $\psi_1\left(\frac{1}{2}\right) = \frac{\pi^2}{2}$, is known, yielding

$$I = \frac{1}{9}\left(4\pi^2 - \pi^2\right) = \frac{\pi^2}{3}.$$

Thus, the problem is solved.

Solution 3 by Michel Bataille, Rouen, France. Let I be the given integral. The change of variables $x = u^{-2}$ leads to

$$I = -4\int_0^\infty \frac{\ln u}{u^4 + u^2 + 1}\,\mathrm{d}u.$$

We now apply a formula deduced from a suitable contour integration, as detailed in [1]: setting $f(z) = \frac{1}{z^4+z^2+1}$, we have

$$\int_0^\infty \frac{\ln u}{u^4 + u^2 + 1}\,\mathrm{d}u = -\frac{1}{2}\mathrm{Re}(\sigma),$$

where σ is the sum of the residues of $f(z)(\log z)^2$ at the poles of $f(z)$ and $\log z = \ln(|z|) + i\theta$ when $0 \le \theta = \arg(z) < 2\pi$.

The poles of $f(z)$ are $-\omega^2$, ω, ω^2, and $-\omega$, where $\omega = \exp(2\pi i/3)$. The residues of $f(z)(\log z)^2$ at these poles, respectively, are

$$\frac{(i\pi/3)^2}{4(-\omega^2)^3 - 2\omega^2} = \frac{\pi^2/9}{4 + 2\omega^2}, \qquad \frac{(2i\pi/3)^2}{4(\omega)^3 + 2\omega} = \frac{-4\pi^2/9}{4 + 2\omega},$$
$$\frac{(4i\pi/3)^2}{4(\omega)^3 + 2\omega^2} = \frac{-16\pi^2/9}{4 + 2\omega^2}, \qquad \frac{(5i\pi/3)^2}{-4 - 2\omega} = \frac{25\pi^2/9}{4 + 2\omega},$$

and their sum is

$$\frac{7\pi^2/3}{4+2\omega} - \frac{5\pi^2/3}{4+2\omega^2} = \frac{\pi^2}{6}\left(\frac{7}{2+\omega} - \frac{5}{2+\omega^2}\right) = \frac{\pi^2}{18}\cdot(3-6i\sqrt{3})$$

so that $\mathrm{Re}(\sigma) = \frac{\pi^2}{6}$. It follows that

$$I = (-4)\cdot\left(-\frac{\pi^2}{12}\right) = \frac{\pi^2}{3}.$$

[1] H. Cartan, *Elementary Theory of Analytic Functions of One or Several Complex Variables*, Dover, 1995, ch. III, p. 109.

Also solved by *Florică Anastase, "Alexandru Odobescu" High School, Lehliu-Gară, Călăraşi, Romania, and Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA.*

A–90. *Proposed by Michel Bataille, Rouen, France.* For $n \in \mathbb{N}$, let $P_n = \prod_{k=1}^{n}(2k-1)$. Find

$$\ell = \lim_{n\to\infty}\frac{\sqrt[n]{P_n}}{n} \qquad \text{and} \qquad \lim_{n\to\infty}(\sqrt[n]{P_n} - n\ell).$$

Solution 1 by Moti Levy, Rehovot, Israel. We have

$$P_n = \prod_{k=1}^{n}(2k-1) = (2n-1)!! = \frac{(2n)!}{2^n n!}.$$

The asymptotic expansion of the factorial is

$$n! = \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}\left(1+O\left(\frac{1}{n}\right)\right).$$

Therefore,

$$P_n = \frac{\sqrt{2\pi}2^{2n+\frac{1}{2}}n^{2n+\frac{1}{2}}e^{-2n}\left(1+O\left(\frac{1}{n}\right)\right)}{2^n\sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}\left(1+O\left(\frac{1}{n}\right)\right)} = 2^{n+\frac{1}{2}}n^n e^{-n}\left(1+O\left(\frac{1}{n}\right)\right)$$

and, thus,

$$\ell = \lim_{n\to\infty} \frac{P_n^{\frac{1}{n}}}{n} = \lim_{n\to\infty} \left(2^{n+\frac{1}{2}}\right)^{\frac{1}{n}} \frac{1}{e} \lim_{n\to\infty} \left(1 + O\left(\frac{1}{n}\right)\right)^{\frac{1}{n}}$$
$$= \frac{2}{e} \lim_{n\to\infty} \left(1 + O\left(\frac{1}{n^2}\right)\right) = \frac{2}{e}.$$

We now have that

$$P_n^{\frac{1}{n}} - nl = 2^{1+\frac{1}{2n}} \frac{n}{e} \left(1 + O\left(\frac{1}{n}\right)\right)^{\frac{1}{n}} - \frac{2n}{e}$$
$$= 2^{1+\frac{1}{2n}} \frac{n}{e} \left(1 + O\left(\frac{1}{n^2}\right)\right) - \frac{2n}{e}$$
$$= 2^{1+\frac{1}{2n}} \frac{n}{e} - \frac{2n}{e} + O\left(\frac{1}{n}\right),$$

and thus

$$\lim_{n\to\infty} P_n^{\frac{1}{n}} - nl = \lim_{n\to\infty} \left(2^{1+\frac{1}{2n}} \frac{n}{e} - \frac{2n}{e}\right)$$
$$= \frac{2}{e} \lim_{n\to\infty} \left(2^{\frac{1}{2n}} - 1\right) n = \frac{2}{e} \lim_{n\to\infty} \left(e^{\frac{\ln 2}{2n}} - 1\right) n$$
$$= \frac{2}{e} \lim_{n\to\infty} \left(1 + \frac{\ln 2}{2n} + O\left(\frac{1}{n^2}\right) - 1\right) n$$
$$= \frac{2}{e} \lim_{n\to\infty} \left(\frac{\ln 2}{2n} + O\left(\frac{1}{n^2}\right)\right) n = \frac{\ln 2}{e}.$$

Solution 2 by the proposer. We have

$$P_n = 1 \cdot 3 \cdot \cdots \cdot (2n-1) = \frac{(2n)!}{2 \cdot 4 \cdot \cdots \cdot (2n)} = \frac{(2n)!}{2^n (n!)}.$$

It follows that

$$\ln(\sqrt[n]{P_n}) = \frac{1}{n}(\ln((2n)!) - \ln(n!) - n\ln(2)).$$

Using the well-known asymptotic expansion as $n \to \infty$:

$$\ln(n!) = n\ln(n) - n + \frac{\ln(n)}{2} + a + o(1)$$

(where $a = \ln(\sqrt{2\pi})$), we successively obtain

$$\ln((2n)!) = 2n\ln(n) + 2n(\ln(2) - 1) + \frac{\ln(n)}{2} + \frac{\ln(2)}{2} + a + o(1)$$

and

$$\ln(\sqrt[n]{P_n}) = \ln(n) + \ln(2/e) + \frac{\ln(2)}{2n} + o(1/n).$$

We deduce that

$$\begin{aligned}\sqrt[n]{P_n} &= \frac{2n}{e}\exp\left(\frac{\ln(2)}{2n} + o(1/n)\right)\\ &= \frac{2n}{e}\left(1 + \frac{\ln(2)}{2n} + o(1/n)\right)\\ &= \frac{2n}{e} + \frac{\ln(2)}{e} + o(1).\end{aligned}$$

Thus,

$$\frac{\sqrt[n]{P_n}}{n} = \frac{2}{e} + \frac{\ln(2)}{ne} + o(1/n)$$

and we can conclude

$$\ell = \frac{2}{e} \qquad \text{and} \qquad \lim_{n\to\infty}(\sqrt[n]{P_n} - n\ell) = \frac{\ln(2)}{e}.$$

Solution 3 by Víctor Martín Chabrera, Barcelona, Spain. We can see that $P_n = \frac{(2n)!}{2^n n!}$.

Using Stirling's approximation: $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)$, we obtain

$$\begin{aligned}\sqrt[n]{P_n} &= \sqrt[n]{\frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)}{2^n\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)}}\\ &= \frac{\left(\frac{2n}{e}\right)^2}{\frac{2n}{e}}\sqrt[n]{\sqrt{2}\left(1 + \mathcal{O}\left(\frac{1}{n}\right)\right)} = \frac{2n}{e}\sqrt[2n]{2}\left(1 + \mathcal{O}\left(\frac{1}{n^2}\right)\right).\end{aligned}$$

From here, we immediately obtain

$$\ell = \lim_{n\to\infty}\frac{\sqrt[n]{P_n}}{n} = \lim_{n\to\infty}\frac{\frac{2n}{e}\sqrt[2n]{2}\left(1 + \mathcal{O}\left(\frac{1}{n^2}\right)\right)}{n} = \boxed{\frac{2}{e}}.$$

Now, we are left with computing the other limit:

$$\begin{aligned}\lim_{n\to\infty} \sqrt[n]{P_n} - n\ell &= \lim_{n\to\infty}\left[\frac{2n}{e}\sqrt[2n]{2}\left(1+\mathcal{O}\left(\frac{1}{n^2}\right)\right) - \frac{2n}{e}\right] \\ &= \lim_{n\to\infty}\frac{2n}{e}\left(\sqrt[2n]{2}-1\right) = \frac{2}{e}\lim_{n\to\infty}\frac{\left(\sqrt[2n]{2}-1\right)}{\frac{1}{n}}.\end{aligned}$$

Let us compute this limit by applying l'Hôpital's rule on functions $f(x) = 2^{\frac{1}{2x}} - 1$ and $g(x) = \frac{1}{x}$, which both tend to 0 as $n \to \infty$:

$$\begin{aligned}\lim_{x\to\infty}\frac{2^{\frac{1}{2x}}-1}{\frac{1}{x}} &= \lim_{x\to\infty}\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(2^{\frac{1}{2x}}-1\right)}{\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x}} = \lim_{x\to\infty}\frac{-\frac{\ln 2}{x^2}2^{\frac{1}{2x}}}{-\frac{1}{x^2}} \\ &= \lim_{x\to\infty}(\ln 2)2^{\frac{1}{2x}} = \ln 2.\end{aligned}$$

And so, we obtain

$$\lim_{n\to\infty}\sqrt[n]{P_n} - n\ell = \frac{2\ln 2}{e} = \frac{\ln 4}{e}.$$

Solution 4 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA. By Stirling's approximation,

$$n! = e^{-n}n^{n+1/2}\sqrt{2\pi}\left(1+O\left(\frac{1}{n}\right)\right).$$

Thus,

$$\begin{aligned}P_n &= \prod_{k=1}^{n}(2k-1) = \frac{(2n)!}{2^n n!} \\ &= \frac{e^{-2n}(2n)^{2n+1/2}\sqrt{2\pi}\left(1+O\left(\frac{1}{n}\right)\right)}{2^n e^{-n}n^{n+1/2}\sqrt{2\pi}\left(1+O\left(\frac{1}{n}\right)\right)} \\ &= \frac{2^{n+1/2}n^n}{e^n}\left(1+O\left(\frac{1}{n}\right)\right),\end{aligned}$$

and

$$\sqrt[n]{P_n} = \frac{2n}{e}\cdot 2^{1/(2n)}\left(1+O\left(\frac{1}{n^2}\right)\right) = \frac{2n}{e}\left(1+\frac{\ln 2}{2n}+O\left(\frac{1}{n^2}\right)\right).$$

It then follows that

$$\ell = \lim_{n\to\infty} \frac{\sqrt[n]{P_n}}{n} = \lim_{n\to\infty} \frac{2}{e}\left(1 + \frac{\ln 2}{2n} + O\left(\frac{1}{n^2}\right)\right) = \frac{2}{e},$$

and

$$\lim_{n\to\infty}\left(\sqrt[n]{P_n} - n\ell\right) = \lim_{n\to\infty}\left(\frac{\ln 2}{e} + O\left(\frac{1}{n}\right)\right) = \frac{\ln 2}{e}.$$

Also solved by *Florică Anastase, "Alexandru Odobescu" High School, Lehliu-Gară, Călăraşi, Romania, and Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA.*

A–91. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be an acute, scalene triangle with orthocenter H and centroid G. Let A_1, B_1, C_1 be the feet of perpendiculars from H on the sidelines BC, CA, AB, respectively. Let G_1 be the centroid of $\triangle A_1B_1C_1$. If HG is parallel to BC, then prove that HG_1 is parallel to B_1C_1.

Solution 1 by Michel Bataille, Rouen, France. We embed the problem in the complex plane and, without loss of generality, we suppose that the circumcircle of ΔABC is the unit circle. We denote by m (lower-case letter) the affix of the point M (upper-case letter). With this notation, we have $a\overline{a} = b\overline{b} = c\overline{c} = 1$ and $g = \frac{a+b+c}{3}$, $h = a + b + c$. The condition HG parallel to BC becomes $\frac{h-g}{b-c} = \frac{\overline{h}-\overline{g}}{\overline{b}-\overline{c}}$, which by an easy calculation gives

$$ab + bc + ca = -a(a + b + c). \tag{1}$$

From the equations $z + bc\overline{z} = b + c$ and $z - bc\overline{z} = a - \frac{bc}{a}$ of BC and the altitude AH, we deduce the affix $a_1 = \frac{1}{2}\left(h - \frac{bc}{a}\right)$. With b_1 and c_1 obtained cyclically, it follows that

$$g_1 = \frac{a_1 + b_1 + c_1}{3} = \frac{h}{2} - \frac{1}{6}\left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c}\right) = \frac{h}{2} - \frac{w}{6}$$

where for convenience we set $w = \frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c}$.
Now, we show that under condition (1), HG_1 is parallel to B_1C_1, that is,

$$(h - g_1)(\overline{b_1 - c_1}) = (b_1 - c_1)(\overline{h - g_1}). \tag{2}$$

From $b_1 - c_1 = \frac{1}{2}\left(\frac{ab}{c} - \frac{ca}{b}\right)$, we deduce that $b_1 - c_1 = -a^2(\overline{b_1 - c_1})$ and (2) writes as $3h + w = -a^2(3\overline{h} + \overline{w})$. We readily obtain

$$abc(3h + w) = (ab + bc + ca)^2 + abc(a + b + c)$$

and

$$-a^2(3\overline{h} + \overline{w})abc = -a^2((a + b + c)^2 + ab + bc + ca)).$$

By subtraction we see that $abc[(3h + w) + a^2(3\overline{h} + \overline{w})]$ equals

$$(ab+bc+ca)(a+b)(a+c)+a(a+b+c)(a+b)(a+c) = 0 \quad \text{(using (1))}$$

so that $3h + w = -a^2(3\overline{h} + \overline{w})$, as desired.

Solution 2 by the proposer. Let M be the midpoint of BC and let D, E, F be the feet of perpendiculars from A_1 on the lines AB, AC, B_1C_1, respectively.

We first establish two lemmas.

Lemma 1. *The orthocenter H is the incenter of $A_1B_1C_1$.*

Proof. Since BA_1HC_1 is a cyclic quadrilateral,

$$\angle A_1C_1H = \angle A_1BH = \angle CBB_1 = 90^\circ - \gamma.$$

Since HC_1AB_1 is a cyclic quadrilateral,

$$\angle HC_1B_1 = \angle HAB_1 = \angle A_1AC = 90^\circ - \gamma.$$

Thus, $\angle A_1C_1H = \angle HC_1B_1$ or C_1H is the bisector of the angle $A_1C_1B_1$. Similarly, B_1H and A_1H bisect the angles at B_1 and A_1. Thus the point H is the incenter of the triangle $A_1B_1C_1$. □

Lemma 2. *Let D_1 and E_1 be the reflections of A_1 in AB and AC, respectively. Then, the points B_1, C_1, D_1, E_1 are collinear.*

Proof. Since cyclic quadrilaterals BA_1HC_1 and HC_1AB_1,

$$\angle BC_1A_1 = \angle BHA_1 = \angle B_1HA = \angle B_1C_1A.$$

But D_1C_1 is the reflection of A_1C_1 in AB, hence $D_1A_1 \perp AB$, $\triangle D_1C_1D \cong \triangle A_1C_1D$ and $\angle D_1C_1D = \angle A_1C_1D = \angle A_1C_1B = \angle AC_1B_1$.

$$\begin{aligned}\angle D_1C_1B_1 &= \angle D_1C_1B + \angle BC_1B_1 \\ &= \angle D_1C_1B + \angle BC_1A - \angle AC_1B_1 = 180^\circ.\end{aligned}$$

Similarly, $\angle E_1B_1C_1 = 180^\circ$ and the points B_1, C_1, D_1, E_1 are collinear. □

From $\triangle D_1C_1D \cong \triangle A_1C_1D$ and $\triangle E_1B_1E \cong \triangle A_1B_1E$ and from Lemma 2, it follows that $D_1C_1 = A_1C_1$, $E_1B_1 = A_1B_1$ or $D_1E_1 = A_1C_1 + C_1B_1 + B_1A_1$.

Let F and I be feet of perpendiculars from A_1 and H, respectively, on D_1E_1. Thus, A_1F is an altitude of $\triangle A_1B_1C_1$. From Lemma 1, HI is the inradius of triangle $\triangle A_1B_1C_1$.

The area of triangle $A_1B_1C_1$ is

$$S_{A_1B_1C_1} = \frac{1}{2}A_1F \cdot B_1C_1 = \frac{1}{2}HI \cdot D_1E_1.$$

DE is a mid-segment of the triangle $A_1D_1E_1$ so $DE = \frac{1}{2}D_1E_1$.

Since $HG \parallel BC$, $\triangle AHG \sim \triangle AA_1M$ or $AH : AA_1 = AG : AM = 2 : 3$. The homothety $\mathsf{h}(A, \frac{2}{3})$ carries $\triangle AA_1M$ to $\triangle AHG$, $\triangle AA_1D$ to $\triangle AHC_1$, $\triangle AA_1E$ to $\triangle AHB_1$, hence $\triangle AC_1B_1$ is the image of triangle ADE under the homothety h. It follows that

$$C_1B_1 = \frac{2}{3}DE = \frac{1}{3}D_1E_1$$

and

$$S_{A_1B_1C_1} = \frac{1}{2}A_1F \cdot B_1C_1 = \frac{1}{2}HI \cdot D_1E_1 = \frac{3}{2}HI \cdot B_1C_1 \text{ or } A_1F = 3HI.$$

Let J, K be the intersection points of the line C_1B_1 with A_1H and A_1G_1 respectively. Since $\triangle JIH \sim \triangle JFA_1$, $JH : JA_1 = HI : A_1F = 1 : 3$ or $A_1H : A_1J = 2 : 3 = A_1G_1 : A_1K$.

It follows $\triangle A_1HG_1$ is the image of $\triangle A_1JK$ under the homothety $(A_1, \frac{2}{3})$, so $HG_1 \parallel JK$, that is, $HG_1 \parallel B_1C_1$.

A–92. *Proposed by Óscar Rivero Salgado, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer, and consider the set of 2^{n^2} matrices whose coefficients are either 0 or 1. Let α_n be the proportion of those matrices whose determinant is an odd number. Prove that $\alpha = \lim_{n\to\infty} \alpha_n$ exists and show that

$$\frac{1}{4} < \alpha < \frac{1}{2\sqrt{e}}.$$

Solution 1 by Michel Bataille, Rouen, France. For any integer m, let $\overline{m}$ denote the class of m modulo 2, so that $\overline{m} \in \mathbb{F}_2 = \{\overline{0}, \overline{1}\}$, the field with two elements. For $A = (a_{i,j}) \in \mathcal{M}_n(\mathbb{Z})$, let $\overline{A} = (\overline{a_{i,j}}) \in \mathcal{M}_n(\mathbb{F}_2)$. Since $\det(\overline{A}) = \overline{\det(A)}$, the integer $\det(A)$ is an odd integer if and only if $\det(\overline{A}) = \overline{1}$, that is, if and only if $\overline{A}$ is invertible. Since there are $(2^n-1)(2^n-2)\cdots(2^n-2^{n-1})$ invertible matrices in $\mathcal{M}_n(\mathbb{F}_2)$, we have

$$\alpha_n = \frac{(2^n-1)(2^n-2)\cdots(2^n-2^{n-1})}{2^{n^2}} = \prod_{k=1}^{n}\left(1-\frac{1}{2^k}\right).$$

Noticing that for all $n \in \mathbb{N}$, $\alpha_n > 0$ and $\frac{\alpha_{n+1}}{\alpha_n} = 1 - \frac{1}{2^{n+1}} < 1$, the sequence (α_n) is decreasing and bounded below by 0, hence it is convergent.

The limit α being $\prod_{k=1}^{\infty}\left(1-\frac{1}{2^k}\right)$, the inequalities to be proved are equivalent to

$$-2\ln 2 < -\ln 2 + \sum_{k=2}^{\infty} \ln\left(1-\frac{1}{2^k}\right) < -\ln 2 - \frac{1}{2},$$

that is, to

$$\frac{1}{2} < \sum_{k=2}^{\infty}\left(-\ln\left(1-\frac{1}{2^k}\right)\right) < \ln 2. \tag{1}$$

Since $-\ln(1-x) > x$ for $x \in (0,1)$, we have

$$\sum_{k=2}^{\infty}\left(-\ln\left(1-\frac{1}{2^k}\right)\right) > \sum_{k=2}^{\infty}\frac{1}{2^k} = \frac{1}{2},$$

which proves the left inequality of (1).

Since

$$-\ln\left(1-\frac{1}{2^k}\right)=\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{1}{2^{kn}},$$

we obtain

$$\begin{aligned}\sum_{k=2}^{\infty}\left(-\ln\left(1-\frac{1}{2^k}\right)\right)&=\sum_{k=2}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{1}{2^{kn}}\\&=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=2}^{\infty}\frac{1}{2^{kn}}\\&=\sum_{n=1}^{\infty}\frac{1}{n}\cdot\frac{1}{2^{2n}}\frac{1}{1-\frac{1}{2^n}}\\&=\sum_{n=1}^{\infty}\frac{1}{n2^n(2^n-1)}.\end{aligned}$$

Observing that $\frac{1}{n2^n(2^n-1)}\leq\frac{1}{n2^n}$ with equality only for $n=1$, we deduce that

$$\sum_{k=2}^{\infty}\left(-\ln\left(1-\frac{1}{2^k}\right)\right)<\sum_{n=1}^{\infty}\frac{1}{n2^n}=-\ln\left(1-\frac{1}{2}\right)=\ln 2,$$

so that the right inequality of (1) is now proved.

Solution 2 by the proposer. The condition of having odd determinant means that the matrix is invertible over the field of 2 elements. We claim that, for a fixed n, the number of matrices with odd determinant is

$$a_n=\prod_{i=0}^{n-1}(2^n-2^i).$$

Indeed, the condition of non-zero determinant means that all the rows are linearly independent, so once we have fixed i rows there are 2^n-2^i options for the $(i+1)$-th row, since we must exclude the 2^i linear combinations obtained by considering the first i rows. Note that all these linear combinations are distinct by the condition of being linearly independent. From the expression for a_n, we have that

$$a_{n+1}=(2^{n+1}-1)2^n a_n.$$

Then,

$$\alpha_{n+1} = (1 - 2^{-n-1})\alpha_n,$$

with $\alpha_1 = 1/2$. In other words,

$$\alpha_n = \prod_{i=1}^{n} (1 - 2^{-i}).$$

Now, the convergence of the sequence α_n follows from the convergence of the sum $\sum_{i=1}^{\infty} 2^{-i}$. To compute its value, we work instead with $-\log(\alpha_n)$, since it admits the simpler expression

$$-\log(\alpha_n) = \sum_{i=1}^{n} \sum_{j=1}^{\infty} \frac{1}{j \cdot 2^{ij}} = \sum_{j=1}^{\infty} \frac{1}{j} \sum_{i=1}^{n} \frac{1}{(2^j)^i}.$$

In particular,

$$-\log(\alpha) = \sum_{j=1}^{\infty} \frac{1}{j(2^j - 1)},$$

where the exchange of the order of summation is now possible since the sequence (α_n) converges (alternatively, in the sum all the terms are positive). From here,

$$\frac{1}{2} + \log(2) = 1 + \sum_{j=2}^{\infty} \frac{1}{j \cdot 2^j} < -\log(\alpha) < 1 + 2 \sum_{j=2}^{\infty} \frac{1}{j \cdot 2^j} = 2\log(2).$$

Taking exponentials in the previous expressions, we obtain

$$\frac{1}{4} < \alpha < \frac{1}{2\sqrt{e}},$$

as claimed.

Remark. With a computer, we see that a numerical approximation of this value is 0.2888, which is of course between 0.25 and 0.3033.

Also solved by *Moti Levy, Rehovot, Israel.*

A–93. *Proposed by José Pérez Cano, CFIS, BarcelonaTech, Barcelona, Spain.* Let $p > 2$ be a prime number such that $p \equiv 2 \pmod 3$ and let a, b, c, d be positive integers. If p divides $ac - 3bd$ and $ad + bc$, prove that either p divides a and b or p divides c and d.

Solution 1 by the proposer. We first prove that either p divides $a^2 + 3b^2$ or p divides $c^2 + 3d^2$.

Proof. The hypothesis is equivalent to saying that

$$\begin{aligned} ac - 3bd &= pr, \\ ad + bc &= ps, \end{aligned}$$

for some integers r, s. Hence,

$$\begin{aligned} p^2(r^2 + 3s^2) &= (pr)^2 + 3(ps)^2 \\ &= (ac - 3bd)^2 + 3(ad + bc)^2 \\ &= a^2c^2 + 9b^2d^2 - 6abcd + 3a^2d^2 + 3b^2c^2 + 6abcd \\ &= a^2c^2 + 9b^2d^2 + 3a^2d^2 + 3b^2c^2 \\ &= (a^2 + 3b^2)(c^2 + 3d^2). \end{aligned}$$

Since p is prime and divides $(a^2 + 3b^2)(c^2 + 3d^2)$ it either divides $a^2 + 3b^2$ or $c^2 + 3d^2$. □

We now know that $p \mid a^2 + 3b^2$ or $p \mid c^2 + 3d^2$. WLOG, assume $p \mid a^2 + 3b^2$. Then, $a^2 \equiv -3b^2 \pmod p$. If $a \not\equiv 0$ or samely $b \not\equiv 0$, then we got -3 is a quadratic residue modulo p. But that is impossible because

$$\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}}\left(\frac{p}{3}\right)(-1)^{\frac{p-1}{2}} = -1,$$

which is true if p is odd, because of quadratic reciprocity and $p \equiv 2 \pmod 3$ is not a quadratic residue. Therefore, -3 is not a quadratic residue modulo p and we have p divides a and b as we wanted.

Solution 2 by Michel Bataille, Rouen, France. Let $u \in \mathbb{Z}$ and $v \in \mathbb{N}$ be such that $ac - 3bd = up$ and $ad + bc = vp$. Solving for c and d gives

$$c(a^2 + 3b^2) = p(au + 3bv), \qquad d(a^2 + 3b^2) = p(av - ub).$$

It follows that if p does not divide a^2+3b^2, then p divides both c and d. Otherwise, p divides a^2+3b^2. We show below that, in addition, p divides b. Therefore, it also divides a^2, hence a, so that p divides both a and b.

With the purpose of a contradiction, assume that p does not divide b. Then, b and p are coprime, hence there exists an integer e such that $be \equiv 1 \pmod{p}$. It follows that $e^2(a^2+3b^2) \equiv (ae)^2+3 \pmod{p}$ and therefore -3 is a square modulo p. However, using the Legendre symbol and the law of quadratic reciprocity, we have

$$\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = (-1)^{\frac{p-1}{2}} \cdot (-1)^{\frac{p-1}{2}\cdot\frac{3-1}{2}}\left(\frac{p}{3}\right) = \left(\frac{p}{3}\right) = -1,$$

where the last equality follows from $p \equiv 2 \pmod{3}$ and the fact that 2 is not a square modulo 3. Thus, -3 is not a square modulo p and we have reached the desired contradiction. We can now conclude that either p divides c and d or p divides a and b.

A–94. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let f be a real function having second derivative in the interval $[x_1, x_2]$ and $f'(x_1) = f'(x_2) = 0$. Prove that there exists at least one point $c \in (x_1, x_2)$ such that

$$\frac{1}{4}\,|f''(c)| \geq \frac{|f(x_2) - f(x_1)|}{(x_2 - x_1)^2}.$$

Solution 1 by the proposer. If f is constant, the claim trivially holds. Moreover, since $f'(x_1) = f'(x_2) = 0$, then f is not linear. Applying Cauchy's theorem to the functions $f(x)$ and $g(x) = \dfrac{(x-x_1)^2}{2}$ in the interval $\left[x_1, \dfrac{x_1+x_2}{2}\right]$, we get

$$\frac{8\left(f\left(\dfrac{x_1+x_2}{2}\right) - f(x_1)\right)}{(x_2-x_1)^2} = \frac{f'(c_1)}{c_1 - x_1}, \quad x_1 < c_1 < \frac{x_1+x_2}{2}.$$

Likewise, applying Cauchy's theorem to the functions $f(x)$ and

$h(x) = \dfrac{(x_2 - x)^2}{2}$ in the interval $\left[\dfrac{x_1 + x_2}{2}, x_2\right]$, we obtain

$$\frac{8\left(f(x_2) - f\left(\dfrac{x_1 + x_2}{2}\right)\right)}{(x_2 - x_1)^2} = \frac{f'(c_2)}{x_2 - c_2}, \quad \frac{x_1 + x_2}{2} < c_2 < x_2.$$

Adding up the above expressions yields

$$\frac{8(f(x_2) - f(x_1))}{(x_2 - x_1)^2} = \frac{f'(c_1)}{c_1 - x_1} + \frac{f'(c_2)}{x_2 - c_2}.$$

On account that $f'(x_1) = f'(x_2) = 0$, we have that

$$\begin{aligned}\frac{f'(c_1)}{c_1 - x_1} + \frac{f'(c_2)}{x_2 - c_2} &= \frac{f'(c_1) - f'(x_1)}{c_1 - x_1} - \frac{f'(x_2) - f'(c_2)}{x_2 - c_2} \\ &= f''(d_1) + f''(d_2),\end{aligned}$$

where $x_1 < d_1 < c_1$ and $c_2 < d_2 < x_2$. Now, it holds that

$$\left|\frac{8(f(x_2) - f(x_1))}{(x_2 - x_1)^2}\right| = \left|\frac{f'(c_1)}{c_1 - x_1} + \frac{f'(c_2)}{x_2 - c_2}\right| \leq |f''(d_1)| + |f''(d_2)|.$$

Since $f(x_1) \neq f(x_2)$, then at least one of the numbers $|f''(d_1)|$ or $|f''(d_2)|$ is nonzero. Let

$$|f''(c)| = \max\{|f''(d_1)|, |f''(d_2)|\};$$

then

$$\frac{8|f(x_2) - f(x_1)|}{(x_2 - x_1)^2} \leq 2\,|f''(c)|$$

and the statement follows. If $|f''(d_1)| = |f''(d_2)|$ then equality holds.

Solution 2 by Ander Lamaison Vidarte, Brno, Czech Republic. Let $m = \frac{x_1 + x_2}{2}$. We claim that the following equality holds:

$$f(x_2) - f(x_1) = \int_{x_1}^{x_2} (m - x) f''(x)\,\mathrm{d}x. \tag{1}$$

Indeed, given $f'(x_1) = 0$, we have

$$\begin{aligned} f(x_2) - f(x_1) &= \int_{x_1}^{x_2} f'(y)\,\mathrm{d}y \\ &= \int_{x_1}^{x_2} \int_{x_1}^{y} f''(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{x_1}^{x_2} \int_{x}^{x_2} f''(x)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{x_1}^{x_2} (x_2 - x) f''(x)\,\mathrm{d}x. \end{aligned}$$

From the previous expression we substract the following:

$$0 = \frac{x_2 - x_1}{2}(f'(x_2) - f'(x_1)) = \int_{x_1}^{x_2} \frac{x_2 - x_1}{2} f''(x)\,\mathrm{d}x,$$

which produces (1). Now assume for contradiction that the statement is false, and apply the triangle inequality to (1):

$$\begin{aligned} |f(x_2) - f(x_1)| &\leq \int_{x_1}^{x_2} |m - x||f''(x)|\,\mathrm{d}x \\ &< \frac{|f(x_2) - f(x_1)|}{\frac{(x_2 - x_1)^2}{4}} \int_{x_1}^{x_2} |m - x|\,\mathrm{d}x \\ &= |f(x_2) - f(x_1)|, \end{aligned}$$

which is a contradiction. Hence the statement holds.

Solution 3 by Víctor Martín Chabrera, Barcelona, Spain. Let us define $K = 4\frac{|f(x_2) - f(x_1)|}{(x_2 - x_1)^2}$. We want to show that there is a point $c \in [x_1, x_2]$ such that $|f''(c)| \geq K$.

Assume there is no such point. Then,

$$f'(x) = f'(x_1) + \int_{x_1}^{x} f''(t)\,\mathrm{d}t = \int_{x_1}^{x} f''(t)\,\mathrm{d}t < K(x - x_1)$$

and, similarly,

$$f'(x) = f'(x_2) - \int_{x}^{x_2} f''(t)\,\mathrm{d}t = -\int_{x}^{x_2} f''(t)\,\mathrm{d}t < K(x_2 - x).$$

So

$$\begin{aligned}
|f(x_2) - f(x_1)| &= \left| \int_{x_1}^{x_2} f'(t)\,\mathrm{d}t \right| \\
&= \left| \int_{x_1}^{\frac{x_1+x_2}{2}} f'(t)\,\mathrm{d}t + \int_{\frac{x_1+x_2}{2}}^{x_2} f'(t)\,\mathrm{d}t \right| \\
&< \left| \int_{x_1}^{\frac{x_1+x_2}{2}} K(x - x_1)\,\mathrm{d}t + \int_{\frac{x_1+x_2}{2}}^{x_2} K(x_2 - x)\,\mathrm{d}t \right| \\
&= K \left| \frac{\left(\frac{x_2-x_1}{2}\right)^2}{2} + \frac{\left(\frac{x_2-x_1}{2}\right)^2}{2} \right| \\
&= K \left(\frac{x_2 - x_1}{2} \right)^2 \\
&= 4 \frac{|f(x_2) - f(x_1)|}{(x_2 - x_1)^2} \frac{(x_2 - x_1)^2}{4} \\
&= |f(x_2) - f(x_1)|,
\end{aligned}$$

leading to a contradiction and completing the proof.

Remark 1. *We can see that the bound is sharp. That is, if we changed the $\geq$ of the statement for a $>$, it would not remain true. The counterexample would come from setting*

$$f''(x) = \begin{cases} 4\frac{f(x_2)-f(x_1)}{(x_2-x_1)^2} & \text{if } x_1 \leq x \leq \frac{x_1+x_2}{2}, \\ -4\frac{f(x_2)-f(x_1)}{(x_2-x_1)^2} & \text{if } \frac{x_1+x_2}{2} < x \leq x_2. \end{cases}$$

Also solved by *Michel Bataille, Rouen, France; Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA, and Moti Levy, Rehovot, Israel.*

MEA CULPA

By mistake, I inadvertently did not credit **Philippe Fondanaiche, Paris, France,** with sending solutions to the following problems: E-83, E-84, E-86, E-88, MH-83, MH-85, MH-87, MH-88. I apologize if this has caused any inconvenience to the author.

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