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Solving quadratic congruences

Pascual Jara and Miguel L. Rodríguez

Abstract

This paper deals with the problem of determining all solutions, in positive integers, of the congruence systems

$$\left.\begin{array}{ll} X^2 \equiv 1 & (\mathrm{mod}\ Y) \\ Y^2 \equiv 1 & (\mathrm{mod}\ X) \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{ll} X^2 \equiv 1 & (\mathrm{mod}\ Y) \\ Y^2 \equiv 1 & (\mathrm{mod}\ Z) \\ Z^2 \equiv 1 & (\mathrm{mod}\ X) \end{array}\right\}$$

For this purpose, we firstly solve the system of equations in two variables. This system and its solutions are well known; the aim of this work is to point out that these solutions have an inner structure in the sense that each solution is uniquely given by consecutive elements of a particular kind of Lucas sequence. Secondly and as for cases of system of equations in three variables, particular families of solutions are shown identifying ways to define triplets (x, y, z) presented as solutions of the system of equations. The inner structure of the set of all solutions of the three-variable quadratic system is also a wide problem to study.

1 Introduction

This work aims at studying the solution, in integer numbers, of the system of congruences featured in (1).

$$\left.\begin{array}{ll} X^2 \equiv 1 & (\mathrm{mod}\ Y) \\ Y^2 \equiv 1 & (\mathrm{mod}\ X) \end{array}\right\}. \tag{1}$$

Firstly, we observe that if $a \geq 1$, then $a^2 \equiv 1 \pmod{a+1}$ and $(a+1)^2 \equiv 1 \pmod{a}$ hold. So, the pairs $(a, a+1)$ and $(1, a)$ are solutions of (1). Besides these *trivial* solutions, we find that (a, a^2-1) is another solution of (1) for $a \geq 2$. We call a solution of the form (a, a^2-1) an *almost trivial* solution. Secondly, we build sequences $\{a_n\}_{n\in\mathbb{N}}$ in which (a_n, a_{n+1}) is a solution of (1). In these examples, the pairs (a_n, a_{n+1}) satisfy, for any $n \in \mathbb{N}$, the relationship

$$a_n a_{n+2} = a_{n+1}^2 - 1. \tag{2}$$

A next step involves the study of these sequences, particularly those that are maximal, and this leads us to consider a particular type of Lucas sequence. In this way, we obtain an array where each column stands as a Lucas sequence, and each row represents a new sequence. This distribution of elements guarantees that each solution for any particular modification of the original problem is one of the pairs of consecutive elements in a column displayed in this array.

The study likewise involves defining solutions for the system of three quadratic congruences

$$\left.\begin{array}{ll} X^2 \equiv 1 & \pmod{Y} \\ Y^2 \equiv 1 & \pmod{Z} \\ Z^2 \equiv 1 & \pmod{X} \end{array}\right\}. \tag{3}$$

In this respect, we find that for each positive integer $x > 2$, there exists a solution (x, y, z) of (3).

The paper is organized as follows.

Section 2 and Section 3 both define the problem of a quadratic congruence system. The analysis allows us to embed each solution in a particular sequence and at the same time shows that each solution appears as consecutive terms in one of these sequences.

In Section 4 we build an array of positive integers. Here, each column are the built sequences that stands as a particular class of Lucas sequence while at the same time characterizing the rows of this array as recurrent sequences.

Finally, in Section 5 we deal with the problem of a quadratic congruence system with three variables. The analysis offers several families of solutions (x, y, z), with $x < y, z$, and proves that for any positive integer number x greater than 2 there exists a solution (x, y, z).

We refer to the works of Hardy and Wright [2] and Niven and Zuckerman [3] as general textbooks on arithmetics of integers.

2 Quadratic congruences in two variables

Let us consider the following problem in order to introduce cases of quadratic congruences:

Problem 1. *Determine all pairs of positive integers,* (x, y)*, which are a solution of* (1)*.*

In theses cases, let P denote the set of all the solutions of (1). It is easy to check that $(x, y) \in P$ if, and only if, $(y, x) \in P$. Furthermore, x and y must be coprime if both of them are greater or equal than 1.

From now on, we only thus study pairs $(x, y) \in P$ satisfying $x < y$. It is therefore possible to derive elements of P by identifying the pairs belonging to it and then, by inverting its components, we obtain all the elements of P.

Besides the trivial solution $(1, 1)$, we notice that for every positive integer number x, the pair $(x, x + 1)$ belongs to P since

$$x^2 - 1 = (x + 1)(x - 1), \text{ hence } x^2 \equiv 1 \pmod{(x + 1)},$$

and

$$(x + 1)^2 = x^2 + 2x + 1, \text{ hence } (x + 1)^2 \equiv 1 \pmod{x}.$$

Pairs of the form $(x, x + 1)$, and $(1, x)$, are called *trivial pairs*. In particular, we have proved that P is a nonempty set.

On the other hand, a sequence $\{x_n\}_{n\in\mathbb{N}}$ in which $x_{n+1} = x_n + 1$ is called a *trivial sequence*. Therefore, every trivial sequence is determined by its first term and the largest trivial sequence always starts with $x_0 = 1$.

We notice that for every $x \in \mathbb{N}\setminus\{0,1\}$, we have that $(x, x^2-1) \in \boldsymbol{P}$. This is derived from the relations

$$x^2 - 1 \equiv 0 \pmod{(x^2-1)}$$

and

$$(x^2-1)^2 - 1 = x^4 - 2x^2 = x(x^3-2) \equiv 0 \pmod{x}.$$

Cases of a pair of the form (x, x^2-1) are thus named an *almost trivial pair*. Consequently, a sequence $\{x_n\}_{n\in\mathbb{N}}$, defined by $x_{n+1} = x_n^2 - 1$ for every $n \in \mathbb{N}$, and $x_0 \geq 2$, is sheer referred to as an *almost trivial sequence*. It is then the case that a unique almost trivial sequence with initial term x results if $x \in \mathbb{N} \setminus \{0,1\}$.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called a $\boldsymbol{P}$*-sequence* if $x_n < x_{n+1}$ and $(x_n, x_{n+1}) \in \boldsymbol{P}$. We can observe that not all $\boldsymbol{P}$-sequences are either trivial or almost trivial. Indeed, it would it possible to build $\boldsymbol{P}$-sequences alternating trivial and almost trivial terms. We also can obtain another class of sequences which are not strictly increasing. For example, we can remove the condition $x_n < x_{n+1}$ while maintaining the condition that each two consecutive terms form a pair in $\boldsymbol{P}$.

An interesting property that every trivial sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfies is that, for every $n \in \mathbb{N}$, the triplet (x_n, x_{n+1}, x_{n+2}) verifies

$$x_n x_{n+2} = x_{n+1}^2 - 1. \tag{4}$$

In contrast, it is evident that almost trivial sequences, which also are $\boldsymbol{P}$-sequences, do not satisfy (4).

We introduce a new sequence in the following definition.

Definition 1. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a P-sequence. We say that it is a PJ-*sequence *if it satisfies the following:*

1. $x_n < x_{n+1}$, *for every* $n \in \mathbb{N}$,
2. $(x_n, x_{n+1}) \in P$, *for every* $n \in \mathbb{N}$, *and*
3. (x_n, x_{n+1}, x_{n+2}) *satisfies* $x_n x_{n+2} = x_{n+1}^2 - 1$, *for every* $n \in \mathbb{N}$.

The following result proves the uniqueness of maximal PJ-sequences, where maximality is defined in the obvious way.

Proposition 1. *Let (x, y, z) be a triplet of positive integers satisfying*

A.1 $x < y < z$,
A.2 $(x, y), (y, z) \in P$,
A.3 $xz = y^2 - 1$.

There exists a unique maximal PJ-sequence $\{x_n\}_{n\in\mathbb{N}}$ and $m \in \mathbb{N}$ such that $x = x_m$, $y = x_{m+1}$ and $z = x_{m+2}$. This means that x, y and z are consecutive terms in this PJ-sequence. Furthermore $x_0 = 1$, $x_2 = x_1^2 - 1$, and so forth.

Proof. Let x, y, z be positive integers verifying the conditions of Proposition 1. We define $y_0 = x$, $y_1 = y$, and $y_2 = z$, and we set $u = \frac{x^2-1}{y}$. As $x < y$ we have that

$$u = \frac{x^2 - 1}{y} < \frac{x^2 - 1}{x} < x - \frac{1}{x} < x.$$

On the other hand, since $y^2 \equiv 1 \pmod{x}$, there exists $h \in \mathbb{N}$ such that $y^2 - 1 = xh$. Hence we deduce that

$$u^2 - 1 = \left(\frac{x^2 - 1}{y}\right)^2 - 1 = \frac{x^4 - 2x^2 + 1 - y^2}{y^2} = \frac{x(x^3 - 2x - h)}{y^2}.$$

Now, as x and y are coprime, then $u^2 - 1 \equiv 0 \pmod{x}$. In consequence, the triplet (u, x, y) satisfies the same properties that the initial triplet, and $u < x$. We define $y_{-1} = u$.

Following this process, we obtain the triplet $(y_{-m}, y_{-m+1}, y_{-m+2})$. By doing this, there will exist m_0 such that we are not able to

obtain y_{-m_0-1} in order to get a new triplet. This happens when $\frac{y_{-m_0}^2-1}{y_{-m_0+1}} = 1$, i. e., if $y_{-m_0+1} = y_{-m_0}^2 - 1$. Now, we define $x_1 = y_{-m_0}$, $x_2 = y_{-m_0+1}$, and in general $x_j = y_{j-m_0-1}$, for every $1 \leq j \leq m_0 + 3$.

In addition, we can extend this sequence by defining $x_j = \frac{x_{j-1}^2-1}{x_{j-2}}$, for every $j > m + 3$. With this construction, and $x_0 = 1$, it is clear that $\{x_n\}_{n\in\mathbb{N}}$ is the only PJ-sequence containing the triplet (x, y, z) as consecutive terms. This proves the result. □

The above construction can also be built whenever we start from a pair $(x, y) \in P$ instead of a triplet.

Theorem 1. *Let $(x, y) \in P$, there exists a unique maximal PJ-sequence, say $\{x_n\}_{n\in\mathbb{N}}$, such that x and y are consecutive terms. Furthermore, in this case $x_1 = x_0^2 - 1$.*

Proof. We define $u = \frac{y^2-1}{x}$. Now, by the same argument developed in Proposition 1 we obtain a triplet (x, y, u) verifying conditions A.1–A.3 in Proposition 1. Hence the result holds. □

We notice that a PJ-sequence is the trivial sequence whenever $y = x + 1$. Otherwise, if $y = x^2 - 1$, there is only one maximal PJ-sequence $\{x_n\}_{n\in\mathbb{N}}$ satisfying $x_1 = x$ and $x_2 = y$.

3 Extended PJ-sequences

This section is devoted to studying the following problem: given a pair $(x, y) \in P$, determine the unique maximal PJ-sequence with consecutive terms x and y.

Our aim is to define new classes of sequences associated to Problem 1. For instance, for each positive integer n, we have $(1, n) \in P$, and each trivial sequence and each P-sequence can be extended by setting 1 as the first term. In consequence, all maximal sequences will start from 1.

Applying Theorem 1 we have one of the following possibilities for every maximal PJ-sequence $\{x_n\}_{n\in\mathbb{N}}$:

1. $x_0 = 1$ and $x_1 = 2$. In this case we obtain a trivial maximal PJ-sequence.
2. $x_0 = 1$ and $x_1 \neq 2$. In this case we obtain a maximal PJ-sequence.
3. $x_0 \neq 1$. Since $x_0x_2 = x_1^2 - 1$, we may extend this sequence $\{x_n\}_{n\in\mathbb{N}}$ to obtain a maximal PJ-sequence $\{y_n\}_{n\in\mathbb{N}}$, being $y_0 = 1$, and there exists $h \in \mathbb{N}$ such that $y_{h+m} = x_m$ for any $m \in \mathbb{N}$.

In consequence, maximal PJ-sequences are parameterized by an element $t \in \mathbb{N} \setminus \{0, 1\}$. So, a unique maximal PJ-sequence will be given by

$$x_0 = 1, \quad x_1 = t, \quad x_2 = t^2 - 1, \quad x_3 = \frac{(t^2-1)^2 - 1}{t}, \ \ldots$$

Indeed, maximal PJ-sequences are a special type of recurrence defined sequences.

Theorem 2. *Let $\{x_n\}_{n\in\mathbb{N}}$ be a maximal PJ-sequence. If we call $t = x_1$, the sequence satisfies the recurrence relation $x_{m+2} = tx_{m+1} - x_m$, with $x_0 = 1$ and $x_1 = t$.*

Proof. For any $m \in \mathbb{N}$, let us suppose that $x_{n+2} = tx_{n+1} - x_n$, for $0 \leq n < m$. Then,

$$\begin{aligned} x_{m+2}x_m &= x_{m+1}^2 - 1 = (tx_m - x_{m-1})^2 - 1 \\ &= t^2x_m^2 - 2tx_mx_{m-1} + x_{m-1}^2 - 1 \\ &= t^2x_m^2 - 2tx_mx_{m-1} + x_mx_{m-2} \\ &= t(tx_m - x_{m-1})x_m - (tx_{m-1} - x_{m-2})x_m \\ &= (tx_{m+1} - x_m)x_m. \end{aligned}$$

Therefore, $x_{m+2} = tx_{m+1} - x_m$ and we have proved the result. □

Thus the problem of determining a maximal PJ-sequence $\{x_n\}_{n\in\mathbb{N}}$ containing a pair $(x, y) \in P$ is reduced to computing the value $t = x_1$. In this context we have the following result.

Theorem 3. *Let x, y be positive integers such that $1 \leq x < y$. Then, the following statements hold:*

1. $a = \dfrac{x^2 + y^2 - 1}{xy} \in \mathbb{N}$ *if, and only if,* $(x, y) \in \boldsymbol{P}$.
2. *If $(x, y) \in \boldsymbol{P}$, then x and y are consecutive terms in the maximal $\boldsymbol{PJ}$-sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by*

$$x_0 = 1, \quad x_1 = a, \quad x_{n+2} = ax_{n+1} - x_n,\ n \in \mathbb{N}.$$

Proof. The result follows immediately if (x, y) is a trivial pair. In this case $y = x + 1$. Hence $a = 2$, and $\boldsymbol{PJ}$-sequence is defined by $y_0 = 1$, $y_1 = 2$ and $y_{n+2} = 2y_{n+1} - y_n = y_{n+1} + 1$. Otherwise, let us assume that (x, y) is not a trivial pair. Then x, y are consecutive terms in a maximal $\boldsymbol{PJ}$-sequence, defined by $x_0 = 1$, $x_1 = b$, $x_{n+2} = bx_{n+1} - x_n$, $n \in \mathbb{N}$. The value of b can be computed as

$$b = \frac{x_{n+2} + x_n}{x_{n+1}} = \frac{x_{n+2} + \frac{x_{n+1}^2 - 1}{x_{n+2}}}{x_{n+1}} = \frac{x_{n+2}^2 + x_{n+1}^2 - 1}{x_{n+2}x_{n+1}}.$$

Therefore, we deduce that $a = b$ and the proof is completed. □

4 Lucas sequences

We have established that for any $t \in \mathbb{N} \setminus \{0, 1\}$ there exists a maximal $\boldsymbol{PJ}$-sequence, denoted by s^t in Table 1. Our aim is to relate all s^t sequences. In this way, every solution to Eq. (1) will be given by two consecutive terms in one column, excluding terms with values 0 and -1.

A table for all maximal PJ-sequences

Let us denote by s^t the maximal $\boldsymbol{PJ}$-sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $x_1 = t$, for every $t \in \mathbb{N} \setminus \{0, 1\}$. Thus $\{s^t\}_{n\in\mathbb{N}}$ is given by

$$s_0^t = 1, \quad s_1^t = t, \quad s_{n+2}^t = ts_{n+1}^t - s_n^t \text{ for every } n \in \mathbb{N}.$$

We may extend the definition of s^t to the case $t = 1$ in this way:

$$s_0^1 = 1, \quad s_1^1 = 1, \quad s_{n+2}^1 = ts_{n+1}^1 - s_n^1 \text{ for every } n \in \mathbb{N}.$$

Therefore, we have the family of sequences $\{s^t\}_{n\in\mathbb{N}}$ which are the previously studied maximal PJ-sequences whenever $t \geq 2$. The generatrix function of s^t is given by $\dfrac{1}{x^2 - tx + 1}$.

As a consequence, $\{s^t\}$ is the Lucas sequence $\{U(t, 1)\}$. So,

$$s_n^t = U(t, 1)(n + 1) = U_{n+1}(t, 1), \quad \forall n \in \mathbb{N}.$$

In Table 1 we can observe that the t-th column contains the ordered terms of the sequence $\{s^t\}$.

	s^1	s^2	s^3	s^4	s^5	s^6	s^7	s^8	s^9	s^{10}
u^0	1	1	1	1	1	1	1	1	1	1
u^1	1	2	3	4	5	6	7	8	9	10
u^2	0	3	8	15	24	35	48	63	80	99
u^3	−1	4	21	56	115	204	329	496	711	980
u^4	−1	5	55	209	551	1189	2255	3905	6319	9701
u^5	0	6	144	780	2640	6930	15456	30744	56160	96030
u^6	1	7	377	2911	12649	40391	105937	242047	499121	950599
u^7	1	8	987	10864	60605	235416	726103	1905632	4435929	9409960
u^8	0	9	2584	40545	290376	1372105	4976784	15003009	39424240	93149001
u^9	−1	10	6765	151316	1391275	7997214	34111385	118118440	350382231	922080050
u^{10}	−1	11	17711	564719	6665999	46611179	233802911	929944511	3114015839	9127651499

Table 1: Sequences in rows and columns

The construction of this table and Theorem 3 allow us to develop an algorithm to compute in which column appears each solution $(x, y) \in P$ with $x < y$.

Theorem 4. *The elements x and y of a pair $(x, y) \in P$, with $x < y$, appear exactly, as consecutive terms, in only one column, and Theorem 3 indicates in which column it appears.*

Example 1. *Let $(x, y) \in P$. Find t such that x and y are consecutive terms of $\{s_n^t \mid n \in \mathbb{N}\}$, and determine n such that $s_n^t = x$ for $x = 39088169$ and $y = 102334155$.*

Solution. We refer to Theorem 3 to compute the value of t. As $t = \dfrac{x^2+y^2-1}{xy}$, we deduce that $t = \dfrac{39088169^2 + 102334155^2 - 1}{39088169 \times 102334155} = 3$. Moreover, since the sequence $\{s^t\}$ is defined by recurrence, it is simple to determine the value of n such that $s^t_n = x$. Finally, we obtain that $s^3_{17} = 14930352$ and $s^3_{18} = 39088169$. □

Following [1], we may restrict to consider prime integers cand we get the following results. Let (x, y) be a solution, and we assume $x < y$.

1. If x and y are prime, then $y \mid x^2 - 1 = (x+1)(x-1)$, hence $y = x + 1$. Thus $x = 2$ and $y = 3$ is the only solution.
2. If only y is prime, then x could be 1, and $(1, y)$ belongs to the sequence s^y. If $x \neq 1$, since $y = x + 1$, then (x, y) belongs to the sequence $\{s^2\}$.
3. If only x is prime, then $x \mid y^2 - 1 = (y+1)(y-1)$. If $x \mid y+1$, there exists $k \in \mathbb{Z}$ such that $y + 1 = xk$, hence $y = xk - 1$, and it belongs to the sequence $\{s^t\}$, being $t = \dfrac{x^2+y^2-1}{xy} = \dfrac{x^2 + x^2k^2 - 2xk}{x(xk-1)} = \dfrac{xk^2 + x - 2k}{xk-1} = k + \dfrac{x-k}{xk-1}$, hence $x = k$, and $y = x^2 - 1$. Therefore, (x, y) belongs to the sequence $\{s^x\}$. If $x \mid y - 1$, there exists $k \in \mathbb{Z}$ such that $y - 1 = xk$, hence $y = xk + 1$, and it belongs to the sequence s^t, being $t = \dfrac{x^2+y^2-1}{xy} = \dfrac{x^2 + x^2k^2 + 2xk}{x(xk+1)} = \dfrac{x + xk^2 + 2k}{xk+1} = k + \dfrac{x+k}{xk+1}$, which is a contradiction, because it is never an integer.

Each row in Table 1 forms a new sequence. We denote the sequence of the i-th row as $\{u^i\}$. Some of them are well known sequences:

1. The sequence u^1 is defined by $u^1_n = n$, for every $n \in \mathbb{N} \setminus \{0\}$. It corresponds to the sequence A000027. This is a recurrent sequence, with recurrence defined by $(2, -1)$, and initial terms $1, 2$.

2. The sequence u^2 is defined by $u_n^2 = (n-1)^2+2(n-1) = n^2-1$, for every $n \in \mathbb{N} \setminus \{0\}$. It is the sequence A005563. This is a recurrent sequence, with recurrence defined by $(3, -3, 1)$, and initial terms $0, 3, 8$.
3. The sequence u^3 is defined by $u_n^3 = n^3 - 2n$, for every $n \in \mathbb{N}\setminus\{0\}$. It is the sequence A242135, it is a recurrent sequence, with recurrence defined by $(4, -6, 4, -1)$, and initial terms $-1, 4, 21, 56$.
4. The sequence u^4 is defined by $u_n^4 = n^4 - 3n^2 + 1$, for every $n \in \mathbb{N}\setminus\{0\}$. It is a recurrent sequence, with recurrence defined by $(5, -10, 10, -5, 1)$, and initial terms $-1, 5, 55, 209, 551$.
5. The sequence u^5 is defined by $u_n^5 = n^5 - 4n^3 + 3n$, for every $n \in \mathbb{N} \setminus \{0\}$. This is a recurrent sequence, with recurrence defined by the sequence $(6, -15, 20, -15, 6, -1)$, and initial terms $0, 6, 144, 780, 2640, 6930$.

We can prove that, in general, the sequence u^h is defined by the recurrence

$$u^h \to \left(\binom{h+1}{1}, -\binom{h+1}{2}, \ldots, (-1)^{j-1}\binom{h+1}{j}, \ldots, (-1)^h\right),$$

with initial values

$$s_h^1, s_h^2, \ldots, s_h^{h+1}.$$

This means that we have the following result.

Theorem 5. *The sequences u^h are defined by the recurrence relation*

$$\sum_{i=0}^{h+1}(-1)^i\binom{h+1}{i}u_{n+i}^h = 0, \text{ for every } n \geq 1.$$

Proof. We do induction on h. If either $h = 0$ or $h = 1$, the result holds. Consider that this formula holds for any index less or equal

than h. We start from the expression with $h+1 \geq 2$.

$$\begin{aligned}
&\sum_{i=0}^{h+1}(-1)^i\binom{h+1}{i}u_{n+i}^h \\
&= u_n^h + \sum_{i=1}^{h}(-1)^i\left[\binom{h}{i} + \binom{h}{i-1}\right]u_{n+i}^h + (-1)^{h+1}u_{n+h+1}^h \\
&= u_n^h + \sum_{i=1}^{h}(-1)^i\binom{h}{i}u_{n+i}^h + \sum_{i=1}^{h}(-1)^i\binom{h}{i-1}u_{n+i}^h + (-1)^{h+1}u_{n+h+1}^h \\
&= \sum_{i=0}^{h}(-1)^i\binom{h}{i}u_{n+i}^h + \sum_{i=1}^{h+1}(-1)^i\binom{h}{i-1}u_{n+i}^h \\
&= \sum_{i=0}^{h}(-1)^i\binom{h}{i}u_{n+i}^h - \sum_{i=0}^{h}(-1)^i\binom{h}{i}u_{n+i+1}^h.
\end{aligned}$$

By using the recurrence $u_{n+i}^h = (n+i)u_{n+i}^{h-1} - u_{n+i}^{h-2}$, we obtain that the previous equals

$$\begin{aligned}
&\sum_{i=0}^{h}(-1)^i\binom{h}{i}(n+i)u_{n+i}^{h-1} - \sum_{i=0}^{h}(-1)^i\binom{h}{i}u_{n+i}^{h-2} \\
&- \sum_{i=0}^{h}(-1)^i\binom{h}{i}(n+i+1)u_{n+i+1}^{h-1} + \sum_{i=0}^{h}(-1)^i\binom{h}{i}u_{n+i+1}^{h-2}.
\end{aligned}$$

On the other hand, by means of the induction hypothesis, and after making use of the former reasoning, we have

$$\sum_{i=0}^{h}(-1)^i\binom{h}{i}u_{n+i}^{h-2} = 0 = \sum_{i=0}^{h}(-1)^i\binom{h}{i}u_{n+i+1}^{h-2}.$$

Therefore, we can continue as follows:

$$\begin{aligned}
&\sum_{i=0}^{h+1}(-1)^i\binom{h+1}{i}u_{n+i}^h \\
&= \sum_{i=0}^{h}(-1)^i\binom{h}{i}(n+i)u_{n+i}^{h-1} - \sum_{i=0}^{h}(-1)^i\binom{h}{i}(n+i+1)u_{n+i+1}^{h-1} \\
&= \sum_{i=0}^{h}(-1)^i\binom{h}{i}(n+i)(u_{n+i}^{h-1} - u_{n+i+1}^{h-1}) + \sum_{i=0}^{h}(-1)^i\binom{h}{i}u_{n+i+1}^{h-1}. \\
&= \sum_{i=0}^{h}(-1)^i\binom{h}{i}(n+i)(u_{n+i}^{h-1} - u_{n+i+1}^{h-1}).
\end{aligned}$$

This process can be iterated to get the relationship

$$\sum_{i=0}^{h+1}(-1)^i\binom{h+1}{i}u_{n+i}^h = \sum_{i=0}^{h-r}(-1)^i i^r(u_{n+i}^{h-r} - u_{n+i+r}^{h-r}),$$

for all $r = 1, 2, \ldots, h$. Finally we obtain

$$\sum_{i=0}^{h+1}(-1)^i\binom{h+1}{i}u_{n+i}^h = \sum_{i=0}^{0}(-1)^i i^h(u_{n+i}^0 - u_{n+i+h}^0) = 0. \qquad \square$$

Remark 1. *In Table 1 we can observe other important facts.*

1. *The sequence $\{s_n^{n+1} \mid n \in \mathbb{N}\}$, with terms $1, 2, 8, 56, 551, \ldots$, is the sequence A097691 in [4].*
2. *The second interesting sequence is defined by the subdiagonal, i.e., the sequence $\{s_n^n \mid n \in \mathbb{N}\}$ with terms $1, 3, 21, 209, \ldots$, is the sequence A097690 in [4].*

The behaviour of Table 1 can be extended to other Lucas sequences. Indeed, we define new sequences s_t^r in the following way:

$$s_t^r(0) = 1,\ s_t^r(1) = t,\ s_t^r(n+2) = s_t^r(n+1)t + s_t^r(n),$$

for every $t, r \in \mathbb{N} \setminus \{0\}$. Hence, for every $r \in \mathbb{N} \setminus \{0\}$ we have a table with top label row $s_1^r, s_2^r, \ldots$ Indeed, this corresponds to

the table for the Lucas sequences $\{U(t,s) \mid t,s\}$ for appropriate indices.

The first row, u_0, is $s_1^r(0), s_2^r(0), \ldots$, the second row, u_1, is given by $s_1^r(1), s_2^r(1), \ldots$, and so on.

Theorem 6. *In the above situation, each sequence u_j in the j-th row satisfies the relationship*

$$\sum_{i=0}^{h+1}(-1)^i\binom{h+1}{i}u_{n+i}^h = 0, \text{ for } n \geq 1.$$

Proof. The proof is similar to the proof of Theorem 5 if we replace -1 by r. □

5 An open problem on quadratic congruences

By studying solutions of the congruence system (1), we find that each solution appears as consecutive terms in a column in Table 1; this completely resolves Problem 1. In the same way, we can consider the congruence system

$$\left.\begin{array}{ll} X^2 \equiv 1 & (\text{mod } Y) \\ Y^2 \equiv 1 & (\text{mod } Z) \\ Z^2 \equiv 1 & (\text{mod } X) \end{array}\right\} \tag{5}$$

and study its solutions in positive integers, i.e., the triplets (x,y,z) of positive integers x,y,z satisfying these congruences. One may observe that, if (x,y,z) is a solution, both (y,z,x) and (z,x,y) are solutions. Otherwise, if $x = 1$, then $y = 1 = z$, hence $(1,1,1)$ is a trivial solution. For a non-trivial solution (x,y,z), we have $x,y,z > 2$, and x, y and z are coprime two by two, so we may assume $x < y,z$.

Let us assume that a solution (x,y,z) satisfies $x = 2$. Then, $y \mid x^2 - 1 = 3$, and $y = 3$. Otherwise, $z \mid y^2 - 1 = 8$, hence z is even, which is a contradiction.

Some questions arise in this study, some of them are:

Q.1 Finding all solutions of the congruence system (5).
Q.2 Is there any solution (n, y, z), for every $n \in \mathbb{N} \setminus \{0, 1, 2\}$?
Q.3 Does any solution (x, y, z) have an even component?
Q.4 Does there exists a structure either of the set of all solutions or of a partial subset that could help us to identify the set of all solutions?

First we deal with the problem Q.2. Consider n, an even positive integer number, let $n = 2a$, for some $a \in \mathbb{N} \setminus \{0, 1\}$.

Proposition 2. *For any* $a \in \mathbb{N} \setminus \{0, 1\}$ *the triplet* $(2a, (2a)^2 - 1, 2a^2 - 1)$ *is a solution of the congruence system* (5).

After proving that each even positive integer number bigger than 2 is part of a solution, we deal with the same problem for odd numbers.

Following the same argument, we can show that $(2^t - 1, 2^{t+1}, 2^{t+1} - 1)$, and $(2^{t-1} + 1, 2^t, 2^t + 1)$ are also solutions of the congruence system for every $t \in \mathbb{N} \setminus \{0, 1\}$. In both cases the smallest element in these triplets is odd.

More in general, we have:

Proposition 3. *For any* $a \in \mathbb{N} \setminus \{0\}$, *both the triplets* $(2a + 1, 4a + 4, 4a + 3)$ *and* $(2a + 1, 4a, 4a + 1)$ *are solutions of the congruence system* (5).

Thus, joining the two propositions, we have:

Theorem 7. *For any* $n \in \mathbb{N} \setminus \{0, 1, 2\}$ *there exists a solution of the system* (5) *of the shape* (n, y, z).

The answer to problem Q.3 is no. The answer appears after exploring different cases. For instance, the triplets $(59, 87, 473)$ and $(77, 741, 265)$ are solutions, and all its components are odd numbers.

There are solutions of the system (5) which are not in these sets of solutions, for instance the triplets $(5, 8, 21)$ and $(8, 21, 11)$, therefore also $(8, 21, 55)$. It would be of interest to establish some

criteria to relate this with other solutions, and, in general, to solve the former problems Q.1 and Q.4.

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Trigonometric Sums Involving the Angle $\pi/21$

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Abstract

In this short note, for $n \in \{1, 2, \ldots, 7\}$ we will calculate sums of the type $\cos^n \frac{2\pi}{21} + \cos^n \frac{8\pi}{21} + \cos^n \frac{10\pi}{21}$ and $\cos^n \frac{4\pi}{21} + \cos^n \frac{16\pi}{21} + \cos^n \frac{20\pi}{21}$.

1 Main results

In the sequel, we will successively use the following identities taken from [1]:

$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}$$

and

$$\sin\frac{3\pi}{7} + \sin\frac{2\pi}{7} - \sin\frac{\pi}{7} = \frac{\sqrt{7}}{2}$$

to establish new trigonometrical identities. We begin with

Theorem 1. *The following identities hold:*

i) $\cos\frac{2\pi}{21} + \cos\frac{8\pi}{21} + \cos\frac{10\pi}{21} = \frac{1+\sqrt{21}}{4}$,

ii) $\cos^2\frac{2\pi}{21} + \cos^2\frac{8\pi}{21} + \cos^2\frac{10\pi}{21} = \frac{13-\sqrt{21}}{8}$,

iii) $\cos^3\frac{2\pi}{21} + \cos^3\frac{8\pi}{21} + \cos^3\frac{10\pi}{21} = \frac{1+3\sqrt{21}}{16}$,

iv) $\cos^4\dfrac{2\pi}{21}+\cos^4\dfrac{8\pi}{21}+\cos^4\dfrac{10\pi}{21}=\dfrac{41-3\sqrt{21}}{32}$,

v) $\cos^5\dfrac{2\pi}{21}+\cos^5\dfrac{8\pi}{21}+\cos^5\dfrac{10\pi}{21}=\dfrac{1+11\sqrt{21}}{64}$,

vi) $\cos^6\dfrac{2\pi}{21}+\cos^6\dfrac{8\pi}{21}+\cos^6\dfrac{10\pi}{21}=\dfrac{139-9\sqrt{21}}{128}$,

vii) $\cos^7\dfrac{2\pi}{21}+\cos^7\dfrac{8\pi}{21}+\cos^7\dfrac{10\pi}{21}=\dfrac{-3+21\sqrt{21}}{128}$.

Proof. i) Denote $S=\cos\dfrac{2\pi}{21}+\cos\dfrac{8\pi}{21}+\cos\dfrac{10\pi}{21}$. Then,

$$\begin{aligned}S&=\cos\Big(\frac{-2\pi}{21}\Big)-\cos\Big(\frac{13\pi}{21}\Big)+\cos\Big(\frac{10\pi}{21}\Big)\\&=\cos\Big(\frac{\pi}{3}-\frac{3\pi}{7}\Big)-\cos\Big(\frac{\pi}{3}+\frac{2\pi}{7}\Big)+\cos\Big(\frac{\pi}{3}+\frac{\pi}{7}\Big).\end{aligned}$$

Using the well-known formulas $\cos(a\pm b)=\cos a\cos b\mp\sin a\sin b$, we obtain, after due simplification,

$$\begin{aligned}S&=\cos\frac{\pi}{3}\Big(\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}\Big)+\sin\frac{\pi}{3}\Big(\sin\frac{3\pi}{7}+\sin\frac{2\pi}{7}-\sin\frac{\pi}{7}\Big)\\&=\frac{1}{2}\cdot\frac{1}{2}+\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{7}}{2}=\frac{1+\sqrt{21}}{4}.\end{aligned}$$

ii) We have

$$\begin{aligned}&\cos^2\frac{2\pi}{21}+\cos^2\frac{8\pi}{21}+\cos^2\frac{10\pi}{21}\\&=\Big(\cos\frac{2\pi}{21}+\cos\frac{8\pi}{21}+\cos\frac{10\pi}{21}\Big)^2-2\cos\frac{2\pi}{21}\cos\frac{8\pi}{21}\\&\quad-2\cos\frac{10\pi}{21}\cos\frac{2\pi}{21}-2\cos\frac{8\pi}{21}\cos\frac{10\pi}{21}\\&=\left(\frac{1+\sqrt{21}}{4}\right)^2-\cos\frac{10\pi}{21}-\cos\frac{6\pi}{21}-\cos\frac{12\pi}{21}-\cos\frac{8\pi}{21}-\cos\frac{18\pi}{21}-\cos\frac{2\pi}{21}\\&=\frac{11+\sqrt{21}}{8}-\Big(\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7}-\cos\frac{\pi}{7}\Big)-\Big(\cos\frac{2\pi}{21}+\cos\frac{8\pi}{21}+\cos\frac{10\pi}{21}\Big)\\&=\frac{11+\sqrt{21}}{8}-\Big(-\frac{1}{2}\Big)-\left(\frac{1+\sqrt{21}}{4}\right)=\frac{13-\sqrt{21}}{8}.\end{aligned}$$

iii) Using $\cos 3a = 4\cos^3 a - 3\cos a$, then we have

$$\begin{aligned} &4\left(\cos^3\frac{2\pi}{21}+\cos^3\frac{8\pi}{21}+\cos^3\frac{10\pi}{21}\right)-3\left(\cos\frac{2\pi}{21}+\cos\frac{8\pi}{21}+\cos\frac{10\pi}{21}\right) \\ &\quad = \cos\frac{2\pi}{7}+\cos\frac{8\pi}{7}+\cos\frac{10\pi}{7} \\ \Longrightarrow\ &4\left(\cos^3\frac{2\pi}{21}+\cos^3\frac{8\pi}{21}+\cos^3\frac{10\pi}{21}\right)-3\left(\frac{1+\sqrt{21}}{4}\right) \\ &\quad = -\left(\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}\right)=-\frac{1}{2}. \end{aligned}$$

Therefore,

$$\cos^3\frac{2\pi}{21}+\cos^3\frac{8\pi}{21}+\cos^3\frac{10\pi}{21}=\frac{1+3\sqrt{21}}{16}.$$

iv) Using identity $8\cos^4 a = \cos 4a + 8\cos^2 a - 1$, we have

$$\begin{aligned} &8\left(\cos^4\frac{2\pi}{21}+\cos^4\frac{8\pi}{21}+\cos^4\frac{10\pi}{21}\right) \\ &=\cos\frac{8\pi}{21}+\cos\frac{32\pi}{21}+\cos\frac{40\pi}{21}+8\left(\cos^2\frac{2\pi}{21}+\cos^2\frac{8\pi}{21}+\cos^2\frac{10\pi}{21}\right)-3, \end{aligned}$$

so

$$\begin{aligned} &8\left(\cos^4\frac{2\pi}{21}+\cos^4\frac{8\pi}{21}+\cos^4\frac{10\pi}{21}\right) \\ &=\cos\frac{2\pi}{21}+\cos\frac{8\pi}{21}+\cos\frac{10\pi}{21}+8\left(\frac{13-\sqrt{21}}{8}\right)-3, \end{aligned}$$

hence

$$\cos^4\frac{2\pi}{21}+\cos^4\frac{8\pi}{21}+\cos^4\frac{10\pi}{21}=\frac{41-3\sqrt{21}}{32}.$$

In the remaining cases, just use the following trigonometric identities and the preceding results:
v) $16\cos^5 a = \cos 5a + 20\cos^3 a - 5\cos a$;
vi) $32\cos^6 a = \cos 6a + 48\cos^4 a - 18\cos^2 a + 1$;
vii) $64\cos^7 a = \cos 7a + 112\cos^5 a - 56\cos^3 a + 7\cos a$. □

It is interesting to note that the following sums involving angles

$$\frac{4\pi}{21}, \frac{16\pi}{21}, \frac{20\pi}{21}$$

are a kind of "conjugate" of the previous sums for angles $\frac{2\pi}{21}, \frac{8\pi}{21}, \frac{10\pi}{21}$.

Theorem 2. *The following identities hold:*

$$\begin{aligned}
&i)\ \cos\frac{4\pi}{21}+\cos\frac{16\pi}{21}+\cos\frac{20\pi}{21}=\frac{1-\sqrt{21}}{4},\\
&ii)\ \cos^2\frac{4\pi}{21}+\cos^2\frac{16\pi}{21}+\cos^2\frac{20\pi}{21}=\frac{13+\sqrt{21}}{8},\\
&iii)\ \cos^3\frac{4\pi}{21}+\cos^3\frac{16\pi}{21}+\cos^3\frac{20\pi}{21}=\frac{1-3\sqrt{21}}{16},\\
&iv)\ \cos^4\frac{4\pi}{21}+\cos^4\frac{16\pi}{21}+\cos^4\frac{20\pi}{21}=\frac{41+3\sqrt{21}}{32},\\
&v)\ \cos^5\frac{4\pi}{21}+\cos^5\frac{16\pi}{21}+\cos^5\frac{20\pi}{21}=\frac{1-11\sqrt{21}}{64},\\
&vi)\ \cos^6\frac{4\pi}{21}+\cos^6\frac{16\pi}{21}+\cos^6\frac{20\pi}{21}=\frac{139+9\sqrt{21}}{128},\\
&vii)\ \cos^7\frac{4\pi}{21}+\cos^7\frac{16\pi}{21}+\cos^7\frac{20\pi}{21}=\frac{-3-21\sqrt{21}}{128}.
\end{aligned}$$

Proof. It is sufficient to prove i) and ii). To prove i), denote by $S=\cos\frac{4\pi}{21}+\cos\frac{16\pi}{21}+\cos\frac{20\pi}{21}$. Then,

$$S=\cos\Big(\frac{\pi}{3}+\frac{3\pi}{7}\Big)+\cos\Big(\frac{\pi}{3}-\frac{\pi}{7}\Big)+\cos\Big(\frac{2\pi}{3}+\frac{2\pi}{7}\Big).$$

Using the well-known formulas $\cos(a\pm b)=\cos a\cos b\mp\sin a\sin b$, we obtain

$$\begin{aligned}
S&=\cos\frac{\pi}{3}\Big(\cos\frac{\pi}{7}-\cos\frac{2\pi}{7}+\cos\frac{3\pi}{7}\Big)+\sin\frac{\pi}{3}\Big(\sin\frac{\pi}{7}-\sin\frac{2\pi}{7}-\sin\frac{3\pi}{7}\Big)\\
&=\frac{1}{2}\cdot\frac{1}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{7}}{2}=\frac{1-\sqrt{21}}{4}.
\end{aligned}$$

ii) We have

$$\begin{aligned}
&\cos^2\frac{4\pi}{21}+\cos^2\frac{16\pi}{21}+\cos^2\frac{20\pi}{21}\\
&=\left(\cos\frac{4\pi}{21}+\cos\frac{16\pi}{21}+\cos\frac{20\pi}{21}\right)^2\\
&\quad-2\cos\frac{4\pi}{21}\cos\frac{16\pi}{21}-2\cos\frac{20\pi}{21}\cos\frac{4\pi}{21}-2\cos\frac{16\pi}{21}\cos\frac{20\pi}{21}\\
&=\left(\frac{1-\sqrt{21}}{4}\right)^2-\cos\frac{20\pi}{21}-\cos\frac{12\pi}{21}-\cos\frac{24\pi}{21}-\cos\frac{16\pi}{21}-\cos\frac{36\pi}{21}-\cos\frac{4\pi}{21}\\
&=\frac{11-\sqrt{21}}{8}-\left(\cos\frac{2\pi}{7}-\cos\frac{3\pi}{7}-\cos\frac{\pi}{7}\right)-\left(\cos\frac{4\pi}{21}+\cos\frac{16\pi}{21}+\cos\frac{20\pi}{21}\right)\\
&=\frac{11-\sqrt{21}}{8}-\left(-\frac{1}{2}\right)-\left(\frac{1-\sqrt{21}}{4}\right)=\frac{13+\sqrt{21}}{8}.
\end{aligned}$$

The remaining identities follow analogously as in Theorem 1. □

Remark 1. *Observe that* $z_1=\cos\frac{2\pi}{21}+\cos\frac{8\pi}{21}+\cos\frac{10\pi}{21}$ *and* $z_2=\cos\frac{4\pi}{21}+\cos\frac{16\pi}{21}+\cos\frac{20\pi}{21}$ *are the zeros of the polynomial* $p(z)=4z^2-2z-5$.

Remark 2. *Using the well-known identity* $a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$, *we obtain*

$$\cos\frac{2\pi}{21}\cos\frac{8\pi}{21}\cos\frac{10\pi}{21}=\frac{25-3\sqrt{21}}{48},$$

and

$$\cos\frac{4\pi}{21}\cos\frac{16\pi}{21}\cos\frac{20\pi}{21}=\frac{25+3\sqrt{21}}{48}.$$

Finally, we propose the following:

Problem. *Prove that*

$$\cos^2\frac{\pi}{5}+\cos^2\frac{2\pi}{5}+\cos^2\frac{4\pi}{5}=\frac{9+\sqrt{5}}{8},$$
$$\cos^2\frac{\pi}{13}+\cos^2\frac{3\pi}{13}+\cos^2\frac{4\pi}{13}=\frac{11+\sqrt{13}}{8},$$
$$\cos^3\frac{2\pi}{13}+\cos^3\frac{6\pi}{13}+\cos^3\frac{8\pi}{13}=\frac{\sqrt{13}-1}{4},$$
$$\cos^2\frac{\pi}{21}+\cos^2\frac{4\pi}{21}+\cos^2\frac{5\pi}{21}=\frac{13+\sqrt{21}}{8},$$
$$\cos\frac{5\pi}{28}+\cos\frac{11\pi}{28}+\cos\frac{13\pi}{28}=\frac{\sqrt{2}+\sqrt{14}}{4},$$
$$\cos\frac{8\pi}{35}+\cos\frac{12\pi}{35}+\cos\frac{17\pi}{35}=\frac{1+\sqrt{5}+\sqrt{70-14\sqrt{5}}}{8}.$$

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Some generalizations and refinements of Gerretsen inequality

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Abstract

In this paper some generalizations of the LHS of the Gerretsen inequality are presented.

1 Introduction

In 1953, J. C. Gerretsen [2] published the following inequality:

$$24Rr - 12r^2 \leq a^2 + b^2 + c^2.$$

The purpose of this paper is to find the best real constants $\alpha_{m,n}$, $m \geq n$ with $m, n \in \{0, 1, 2, 3, 4, 5\}$, for which the inequality

$$\alpha_{m,n}R^m r^{6-m} + (2^{6-n} \cdot 81 - 2^{m-n}\alpha_{m,n})R^n r^{6-n} \leq a^6 + b^6 + c^6 \quad (1)$$

holds in any triangle ABC. First, we find the best homogenous function $f(R, r)$ for which the inequality $f(R, r) \leq a^6 + b^6 + c^6$ holds in any triangle ABC. Also we shall find the best integer constants $\alpha_{m,n}$ for which the inequality (1) is true in any triangle ABC. After this replacing $\alpha_{m,n} \in \mathbb{Z}$ in (1) we compare all this inequalities.

To prove our results we use the following:

Theorem 1 (Blundon). *For any triangle ABC, the inequality $s_1 \le s \le s_2$ holds, where s_1, s_2 represent the semiperimeter of two isosceles triangles $A_1B_1C_1$ and $A_2B_2C_2$ which have the same circumradius R and inradius r as the triangle ABC with the sides*

$$a_1 = 2\sqrt{(R+r-d)(R-r+d)}, \quad b_1 = c_1 = \sqrt{2R(R+r-d)},$$
$$a_2 = 2\sqrt{(R+r+d)(R-r-d)}, \quad b_2 = c_2 = 2\sqrt{2R(R+r+d)},$$

where $d = \sqrt{R^2-2Rr}$.

A proof of this theorem is given in [1].

Next, some minimal and maximal bounds for the sum $a^6+b^6+c^6$ are presented

Theorem 2. *In any triangle ABC, the following inequality holds:*

$$2^4(R+r-d)^3\left[4(R-r+d)^3+R^3\right] \le a^6+b^6+c^6 \le 2^4(R+r+d)^3\left[4(R-r-d)^3+R^3\right]. \tag{2}$$

Proof. From the identity

$$\sum_{\text{cyc}} x^3 = 3xyz + \sum_{\text{cyc}} x\left(\sum_{\text{cyc}} x^2 - \sum_{\text{cyc}} yz\right),$$

if we replace $x = a^2$, $y = b^2$, $z = c^2$, then we obtain

$$\sum_{\text{cyc}} a^6 = 3a^2b^2c^2 + \sum_{\text{cyc}} a^2\left(\sum_{\text{cyc}} a^4 - \sum_{\text{cyc}} b^2c^2\right).$$

We consider the functions $f, g, h, F\colon [s_1, s_2] \to \mathbb{R}$ given by

$$\begin{aligned}
f(s) &= 3a^2b^2c^2 = 3(4Rrs)^2,\\
g(s) &= 2(s^2-r^2-4Rr),\\
h(s) &= \sum_{\text{cyc}} a^4 - \sum_{\text{cyc}} (bc)^2\\
&= \left(\sum_{\text{cyc}} a\right)^4 - 4\sum_{\text{cyc}} ab\left(\sum_{\text{cyc}} a\right)^2 + 6abc\sum_{\text{cyc}} a + \left(\sum_{\text{cyc}} ab\right)^2\\
&= s^4 - 2r(4R+7r)s^2 + (4R+r)^2r^2,\\
F(s) &= f(s) + g(s)h(s).
\end{aligned}$$

Note that $h'(s) = 4s(s^2-4Rr-7r^2)$. Since $s^2 \geq s_1^2 \geq 16Rr-5r^2 \geq 4Rr+7r^2$, it results that h is increasing on $[s_1, s_2]$. Furthermore, f and g are also increasing on $[s_1, s_2]$. Hence, F is increasing on $[s_1, s_2]$, and thus $F(s_1) \leq F(s) \leq F(s_2)$, or

$$a_1^6 + b_1^6 + c_1^6 \leq a^6 + b^6 + c^6 \leq a_2^6 + b_2^6 + c_2^6. \tag{3}$$

If we replace $a_1, b_1, c_1, a_2, b_2, c_2$ from Theorem 1 in (3), then we obtain (2). □

2 Main results

Let $m, n \in \{0, 1, 2, 3, 4, 5\}$. In what follows we shall find the best real constant $\alpha_{m,n}$ such that the inequality (1) holds in any triangle ABC. From (2) we have that

$$\begin{aligned} &\alpha_{m,n}R^m r^{6-m} + (2^{6-n} \cdot 81 - 2^{m-n}\alpha_{m,n})R^n r^{6-n} \\ &\leq 2^4(R+r-d)^3[4(R-r+d)^3 + R^3]. \end{aligned} \tag{4}$$

If we denote $x = \dfrac{R}{r} \geq 2$, then the inequality (4) is equivalent to

$$\begin{aligned} &\alpha_{m,n}x^m + (2^{6-n} \cdot 81 - 2^{m-n}\alpha_{m,n})x^n \\ &\leq 16(x+1-\sqrt{x^2-2x})^3\left[4(x-1+\sqrt{x^2-2x})^3 + x^3\right] \end{aligned}$$

or

$$\alpha_{m,n} \leq \frac{16(x+1-\sqrt{x^2-2x})^3(4(x-1+\sqrt{x^2-2x})^3+x^3) - 81\cdot 2^{6-n}x^n}{x^n(x^{m-n}-2^{m-n})}, \tag{5}$$

for all $x \geq 2$.

Case 1. If $m > n$, then we have

$$\begin{aligned} (m,n) \in \{&(5,4),(5,3)(5,2),(5,1),(5,0),(4,3),(4,2),(4,1),(4,0),\\ &(3,2),(3,1),(3,0),(2,1),(2,0),(1,0)\}. \end{aligned}$$

So, from (5) it follows that the best constant for (1) is the infimum of the function $F_{m,n}\colon [0,\infty) \to \mathbb{R}$,

$$F_{m,n}(x) = \frac{16(x+1-\sqrt{x^2-2x})^3(4(x-1+\sqrt{x^2-2x})^3+x^3)-81\cdot 2^{6-n}x^n}{x^n(x^{m-n}-2^{m-n})},$$

i.e.,

$$\alpha_{m,n} = \inf_{x\in[2,\infty)} F_{m,n}(x).$$

Using Wolfram Alpha, we obtain

$$\begin{aligned} \alpha_{5,4} = \alpha_{5,3} = \alpha_{5,2} = \alpha_{5,1} = \alpha_{5,0} &\cong -7.53288; \\ \alpha_{4,3} = \alpha_{4,2} = \alpha_{4,1} = \alpha_{4,0} = 0;\ \alpha_{3,2} &\cong 3357.74; \\ \alpha_{3,1} &\cong 2020.5; \\ \alpha_{3,0} &\cong 1568.17; \\ \alpha_{2,1} &\cong 8123.05; \\ \alpha_{2,0} &\cong 4718.54; \\ \alpha_{1,0} &\cong 18904.2. \end{aligned}$$

Case 2. If $m = n \in \{0,1,2,3,4,5\}$, then (1) can be written as

$$2^{6-m}R^m r^{6-m} \le a^6 + b^6 + c^6.$$

If $m = n \in \{0,1,2,3\}$, we obtain the inequalities (10), (9), (8) and (7) from below.

If $m = n = 4$, we obtain that $2^2R^4r^2 \le a^6+b^6+c^6$. Then, by Theorem 2, it suffices to prove that $2^2x^4 \le 16(x+1-\sqrt{x^2-2x})^3\left[4(x-1+\sqrt{x^2-2x})^3+x^3\right]$, which is true for $2 \le x \le 1053.84$; so not for each $x \ge 2$.

If $m = n = 5$, it suffices to prove that

$$2x^5 \le 16(x+1-\sqrt{x^2-2x})^3\left[4(x-1+\sqrt{x^2-2x})^3+x^3\right],$$

which is true for $2 \le x \le 44.8476$; so not for each $x \ge 2$.

Hence in the case $m = n \in \{4,5\}$, the inequality (1) is not true in any triangle ABC.

Remark 1. *Since the function* $f\colon (0,\infty)\to\mathbb{R}$ *given by*

$$f(\alpha_{m,n}) = (R^{m-n} - (2r)^{m-n})\alpha_{m,n}R^n r^{6-m} + 2^{6-n}\cdot 81R^n$$

is increasing, from (1) *we have that* $f([\alpha_{m,n}]) \le f(\alpha_{m,n})$ *or*

$$[\alpha_{m,n}]R^m r^{6-m} + (2^{6-n}\cdot 81 - 2^{m-n}[\alpha_{m,n}])R^n r^{6-n} \le a^6+b^6+c^6$$

holds in any triangle ABC.

We obtain the following inequalities

$$I_0 = 340R^4r^2 - 8R^5r \le a^6+b^6+c^6; \tag{6}$$
$$I_1 = 2^3\cdot 81\cdot R^3r^3 \le a^6+b^6+c^6; \tag{7}$$
$$I_2 = 2^4\cdot 81\cdot R^2r^4 \le a^6+b^6+c^6; \tag{8}$$
$$I_3 = 2^5\cdot 81\cdot Rr^5 \le a^6+b^6+c^6; \tag{9}$$
$$I_4 = 2^6\cdot 81\cdot r^6 \le a^6+b^6+c^6, \tag{10}$$

i.e., the problem 11984 from AMM, vol. 124, No. 5 (may 2017), p. 466;

$$I_5 = 3357R^3r^3 - 5418R^2r^4 \le a^6+b^6+c^6; \tag{11}$$
$$I_6 = 2020R^3r^3 - 5488Rr^5 \le a^6+b^6+c^6; \tag{12}$$
$$I_7 = 1568R^3r^3 - 7360r^6 \le a^6+b^6+c^6; \tag{13}$$
$$I_8 = 8123R^2r^4 - 13654Rr^5 \le a^6+b^6+c^6; \tag{14}$$
$$I_9 = 4718R^2r^4 - 13688r^6 \le a^6+b^6+c^6; \tag{15}$$
$$I_{10} = 18904Rr^5 - 32624r^6 \le a^6+b^6+c^6. \tag{16}$$

From Euler's inequality $R\ge 2r$ and (7) we have that

$$2^6\cdot 81\cdot r^6 \le 2^5\cdot 81\cdot Rr^5 \le 2^4\cdot 81\cdot R^2r^4 \le 2^3\cdot 81\cdot R^3r^3 \le a^6+b^6+c^6,$$

i.e.,

$$I_4 \le I_3 \le I_2 \le I_1 \le a^6+b^6+c^6. \tag{17}$$

We denote

$$x_1 = \frac{4734}{2359} \cong 2.006;$$
$$x_2 = \frac{16312}{8123} \cong 2.008;$$
$$x_3 = \frac{5\sqrt{71} - 14}{14} \cong 2.0093;$$
$$x_4 = \frac{6844}{3405} \cong 2.0099;$$
$$x_5 = \frac{113 + \sqrt{114017}}{224} \cong 2.011;$$
$$x_6 = \frac{\sqrt{2314415} - 505}{505} \cong 2.012;$$
$$x_7 = \frac{339 + \sqrt{1393801}}{2020} \cong 2.0161;$$
$$x_8 = \frac{4987 + 9\sqrt{22089}}{3136} \cong 2.0167;$$
$$x_9 = \frac{2(\sqrt{1532758} - 108)}{119} \cong 2.019;$$
$$x_{10} = \frac{4083}{2020} \cong 2.021;$$
$$x_{11} = \frac{\sqrt{116729} - 113}{113} \cong 2.023;$$
$$x_{12} = \frac{1711 + \sqrt{25902829}}{3357} \cong 2.025;$$
$$x_{13} = \frac{6827}{3359} \cong 2.033;$$
$$x_{14} = \frac{4(230 + \sqrt{464370})}{1789} \cong 2.037;$$
$$x_{15} = \frac{392}{191} \cong 2.05;$$
$$x_{16} = \frac{2\sqrt{530} - 9}{9} \cong 4.11;$$
$$x_{17} = \frac{1711 + \sqrt{7362433}}{648} \cong 6.8;$$

$$x_{18} = \frac{6827}{648} \cong 10.5;$$
$$x_{19} = \frac{2039}{162} \cong 12.58.$$

We note that $x_i < x_j$ for all $i, j \in \{1, 2, \ldots, 19\}$, $i < j$. If we denote $x = \dfrac{R}{r}$, then

$$I_1 \leq I_5 \ \forall x \geq 2;$$
$$I_1 \leq I_6 \ \forall x \geq 2;$$
$$I_1 \leq I_7 \ \forall x \geq 2;$$
$$I_1 \leq I_8 \ \forall x \in [2, x_{18}] \text{ and } I_1 \geq I_8 \ \forall x \geq x_{18};$$
$$I_1 \leq I_9 \ \forall x \in [2, x_{17}] \text{ and } I_1 \geq I_9 \ \forall x \geq x_{17};$$
$$I_1 \leq I_{10} \ \forall x \in [2, x_{16}] \text{ and } I_1 \geq I_{10} \ \forall x \geq x_{16};$$
$$I_5 \leq I_6 \ \forall x \in [2, x_{15}] \text{ and } I_5 \geq I_6 \ \forall x \geq x_{15};$$
$$I_5 \leq I_7 \ \forall x \in [2, x_{14}] \text{ and } I_5 \geq I_7 \ \forall x \geq x_{14};$$
$$I_5 \leq I_8 \ \forall x \in [2, x_{13}] \text{ and } I_5 \geq I_8 \ \forall x \geq x_{13};$$
$$I_5 \leq I_9 \ \forall x \in [2, x_{12}] \text{ and } I_5 \geq I_9 \ \forall x \geq x_{12};$$
$$I_5 \leq I_{10} \ \forall x \in [2, x_9] \text{ and } I_5 \geq I_{10} \ \forall x \geq x_9;$$
$$I_6 \leq I_7 \ \forall x \in [2, x_{11}] \text{ and } I_6 \geq I_7 \ \forall x \geq x_{11};$$
$$I_6 \leq I_8 \ \forall x \in [2, x_{10}] \text{ and } I_6 \geq I_8 \ \forall x \geq x_{10};$$
$$I_6 \leq I_9 \ \forall x \in [2, x_7] \text{ and } I_6 \geq I_9 \ \forall x \geq x_7;$$
$$I_6 \leq I_{10} \ \forall x \in [2, x_6] \text{ and } I_6 \geq I_{10} \ \forall x \geq x_6;$$
$$I_7 \leq I_8 \ \forall x \in [2, x_8] \text{ and } I_7 \geq I_8 \ \forall x \geq x_8;$$
$$I_7 \leq I_9 \ \forall x \in [2, x_5] \text{ and } I_7 \geq I_9 \ \forall x \geq x_5;$$
$$I_7 \leq I_{10} \ \forall x \in [2, x_3] \text{ and } I_7 \geq I_{10} \ \forall x \geq x_3;$$
$$I_8 \leq I_9 \ \forall x \in [2, x_4] \text{ and } I_8 \geq I_9 \ \forall x \geq x_4;$$
$$I_8 \leq I_{10} \ \forall x \in [2, x_2] \text{ and } I_8 \geq I_{10} \ \forall x \geq x_2;$$
$$I_9 \leq I_{10} \ \forall x \in [2, x_1] \text{ and } I_9 \geq I_{10} \ \forall x \geq x_1;$$
$$I_2 \leq I_{10} \ \forall x \in [2, x_{19}] \text{ and } I_2 \geq I_{10} \ \forall x \geq x_{19}.$$

From the above inequalities and (17) we obtain the following chains of inequalities:

1) $I_4 \le I_3 \le I_2 \le I_1 \le I_5 \le I_6 \le I_7 \le I_8 \le I_9 \le I_{10} \le \sum_{\text{cyc}} a^6$ $\forall\, x \in [2, x_1]$;
2) $I_4 \le I_3 \le I_2 \le I_1 \le I_5 \le I_6 \le I_7 \le I_8 \le I_{10} \le I_9 \ \ \forall\, x \in [x_1, x_2]$;
3) $I_4 \le I_3 \le I_2 \le I_1 \le I_5 \le I_6 \le I_7 \le I_{10} \le I_8 \le I_9 \ \ \forall\, x \in [x_2, x_3]$;
4) $I_4 \le I_3 \le I_2 \le I_1 \le I_5 \le I_6 \le I_{10} \le I_7 \le I_8 \le I_9 \ \ \forall\, x \in [x_3, x_4]$;
5) $I_4 \le I_3 \le I_2 \le I_1 \le I_5 \le I_6 \le I_{10} \le I_7 \le I_9 \le I_8 \ \ \forall\, x \in [x_4, x_5]$;
6) $I_4 \le I_3 \le I_2 \le I_1 \le I_5 \le I_6 \le I_{10} \le I_9 \le I_7 \le I_8 \ \ \forall\, x \in [x_5, x_6]$;
7) $I_4 \le I_3 \le I_2 \le I_1 \le I_5 \le I_{10} \le I_6 \le I_9 \le I_7 \le I_8 \ \ \forall\, x \in [x_6, x_7]$;
8) $I_4 \le I_3 \le I_2 \le I_1 \le I_5 \le I_{10} \le I_9 \le I_6 \le I_7 \le I_8 \ \ \forall\, x \in [x_7, x_8]$;
9) $I_4 \le I_3 \le I_2 \le I_1 \le I_5 \le I_{10} \le I_9 \le I_6 \le I_8 \le I_7 \ \ \forall\, x \in [x_8, x_9]$;
10) $I_4 \le I_3 \le I_2 \le I_1 \le I_{10} \le I_5 \le I_9 \le I_6 \le I_8 \le I_7 \ \ \forall\, x \in [x_9, x_{10}]$;
11) $I_4 \le I_3 \le I_2 \le I_1 \le I_{10} \le I_5 \le I_9 \le I_8 \le I_6 \le I_7 \ \ \forall\, x \in [x_{10}, x_{11}]$;
12) $I_4 \le I_3 \le I_2 \le I_1 \le I_{10} \le I_5 \le I_9 \le I_8 \le I_7 \le I_6 \ \ \forall\, x \in [x_{11}, x_{12}]$;
13) $I_4 \le I_3 \le I_2 \le I_1 \le I_{10} \le I_9 \le I_5 \le I_8 \le I_7 \le I_6 \ \ \forall\, x \in [x_{11}, x_{12}]$;
14) $I_4 \le I_3 \le I_2 \le I_1 \le I_{10} \le I_9 \le I_8 \le I_5 \le I_7 \le I_6 \ \ \forall\, x \in [x_{12}, x_{13}]$;
15) $I_4 \le I_3 \le I_2 \le I_1 \le I_{10} \le I_9 \le I_8 \le I_7 \le I_5 \le I_6 \ \ \forall\, x \in [x_{13}, x_{14}]$;
16) $I_4 \le I_3 \le I_2 \le I_1 \le I_{10} \le I_9 \le I_8 \le I_7 \le I_6 \le I_5 \ \ \forall\, x \in [x_{14}, x_{15}]$;
17) $I_4 \le I_3 \le I_2 \le I_{10} \le I_1 \le I_9 \le I_8 \le I_7 \le I_6 \le I_5 \ \ \forall\, x \in [x_{15}, x_{16}]$;
18) $I_4 \le I_3 \le I_2 \le I_{10} \le I_9 \le I_1 \le I_8 \le I_7 \le I_6 \le I_5 \ \ \forall\, x \in [x_{16}, x_{17}]$;
19) $I_4 \le I_3 \le I_2 \le I_{10} \le I_9 \le I_8 \le I_1 \le I_7 \le I_6 \le I_5 \ \ \forall\, x \in [x_{17}, x_{18}]$.

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Generalization of Problem EM-55

Toyesh Prakash Sharma

Abstract

In this short note, we generalize an inequality of Berindeanu appeared in this journal as Problem EM-55.

1 Introduction

In 2018, in issue 1 of the AMJ journal, Mihaela Berindeanu proposed the following problem [2]:

Let x, y, z, t be positive real numbers such that $x+y+z+t=2$. Show that

$$\left(\frac{4}{x^2}-1\right)\left(\frac{4}{y^2}-1\right)\left(\frac{4}{z^2}-1\right)\left(\frac{4}{t^2}-1\right)\geq 15^4.$$

Its solution was published in [1]. The main goal of this note is to present a generalization of the preceding inequality.

2 Main result

A general result from which problem EM-55 can be obtained as a particular case is presented in the following theorem.

Theorem 1. *Let $x_1, x_2, \ldots, x_n$ be $n \geq 2$ positive real numbers. Then, the inequality*

$$\left(\frac{a^2}{x_1^2} - 1\right)\left(\frac{a^2}{x_2^2} - 1\right)\ldots\left(\frac{a^2}{x_n^2} - 1\right) \geq (n^2 - 1)^n,$$

where $a = x_1 + x_2 + \ldots + x_n$, holds.

Proof. On account of the AM-GM inequality, we have

$$\frac{a + x_1}{n+1} = \frac{x_1 + x_2 + \ldots + x_n + x_1}{n+1} \geq \sqrt[n+1]{x_1^2 x_2 \ldots x_n}$$

and

$$\frac{a - x_1}{n-1} = \frac{x_1 + x_2 + \ldots + x_n - x_1}{n-1} \geq \sqrt[n-1]{x_2 x_3 \ldots x_n}.$$

Multiplying up the preceding inequalities yields

$$\frac{a^2 - x_1^2}{n^2 - 1} \geq \sqrt[n+1]{x_1^2 x_2 \ldots x_n}\,\sqrt[n-1]{x_2 x_3 \ldots x_n},$$

or equivalently,

$$a^2 - x_1^2 \geq (n^2 - 1)\sqrt[n+1]{x_1^2 x_2 \ldots x_n}\,\sqrt[n-1]{x_2 x_3 \ldots x_n}.$$

After division by x_1^2, we obtain

$$\left(\frac{a^2}{x_1^2} - 1\right) \geq (n^2 - 1)\frac{\sqrt[n+1]{x_1^2 x_2 \ldots x_n}\,\sqrt[n-1]{x_2 x_3 \ldots x_n}}{x_1^2}.$$

Likewise, we have

$$\left(\frac{a^2}{x_2^2} - 1\right) \geq (n^2 - 1)\frac{\sqrt[n+1]{x_1 x_2^2 \ldots x_n}\,\sqrt[n-1]{x_1 x_3 \ldots x_n}}{x_2^2},$$

$$\vdots$$

$$\left(\frac{a^2}{x_n^2} - 1\right) \geq (n^2 - 1)\frac{\sqrt[n+1]{x_1 x_2 \ldots x_n^2}\,\sqrt[n-1]{x_1 x_3 \ldots x_{n-1}}}{x_n^2}.$$

Multiplying up the preceding inequalities, yields

$$\left(\frac{a^2}{x_1^2}-1\right)\left(\frac{a^2}{x_2^2}-1\right)\cdots\left(\frac{a^2}{x_n^2}-1\right)\geq(n^2-1)^n\,\frac{x_1^2x_2^2\ldots x_n^2}{x_1^2x_2^2\ldots x_n^2}=(n^2-1)^n,$$

and the proof is complete. □

Hence, for $n=4$, the last inequality becomes

$$\left(\frac{a^2}{x_1^2}-1\right)\left(\frac{a^2}{x_2^2}-1\right)\left(\frac{a^2}{x_3^2}-1\right)\left(\frac{a^2}{x_4^2}-1\right)\geq 15^4.$$

Putting $a=x+y+z+t=2$ in the above result, Berindeanu's inequality follows.

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[1] Amengual, M. "Solution to Problem EM-55". *Arhimede math. j.* 5.2 (2018), pp. 159–160.

[2] Berindeanu, M. "Problem EM-55". *Arhimede math. j.* 5.1 (2018), p. 33.

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Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.
2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

April 30, 2021

Elementary Problems

E–83. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $a_1, a_2, a_3, a_4, a_5, a_6$ be six distinct positive integers. Show that

$$\prod_{1 \le i < j \le 6} (a_i - a_j)$$

is a multiple of 320.

E–84. *Proposed by Henry Ricardo, Westchester Area Math Circle, Purchase New York, USA.* The following function is constant on the interval I:

$$f(x) = \sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}}.$$

Determine the interval I and the constant value of f on I.

E–85. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be distinct positive numbers such that

$$\frac{\ln a}{b^2 - c^2} = \frac{\ln b}{c^2 - a^2} = \frac{\ln c}{a^2 - b^2}.$$

Compute the value of $a^{a^2} b^{b^2} c^{c^2}$.

E–86. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Show that no prime of the form $4n + 27$ can be written as the sum of two squares.

E–87. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let ℓ_a, ℓ_b, ℓ_c be the length of the bisectors of the angles of triangle ABC. If a, b, c are the length of its sides, then prove that

$$\frac{\ell_a}{\cos A/2} + \frac{\ell_b}{\cos B/2} + \frac{\ell_c}{\cos C/2} \le \frac{ab}{c} + \frac{bc}{a} + \frac{ca}{b}.$$

E–88. *Proposed by Mihaela Berindeanu, Bucharest, România.* Let AB be the diameter of a circle Γ. Points C and D lie on either side of AB, so that $AD = AC$. Knowing that $E \in BD$, $F \in BC$ so that $DF \perp AE$, $AF \cap EC = \{X\}$ and Y is the middle point of AE, show that XYD is an isosceles triangle.

Easy–Medium Problems

EM–83. *Proposed by Oriol Baeza Guasch, CFIS, BarcelonaTech, Terrassa, Spain.* Let Γ_1 and Γ_2 be two non-intersecting, non-overlapping circles, whose common external tangents meet at O. One of the internal common tangents meets Γ_1 at P, and Γ_2 at R. Denote by S the intersection of ray OP with Γ_1 closest to P, and denote by Q the intersection of ray OR with Γ_2 closest to R. Prove that PR, QS and the line joining the centers of Γ_1 and Γ_2 concur.

EM–84. *Proposed by Mihaela Berindeanu, Bucharest, România.* Let a, b, c be positive real numbers. Prove that

$$\frac{a^9}{bc} + \frac{b^9}{ca} + \frac{c^9}{ab} \geq a^2b^2c^2\,(a+b+c).$$

EM–85. *Proposed by Henry Ricardo, Westchester Area Math Circle, Purchase New York, USA.* A group of people joined a parade. If they formed the same three lines, then there was one person left (not included in the line). If they formed four lines, there were two people left. If they formed five lines, there were three people left. What is the minimum number of people in the group?

EM–86. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $y = ax^2 - bx - c$, $y = bx^2 - cx - a$ and $y = cx^2 - ax - b$ be the equations of three parabolas where a, b, c are non-zero real numbers. If their graphics have a common point, then prove that $a = b = c$.

EM–87. *Proposed by Henry Ricardo, Westchester Area Math Circle, Purchase New York, USA.* For any positive integer N, prove that

$$\frac{\pi^2}{6} - \frac{1}{N} < \sum_{n=1}^{N} \frac{1}{n^2} < \frac{\pi^2}{6} - \frac{1}{N+1}.$$

EM–88. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let x_1, x_2, x_3, x_4 be the roots of the equation $x^4 - x^3 + 5x^2 + 2x + 1 = 0$. Find an equation of fourth degree with roots t_1, t_2, t_3, t_4 such that $t_1 = (x_2 + x_3 + x_4)\, x_2x_3x_4$, $t_2 = (x_3 + x_4 + x_1)\, x_3x_4x_1$, $t_3 = (x_4 + x_1 + x_2)\, x_4x_1x_2$ and $t_4 = (x_1 + x_2 + x_3)\, x_1x_2x_3$, respectively.

Medium–Hard Problems

MH–83. *Proposed by Dorin Mărghidanu, Colegiul National "A.I. Cuza", Corabia, Romania.* Let a, b, c, d be the lengths of the sides of a quadrilateral. Prove that

$$\sum_{\text{cyclic}} \sqrt{\frac{b+c+d-a}{a}} > 4.$$

MH–84. *Proposed by Nicolae Papacu, Slobozia, Romania.* Let z_1, z_2, z_3 be three distinct nonzero complex numbers.

(a) If $|z-z_k| = |z_k|$ for $1 \leq k \leq 3$, then prove that $z_1+z_2+z_3 = 0$.
(b) Let p be an integer that it is not a multiple of 3. If $|z_1| = |z_2| = |z_3|$, then prove that $z_1^p + z_2^p + z_3^p = 0$.

MH–85. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all triples (x, y, z) of nonnegative integers such that $x^3 + 4y^3 = 2z^3$.

MH–86. *Proposed by Mihály Bencze, Braşov, Romania.* Let $F, G\colon \mathbb{R} \to \mathbb{R}$ be the primitives of the functions $f, g\colon \mathbb{R} \to \mathbb{R}$, respectively. Determine $f(x)$ and $g(x)$ if for all $x \in \mathbb{R}$ it holds that $f(x) + G(x) = x$ and $F(x) + g(x) = -x$.

MH–87. *Proposed by Oriol Baeza Guasch, CFIS, BarcelonaTech, Terrassa, Spain.* Let ABC be a triangle, and let D, E be the contact points of the incircle and the A-excircle on side BC, respectively. Let F be the intersection of ray AD with the A-excircle, closest to D; and let G be the intersection of ray AE with the incircle, closest to E. Prove that FG, side BC and the bisector of A concur.

MH–88. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* The 2020-th prime number is 17573, and the next prime is 17579. Show that there exists a sequence of 2020^{2020} consecutive positive integers that contains exactly 2020 prime numbers.

Advanced Problems

A–83. *Proposed by Henry Ricardo, Westchester Area Math Circle, NY, USA.* For any nonnegative integer n, define

$$I_n = \int_0^{\pi/2} \frac{\sin^2 nt}{\sin t}\,\mathrm{d}t.$$

Find $\lim\limits_{n\to\infty}(2I_n - \ln n)$.

A–84. *Proposed by Mihaela Berindeanu, Bucharest, România.* Let $A \in M_2(\mathbb{R})$ be a matrix with $\det A = 3$ and

$$\det\left(A^4 + 5A^2 + 9I_2\right) = 81.$$

Show that at least one of the matrices $A + I_2$ or $A - I_2$ is invertible.

A–85. *Proposed by Vasile Mircea Popa, "Lucian Blaga" University of Sibiu, Romania.* Calculate

$$\int_{-1}^{1} \frac{\arccos x}{\sqrt{3x^4 + 2x^2 + 3}}\,\mathrm{d}x.$$

A–86. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Suppose V is a finite inner-product space and $v, w \in V$. Let $T\colon V \to V$ be a linear transformation defined by $Tu = \langle u, v\rangle w$. Find a formula for the trace of T.

A–87. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $n \geq k \geq 0$ be integers. If $\gamma = \{z \in \mathbb{C} : |z| = 1\}$, then prove that

$$\frac{1}{2^{n+1}\pi i} \sum_{k=0}^{n} \oint_\gamma \frac{(1+z)^n}{z^{k+1}}\,\mathrm{d}z$$

is an integer and determine its value.

A–88. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $(\boldsymbol{A}, +, \circ)$ be a commutative ring. If $\boldsymbol{I}$ and $\boldsymbol{J}$ are two ideals of $\boldsymbol{A}$, then prove that

$$\sqrt{\boldsymbol{I} \cap \boldsymbol{J}} = \sqrt{\boldsymbol{I}} \cap \sqrt{\boldsymbol{J}}$$

and compute $\sqrt{\mathbf{4320}\mathbb{Z}} \cap \sqrt{\mathbf{3024}\mathbb{Z}}$.

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

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The Versatile Mean Value Theorem

Henry Ricardo

Abstract

This article highlights the versatility of the humble Mean Value Theorem by providing examples that go beyond the usual calculus course illustrations.

1 Introduction

As presented traditionally in calculus textbooks, the Mean Value Theorem (MVT) is viewed as a consequence of Rolle's Theorem:

Theorem (Rolle). *Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then for some $c \in (a,b)$, $f'(c) = 0$.*

Geometrically, this says that if a continuous curve passes through the same y-value twice and has a unique tangent line at every point of the interval, then somewhere between the endpoints it has a tangent parallel to the x-axis (figure 1).

This result was given in the context of locating the zeros of polynomials—and without a calculus-based proof—by Michel Rolle in 1691, although the basic idea has been found in a 12th-century treatise by the Indian mathematician Bhāskara II (see [11, §3.2] for further details). By applying Rolle's Theorem to the function

$$\varphi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}, \qquad (*)$$

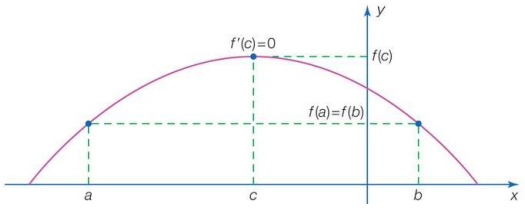

Figure 1: Rolle's Theorem

which is the difference of the ordinate of a point on the graph of $f(x)$ and the ordinate to the secant line for the same x, we obtain the Mean Value Theorem (MVT) (figure 2). To many students, the appearance of the auxiliary function $(*)$ is mathematical sleight of hand, and there has been much discussion in the educational literature about this.

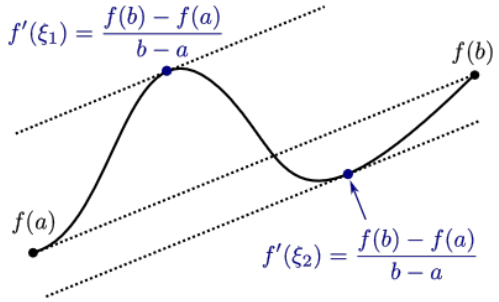

Figure 2: The Mean Value Theorem (MVT)

Theorem (Lagrange, 1797). *Let $f\colon [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Then, there is a $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

or, equivalently,

$$f(b) = f(a) + f'(c)(b-a).$$

Note that Rolle's Theorem is now a special case of the MVT, and we shall treat it as such in what follows.

The following quote from a textbook [9],

> *The Mean Value Theorem is the midwife of calculus—not very important or glamorous by itself, but often helping to deliver other theorems that are of major significance*,

emphasizes that, in most calculus courses, the dynamic duo of Rolle's Theorem and the MVT are usually employed to obtain results that are considered more important (and sometimes wrongly taken as 'obvious')—for example (in no particular order),

- If $f'(x) > 0 (< 0)$ on an interval, then f is strictly increasing (decreasing) on the interval.
- If $f'(x) = 0$ for all $x \in [a, b]$, then f is constant on $[a, b]$.
- If $f''(c) > 0 (< 0)$ for $c \in (a, b)$, then the graph of f is concave upward (downward) at $(c, f(c))$.
- L'Hôpital's Rule.
- The formula for the arc length of a smooth function over an interval $[a, b]$.
- Functions with the same derivative differ by a constant.

The MVT has many extensions and generalizations, some of which will be discussed briefly in Section 3. For now, we note that **Taylor's Theorem**, which allows us to approximate functions by polynomials with an estimate of the error involved, is a very important extension of the MVT, sometimes referred to as the **Higher Order Mean Value Theorem**.

Our goal now is to demonstrate the usefulness of the MVT in solving a variety of problems involving algebra, number theory, and analysis and to provide readers with some opportunities to exhibit their problem-solving skill.

2 Some Applications

We start with perhaps the simplest "practical" use of the MVT—interesting at least to students in countries with toll roads and their accompanying traffic laws.

A trucker travels m miles on a toll road with a speed limit of M miles per hour. The trucker completes the m mile journey in h hours. At the end of the toll road, the trucker is issued a speeding ticket. Why?

Of course, this is the quintessential use of the Mean Value Theorem, connecting average velocity with instantaneous velocity: Let $f(t)$ represent the truck's odometer reading at time t on its journey. Then, $f(0)$ denotes the reading when the truck enters the toll road and $f(h)$ is the reading h hours later when the truck exits. The Mean Value Theorem then states that there is a value t^*, $0 < t^* < h$, for which

$$f'(t^*) = \frac{f(h) - f(0)}{h - 0} = \frac{m}{h} \text{ (miles per hour).}$$

In our situation, it seems that $f'(t^*) > M$—that is, at some time during its trip, the truck's speed exceeded the posted speed limit. While local traffic laws might be lenient toward someone traveling only a little faster than the speed limit, our unfortunate trucker has apparently exceeded the speed limit tolerance and must suffer the financial consequences.

Zipping past this simple example and continuing our mathematical journey, we shall highlight the key role played by the MVT in establishing inequalities and solving equations. As we'll see, choosing the right function(s) to which to apply the MVT is the important consideration here. It is rare that we get to apply the MVT directly to the given function in a problem. Usually a more

sophisticated function (exploiting the conditions specified in the problem) must be devised—as was the case in Section 1 where we used the auxiliary function $(*)$ to derive Lagrange's theorem from Rolle's Theorem.

The first nontrivial example we consider is in the spirit of Rolle's investigations into the location of zeros of polynomials.

Example 1. *Prove that* $6x^3 - 2x^2 + 4x - 3$ *has exactly one real root.*

Proof. First we realize, since complex roots occur in pairs, that a cubic polynomial has either one or three real roots. Let $p(x) = 6x^3 - 2x^2 + 4x - 3$, and suppose that p has (at least) the two real roots r_1 and r_2. Then $p(r_1) = 0 = p(r_2)$ and Rolle's Theorem implies that there exists $\xi \in (r_1, r_2)$ such that $p'(\xi) = 0$. But $p'(x) = 18x^2 - 4x + 4 = 0$ has the same roots as $9x^2 - 2x + 2 = 0$—namely, $1 \pm \sqrt{17}\, i$. Thus no real number ξ exists, and this contradiction shows that our polynomial equation has only one real root. □

The next problem features *two* applications of the MVT.

Example 2 ([5], problem 234). *Prove that*

$$\sqrt[2015]{2} + 2\sqrt[2015]{6} < \sqrt[2015]{4} + 2\sqrt[2015]{5}.$$

Proof. Consider $f(x) = \sqrt[2015]{x}$ for $x > 0$. We can easily calculate that $f''(x) = -2014/(2015)^2\, x^{-4029/2015} < 0$, so that f' is decreasing. The MVT applied to $f\colon [5,6] \to \mathbb{R}$ and then $f\colon [2,4] \to \mathbb{R}$ states that there exist $c_1 \in (5,6)$ such that

$$f'(c_1) = (f(6) - f(5))/(6-5)$$

and $c_2 \in (2,4)$ such that

$$f'(c_2) = (f(4) - f(2))/(4-2).$$

Thus $2 < c_2 < 4 < 5 < c_1 < 6$, and $c_2 < c_1$ implies that $f'(c_1) < f'(c_2)$ and so $f(6) - f(5) < (f(4) - f(2))/2$. This can

be written as $f(2) + 2f(6) < f(4) + 2f(5)$, which is the desired result. □

The logarithmic inequality that follows is important in many limit calculations.

Example 3. *Prove that* $\dfrac{x-1}{x} < \ln x < x - 1$ *for* $x > 1$.

Proof. Consider $f(t) = \ln t$ on the interval $[1, x]$, where $x > 1$. By the MVT, there exists $c \in (1, x)$ such that

$$f'(c) = \frac{f(x) - f(1)}{x - 1}, \quad \text{or} \quad \frac{1}{c} = \frac{\ln x}{x - 1}.$$

Therefore, $\ln x = (x-1)/c < x - 1$ since $c > 1$. Similarly, $\ln x = (x-1)/c > (x-1)/x$ since $c < x$. □

Example 4. *Prove that* $a^b + b^a > 1$ *for all* $a, b > 0$.

Proof. If $a \geq 1$ or $b \geq 1$, the result is obvious. Thus we assume that $a, b \in (0, 1)$, and we can write $a = 1 - c$, $b = 1 - d$, with $c, d \in (0, 1)$. If we apply the MVT to the function defined by $f(x) = (1 - cx)^d$, $x \in [0, 1]$, we see that there exists $\xi \in (0, 1)$ such that

$$(1 - c)^d = 1 - \frac{cd}{(1 - c\xi)^{1-d}} < 1 - cd.$$

Therefore,

$$a^b + b^a = \frac{1-c}{(1-c)^d} + \frac{1-d}{(1-d)^c} > \frac{1-c}{1-cd} + \frac{1-d}{1-cd} = \frac{(1-c)(1-d)}{1-cd} + 1.$$

□

The next problem expands on the technique used in Example 1 and reminds us that multiple applications of Rolle's Theorem may be necessary to derive a result.

Example 5 (Putnam Competition, 1973). *How many real zeros does the function* $f(x) = 2^x - 1 - x^2$ *have?*

Solution. We claim there are three real zeros. A brief examination of the function yields $x = 0$ and $x = 1$ as solutions of $f(x) = 0$.

Since $f(4) = -1 < 0$ and $f(5) = 6 > 0$, the Intermediate Value Theorem implies that there is a zero of f between 4 and 5. Now we prove that there are no additional zeros.

Supposing that there were another zero, we can arrange the four zeros in ascending order: $x_1 < x_2 < x_3 < x_4$. Then, the following implications result from Rolle's Theorem:

$$\begin{aligned} f(x_1) = 0 = f(x_2) &\implies f'(c_1) = 0,\ x_1 < c_1 < x_2; \\ f(x_2) = 0 = f(x_3) &\implies f'(c_2) = 0,\ x_2 < c_2 < x_3; \\ f(x_3) = 0 = f(x_4) &\implies f'(c_3) = 0,\ x_3 < c_3 < x_4. \end{aligned}$$

But

$$f'(c_1) = 0 = f'(c_2) \implies f''(d_1) = 0,\ c_1 < d_1 < c_2$$

and

$$f'(c_2) = 0 = f'(c_3) \implies f''(d_2) = 0,\ c_2 < d_2 < c_3$$

since f' satisfies the conditions for Rolle's Theorem. Finally, $f''(d_1) = 0 = f''(d_2)$ implies $f'''(\xi) = 0$ for $\xi \in (d_1, d_2)$. But $f'''(x) = (\ln x)^3 2^x \neq 0$ for any real x not equal to 1. This contradiction establishes that there can be only three zeros. □

We now present an interesting analysis problem.

Example 6 (Dorin Andrica [1]). *Let $f : \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function with positive second derivative. Prove that $f(x + f'(x)) \geq f(x)$ holds for any real number x.*

Solution. If there is a point x^* such that $f'(x^*) < 0$, apply the MVT on the interval $[x^* + f'(x^*), x^*]$ to get

$$\frac{f(x^*) - f(x^* + f'(x^*))}{x^* - (x^* + f'(x^*))} = f'(c),$$

or

$$f(x^*) - f(x^* + f'(x^*)) = f'(c)(-f'(x^*))$$

for some c with $x^* + f'(x^*) < c < x^*$). Since we are given that $f''(x) > 0$, we see that f' is increasing. Hence $f'(c) < f'(x^*) < 0$, and this implies $f(x^*) - f(x^* + f'(x^*)) < 0$.

On the other hand, if $f'(x^*) > 0$ for some x^*, the same argument yields $f(x^* + f'(x^*)) - f(x^*) = f'(x^*)f'(c)$ for c between x^* and $x^* + f'(x^*)$, leading to $f'(c) > f'(x^*) > 0$. Again we obtain $f(x^*) - f(x^* + f'(x^*)) < 0$. □

The next result, essentially a differential equations theorem, plays an important role in many areas of applied mathematics.

Example 7 (Lyapunov's inequality, 1907). *Let p be a real-valued continuous function on $[a, b]$ ($p \not\equiv 0$) and let f be a non-trivial function of class C^2 [functions having continuous second derivatives] such that $f''(x) + p(x)f(x) = 0$ for all $x \in [a, b]$ and $f(a) = f(b) = 0$. Then,*

$$\int_a^b \left| \frac{f''(x)}{f(x)} \right| \mathrm{d}x > \frac{4}{b-a}.$$

Proof. Since f is continuous on a closed interval, it attains a maximum value M. Let $M = \max_{a \le x \le b} |f(x)| < \infty$. Then,

$$\int_a^b \left| \frac{f''(x)}{f(x)} \right| \mathrm{d}x > \frac{\displaystyle\int_a^b |f''(x)|\,\mathrm{d}x}{M} > \frac{\left|\displaystyle\int_c^d f''(x)\,\mathrm{d}x\right|}{M} = \frac{|f'(d) - f'(c)|}{M} \tag{1}$$

for arbitrary $a \le c < d \le b$. Take $x_0 \in (a, b)$ such that $f(x_0) = M$. Then, the MVT implies the existence of $\xi_1 \in (a, x_0)$ and $\xi_2 \in (x_0, b)$ such that $M = (x_0 - a)f'(\xi_1) = -(b - x_0)f'(\xi_2)$. Combining this last relation with (1), we get

$$\int_a^b \left| \frac{f''(x)}{f(x)} \right| \mathrm{d}x > \frac{1}{x_0 - a} + \frac{1}{b - x_0} > \frac{4}{b-a},$$

where the last inequality follows from finding the minimum value of $g(x) = (x - a)^{-1} + (b - x)^{-1}$, occurring at $x = (a + b)/2$. □

Example 8 (J. L. Díaz-Barrero [2]). *Let $0 < a < b$ be real numbers, and let $f, g\colon [a, b] \to (0, \infty)$ be continuous functions. Prove that there exists $c \in (a, b)$ such that*

$$\left(\frac{1}{f(c)} + \frac{1}{\int_c^b g(t)\,\mathrm{d}t} \right) \left(g(c) + \int_a^c f(t)\,\mathrm{d}t \right) \ge 4.$$

Proof. Define $F(x) = \int_a^x f(t)\,\mathrm{d}t \cdot \int_x^b g(t)\,\mathrm{d}t$. Since $F(a) = F(b) = 0$, Rolle's theorem tells us that there exists $c \in (a,b)$ such that $0 = F'(c) = f(c)\int_c^b g(t)\,\mathrm{d}t - g(c)\int_a^c f(t)\,\mathrm{d}t$, or

$$f(c)\int_c^b g(t)\,\mathrm{d}t = g(c)\int_a^c f(t)\,\mathrm{d}t. \qquad (\star)$$

Because f and g are nonnegative, the AM-GM inequality yields

$$\left(\frac{1}{f(c)} + \frac{1}{\int_c^b g(t)\,\mathrm{d}t}\right)\left(g(c) + \int_a^c f(t)\,\mathrm{d}t\right)$$
$$\geq \frac{2}{\sqrt{f(c)\int_c^b g(t)\,\mathrm{d}t}} \cdot 2\sqrt{g(c)\int_a^c f(t)\,\mathrm{d}t} = 4$$

by statement $(\star)$. □

Example 9 (Duong Viet Thong). *Let f be a continuously differentiable function on $[a,b]$ such that $\int_a^b f(x)\,\mathrm{d}x = 0$. Prove that*

$$\left|\int_a^b xf(x)\,\mathrm{d}x\right| \leq \frac{(b-a)^3}{12}\max\{|f'(x)| : x \in [a,b]\}.$$

Proof. Integration by parts gives us

$$\int_a^b xf(x)\,\mathrm{d}x = -\int_a^b \int_a^x f(t)\,\mathrm{d}t\,\mathrm{d}x,$$

so $\left|\int_a^b xf(x)\,\mathrm{d}x\right| \leq \int_a^b \left|\int_a^x f(t)\,\mathrm{d}t\right|\mathrm{d}x$. Set $F(x) = \int_a^x f(t)\,\mathrm{d}t$ for $x \in [a,b]$, noting that $F(a) = 0 = F(b)$, $F'(x) = f(x)$, and $F''(x) = f'(x)$. Now, for $a < x < b$, define

$$g(t) = (t-a)(t-b)F(x) - (x-a)(x-b)F(t).$$

Since $g(a) = g(x) = g(b) = 0$, we can apply Rolle's Theorem to see that there exist $c_1 \in (a,x)$ such that $g'(c_1) = 0$ and $c_2 \in (x,b)$

such that $g'(c_2) = 0$. Applying Rolle's Theorem again, this time to g', we know that there is a point $c \in (c_1, c_2) \subset (a, b)$ such that $g''(c) = 0$—that is, $g''(c) = 2F(x) - (x-a)(x-b)F''(c) = 0$ for some $c \in (a, b)$. This gives the estimate

$$|F(x)| \;=\; \frac{|(x-a)(x-b)F''(c)|}{2} \;\leq\; \frac{M(x-a)(b-x)}{2},$$

where $M = \max_{a\leq x\leq b} |F''(x)| = \max_{a\leq x\leq b} |f'(x)|$. Then, integration yields the estimate $\displaystyle\int_a^b |F(x)|\,\mathrm{d}x \leq \frac{(b-a)^3}{12} \cdot M$, which finishes our solution. □

The next example is a famous theorem that shows that irrational *algebraic* numbers cannot be approximated well by rational numbers. (Recall that an algebraic number is the root of a nonzero polynomial with integer coefficients.) For a more encouraging result about rational approximation to irrational numbers in general, see Problem 6 in [3].

Example 10 (Liouville). *Let P be an n-th degree nonzero polynomial with integer coefficients, and suppose that α is an irrational zero of P. Then, there exists a constant $C > 0$ such that, for all rational numbers p/q with $p, q \in \mathbb{Z}$ and $q \neq 0$, we have*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{C}{|q|^n}.$$

Proof. Let $P(x) = \sum_{k=0}^{n} a_k x^k$. Since P' is continuous on I, it is bounded, and we can choose M_α such that $M_\alpha \geq |P'(x)|$ for all $x \in I = (\alpha - 1, \alpha + 1)$. Suppose now that $p/q \neq \alpha$ is an approximation to α. We may assume the approximation is close enough that p/q lies in I and is nearer to α than any other root of $P(x) = 0$, so that $P(p/q) \neq 0$. Now we show that $|\alpha - p/q| \geq |P(p/q)|/M$ for all $x = p/q \in I$.

We have

$$\begin{aligned}\left|P\left(\frac{p}{q}\right)\right| &= \left|a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \ldots + a_1\left(\frac{p}{q}\right) + a_0\right| \\ &= \frac{|a_n p^n + a_{n-1} q p^{n-1} + \ldots + a_1 q^{n-1} p + a_0 q^n|}{|q|^n} \\ &\geq \frac{1}{|q|^n}\end{aligned}$$

since the numerator of the fraction must be a positive integer. Then, applying the MVT, we have $P'(c) = (P(x) - P(\alpha))/(x - \alpha) = P(x)/(x - \alpha)$ for some $c \in I$. Thus,

$$M_\alpha \geq |P'(c)| = \frac{|P(p/q)|}{|\alpha - p/q|}, \quad \text{or} \quad \left|\alpha - \frac{p}{q}\right| \geq \frac{|P(p/q)|}{M_\alpha} \geq \frac{C}{|q|^n},$$

where $C = 1/M_\alpha$. □

Descartes's rule of signs, which puts an upper bound on the number of positive roots of polynomials, is expressed as an inequality. Rolle's Theorem has been used to provide a simple proof of a generalization of this well-known algebraic criterion.

Example 11 (Komornik [6]). *Let $p(x) = a_0 x^{b_0} + a_1 x^{b_1} + \ldots + a_n x^{b_n}$ be a function with nonzero real coefficients $a_0, a_1, \ldots, a_n$ and real exponents $b_0, b_1, \ldots, b_n$ satisfying $b_0 > b_1 > \ldots > b_n$. Then, p cannot have more positive roots (even counted with multiplicity) than the number of sign changes in the sequence $a_0, a_1, \ldots, a_n$.*

Proof. Denote by $z(p)$ the number of positive roots of p and by $v(p)$ the number of sign changes in the sequence $a_0, \ldots, a_n$. We prove that $z(p) \leq v(p)$ by induction on $v(p)$. The case $v(p) = 0$ is obvious because in this situation all terms of $p(x)$ have the same sign for all positive values of x, so $z(p) = 0$.

If $v(p) > 0$, then we choose an index i for which $a_i a_{i+1} < 0$. Since dividing $p(x)$ by some power of x changes neither $v(p)$ nor $z(p)$, we may assume that $b_i > 0$ and $b_{i+1} < 0$. If we differentiate p_i, then none of the coefficients $a_0, \ldots, a_i$ changes sign, whereas

$a_{i+1}, \ldots, a_n$ all change signs. It follows that $v(p') = v(p) - 1$. Furthermore, we apply Rolle's Theorem to show that

$$z(p') \geq z(p) - 1.$$

To do this, we first note that, if $x_1 < \ldots < x_k$ are the positive roots of p with multiplicities $m_1, \ldots, m_k$, respectively, then they are also roots of p' with multiplicities $m_1 - 1, \ldots, m_k - 1$. Now Rolle's Theorem ensures that each of the $k - 1$ open intervals (x_i, x_{i+1}) contains at least one root of p'. Hence,

$$\begin{aligned} z(p') &\geq (m_1 - 1) + \cdots + (m_k - 1) + (k - 1) \\ &= (m_1 + \cdots + m_k) - 1 = z(p) - 1. \end{aligned}$$

Since $z(p') \leq v(p')$ by the induction hypothesis, we conclude that

$$z(p) \ \leq z(p') + 1 \ \leq v(p') + 1 \ = \ v(p). \qquad \square$$

3 Extensions and Generalizations

One of the first extensions/generalizations of the basic MVT is a result of Cauchy.

Theorem (Cauchy 1823). *Let f and g be continuous real-valued functions defined on $[a, b]$ that are differentiable on (a, b) and are such that $g(a) \neq g(b)$. Then, there is a point $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

This version of a mean value theorem is the one usually used to prove L'Hôpital's rule. Another variant of Lagrange's mean value theorem is a fairly modern one.

Theorem (Flett [4]). *If f is differentiable in $[a, b]$ and $f'(a) = f'(b)$, then there is a point $\xi \in (a, b)$ such that*

$$f'(\xi) = \frac{f(\xi) - f(a)}{\xi - a}.$$

Finally, we recall that the *mean* or *average* value of a continuous function f on an interval $[a, b]$ is defined as

$$\overline{f} = \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x.$$

The First Mean Value Theorem for Integrals can be stated as follows:

Theorem (Cauchy 1823). *Let $f\colon [a, b] \to \mathbb{R}$ be a continuous function. Then, there exists $\xi \in [a, b]$ such that*

$$f(\xi) = \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x.$$

The conclusion of this theorem can be written as

$$\int_a^b f(x)\,\mathrm{d}x = f'(\xi)(b-a)$$

for some $\xi \in [a, b]$. Geometrically, this says that the (signed) area under the curve $y = f(x)$ equals the signed area of a rectangle of 'width' $f(\xi)$ and length $b - a$ for some $\xi \in [a, b]$.

As the name implies, there is a Second Mean Value Theorem for Integrals. In fact, each of the theorems in this section has been extended and generalized considerably, for both real-valued and vector-valued functions. (See, for example, [7], [11] or Chapter 5 of [9].)

4 Some Exercises

We end this excursion by providing a few problems for the reader. The examples given in the last section (and a hint or two) should provide a template adequate to solve these exercises.

Problem 1. *The function $f(x)$ is continuous and differentiable on the interval $[-7, 0]$, $f(-7) = -3$, and $f'(x) \leq 2$. What is the largest possible value for $f(0)$?*

Problem 2. *Prove that* $f(x) = x^3 - 7x^2 + 25x + 8$ *has exactly one real root.*

Problem 3. *Prove that* $|\cos a - \cos b| \leq |a - b|$ *for all real* a, b.

[Note: This result shows the uniform continuity of the cosine on any closed interval.]

Problem 4. *Let* $\varphi(x) = (x - a)^m (x - b)^n$, *where* m *and* n *are positive integers. Show that the 'c' of Rolle's Theorem divides the interval* $[a, b]$ *in the ratio* m/n.

Problem 5. *Solve the equation* $4^x - 3^x = x$ *in* $\mathbb{R}$.

Problem 6. *Let* $f\colon [0, 1] \to \mathbb{R}$ *be differentiable such that* $|f'(x)| < 1$ *for all* $x \in [0, 1]$. *Show that there exists at most one* $c \in [0, 1]$ *such that* $f(c) = c$.

Problem 7. *Let* $f\colon [1, 2] \to \mathbb{R}$ *be a continuous function such that* $\int_1^2 f(x)\,\mathrm{d}x = 0$. *Prove that there exists a real number* $c \in (1, 2)$ *such that*

$$cf(c) = \int_c^2 f(x)\,\mathrm{d}x.$$

[Hint: Consider $F(t) = t \int_1^t f(x)\,\mathrm{d}x$.]

Problem 8. *Consider a continuous function* $f\colon [a, b] \to \mathbb{R}$ *satisfying* $\int_a^b f(x)\,\mathrm{d}x = 0$. *Show that there exists a real number* $c \in (a, b)$ *such that*

$$\int_a^c f(x)\,\mathrm{d}x = \left(\frac{a+b}{2} - c\right) f(c).$$

[Hint: Consider the auxiliary function $g(x) = \left(\frac{a+b}{2} - c\right) \int_a^x f(t)\,\mathrm{d}t$.]

Problem 9. *Let* $f(x)$ *be differentiable on* $[0, 1]$ *with* $f(0) = 0$ *and* $f(1) = 1$. *For each positive integer* n, *show that there exist distinct* $x_1, x_2, \ldots, x_n$ *such that* $\sum_{i=1}^n 1/f'(x_i) = n$.

[Hint: Use the Intermediate Value Theorem to construct a sequence of points $0 = c_0 < c_1 < c_2 < \ldots < c_{n-1} < c_n = 1$ such that $f(c_i) = i/n$. Then apply the MVT on each interval $[c_{i-1}, c_i]$ for $i = 1, 2, \ldots, n$.]

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Inversion

Marc Felipe i Alsina

1 Introduction

Inversion is a transformation of the plane that acts like a reflection with respect to a circle. Unlike other well-known planar transformations, an inversion does not preserve distance, nor proportions nor parallelism. However, it has some nice properties that we will explore.

First of all, let us start with the definition:

Definition. *An inversion with center O and power k is a transformation of the plane that sends each point P to a point P' on the ray OP such that $OP \cdot OP' = k$.*

We can see directly from the definition that inversions are involutions, that is, when we do the same inversion twice, we end up where we started.

In the case where the power k is a negative number, we get that OP' is negative, which just means that P' should be put to the other side of O. Since the only difference between an inversion with positive power and one with negative power is a central symmetry with respect to the center, we will focus only on inversions of positive power from now on. We will set $k = R^2$, and we will call R the radius of inversion. If Γ is a circle with center O and radius R, we may say 'inversion with respect to Γ' instead of 'inversion with center O and power R^2'.

Since $OP \cdot OP' = R^2$ we see that if P is inside the circle of inversion, we have that $OP < R$ and $OP' > R$, so P' lies outside

the circle. The closer P is to O, the further is P'. Similarly, if P is outside the circle, P' ends up inside. The points on the circumference are the only ones fixed by this transformation.

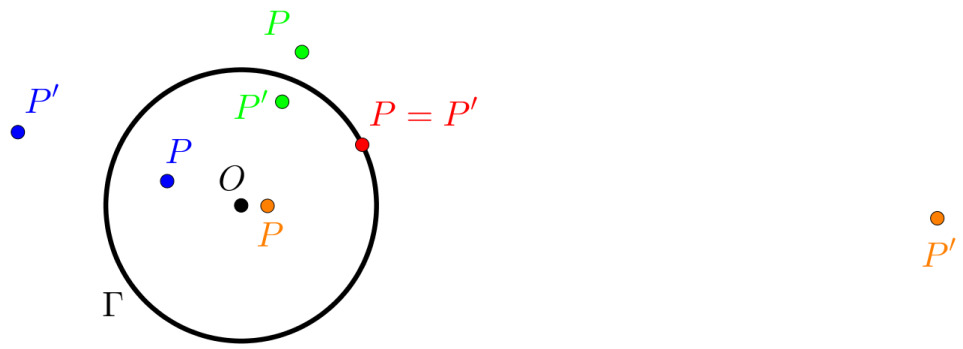

Figure 1: Inversion with respect to Γ illustrated by some points and their images.

The points that are close to the center of inversion are sent to points very far away. The center itself should be sent to a point infinitely far in every direction, so the image of the center is left undefined.

2 Lines and circles

One of the key properties of the inversion is that it transforms lines and circles into lines and circles. For example, if r is a line that passes through the center of Γ, when we invert the points of r with respect to Γ, they land also on r: the two rays outside of Γ are sent to the two segments inside Γ and vice versa, so we can say that r is invariant by this inversion even though the points themselves move around.

Before we tackle the case were the line does not pass through the center, observe what happens when we invert two points which are not aligned with the center: let A and B be such points

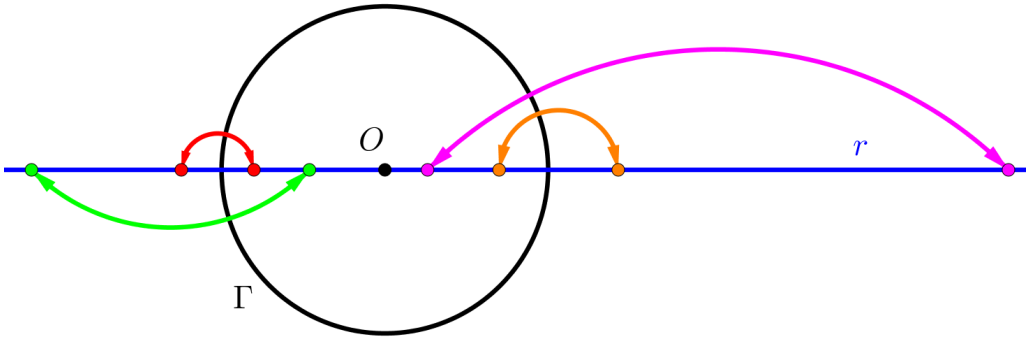

Figure 2: A line through the center is sent to itself.

and let A' and B' be their inverses. By definition, we have that $OA \cdot OA' = R^2 = OB \cdot OB'$, so $\dfrac{OA}{OB} = \dfrac{OB'}{OA'}$ and triangles OAB and $OB'A'$ are similar by the side-angle-side criterion. Notice that the angle at the vertex A is the same as the angle at the vertex B', so the triangle is not only scaled but also flipped.

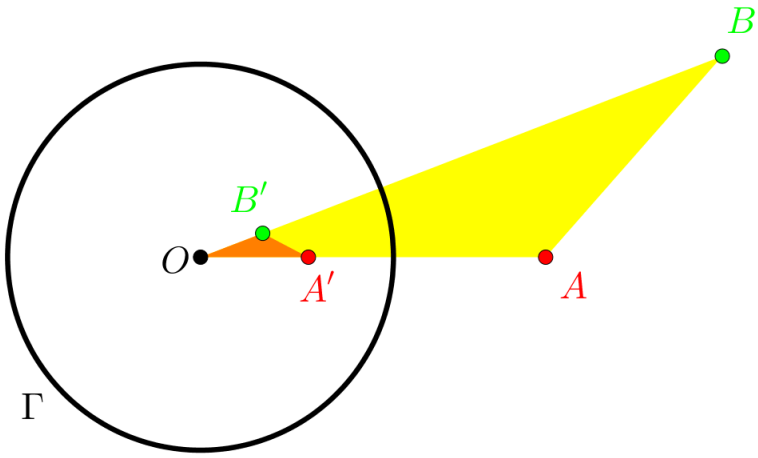

Figure 3: Similarity of triangles with vertex at O.

Now consider a line r not passing through the center O of Γ. We would like to know what curve we obtain when we invert each of the points of r. Consider A to be the foot of the perpendicular

through O. Any point B on r satisfies that $\widehat{OAB}$ is a right angle, so we have that $\widehat{OB'A'}$ is also a right angle, which means that all the points B' must lie on the circle with diameter OA'. It can be checked that all points of that circle, with the exception of O, are inverse to some point on r. Therefore, we can say that the inverse of the line r is a circle passing through O whose diameter containing O is perpendicular to r.

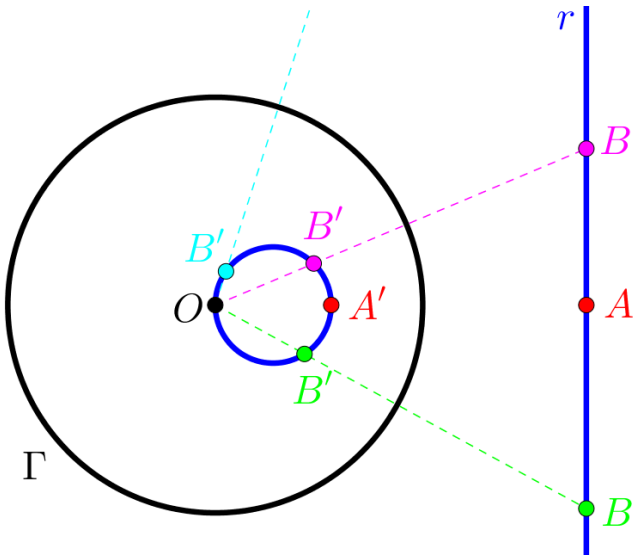

Figure 4: A line not incident with O is sent to a circle through O.

Note that, in the case where the line intersects Γ at two points A and B, we have that its inverse is a circle through O that must pass through A' and B', but A and B are in Γ and are their own inverses, so the inverse of the line is the circle through A, B and O.

So now we know what inversion does to lines, both through and not through the center of inversion. We also know that inverting a circle through O results in a line that does not pass through O, because of the same exact reasoning. What remains is to know what inversion does to circles that do not pass through O.

Let Γ be a circle with radius R centered at O and let ω be another circle that does not pass through O. Take a line through O and let it intersect ω at points A and B. When we invert A and B with respect to Γ, we obtain points A' and B' that satisfy $OA \cdot OA' = OB \cdot OB' = R^2$. Dividing by $OA \cdot OB$ we obtain $\frac{OA'}{OB} = \frac{OB'}{OA} = \frac{R^2}{OA \cdot OB}$. Note that the quantity $\lambda = \frac{R^2}{OA \cdot OB}$ does not depend on the choice of the line through O, since the denominator is the power of O with respect to ω. This means that $OA' = \lambda \cdot OB$ and $OB' = \lambda \cdot OA$. A' and B' lie on circle obtained applying an homothety to ω of center O and ratio λ, and so do all the inverses of the points of ω, obtaining thus the whole circle. Therefore, circles are sent to circles if they do not go through the center of inversion.

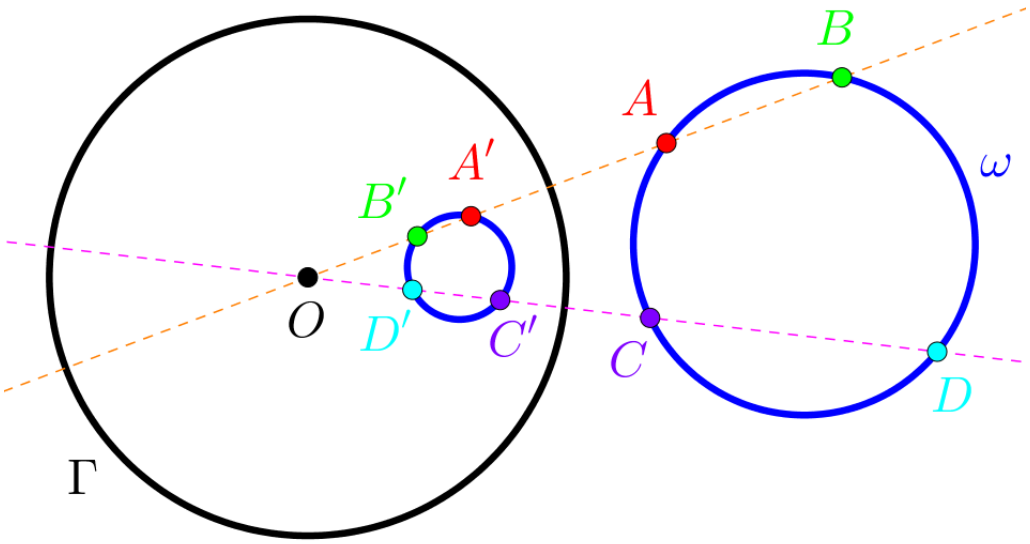

Figure 5: A circle not passing through with O is sent to another circle of the same kind.

3 Some additional observations and problems

Since inversion is defined pointwise, the number of intersections of two curves is preserved. This means that if two circles were tangent, their inverses will also be, since they will intersect at exactly one point. Parallel lines only meet at infinity, so their inverses will be tangent at the center of inversion.

We can also ask the question of when a circle ω (different from Γ) is sent to itself. For this to happen, we need it not to pass through the center for it to be sent to an homothetic circle, and we need that homothety to be at ratio $\lambda = 1$. Therefore, the power of O with respect to ω has to be R^2, which implies that the tangents from O are radii and the circles are orthogonal. This is a special case of a more general theorem that states that if two curves intersect at an angle α (meaning that the tangents at the point of intersection make an angle of α), then the inverses of the two curves also intersect at the same angle.

Sometimes, we may say 'inversion preserves angles' referring to this general theorem, but this sentence should not be interpreted as $\widehat{ABC} = \widehat{A'B'C'}$ for general points A, B and C, which does not hold. Another misconception about inversions is that if a circle is sent to another circle, then the center of the first maps to the center of the second, which is also not true.

We will now present a couple of problems that can be solved using inversion.

Problem 1. *Let Γ_1 and Γ_2 be internally tangent circles respectively centered at O_1 and O_2. Draw a circle centered at the point of tangency T that intersects Γ_1 at P_1 and Q_1 and intersects Γ_2 at P_2 and Q_2. Prove that O_1 lies on the line P_2Q_2 if and only if O_2 lies on the line P_1Q_1.*

Solution. Let Ω be the circle centered at T mentioned in the statement. We are going to invert with respect to Ω. Since Γ_1 and

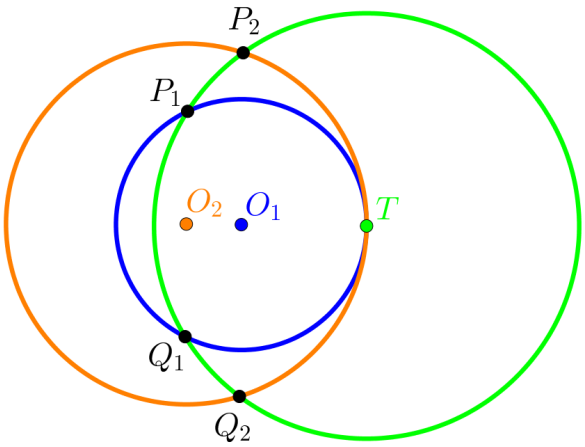

Figure 6: Scheme for the solution of Problem 1.

Γ_2 pass through the center of Ω, they are going to invert to lines. Moreover, since P_1, Q_1, P_2 and Q_2 lie in Ω, we have that Γ_1' is the line P_1Q_1 and Γ_2' is the line P_2Q_2. Now, if O_1 lies on P_2Q_2, then P_2Q_2 contains a diameter and so is orthogonal to Γ_1. This means that their inverses, Γ_2 and the line P_1Q_1, must also be orthogonal, which can only happen if P_1Q_1 passes through O_2. The other implication analogously holds. □

Problem 2. *Draw five different points in the plane such that no three of them are collinear and no four of them are concyclic. We say that a circle is separating if it passes through three of those five points, and the other two points lie one inside of the circle and one outside. Prove that there are exactly four separating circles.*

Solution. There are $\binom{5}{3} = 10$ circles that pass through three of the points and we need to tell which are and which are not separating. Label the points A, B, C, D and E and invert everything with respect to a circle centered at A and arbitrary radius. Six of the circles passed through A, so they will become lines, while the other four will remain circles. We need to see how the condition of being separating is translated once we invert.

On the one hand, imagine a circle ω passing through A and being inverted into a line r. Let P be a interior point of ω. The line AP

intersects ω again after having hit P, so P' will be beyond the intersection of that line with r. This means that the points interior to the ω are sent to a specific half-plane delimited by r, while the points exterior to ω are sent to the other half plane. Therefore, if ω was a separating circle, then the two points among B', C', D' and E' that do not belong to r lie in different half-planes.

On the other hand, suppose now that ω does not pass through A and is sent to another circle ω'. Using a similar reasoning, we see that the interior of ω must be sent to either the interior or the exterior of ω'. However, since A is sent infinitely far away, if A is interior to ω, that interior is sent to the exterior of ω', and if A is exterior to ω, the interior is sent to the interior. In either case, A and P are separated by ω if and only if P' lies inside ω'. This means that a circle will be separating if its inverse is a circle passing through three points and containing the fourth, or a line passing through two of them and separating the other two.

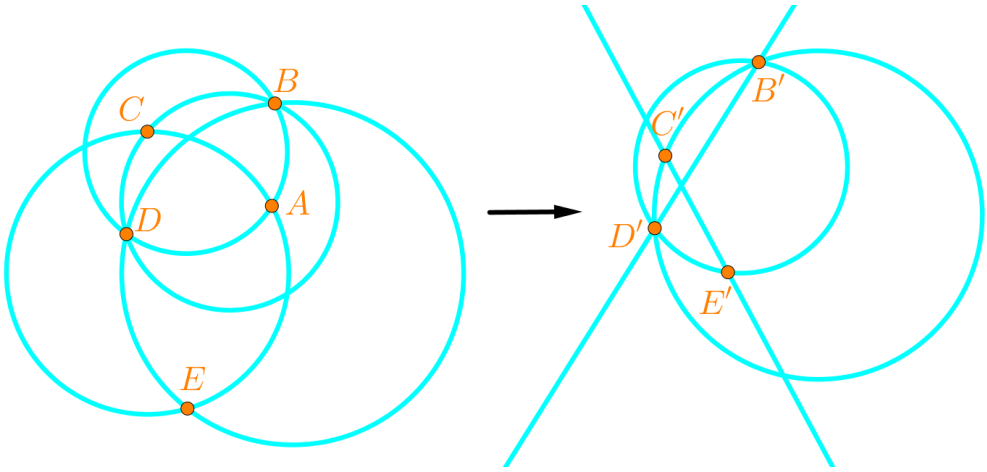

Figure 7: Scheme for the solution of Problem 2.

Now the four inverted points either form a convex quadrilateral $B'C'D'E'$ (relabel the points if necessary) or they do not. If they do, then exactly two of the six lines are separating, the two diagonals, so we need to see that there are exactly two separating circles. Since $B'C'D'E'$ is not cyclic, we have that $\widehat{B'C'E'} \neq \widehat{B'D'E'}$, so one of them is smaller. Whichever is the smaller angle will force the

other point to lie on the inside, so exactly one of the circumcircles of $B'C'E'$ and $B'D'E'$ will be separating. The same reasoning can be applied to $\widehat{C'B'D'} \neq \widehat{C'E'D'}$ to conclude that exactly two of the four circles fulfill the condition.

However, if the four inverted points do not form a quadrilateral, we see that they form a triangle with an interior point. Exactly three of the six lines, the ones through the interior point, count as separating, so we need only one valid circle, which will be the circumcircle of the triangle. Since the angle subtending a side from an interior point to a triangle is always bigger than the angle subtended from the other vertex, the three circles involving the interior point will not work. Adding up, there will be exactly four separating circles in either case. □

Problem 3. *Let s be a line which intersects a pair r, r' of parallel lines at X and Y. The circle ω centered at X that passes through Y intersects r at M and N. The circle Ω, with center M passing through N, and Γ, with center N passing through Y, intersect at P and Q. Show that the line PQ passes through the center of the circle tangent to r, r' and s.*

Solution. Let T_r, $T_{r'}$ and T_s be the three points of tangency of the last circle to the respective lines. Let Z be the projection of Y to r. We have that $XT_rT_{r'}Y$ is a right trapezoid, so $XT_r - YT_{r'} = XZ$. We also have $XT_r + YT_{r'} = XT_s + YT_s = XY = XN$, which implies that $XT_r = \dfrac{XZ + XN}{2}$ and T_r is the midpoint of ZN. Now we will consider an inversion with respect to Γ. Since ω passes through N and Y, its inverse must be a line perpendicular to NX that passes through Y. That line passes through Z, which lies on the invariant line r. This means that the inverse of Z is the other intersection of ω and r, which is M.

Let K be the other point of intersection of Ω and r. Since $MK = MN$ we have that $KN = 2MN$. The point K is twice as far as M, so its inverse will be at half the distance of Z from N, which is T_r. Since Ω also passes through N, its inverse is a line, which is perpendicular to MN and so to r. Since Ω passes through K, the line will pass through T_r, and so it will also pass through the

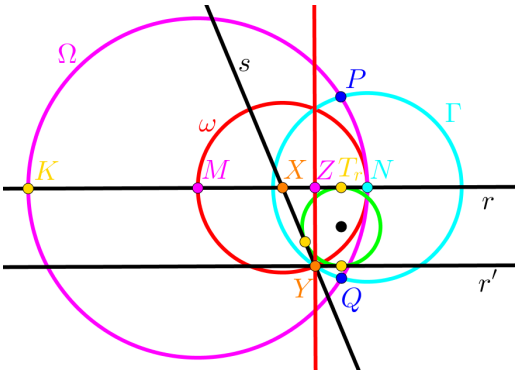

Figure 8: Scheme for the solution of Problem 3.

center of the circle tangent to the three lines. On the other hand, Ω passes through P and Q, which are their own inverses, so the inverse of Ω is the line PQ, and this finishes the proof. □

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Contests

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to find interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Problems and solutions from the 7th edition of the BarcelonaTech Mathcontest

O. Rivero Salgado and J. L. Díaz-Barrero

1 Problems and solutions

Hereafter, we present the four problems that appeared in the paper given to the contestants of Mathcontest 2020, as well as their official solutions.

Problem 1. Determine the positive integers a_n such that, for all integers $n \geq 1$, the following holds:

$$\left(1-\frac{1}{a_1^2}\right)\left(1-\frac{1}{a_2^2}\right)\cdots\left(1-\frac{1}{a_n^2}\right)=\frac{5n+30}{6n+30}.$$

Solution. Let $P_n = \prod_{k=1}^{n}\left(1-\frac{1}{a_k^2}\right)$ for all $n \geq 1$. From the fact that $P_n = \frac{5n+30}{6n+30}$ we have $P_{n+1} = \frac{5n+35}{6n+36}$, and dividing P_{n+1} by P_n yields

$$1-\frac{1}{a_{n+1}^2}=\frac{P_{n+1}}{P_n}=\frac{(n+5)(n+7)}{(n+6)^2}.$$

Then,

$$\frac{1}{a_{n+1}^2}=1-\frac{(n+5)(n+7)}{(n+6)^2}=\frac{1}{(n+6)^2},$$

from which we get $a_{n+1} = n+6$ and $a_n = n+5$ for all $n \geq 1$.

Problem 2. Consider two integer numbers a and b such that $0 \leq a < b$. Determine how many n-tuples of positive integers $(\alpha_1, \ldots, \alpha_n)$ satisfy that

$$\gcd(\alpha_1, \ldots, \alpha_n) = 6^a \qquad \text{and} \qquad \operatorname{lcm}(\alpha_1, \ldots, \alpha_n) = 6^b.$$

Solution. Let p be a prime and let r stand for a natural number. We write $v_p(r)$ for the greatest non-negative integer such that $p^{v_p(r)}$ divides r. That way, we have that

$$a \leq v_2(\alpha_k), v_3(\alpha_k) \leq b, \quad \text{for all } 1 \leq k \leq n.$$

Further, there must exist indexes i, i' such that $v_2(\alpha_i) = v_3(\alpha_{i'}) = a$, and indexes j, j' such that $v_2(\alpha_j) = v_3(\alpha_{j'}) = b$. Hence, for $v_2(\alpha_k)$ (resp. $v_3(\alpha_k)$) we have $b-a+1$ options, with the restrictions that at least one of them has to take the minimum value a and at least other the maximum value b. Using the inclusion-exclusion principle, this gives a total of

$$(b-a+1)^n - 2(b-a)^n + (b-a-1)^n$$

options for $(v_2(\alpha_1), \ldots, v_2(\alpha_n))$ (resp. $(v_3(\alpha_1), \ldots, v_3(\alpha_n))$), since from the total number we have to remove those choices not including neither a nor b, and then add the term corresponding to those options excluding both.

Therefore, writing $d = a - b$, the answer is

$$((d+1)^n + (d-1)^n - 2d^n)^2.$$

Problem 3. Two exterior circles $\gamma_1(O, R)$ and $\gamma_2(O', R')$ are tangent at point A. A straight line through A meets circles γ_1 and γ_2 at points M and M', respectively. Let $N \in \gamma_1$ and $N' \in \gamma_2$ such that $MN \parallel M'N'$. Prove that N, A, and N' are collinear.

Solution. Since isosceles triangles OAM and $O'AM'$ have $\angle OMA = \angle OAM = \angle O'AM' = O'M'A$, then they are similar. We have

$$\frac{R}{R'} = \frac{AM}{AM'}.$$

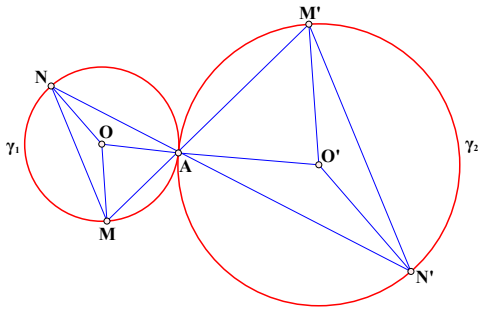

Figure 1: Scheme for solving Problem 3.

From the fact that $\angle MOA = \angle M'O'A$ we have $OM \parallel O'M'$, which jointly with $MN \parallel M'N'$ allows us to obtain $\angle OMN = \angle O'M'N'$. Thus, isosceles triangles OMN and $O'M'N'$ are similar and

$$\frac{R}{R'} = \frac{MN}{M'N'}.$$

Since

$$\frac{AM}{AM'} = \frac{MN}{M'N'},$$

then $\triangle AMN \sim \triangle AM'N'$ and $\angle AM'N' = \angle AMN$.

Finally, since $\angle MAN = \angle M'AN'$ and $\angle OAM = \angle O'AM'$, then we obtain that $\angle OAN = \angle O'AN'$. That is, points N, A, and N' are collinear.

Problem 4. Let a, b, c, d be positive integers. If $a^2 + b^2 + ab = c^2 + d^2 + cd$, then prove that $a + b + c + d$ is composite.

Solution. Note that from $a^2 + b^2 + ab = c^2 + d^2 + cd$, we will have

$$\begin{aligned} & (a+b)^2 - ab = (c+d)^2 - cd \\ \iff & (a+b)^2 - (c+d)^2 = ab - cd \\ \iff & (a+b+c+d)(a+b-c-d) = ab - cd. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & (a-b)^2+3ab=(c-d)^2+3cd \\ \iff & (a-b)^2-(c-d)^2=-3(ab-cd) \\ \iff & (a-b+c-d)(a-b-c+d)=-3(ab-cd). \end{aligned}$$

Combining the preceding, we get

$$-3(a+b+c+d)(a+b-c-d)=(a-b+c-d)(a-b-c+d).$$

Now, we have that $S=a+b+c+d$ is a positive integer that divides the RHS of the last expression, $(a-b+c-d)(a-b-c+d)$. Suppose S is prime; then, $S \mid (a-b+c-d)$ or $S \mid (a-b-c+d)$, but $|S|>|a-b+c-d|$ and $|S|>|a-b-c+d|$. So $a-b+c-d=0$ or $a-b-c+d=0$, which implies $a+b+c+d=2(a+c)$ or $a+b+c+d=2(b+d)$, respectively, and hence it is composite, which leads to a contradiction. If $a+b+c+d$ is not prime, then it is composite.

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Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

jose.luis.diaz@upc.edu

Elementary Problems

E–77. *Proposed by Gabriel T. Prăjitură, SUNY Brockport, Brockport, NY, USA.* Prove that in every triangle we have

$$P \geq 2\sqrt{3\sqrt{3}}\,\sqrt{A},$$

where P is the perimeter and A is the area, with equality for the case of the equilateral triangle.

Solution 1 by Michel Bataille, Rouen, France. Let $s = \frac{P}{2}$ and let r denote the inradius of the triangle. Then we have $A = rs$ and also

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where a, b, c are the sides of the triangle (Heron's formula).

The arithmetic mean-geometric mean inequality gives

$$\frac{(s-a)+(s-b)+(s-c)}{3} \geq \sqrt[3]{(s-a)(s-b)(s-c)}, \qquad (1)$$

that is,

$$\frac{s}{3} \geq \sqrt[3]{\frac{A^2}{s}} = \sqrt[3]{\frac{r^2 s^2}{s}} = \sqrt[3]{r^2 s}.$$

It follows that $s^3 \geq 27 r^2 s$, hence $s \geq 3\sqrt{3} r$ and so $s^2 \geq 3\sqrt{3} A$, and finally

$$\frac{P}{2} \geq \sqrt{3\sqrt{3}}\sqrt{A},$$

as required.

Equality holds if and only if it holds in (1) if and only if $s - a = s - b = s - c$. Thus, equality holds if and only if the triangle is equilateral.

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. As usual, let a, b, c denote the sides of $\triangle ABC$. Then,

$$\begin{aligned} P &= 2\left[\left(\frac{P}{2} - a\right) + \left(\frac{P}{2} - b\right) + \left(\frac{P}{2} - c\right)\right] \\ &\geq 2 \cdot 3\sqrt[3]{\left(\frac{P}{2} - a\right)\left(\frac{P}{2} - b\right)\left(\frac{P}{2} - c\right)} && \text{(AM-GM inequality)} \\ &= 6 \cdot \sqrt[3]{\frac{A^2}{P/2}}, && \text{(Heron's formula)} \end{aligned}$$

giving

$$P^{\frac{4}{3}} \geq 6 \cdot 2^{\frac{1}{3}} \cdot A^{\frac{2}{3}}.$$

Solving for P,

$$P \geq 6^{\frac{3}{4}} \cdot 2^{\frac{1}{4}} \cdot A^{\frac{1}{2}} = 2\sqrt{3\sqrt{3}}\sqrt{A},$$

as desired. Equality occurs only if $a = b = c$.

Solution 3 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. Let a, b, and c be the lengths of the sides of the triangle. Substituting

$$P = a + b + c$$

and

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = (a+b+c)/2$, into $P \geq 2\sqrt{3\sqrt{3}}\sqrt{A}$ and rearranging yields the equivalent inequality

$$\frac{a+b+c}{3} \geq \sqrt[3]{(a+b-c)(b+c-a)(c+a-b)},$$

which follows immediately from the arithmetic mean - geometric mean inequality:

$$\begin{aligned}\frac{a+b+c}{3} &= \frac{(a+b-c)+(b+c-a)+(c+a-b)}{3} \\ &\geq \sqrt[3]{(a+b-c)(b+c-a)(c+a-b)}.\end{aligned}$$

Equality holds if and only if $a+b-c = b+c-a = c+a-b$, or $a = b = c$; that is, equality holds if and only if the triangle is an equilateral triangle.

Solution 4 by the proposer. Let a, b, c denote the lengths of the sides of the triangle and $p = P/2$.

By Heron's formula,

$$A^2 = p(p-a)(p-b)(p-c).$$

Therefore,

$$\begin{aligned}\sqrt[3]{A^2} &= \sqrt[3]{p(p-a)(p-b)(p-c)} = \sqrt[3]{p}\sqrt[3]{(p-a)(p-b)(p-c)} \\ &\leq \sqrt[3]{p}\,\frac{p-a+p-b+p-c}{3} = \frac{1}{3}p\sqrt[3]{p}\end{aligned}$$

and thus

$$A^2 \leq \frac{1}{27}p^4 \iff P^4 \geq 16 \cdot 27A^2 \iff P \geq 2\sqrt{3\sqrt{3}}\sqrt{A}.$$

Equality holds when $a = b = c$. That is, when triangle ABC is equilateral.

Also solved by *Scott H. Brown, Auburn University Montgomery, Montgomery, AL, USA; Sarah B. Seales, Prescott, AZ, USA; Rovsen Pirkuliyev, Sumgayit City, Azerbaijan; Daniel Văcaru, Piteşti, Romania (two solutions), and SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia.*

E–78. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let p, q, r be three prime numbers. Determine the number of ordered pairs of positive integers (a, b) such that the least common multiple of a and b is $p^q\, q^r\, r^p$.

Solution 1 by SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia. Note that a and b have to take the form $a = p^{a_1} q^{b_1} r^{c_1}$ and $b = p^{a_2} q^{b_2} r^{c_2}$, $\max(a_1, a_2) = q$, $\max(b_1, b_2) = r$ and $\max(c_1, c_2) = p$. In these cases, we have the number of pairs is

$$(2p + 1)\,(2q + 1)\,(2r + 1).$$

Solution 2 by the proposer. Both a and b are factors of $p^q\, q^r\, r^p$, and so $a = p^{x_1}\, q^{x_2}\, r^{x_3}$ and $b = p^{y_1}\, q^{y_2}\, r^{y_3}$ for some non negative integers x_i, y_i, $1 \le i \le 3$. Since $p^q\, q^r\, r^p$ is the least common multiple of a and b, then $\max\{x_1, y_1\} = q$, $\max\{x_2, y_2\} = r$, $\max\{x_3, y_3\} = p$. Hence (x_1, y_1) may be equal to

$$(0, q), (1, q), \ldots (q-1, q), (q, q), (q, q-1), (q, q-2), \ldots, (q, 1), (q, 0),$$

so there are $2q + 1$ choices for (x_1, y_1). Likewise, there are $2r + 1$ choices for (x_2, y_2), and $2p + 1$ choices for (x_3, y_3), respectively. Then, by the multiplication principle, there are

$$(2p + 1)\,(2q + 1)\,(2r + 1)$$

ordered pairs of positive integers (a, b) having $p^q\, q^r\, r^p$ as their least common multiple.

Also solved by *Oriol Baeza Guasch, CFIS, Terrassa, Spain, and Alberto Espuny Díaz, Technische Universität Ilmenau, Ilmenau, Germany.*

E–79. *Proposed by Mihály Bencze, Braşov, Romania.* Let a, x be real numbers and let $f, g\colon \mathbb{R} \to \mathbb{R}$ be two functions defined by $f(x) = e^{ax} \sin x$ and $g(x) = e^{ax} \cos x$, respectively. Show that

$$(f^{(n)}(x))^2 + (g^{(n)}(x))^2 = (a^2 + 1)^n e^{2ax}.$$

Here, $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$.

Solution 1 by Michel Bataille, Rouen, France. For the sake of simplicity, we write $f^{(n)}$ instead of $f^{(n)}(x)$.

The desired result is contained in the following one. For every positive integer n, the property (P_n) below holds:

$$\begin{aligned} &(f^{(n-1)})^2 + (g^{(n-1)})^2 = (a^2+1)^{n-1}e^{2ax}, \\ &(f^{(n)})^2 + (g^{(n)})^2 = (a^2+1)^n e^{2ax}, \\ \text{and} \qquad &f^{(n-1)}f^{(n)} + g^{(n-1)}g^{(n)} = a(a^2+1)^{n-1}e^{2ax}. \end{aligned}$$

The proof is by induction. First, (P_1) holds because $f^2 + g^2 = e^{2ax} = (a^2+1)^0 e^{2ax}$,

$$\begin{aligned} (f')^2 + (g')^2 &= (ae^{ax}\sin x + e^{ax}\cos x)^2 + (ae^{ax}\cos x - e^{ax}\sin x)^2 \\ &= (a^2+1)e^{2ax} \end{aligned}$$

and $ff' + gg' = ae^{2ax} = a(a^2+1)^0 e^{2ax}$.

Now, assume that (P_n) holds for some positive integer n. To prove (P_{n+1}), we use the following known or easily checked results,

$$f'' = 2af' - (a^2+1)f, \qquad g'' = 2ag' - (a^2+1)g,$$

from which, by taking the $(n-1)$-th derivatives, we deduce that

$$f^{(n+1)} = 2af^{(n)} - (a^2+1)f^{(n-1)}, \qquad g^{(n+1)} = 2ag^{(n)} - (a^2+1)g^{(n-1)}.$$

We calculate

$$A = \left(f^{(n+1)}\right)^2 + \left(g^{(n+1)}\right)^2 \quad \text{and} \quad B = f^{(n+1)}f^{(n)} + g^{(n+1)}g^{(n)}$$

as follows:

$$\begin{aligned} A = 4a^2\left(\left(f^{(n)}\right)^2 + \left(g^{(n)}\right)^2\right) + (a^2+1)^2\left(\left(f^{(n-1)}\right)^2 + \left(g^{(n-1)}\right)^2\right) \\ - 4a(a^2+1)\left(f^{(n-1)}f^{(n)} + g^{(n-1)}g^{(n)}\right), \end{aligned}$$

that is,

$$\begin{aligned} A &= e^{2ax}\left(4a^2(a^2+1)^n + (a^2+1)^{n+1} - 4a(a^2+1)\cdot a(a^2+1)^{n-1}\right) \\ &= (a^2+1)^{n+1}e^{2ax} \end{aligned}$$

and

$$\begin{aligned} B &= 2a(f^{(n)})^2 - (a^2+1)f^{(n-1)}f^{(n)} + 2a(g^{(n)})^2 - (a^2+1)g^{(n-1)}g^{(n)} \\ &= 2a(a^2+1)^n e^{2ax} - (a^2+1)\cdot a(a^2+1)^{n-1}e^{2ax} \\ &= a(a^2+1)^n e^{2ax}. \end{aligned}$$

It follows that (P_{n+1}) holds, completing the induction step and the proof.

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. By using the derivative rules, we easily find

$$f'(x) = a\cdot f(x) + g(x), \qquad g'(x) = a\cdot g(x) - f(x).$$

Therefore,

$$\begin{aligned} f''(x) &= (a\cdot f(x) + g(x))' = a\cdot f'(x) + g'(x), \\ g''(x) &= (a\cdot g(x) - f(x))' = a\cdot g'(x) - f'(x), \end{aligned}$$

and so on. That is, for $k \geq 1$ we have

$$\begin{aligned} f^{(k)}(x) &= a\cdot f^{(k-1)}(x) + g^{(k-1)}(x), \\ g^{(k)}(x) &= a\cdot g^{(k-1)}(x) - f^{(k-1)}(x). \end{aligned}$$

Hence, for $k \geq 1$, we have

$$\begin{aligned} \left(f^{(k)}(x)\right)^2 + \left(g^{(k)}(x)\right)^2 = & \left(a\cdot f^{(k-1)}(x) + g^{(k-1)}(x)\right)^2 \\ + & \left(a\cdot g^{(k-1)}(x) - f^{(k-1)}(x)\right)^2 \\ = & \ (a^2+1)\left(\left(f^{(k-1)}(x)\right)^2 + \left(g^{(k-1)}(x)\right)^2\right). \end{aligned}$$

We now multiply from $k=1$ to $k=n$, simplify, and obtain

$$\begin{aligned} \left(f^{(n)}(x)\right)^2 + \left(g^{(n)}(x)\right)^2 &= (a^2+1)^n\left((f(x))^2 + (g(x))^2\right) \\ &= (a^2+1)^n\left((e^{ax}\sin x)^2 + (e^{ax}\cos x)^2\right) \\ &= (a^2+1)^n e^{2ax}\left(\sin^2 x + \cos^2 x\right) \\ &= (a^2+1)^n e^{2ax}, \end{aligned}$$

the desired result.

Solution 3 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. Write

$$f(x) = e^{ax}\sin x = \frac{1}{2i}(e^{(a+i)x} - e^{(a-i)x}).$$

Then,

$$f^{(n)}(x) = \frac{1}{2i}\Big((a+i)^n e^{(a+i)x} - (a-i)^n e^{(a-i)x}\Big), \text{ and}$$

$$(f^{(n)}(x))^2 = -\frac{1}{4}\Big((a+i)^{2n}e^{2(a+i)x} - 2(a^2+1)^n e^{2ax} + (a-i)^{2n}e^{2(a-i)x}\Big).$$

Similarly, write

$$g(x) = e^{ax}\cos x = \frac{1}{2}(e^{(a+i)x} + e^{(a-i)x}),$$

so that

$$g^{(n)}(x) = \frac{1}{2}\Big((a+i)^n e^{(a+i)x} + (a-i)^n e^{(a-i)x}\Big), \text{ and}$$

$$(g^{(n)}(x))^2 = \frac{1}{4}\Big((a+i)^{2n}e^{2(a+i)x} + 2(a^2+1)^n e^{2ax} + (a-i)^{2n}e^{2(a-i)x}\Big).$$

Finally,

$$(f^{(n)}(x))^2 + (g^{(n)}(x))^2 = (a^2+1)^n e^{2ax}.$$

Also solved by *José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain; Rovsen Pirkuliyev, Sumgayit City, Azerbaijan; SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia, and the proposer.*

E–80. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.* The lengths of the sides of a triangle are in arithmetic progression. If a denotes the side having a medium length or "middle" side, then with the usual notations, prove that

$$a^2 \geq 6Rr.$$

Solution 1 by Todor Zaharinov, Sofia, Bulgaria. For the lengths of the sides b, a, c of a triangle in arithmetic progression we have

$$a = \frac{b+c}{2}.$$

Using the arithmetic-geometric mean inequality, we have

$$bc \leq \left(\frac{b+c}{2}\right)^2 = a^2,$$

with equality if and only if $b = c$.

Using well-known formulas for the area S of a triangle, we have

$$R = \frac{abc}{4S}, \quad r = \frac{2S}{a+b+c},$$

from which it follows that

$$6Rr = 6 \cdot \frac{abc}{4S} \cdot \frac{2S}{a+b+c} = \frac{3abc}{a+b+c} = \frac{3abc}{a+2a} \leq \frac{3aa^2}{3a} = a^2,$$

with equality if and only if $a = b = c$.

Solution 2 by Oriol Baeza Guasch, CFIS, Terrassa, Spain. Let us denote by d be the difference of the arithmetic progression, so the sides of the triangle have length $a-d, a, a+d$. Expressing the area of the triangle in two different ways, say,

$$S = pr = \frac{abc}{4R},$$

we get

$$\frac{(a-d)+a+(a+d)}{2}\, r = \frac{(a-d)a(a+d)}{4R}$$

or

$$6aRr = a(a+d)(a-d) \implies 6Rr = a^2 - d^2 \leq a^2,$$

just as we wanted to show. The equality occurs when $d = 0$. That is, when the triangle is equilateral.

Solution 3 by the proposer. Let b, c be the remaining sides. Then $2a = b + c$ and $s = \frac{3a}{2}$.

By the arithmetic-geometric mean and the well known relation $abc = 4Rrs$,

$$a^2 = \left(\frac{b+c}{2}\right)^2 \geq bc = \frac{4Rrs}{a} = \frac{4Rr \cdot \frac{3a}{2}}{a} = 6Rr.$$

Solution 4 by the proposer. Let the semiperimeter be denoted by s; then, $s = \frac{3a}{2}$.

Since the sides of a triangle are in arithmetic progression if and only if $s^2 = 9r(2R - r)$ [1], we have

$$\left(\frac{3a}{2}\right)^2 = 9r(2R - r),$$

whence

$$a^2 = 4r(2R - r),$$

which we rewrite as

$$a^2 = 2r(R - 2r) + 6Rr,$$

implying

$$a^2 \geq 6Rr,$$

since the first term of the RHS is non-negative by the Euler's inequality $(R - 2r \geq 0)$.

Equality occurs only if the triangle is equilateral.

[1] Blundon, W. J. "Problem 213". *Crux Mathematicorum* 3 (1977) p. 166.

Solution 5 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. Let d denote the common difference in the arithmetic progression of side lengths. Because a is the side having medium length, the lengths of the other two sides of the triangle are

$$b = a - d \quad \text{and} \quad c = a + d.$$

The circumradius R and the innards r are related by

$$R = \frac{abc}{4rs},$$

where $s = (a + b + c)/2$ is the semiperimeter. Thus,

$$6Rr = \frac{3abc}{a + b + c} = \frac{3a(a - d)(a + d)}{a + a - d + a + d} = a^2 - d^2 \leq a^2.$$

Equality holds if and only if $d = 0$; that is, if and only if the triangle is equilateral.

Also solved by *Michel Bataille, Rouen, France; José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain; Rovsen Pirkuliyev, Sumgayit City, Azerbaijan; Daniel Văcaru, Piteşti, Romania, and SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia.*

E–81. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* [Correction] Let a, b, c, d be four distinct positive real numbers in harmonic progression. Prove that $(a+d)(b+c) > 4bc$.

Solution 1 by Oriol Baeza Guasch, CFIS, Terrassa, Spain. If a, b, c, d form an harmonic progression, then their inverses are in arithmetic progression and we may write them as

$$\frac{1}{x-2y}, \quad \frac{1}{x-y}, \quad \frac{1}{x+y}, \quad \frac{1}{x+2y}.$$

So, we must prove

$$\left(\frac{1}{x-2y}+\frac{1}{x+2y}\right)\left(\frac{1}{x-y}+\frac{1}{x+y}\right) > \frac{4}{(x-y)(x+y)}.$$

Clearing denominators and operating, we get

$$\frac{2x}{x^2-4y^2}\cdot\frac{2x}{x^2-y^2} > \frac{4}{x^2-y^2} \quad \Longleftrightarrow \quad x^2 > x^2-4y^2,$$

which is clearly true. This is a strict inequality given that a, b, c, d are distinct, and therefore $y \neq 0$.

Solution 2 by the proposer. If a, b, c, d are in harmonic progression, then their inverses are in arithmetic progression. WLOG, we may suppose that $\frac{1}{b} = \frac{1}{a} + r$, $\frac{1}{c} = \frac{1}{a} + 2r$, $\frac{1}{d} = \frac{1}{a} + 3r$ with $r > 0$. Then, we have

$$\frac{1}{b} = \frac{1+ar}{a}, \qquad \frac{1}{c} = \frac{1+2ar}{a}, \qquad \frac{1}{d} = \frac{1+3ar}{a},$$

and

$$b = \frac{a}{1+ar}, \qquad c = \frac{a}{1+2ar}, \qquad d = \frac{a}{1+3ar}.$$

Form the preceding, we get

$$a+d=\frac{2a+3ar^2}{1+3ar} \quad \text{and} \quad b+c=\frac{2a+3a^2r}{1+3ar+2a^2r^2}.$$

Since $1+3ar<1+3ar+2a^2r^2$, then $a+d>b+c$, and on account of means inequalities, we have

$$\frac{a+d}{2}>\frac{b+c}{2}\geq\frac{2bc}{b+c},$$

from which $(a+d)(b+c)>4bc$ follows.

Comment by Todor Zaharinov, Sofia, Bulgaria. A harmonic progression is defined as a sequence of real numbers which is determined by taking the reciprocals of the arithmetic progression that does not contain 0,

$$a=\frac{1}{x}, \quad b=\frac{1}{x+y}, \quad c=\frac{1}{x+2y}, \quad d=\frac{1}{x+3y}, \quad x\neq 0,\ y\neq 0.$$

Let

$$\begin{aligned}
A&=(a+d)(b+c)-4bc\\
&=\left(\frac{1}{x}+\frac{1}{x+3y}\right)\left(\frac{1}{x+y}+\frac{1}{x+2y}\right)-4\cdot\frac{1}{x+y}\cdot\frac{1}{x+2y}\\
&=\frac{2x+3y}{x(x+3y)}\cdot\frac{2x+3y}{(x+y)(x+2y)}-\frac{4}{(x+y)(x+2y)}\\
&=\frac{(2x+3y)^2-4x(x+3y)}{x(x+y)(x+2y)(x+3y)}\\
&=\frac{9y^2}{x(x+y)(x+2y)(x+3y)}=9y^2abcd
\end{aligned}$$

and $A>0$ if and only if $abcd>0$, because $y^2>0$.

But $abcd>0$ if either two or all of a,b,c,d have the same sign. If one of a,b,c,d has sign opposite than others, $abcd<0$, hence $A<0$.

The problem turns out to be incorrect.

Example: if $x = -5$ and $y = 2$, then

$$a = \frac{1}{-5} = -\frac{1}{5}, \qquad b = \frac{1}{-5+2} = -\frac{1}{3},$$
$$c = \frac{1}{-5+4} = -1, \qquad d = \frac{1}{-5+6} = 1,$$

and we have that

$$\begin{aligned}(a+d)(b+c) - 4bc &= \left(-\frac{1}{5}+1\right)\left(-\frac{1}{3}-1\right) - 4\left(-\frac{1}{3}\right)(-1) \\ &= \left(\frac{4}{5}\right)\left(-\frac{4}{3}\right) - \left(\frac{4}{3}\right) < 0.\end{aligned}$$

Also solved by *SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia.*

E–82. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Prove that, from a set of fourteen distinct three-digit numbers in the decimal system, it is possible to select two disjoint subsets whose elements have the same sum.

Solution by SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia and the proposer. Note that there are $2^{14} - 2 = 16382$ distinct nonempty proper subsets of our set of 14 three-digit numbers. Also note that the sum of the elements of any subset of our set of 14 three-digit numbers must be between 100 and $986+987+988+989+\cdots+996+997+998+999 = 13985 < 16382$. The Pigeonhole Principle then implies that there are two distinct subsets whose members have the same sum. Let these sets be A and B. Note that $A-(A\cap B)$ and $B-(A\cap B)$ are two disjoint sets whose members have the same sum. These two sets are disjoint subsets of our set of 14 distinct three-digit numbers, so this proves the claim.

Also solved by *Oriol Baeza Guasch, CFIS, Terrassa, Spain; Alberto Espuny Díaz, Technische Universität Ilmenau, Ilmenau, Germany, and the proposer.*

Easy–Medium Problems

EM–77. *Proposed by Toyesh Prakash Sharma, Class XI, St. C.F. Andrews School, Agra, India.* Show that there are infinitely many integers $n \geq 1$ for which

$$\frac{\sin\frac{\pi}{4} + \sin\frac{2\pi}{4} + \sin\frac{3\pi}{4} + \ldots + \sin\frac{n\pi}{4}}{\cos\frac{\pi}{4} + \cos\frac{2\pi}{4} + \cos\frac{3\pi}{4} + \ldots + \cos\frac{n\pi}{4}}$$

is a positive integer.

Solution 1 by Michel Bataille, Rouen, France. Let R_n be the given ratio. We answer the problem by showing that $R_n = 1$ whenever $n = 1 + 8k$ where k is any nonnegative integer.

Let S_n and C_n denote the numerator and the denominator of R_n, respectively. Applying repeatedly the formula

$$2\sin a\cos b = \sin(a+b) + \sin(a-b)$$

we obtain

$$\left(2\cos\frac{(n+1)\pi}{8}\right)S_n = \sum_{k=1}^{n}\sin\left(\frac{(n+1)\pi}{8} + \frac{k\pi}{4}\right) - A_n$$

and

$$\left(2\sin\frac{(n+1)\pi}{8}\right)C_n = \sum_{k=1}^{n}\sin\left(\frac{(n+1)\pi}{8} + \frac{k\pi}{4}\right) + A_n,$$

where

$$A_n = \sum_{k=1}^{n}\sin\left(\frac{(n+1)\pi}{8} - \frac{k\pi}{4}\right).$$

Now, the change of index $k \to j = n+1-k$ gives

$$A_n = \sum_{j=1}^{n}\sin\left(\frac{(n+1)\pi}{8} - \frac{(n+1-j)\pi}{4}\right) = \sum_{j=1}^{n}\sin\left(\frac{j\pi}{4} - \frac{(n+1)\pi}{8}\right),$$

hence $A_n = -A_n$ so that $A_n = 0$. It follows that

$$\left(2\cos\frac{(n+1)\pi}{8}\right)S_n = \left(2\sin\frac{(n+1)\pi}{8}\right)C_n$$

and therefore

$$R_n = \tan\frac{(n+1)\pi}{8}.$$

If $n = 1 + 8k$, then $R_n = \tan\left(\frac{\pi}{4} + k\pi\right) = 1$.

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. We will use the formula for the sum of a geometric series,

$$\begin{aligned}\sum_{j=1}^{n} e^{ij\pi/4} &= \frac{e^{i\pi/4} - e^{i(n+1)\pi/4}}{1 - e^{i\pi/4}} = \frac{e^{-in\pi/8} - e^{in\pi/8}}{e^{-i\pi/8} - e^{i\pi/8}} e^{i(n+1)\pi/8} \\ &= \frac{\sin\frac{n\pi}{8}}{\sin\frac{\pi}{8}} e^{i(n+1)\pi/8}.\end{aligned}$$

Then,

$$\begin{aligned}\sum_{j=1}^{n} \sin\frac{j\pi}{4} &= \operatorname{Im}\sum_{j=1}^{n} e^{ij\pi/4} = \frac{\sin\frac{n\pi}{8}}{\sin\frac{\pi}{8}} \sin\frac{(n+1)\pi}{8}, \\ \sum_{j=1}^{n} \cos\frac{j\pi}{4} &= \operatorname{Re}\sum_{j=1}^{n} e^{ij\pi/4} = \frac{\sin\frac{n\pi}{8}}{\sin\frac{\pi}{8}} \cos\frac{(n+1)\pi}{8},\end{aligned}$$

and

$$\frac{\displaystyle\sum_{j=1}^{n} \sin\frac{j\pi}{4}}{\displaystyle\sum_{j=1}^{n} \cos\frac{j\pi}{4}} = \tan\frac{(n+1)\pi}{8}.$$

Now, $\tan\frac{(n+1)\pi}{8}$ is a positive integer when

$$\frac{n+1}{8} = m + \frac{1}{4} \quad \text{or} \quad n = 8m + 1,$$

for any nonnegative integer m. Thus,

$$\frac{\sin\frac{\pi}{4}+\sin\frac{2\pi}{4}+\sin\frac{3\pi}{4}+\cdots+\sin\frac{n\pi}{4}}{\cos\frac{\pi}{4}+\cos\frac{2\pi}{4}+\cos\frac{3\pi}{4}+\cdots+\cos\frac{n\pi}{4}}$$

is a positive integer whenever $n \equiv 1 \pmod 8$.

Also solved by *Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain; Rovsen Pirkuliyev, Sumgayit City, Azerbaijan; SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia, and the proposer.*

EM–78. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Find all pairs of real numbers (x, y) such that $x \geq 8$, $y \geq 7$ and

$$\frac{(x+y)^2}{\sqrt{x^2-64}+\sqrt{y^2-49}} = 30.$$

Solution 1 by Michel Bataille, Rouen, France. The only solution is $(x, y) = (8\sqrt{2}, 7\sqrt{2})$.

It is readily checked that this pair is a solution. Conversely, let (x, y) be a solution and let $a = \frac{x}{8}$ and $b = \frac{y}{7}$. The equation shows that the pair (a, b) satisfies $\frac{(8a+7b)^2}{30} = 8\sqrt{a^2-1}+7\sqrt{b^2-1}$, that is

$$\frac{1}{2}\left(\frac{8a+7b}{15}\right)^2 = \frac{8\sqrt{a^2-1}+7\sqrt{b^2-1}}{15}. \tag{1}$$

Now, the function $f\colon x \mapsto (x^2-1)^{1/2}$ being concave on $[1, \infty)$ (since $f''(x) = -(x^2-1)^{-3/2} \leq 0$), Jensen's inequality gives

$$\frac{8\sqrt{a^2-1}+7\sqrt{b^2-1}}{15} \leq \sqrt{\left(\frac{8a+7b}{15}\right)^2-1} \tag{2}$$

with equality if and only if $a = b$.

From (1), we now deduce that $g(u) \geq 0$ where g is the function defined by $g(x) = \sqrt{x-1} - \frac{x}{2}$ and $u = \left(\frac{8a+7b}{15}\right)^2$. However, a quick study of the function g shows that $g(x) \leq 0$ for any $x \in [1, \infty)$,

with $g(x) = 0$ if and only if $x = 2$. Thus, we must have $g(u) = 0$, hence $u = 2$ and, from equality in (2), $a = b$. As a result, we obtain that

$$\frac{x}{8} = \frac{y}{7} = \frac{x+y}{15} = \sqrt{2},$$

and so $x = 8\sqrt{2}$, $y = 7\sqrt{2}$. This completes the proof.

Solution 2 by the proposer. We have at the denominator of the fraction

$$\sqrt{x^2-64} + \sqrt{y^2-49} = \sqrt{(x+8)(x-8)} + \sqrt{(x+7)(y-7)}.$$

Note that

$$\begin{aligned}\left(\sqrt{x^2-64} + \sqrt{y^2-49}\right)^2 &= \left(\sqrt{(x+8)(x-8)} + \sqrt{(x+7)(y-7)}\right)^2 \\ &\leq (x-8+y-7)(x+8+y+7).\end{aligned}$$

So,

$$\sqrt{x^2-64} + \sqrt{y^2-49} \leq \sqrt{(x+y-15)(x+y+15)},$$

with equality for $\dfrac{x-8}{y-7} = \dfrac{x+8}{y+7}$, or

$$7x = 8y. \tag{3}$$

Now,

$$\begin{aligned}30 &= \frac{(x+y)^2}{\sqrt{x^2-64} + \sqrt{y^2-49}} \\ &\geq \frac{(x+y)^2}{\sqrt{(x+y)^2-225}} \\ &= \frac{(x+y)^2-225}{\sqrt{(x+y)^2-225}} + \frac{225}{\sqrt{(x+y)^2-225}} \\ &= \sqrt{(x+y)^2-225} + \frac{225}{\sqrt{(x+y)^2-225}}.\end{aligned} \tag{4}$$

By setting $A = \sqrt{(x+y)^2 - 225}$, for $A > 0$ we have that $A + \frac{225}{A} \geq 2\sqrt{A\frac{225}{A}} = 30$, so in (4) the equality is reached for

$$A = \frac{225}{A} \implies A^2 = 225 \implies A = 15,$$

that is,

$$\sqrt{(x+y)^2 - 225} = 15 \implies (x+y)^2 = 450 \implies x + y = 15\sqrt{2}.$$

Combining this with (3) yields

$$\frac{8y}{7} + y = 15\sqrt{2},$$

which means that $y = 7\sqrt{2}$ and $x = \frac{8y}{7} = \frac{8 \cdot 7\sqrt{2}}{7} = 8\sqrt{2}$. In conclusion: the solution (x, y) is $\left(8\sqrt{2}, 7\sqrt{2}\right)$.

EM–79. *Proposed by Andrés Sáez Schwedt, Universidad de León, Spain.* Let ABC be a triangle with circumcenter O, circumcircle Γ and incircle ω. Let E and F be the tangency points of ω with AC and AB. A circle Ω is internally tangent to Γ at A and externally tangent to ω. Show that the radius of Ω is equal to

$$\frac{AO \cdot AE \cdot AF}{AB \cdot AC}.$$

Solution 1 by Michel Bataille, Rouen, France. Let γ be the circle with center A and radius $AE = AF$ and let I denote the inversion in γ. Let $B' = \mathrm{I}(B)$ and $C' = \mathrm{I}(C)$. Then the inverse of Γ is the line $B'C'$ (which passes through the points of intersection of γ and Γ since these points are invariant under I). Note that the line $B'C'$ is perpendicular to the line AO.

Since AE^2 is the power of A with respect to ω, ω is its own inverse. It follows that the inverse of Ω is a line tangent to ω and parallel to $B'C'$. There are two such lines, but $\mathrm{I}(\Omega)$ is the one which is exterior to ω (ω is internally tangent to the inverse of the other

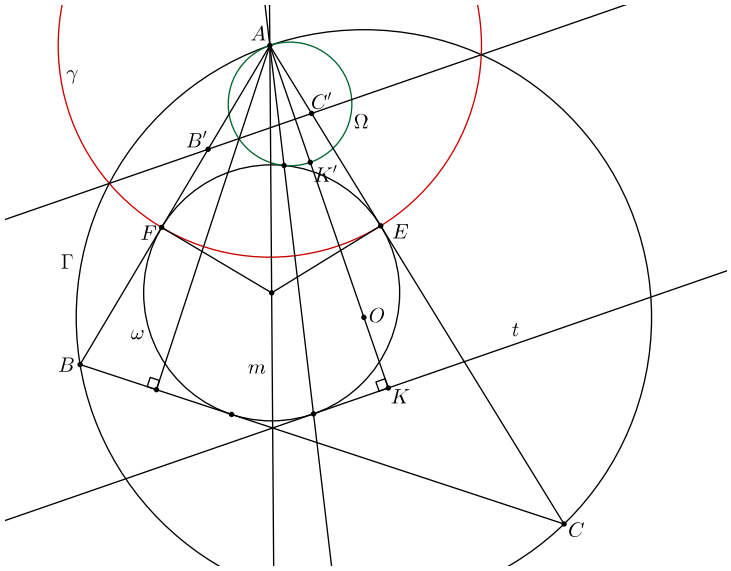

Figure 1: Scheme for Solution 1 of Problem EM–79.

one). Actually, $\mathrm{I}(\Omega)$ is $t = \mathrm{R}_m(BC)$ where R_m is the reflection in the internal bisector m of $\angle BAC$. Indeed, $\mathrm{R}_m(BC)$ is a line tangent to ω (since BC is tangent to ω and $\mathrm{R}_m(\omega) = \omega$) and since BC is perpendicular to the altitude from A, t is perpendicular to the image AO of this altitude under R_m, hence t is parallel to $B'C'$. Moreover, t is exterior to γ, as BC is, since the exterior of γ is invariant under R_m [note that $\mathrm{R}_m(\gamma) = \gamma$].

Now, if K is the orthogonal projection of A onto t and $K' = \mathrm{I}(K)$, then $AK' = 2\rho$ where ρ is the required radius. We observe that AK is equal to the length h_a of the altitude from A. It follows that $2\rho \cdot h_a = AE^2$. But $h_a = \frac{AB \cdot AC \cdot \sin A}{BC} = \frac{AB \cdot AC}{2AO}$ (twice the area of ΔABC is $h_a \cdot BC$ as well as $AB \cdot AC \sin A$), and we obtain

$$\rho = \frac{AE^2}{2h_a} = \frac{AE \cdot AF}{AB \cdot AC/AO} = \frac{AO \cdot AE \cdot AF}{AB \cdot AC}.$$

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Tangency invites inversion, so invert in circle with center A and radius $AE(= AF)$. Let Ω circumscribes triangle APQ with P on AC and Q on AB. The incircle ω is invariant, the circumcircle Γ becomes the straight line through the maps B', C' of B, C and

Ω is inverted into a straight line parallel to $B'C'$. Tangency is preserved, so this line parallel to $B'C'$ is tangent to ω and goes through the maps P', Q' of P, Q.

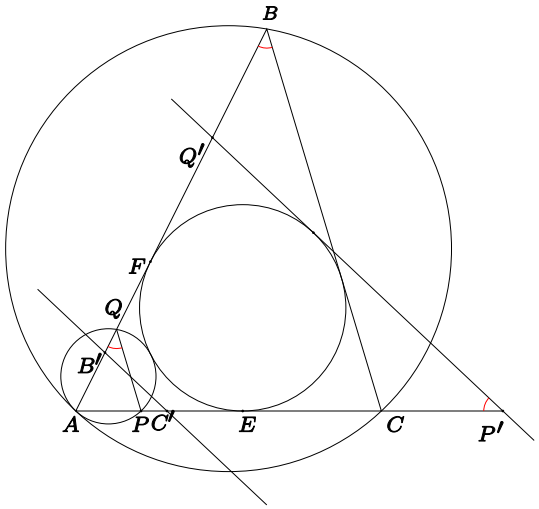

Figure 2: Construction for Solution 2 of Problem EM–79.

The pairs of inverse points B and B', C and C' make $B'C'CB$ cyclic, and

$$\angle ABC = \angle AC'B' = (\text{since } B'C' \parallel P'Q') = \angle AP'Q'.$$

Therefore, triangles ABC and $AP'Q'$ which have a common angle at A are similar. Since ω is the common incircle of $\triangle ABC$ and $\triangle AP'Q'$, these similar triangles are, in fact, congruent with

$$AP' = AB. \tag{1}$$

Similarly, on cyclic quadrilateral $QPP'Q'$

$$\angle AQP = \angle PP'Q' = \angle AP'Q'.$$

Hence triangles AQP and $AP'Q'$ are similar. We conclude that $\triangle AQP$ is similar to $\triangle ABC$ and the corresponding sides AP and

AC are in the ratio

$$\frac{AP}{AC} = (\text{since } AP \cdot AP' = AE^2) = \frac{\frac{AE^2}{AP'}}{AC} = (\text{from (1)})$$
$$= \frac{\frac{AE^2}{AB}}{AC} = (\text{since } AE = AF) = \frac{AE \cdot AF}{AB \cdot AC}.$$

Hence their circumradius are in this ratio, i.e.,

$$\frac{\text{radius of } \Omega}{AO} = \frac{\text{circumradius of } \triangle AQP}{\text{circumradius of } \triangle ABC} = \frac{AE \cdot AF}{AB \cdot AC}$$

and

$$\text{Radius of } \Omega = \frac{AO \cdot AE \cdot AF}{AB \cdot AC},$$

as desired.

Also solved by *the proposer*

EM–80. *Proposed by Nicolae Papacu, Slobozia, Romania.* Determine all positive integers x, y, z such that $x < y < z$ and $xyz = ax + by + cz$, where $a, b, c \in \{1, 2, 3\}$.

Solution by José Luis Díaz-Barrero, BracelonaTech, Barcelona, Spain. From $xyz = ax + by + cz$ we get $\frac{a}{yz} + \frac{b}{zx} + \frac{c}{xy} = 1$ with $1 \leq x < y < z$. Now, we claim that $x = 1$. Indeed, assume that $x \geq 2$ from $2 \leq x < y < z$ we get $y \geq 3$, $z \geq 4$ and

$$1 = \frac{a}{yz} + \frac{b}{zx} + \frac{c}{xy} \leq \frac{a}{12} + \frac{b}{8} + \frac{c}{6}.$$

Since $a + b + c = 6$, then

$$\frac{a}{12} + \frac{b}{8} + \frac{c}{6} = \frac{2a + 3b + 4c}{24} = \frac{2(a + b + c) + b + 2c}{24} = \frac{12 + b + 2c}{24}.$$

On account that $4 \leq b + 2c \leq 8$, we obtain

$$1 \leq \frac{a}{12} + \frac{b}{8} + \frac{c}{6} = \frac{12 + b + 2c}{24} \leq \frac{5}{6} < 1.$$

Contradiction, and the claim is proven.

Thus, $x = 1$ and $yz = a + by + cz$ or $(y - c)(z - b) = a + bc$. Since for any $a, b, c \in \{1, 2, 3\}$ is $a + bc \in \{5, 7\}$, then we have $y - c = 1$ and $z - b = a + bc$ or $y - c = a + bc$ and $z - b = 1$, respectively. Therefore, possible solutions are $x = 1$, $y = 1 + c$, $z = a + b + bc$ and $x = 1$, $y = c + a + bc$, $z = 1 + b$. Since $x < y < z$ then the only valid solution is $x = 1$, $y = 1 + c$, $z = a + b + bc$, and we are done.

Also solved by *the proposer*

EM–81. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $n \geq 1$ be an integer number. Determine the coefficients a_k $(1 \leq k \leq 2n)$ such that the points of the z-plane corresponding to the roots of the equation

$$z^{2n+1} + a_{2n} z^{2n} + a_{2n-1} z^{2n-1} + \ldots + a_2 z^2 + a_1 z = 0$$

represent the vertices of a regular $(2n + 1)$-gon with center at the point $z = 1$.

Solution 1 by Michel Bataille, Rouen, France. One of the roots is 0, hence the circumcircle of the regular $(2n + 1)$-gon has radius 1. If w denotes $\exp(2\pi i/(2n + 1))$, the roots of the polynomial are $0 = 1 + (-1), 1 + (-1)w, 1 + (-1)w^2, \ldots, 1 + (-1)w^{2n}$.

Since $1, w, w^2, \ldots, w^{2n}$ are the roots of $1 - X^{2n+1}$, the numbers

$$0, 1 - w, 1 - w^2, \ldots, 1 - w^{2n}$$

are the roots of the polynomial

$$1 - (1 - X)^{2n+1} = \sum_{k=1}^{2n+1} \binom{2n+1}{k} (-1)^{k+1} X^k$$

and we conclude that

$$a_k = (-1)^{k+1} \binom{2n+1}{k} \quad (1 \leq k \leq 2n).$$

Solution 2 by the proposer. The points corresponding to the roots of the equation

$$z^{2n+1} + 1 = 0$$

are the vertices of a regular $(2n+1)$-gon with center at the origin and with a vertex at the point $z = -1$. Putting $z = u - 1$, we obtain from the last equation

$$(u-1)^{2n+1} + 1 = 0.$$

Its roots are the vertices of a regular $(2n+1)$-gon with center at the point $u = 1$ and with a vertex at the point $w = 0$.

Note

$$\begin{aligned}(u-1)^{2n+1} + 1 &= \sum_{k=0}^{2n+1} (-1)^k \binom{2n+1}{k} u^{2n+1-k} + 1 \\ &= u^{2n+1} - \binom{2n+1}{1} u^{2n} + \binom{2n+1}{2} u^{2n-1} + \dots \\ &\quad + \binom{2n+1}{2n-1} (-1)^{2n-1} u^2 + (-1)^{2n} \binom{2n+1}{2n} u = 0.\end{aligned}$$

The given equation is equivalent to the last equation if and only if, for $1 \le k \le 2n$,

$$a_k = (-1)^{2n+1-k} \binom{2n+1}{2n+1-k} = (-1)^{2n+1-k} \binom{2n+1}{k},$$

and we are done.

EM–82. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Prove that there exist infinitely many positive integer powers of 19 ending up in 0001.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA. We must show there exist infinitely many positive integers n such that

$$19^n \equiv 1 \pmod{10000}.$$

Now,

$$19^n = (20-1)^n = \sum_{j=0}^{n} \binom{n}{j} 20^j (-1)^{n-j} = (-1)^n + \sum_{j=1}^{n} \binom{n}{j} 20^j (-1)^{n-j}.$$

Take $n = 500$. Then

$$\binom{500}{j} 20^j \equiv 0 \pmod{10000}$$

for $j = 1, 2, \ldots, 500$, and

$$19^{500} \equiv (-1)^{500} \pmod{10000} = 1 \pmod{10000}.$$

Finally,

$$19^{500k} \equiv 1 \pmod{10000}$$

for every positive integer k.

Solution 2 by the proposer. First, we observe that if a positive integer N ends in 0001 it is of the form $N = k \cdot 10^4 + 1$ or $N \equiv 1 \pmod{10^4}$, where k is a positive integer.

Now, we consider the remainders $\pmod{10^4}$ of the numbers

$$19^1, 19^2, 19^3, 19^4, 19^5 \ldots 19^{10^4}, 19^{10^4+1}.$$

By the Pigeonhole Principle there are two integers $m < n$ such that $19^n \equiv 19^m \pmod{10^4}$. This means that $19^n - 19^m \equiv 0 \pmod{10^4}$ or, equivalently, that there exists a positive integer j such that $19^n - 19^m = j \cdot 10^4$. Then dividing both terms of the last expression by 19^m, yields

$$19^{n-m} - 1 = \frac{j}{19^m} \cdot 10^4.$$

Since $19^{n-m} - 1$ is an integer number and $(19, 10^4) = 1$, then $19^m \mid j$ and

$$19^{n-m} - 1 \equiv 0 \pmod{10^4}.$$

We may actually find such an exponent using Euler's theorem. It states that if $(a, \alpha) = 1$, then

$$a^{\phi(\alpha)} \equiv 1 \pmod{\alpha},$$

as is well-known. Let $\alpha = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$. Then,

$$\phi(\alpha) = (p_1-1)p_1^{e_1-1}(p_2-1)p_2^{e_2-1}\dots(p_k-1)p_k^{e_k-1} = \alpha \prod_{i=1}^{k}\Big(1-\frac{1}{p_i}\Big),$$

as is well-known. Finally, the furnished exponent is $\phi(10^4) = 4000$ and 19^{4000} ends up in 0001. Since for any integer $k \geq 1$, numbers 19^k and 10^4 are coprime, then all the powers of the form $19^{4000\,k}$ end up in 0001. This completes the proof.

Also solved by *Alberto Espuny Díaz, Technische Universität Ilmenau, Ilmenau, Germany; Philippe Fondanaiche, Paris, France, and SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia.*

Medium–Hard Problems

MH–77. *Proposed by Mihály Bencze, Brasov, Romania, and Gabriel T. Prăjitură, SUNY Brockport, Brockport, NY, USA.* Let z be a nonzero complex number. Prove that

$$\max\left\{|z|, \frac{1}{|z|}\right\} \leq \min\left\{\frac{|z^2+1| + \sqrt{|z^2+1|^2 + 4|z|^2}}{2|z|}, \frac{|z^2-1| + \sqrt{|z^2-1|^2 + 4|z|^2}}{2|z|}\right\}.$$

Solution 1 by the proposers. We have

$$\begin{aligned} & |z| = \left|z + \frac{1}{z} - \frac{1}{z}\right| \leq \left|z + \frac{1}{z}\right| + \frac{1}{|z|} \\ \iff & |z|^2 - \left|z + \frac{1}{z}\right||z| - 1 \leq 0 \\ \iff & |z|^2 - 2\frac{1}{2}\left|z + \frac{1}{z}\right||z| + \frac{1}{4}\left|z + \frac{1}{z}\right|^2 \leq \frac{1}{4}\left|z + \frac{1}{z}\right|^2 + 1 \\ \iff & \left(|z| - \frac{1}{2}\left|z + \frac{1}{z}\right|\right)^2 \leq \frac{1}{4}\left|z + \frac{1}{z}\right|^2 + 1. \end{aligned}$$

Therefore,

$$\begin{aligned} & |z| - \frac{1}{2}\left|z + \frac{1}{z}\right| \leq \sqrt{\frac{1}{4}\left|z + \frac{1}{z}\right|^2 + 1} \\ \iff & |z| \leq \frac{|z^2+1| + \sqrt{|z^2+1|^2 + 4|z|^2}}{2|z|}. \end{aligned}$$

In the same way, starting with

$$|z| = \left|z - \frac{1}{z} + \frac{1}{z}\right| \leq \left|z - \frac{1}{z}\right| + \frac{1}{|z|}$$

we get

$$|z| \leq \frac{|z^2-1| + \sqrt{|z^2-1|^2 + 4|z|^2}}{2|z|}.$$

If we replace z by $\frac{1}{z}$ on both inequalities we get $\frac{1}{|z|}$ on the left hand side while the right hand side does not change.

Solution 2 by Michel Bataille, Rouen, France. Let

$$\rho_{+1} = \frac{|z^2+1| + \sqrt{|z^2+1| + 4|z|^2}}{2|z|},$$
$$\rho_{-1} = \frac{|z^2-1| + \sqrt{|z^2-1| + 4|z|^2}}{2|z|}$$

and let $\varepsilon \in \{+1, -1\}$.

We have

$$\rho_\varepsilon = \frac{|z||z+\frac{\varepsilon}{z}| + |z|\sqrt{|z+\frac{\varepsilon}{z}|^2 + 4}}{2|z|} = \frac{|z+\frac{\varepsilon}{z}| + \sqrt{|z+\frac{\varepsilon}{z}|^2 + 4}}{2},$$

hence the inequality $\rho_\varepsilon \geq |z|$ rewrites as

$$\sqrt{\left|z+\frac{\varepsilon}{z}\right|^2 + 4} \geq 2|z| - \left|z+\frac{\varepsilon}{z}\right|.$$

This inequality certainly holds if its right side is negative and otherwise is equivalent to

$$\left|z+\frac{\varepsilon}{z}\right|^2 + 4 \geq 4|z|^2 + \left|z+\frac{\varepsilon}{z}\right|^2 - 4|z|\left|z+\frac{\varepsilon}{z}\right|,$$

that is, to

$$\left|z+\frac{\varepsilon}{z}\right| + \frac{1}{|z|} \geq |z|.$$

This holds since, by the triangular inequality,

$$|z| = \left|\left(z+\frac{\varepsilon}{z}\right) - \frac{1}{z}\right| \leq \left|z+\frac{\varepsilon}{z}\right| + \frac{1}{|z|}.$$

We conclude that $\rho_\varepsilon \geq |z|$ for any nonzero z.

We observe that replacing z by $\frac{1}{z}$ keeps ρ_ε invariant, hence $\rho_\varepsilon \geq |\frac{1}{z}| = \frac{1}{|z|}$ holds as well. Thus, both ρ_{+1} and ρ_{-1} are not less than both $|z|$ and $\frac{1}{|z|}$ and the result follows.

Also solved by *José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.*

MH–78. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* From point P inside equilateral triangle ABC, perpendiculars are drawn to the sides meeting BC, CA, and AB at points D, E, and F, respectively. Compute the value of

$$\frac{DE + EF + FD}{PA + PB + PC}.$$

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. The answer is $\frac{\sqrt{3}}{2}$. The right angles at D and F make $BDPF$ cyclic. We find, by the Ptolemy's theorem, that

$$FD \cdot PB = BD \cdot PF + FB \cdot PD. \tag{1}$$

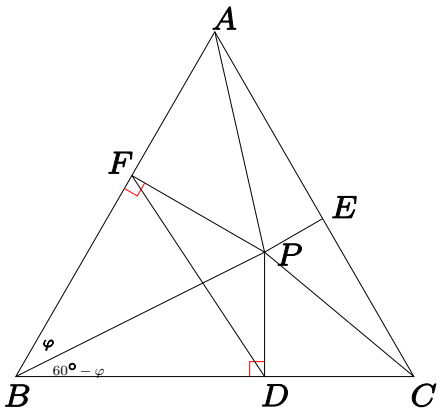

Figure 3: Construction for Solution 1 of Problem MH–78.

We put $\angle FBP = \varphi$. Then, $\angle PBD = 60^\circ - \varphi$ and, therefore,

$$BD = PB \cdot \cos(60^\circ - \varphi), \qquad PD = PB \cdot \sin(60^\circ - \varphi),$$
$$FB = PB \cdot \cos\varphi, \qquad PF = PB \cdot \sin\varphi.$$

We substitute these expressions into (1), simplify, and obtain

$$FD = PB(\cos(60^\circ - \varphi)\sin\varphi + \cos\varphi\sin(60^\circ - \varphi)).$$

That is to say,

$$\frac{FD}{PB} = \sin(\varphi + (60^\circ - \varphi)) = \sin 60^\circ = \frac{\sqrt{3}}{2}.$$

Similarly,

$$\frac{DE}{PC} = \frac{\sqrt{3}}{2} = \frac{EF}{PA}.$$

Thus

$$\frac{\sqrt{3}}{2} = \frac{FD}{PB} = \frac{DE}{PC} = \frac{EF}{PA} = \frac{FD + DE + EF}{PB + PC + PA},$$

as claimed.

Solution 2 by the proposer. We claim that quadrilaterals $AFPE$, $BDPF$, and $CDPE$ are inscribed in the circles whose diameters are, respectively, PA, PB, and PC. Indeed, since $\angle AFP$ and $\angle AEP$ are right and opposite, then quadrilateral $AFPE$ is cyclic. The diameter of this circle is PA because it sees the right angles $\angle AEP$ and $\angle AFP$, respectively.

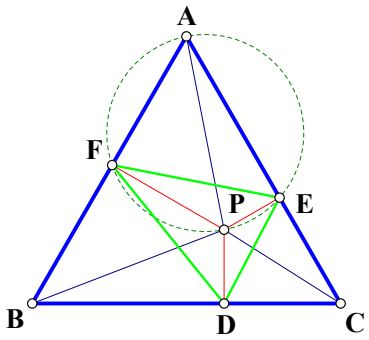

Figure 4: Construction for Solution 2 of Problem MH–78.

Using the sine law in $\triangle AFE$, we have that

$$\frac{EF}{\sin 60^\circ} = \frac{EA}{\sin \angle AFE} = \frac{EA}{\sin \angle APE} = PA$$

on account that $\angle AFE$ sees the same chord as $\angle APE$. Then, we have

$$\frac{EF}{PA} = \sin 60^\circ = \frac{\sqrt{3}}{2}.$$

Likewise, $\dfrac{FD}{PB} = \dfrac{\sqrt{3}}{2}$ and $\dfrac{DE}{PC} = \dfrac{\sqrt{3}}{2}$. Finally, we conclude that

$$\frac{EF}{PA} = \frac{FD}{PB} = \frac{DE}{PC} = \frac{\sqrt{3}}{2},$$

from which

$$\frac{DE + EF + FD}{PA + PB + PC} = \frac{\sqrt{3}}{2}$$

follows.

Also solved by *Michel Bataille, Rouen, France; Philippe Fondanaiche, Paris, France, and SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia (two solutions).*

MH–79. *Proposed by Oriol Baeza Guasch, CFIS, BarcelonaTech, Terrassa, Spain.* Let ABC be a triangle, and let M be the midpoint of arc BC not containing A. Let the circumference with center M that goes through B, C meet rays AB, AC again at points $J \neq B$, $K \neq C$, respectively. Finally, let the line JK intersect the circumcircle of $\triangle ABC$ at points P, Q. Show that $\triangle APQ$ is isosceles.

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. In triangles AJK and ABC (see figure 5), we have

$$\angle AJK = \angle BJK = (\text{on chord } BK) = \angle BCK = \angle BCA.$$

Therefore, these triangles which have a common angle at A are similar. Consequently, their third angles are equal, that is,

$$\angle JKA = \angle ABC. \tag{1}$$

Hence,

$$\begin{aligned}
\angle PQA &= \angle KQA \\
&= \angle JKA - \angle KAQ \quad \text{(by the exterior angle theorem applied to } \triangle KQA \text{ at } K) \\
&= \angle ABC - \angle CAQ \quad \text{(by (1))} \\
&= \angle APC - \angle QPC \quad \text{(on chords } CA \text{ and } CQ, \text{ respectively)} \\
&= \angle APQ,
\end{aligned}$$

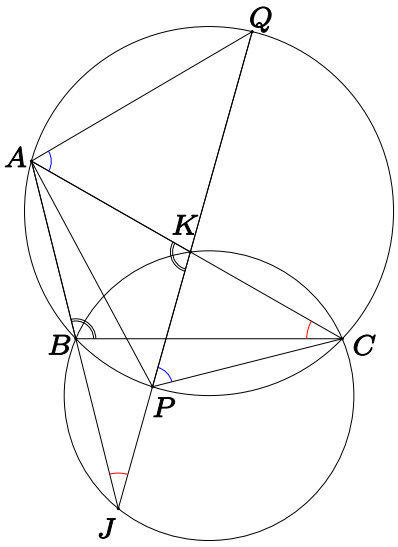

Figure 5: Scheme for Solution 1 of Problem MH–79.

making triangle APQ isosceles, as desired.

Solution 2 by Michel Bataille, Rouen, France. [We assume that the arc BC with mid-point M is an arc of the circumcircle of ΔABC.]

Let $\angle(XY, ZT)$ denote the directed angle from the line XY to the line ZT. Considering the angles of ΔAKQ, we obtain

$$\angle(QA, QP) + \angle(KQ, KA) + \angle(AK, AQ) = 0,$$

from which we successively deduce that

$$\angle(QA, QP) + \angle(KJ, KC) + \angle(AC, AQ) = 0$$

(since K, J, Q and A, K, C are collinear),

$$\angle(QA, QP) + \angle(BJ, BC) + \angle(BC, BQ) = 0$$

(since B, K, J, C and A, B, C, Q are concyclic),

$$\angle(QA, QP) + \angle(BJ, BQ) = 0,$$

$$\angle(QA, QP) + \angle(BA, BQ) = 0$$

(since A, J, B are collinear), and

$$\angle(QA, QP) + \angle(PA, PQ) = 0$$

(since B, A, P, Q are concyclic), and so $\angle(QA, QP) = \angle(PQ, PA)$, proving that ΔAPQ is isosceles.

Solution 3 by Philippe Fondanaiche, Paris, France. As M is the midpoint of arc BC, the line AM is the angle bisector of $\angle BAC$. Let us say it intersects side BC at point D. Also, we denote $\alpha = \angle BAC$, $\beta = \angle ABC$ and $\gamma = \angle ACB$ with $\alpha + \beta + \gamma = 180°$.

Lemma 1. *The four points B, D, M, J are concyclic, as well as C, K, D, M.*

Proof. An angle chase shows that

$$\angle DBJ = 180° - \beta = \angle DBM + \angle MBJ = \alpha/2 + \angle BJM,$$

where we used that ΔMBJ is isosceles. Then, $\angle BJM = 180° - \beta - \alpha/2$.

Or also on the triangle ΔABD, we have $\angle ADB = 180° - \beta - \alpha/2$. From here, it follows $\angle BJM = \angle ADB$.

The same proof shows C, K, D, M are concyclic as well. □

Lemma 2. *Points J, D, K are aligned.*

Proof. From Lemma 1, we have $\angle BDJ = \angle BMJ = 180° - 2\angle MBJ$. Then

$$\angle BDJ = 180° - 2(180° - \beta - \alpha/2) = \beta - \gamma.$$

On the other side, $\angle CDK = \angle CMK = 180° - 2\angle KCM$, so

$$\angle CDK = 180° - 2(\gamma + \alpha/2) = \beta - \gamma.$$

Then, the points are aligned. □

Lemma 3. *Triangles ΔACP and ΔQCK are similar*

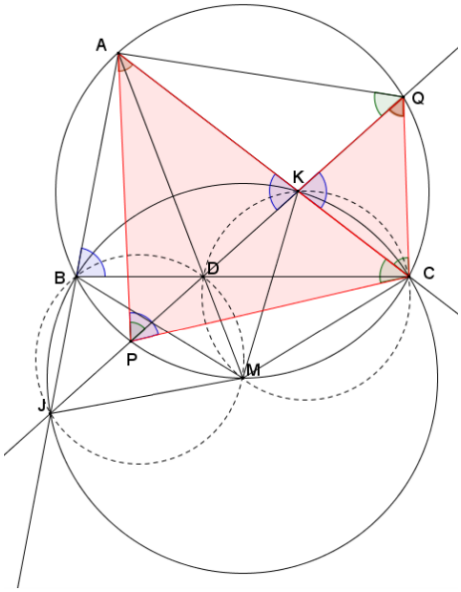

Figure 6: Scheme for Solution 3 of Problem MH–79.

Proof. We have the angle relations $\angle CAP = \angle CQK = \angle CQP$, as well as $\angle APC = \angle ABC = 180^\circ - \angle DBJ$.

Also, after Lemma 2, $\angle DBJ = \angle DKC = 180^\circ - \angle AKP$.

From all this, we have

$$\angle APC = \angle ABC = 180^\circ - \angle DBJ = 180^\circ - \angle DKC = \angle CKQ,$$

so the triangles are similar, and we also deduce $\angle ACP = \angle QCK$. □

Lemma 4. *Triangle ΔAPQ is isosceles.*

Proof. Using the previous lemmas and the similarity found, we have the angle relations

$$\begin{aligned} \angle AQP = \angle ACP \text{ (angle which looks arc } AP) &= \angle QCK, \\ \angle APQ = \angle ACQ \text{ (angle which looks arc } AQ) &= \angle QCK. \end{aligned}$$

Thus, $\angle APQ = \angle AQP$, which is what we wanted to prove. □

Solution 4 by the proposer. First of all, let O be the circumcenter of $\triangle ABC$. Since O is also the circumcenter of $\triangle APQ$, showing that it is isosceles is equivalent to showing that $AO \perp PQ$.

Next, we will show that line PQ is tangent to the incircle of $\triangle ABC$. First, it is well known that the incenter I belongs to the circumference with center M that goes through B and C, so $JBIKC$ is cyclic. Then, we have

$$\begin{aligned}\angle IKJ &= 180^\circ - \angle IBJ = \angle IBA \\ &= \angle IBC = 180^\circ - \angle IKC = \angle IKA.\end{aligned}$$

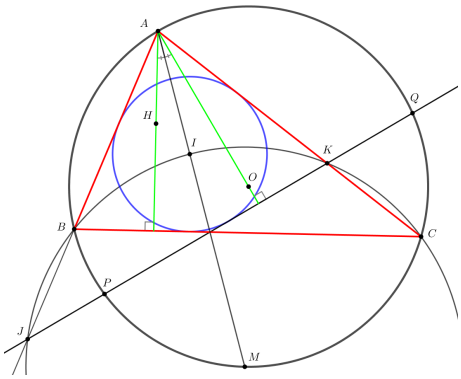

Figure 7: Scheme for Solution 4 of Problem MH–79.

So KI is the bisector of $\angle AKJ$ and, hence, I is also the incenter of triangle $\triangle AKJ$. Even more, we have seen halfway through our argument that $\angle IKA = \angle IBA$, so $\angle AKJ = \angle ABC$. Therefore, the triangles $\triangle AKJ \cong \triangle ABC$ are congruent, since they have the same angles and the same inradius.

Now, consider φ the reflection of the figure over AI. Clearly, given that reflection preserve distances, $\varphi(B) = K$ and $\varphi(C) = J$, so we have that $\varphi(BC) = JK$. On the other hand, it is well known that $\varphi(AH) = AO$. Therefore, since reflection preserves angles,

$$AH \perp BC \implies \varphi(AH) \perp \varphi(BC) \implies AO \perp JK,$$

as we wanted to show.

MH–80. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Find all functions $f\colon \mathbb{N}^* \to \mathbb{N}^*$ such that there exists $k \in \mathbb{N}^*$ such that, for all positive integers x, y,

$$\frac{f(x+y)+f(x)}{ky+f(x)} = \frac{kx+f(y)}{f(x+y)+f(y)}.$$

Solution by Michel Bataille, Rouen, France. The solutions are the functions $f_a\colon x \mapsto f_a(x) = ax$ where $a \in \mathbb{N}^*$.

Every function f_a is a solution since taking $k = 2a$, we have

$$\frac{f(x+y)+f(x)}{2ay+f(x)} = \frac{2ax+f(y)}{f(x+y)+f(y)} = \frac{2ax+ay}{ax+2ay}$$

for all x, y. Conversely, suppose that $f\colon \mathbb{N}^* \to \mathbb{N}^*$ satisfies the given equality, that we denote by $E(x,y)$, and let $a = f(1)$. We first show that $f(2n) = kn$ for all $n \in \mathbb{N}^*$. The proof is by induction: from $E(1,1)$ we immediately deduce that $(f(2)+a)^2 = (k+a)^2$ and, since $f(2)+a, k+a > 0$, we obtain $f(2) = k$.

Now, assume $f(2n) = kn$ for some positive integer n. Then, $E(2n,2)$ easily leads to

$$[f(2(n+1))]^2 + k(n+1)f(2(n+1)) - 2k^2(n+1)^2 = 0.$$

Solving for $f(2(n+1))$ we obtain

$$f(2(n+1)) = \frac{-k(n+1)+3k(n+1)}{2} = k(n+1).$$

This completes the induction step and we conclude that $f(2n) = kn$ for all $n \in \mathbb{N}^*$.

The relation $E(2n+1,1)$ leads to

$$\frac{k(n+1)+f(2n+1)}{k+f(2n+1)} = \frac{k(2n+1)+a}{k(n+1)+a},$$

from which we deduce that $f(2n+1) = a+kn$.

From $E(1,2)$ we readily obtain $k = 2a$ and so $f(2n) = 2an$ and $f(2n+1) = a+2an = a(2n+1)$ for all n. Thus, $f(x) = ax$ for all positive integer x and $f = f_a$. This completes the proof.

Also solved by *the proposer.*

MH–81. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* A group of 2021 students sit in a circle, consecutively counting numbers 1, 2 and 3 and repeating. Starting from some student A with number 1, and counting clockwise round the remaining students, students that count numbers 2 and 3 must leave the circle until only one remains. Determine, after this process, who is the last student remaining in the circle?

Solution by the proposer. We consider two cases:

- **Case 1:** There are 3^n students sitting in a circle, then after the first counting there remain 3^{n-1} students and student A who counts 1 first at the first round will count the same 1 first at the second round. So A will remain the last one.
- **Case 2:** There are 2021 students. Since $3^6 = 729 < 2021 < 3^7 = 2187$, we will reduce the problem to Case 1 until there remain 3^6 students, then student A should be the one who counts 1 first among 3^6 students. If so, we need to remove $2021 - 729 = 1292$ students, that corresponds to 646 groups (each group of three students of which two leave). So we need $646 \cdot 3 = 1938$ students sitting before A. That is, the winner should choose the 1939-th place counting clockwise from A, who counts the first 1 among 2021 students.

MH–82. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be three positive numbers such that $ab + bc + ca = 3abc$. Prove that

$$\frac{bc \arctan^2 a}{b+c} + \frac{ca \arctan^2 b}{c+a} + \frac{ab \arctan^2 c}{a+b} \geq \frac{3\pi^2}{32}.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA. The condition

$$ab + bc + ca = 3abc$$

is equivalent to

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 3.$$

By the Cauchy-Schwarz inequality,

$$\begin{aligned}
&\frac{bc\arctan^2 a}{b+c}+\frac{ca\arctan^2 b}{c+a}+\frac{ab\arctan^2 c}{a+b}\\
&=\frac{\arctan^2 a}{\frac{1}{b}+\frac{1}{c}}+\frac{\arctan^2 b}{\frac{1}{a}+\frac{1}{c}}+\frac{\arctan^2 c}{\frac{1}{a}+\frac{1}{b}}\\
&\geq\frac{(\arctan a+\arctan b+\arctan c)^2}{2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\\
&=\frac{(\arctan a+\arctan b+\arctan c)^2}{6}.
\end{aligned}$$

Let $f(x)=\arctan\frac{1}{x}$. Then,

$$f''(x)=\frac{2x}{(1+x^2)^2}>0$$

for $x>0$, so f is convex for $x>0$. By Jensen's inequality,

$$\begin{aligned}
&\arctan a+\arctan b+\arctan c\\
&=\arctan\frac{1}{1/a}+\arctan\frac{1}{1/b}+\arctan\frac{1}{1/c}\\
&\geq 3\arctan\left(\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\right)^{-1}=3\arctan 1=\frac{3\pi}{4}.
\end{aligned}$$

Thus,

$$\frac{bc\arctan^2 a}{b+c}+\frac{ca\arctan^2 b}{c+a}+\frac{ab\arctan^2 c}{a+b}\geq\frac{(3\pi/4)^2}{6}=\frac{3\pi^2}{32}.$$

Solution 2 by Michel Bataille, Rouen, France. Let $u=\frac{1}{a}$, $v=\frac{1}{b}$, $w=\frac{1}{c}$. The hypotheses become $u,v,w>0$ and $u+v+w=3$, while the inequality to be proved now is

$$\frac{(\pi/2-\arctan u)^2}{v+w}+\frac{(\pi/2-\arctan v)^2}{w+u}+\frac{(\pi/2-\arctan w)^2}{u+v}\geq\frac{3\pi^2}{32}$$

(using the well-known relation $\arctan(1/x)=\pi/2-\arctan x$ for positive x).

Let $\mathcal{L}$ denote the left-hansd side of this inequality. Then, from the Cauchy-Schwarz inequality

$$\begin{aligned}&\mathcal{L}\cdot((v+w)+(w+u)+(u+v))\\&\geq((\pi/2-\arctan u)+(\pi/2-\arctan v)+(\pi/2-\arctan w))^2,\end{aligned}$$

that is,

$$6\mathcal{L}\geq(3\pi/2-(\arctan u+\arctan v+\arctan w))^2.$$

Now, from Jensen's inequality applied to the concave function arctan on $(0,\infty)$, we obtain

$$\arctan u+\arctan v+\arctan w\leq 3\arctan\left(\frac{u+v+w}{3}\right)=\frac{3\pi}{4}$$

(since $\arctan(1)=\frac{\pi}{4}$) and, therefore,

$$(3\pi/2-(\arctan u+\arctan v+\arctan w))^2\geq\left(\frac{3\pi}{4}\right)^2=\frac{9\pi^2}{16}.$$

We conclude that $6\mathcal{L}\geq\frac{9\pi^2}{16}$ and so $\mathcal{L}\geq\frac{3\pi^2}{32}$, as required.

Solution 3 by Daniel Văcaru, Piteşti, Romania. First, We observe that $ab+bc+ca=3abc\iff\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$. Then we have

$$\begin{aligned}&\frac{bc\arctan^2 a}{b+c}+\frac{ca\arctan^2 b}{c+a}+\frac{ab\arctan^2 c}{a+b}\\&=\frac{\arctan^2 a}{\frac{1}{b}+\frac{1}{c}}+\frac{\arctan^2 b}{\frac{1}{c}+\frac{1}{a}}+\frac{\arctan^2 c}{\frac{1}{a}+\frac{1}{b}}\\&\geq\frac{(\arctan a+\arctan b+\arctan c)^2}{2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\\&=\frac{(\arctan a+\arctan b+\arctan c)^2}{6}\\&=\frac{1}{6}\left(\frac{3\pi}{2}-\arctan\frac{1}{a}-\arctan\frac{1}{b}-\arctan\frac{1}{c}\right)^2,\end{aligned}\tag{1}$$

where the inequality holds by Bergstrom's inequality and for the last line we have used that, for all $x > 0$,

$$\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}.$$

In the paper "The extremes of the sums of arctangents with restrictions for arguments", by Florentin Vişescu, published in *Romanian Mathematical Magazine*, 28, the followin claim is proved.

Claim. *If a, b, c are positive numbers such that $a + b + c = k$, then*

$$\arctan a + \arctan b + \arctan c \leq 3 \arctan \frac{k}{3}.$$

On account of it, we have that

$$\begin{aligned} &\arctan \frac{1}{a} + \arctan \frac{1}{b} + \arctan \frac{1}{c} \leq \frac{3\pi}{4} \\ \Longrightarrow\ &\frac{3\pi}{2} - \arctan \frac{1}{a} - \arctan \frac{1}{b} - \arctan \frac{1}{c} \geq \frac{3\pi}{4}. \end{aligned} \tag{2}$$

Finally, using (1) and (2) we have

$$\begin{aligned} &\frac{bc \arctan^2 a}{b+c} + \frac{ca \arctan^2 b}{c+a} + \frac{ab \arctan^2 c}{a+b} \\ &\geq \frac{1}{6}\left(\frac{3\pi}{2} - \arctan \frac{1}{a} - \arctan \frac{1}{b} - \arctan \frac{1}{c}\right)^2 \geq \frac{\left(\frac{3\pi}{4}\right)^2}{6} = \frac{3\pi^2}{32}, \end{aligned}$$

as desired.

Solution 4 by the proposer. We begin with the following claim:

Claim. *Let $f\colon (0, +\infty) \to (0, +\infty)$ be a convex function. If $x_1, x_2, \ldots, x_n$ and $a_1, a_2, \ldots, a_n$ are positive numbers, then*

$$\left(\frac{1}{n}\sum_{k=1}^{n} \frac{f^2(x_k)}{a_k}\right)\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right) \geq \left[f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right]^2.$$

Proof. Indeed, applying the CBS inequality to the vectors $\vec{u} = \left(\frac{f(x_1)}{\sqrt{a_1}}, \frac{f(x_2)}{\sqrt{a_2}}, \ldots, \frac{f(x_n)}{\sqrt{a_n}}\right)$ and $\vec{v} = (\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n})$ yields

$$\left(\sum_{k=1}^{n} \frac{f^2(x_k)}{a_k}\right)\left(\sum_{k=1}^{n} a_k\right) \geq \left[f\left(\sum_{k=1}^{n} x_k\right)\right]^2.$$

Since f is convex, then, on account of Jensen's inequality, we have

$$f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right) \leq \frac{1}{n}\sum_{k=1}^{n} f(x_k).$$

Substituting the preceding inequality in the previous one, we get

$$\left(\sum_{k=1}^{n} \frac{f^2(x_k)}{a_k}\right)\left(\sum_{k=1}^{n} a_k\right) \geq n^2\left[f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)\right]^2,$$

from which the claim immediately follows. □

Putting $x_1 = 1/a$, $x_2 = 1/b$, $x_3 = 1/c$ and $a_1 = \frac{1}{b} + \frac{1}{c}$, $a_2 = \frac{1}{c} + \frac{1}{a}$, $a_3 = \frac{1}{a} + \frac{1}{b}$, and considering the function $f\colon (0, +\infty) \to (0, +\infty)$ defined by $f(x) = \arctan(1/x)$ which is convex for $x > 0$, we have on account of the claim and the constraint that

$$\begin{aligned}
&\frac{bc\,\arctan^2 a}{b+c} + \frac{ca\,\arctan^2 b}{c+a} + \frac{ab\,\arctan^2 c}{a+b} \\
&= \frac{\arctan^2 a}{\frac{1}{b}+\frac{1}{c}} + \frac{\arctan^2 b}{\frac{1}{c}+\frac{1}{a}} + \frac{\arctan^2 c}{\frac{1}{a}+\frac{1}{b}} \\
&\geq \frac{9}{2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \arctan^2\left(\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\right) = \frac{3\pi^2}{32}.
\end{aligned}$$

Equality holds when $a = b = c = 1$, and we are done.

Also solved by *Philippe Fondanaiche, Paris, France.*

Advanced Problems

A–77. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In the ring $\mathbb{Z}_{19}[X]$, find a monic polynomial of degree four knowing that its zeros x_1, x_2, x_3, x_4 satisfy

$$\sum_{i=1}^{4} x_i = 13, \quad \sum_{i=1}^{4} x_i^2 = 1, \quad \sum_{i=1}^{4} x_i^3 = 13, \quad \text{and} \quad \sum_{i=1}^{4} x_i^4 = 18.$$

Solution 1 by Michel Bataille, Rouen, France. In what follows, all equalities are in $\mathbb{Z}_{19}$ (or in $\mathbb{Z}_{19}[X]$).

We show that the desired polynomial is

$$X^4 + 6X^3 + 8X^2 + 16X + 17 = (X-3)(X-5)(X-11)(X-13).$$

Write the required polynomial as

$$p(X) = X^4 - e_1X^3 + e_2X^2 - e_3X + e_4,$$

where e_1, e_2, e_3, e_4 are the usual symmetric functions of x_1, x_2, x_3, x_4, and let $s_k = \sum_{i=1}^{4} x_i^k$ so that $s_1 = 13 = s_3$, $s_2 = 1$, $s_4 = 18$.

Clearly, $e_1 = s_1 = 13$ and so $-e_1 = 6$. Then, we use Newton's formulas, which yield

$$2e_2 = -s_2 + e_1s_1 = s_1^2 - s_2 = 35 = -3$$

and, since $2^{-1} = 10$, $e_2 = -30 = 8$. Then,

$$3e_3 = s_3 - e_1s_2 + e_2s_1 = 13 - 13 + 8 \times 13 = 9,$$

hence $e_3 = 9 \times 13 = 3$, $-e_3 = 16$, and finally

$$4e_4 = -s_4 + e_1s_3 - e_2s_2 + e_3s_1 = 1 + 13 \times 13 - 8 + 3 \times 13 = 11$$

and so $e_4 = 5 \times 11 = 55 = 17$. In conclusion, $p(X) = X^4 + 6X^3 + 8X^2 + 16X + 17$.

The roots of $p(X)$ in $\mathbb{Z}_{19}$ are easily found to be $3, 5, -6, -8$ and therefore $p(X) = (X-3)(X-5)(X-11)(X-13)$.

Solution 2 by the proposer. Let $A(X) = X^4 + aX^3 + bX^2 + cX + d \in \mathbb{Z}_{19}[X]$ be the polynomial to be determined. For $k = 1, 2, 3, 4$ we will denote the sum of the powers and the symmetric functions of the zeros by

$$P_k = \sum_{i=1}^{4} x_i^k \qquad \text{and} \qquad S_k = \sum x_{i_1} x_{i_2} \dots x_{i_k}.$$

Then, on account Viète's formulae we have $S_1 = -a$, $S_2 = b$, $S_3 = -c$, $S_4 = d$. On the other hand, on account of Newton's formulas for $k = 1, 2, 3, 4$, we have

$$(-1)^{k-1} P_k + (-1)^{k-2} P_{k-1} S_1 + \ldots + P_1 S_{k-1} = k S_k.$$

Since $P_1 = 13$, $P_2 = 1$, $P_3 = 13$ and $P_4 = 18$, then for $k = 1$ we have $P_1 = S_1 = -a = 13$ and working modulo 19 we get $a = 6$. For $k = 2$ we have $-P_2 + P_1 S_1 = 2S_2$ or $-1 + 13(-a) = 2S_2 \pmod{19}$ and $b = 8$. For $k = 3$, we have $P_3 - P_2 S_1 + P_1 S_2 = 3S_3 \pmod{19}$ from which $c = 16$ follows. For $k = 4$ we have $-P_4 + P_3 S_1 - P_2 S_2 + P_1 S_3 = 4S_4 \pmod{19}$ and $d = 17$. Thus,

$$A(X) = X^4 + 6X^3 + 8X^2 + 16X + 17.$$

Its zeros are $3, 5, 11, 13 \in \mathbb{Z}_{19}$, as can be easily checked.

Also solved by *Moti Levy, Rehovot, Israel, and SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia.*

A–78. *Proposed by Mihály Bencze, Braşov, Romania.* Let A be an $n \times n$ matrix with real entries such that $A^3 + 4A^2 + 6A + 3I = 0$ and let $B = 2A^2 + 7A + 8I$. Prove that B is invertible and compute its determinant knowing that $\det A = 1/3$.

Solution 1 by Michel Bataille, Rouen, France. We observe that

$$4A^2B = 4BA^2 = (8A - 4I)(A^3 + 4A^2 + 6A + 3I) + 12I = 12I.$$

It follows that $A^2B = BA^2 = 3I$, proving that B is invertible with $B^{-1} = (1/3)A^2$.

Also, from $(\det(A))^2 \det(B) = \det(3I) = 3^n$ we deduce that

$$\det(B) = 3^{n+2}.$$

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. First,

$$A^2B = BA^2 = 2A^4 + 7A^3 + 8A^2.$$

Next, the relation

$$A^3 + 4A^2 + 6A + 3I = 0$$

implies

$$A^4 + 4A^3 + 6A^2 + 3A = 0, \quad \text{or} \quad A^4 = -4A^3 - 6A^2 - 3A.$$

Thus,

$$A^2B = BA^2 = -A^3 - 4A^2 - 6A = 3I,$$

or

$$\left(\frac{1}{3}A^2\right)B = B\left(\frac{1}{3}A^2\right) = I.$$

The matrix B is therefore invertible, with $B^{-1} = \frac{1}{3}A^2$. Moreover,

$$\det B(\det A)^2 = \det(3I) = 3^n,$$

so

$$\det B = \frac{3^n}{(1/3)^2} = 3^{n+2}.$$

Solution 3 by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. From the equation $A^3 + 4A^2 + 6A + 3I = 0$ we get that $A(A^2 + 4A + 3) = -3I$ and $-\frac{1}{3}A(A^2 + 4A + 6I) = I$. Then, $A^2 + 4A + 6I$ is invertible and therefore $(A^2 + 4A + 6I)^2$ is also invertible. Furthermore,

$$\begin{aligned}(A^2 + 4A + 6I)^2 &= A^4 + 8A^3 + 28A^2 + 48A + 36I \\ &= -(4A^3+6A^2+3A) + 8A^3 + 28A^2 + 48A + 36I \\ &= 4A^3 + 22A^2 + 45A + 36I \\ &= -4(4A^2 + 6A + 3I) + 22A^2 + 45A + 36I \\ &= 3(2A^2 + 7A + 8I) = 3B.\end{aligned}$$

Thus, B is invertible, as we wanted to prove.

Now we claim that $\det B = 3^{n+2}$. Indeed, from the fact that $-\frac{1}{3}A(A^2+4A+6I) = I$ we get $A^{-1} = -\frac{1}{3}(A^2+4A+6I)$ and

$$\begin{aligned} A^{-2} &= \frac{1}{9}(A^2+4A+6I)^2 = \frac{1}{9}(A^4+8A^3+28A^2+48A+36I) \\ &= \frac{1}{3}(2A^2+7A+8I) = \frac{1}{3}B. \end{aligned}$$

Thus, $B \cdot A^2 = 3I$ and computing determinants in both sides we get $\det B \cdot \det A^2 = 3^n$ from which $\det B = 3^{n+2}$ follows.

Comment by Ander Lamaison Vidarte, Masaryk University, Brno, Czech Republic. From the equation $A^3+4A^2+6A+3I = 0$, it follows that all eigenvalues of A have absolute value at least 1. Therefore, the determinant of A cannot be $1/3$. However, this does not technically change the solutions for the problem.

This was also remarked by Moti Levy, Rehovot, Israel.

Also solved by *SQ Mathematical Problem Solving Group, Yogyakarta, Indonesia.*

A–79. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $n \geq 2$ be a positive integer. Compute

$$\int_1^2 \frac{\mathrm{d}x}{x\left(1+x\sqrt{x\sqrt[3]{x\sqrt[4]{x\ldots\sqrt[n]{x}}}}\right)}.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. First,

$$x\sqrt{x\sqrt[3]{x\sqrt[4]{x\cdots\sqrt[n]{x}}}} = x^{a_n},$$

where

$$a_n = \sum_{k=1}^{n} \frac{1}{k!}.$$

Next,

$$\frac{1}{x(1+x^{a_n})} = \frac{1}{x} - \frac{x^{a_n-1}}{1+x^{a_n}}.$$

Thus,

$$\int_1^2 \frac{\mathrm{d}x}{x\left(1+x\sqrt{x\sqrt[3]{x\sqrt[4]{x\cdots\sqrt[n]{x}}}}\right)} = \ln x - \frac{1}{a_n}\ln(1+x^{a_n})\Big|_1^2$$
$$= \ln 2 - \frac{1}{a_n}\ln(1+2^{a_n}) + \frac{1}{a_n}\ln 2$$
$$= \frac{a_n+1}{a_n}\ln 2 - \frac{1}{a_n}\ln(1+2^{a_n}).$$

Solution 2 by Michel Bataille, Rouen, France. First, we caculate $x\sqrt{x\sqrt[3]{x\sqrt[4]{x}\cdots\sqrt[n]{x}}}$ step by step starting from $\sqrt[n]{x}$, successively obtaining

$$x^{1/n},$$
$$\left(x\cdot x^{1/n}\right)^{1/(n-1)} = x^{1/(n-1)}\cdot x^{1/(n(n-1))},$$
$$\left(x\cdot x^{1/(n-1)}\cdot x^{1/(n(n-1))}\right)^{1/(n-2)} = x^{1/(n-2)}\cdot x^{1/((n-1)(n-2))}\cdot x^{1/(n(n-1)(n-2))},$$

and continuing this way we arrive at

$$x\left(x\cdot x^{1/3}\cdot x^{1/(3\cdot 4)}\cdot x^{1/(3\cdot 4\cdot 5)}\cdot\ldots\cdot x^{1/(3\cdot 4\cdot\ldots\cdot n)}\right)^{1/2},$$

that is,

$$x\sqrt{x\sqrt[3]{x\sqrt[4]{x}\cdots\sqrt[n]{x}}} = x^{s_n},$$

where $s_n = 1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$.

Now, let I_n be the integral to be computed. The substitution $x = u^{1/s_n}$, $\mathrm{d}x = \frac{1}{s_n}u^{(1/s_n)-1}\,\mathrm{d}u$ yields

$$I_n = \int_1^{2^{s_n}} \frac{1}{s_n}\cdot\frac{u^{(1/s_n)-1}\,\mathrm{d}u}{u^{1/s_n}(1+u)} = \frac{1}{s_n}\int_1^{2^{s_n}} \frac{\mathrm{d}u}{u(1+u)}.$$

Since $\int_1^a \frac{\mathrm{d}u}{u(1+u)} = \ln\left(\frac{2a}{1+a}\right)$, we finally get

$$I_n = \frac{1}{s_n}\ln\left(\frac{2^{s_n+1}}{1+2^{s_n}}\right) \quad \text{with} \quad s_n = \sum_{k=1}^{n}\frac{1}{k!}.$$

Also solved by *Moti Levy, Rehovot, Israel, and the proposer.*

A–80. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $a < b$ be two positive real numbers and let X be a random variable with support the $r \geq 2$ points $a_1, a_2, \ldots, a_r$, lying on the closed interval $[a, b]$, with probabilities $p_1, p_2, \ldots, p_r$, respectively. Prove that

$$\sigma_X \leq \frac{b-a}{2}.$$

When does equality occur? (Here σ_X represents the standard deviation of random variable X.)

Solution 1 by Michel Bataille, Rouen, France. Without loss of generality, we suppose that $a_1 \leq a_2 \leq \ldots \leq a_r$. Let $m = \sum_{i=1}^{r} p_i a_i$ be the mean of X so that

$$\sigma_X^2 = \sum_{i=1}^{r} p_i a_i^2 - m^2.$$

We first prove that $(a_r - m)(m - a_1) \geq \sigma_X^2$. To this aim, we calculate

$$(a_r - m)(m - a_1) - \sigma_X^2 = m(a_1 + a_r) - a_1 a_r - \sum_{i=1}^{r} p_i a_i^2$$

or, since $p_1 + p_2 + \cdots + p_r = 1$,

$$(a_r - m)(m - a_1) - \sigma_X^2 = \sum_{i=1}^{r} p_i (a_1 + a_r) a_i - \sum_{i=1}^{r} p_i a_1 a_r - \sum_{i=1}^{r} p_i a_i^2,$$

which immediately rewrites as

$$(a_r - m)(m - a_1) - \sigma_X^2 = \sum_{i=1}^{r} p_i (a_r - a_i)(a_i - a_1) \geq 0$$

(the inequality since $a_1 \leq a_i \leq a_r$ for $i = 1, 2, \ldots, r$).

Now, since $xy \leq \frac{(x+y)^2}{4}$ for all real numbers x, y, we obtain

$$\begin{aligned} \sigma_X^2 \leq (a_r - m)(m - a_1) &\leq \frac{(a_r - m + m - a_1)^2}{4} \\ &= \frac{(a_r - a_1)^2}{4} \leq \frac{(b-a)^2}{4}, \end{aligned}$$

hence $\sigma_X \leq \frac{b-a}{2}$, with equality if and only if above equality holds throughout, that is, if and only if a_i is equal to a_1 or to a_r for all i, $a_1 - m = m - a_r$ and $a_1 = a$, $a_r = b$. We readily deduce that equality holds exactly if and only if $a_1 = a_2 = \cdots = a_k = a$, $a_{k+1} = \cdots = a_r = b$ for some $k \in \{1, 2, \ldots, r-1\}$ with $p_1 + \cdots + p_k = p_{k+1} + \cdots + p_r = \frac{1}{2}$ (the latter to ensure $m = \frac{a+b}{2}$).

Solution 2 by the proposer. We will find an upper bound for the variance of X. Indeed, we have that the following relations hold:

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{Var}\Big(X - \frac{a+b}{2}\Big) \leq \mathbb{E}\left\{\Big(X - \frac{a+b}{2}\Big)^2\right\} \\ &= \sum_{k=1}^{r}\Big(x_k - \frac{a+b}{2}\Big)^2 p_k \leq \sum_{k=1}^{r}\Big(\frac{b-a}{2}\Big)^2 p_k \\ &= \Big(\frac{b-a}{2}\Big)^2 \sum_{k=1}^{r} p_k = \Big(\frac{b-a}{2}\Big)^2\end{aligned}$$

on account that

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{Var}\Big(X - \frac{a+b}{2}\Big) \\ &= \mathbb{E}\left\{\Big(X - \frac{a+b}{2}\Big)^2\right\} - \left\{\mathbb{E}\Big(X - \frac{a+b}{2}\Big)\right\}^2\end{aligned}$$

and that $p_1 + p_2 + \ldots + p_r = 1$, as is well known. Equality holds when $r = 2$, with $a_1 = a$, $a_2 = b$ and $p(X = a) = p(X = b) = 1/2$, as can be easily proven. Since $\sigma_X = \sqrt{\mathrm{Var}(X)}$ then from the preceding the statement follows.

A–81. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let A be an $n \times n$ real matrix such that $4A^4 = A^3 + A^2 + A + I$, where I is the identity matrix. Let

$$\lim_{j\to\infty} A^j = B.$$

Show that $B^2 = B$.

Solution 1 by the proposer. The minimal polynomial of A is a divisor of $4x^4 - x^3 - x^3 - x - 1$. This polynomial has four distinct roots. This implies that A is diagonalizable and $A = P^{-1}DP$, where P is a diagonal matrix. The eigenvalues of the matrices A and D are all roots of polynomial $4x^4 - x^3 - x^3 - x - 1$, as is well-known. One of the three roots is 1, the remaining three roots have smaller absolute value than 1. Hence, the diagonal elements of D^j, which are the j-th powers of the eigenvalues, tend to either 0 or 1 and

$$C = \lim_{j\to\infty} D^j$$

is idempotent. Then,

$$B = \lim_{j\to\infty} A^j = \lim_{j\to\infty} P^{-1}D^jP$$

is idempotent as well. That is, $B^2 = B$, and we are done.

Solution 2 by Michel Bataille, Rouen, France. Let $p(X) = 4X^4 - X^3 - X^2 - X - 1$ and let $z_1 = 1, z_2, z_3, z_4$ be a list of its complex roots. We use the following lemma, proved at the end:

Lemma. *The roots of $p(X)$ are distinct and $|z_k| < 1$ for $k = 2, 3, 4$.*

Since $p(A) = 0$, the minimum polynomial $m(X)$ of A divides $p(X)$, hence $m(X)$ has no multiple roots. It follows that A is diagonalizable and that the eigenvalues of A, that is, the roots of $m(X)$, belong to the set $\{1, z_2, z_3, z_4\}$. As a result, for $j \in \mathbb{N}^*$, the matrix A^j is similar to a diagonal matrix whose diagonal entries belong to the set $\{1, z_2^j, z_3^j, z_4^j\}$. Since $\lim\limits_{j\to\infty} z_k^j = 0$ for $k = 2, 3, 4$, the matrix $B = \lim\limits_{j\to\infty} A^j$ exists and B is similar to a diagonal matrix D whose diagonal entries belong to $\{0, 1\}$. Such a diagonal matrix D satisfies $D^2 = D$, hence we also have $B^2 = B$, as desired.

Proof of the lemma. We have $p(X) = (X - 1)q(X)$ where $q(X) = 4X^3 + 3X^2 + 2X + 1$. Since $q(1) \neq 0$, $z_2, z_3, z_4 \neq 1$. Moreover, $q'(X) = 2(6X^2 + 3X + 1)$ and

$$q(X) = \left(\frac{2X}{3} + \frac{1}{6}\right)(6X^2 + 3X + 1) + \frac{5}{6}(X + 1), \tag{1}$$

hence $q(X)$ and $q'(X)$ have no common root [from (1), the only possible one is -1 and $q(-1) \neq 0$]. It follows that the roots z_2, z_3, z_4 of $q(X)$ are simple. We can conclude that the roots of $p(X)$ are distinct.

If z were a root of $p(X)$ with $|z| > 1$, then we would have $4 = \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}$ and so

$$4 = \left|\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4}\right| \leq \frac{1}{|z|} + \frac{1}{|z|^2} + \frac{1}{|z|^3} + \frac{1}{|z|^4} < 4,$$

a contradiction. Therefore, $|z_k| \leq 1$ for $k = 2, 3, 4$.

Moreover, if a root z of $p(X)$ satisfied $z \neq 1$ and $|z| = 1$, then $\frac{1}{z} = e^{i\theta}$ (with $\theta \in (0, 2\pi)$) is a root of $X^4 + X^3 + X^2 + X - 4$, hence also of $(X-1)(X^4 + X^3 + X^2 + X - 4) = X^5 - 5X + 4$ and we would have

$$1 = |e^{i\theta}|^5 = 5\left|e^{i\theta} - \frac{4}{5}\right|. \tag{2}$$

However,

$$\left|e^{i\theta} - \frac{4}{5}\right|^2 - \frac{1}{25} = \left(\cos\theta - \frac{4}{5}\right)^2 + \sin^2\theta - \frac{1}{25} = \frac{8(1-\cos\theta)}{5} > 0,$$

hence $\left|e^{i\theta} - \frac{4}{5}\right| > \frac{1}{5}$, in contradiction with (2). Thus, $|z_k| < 1$ for $k = 2, 3, 4$. □

Solution 3 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. Because $4A^4 = A^3 + A^2 + A + I$, it is possible to express the matrix A^j in the form

$$A^j = \alpha_j A^3 + \beta_j A^2 + \gamma_j A + \delta_j I,$$

for each $j \geq 4$. Multiplying this last expression by A yields

$$\begin{aligned} A^{j+1} &= \alpha_j A^4 + \beta_j A^3 + \gamma_j A^2 + \delta_j A \\ &= \left(\frac{1}{4}\alpha_j + \beta_j\right)A^3 + \left(\frac{1}{4}\alpha_j + \gamma_j\right)A^2 + \left(\frac{1}{4}\alpha_j + \delta_j\right)A + \frac{1}{4}\alpha_j I. \end{aligned}$$

Thus,

$$\begin{bmatrix} \alpha_{j+1} \\ \beta_{j+1} \\ \gamma_{j+1} \\ \delta_{j+1} \end{bmatrix} = \begin{bmatrix} 1/4 & 1 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 1/4 & 0 & 0 & 1 \\ 1/4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_j \\ \beta_j \\ \gamma_j \\ \delta_j \end{bmatrix}.$$

Let

$$M = \begin{bmatrix} 1/4 & 1 & 0 & 0 \\ 1/4 & 0 & 1 & 0 \\ 1/4 & 0 & 0 & 1 \\ 1/4 & 0 & 0 & 0 \end{bmatrix}.$$

Then,

$$B = \lim_{j\to\infty} A^j = \alpha A^3 + \beta A^2 + \gamma A + \delta I,$$

where

$$\begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \lim_{j\to\infty} M^j \begin{bmatrix} \alpha_4 \\ \beta_4 \\ \gamma_4 \\ \delta_4 \end{bmatrix} = \lim_{j\to\infty} M^j \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}.$$

Because the sum of the entries in each column of M is equal to 1 and

$$M^4 = \begin{bmatrix} 125/256 & 25/64 & 5/16 & 1/4 \\ 61/256 & 25/64 & 5/16 & 1/4 \\ 45/256 & 9/64 & 5/16 & 1/4 \\ 25/256 & 5/64 & 1/16 & 1/4 \end{bmatrix}$$

has only positive entries, M is a regular stochastic matrix. This implies that

$$\lim_{j\to\infty} M^j = \begin{bmatrix} \mathrm{v} & \mathrm{v} & \mathrm{v} & \mathrm{v} \end{bmatrix},$$

where v is the unique solution to the equation $M\mathrm{v} = \mathrm{v}$ with the property that the sum of the entries in v is equal to 1. Now,

$$M \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix},$$

so

$$\mathrm{v} = \frac{1}{10}\begin{bmatrix} 4 & 3 & 2 & 1 \end{bmatrix}^T,$$

and

$$B = \frac{2}{5}A^3 + \frac{3}{10}A^2 + \frac{1}{5}A + \frac{1}{10}I.$$

Next,

$$B^2 = \frac{4}{25}A^6 + \frac{6}{25}A^5 + \frac{1}{4}A^4 + \frac{1}{5}A^3 + \frac{1}{10}A^2 + \frac{1}{25}A + \frac{1}{100}I.$$

With

$$A^4 = \frac{1}{4}A^3 + \frac{1}{4}A^2 + \frac{1}{4}A + \frac{1}{4}I,$$

it follows that

$$A^5 = \frac{5}{16}A^3 + \frac{5}{16}A^2 + \frac{5}{16}A + \frac{1}{16}I$$

and

$$A^6 = \frac{25}{64}A^3 + \frac{25}{64}A^2 + \frac{9}{64}A + \frac{5}{64}I.$$

Then,

$$\begin{aligned} B^2 &= \Big(\frac{1}{16} + \frac{3}{40} + \frac{1}{16} + \frac{1}{5}\Big)A^3 + \Big(\frac{1}{16} + \frac{3}{40} + \frac{1}{16} + \frac{1}{10}\Big)A^2 \\ &\quad + \Big(\frac{9}{400} + \frac{3}{40} + \frac{1}{16} + \frac{1}{25}\Big)A + \Big(\frac{1}{80} + \frac{3}{200} + \frac{1}{16} + \frac{1}{100}\Big)I \\ &= \frac{2}{5}A^3 + \frac{3}{10}A^2 + \frac{1}{5}A + \frac{1}{10}I = B. \end{aligned}$$

Solution 4 by Moti Levy, Rehovot, Israel. Let λ an eigenvalue of A, then it must be a root of the polynomial $p(x)$,

$$\begin{aligned} p(x) &= x^4 - \frac{1}{4}x^3 - \frac{1}{4}x^2 - \frac{1}{4}x - \frac{1}{4} \\ &= \frac{1}{4}(x-1)\big(4x^3 + 3x^2 + 2x + 1\big). \end{aligned}$$

The polynomial $p(x)$ is separable over $\mathbb{C}$ (this can be verified by evaluating the roots of $(4x^3 + 3x^2 + 2x + 1)' = 12x^2 + 6x + 2$ and checking that the roots of $12x^2 + 6x + 2$ are not roots of $4x^3 + 3x^2 + 2x + 1$).

The minimal polynomial of A must divide $p(x)$, hence it is also separable. It follows that A is diagonalizable over $\mathbb{C}$.

Thus, A can be diagonalized $A = P^{-1}DP$, where D is diagonal matrix with the roots of $p(x)$ on its diagonal (not necessarily all the roots).

Clearly, $A^j = P^{-1}D^jP$, and D^j is diagonal matrix with the j-th power of roots of $p(x)$ on its diagonal.

Now we use the Lagranges's upper bounds on all complex roots of $p(x)$. Lagrange's bound is

$$\max\left\{1, \sum_{k=0}^{3}\left|\frac{a_k}{a_4}\right|\right\} = \max\left\{1, 4\left|\frac{1}{4}\right|\right\} = 1.$$

We conclude that the roots of $p(x)$ are 1 and three more roots with absolute value at most 1.

Now we want to verify that the roots of $4x^3 + 3x^2 + 2x + 1$ lie inside the unit disk. To this end, we find the minimum of

$$\left|4e^{i3t} + 3e^{i2t} + 2e^{it} + 1\right|^2 = 32\cos^3 t + 44\cos^2 t + 16\cos t + 8,$$

for $0 \leq t < 2\pi$. The minimum is attained at $t = \pi$ and is equal to 2. It follows that $4x^3 + 3x^2 + 2x + 1$ does not have a root on the unit circle, hence all its roots lies inside the unit disk.

The matrix $D_\infty := \lim_{j\to\infty} D^j$ is a diagonal matrix whose entries on the diagonal are 1 or 0.

This implies that D_∞ is idempotent and hence $B = P^{-1}D_\infty P$ is also idempotent.

A–82. *Proposed by Vasile Mircea Popa, "Lucian Blaga" University of Sibiu, Romania.* Calculate the following integral:

$$\int_{-1}^{\infty} \frac{\ln(x+2)}{x^2+3x+3}\,\mathrm{d}x.$$

Solution by the proposer. Let $I = \displaystyle\int_{-1}^{\infty} \frac{\ln(x+2)}{x^2+3x+3}\,\mathrm{d}x$. Making the change of variables

$$x = \frac{1}{t} - 2,$$

we have

$$\ln(x+2) = -\ln t, \quad x^2+3x+3 = \frac{t^2-t+1}{t^2}, \quad \mathrm{d}x = -\frac{1}{t^2}\,\mathrm{d}t,$$

and we obtain

$$I = -\int_0^1 \frac{\ln t}{t^2 - t + 1}\,\mathrm{d}t.$$

To calculate this integral we use the notation

$$f(t) = \frac{1}{t^2 - t + 1}$$

and we develop the function in power series. For $t \in (0, 1)$ we have

$$f(t) = 1 + t - t^3 - t^4 + t^6 + t^7 - t^9 - t^{10} + t^{12} + t^{13} - t^{15} - t^{16} + t^{18} + t^{19} - \dots$$

and

$$\begin{aligned} I_1 &= \int_0^1 f(t) \ln t \,\mathrm{d}t \\ &= -1 - \frac{1}{4} + \frac{1}{16} + \frac{1}{25} - \frac{1}{49} - \frac{1}{64} + \frac{1}{100} + \\ &\quad + \frac{1}{121} - \frac{1}{169} - \frac{1}{196} + \frac{1}{256} + \frac{1}{289} - \frac{1}{361} - \frac{1}{400} + \dots \\ &= -\frac{1}{1^2} - \frac{1}{7^2} - \frac{1}{13^2} - \frac{1}{19^2} - \dots \\ &\quad - \frac{1}{2^2} - \frac{1}{8^2} - \frac{1}{14^2} - \frac{1}{20^2} - \dots \\ &\quad + \frac{1}{4^2} + \frac{1}{10^2} + \frac{1}{16^2} + \dots \\ &\quad + \frac{1}{5^2} + \frac{1}{11^2} + \frac{1}{17^2} + \dots \end{aligned}$$

Now we will use the trigamma function, defined by

$$\psi_1(x) = \sum_{n=0}^{\infty} \frac{1}{(x+n)^2}.$$

We obtain

$$I_1 = \frac{1}{36}\left[-\psi_1\left(\frac{1}{6}\right) - \psi_1\left(\frac{1}{3}\right) + \psi_1\left(\frac{2}{3}\right) + \psi_1\left(\frac{5}{6}\right)\right]$$

and we can write

$$I = -I_1 = \frac{1}{36}\left[\psi_1\left(\frac{1}{6}\right) + \psi_1\left(\frac{1}{3}\right) - \psi_1\left(\frac{2}{3}\right) - \psi_1\left(\frac{5}{6}\right)\right].$$

But we have using the reflection formula that

$$\psi_1\left(\frac{1}{6}\right) + \psi_1\left(\frac{5}{6}\right) = 4\pi^2 \quad \text{and} \quad \psi_1\left(\frac{1}{3}\right) + \psi_1\left(\frac{2}{3}\right) = \frac{4\pi^2}{3}.$$

We obtain that

$$I = \frac{1}{18}\left[\psi_1\left(\frac{1}{6}\right) + \psi_1\left(\frac{1}{3}\right) - \frac{8\pi^2}{3}\right].$$

But we also have using the reflection and the duplication formulas that

$$\psi_1\left(\frac{1}{3}\right) = \frac{1}{5}\psi_1\left(\frac{1}{6}\right) + \frac{4\pi^2}{15}.$$

We thus obtain the value of the integral from the problem statement:

$$I = \frac{1}{15}\left[\psi_1\left(\frac{1}{6}\right) - 2\pi^2\right].$$

Thus, the problem is solved.

Also solved by *Brian Bradie, Christopher Newport University, Newport News, VA, USA, and Moti Levy, Rehovot, Israel.*

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