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On two problems on random walks

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Abstract

This note serves as an introduction to some of the techniques related to random walks. We present some of the basic concepts, such as probability generating functions, through a series of problems and solutions. The traditional one-dimensional random walk is presented and thoroughly studied using these techniques, and Pólya's transient theorem is proved in this particular case. First passage times are presented and computed for one-dimensional walks. Two problems on areas enclosed by random walks are presented, and solutions based on reducing the problem to the one-dimensional random walk are produced. Lastly, a problem about areas is stated and left open.

1 Introduction

There are many problems that can be modelled through a sequence of random variables. These are problems that sometimes require strong techniques to deal with, but that may seem very simple and accessible at first. Here, we show an example of these problems, and how its solution can be used to face different problems. The reader should have a basic knowledge about formal power series, random variables and probability concepts in order to fully understand this note.

2 The drunken sailor problem

One of the problems that can be modelled through a sequence of random variables is that of a person who moves at random. This might happen, for example, if said person is totally drunk. If we assume perfect randomness, what can we say about this person's behaviour?

2.1 A finite case

Problem 1 (The drunken sailor). *A drunken sailor gets out of a bar near the coast. One step to his left there is a cliff. His house is d steps to his right, but he is so drunk that he does no longer recognize his surroundings, and he just moves at random. If the probability that he takes a step to the left or to the right is $\frac{1}{2}$ each, what is the probability that he will fall off the cliff and die?*

This is the most typical example of a problem that can be modelled through a random walk in one dimension (we are assuming that the sailor can only move right or left, and that he will not remain in the same place). We say that a *random walk* is a sequence of random variables $X_1, X_2, \ldots$ which are independent and identically distributed. In this case, we have that the possible values for each of the random variables are 1 and -1 (which would correspond to a step to the right or to the left, respectively), and that

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2} \quad \forall \ i.$$

We also write

$$S_n = \sum_{i=1}^{n} X_i$$

to denote the position of the man at the n-th step. If at any given time we have $S_n = -1$, the sailor will die, and if $S_n = d$ he will save himself. For anything in between, he will have to keep walking. Let us see how we can compute the actual probability that he will not survive.

Solution. First of all, notice that the statement could be written supposing that the bar is in any other point between the cliff and the house. We would then have random walks with different starting points, all of them between 0 and $d-1$, since at position -1 the sailor would die and at position d he would be safely at home. For each of these problems we could find the probability that the sailor dies.

In order to define this probability, first write

$$T = \min\{n \geq 0 : S_n = -1 \text{ or } S_n = d\},$$

that is, T is a random variable that measures the time it takes for the sailor's walk to end. Then, the probability that the sailor dies, given that the random walk starts at point i, can be written as $P(S_T = -1 | S_0 = i)$ for any $i \in \{0, 1, \ldots, d-1\}$. For simplicity, write $u_i = P(S_T = -1 | S_0 = i)$. We can also define $u_{-1} = 1$ and $u_d = 0$ for the border cases in which the sailor dies or arrives at home.

We are going to use difference equations in order to solve this problem. Assume that the walk starts at some point i. Then, only two things can happen: either the sailor moves to the left or he moves to the right. In either case, we can suppose that we start a new random walk at position $i-1$ or $i+1$, respectively. This is because the sailor is absolutely drunk and he does not remember where he comes from, which makes the random variables independent. Hence, the probability that he dies will be equal to the probability that he dies when starting one step to his left multiplied by the probability that he moves to his left plus the probability that he dies when starting one step to his right multiplied by the probability that he moves to his right, that is,

$$u_i = \frac{1}{2}u_{i-1} + \frac{1}{2}u_{i+1},$$

and this can be done for any starting position i. We can rewrite the above expression as

$$\frac{1}{2}(u_i - u_{i-1}) = \frac{1}{2}(u_{i+1} - u_i) \Longrightarrow u_i - u_{i-1} = u_{i+1} - u_i,$$

that is, the difference between any two consecutive values is the same. This is the characterization of an arithmetic progression!

Now we know that these values follow an arithmetic progression, and we also have the specific values $u_{-1} = 1$ and $u_d = 0$. Since there are $d+1$ consecutive differences, we must have that

$$u_i = 1 - \frac{i+1}{d+1} = \frac{d-i}{d+1}.$$

This means that, in our initial problem, the sailor will die with probability $\dfrac{d}{d+1}$. □

In a similar way, we could consider a very similar problem, slightly different from this: *What is the probability that the sailor will die if he takes a step towards his house with probability p and towards the cliff with probability $1-p$?* In this case, the solution is obtained in the same way, but the resulting progression will be geometric instead of arithmetic. This can be done as an exercise by the reader.

One can easily observe that the further the sailor's house is, the higher the probability that he will die. This leads us to a slightly different problem, and we are forced to introduce infinite random walks: what happens if the sailor's house is infinitely far away?

2.2 Infinite random walk

This time, let us not kill the sailor. We can write the problem in a different setting, although they are very similar in the mathematical model:

Problem 2 (The drunken sailor 2). *A drunken sailor gets out of a bar into an infinitely long street. He is so drunk that he does not recognize his surroundings, so he starts moving at random. If the probability that he takes a step to the left or to the right is $\frac{1}{2}$ each, what is the probability that he will come back to the bar?*

Observe that this problem can be easily compared to the previous one: at the first step, the sailor will move either right or left, and then he will have the origin one step to one of his sides, and we want to know the probability that he will reach that position, which would correspond to the probability of his death in Problem 1.

Solution. Say that the sailor starts his walk at 0, and we want to know the probability that he will come back to 0 again. His walk is modelled through a random walk $X_1, X_2, \ldots$, where X_i take values 1 and -1 with probability $\frac{1}{2}$ and are independent. Define the probabilities

$$\begin{aligned} u_n &= P(\text{"the walk starts at 0 and is at 0 at the } n\text{-th step"}) \\ &= P(S_n = 0). \end{aligned}$$

Observe that $u_{2k+1} = 0$ for any k, since we cannot return to the origin with an odd number of steps. Define the events

$$F_n = \{S_n = 0\} \cap \bigcap_{i=1}^{n-1} \{S_i \neq 0\},$$

that is, the event that the random walk goes back to the origin for the first time at the n-th step. Observe that all these events are disjoint. Now, define the probability of escape, P_{esc}, as the probability that the sailor will never return to the bar, and then we have that

$$P_{\text{esc}} = P\left(\overline{\bigcup_{n\geq 1} F_n}\right),$$

so

$$1 - P_{\text{esc}} = \sum_{n\geq 1} P(F_n).$$

With this, it is enough to study the sum of $P(F_n)$ to obtain results about the probability of escape. For convenience, write $f_n = P(F_n)$. Now, observe that any walk that starts at the origin and returns to the origin at the n-th step can be defined in terms of its first return to the origin. That is, any walk that starts at the origin, ends at the origin after n steps, and goes through the origin

k times, can be understood as the composition of a walk of length l that goes back to the origin only once (these walks are known as Dyck paths) and a walk of length $n-l$ that goes back to the origin $k-1$ times. In this sense, for any n we may write

$$u_n = f_0 u_n + f_1 u_{n-1} + f_2 u_{n-2} + \ldots + f_{n-1} u_1 + f_n u_0, \qquad (1)$$

which means that the probability that we return to the origin after n steps can be computed as the sum of the products of the probability of returning for the first time to the origin after i steps and the probability of being again at the origin after $n-i$ steps. By convention, we may say that $f_0 = 0$ (there is no return if we still did not leave) and $u_0 = 1$ ($S_0 = 0$ by definition). To obtain a relation between the u_n's and the f_n's, define

$$U(s) = \sum_{n\geq 0} u_n s^n \qquad \text{and} \qquad F(s) = \sum_{n\geq 0} f_n s^n.$$

It is easy to check that

$$U(s)F(s) = \sum_{n\geq 0} \sum_{r=0}^{n} f_r u_{n-r} s^n.$$

Using the recurrence defined in (1), we can write

$$\begin{aligned} u_0 &= 1, \\ s u_1 &= (f_0 u_1 + f_1 u_0) s, \\ s^2 u_2 &= (f_0 u_2 + f_1 u_1 + f_2 u_0) s^2, \\ s^3 u_3 &= (f_0 u_3 + f_1 u_2 + f_2 u_1 + f_3 u_0) s^3, \\ &\vdots \end{aligned}$$

Adding up all these expressions and taking into account the observation about the product of $U(s)$ and $F(s)$, we have

$$U(s) = U(s)F(s) + 1,$$

and from this

$$F(s) = \frac{U(s) - 1}{U(s)}.$$

We know that $F(1) = 1 - P_{\text{esc}}$, so if we can compute $U(1) = \sum_{n\geq 0} u_n$ we will find the probability of escape.

We already know that $u_{2k+1} = 0 \ \ \forall\, k$. On the other hand,

$$u_{2k} = \left(\frac{1}{2}\right)^{2k} \cdot |\{\text{paths of length } 2k \text{ that start at the origin and end at the origin}\}| = \left(\frac{1}{2}\right)^{2k} \binom{2k}{k},$$

where the combinatorial number comes from choosing the steps in which we move to the right. Using Stirling's formula

$$k! \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^{k}$$

we have that

$$u_{2k} = \frac{1}{2^{2k}} \frac{(2k)!}{k!k!} \sim \frac{1}{2^{2k}} \frac{\sqrt{4\pi k}\left(\frac{2k}{e}\right)^{2k}}{2\pi k\left(\frac{k}{e}\right)^{2k}} = \frac{1}{\sqrt{\pi k}}.$$

With this, we have that

$$U(1) = \sum_{k\geq 0} u_{2k} \sim \sum_{k\geq 0} \frac{1}{\sqrt{\pi k}} = +\infty,$$

so $F(1) = 1$ and $P_{\text{esc}} = 0$. This means that the sailor will come back to the bar with probability 1. □

Notice the following: we know that the sailor will return to the bar, and at that point in time everything will be as it was at the beginning of the problem. This means that he will leave the bar in one more step, and then come back with probability one. This will happen over and over again, so we have that the sailor will come back to the bar an infinite number of times! This is a property known as *recurrence*, and random walks that present this behaviour are said to be *recurrent*. Other random walks, for which the probability of escape is positive, are said to be *transient*.

What we have proved in this problem is a particular case of the one-dimensional case of a result known as *Pólya's transient theorem*. This theorem states that a random walk with expectation

zero in a d-dimensional lattice is recurrent when $d = 1, 2$ (in the case $d = 2$ one also needs to impose that the variance of the random variables X_i is finite) and transient $\forall\ d \geq 3$. The proof of case $d = 2$ is done in the same way as the case we showed, and is left as an exercise for the reader. The proof of the transience of higher dimension random walks becomes a bit harder.

Once this is done, one may wonder what happens if the random walk is not symmetric, that is, if the probability to go to each of the sides is different. In the one dimensional case, we would have probability p to go to the left and $1 - p$ to go to the right. In this case, a study using the same proof of Pólya's theorem tells us that the random walks are always transient when $p \neq \frac{1}{2}$, that is, there is a positive probability that the sailor will never come back to the bar again. We can see this as the following problem.

Problem 3 (The drunken sailor 3). *A drunken sailor gets out of a bar into an infinitely long street. He sniffs around and, to his right, he smells the most awesome pie he ever has, so he decides to go look for it. However, he is so drunk that sometimes he does not move in the direction he wants to. If the probability that he takes a step to the right is $p > \frac{1}{2}$, and otherwise he takes a step to the left, will he always come back to the bar?*

Solution. We may proceed in the exact same way as in Problem 2 until we have the relation between $U(s)$ and $F(s)$. Now, we want to compute $U(1)$ in order to be able to know the probability of escape. As before, $u_{2k+1} = 0\ \ \forall\ k$, and now

$$u_{2k} = \binom{2k}{k} p^k (1-p)^k.$$

Using Stirling's formula we have that

$$u_{2k} = \frac{(2k)!}{k!k!} p^k (1-p)^k \sim \frac{2^{2k}}{\sqrt{\pi k}} p^k (1-p)^k.$$

With this, we have that

$$U(1) = \sum_{k \geq 0} u_{2k} \sim \sum_{k \geq 0} \frac{2^{2k}}{\sqrt{\pi k}} p^k (1-p)^k.$$

In order to see if this sum is convergent, we apply the ratio test:

$$\lim_{k\to+\infty}\frac{u_{2k+2}}{u_{2k}}=\lim_{k\to+\infty}\frac{\sqrt{k}}{\sqrt{k+1}}2^2p(1-p)=4p(1-p),$$

and this is smaller than 1 for any value of p distinct from $\frac{1}{2}$. This means that the sum is convergent, so $U(1)<+\infty$, $F(1)<1$ and the probability of escape is strictly positive. That is, there is a chance that the man will never come back to the bar. □

One must note that the sailor will escape to the right. The probability that he escapes to the left is 0. This is because the probability that he goes to the right is greater than the probability that he moves to the left. This means that if at any given point in time the sailor is to the left of the bar, then he will go back to the bar with probability 1.

2.3 First passage time and probability generating functions

Going back to our original problem, we already know that the sailor will visit the bar many times over. However, how long will it take for him to visit the bar each time he leaves? Since we are working with random variables, we cannot give an exact answer, but we can try to find an expectation.

Problem 4. *With the same statement of Problem 2, what is the expected time it will take the sailor to go back to the bar?*

In order to solve this problem we are going to use a tool known as probability generating functions. Let us first introduce this concept and some of its basic properties.

Definition 1. *Given a discrete random variable X that takes values on the integers $\{0,1,2,\ldots\}$, the probability generating function of X is*

$$G_X(s)=\sum_{r\geq 0}P(X=r)s^r=\mathbb{E}[s^X].$$

This is a formal power series, so it can be studied by analytic means. In particular, we have the following properties:

Proposition 1. *Let X be a discrete random variable, and let $G_X(s)$ be its probability generating function. Then,*

i) $G_X(0) = P(X = 0)$.
ii) $G_X(1) = 1$.
iii) $\mathbb{E}[X] = G'_X(1)$.
iv) $\mathbb{E}[X(X-1)] = G''_X(1)$.

The proof of all of these is elementary and comes from the definition of probability generating functions.

Proposition 2. *Let X and Y be two independent discrete random variables. Then,*

$$G_{X+Y}(s) = G_X(s)G_Y(s).$$

Proof. We have that

$$\begin{aligned} P(X+Y=n)s^n &= \sum_{r=0}^{n} P(X=r, Y=n-r)s^n \\ &= \sum_{r=0}^{n} P(X=r)s^r P(Y=n-r)s^{n-r}, \end{aligned}$$

so

$$\begin{aligned} G_{X+Y}(s) &= \sum_{n\geq 0}\sum_{r=0}^{n} P(X=r)s^r P(Y=n-r)s^{n-r} \\ &= \sum_{u\geq 0}(P(X=u)s^u)\sum_{v\geq 0}(P(Y=v)s^v) = G_X(s)G_Y(s). \square \end{aligned}$$

With this, we can already solve Problem 4.

Solution. We start with $S_0 = 0$. Thanks to the symmetry of the problem, it does not matter which way we move in the first step, so assume that we move to the right ($S_1 = 1$). The total amount of time it takes to reach the origin again will be the time it takes

to go from 1 to 0 plus 1 (this is the first step we already took). Now, the time it takes to go from 1 to 0 can be modelled by a random variable which is identical to that of the time it takes to go from 0 to 1 (or, in general, from i to $i+1$). This is true, again, because of the symmetry of the problem and the fact that all the random variables X_i are independent. We will work with the time it takes to go from 0 to 1, but bear in mind that the distribution of the time it takes to reach any state conditioned to starting at a state that is next to it is the same. Let the random variable that measures this time be τ,

$$\tau = \min\{n \geq 0 : S_n = 1\}.$$

This random variable is called the *first passage time* and, in general, one can define the random variables

$$\tau(m) = \min\{n \geq 0 : S_n = m\}.$$

Now, in order to be able to compute its expectancy, we want to know the distribution of the random variable τ. To do so, we resort to probability generating functions. Let $G_\tau(s)$ be the probability generating function associated to τ.

In order to obtain the distribution, we can condition the events on the first step. There are two possibilities: either $S_1 = 1$ or $S_1 = -1$. If $S_1 = 1$ we have reached our destination and $\tau = 1$. This happens with probability $\frac{1}{2}$. If $S_1 = -1$, the random walk must return to 0 before it can reach 1. But the time it takes to reach 0 from -1 is modelled with a random variable, τ', which is identical to τ, and the time it would take then to reach 1 from 0 is again modelled by a random variable, τ'', with the same distribution. Furthermore, these variables are independent from each other, so using the expectancy notation we can write

$$G_\tau(s) = \mathbb{E}[s^\tau] = \frac{s}{2} + \frac{s}{2}\mathbb{E}[s^{\tau'+\tau''}] = \frac{s + sG_\tau(s)^2}{2}.$$

This is a quadratic equation on $G_\tau(s)$. Solving it yields

$$G_\tau(s) = \frac{1 \pm \sqrt{1-s^2}}{s}.$$

In order to know which of the two signs is the correct one, we need to consider the properties of probability generating functions. In particular, we know that probability generating functions are positive and increasing (because all their coefficients are non-negative) for any $0 \leq s \leq 1$, and that in this same interval it is bounded by 1. The plus sign would yield values greater than 1 in this interval, so the only possible solution is

$$G_\tau(s) = \frac{1 - \sqrt{1 - s^2}}{s}.$$

Indeed, one can check that the mentioned properties hold.

Once we have this, we can compute the expectation of the random variable τ through the derivative of its probability generating function. We have

$$G'_\tau(s) = \frac{1}{\sqrt{1 - s^2}} - \frac{1 - \sqrt{1 - s^2}}{s^2},$$

which tends to infinity as s approaches 1. This means that $\mathbb{E}[\tau] = +\infty$ (if a random variable has finite expectancy, it is determined by the derivative of the probability generating function). With this, the plus 1 we had to consider from before does not change the result. □

So what we have is that the sailor will visit the bar an infinite number of times, but the expected time that will pass between any two of his visits is infinite!

This is what happens in the symmetric case, so one would now like to know what happens in the asymmetric one. We can consider the following statement.

Problem 5. *Consider the same situation as in Problem 3. If at any given point the man is one step to the left of the bar, what is the expected time it will take him to reach it?*

Solution. As happened between Problem 2 and Problem 3, in this case the solution is similar to that of Problem 4. We define again the random variable τ that measures the time it takes to go from

a certain point to its immediate point to the right. Note that now this is different from the expected time to move one step to the left. Using the exact same argument as before, when we are at position -1, there are two possibilities: either we move right or left. If we move right, with probability p, we have already reached our destination. If we move left, then we have to arrive to a position to the right two times, which is measured by τ again. With this, the probability generating function follows the equation

$$G_\tau(s) = \mathbb{E}[s^\tau] = ps + (1-p)s\mathbb{E}[s^{\tau'+\tau''}] = ps + (1-p)sG_\tau(s)^2,$$

and solving this equation yields

$$G_\tau(s) = \frac{1 \pm \sqrt{1-4p(1-p)s^2}}{2(1-p)s}.$$

In order to choose the sign of the solution, just observe that if we consider the positive sign, then $G_\tau(1) > 1$, which cannot happen, so the correct sign must be the negative one. Hence,

$$G_\tau(s) = \frac{1 - \sqrt{1-4p(1-p)s^2}}{2(1-p)s}.$$

Remember that this is only true for $p \geq \frac{1}{2}$. If we try to make it work for $p < \frac{1}{2}$ we will see that the result is not a probability generating function. This is also a way to see that the walk is transient: the sum of all the probabilities for $p < \frac{1}{2}$, given by $G_\tau(1)$, is smaller than 1, so there is a chance that the walk never returns to the bar (in fact, the probability is $1 - G_\tau(1)$). Now, in order to obtain the expectancy we first find the derivative. After some calculations, one can check that

$$G'_\tau(s) = \frac{2p}{\sqrt{1-4p(1-p)s^2}} - \frac{1 - \sqrt{1-4p(1-p)s^2}}{2(1-p)s^2},$$

so the expected time it takes to reach the bar is

$$\mathbb{E}[\tau] = G'_\tau(1) = \frac{2p}{\sqrt{1-4p(1-p)}} - \frac{1 - \sqrt{1-4p(1-p)}}{2(1-p)}.$$

Since $p \geq \frac{1}{2}$, observe that

$$(1-2(1-p))^2 = 1-4(1-p)+4(1-p)^2 = 1-4p(1-p),$$

so $1-2(1-p) = 2p-1$ is the positive square root of what we had above. We then can write

$$\mathbb{E}[\tau] = \frac{1}{2p-1}. \qquad \square$$

For each value of p we will obtain a different expectancy. For example, if $p = \frac{2}{3}$, then $\mathbb{E}[\tau] = 3$, and if $p = \frac{3}{4}$ then $\mathbb{E}[\tau] = 2$. We observe here the huge difference between the symmetric and the asymmetric problems.

3 An application

Here we present a problem that looks a lot harder than the one-dimensional random walk we have treated so far. However, we will see that the previous work can be used again in this case.

Problem 6. *God and the Devil used to be very good friends, and they always walked hand in hand. They live in a 2-dimensional world where they can only travel North or East, and they always move one unit of space in one of these directions for every unit of time. However, one day (let us say at time $t = 0$) they let go of each other's hand by mistake, and went separate ways: God travelled North, and the Devil travelled East. Since then, they want to meet each other again, but their memory is so bad that they do not remember which way they travelled when they first got separated. They both decided that the best strategy in order to find each other is to travel randomly (that is, at each step each of them will move either East or North with probability $1/2$ independently), and hope for the best. Let us say that God and the Devil each walk through a path, and that both paths start at the same point, which is the point when they first got separated. Considering the strategy that they both have taken, prove that the expected area enclosed by their paths when they meet (if they do) is infinite.*

Solution. In Figure 1 we can see an example of the paths they may take. Let $\mathcal{A}$ be a random variable, its value being the area

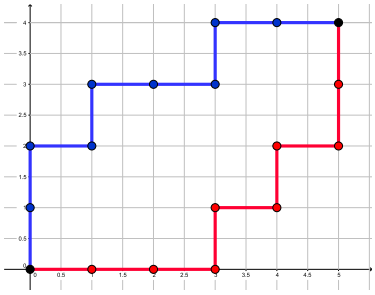

Figure 1: Possible paths God and the Devil may take.

enclosed by the two paths. First of all, we should prove that this area exists, that is, the paths do eventually meet. To prove this and that the expected area is infinite, we first reduce this problem to a simpler one. We do so by considering an application between the paths described in the statement and a generalization of Dyck paths as follows. Consider, in a lattice, paths that, at each unit of time (represented as a movement to the right), move one space up $\nearrow$, down $\searrow$, or straight $\rightarrow$ with a different probability. The probabilities are $1/4$ to go up or down and $1/2$ to go straight.

In this way, these new paths somehow measure the distance between God and the Devil: when they grow apart, the path will go up, and when they approach each other, it will go down. The path will meet the x-axis for the first time when God and the Devil meet each other. In such a way we obtain random paths that start at point $(1, 1)$ in the lattice (the first step from $(0, 0)$ to $(1, 1)$ will be imposed by the statement) for which the area enclosed by them and the x-axis is the same as the area enclosed by the two paths. An example of this application is shown between Figure 2 and Figure 3.

The fact that each of the areas of each of the possible paths are equivalent can be checked easily. The diagonal lines in Figure 2

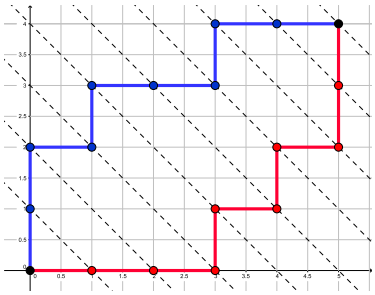

Figure 2: Possible paths followed by God and the Devil.

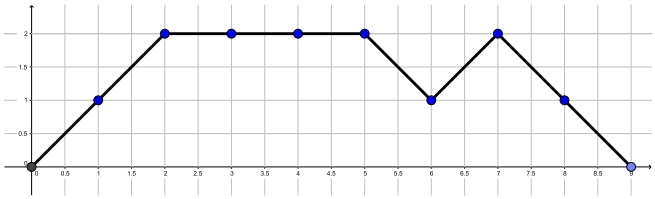

Figure 3: Injection of the paths from Figure 2 into a new path.

help us visualize this: when the distance between God and the Devil increases, the area enclosed by their paths will grow, and when they approach each other, the area decreases. To be more precise, at each step the area of the generalized Dyck path (considered as the area from the start to the last movement and the vertical line at that point) must increase from that in the previous step as much as the area of the trapezoid given in the original path by the two diagonal lines that surround the last movement. And this is easily checked.

We must also see that the probability of obtaining each of these areas is the same in both cases. This is a consequence of the choice of the probabilities: since the paths in the original problem have probability $1/2$, there is a $1/4$ chance that they will approach each other, a $1/4$ chance that the distance will increase, and a $1/2$ chance that they will move in the same direction.

Now, consider another application. In this case, we go from these generalized Dyck paths to normal Dyck paths by deleting the horizontal steps, and achieve something as shown in Figure 4.

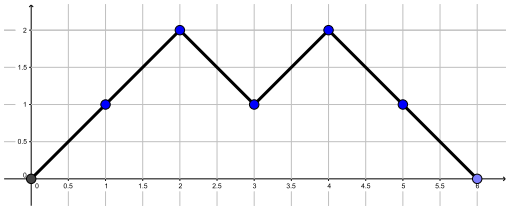

Figure 4: Last reduction of our problem.

In this case, it is obvious that we may obtain paths whose area is at most the same as the area of the previous paths, and that will in general be smaller. Hence, we have that the expected area between the two paths, $\mathbb{E}[\mathcal{A}]$, is bounded below by the expected area of these Dyck paths, which we will model with a random variable $\mathcal{A}'$. Furthermore, for these paths we have that the area is at least the length of the path minus 1, attained, for example, when the path has length 2. Hence, if the length of the Dyck paths is denoted by a random variable $\mathcal{L}'$, $\mathbb{E}[\mathcal{A}] \geq \mathbb{E}[\mathcal{A}'] \geq \mathbb{E}[\mathcal{L}' - 1]$. Now, by linearity of expectation, if we prove that $\mathbb{E}[\mathcal{L}']$ is infinite, we will be done. But Dyck paths are representations of a one-dimensional random walk (considering that their movement is done in the y-axis and that the time is represented by the x-axis). Hence, we can use everything we proved in the previous section and claim that God and the Devil will meet again, but the expected time it will take them to meet (which would correspond to the length of the generalized Dyck path, that is bounded below by the length of Dyck paths) is infinite, and so is the expected area enclosed by their paths. □

This is a very particular result. The only reason why this works is that the expected time of return in the one-dimensional walk is infinite; otherwise, the inequalities that we have used would only give us a lower bound, which would not even tell us whether the area is infinite or not.

Now consider the case of asymmetric one-dimensional random

walks. We saw that if the random walk starts at one side of a certain point and the probability to move to the other side is greater than $\frac{1}{2}$, then the expected first passage time through that point will be finite. As this happens for any $p > \frac{1}{2}$, it raises a question related to Problem 6:

Problem 7. *Consider the setting given in Problem 6. If the probability that God travels East is the same as the probability that the Devil travels North, what is the minimum value for this probability such that the expected area enclosed by their paths will be finite?*

Solution. The reduction from this problem to generalized Dyck paths is exactly the same as in the previous problem. There is a difference with the probabilities, though: if the probability that God travels East is p, then in the generalized Dyck path we will have a p^2 chance to go down, a $2p(1-p)$ chance to remain at the same height, and a $(1-p)^2$ chance to go up. Equivalently, we would have a one dimensional random walk, starting at position -1, with probability p^2 to go right, $(1-p)^2$ to go left, and $2p(1-p)$ to remain in the same place for a unit of time. Now, let us work in this setting.

Let $\mathcal{A}$ be the random variable that measures the area enclosed by one of these random paths, and let τ be the random variable that measures their length (it is the time it takes to reach the position 0, so we use the same notation as before). Remember that we have to add one unit to the distance later, since we are starting at position -1 instead of 0, so the real length is $\tau + 1$. If we have a path of a certain length, we can bound its area by the area of its highest possible path, that is, the one that forms a right isosceles triangle (this is true for even lengths; for odd lengths, we would have $\frac{\tau}{2}$ steps up, 1 horizontal, and $\frac{\tau}{2}$ steps down, but this can again be bounded above by a right isosceles triangle of base $\tau + 1$). That is, given a path of length τ, the area enclosed by this path is bounded by

$$\mathcal{A} \leq \frac{1}{2}(\tau+1)\frac{\tau+1}{2} = \frac{(\tau+1)^2}{4}.$$

If we find the expectancy of this last part we will have a bound for the expected area.

We can use probability generating functions, as we did in Problem 4 and Problem 5. We have a random walk starting at position -1, and we want to know the expected time it will take to reach position 0, as happened in Problem 5. τ is the random variable that measures this time, and we can try to find a recurrence as we did before. In this case, we have three possibilities: either we move right, or left, or stay at the same place. If we move right, we have already reached our destination. If we move left, we need to move two spaces to the right, which is modelled by two random variables that behave like τ. Finally, if we stay at the same place, we still need to move one space to the right, which is again modelled by a random variable distributed as τ. Using this, we can write

$$G_\tau(s) = p^2 s + 2p(1-p)sG_\tau(s) + (1-p)^2 sG_\tau(s)^2,$$

and solving this quadratic equation yields

$$G_\tau(s) = \frac{1 - 2p(1-p)s \pm \sqrt{1 - 4p(1-p)s}}{2(1-p)^2 s}.$$

As before, we have to check which sign is the correct one. There are several ways to check this, and as before, we have that the correct one is the negative sign. Hence, our probability generating function is

$$G_\tau(s) = \frac{1 - \sqrt{1 - 4p(1-p)s}}{2(1-p)^2 s} - \frac{p}{1-p}.$$

It is important to note that this is only true if $p \geq \frac{1}{2}$. Otherwise, its value at $s = 1$ is different from 1, so it is not a probability generating function. As before, we can find the expected value for our random variable τ by differenciating and substituting $s = 1$. In this case,

$$G'_\tau(s) = \frac{p}{(1-p)s\sqrt{1 - 4p(1-p)s}} - \frac{1 - \sqrt{1 - 4p(1-p)s}}{2(1-p)^2 s^2}$$

and

$$\begin{aligned}\mathbb{E}[\tau] &= \frac{p}{(1-p)\sqrt{1-4p(1-p)}} - \frac{1-\sqrt{1-4p(1-p)}}{2(1-p)^2} \\ &= \frac{p}{(1-p)(2p-1)} - \frac{1}{1-p} = \frac{1}{2p-1}.\end{aligned}$$

We thus reach a rather interesting result: even though this random walk and the one from Problem 5 are clearly different, and so are their probability generating functions, the expected time it takes to reach the desired position is the same.

With this we have the expected length, but we need the square of this quantity for our purposes. In order to obtain this, we must differenciate one more time. Then,

$$\begin{aligned}G''_\tau(s) =& \frac{2p^2}{s\sqrt{1-4p(1-p)s}^3} - \frac{2p}{(1-p)s^2\sqrt{1-4p(1-p)s}} \\ &+ \frac{1-\sqrt{1-4p(1-p)s}}{(1-p)^2s^3}\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}[\tau^2-\tau] =& \mathbb{E}[\tau^2] - \mathbb{E}[\tau] = G''_\tau(1) \\ =& \frac{2p^2}{\sqrt{1-4p(1-p)}^3} - \frac{2p}{(1-p)\sqrt{1-4p(1-p)}} \\ &+ \frac{1-\sqrt{1-4p(1-p)}}{(1-p)^2} \\ =& \frac{2p^2}{(2p-1)^3} - \frac{2p}{(1-p)(2p-1)} + \frac{2}{1-p} \\ =& \frac{(6p-2)(1-p)}{(2p-1)^3}.\end{aligned}$$

By linearity, we have that

$$\mathbb{E}\left[\tau^2\right] = \frac{(6p-2)(1-p)}{(2p-1)^3} + \mathbb{E}[\tau] = \frac{4p-2p^2-1}{(2p-1)^3}.$$

If we take $p = \frac{2}{3}$, then $\mathbb{E}[\tau^2] = 21$, and if $p = \frac{3}{4}$ then $\mathbb{E}[\tau^2] = 7$. We can compare these to the expectancies we obtained in Problem 5 and see that they are clearly bigger than their squares.

Now that we have these expectancies, let us compute the one we are interested in. We had $\mathcal{A} \leq \frac{(\tau+1)^2}{4}$, so

$$\mathbb{E}[\mathcal{A}] \leq \mathbb{E}\left[\frac{(\tau+1)^2}{4}\right] = \frac{\mathbb{E}[(\tau+1)^2]}{4} = \frac{\mathbb{E}[\tau^2] + 2\mathbb{E}[\tau] + 1}{4}.$$

Adding up the different terms that we already know, we have that

$$\mathbb{E}[\mathcal{A}] \leq \frac{p(4p^2 - 3p + 1)}{2(2p-1)^3},$$

which is finite for every $p > \frac{1}{2}$. Hence, the expected area is finite for these values of p, and we already proved that it is infinite for $p = \frac{1}{2}$. □

We have that $\mathbb{E}[\mathcal{A}] \leq 7$ for $p = \frac{2}{3}$ and $\mathbb{E}[\mathcal{A}] \leq 5$ for $p = \frac{3}{4}$.

In the case of the lengths of non-symmetric random walks, we were able to find the exact expectancy of the length of the random walks. This opens a new question about the problem we presented here.

Problem 8. *In the setting of Problem 7, can you find the exact value of the expected area?*

The tools we have presented so far do not seem to help. This is left as an open problem.

References

[1] Moore, C. N. *Random Walks*. URL: `https://www.math.ksu.edu/~cnmoore/randomwalk.pdf` (visited on 2015-10-03).

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Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.
2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to: **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to: jose.luis.diaz@upc.edu.

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

May 31, 2016

Elementary Problems

E–23. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let M be the midpoint of side BC in a triangle ABC. Show that

$$\frac{AB^2 + AC^2}{MA^2 + MB^2}$$

is an integer and determine its value.

E–24. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 a_2 \ldots a_n = 1$. Prove that

$$\frac{1}{n}\left[\left(1 + \frac{1}{a_1 a_n}\right)^2 + \left(1 + \frac{1}{a_2 a_{n-1}}\right)^2 + \ldots + \left(1 + \frac{1}{a_n a_1}\right)^2\right] \geq 4.$$

E–25. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find the smallest positive integer whose remainder is 2 when divided by 5, 3 when divided by 7, and 4 when divided by 11.

E–26. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Compute

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right) \ldots \left(1 - \frac{1}{2015^2}\right)\left(1 - \frac{1}{2016^2}\right).$$

E–27. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $n \geq 1$ be an integer. Show that the polynomial

$$(x-1)\left[(x-3)^{2(n+1)} + 3\right] - x^2$$

is a multiple of $x - 2$.

E–28. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let x, y, z, t be positive real numbers such that $xyzt = 1$. Prove that

$$t(x^2 + yz)(y + z) \geq 4.$$

Easy–Medium Problems

EM–23. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In any triangle ABC, with the usual notations, prove that the identity

$$\sum_{\text{cyclic}} \sqrt{\frac{\sin A}{\sin B \sin C}} = \sqrt{\frac{2R}{r}\left(\sum_{\text{cyclic}} \sin A\right)}$$

holds.

EM–24. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $p \neq 5$ be an odd prime number. Show that there exists a positive integer k for which the number

$$N = \underbrace{1111\ldots111}_{k}$$

satisfies that $N \equiv 0 \pmod{p}$.

EM–25. *Proposed by Alberto Espuny-Díaz, CFIS, BarcelonaTech, Barcelona, Spain.* Consider a rhombus. Let one of its diagonals be called a_1, and the other, b_1. Each of these diagonals divides the rhombus in two isosceles triangles, for each of which the incenter can be obtained. The four incenters determine a new rhombus of diagonals a_2 and b_2. Prove that by repeating this process, the figure tends to a square.

EM–26. *Proposed by Damià Torres Latorre, CFIS, BarcelonaTech, Barcelona, Spain.* A new bookstore sells one or two books per day, and the day it opened it sold exactly one book. Prove that, for any positive integer a, there exists a period of consecutive days in which a books were sold.

EM–27. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be three positive numbers such that $a + b + c = 1$. Determine the maximum value of

$$(a+b)^c(b+c)^a(c+a)^b.$$

EM–28. *Proposed by Alberto Espuny-Díaz, BarcelonaTech, Barcelona, Spain.* Compute the value of the series

$$\sum_{n=1}^{\infty} 2\sin^4\frac{\pi}{2^n} - 6\cos^2\frac{\pi}{2^n}\sin^2\frac{\pi}{2^n}.$$

Medium–Hard Problems

MH–23. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Prove that

$$\frac{1}{n}\left[\sum_{k=1}^{n}\left(\frac{1+a^2}{a}\right)^{2k}\prod_{\substack{j=1\\ j\neq k}}^{n}\left(\frac{1+a^2}{a}\right)^{j}\right] \geq 2^{\frac{(n+1)^2}{2}}$$

holds for all $a > 0$.

MH–24. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $A(z) = z^3 - a_2z^2 + a_1z - a_0$ be a polynomial with real coefficients. If all the zeros of $A(z)$ are positive, then prove that

$$9a_0^2 + a_1^2a_2^2 \geq \frac{4}{3}a_1^3 + 6a_0a_1a_2.$$

MH–25. *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.* Show that for any numbers a, b, $c > 0$ such that $a + b + c = 3$ we have

$$\left(a^3 + 2a^2 + 2a + 1\right)\left(b^3 + 2b^2 + 2b + 1\right)\left(c^3 + 2c^2 + 2c + 1\right) \leq 6^3.$$

MH–26. *Proposed by Mihály Bencze, Braşov, Romania.* Find all real solutions of the following system of equations:

$$\begin{cases} (x_1^2+1)(x_1+1)^2 + 7x_2^2 = 5x_3(x_3^2+x_3+1), \\ (x_2^2+1)(x_2+1)^2 + 7x_3^2 = 5x_4(x_4^2+x_4+1), \\ \qquad\qquad\qquad\qquad\vdots \\ (x_n^2+1)(x_n+1)^2 + 7x_1^2 = 5x_2(x_2^2+x_2+1). \end{cases}$$

MH–27. *Proposed by Mihály Bencze, Braşov, Romania.* Let n be a positive integer. Show that there exists a positive integer x such that the sum of the digits of x^2 is 4^n.

MH–28. *Proposed by Damià Torres Latorre, CFIS, BarcelonaTech, Barcelona, Spain.* Let $\tau(n)$ be the number of positive divisors of the positive integer n, and let $\sigma(n)$ be the sum of these divisors. Prove that

$$\frac{\tau(n)^2}{n} < \frac{2\sigma(n)}{n} \leq \ln n + 4.$$

Advanced Problems

A–23. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Compute

$$\lim_{n\to+\infty}\frac{1}{n^3}\sum_{1\le i<j\le n}\frac{n+\sqrt{n(i+j)}+ij}{n^2+n(i+j)+ij}.$$

A–24. *Proposed by Mihály Bencze, Braşov, Romania.* Let $ABCD$ be a quadrilateral. Let us denote by E the midpoint of side AB, F the centroid of triangle ABC, K the centroid of triangle BCD, and G the centroid of quadrilateral $ABCD$. Prove that

$$\frac{6\,MB}{MA\cdot ME}+\frac{2\,MC}{ME\cdot MF}+\frac{MD}{MF\cdot MG}\ge\frac{9\,MK}{MA\cdot MG}$$

holds for all points M in the plane of $ABCD$ different from A, E, F, G.

A–25. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $z_1, z_2, \ldots, z_n$ be the zeros of the polynomial with complex coefficients $A(z)=\sum_{k=0}^{n}a_kz^k$ $(a_n=1)$. Prove that z_i^2, $1\le i\le n$, lie in the disk $\mathcal{C}=\{z\in\mathbb{C}:|z|\le r\}$, where

$$r=2\max_{0\le k\le n-1}\left\{\frac{|a_k|^2+\sum_{j=1}^{n-k+1}|a_{k-j}a_{k+j}|}{|a_{k+1}|^2}\right\}$$

with $a_\ell=0$ if $\ell<0$ or $\ell>n$.

A–26. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Prove the following identity:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n}}{m^2n^2}=\left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\right\}^2.$$

A–27. *Proposed by Alberto Espuny-Díaz, CFIS, BarcelonaTech, Barcelona, Spain.* Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers defined by

$$\frac{1}{a_n} = \frac{1}{n}\sum_{k=1}^{n-1} a_k$$

with a_1 a nonzero number. Determine its convergence and, if it is convergent, compute its limit depending on a_1.

A–28. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Prove that

$$\frac{1}{2n}\left[\prod_{k=1}^{n} F_k^{2/n} + \left(\frac{1}{n}\sum_{k=1}^{n} F_k^n\right)^{1/n}\right] \geq \frac{(F_1F_2\ldots F_n)^{4/n}}{F_nF_{n+1}},$$

where F_n represents the n^{th} Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$.

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to: **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to: jose.luis.diaz@upc.edu.

Dividing sines of ugly angles in Geometry

A. Sáez-Schwedt

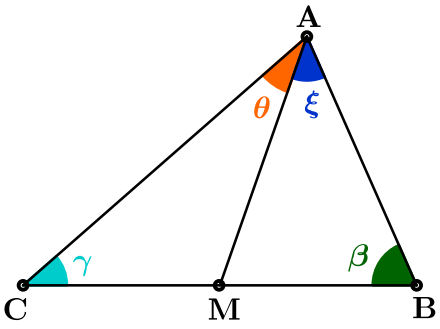
1 Introduction

Sometimes, the solution of a geometric problem requires proving that some "ugly" angle θ is equal to some other "ugly" angle θ'. By "ugly" we mean angles which are difficult to treat with the usual angle-chasing techniques (similarities, cyclic quadrilaterals, etc.). For example, consider the following situation, where we have to prove that a certain line is a symmedian, i.e., the reflection of a median through the angle bisector at the same vertex.

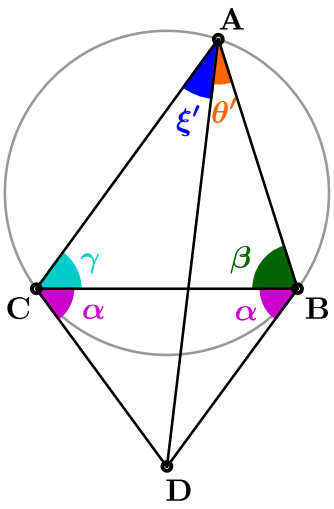
Problem 1 (median and symmedian). *Let ABC be a triangle such that the tangents at B, C to its circumcircle meet at D. If M is the midpoint of BC, prove that $\angle CAM = \angle DAB$.*

Solution. With the notations of Figure 1, one has to prove that $\theta = \theta'$. At first sight, the angles into which a median divides a vertex are ugly, they are not expressible as a simple combination of the angles of ABC, like $\frac{\alpha}{2} + \frac{\beta}{2}$, $90° - \gamma$, etc. (except, of course, in very special cases).

Let us consider the (also ugly!) angles ξ, ξ' such that $\theta + \xi = \theta' + \xi' = \alpha$ (see Figure 1). We will prove that $\frac{\sin(\theta)}{\sin(\xi)} = \frac{\sin(\theta')}{\sin(\xi')}$, and this will ensure the double equality $\theta = \theta'$ and $\xi = \xi'$ (yes, proving a double equality can sometimes be easier than proving just one!).



(a) Angles related to M.



(b) Angles related to D.

Figure 1: Elements of Problem 1.

Computing the ratio of the areas of triangles ACM and ABM in two different ways, we get

$$1 = \frac{CM}{BM} = \frac{\frac{1}{2}AC \cdot AM \cdot \sin(\theta)}{\frac{1}{2}AB \cdot AM \cdot \sin(\xi)} \Longrightarrow \frac{\sin(\theta)}{\sin(\xi)} = \frac{AB}{AC}.$$

Similarly, dividing the areas of ABD and ACD yields

$$\frac{\frac{1}{2}AB \cdot AD \cdot \sin(\theta')}{\frac{1}{2}AC \cdot AD \cdot \sin(\xi')} = \frac{\frac{1}{2}BA \cdot BD \cdot \sin(\alpha+\beta)}{\frac{1}{2}CA \cdot CD \cdot \sin(\alpha+\gamma)}$$

and this implies that

$$\frac{\sin(\theta')}{\sin(\xi')} = \frac{\sin(\gamma)}{\sin(\beta)},$$

which is equal to $\frac{AB}{AC}$ by the law of sines. This proves what we wanted. □

There are many other solutions to the above problem. Try to find other solutions yourself, and then see for example the document of Yufei Zhao [3] with training material for IMOs.

Remember for future problems: in a triangle with angles α, β, γ, we know how to compute the ratio of the sines of the angles into which a median or a symmedian divides a given vertex. The expression is a ratio between two of the sines of α, β, γ.

This will be the strategy: to prove that two angles θ, θ' coincide, look for some companions (mates) ξ, ξ' such that $\theta + \xi = \theta' + \xi'$ and try to prove that $\frac{\sin(\theta)}{\sin(\xi)} = \frac{\sin(\theta')}{\sin(\xi')}$. The same will work in the case $\theta - \xi = \theta' - \xi'$.

2 Validity of this technique

The following result will complete the solution of the introductory problem, and will be the key to solving many other questions.

Lemma 1. *Let $\theta, \theta', \xi, \xi', \alpha < 180°$ be angles such that either $\theta + \xi = \theta' + \xi' = \alpha$ or $\theta - \xi = \theta' - \xi' = \alpha$. If $\frac{\sin(\theta)}{\sin(\xi)} = \frac{\sin(\theta')}{\sin(\xi')}$, then $\theta = \theta'$ and $\xi = \xi'$.*

Proof. If $\theta + \xi = \theta' + \xi' = \alpha$, we can imagine two triangles with a common angle $180° - \alpha$, as in Figure 2.

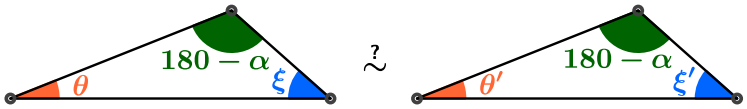


Figure 2: Triangles which we expect to be similar.

By the law of sines, the ratios between the sides adjacent to the common angle are $\frac{\sin(\theta)}{\sin(\xi)}$ and $\frac{\sin(\theta')}{\sin(\xi')}$. If they coincide, the triangles are similar and it follows that their angles must be equal.

If $\theta - \xi = \theta' - \xi' = \alpha$, a similar reasoning applies by considering triangles with angles $\{\alpha, 180° - \theta, \xi\}$ and $\{\alpha, 180° - \theta', \xi'\}$. □

Remark 1 (sines vs cotangents). *Note that if $\theta \pm \xi = \alpha$, one can*

compute $\frac{\sin(\theta)}{\sin(\xi)} = \frac{\sin(\alpha \mp \xi)}{\sin(\xi)}$ *and operate further:*

$$\frac{\sin(\alpha)\cos(\xi) \mp \cos(\alpha)\sin(\xi)}{\sin(\xi)} = \sin(\alpha) \cdot \cot(\xi) \mp \cos(\alpha).$$

Therefore, knowing the value of such ratio allows one to solve for $\cot(\xi)$ *in the above equation. Since for angles* $< 180^\circ$ *the equality* $\cot(\xi) = \cot(\xi')$ *implies* $\xi = \xi'$*, this reproves the previous lemma. For more details about the "cotangent trick", see [2], which contains training material for the brazilian olympic team. Our method has the advantage that we never need to expand trigonometric formulas. On the other hand, with the cotangent trick we do not need to have a common angle* α*: for example, we can compute* $\cot(\xi)$ *from* $\frac{\sin(\alpha-\xi)}{\sin(\xi)}$ *and* $\cot(\xi')$ *from* $\frac{\sin(\beta+\xi')}{\sin(\xi')}$*, with* $\beta \neq \alpha$*, and then check if* $\cot(\xi) = \cot(\xi')$*.*

3 Ceva-like theorems about sines ratios

The following result is a trigonometric version of Ceva's theorem in terms of ratios of sines instead of points, see e.g. [1].

Theorem 1 (trigonometric Ceva's theorem). *Let* ABC *be a triangle, and consider lines starting from* A, B, C *defining angles* α_1*,* α_2*,* β_1*,* β_2*,* γ_1 *and* γ_2 *as indicated in Figure 3. Then, the lines are concurrent if and only if*

$$\frac{\sin(\alpha_1)}{\sin(\alpha_2)} \cdot \frac{\sin(\beta_1)}{\sin(\beta_2)} \cdot \frac{\sin(\gamma_1)}{\sin(\gamma_2)} = 1.$$

Proof. There are two possible situations: either all lines are pointing towards the inside of the triangle, or one of them points inside and the other two outside. The proof can be done directly by multiple uses of the law of sines. Alternatively, if we extend the lines to touch the sides of ABC at D, E, F (which is possible if we admit that some of the points may be at infinity), the result is easily reduced to the classical version of Ceva's theorem. □

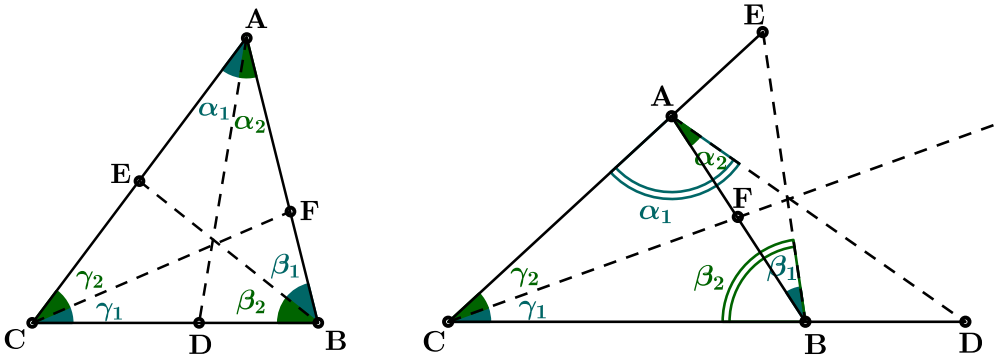


Figure 3: Possible configurations in Theorem 1.

A complete understanding of Ceva's theorem requires the use of signed segments, angles or areas. However, in this lesson we will only use positive distances, and angles between 0 and 180°. In every problem it will be clear from the context if a point lies inside or outside a segment, and different figures will be drawn, if necessary.

Problem 2. *Solve again the problem of the symmedian, now applying this version of Ceva's theorem.*

As an immediate consequence of Ceva's theorem, one can determine missing slopes in a quadrilateral.

Example 1 (missing slopes in a quadrilateral). *In the most general situation, we want to determine the slope of the line joining the vertices of two known triangles with a common side. When this situation is translated to quadrilaterals, the slope of a side or a diagonal is missing, and all the other slopes are known.*

In Figure 4a, one can compute $\frac{\sin(\beta_1)}{\sin(\beta_2)}$ *by applying Ceva's theorem to triangle* ABC *with* D *outside:*

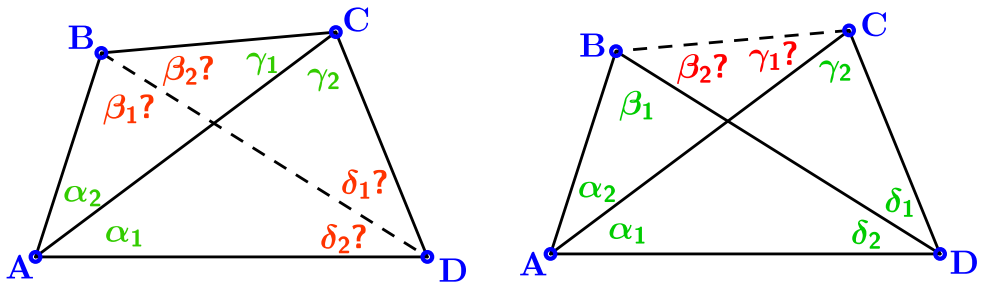
$$\frac{\sin(\alpha_1+\alpha_2)}{\sin(\alpha_1)}\cdot\frac{\sin(\gamma_2)}{\sin(\gamma_1+\gamma_2)}\cdot\frac{\sin(\beta_2)}{\sin(\beta_1)}=1$$

Since the value of $\beta_1+\beta_2$ *is known, this solves the unknown angles at* B*. Alternatively, by using the triangle* BCD *and the point* A

outside we get:

$$\frac{\sin(\beta_1+\beta_2)}{\sin(\beta_1)}\cdot\frac{\sin(\delta_2)}{\sin(\delta_1+\delta_2)}\cdot\frac{\sin(\gamma_2)}{\sin(\gamma_1)}=1,$$

from which the value of $\frac{\sin(\delta_2)}{\sin(\beta_1)}$ *is obtained. Similar considerations can be made in Figure 4b.*



(a) Diagonal. (b) Side.

Figure 4: Problems of missing slopes in a quadrilateral.

Next, we present a version of Ceva's theorem valid for arbitrary convex quadrilaterals (and of course, without much effort it can be easily adapted to any convex n-gon).

Theorem 2 (Ceva's theorem for quadrilaterals). *Let* P *be a point inside a convex quadrilateral* $ABCD$*, and let* u *be a line starting from* A*, pointing inside the quadrilateral. With the angles indicated in Figure 5, the line* u *passes through* P *if and only if*

$$\frac{\sin(\alpha_1)}{\sin(\alpha_2)}\cdot\frac{\sin(\beta_1)}{\sin(\beta_2)}\cdot\frac{\sin(\gamma_1)}{\sin(\gamma_2)}\cdot\frac{\sin(\delta_1)}{\sin(\delta_2)}=1$$

Proof. Assume that P belongs to u. Applying the sines law to the triangles PAD, PDC, PCB and PBA we obtain four equalities

$$\frac{\sin(\beta_1)}{\sin(\alpha_2)}=\frac{PA}{PD},\ \frac{\sin(\gamma_1)}{\sin(\beta_2)}=\frac{PD}{PC},\ \frac{\sin(\delta_1)}{\sin(\gamma_2)}=\frac{PC}{PB},\ \frac{\sin(\alpha_1)}{\sin(\delta_2)}=\frac{PB}{PA},$$

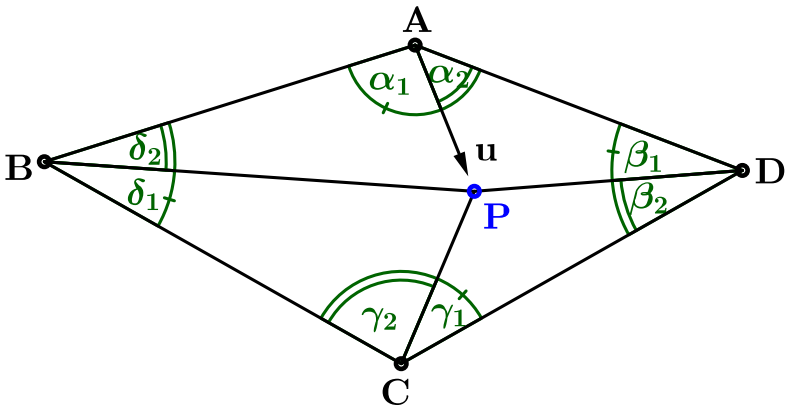


Figure 5: Elements of Theorem 2.

whose multiplication gives the desired result. Conversely, assume that

$$\frac{\sin(\alpha_1)}{\sin(\alpha_2)} \cdot \frac{\sin(\beta_1)}{\sin(\beta_2)} \cdot \frac{\sin(\gamma_1)}{\sin(\gamma_2)} \cdot \frac{\sin(\delta_1)}{\sin(\delta_2)} = 1$$

and let $\alpha_1' = \angle PAB$, $\alpha_2' = \angle PAD$. By the first part already proved, one has

$$\frac{\sin(\alpha_1')}{\sin(\alpha_2')} \cdot \frac{\sin(\beta_1)}{\sin(\beta_2)} \cdot \frac{\sin(\gamma_1)}{\sin(\gamma_2)} \cdot \frac{\sin(\delta_1)}{\sin(\delta_2)} = 1,$$

in particular $\frac{\sin(\alpha_1')}{\sin(\alpha_2')} = \frac{\sin(\alpha_1)}{\sin(\alpha_2)}$. Since $\alpha_1 + \alpha_2 = \alpha_1' + \alpha_2'$, we are in the situation of Lemma 1, so $\alpha_1 = \alpha_1'$ and $\alpha_2 = \alpha_2'$, and hence u passes through P. □

4 Solving problems

Let us illustrate how the technique of ratios of sines works in solving various problems, the first of which was proposed in the Spanish Mathematical Olympiad. The solutions may not be complete, but we will give at least sufficient hints to allow the readers to complete the solutions themselves.

Problem 3 (OME 2010, nr. 5). *In the scalene triangle ABC, a point P is chosen on the internal bisector of A and A', B' and C' are the feet of the perpendiculars drawn from P to the sides BC, CA and AB, respectively. If PA' meets $B'C'$ at Q, prove that Q belongs to the median starting from vertex A.*

Solution. Let α, β and γ be the angles of ABC. Let $\theta = \angle QAC$ and $\xi = \angle QAB$. The other angles indicated in Figure 6 are easily deduced. Applying Ceva's theorem to the quadrilateral $AB'PC'$ gives $\frac{\sin(\theta)}{\sin(\xi)} = \frac{\sin(\gamma)}{\sin(\beta)}$. Sounds familiar? Yes, this is precisely the ratio of sines which produces the median AM.

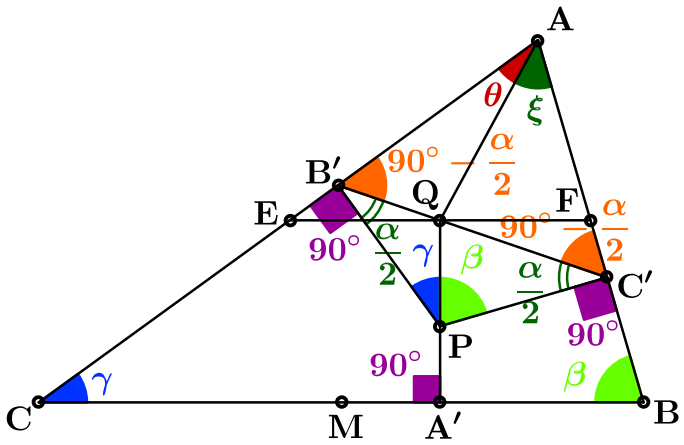


Figure 6: Solution to Problem 3.

□

The next problem takes profit of the fact that the lines joining a vertex of a triangle with the orthocenter and circumcenter are symmetric with respect to the bisector of that vertex, and so it is possible that cancellations occur when multiplying and dividing sines.

Problem 4. *Let ABC be an acute triangle with circumcenter O, orthocenter H and incenter I. If AI meets CO at P and CH at Q, prove that $\angle PBA = \angle QBC$. Use this to prove that AI, BO and CH concur if and only if AI, BH and CO concur.*

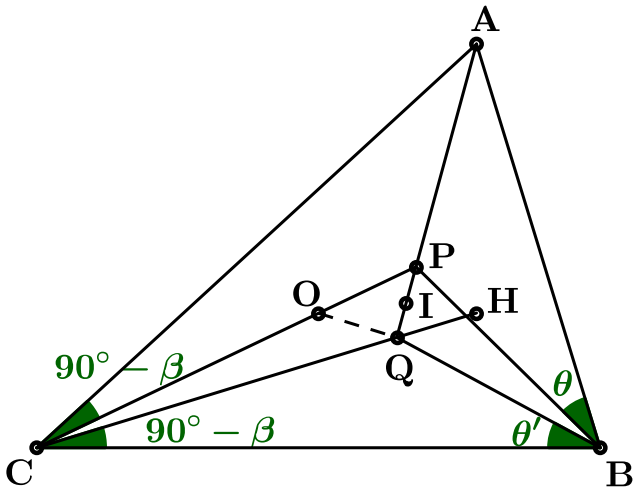


Figure 7: Elements of the solution of Problem 4.

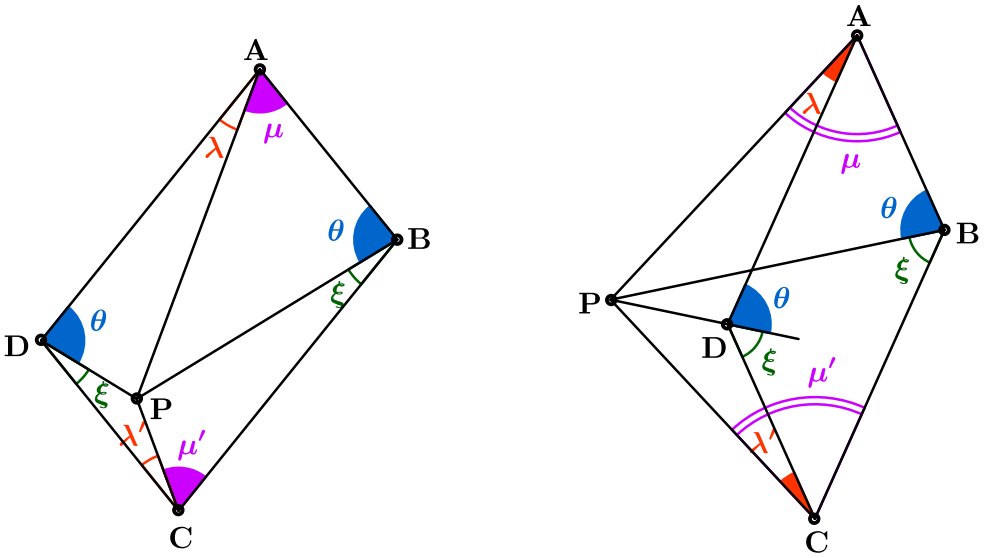
Solution. With the notations of Figure 7, one has to prove that $\theta = \theta'$. As usual, we denote by ξ, ξ' their mates or companions such that $\theta + \xi = \theta' + \xi' = \beta$ and we will prove that $\frac{\sin(\theta)}{\sin(\xi)} = \frac{\sin(\theta')}{\sin(\xi')}$, which will allow us to conclude that $\theta = \theta'$ and $\xi = \xi'$. Applying Ceva's theorem to the triangle ABC (once with point P and another time with point Q) solves the first part.

For the second part, note that the concurrence of AI, BO and CH is equivalent to the condition that BO passes through Q, i.e. θ' must be equal to $90° - \alpha$, the value of $\angle CBO$. On the other hand, AI, BH and CO concur if and only if BH passes through P, which is equivalent to $\theta = \angle ABH$, which is also $90° - \alpha$. Since $\theta = \theta'$ by the first part, both conditions of concurrence are clearly equivalent. □

Problem 5 (symmetric directions in a parallelogram). *Consider a parallelogram $ABCD$. If a point P is such that the lines PB and PD have symmetric directions with respect to the sides ($\theta = \theta$ and $\xi = \xi$ in Figure 8), then PA and PC have also symmetric directions ($\lambda = \lambda'$).*

Solution. Let α be the common angle at A and C. Consider an-

gles μ, μ' such that $\lambda + \mu = \lambda' + \mu' = \alpha$ in Figure 8a (P inside) or $\mu - \lambda = \mu' - \lambda' = \alpha$ in Figure 8b (P outside). As usual, we will prove that $\frac{\sin(\lambda)}{\sin(\mu)} = \frac{\sin(\lambda')}{\sin(\mu')}$. The mission is accomplished with one single application of Ceva's theorem in a quadrilateral, at least in Figure 8a, with P inside $ABCD$.



(a) P is interior. (b) P is exterior.

Figure 8: Possible configurations in Problem 5.

Although Theorem 2 does not consider the case when P lies outside the quadrilateral, if we follow carefully its proof, applying the law of sines to the triangles PAB, PBC, PCD and PDA, we see that it works, and we prove what we want. In fact, the only (trivial) difficulty consists in making cancellations of the type $\sin(180^\circ - \theta) = \sin(\theta)$, etc.

By the way, a more difficult problem is to find the locus of points P satisfying the above conditions (the result is a hyperbola). □

Problem 6. *Let $ABCD$ be a cyclic quadrilateral, with acute angles at A and B. The diagonals AC and BD meet at E, and the lines AD and BC meet at F. Prove that E belongs to the line joining the orthocenters of FAB and FCD.*

Solution. Let G be the orthocenter of ABF, and H that of CDF. Denote by θ, ξ the angles formed by GE at E, and by θ', ξ' the angles formed by HE at E on the opposite side, as indicated in Figure 9. Clearly, the collinearity of E, G and H is equivalent to the double equality $\theta = \theta'$ and $\xi = \xi'$. But $\theta + \xi$ is trivially equal to $\theta' + \xi'$. Therefore, our techniques can be applied: we only need to prove the equality of the ratios of sines.

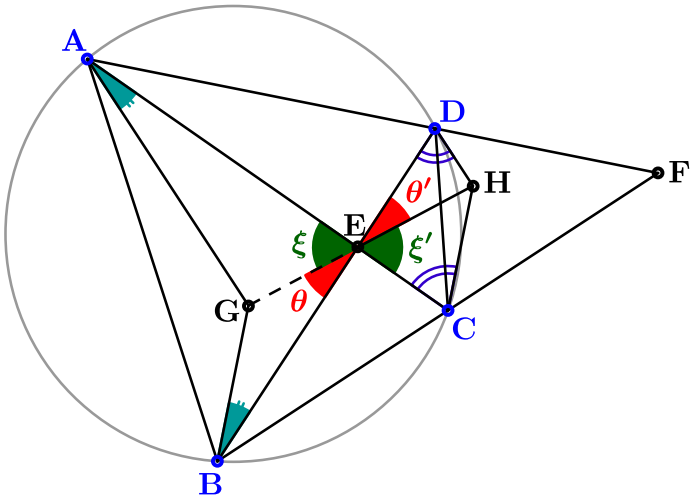


Figure 9: Elements of the solution to Problem 6.

First, note that $\angle ADB = \angle ACD$ ($ABCD$ is cyclic) and $\angle FDH = \angle FCH$ (both $90° - \angle CFD$), therefore we deduce that $\angle EDH = \angle ECH$. As a consequence, the ratio of the areas of EDH and ECH is equal to $\frac{DE \cdot DH}{CE \cdot CH}$, but it is also $\frac{DE \cdot \sin(\theta')}{CE \cdot \sin(\xi')}$, from which it follows that $\frac{\sin(\theta')}{\sin(\xi')} = \frac{DH}{CH}$.

On the other hand, $\angle GAD = \angle GBC$ (both $90° - \angle AFB$) and $\angle EAD = \angle EBC$ (because $ABCD$ is cyclic). Subtracting the equalities one gets $\angle EAG = \angle EBG$. Now, computing the ratio of areas of GBE and GAE in two different ways we obtain $\frac{\sin(\theta)}{\sin(\xi)} = \frac{BG}{AG}$, which is equal to $\frac{DH}{CH}$ because G and H are corresponding points under the similarity which sends FAB to FCD. Applying Lemma 1 completes the solution. Another figure is necessary if G and H lie both on the same side of CD. □

Problem 7. *Let $ABCD$ be a quadrilateral inscribed in a circle Γ, such that the tangents at B and D meet at X, a point lying on AC. The tangent to Γ at C meets BX at P and DX at Q. AQ meets Γ again at E. Prove that the lines AD, BE and CQ are concurrent.*

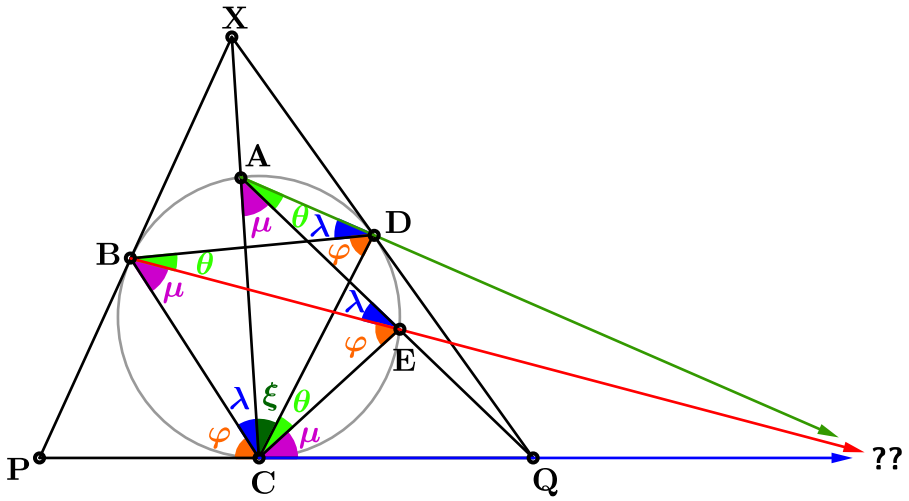


Figure 10: Elements of the solution to Problem 7.

Solution. Denote by $\varphi, \lambda, \xi, \theta, \mu$ the five angles formed at C, with sum $180°$. With these notations, various other angles are deduced, by using elementary properties of inscribed angles, tangents, etc. (See Figure 10.)

We have three lines starting from the vertices of AEC. To prove that they concur, by virtue of Ceva's theorem, it must be shown that

$$\frac{\sin(\mu+\theta)}{\sin(\theta)} \cdot \frac{\sin(\lambda)}{\sin(\varphi)} \cdot \frac{\sin(\mu)}{\sin(\xi+\theta+\mu)} = 1.$$

Now, as we saw in Problem 1, AQ is the symmedian of A in triangle ACD, and CX is the symmedian of C in BCD, so the following relations hold:

$$\frac{\sin(\mu)}{\sin(\theta)} = \frac{\sin(\lambda+\varphi)}{\sin(\xi)} \quad \text{and} \quad \frac{\sin(\lambda)}{\sin(\xi)} = \frac{\sin(\varphi)}{\sin(\mu+\theta)}.$$

Multiplying these identities yields immediately what we wanted to prove. □

Of course, the technique of dividing sines does not replace the usual synthetic reasoning in Geometry, but I hope that the examples illustrated here have shown that this is a useful tool, and that it is worth having at hand, as a complement of all our machinery. Now, have fun solving some more problems!

5 Exercises

1. Prove that the three symmedians of a triangle concur.
2. Let ABC be a triangle with $\angle B = 60^\circ$ and $\angle C = 80^\circ$. An interior point D satisfies $\angle BCT = 70^\circ$ and $\angle CBT = 40^\circ$. Prove that AT and BC are perpendicular.
3. Consider a cyclic quadrilateral such that the tangents at the endpoints of one of the diagonals meet on the extension of the other diagonal. Prove that the quadrilateral has equal product of opposite sides.
4. "The world's hardest easy Geometry problem". Let ABC be a triangle with $\angle B = \angle C = 80^\circ$. Let D and E be points on AC and AB, respectively, such that $\angle CBD = 60^\circ$ and $\angle BCE = 70^\circ$. Find $\angle CED$.
5. $ABCD$ is a cyclic quadrilateral with $DA = AB$ and $BC = CD$. Let E, F and G be points on DB, BC and CD, respectively, such that $CGEF$ is a parallelogram. Prove that FG is perpendicular to AE.
   Hint: Prove a version of Ceva's theorem in a pentagon and use it in $ABFGD$.
6. In a cyclic quadrilateral, prove the concurrence of the four lines starting from the midpoints of the sides and perpendicular to the opposite sides.

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Solutions

No problem is ever permanently closed. We will be very pleased to consider for publication new solutions or comments on the past problems.

Please, send submittals to: **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to: jose.luis.diaz@upc.edu.

Elementary Problems

E–17. *Proposed by Nicolae Papacu, Slobozia, Romania.* Solve the equation $[x \cdot [x]] = 2014$. Here, $[t]$ represents the integer part of $t \in \mathbb{R}$.

Solution by the proposer. We consider two cases:

1. $x \geq 0$. If $x \geq 45$, then $[x] \geq 45$ and $[x\,[x]] \geq 45^2 = 2025$, from which $2014 < 2025 \leq [x\,[x]]$ follows. If $x \in [0, 45)$, then $[x] \in [0, 44]$ and $[x\,[x]] < 45 \cdot 44 = 1980$, from which it follows that $[x\,[x]] < 1980 < 2014$, and we conclude that the equation does not have any nonnegative solutions.
2. $x < 0$. If $x < -45$, then $[x] \leq -46$ and $[x\,[x]] \geq 45 \cdot 46 = 2070$, which means that $2014 < 2070 \leq [x\,[x]]$. If $x \geq -44$ then $[x] \geq -44$ and $[x\,[x]] \leq 44^2 = 1936$. From this, it follows that $[x\,[x]] \leq 1936 < 2014$. It remains to analyze the

case when $x \in [-45, -44)$. In this case $[x] = -45$ and the equation becomes $[-45x] = 2014$, and $-45x \in [2014, 2015)$. From this, we obtain

$$x \in \left(-\frac{2015}{45}, -\frac{2014}{45}\right] \cap [-45, -44] \Leftrightarrow x \in \left(-\frac{2015}{45}, -\frac{2014}{45}\right].$$

Also solved by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain; Alberto Espuny-Díaz, BarcelonaTech, Barcelona, Spain; Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain.

E–18. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be the side lengths of a triangle with area $\mathcal{A}$ and inradius r. Characterize all triangles for which

$$r\left(\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}\right) = \mathcal{A}$$

holds.

Solution 1 by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain. For any triangle, $\mathcal{A} = sr$, where s is the semiperimeter and r is the radius of the incircle. Therefore,

$$r\left(\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}\right) = sr \Longrightarrow s = \frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}.$$

As $s = \frac{a+b+c}{2}$, we have

$$\frac{a+b+c}{2} = \frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}.$$

$\frac{2xy}{x+y}$ is the harmonic mean of two numbers x and y, so applying the HM–AM inequality yields

$$a+b+c = \frac{2ab}{a+b} + \frac{2bc}{b+c} + \frac{2ca}{c+a} \leq \frac{a+b}{2} + \frac{b+c}{2} + \frac{a+c}{2} = a+b+c,$$

so we must have equality in the HM–AM inequality, and this only happens when $a = b = c$. Then, the triangles for which this equality holds are equilateral triangles.

Solution 2 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. By the AM–GM inequality, we know that

$$\frac{a+b}{2} \geq \sqrt{ab}$$

or, what is the same,

$$ab \leq \frac{(a+b)^2}{4}$$

(with equality if and only if $a = b$). Analogous inequalities can be obtained by replacing a and b by their images under cyclic permutations of a, b, c. Therefore,

$$\begin{aligned} r\left(\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}\right) &\leq r\left(\frac{a+b}{4} + \frac{b+c}{4} + \frac{c+a}{4}\right) \\ &= r\frac{a+b+c}{2} = \mathcal{A} \end{aligned}$$

and equality holds precisely when $a = b = c$ (that is to say, for equilateral triangles).

Also solved by Alberto Espuny-Díaz, BarcelonaTech, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain, and the proposer.

E–19. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be three different numbers such that

$$\begin{cases} a^3 + 7a + 11 = 0, \\ b^3 + 7b + 11 = 0, \\ c^3 + 7c + 11 = 0. \end{cases}$$

Find the value of $a + b + c$.

Solution by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain. From the statement, it can be deduced that a, b and c are the three different roots of the polynomial

$x^3 + 7x + 11$. Applying the Cardano–Viète's formulas, it can be obtained that

$$\begin{cases} a + b + c = 0, \\ ab + bc + ac = 7, \\ abc = -11. \end{cases}$$

Therefore, $a + b + c = 0$.

Also solved by Alberto Espuny-Díaz, BarcelonaTech, Barcelona, Spain; Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain, and the proposer.

E–20. *Proposed by Alberto Espuny-Díaz, CFIS, BarcelonaTech, Barcelona, Spain.* Let p and q be two prime numbers. In which cases is $p + q$ a prime number?

Solution by the proposer. First, let us suppose that p and q are both odd prime numbers. Then, their sum is an even number, which cannot be a prime number because the only even prime number is 2, which is smaller than any other prime number. With this, we have that either $p = 2$ or $q = 2$.

Now let us assume, without loss of generality, that $p = 2$. We now want to find the prime numbers q such that $q + 2$ is a prime number, and this is the definition of twin primes. So, the answer is that the sum of two primes is a new prime only when one of them is 2 and the other is the first of a pair of twin primes.

Also solved by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain; José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain; Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and Alejandro Suárez Hernández, CFIS, BarcelonaTech, Barcelona, Spain.

E–21. *Proposed by Francesc Gispert Sánchez, CFIS, Barcelona-Tech, Barcelona, Spain.* Let ABC be a triangle and let D be the point where the median from A meets the circumcircle of the triangle. Prove that the triangles ABD and ACD have the same area.

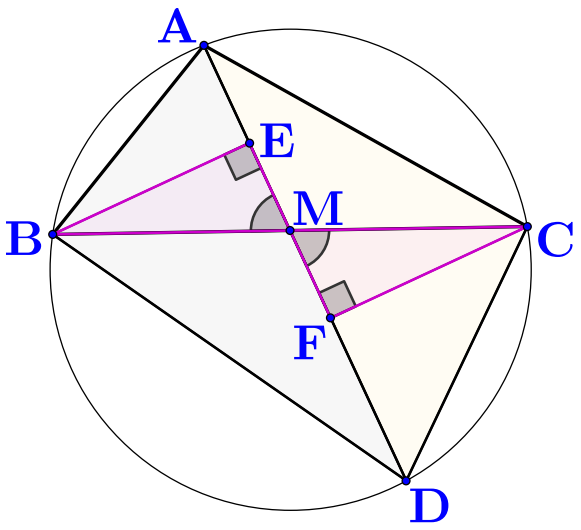


Figure 1: Construction for Solution 1 to Problem E–21.

Solution 1 by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain. We need to prove that $[ABD] = [ACD]$. Let M be the point where the median from A intersects BC. Let BE be the altitude from B of triangle ABD, and CF be the altitude from C of triangle ACD.

Triangles BME and CMF are congruent because their angles are equal and, as AM is the median, $BM = MC$, so they have an equal side. Hence, the altitudes BE and CF are also equal. So, triangles ABD and ACD have a common side AD and the same height with respect to this side. Then, their area is equal,

$$[ABD] = \frac{AD \cdot BE}{2} = \frac{AD \cdot CF}{2} = [ACD].$$

Solution 2 by the proposer. We consider M the midpoint of BC. Since all angles inscribed in a circle and subtended by the

same arc are equal, we have that $\angle DAC = \angle DBC$, $\angle ACB = \angle ADB$, $\angle BCD = \angle BAD$ and $\angle CDA = \angle CBA$, as can be seen in Figure 2. Hence, $\triangle ABM \sim \triangle CDM$ and $\triangle AMC \sim \triangle BMD$. Thus, we obtain the following relations:

$$\frac{AM}{CM} = \frac{BM}{DM} = \frac{AB}{CD};$$
$$\frac{AM}{BM} = \frac{AC}{BD} = \frac{MC}{MD}.$$

From these expressions, and taking into account that $BM = CM$, it follows that $AB \times BD = AC \times CD$.

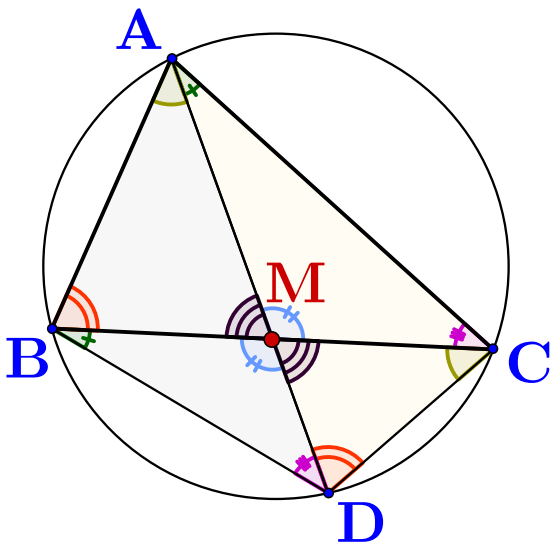


Figure 2: Construction for Solution 2 to Problem E–21.

Furthermore, we have that $\angle ABD + \angle ACD = \pi$ because the quadrilateral $ABCD$ is cyclic.

Therefore,

$$\mathcal{A}(ABD) = \frac{AB \times BD \times \sin(\angle ABD)}{2}$$
$$= \frac{AC \times CD \times \sin(\angle ACD)}{2} = \mathcal{A}(ACD).$$

Solution 3 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. Let M be the midpoint between B and C, which is also the intersection point of lines BC and AD. Let $\alpha = \widehat{AMB}$. Then, $\alpha = \widehat{CMD}$ and $\widehat{BMD} = \widehat{AMC} = \pi - \alpha$. If we denote the area of a triangle with vertices X, Y and Z as $|XYZ|$ we have that $|ABD| = |ABM| + |BMD|$ and $|ACD| = |ACM| + |CMD|$. If we denote the distance between two points X and Y as $|XY|$, since M is the midpoint between B and C, we have $|BM| = |CM|$. So, using the fact that $\sin\alpha = \sin(\pi - \alpha)$,

$$\begin{aligned} |ABD| &= |ABM| + |BMD| \\ &= \frac{|AM||BM|\sin\alpha}{2} + \frac{|DM||BM|\sin(\pi-\alpha)}{2} \\ &= \frac{|AM||CM|\sin(\pi-\alpha)}{2} + \frac{|DM||CM|\sin\alpha}{2} \\ &= |ACM| + |CMD| = |ACD|, \end{aligned}$$

as we wanted to see.

Also solved by Alberto Espuny-Díaz, BarcelonaTech, Barcelona, Spain, and Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.

E–22. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find the number of zeros at the end of the positive integer

$$1 + 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + 2014 \cdot 2014!\,.$$

Solution 1 by the proposer. We begin with the following general question: *What is the maximum power of a prime p that divides $n!$?* To answer this question we consider, for instance, $n = 12$ and $p = 2$. We have

1	2	3	4	5	6	7	8	9	10	11	12
0	1	0	1	0	1	0	1	0	1	0	1
0	0	0	1	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	1	0	0	0	0

where we have written in each column beginning with a number from 1 to 12 as many $1's$ as the maximum exponent of 2 that divides it, and otherwise we have written 0. So, the maximum exponent of 2 that divides 12! equals the number of ones in the previous table. That is, $e(2) = 10$. Now we count this number by rows and we get

$$e(2) = \left\lfloor \frac{12}{2} \right\rfloor + \left\lfloor \frac{12}{2^2} \right\rfloor + \left\lfloor \frac{12}{2^3} \right\rfloor = 6 + 3 + 1 = 10.$$

In general for a prime p and any integer $n \geq p$, we have

$$e(p) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots + \left\lfloor \frac{n}{p^k} \right\rfloor + \ldots$$

Let us compute the number $N = 1 + 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + 2014 \cdot 2014!$. Since

$$(k+j)! - (k+j-1)! = (k+j-1)!(k+j-1),$$

then

$$\sum_{j=1}^{n} (k+j)! - \sum_{j=1}^{n} (k+j-1)! = \sum_{j=1}^{n} (k+j-1)!(k+j-1),$$

or equivalently,

$$(k+n)! - k! = \sum_{j=1}^{n} (k+j-1)!(k+j-1).$$

Putting $k = 0$ in the last identity yields $n! - 1 = 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + (n-1) \cdot (n-1)!$, from which $N = 2015!$ follows.

To find the number of zeros ending 2015!, it suffices to know $e(5)$ because a zero is obtained multiplying 2×5 and $e(2) \geq e(5)$. That is,

$$\begin{aligned} e(5) &= \left\lfloor \frac{2015}{5} \right\rfloor + \left\lfloor \frac{2015}{5^2} \right\rfloor + \left\lfloor \frac{2015}{5^3} \right\rfloor + \left\lfloor \frac{2015}{5^4} \right\rfloor \\ &= 403 + 80 + 16 + 3 = 502. \end{aligned}$$

Solution 2 by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain. First, we find a formula and prove it by induction. We present three base cases:

$$\begin{aligned} n=1: &\quad 1+1\cdot 1!=2=2!,\\ n=2: &\quad 1+1\cdot 1!+2\cdot 2!=6=3!,\\ n=3: &\quad 1+1\cdot 1!+2\cdot 2!+3\cdot 3!=24=4!. \end{aligned}$$

Let us prove the general case. For $n=k$, assume that

$$1+1\cdot 1!+2\cdot 2!\ldots+k\cdot k!=(k+1)!.$$

We need to prove that it is also true for $n=k+1$. We have

$$\begin{aligned} &1+1\cdot 1!+2\cdot 2!+\ldots+k\cdot k!+(k+1)\cdot(k+1)!\\ &\quad=(k+1)!+(k+1)\cdot(k+1)!=(k+1)!(k+1+1)\\ &\quad=(k+1)!(k+2)=(k+2)!. \end{aligned}$$

Now it is easier to find the number of zeros at the end of that expression, as we know its value,

$$1+1\cdot 1!+2\cdot 2!+3\cdot 3!\ldots+2014\cdot 2014!=2015!.$$

Finding the number of zeros at the end of $2015!$ implies counting the number of fives that appear in its decomposition, which is

$$\left\lfloor\frac{2015}{5}\right\rfloor+\left\lfloor\frac{2015}{5^2}\right\rfloor+\left\lfloor\frac{2015}{5^3}\right\rfloor+\left\lfloor\frac{2015}{5^4}\right\rfloor=403+80+16+3=502.$$

So, the number of zeros at the end of $1+1\cdot 1!+2\cdot 2!+3\cdot 3!\ldots+2014\cdot 2014!$ is 502.

Also solved by Alberto Espuny-Díaz, BarcelonaTech, Barcelona, Spain; Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain.

Easy–Medium Problems

EM–17. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all positive solutions to the following system of equations:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = x_5^2, \\ x_2 + x_3 + x_4 + x_5 = x_6^2, \\ x_3 + x_4 + x_5 + x_6 = x_7^2, \\ x_4 + x_5 + x_6 + x_7 = x_8^2, \\ x_5 + x_6 + x_7 + x_8 = x_1^2, \\ x_6 + x_7 + x_8 + x_1 = x_2^2, \\ x_7 + x_8 + x_1 + x_2 = x_3^2, \\ x_8 + x_1 + x_2 + x_3 = x_4^2. \end{cases}$$

Solution 1 by the proposer. Setting $x_1 = x_9$, $x_2 = x_{10}$, $x_3 = x_{11}$ and $x_4 = x_{12}$, we can write the system in the more convenient form

$$x_k + x_{k+1} + x_{k+2} + x_{k+3} = x_{k+4}^2,\ k = 1, 2, \ldots, 8.$$

Let $x_i = \min\limits_{1 \le k \le 8} \{x_k\}$ and let $x_j = \max\limits_{1 \le k \le 8} \{x_k\}$. Then,

$$x_i^2 = x_{i-1} + x_{i-2} + x_{i-3} + x_{i-4} \ge 4x_i,$$

from which $x_i \ge 4$ follows. On the other hand, we have

$$x_j^2 = x_{j-1} + x_{j-2} + x_{j-3} + x_{j-4} \le 4x_j,$$

which means that $x_j \le 4$. On account of the fact that $4 \le x_i \le x_j \le 4$, we conclude that $x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = x_8 = 4$. Thus, the only positive solution of the system is $(4, 4, 4, 4, 4, 4, 4, 4)$, and we are done.

Solution 2 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. We have one "trivial" solution: $x_1 = x_2 = \ldots = x_8 = 4$. Let us prove that it is the only one. If we sum the 8 equations, we get

$$4 \sum_{i=1}^{8} x_i = \sum_{i=1}^{8} x_i^2,$$

which can be written as $32AM = 8QM^2$, where AM represents the arithmetic mean of the eight unknown variables, and QM, their quadratic mean. As $x_1, \ldots, x_8$ are positive, $AM \leq QM$, and thus, we get $QM \leq 4$.

Now, using the AM-GM inequality for each of the eight equations yields

$$x_{n+4}^2 = x_n + x_{n+1} + x_{n+2} + x_{n+3} \geq 4\sqrt[4]{x_n x_{n+1} x_{n+2} x_{n+3}},$$

where it is understood that $x_9 \equiv x_1, \ldots, x_{12} \equiv x_4$. Now, multiplying all the equations, as both sides of each of them are positive, we get

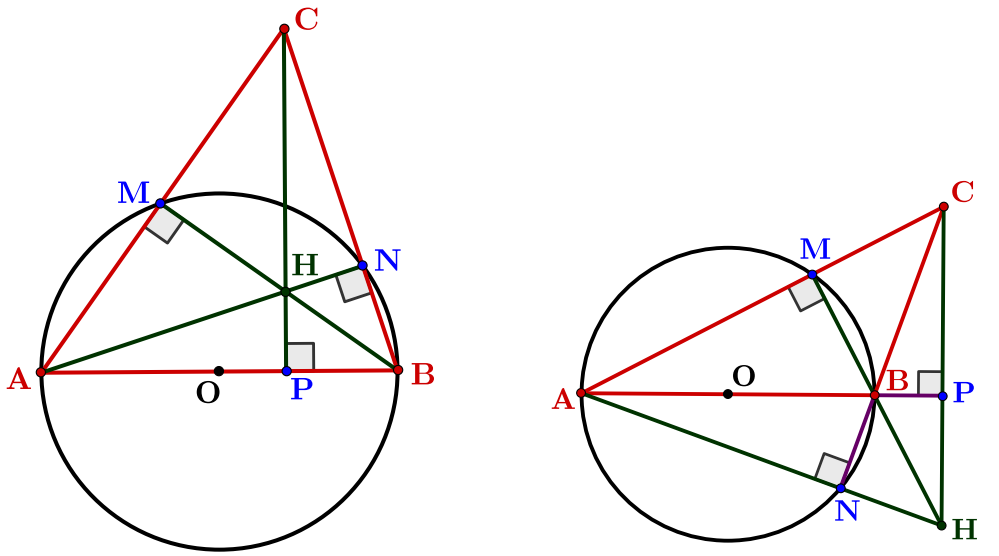
$$\prod_{i=1}^{8} x_i^2 \geq 4^8 \prod_{i=1}^{8} \sqrt[4]{x_i^4} \Longrightarrow \prod_{i=1}^{8} x_i \geq 4^8 \Longrightarrow \sqrt[8]{\prod_{i=1}^{8} x_i} = GM \geq 4,$$

where GM is the geometric mean of $x_1, \ldots, x_8$. As $GM \leq QM$, reaching equality if and only if $x_1 = \ldots = x_8$, and $QM \leq 4 \leq GM$, then $GM = QM = 4$, which means that $x_1 = \ldots = x_8 = 4$, as we wanted to show.

Also solved by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain; Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain, and Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.

EM–18. *Proposed by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain.* Let AB be a diameter of a circle Γ. Let us consider M and N two points on Γ different from A and B and such that AN and BM are not parallel. Let H be the point of intersection of AN and BM and let P be the foot of the perpendicular from H to AB. Prove that the lines AM, BN and PH are concurrent.

Solution by the proposer. Since AB is a diameter of Γ and M and N lie on Γ, we have that $\angle AMB = \angle ANB = \frac{\pi}{2}$ by Thales' theorem. From this, we deduce that AM and BN are



(a) Acute triangle. (b) Obtuse triangle.

Figure 3: Two possible (non-degenerate) configurations in Problem EM–18.

not parallel: for if they were parallel, then $AMBN$ would be a rectangle and AN and BM would be parallel as well.

Let C be the point of intersection of AM and BN. Now we have that AN and BM are the heights of the triangle ABC from A and B, respectively. Therefore, H must be the orthocentre of ABC. Consequently, PH is a line which is perpendicular to AB and passes through the orthocentre of ABC; that is, PH must be the height of ABC from C. In conclusion, AM, BN and PH meet at C.

Also solved by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.

EM–19. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Show that

$$\left[\prod_{k=1}^{n}\left(k+\frac{1}{n}\right)\right]^{\frac{1}{n+1}} < \left[\prod_{k=1}^{n}\left(k+\frac{1}{n+1}\right)\right]^{\frac{1}{n}}.$$

Solution 1 by the proposer. Let $0 < x < y$. From the identity

$$\sum_{k=0}^{n} x^k y^{n-k} = \frac{y^{n+1} - x^{n+1}}{y - x}$$

we have

$$\frac{y^{n+1} - x^{n+1}}{y - x} < (n+1)y^n \quad \text{or} \quad y^n[(n+1)x - ny] < x^{n+1}. \quad (1)$$

Setting $x = k + \dfrac{1}{n+1}$ and $y = k + \dfrac{1}{n}$, $1 \le k \le n$, by (1) we have that

$$\left(k + \frac{1}{n}\right)^n \le k\left(k + \frac{1}{n}\right)^n < \left(k + \frac{1}{n+1}\right)^{n+1}, k = 1,2,\cdots,n. \quad (2)$$

From (2), we have

$$\left(k + \frac{1}{n}\right)^{1/n+1} < \left(k + \frac{1}{n+1}\right)^{1/n}, 1 \le k \le n.$$

Multiplying the above inequalities, we get the claimed inequality.

Solution 2 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. We will prove that the inequality is true for each factor, that is:

$$\left(k + \frac{1}{n}\right)^{\frac{1}{n+1}} < \left(k + \frac{1}{n+1}\right)^{\frac{1}{n}} \quad \forall\, k, n \ge 1.$$

Let us start by raising both sides to the power of $kn(n+1)$. We can rewrite the inequality as

$$\left(k + \frac{1}{n}\right)^{kn} < \left(k + \frac{1}{n+1}\right)^{k(n+1)}.$$

Taking common factors, we get

$$k^{kn}\left(1 + \frac{1}{kn}\right)^{kn} < k^{k(n+1)}\left(1 + \frac{1}{k(n+1)}\right)^{k(n+1)},$$

and simplifying,

$$\left(1+\frac{1}{kn}\right)^{kn} < k^k\left(1+\frac{1}{k(n+1)}\right)^{k(n+1)}.$$

It is well known that the function $f(x) = (1+1/x)^x$ is strictly increasing over the positive real numbers, so

$$\left(1+\frac{1}{kn}\right)^{kn} < \left(1+\frac{1}{k(n+1)}\right)^{k(n+1)}.$$

As $k \geq 1$,

$$\left(1+\frac{1}{kn}\right)^{kn} < k^k\left(1+\frac{1}{k(n+1)}\right)^{k(n+1)}$$

also holds, thus completing the proof.

Solution 3 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. We want to prove that

$$\left[\prod_{k=1}^{n}\left(k+\frac{1}{n}\right)\right]^n < \left[\prod_{k=1}^{n}\left(k+\frac{1}{n+1}\right)\right]^{n+1}.$$

To this aim, we are going to prove that

$$\left(k+\frac{1}{n}\right)^n < \left(k+\frac{1}{n+1}\right)^{n+1}$$

for each $1 \leq k \leq n$ (the desired inequality is obtained thus by multiplying all these inequalities).

We fix some k between 1 and n. Hence, the claimed inequality can be rewritten equivalently as

$$\begin{aligned}\left(k+\frac{1}{n+1}\right)\left(\frac{k+\frac{1}{n+1}}{k+\frac{1}{n}}\right)^n &= \frac{kn+k+1}{n+1}\left(\frac{kn^2+kn+n}{kn^2+kn+n+1}\right)^n \\ &= \frac{kn+k+1}{n+1}\left(1-\frac{1}{kn^2+kn+n+1}\right)^n > 1.\end{aligned}$$

Now, by Bernoulli's inequality,

$$\left(1-\frac{1}{kn^2+kn+n+1}\right)^n \geq 1-\frac{n}{kn^2+kn+n+1};$$

therefore, we only need to check that

$$\frac{kn+k+1}{n+1}\frac{kn^2+kn+1}{kn^2+kn+n+1} > 1\,.$$

But this is equivalent to

$$\begin{aligned} k^2n^3+(2k^2+k)n^2+(k^2+2k)n+(k+1) \\ > kn^3+(2k+1)n^2+(k+2)n+1\,, \end{aligned}$$

which is clearly true.

EM–20. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let H be the orthocenter of an acute triangle ABC with altitudes of length h_a, h_b, and h_c respectively. Compute

$$\frac{HA}{h_a}+\frac{HB}{h_b}+\frac{HC}{h_c}.$$

Solution 1 by the proposer. We claim that

$$\frac{HA}{h_a}+\frac{HB}{h_b}+\frac{HC}{h_c}=2.$$

The statement follows from the following result. *If M is a point interior to an acute triangle $\triangle ABC$ with altitudes of length h_a, h_b, and h_c, and x, y, z are the distances from M to the sides BC, AC, and AB respectively, then*

$$\frac{x}{h_a}+\frac{y}{h_b}+\frac{z}{h_c}=1.$$

Indeed, if a, b, c are the lengths of the sides of $\triangle ABC$ and S is its area, then $h_a=\dfrac{2S}{a}$, $h_b=\dfrac{2S}{b}$, $h_c=\dfrac{2S}{c}$. Moreover,

$$S=\frac{ax}{2}+\frac{by}{2}+\frac{cz}{2} \Leftrightarrow 2S=ax+by+cz.$$

Combining the preceding expressions, we get

$$\frac{ax}{2S}+\frac{by}{2S}+\frac{cz}{2S}=1 \Leftrightarrow \frac{x}{h_a}+\frac{y}{h_b}+\frac{z}{h_c}=1.$$

Let A', B' and C' be the feet of the altitudes drawn from the vertices A, B, C. Then $\dfrac{HA'}{h_a}+\dfrac{HB'}{h_b}+\dfrac{HC'}{h_c}=1$ on account of the preceding result. Since $HA'=h_a-HA$, $HB'=h_b-HB$ and $HC'=h_c-HC$, then we have

$$\left(1-\frac{HA}{h_a}\right)+\left(1-\frac{HB}{h_b}\right)+\left(1-\frac{HC}{h_c}\right)=1,$$

from which the statement follows.

Solution 2 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. Let α, β and γ be the angles $\angle CAB$, $\angle ABC$ and $\angle BCA$, respectively. Furthermore, let A', B' and C' denote the feet of the altitudes from A, B and C, respectively. We can express

$$\begin{aligned}\frac{HA}{h_a}+\frac{HB}{h_b}+\frac{HC}{h_c}&=1-\frac{HA'}{AA'}+1-\frac{HB'}{BB'}+1-\frac{HC'}{CC'}\\&=3-\left(\frac{1}{\tan\beta\tan\gamma}+\frac{1}{\tan\gamma\tan\alpha}+\frac{1}{\tan\alpha\tan\beta}\right)\end{aligned}$$

because

$$AA'=A'B\tan\beta \quad\text{and}\quad HA'=A'B\tan\left(\frac{\pi}{2}-\gamma\right)=\frac{A'B}{\tan\gamma}$$

(as shown in Figure 4), and analogous formulas are obtained for the other addends.

Now, we have that

$$-\tan\alpha=\tan(\beta+\gamma)=\frac{\tan\beta+\tan\gamma}{1-\tan\beta\tan\gamma}$$

and, from this identity, we obtain that

$$\tan\beta\tan\gamma=1+\frac{\tan\beta+\tan\gamma}{\tan\alpha}=\frac{\tan\alpha+\tan\beta+\tan\gamma}{\tan\alpha}.$$

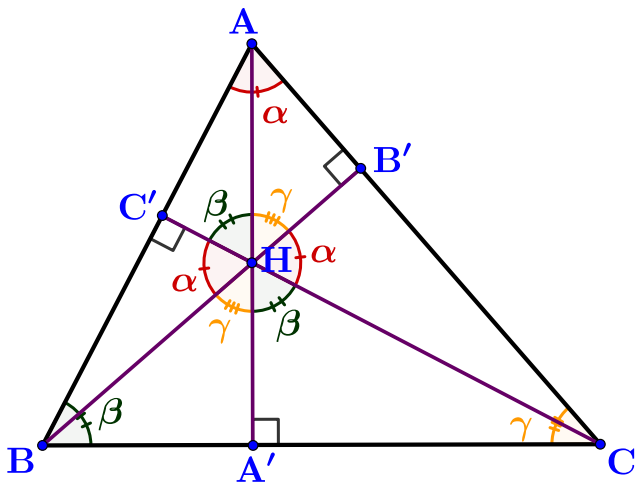


Figure 4: Angles needed to compute $\frac{HA'}{AA'}$ in Solution 2 to Problem EM–20.

Therefore,

$$\frac{HA}{h_a} + \frac{HB}{h_b} + \frac{HC}{h_c} = 3 - \frac{\tan\alpha + \tan\beta + \tan\gamma}{\tan\alpha + \tan\beta + \tan\gamma} = 2\,.$$

Solution 3 by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain. Let $h_a = AD$, $h_b = BE$ and $h_c = CF$ be the altitudes of triangle ABC.We have that

$$\frac{HA}{h_a} = \frac{h_a - HD}{h_a} = 1 - \frac{HD}{h_a} = 1 - \frac{\frac{1}{2}BC \cdot HD}{\frac{1}{2}BC \cdot h_a},$$

$$\frac{HB}{h_b} = \frac{h_b - HE}{h_b} = 1 - \frac{HE}{h_b} = 1 - \frac{\frac{1}{2}AC \cdot HE}{\frac{1}{2}AC \cdot h_b},$$

$$\frac{HC}{h_c} = \frac{h_c - HF}{h_c} = 1 - \frac{HF}{h_c} = 1 - \frac{\frac{1}{2}AB \cdot HF}{\frac{1}{2}AB \cdot h_c}.$$

On the other hand,

$$[ABC] = \frac{1}{2}BC \cdot h_a = \frac{1}{2}AC \cdot h_b = \frac{1}{2}AB \cdot h_c,$$

so the denominator is always the same. Adding up the three expressions, we obtain

$$\frac{HA}{h_a}+\frac{HB}{h_b}+\frac{HC}{h_c}=3-\frac{\frac{1}{2}BC\cdot HD+\frac{1}{2}AC\cdot HE+\frac{1}{2}AB\cdot HF}{[ABC]}.$$

As $\frac{1}{2}BC\cdot HD+\frac{1}{2}AC\cdot HE+\frac{1}{2}AB\cdot HF=[HBC]+[HAC]+[HAB]=[ABC]$, we conclude that

$$\frac{HA}{h_a}+\frac{HB}{h_b}+\frac{HC}{h_c}=3-\frac{[ABC]}{[ABC]}=2.$$

Also solved by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.

EM–21. *Proposed by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain.* Find all real solutions of the system of equations

$$\begin{cases} x^4+y^2=2z(x+y)-4,\\ y^4+z^2=2x(y+z)-4,\\ z^4+x^2=2y(z+x)-4. \end{cases}$$

Solution by the proposer. We claim that there are no real solutions. The system can be written as

$$\begin{cases} x^4+y^2-2zx-2yz+4=0,\\ y^4+z^2-2xy-2zx+4=0,\\ z^4+x^2-2yz-2xy+4=0. \end{cases}$$

Adding up the three equations, we obtain that

$$x^4+y^4+z^4+x^2+y^2+z^2-4xy-4yz-4zx+12=0$$

or, equivalently,

$$\begin{aligned} 0&=(x^4-4x^2+4)+(y^4-4y^2+4)+(z^4-4z^2+4)\\ &\quad+(x^2-4xy+4y^2)+(y^2-4yz+4z^2)+(z^2-4zx+4x^2)\\ &=(x^2-2)^2+(y^2-2)^2+(z^2-2)+(x-2y)^2+(y-2z)^2\\ &\quad+(z-2x)^2. \end{aligned}$$

Since this sum of squares equals zero, each of the squares must be zero:

$$\begin{cases} x^2 - 2 = 0, \\ y^2 - 2 = 0, \\ z^2 - 2 = 0, \\ x - 2y = 0, \\ y - 2z = 0, \\ z - 2x = 0. \end{cases}$$

Now, from the first three equations we deduce that x, y and z must be $\pm\sqrt{2}$, but from the last three equations we deduce that x, y and z must be 0. In conclusion, there are no real numbers x, y and z which satisfy these conditions.

Also solved by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain, and Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.

EM–22. *Proposed by Félix Gimeno Gil, CFIS, BarcelonaTech, Barcelona, Spain.* Find all strictly increasing functions $f\colon [0,1] \to [0,1]$ such that for all $x \in [0,1]$ and for $r > 0$

$$f(f(x)) = x^2 \quad \text{and} \quad f(x^r) = f(x)^r\,.$$

Solution 1 by the proposer. The only function is $f(x) = x^{\sqrt{2}}$ for all x. In order to prove this, let $\psi(x) = f(x) - x^{\sqrt{2}}$.

First, we prove that $\psi(x) \geq 0$. By contradiction, suppose there exists a value t such that $\psi(t) < 0$. Then $f(t) < t^{\sqrt{2}}$, and as f is strictly increasing, $f(f(t)) = t^2 < f(t^{\sqrt{2}}) = f(t)^{\sqrt{2}}$. Then, taking $\sqrt{2}$–roots we have $t^{\frac{2}{\sqrt{2}}} = t^{\sqrt{2}} < f(t)^{\frac{\sqrt{2}}{\sqrt{2}}} = f(t)$, reaching thus a contradiction.

Now observe that

$$\begin{aligned} \psi(f(x)) + \psi\left(x^{\sqrt{2}}\right) &= f(f(x)) - f(x)^{\sqrt{2}} + f\left(x^{\sqrt{2}}\right) - x^2 \\ &= -f(x)^{\sqrt{2}} + f(x)^{\sqrt{2}} = 0. \end{aligned}$$

If the sum of two non-negative values is zero, the summed values must be zero, so $\psi(f(x)) = 0 \ \ \forall x$, but as f is surjective (indeed, $\forall y, \ f(f(\sqrt{y})) = y$), then $\psi(x) = 0 \ \ \forall x$ and the proof is complete.

Solution 2 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. First, we observe that, if one such f exists, then $f(0) = 0$. Otherwise, we would have $f(0) > 0$ and, consequently, $0 = f(f(0)) > f(0) > 0$ (because f is strictly increasing), which is impossible. From this, since f is strictly increasing, we deduce that $f(x) > f(0) = 0$ for all $x \in (0, 1]$. With a similar argument, we can conclude that $f(1) = 1$ and $f(x) < f(1)$ for all $x \in [0, 1)$.

Thus, given $0 < x < 1$, it makes sense to consider $\log f(x) \neq 0$. We fix one such x and also $0 < y = f(x) < 1$. By definition of f, we have that $f(y) = f(f(x)) = x^2$. Similarly,

$$f(y) = f\left(x^{\frac{\log y}{\log x}}\right) = y^{\frac{\log y}{\log x}} = y^{\frac{\log y}{\frac{1}{2}\log f(y)}}$$

and, taking logarithms of both sides of the equality, we obtain that

$$\log f(y) = 2\frac{\log y}{\log f(y)} \log y \, .$$

This is equivalent to

$$\log f(y) = \sqrt{2} \log y$$

and

$$f(y) = y^{\sqrt{2}}$$

for $y = f(x)$ with $0 < x < 1$ (and we can check that this is also true for $x = y = 0$ and $x = y = 1$).

Finally, we observe that f is surjective: for any $y \in [0, 1]$, we can choose $x = f\left(\sqrt{y}\right)$ and we have that $f(x) = f\left(f\left(\sqrt{y}\right)\right) = y$. In conclusion, there is exactly one function f which satisfies all the requirements in the statement, and is defined by $f(x) = x^{\sqrt{2}}$.

Also solved by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.

Medium–Hard Problems

MH–17. *Proposed by Ander Lamaison Vidarte, CFIS, Barcelona-Tech, Barcelona, Spain.* Find all pairs of non-negative integers (a, b) satisfying

$$a^3b^2 = (a^2 + b)^3 .$$

Solution 1 by the proposer. For $a = 0$, we substitute in the original equation to obtain $0 = b^3$, and $b = 0$. Similarly, $b = 0 \Rightarrow a = 0$. Assume therefore that a and b are positive. Let $c = a^2$, which allows us to write $acb^2 = (c + b)^3$. Let $d = \gcd(b, c)$ and $b = dx$, $c = dy$, being x and y such that $gcd(x, y) = 1$. Dividing both sides of $acb^2 = (c+b)^3$ by d^3 one gets $ayx^2 = (y+x)^3$. Since $x|y^3$ and x and y are relatively prime, then $x = 1$. On the other hand, from $y|x^3$ we get $y = 1$ in the same way. This leads us to conclude that $a^2 = b = d$, and substituting in the initial equation $a^3(a^2)^2 = (a^2 + a^2)^3$ yields $a^7 = 8a^6$. Then $a = 8$ and $b = 64$ is also a solution. Therefore, the only solutions are

$$(0, 0) \quad \text{and} \quad (8, 64) .$$

Solution 2 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. First, we observe that both a and b must be even numbers. To see that, we consider all the possible cases.

- If both a and b were odd, the LHS would be odd while the RHS would be even.
- If a were odd and b were even, the LHS would be even but the RHS would be odd. The same would happen if a were even and b were odd.

In conclusion, the only remaining possibility is that both a and b be even.

Moreover, it is easy to see that b must be a perfect cube, because

$$b^2 = \left(\frac{a^2 + b}{a}\right)^3 .$$

We know, thus, that $2 \mid a$ and $2^3 \mid b$. From this, we obtain that

$$2^9 \mid a^3b^2 = \left(a^2+b\right)^3 \implies 2^3 \mid a^2+b$$

and, since $2^3 \mid b$, also $2^3 \mid a^2$ and, consequently, $2^2 \mid a$.

We can repeat the same reasoning a few times. In particular, now we know that

$$2^{12} \mid a^3b^2 = \left(a^2+b\right)^3$$

and deduce that $2^4 \mid b$, so $2^6 \mid b$ (because b is the cube of an integer). Again, from

$$2^{18} \mid a^3b^2 = \left(a^2+b\right)^3,$$

we get that $2^3 \mid a$. Having reached this point, the same argument does not apply anymore.

We can express, however, $a = 2^3x$ and $b = 2^6y^3$ and our equation thus turns into

$$2xy^2 = x^2+y^3 \iff y^3 = x\left(2y^2-x\right)$$

(for non-negative integers x and y). From this equation, we deduce easily that $y \mid x$ and $x \mid y^3$, so the prime numbers which divide x and y coincide (maybe with different multiplicities).

Consider then a solution (x,y) and a prime p with $v_p(y) = \alpha > 0$. Since $y \mid x$, we know that $v_p(x) = \alpha+m$ for some $m \geq 0$. Now, we can express $x = p^{\alpha+m}X$ and $y = p^\alpha Y$ with $p \nmid X, Y$ and we have that

$$\begin{aligned} p^{3\alpha}Y^3 = y^3 = x\left(2y^2-x\right) &= p^{\alpha+m}X\left(2p^{2\alpha}Y^2 - p^{\alpha+m}X\right) \\ &= p^{2\alpha+2m}X\left(2p^{\alpha-m}Y^2 - X\right), \end{aligned}$$

which is only possible if $2m = \alpha$. Therefore, if we consider a non-trivial prime factorisation $y = \prod_i p_i^{\alpha_i}$, the equation becomes

$$2\prod_i p_i^{\left(3+\frac{1}{2}\right)\alpha_i} = \prod_i p_i^{3\alpha_i} + \prod_i p_i^{3\alpha_i} = 2\prod_i p_i^{3\alpha_i},$$

which is impossible. Hence, the solutions to the problem must come from values of x and y wich have no non-trivial prime factorisation (that is, 0 or 1). We can find those solutions by inspection and recover the corresponding values for a and b. In particular, we verify that $x = 0$ if and only if $y = 0$ and, in a similar fashion, $x = 1$ if and only if $y = 1$.

In conclusion, $(0, 0)$ and $(8, 64)$ are the only solutions to the problem.

Also solved by Miguel Cidrás Senra, CFIS, BarcelonaTech, Barcelona, Spain; Marc Felipe Alsina, CFIS, BarcelonaTech, Barcelona, Spain; Gerard Orriols Giménez, CFIS, BarcelonaTech, Barcelona, Spain; Alejandro Suárez Hernández, CFIS, BarcelonaTech, Barcelona, Spain, and Damià Torres Latorre, CFIS, BarcelonaTech, Barcelona, Spain.

MH–18. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In a triangle ABC, let a, b and c denote the lengths of the sides BC, CA and AB, respectively. If $abc \geq 1$, then show that

$$\sqrt{\frac{\sin A}{a^3 + b^6 + c^6}} + \sqrt{\frac{\sin B}{b^3 + c^6 + a^6}} + \sqrt{\frac{\sin C}{c^3 + a^6 + b^6}} \leq \sqrt[4]{\frac{27}{4}}.$$

Solution 1 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain, and Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. By the Cauchy–Schwarz inequality,

$$\sum_{\text{cyc}} \sqrt{\frac{\sin A}{a^3 + b^6 + c^6}} \leq \left(\frac{1}{3}\sum_{\text{cyc}} \sin A\right)^{\frac{1}{2}} \left(3\sum_{\text{cyc}} \frac{1}{a^3 + b^6 + c^6}\right)^{\frac{1}{2}}.$$

We are going to find upper bounds for these last terms.

First, observe that $\sin(x)$ is a concave function for $0 \leq x \leq \pi$. Therefore, we can apply Jensen's inequality to obtain

$$\sqrt{\frac{\sin A + \sin B + \sin C}{3}} \leq \sqrt{\sin\left(\frac{A + B + C}{3}\right)} = \sqrt{\frac{\sqrt{3}}{2}} = \sqrt[4]{\frac{3}{4}}.$$

All that is left is to prove that

$$\frac{1}{a^3+b^6+c^6}+\frac{1}{b^3+c^6+a^6}+\frac{1}{c^3+a^6+b^6}\le 1$$

and we have not used the hypothesis $abc\ge 1$ yet. For simplicity, we define $x=\sqrt[3]{a}$, $y=\sqrt[3]{b}$ and $z=\sqrt[3]{c}$ (and so, $xyz=\sqrt[3]{abc}\ge 1$). Using the Cauchy–Schwarz inequality again, this time with vectors $\left(x,y,\sqrt{z}\right)$ and $\left(1,1,\sqrt{z}\right)$, we get

$$(x+y+z)^2\le\left(x^2+y^2+z\right)(z+2)$$

and thus

$$\frac{1}{x^2+y^2+z}\le\frac{z+2}{(x+y+z)^2}\,.$$

We obtain analogous results for the cyclic permutations of x, y and z.

In this way, we find that

$$\sum_{\text{cyc}}\frac{1}{x^2+y^2+z}\le\frac{x+y+z+6}{(x+y+z)^2}=\frac{1}{x+y+z}+\frac{6}{(x+y+z)^2}\,.$$

But using the AM–GM inequality we see that

$$x+y+z\ge 3\sqrt[3]{xyz}\ge 3$$

and, therefore,

$$\frac{1}{x+y+z}+\frac{6}{(x+y+z)^2}\le\frac{1}{3}+\frac{6}{9}=1$$

as desired.

The stated inequality is an immediate consequence of the given bounds. Equality occurs precisely when $\angle A=\angle B=\angle C$ and $a=b=c=1$ (that is, for the equilateral triangle with unit side length).

Solution 2 by the proposer. Applying Cauchy's inequality to the vectors

$$\vec{u}=(\sqrt{\sin A},\sqrt{\sin B},\sqrt{\sin C})$$

and

$$\vec{v} = \left(\sqrt{\frac{1}{a^3+b^6+c^6}}, \sqrt{\frac{1}{b^3+c^6+a^6}}, \sqrt{\frac{1}{c^3+a^6+b^6}} \right)$$

we obtain

$$\begin{aligned} \left(\sum_{\text{cyclic}} \sqrt{\frac{\sin A}{a^3+b^6+c^6}} \right)^2 &\leq \left(\sum_{\text{cyclic}} \sin A \right) \left(\sum_{\text{cyclic}} \frac{1}{a^6+b^6+c^3} \right) \\ &\leq \frac{3\sqrt{3}}{2} \left(\sum_{\text{cyclic}} \frac{1}{a^6+b^6+c^3} \right) \end{aligned}$$

on account of the well-known inequality

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$$

(which comes from a direct application of Jensen's inequality).

Now, it suffices to prove that

$$\frac{1}{a^6+b^6+c^3} + \frac{1}{b^6+c^6+a^3} + \frac{1}{c^6+a^6+b^3} \leq 1$$

holds. Indeed, let x, y be two positive real numbers. Then,

$(x-y)^2(x^4+x^3y+x^2y^2+xy^3+y^4) \geq 0 \Leftrightarrow x^6+y^6 \geq xy(x^4+y^4)$.

On the other hand, if $xyz \geq 1$, then $x+y+z \geq 3\sqrt[3]{xyz} \geq 3$ and $x^4+y^4+z^4 \geq (x+y+z)^4/27$ on account of mean inequalities.

Next, we have

$$\begin{aligned} \sum_{\text{cyclic}} \frac{1}{a^6+b^6+c^3} &\leq \sum_{\text{cyclic}} \frac{1}{ab(a^4+b^4)+c^3} \\ &\leq \sum_{\text{cyclic}} \frac{1}{\frac{a^4+b^4}{c}+c^3} = \sum_{\text{cyclic}} \frac{c}{a^4+b^4+c^4} \\ &= \frac{a+b+c}{a^4+b^4+c^4} \leq \frac{27}{(a+b+c)^3} \leq 1. \end{aligned}$$

Equality holds when $a=b=c=1$, and we are done.

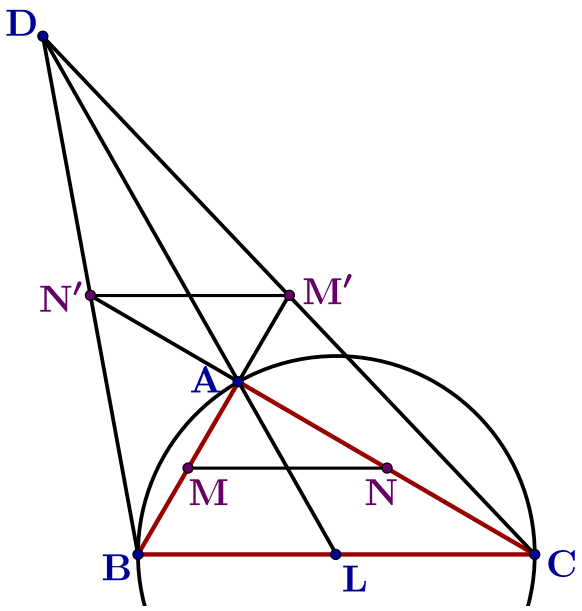


Figure 5: Construction of Solution 1 to Problem MH–19.

MH–19. *Proposed by Andrés Sáez-Schwedt, Universidad de León, León, Spain.* Triangle ABC is rectangle in A, with B, C fixed and A variable. Let M, N be the midpoints of AB, AC, respectively, and construct M', N' such that A is the common midpoint of MM' and NN'. Find the locus of the point of intersection of lines BN' and CM'.

Solution 1 by the proposer. Since $MNM'N'$ is a parallelogram, $M'N'$ is parallel to MN and equal in length, so it is parallel to BC and has half its length. Therefore, M', N' must be the midpoints of segments DC, DB respectively (D is the center of the dilation of ratio 2 which sends $M'N'$ into CB), i.e. A is the centroid of triangle BCD, and so the third median DL also passes through A. But L is the circumcenter of triangle ABC, hence $AL = \frac{BC}{2}$. Moreover, $DL = 3AL$, which means that the triangle BCD has the property that the median DL is $\frac{3}{2}$ times the side BC. Therefore, the point D varies on a fixed circle of center L and radius $\frac{3}{2}BC$.

Solution 2 by Francesc Gispert Sánchez, CFIS, BarcelonaTech,

Barcelona, Spain. Since $\angle CAB$ is right, the variable point A is always on the circle Γ with diameter BC: let O be its centre (which is also the midpoint of BC). For a fixed such choice of A, we consider the construction described in the statement and let I denote the point of intersection of BN' and CM', as in Figure 6. We claim that I lies on a circle concentric with Γ and whose diameter is triple the length of BC. More precisely, I is the point on OA such that $OI = 3OA$.

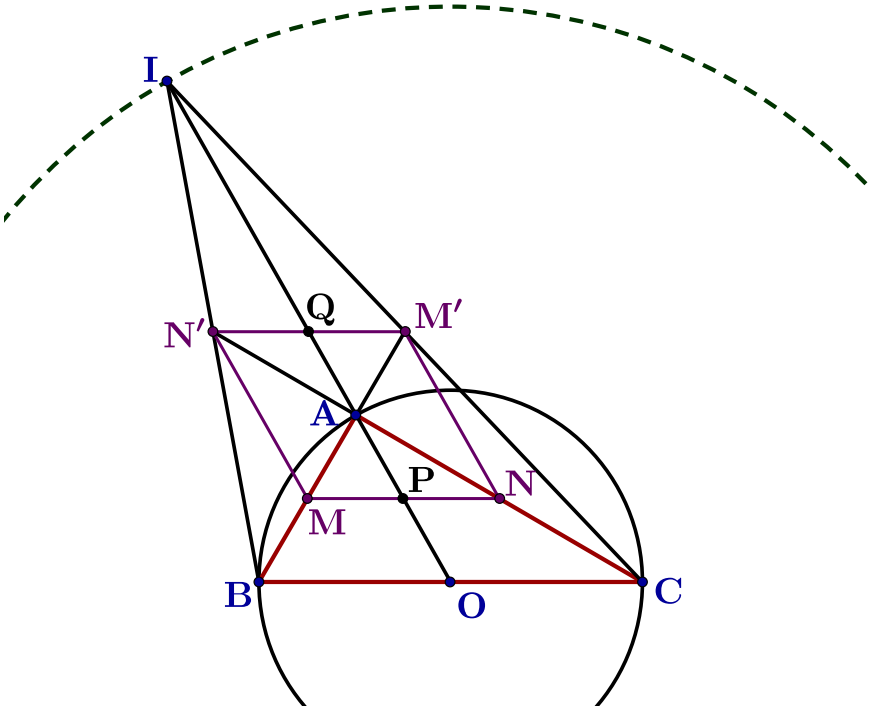


Figure 6: Construction of Solution 2 to Problem MH–19.

By construction, $AM' = AM = \frac{1}{2}AB$, $AN' = AN = \frac{1}{2}AC$ and $M'N' = MN = \frac{1}{2}BC$. Moreover, since $\angle M'AN' = \angle MAN = \angle BAC = \frac{\pi}{2}$, we deduce that the triangles $AM'N'$ and AMN are congruent, and both are similar to the triangle ABC (and the ratio of the corresponding side lengths is $\frac{1}{2}$). In particular, $MNM'N'$ is a rhombus with side length $\frac{1}{2}BC$, and A is the midpoint of its diagonals (the centre of the rhombus).

Consequently, $M'N'$ is parallel to MN and, so, to BC as well.

Therefore, the triangles $IN'M'$ and IBC are similar and

$$\frac{IN'}{IB} = \frac{IM'}{IC} = \frac{N'M'}{BC} = \frac{1}{2}.$$

Now we define Q to be the point of intersection of OI and $M'N'$. Again, by the similarity of the triangles $IN'Q$ and IBO and by that of the triangles $IM'Q$ and ICO as well, we obtain that

$$\frac{N'Q}{BO} = \frac{IN'}{IB} = \frac{IM'}{IC} = \frac{M'Q}{CO},$$

so Q is the midpoint of $N'M'$ (because O is the midpoint of BC). Therefore, Q is the image of O by the homothety of centre A and ratio $\frac{1}{2}$, which means that Q, A and O are collinear. That is, A belongs to the line QO (which is the same as IO).

Furthermore, by the previous arguments, we can also conclude that

$$\frac{IQ}{IO} = \frac{IN'}{IB} = \frac{1}{2}$$

and

$$QA = \frac{1}{2}AO \implies QO = \frac{3}{2}AO,$$

whence $IO = 2IQ = 2QO = 3AO = 3BO = 3CO$.

In conclusion, the locus of I (as A takes different values on Γ) is the circle centred at O and having diameter $3BC$.

Also solved by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.

MH–20. *Proposed by Andrés Sáez-Schwedt, Universidad de León, León, Spain.* Let ABC be an acute triangle. The circles having the medians BN and CM as diameters intersect at two points P and Q. The line through the orthocenter of ABC parallel to BC meets AC at R and AB at S. If X and Y are the orthocenters of PQR and PQS, respectively, show that $XY = BC$.

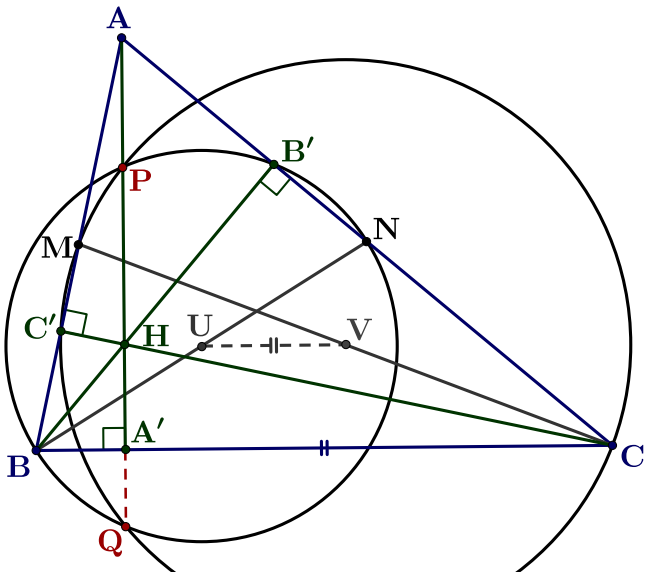


Figure 7: Necessary elements of the first part of Problem 3 to Problem MH–20.

Solution by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. Let U and V be the midpoints of BN and CM, respectively. Since $MN \parallel BC$ and U and V are the midpoints of BN and CM, $UV \parallel BC$ as well. But U and V are precisely the centres of the circles Γ_U and Γ_V with diameters BN and CM; consequently, their radical axis PQ is perpendicular to BC.

In fact, PQ coincides with the altitude AH. To prove this, it suffices to show that the power of A with respect to each of these two circles is the same and so A belongs to their radical axis PQ (because we already know that $AH \parallel PQ$). Let A', B' and C' be the feet of the altitudes of ABC from A, B and C, respectively. Since $\angle NB'B = \frac{\pi}{2}$ (because BB' is an altitude), the point B' belongs to Γ_U by Thales's theorem. Therefore, the power of A with respect to Γ_U is $AN \cdot AB' = \frac{1}{2}AC \cdot AB'$. Similarly, the power of A with respect to Γ_V is $\frac{1}{2}AB \cdot AC'$. These two values are the same because

$$\frac{AB'}{AB} = \frac{AC'}{AC} = \cos(\angle CAB)\,.$$

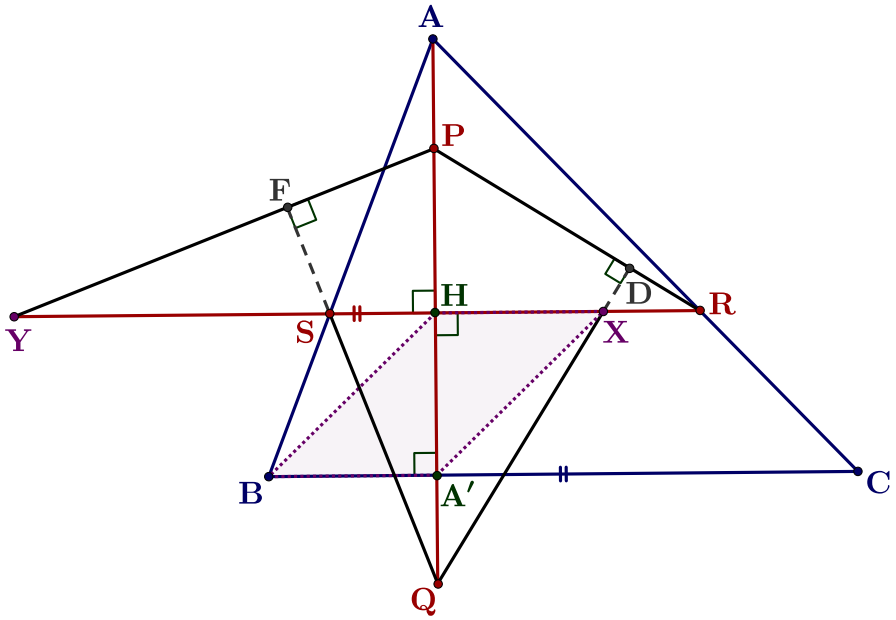


Figure 8: Possible configurations in the second part of Problem 3 to Problem MH–20.

In particular, A, A', H, P and Q are collinear. The power of H with respect to Γ_U can be written as $HP \cdot HQ$ and as $HB \cdot HB'$. Furthermore, since $\angle AB'B = \angle AA'B = \frac{\pi}{2}$, $ABA'B'$ is a cyclic quadrilateral and thus $HA \cdot HA' = HB \cdot HB'$. In conclusion, $HP \cdot HQ = HA \cdot HA'$.

It is clear from what we have seen that R, S, X, Y and H are collinear. There are two possible configurations depending on the relative position of X and R: either $HX < HR$ or $HR \leq HX$ (the two configurations are depicted in Figure 8 by the positions of X and Y, respectively, so we base the rest of the proof on it).

We have that $\angle PRH = \angle XQH$ because $\angle RHP = \angle QDP = \frac{\pi}{2}$. Therefore, the triangles HPR and HXQ are similar. We obtain that

$$\frac{HP}{HR} = \frac{HX}{HQ}$$

and, consequently, $HR \cdot HX = HP \cdot HQ = HA \cdot HA'$ and

$$\frac{HA}{HR} = \frac{HX}{HA'}.$$

This, together with the fact that $AA' \perp HR$, means that the triangles HAR and HXA' are similar (side-angle-side criterion). Hence, as $HA \perp HX$ and $HR \perp HA'$, $AR \perp XA'$ too. That is to say, XA' is perpendicular to AC and, so, parallel to BH (which is an altitude).

Analogously, $\angle PYH = \angle SQH$ because $\angle YHP = \angle QHS = \frac{\pi}{2}$. Therefore, $HYP \sim HQS$, whence $HY \cdot HS = HP \cdot HQ = HA \cdot HA'$ or, equivalently, $HAS \sim HYA'$. Again, $HA \perp HY$ and $HS \perp HA'$ imply that $AS \perp YA'$. That is, $YA' \parallel CH$ (we know that $CH \perp AB$ because it is the corresponding altitude).

From the previous paragraphs, we conclude that $BA'XH$ and $CA'YH$ are parallelograms. In conclusion,

$$XY = XH + HY = BA' + A'C = BC.$$

MH–21. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer with prime factorization $n = p_1^{k_1} \dots p_r^{k_r}$. Prove that

$$\frac{n^2}{2} < \prod_{i=1}^{r} \left(p_i^{k_i} - p_i^{k_i - 1}\right)\left(1 + p_i + \dots + p_i^{k_i}\right) < n^2.$$

Solution 1 by the proposer. We have

$$\begin{aligned}
&\prod_{j=1}^{r} \left(p_j^{k_j} - p_j^{k_j - 1}\right)\left(1 + p_j + \dots + p_j^{k_j}\right) \\
&\quad = \prod_{j=1}^{r} p_j^{k_j} \left(1 - \frac{1}{p_j}\right) \frac{p_j^{k_j+1} - 1}{p_j - 1} = n \prod_{j=1}^{r} \frac{p_j - 1}{p_j} \frac{p_j^{k_j+1} - 1}{p_j - 1} \\
&\quad = n \prod_{j=1}^{r} \frac{p_j^{k_j+1} - 1}{p_j} = n \prod_{j=1}^{r} p_j^{k_j} \left(1 - \frac{1}{p_j^{k_j+1}}\right) \\
&\quad = n^2 \prod_{j=1}^{r} \left(1 - \frac{1}{p_j^{k_j+1}}\right).
\end{aligned}$$

Now we need to show that

$$\frac{1}{2} < \prod_{j=1}^{r}\left(1 - \frac{1}{p_j^{k_j+1}}\right) < 1.$$

The RHS inequality trivially holds. To prove the LHS inequality, we observe that

$$\prod_{j=1}^{r}\left(1 - \frac{1}{p_j^{k_j+1}}\right) > \prod_{p}\left(1 - \frac{1}{p^2}\right),$$

where the product is taken over all prime numbers (each factor is less than 1). Since the n^{th} prime number does not exceed $n+1$, we have

$$\begin{aligned}\prod_{p}\left(1 - \frac{1}{p^2}\right) \geq \prod_{n\geq 2}\left(1 - \frac{1}{n^2}\right) &= \prod_{n\geq 2}\frac{(n-1)(n+1)}{n^2} \\ &= \frac{1\cdot 3}{2\cdot 2}\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{4\cdot 6}{5\cdot 5}\cdots = \frac{1}{2}\end{aligned}$$

and we are done.

Solution 2 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. Let us rewrite the product in a more convenient way:

$$\begin{aligned}&\prod_{i=1}^{r}\left(p_i^{k_i} - p_i^{k_i-1}\right)\left(1 + p_i + \ldots + p_i^{k_i}\right) \\ =&\prod_{i=1}^{r} p_i^{k_i-1}(p_i-1)\left(\frac{p_i^{k_i+1}-1}{p_i-1}\right) \\ =&\prod_{i=1}^{r} p_i^{k_i-1}\left(p_i^{k_i+1}-1\right) = \prod_{i=1}^{r} p_i^{2k_i} - p_i^{k_i-1}.\end{aligned}$$

With this, it is trivial to prove the RHS inequality:

$$\prod_{i=1}^{r} p_i^{2k_i} - p_i^{k_i-1} < \prod_{i=1}^{r} p_i^{2k_i} = n^2.$$

Now we need to prove the other inequality. Taking into account that $k_i \geq 1\ \forall i = 1, \ldots, r$ and Euler's formula for Riemann's zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}}$$

with $\zeta(2) = \frac{\pi^2}{6}$ (here, $\mathbb{P}$ represents the set of prime numbers), we conclude the proof with

$$\begin{aligned}\prod_{i=1}^{r} p_i^{2k_i} - p_i^{k_i - 1} &= \prod_{i=1}^{r} p_i^{2k_i}\left(1 - p_i^{-k_i - 1}\right) = n^2 \prod_{i=1}^{r}\left(1 - p_i^{-k_i - 1}\right) \\ &\geq n^2 \prod_{i=1}^{r}\left(1 - p_i^{-2}\right) > n^2 \prod_{p \in \mathbb{P}}\left(1 - p^{-2}\right) \\ &= \frac{n^2}{\zeta(2)} = \frac{6}{\pi^2} n^2 > \frac{n^2}{2}.\end{aligned}$$

Also solved by Jordi Bosch Bosch, CFIS, BarcelonaTech, Barcelona, Spain, and Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain.

MH–22. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Prove that

$$\sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^{\binom{n+1}{2}}$$

is an irrational number.

Solution by Alejandro Suárez Hernández, CFIS, BarcelonaTech, Barcelona, Spain. First, let us notice that the exponents of the addends are the triangular numbers from 1 on:

$$\binom{n+1}{2} = \frac{n(n+1)}{2} \in \{1, 3, 6, 10, 15, 21, 28, \cdots\}, \quad n \in \mathbb{N}.$$

We can express in base 5

$$\frac{1}{5} = [0.1]_5.$$

Therefore, the result of the sum follows the pattern

$$\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^{\binom{n+1}{2}} = [0.101001000100001000001\cdots]_5\,.$$

Each addend adds a "1" to the next triangular-number position. There are infinite addends, so the fractional part also has infinite digits. We can observe that the n^{th} "1" is separated from the next "1" by n "0"s, so there does not exist any periodical sequence. These two reasons lead to the conclusion that the proposed summation has an irrational number as a result.

Also solved by Alberto Espuny Díaz, BarcelonaTech, Barcelona, Spain; Marc Felipe Alsina, CFIS, BarcelonaTech, Barcelona, Spain; Guillermo Girona San Miguel, Barcelona, Spain; Francesc Gispert Sánchez, BarcelonaTech, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Eric Milesi Vidal, CFIS, BarcelonaTech, Barcelona, Spain; Gerard Orriols Giménez, CFIS, BarcelonaTech, Barcelona, Spain; Óscar Rivero Salgado, CFIS, BarcelonaTech, Barcelona, Spain; Isaac Sánchez Barrera, BarcelonaTech, Barcelona, Spain; Damià Torres Latorre, CFIS, BarcelonaTech, Barcelona, Spain, and the proposer.

Advanced Problems

A–17. *Proposed by Ovidiu T. Pop, Satu Mare, Romania, and Mihály Bencze, Braşov, Romania.* Show that there do not exist invertible matrices A, $B \in \mathcal{M}_n(\mathbb{C})$ such that $(A+B)^2 = A^2 + B^2$ and $(A+B)^3 = A^3 + B^3$.

Solution 1 by Alejandro Suárez Hernández, CFIS, BarcelonaTech, Barcelona, Spain. We start assuming that there exist two matrices A, B that accomplish the given identities. Let us expand these expressions:

$$(A+B)^2 = A^2 + AB + BA + B^2.$$

For $(A+B)^2 = A^2+B^2$ to hold, we impose that $AB+BA = 0_{n\times n}$, so

$$AB = -BA. \tag{3}$$

We shall continue with the expression

$$(A+B)^3 = (A^2 + AB + BA + B^2)(A+B) =$$

$$(A^2 + B^2)(A+B) = A^3 + A^2B + B^2A + B^3.$$

Now we impose that $A^2B + B^2A = 0_{n\times n}$. By (3), this implies that $A^2B + B^2A = A^2B - BAB = (A-B)AB = 0_{n\times n}$. Since we have assumed that A and B are invertible, AB is also invertible and $A - B = 0_{n\times n} \Rightarrow A = B$. If we return to (3) applying this new identity, we get: $A^2 + A^2 = 2A^2 = 0_{n\times n}$. This means that A is not invertible, which invalidates our first assumption and leads to the desired proof.

Solution 2 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. Suppose, for the sake of contradiction, that there exist A, $B \in \mathcal{M}_n(\mathbb{C})$ satisfying the required conditions. In this situation,

$$A^2 + B^2 = (A+B)^2 = A^2 + B^2 + AB + BA \implies AB = -BA$$

and also

$$\begin{aligned} A^3 + B^3 &= (A+B)^3 = (A+B)^2(A+B) = (A^2+B^2)(A+B) \\ &= A^3 + B^3 + A^2B + B^2A \implies \\ \implies A^2B &= -B^2A. \end{aligned}$$

However, using that $AB = -BA$, we deduce that

$$-B^2A = A^2B = A \cdot AB = -A \cdot BA = -AB \cdot A = BA \cdot A = BA^2.$$

This means that $B^2A = 0$, but this is impossible because both A and B are invertible (and so must be the product B^2A).

Also solved by Alberto Espuny Díaz, BarcelonaTech, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain, and the proposers.

A–18. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $n \geq 1$ be a positive integer, and let $S \subset \{0, 1, 2, \ldots, n\}$ such that

$$|S| \geq \frac{n}{2} + 1.$$

Show that some power of 2 is either an element of S or the sum of two distinct elements of S.

Solution 1 by the proposer. The proof is by induction on n. It is easy to check that the result is true for $n = 1$, 2, 3, and 4. Let $n > 4$, and assume that the result holds for all positive integers $m < n$. Now choose $s > 2$ such that

$$2^s \leq n < 2^{s+1},$$

and let $r = n - 2^s \in \{0, 1, \ldots, 2^s - 1\}$. Let

$$S_1 = S \cap \{0, \ldots, 2^s - r - 1\}$$

and

$$S_2 = S \cap \{2^s - r, 2^s - r + 1, \ldots, 2^s + r\}.$$

Then S is the disjoint union of S_1 and S_2, and $|S| = |S_1| + |S_2|$.

Suppose that the statement is false for the set S. Then $|S| \geq \frac{n}{2} + 1$, but no power of 2 belongs to S or is the sum of two distinct elements of S. It follows that $2^s \notin S_2$ and, for each $i = 1, 2, \ldots, r$, the set S_2 contains at most one of the two integers $2^s - i$, $2^s + i$. Therefore, $|S_2| \leq r$.

If $n = 2^{s+1} - 1$, then $r = 2^s - 1$ and $S_1 \subseteq \{0\}$. Thus, $|S_1| \leq 1$. Now, it follows that

$$\frac{n}{2} + 1 \leq |S| \leq 1 + r = 2^s = \frac{n+1}{2},$$

which is impossible.

Similarly, if $2^s \leq n < 2^{s+1} - 1$, then $0 \leq r < 2^s - 1$ and $m = 2^s - r - 1 \geq 1$. Since the set S contains S_1, then no power of 2 belongs to S_1 or is the sum of two distinct elements of S_1. By the induction hypothesis, we have

$$|S_1| < \frac{m}{2} + 1 = \frac{2^s - r - 1}{2} + 1,$$

and

$$\frac{n}{2} + 1 \leq |S| = |S_1| + |S_2| < \frac{2^s - r - 1}{2} + 1 + r = \frac{n+1}{2},$$

which is also impossible.

Solution 2 by Diego Goldsztajn, Universidad de la República Uruguay, Montevideo, Uruguay. We are going to prove that for every integer $n \geq 1$ we can find a collection $\mathcal{C}$ satisfying the following properties:

1. The elements of $\mathcal{C}$ are pairs of two different elements of the set $\{0, 1, \ldots, n\}$,

$$\mathcal{C} \subset \{\{x, y\} : 1 \leq x, y \leq n \text{ and } x \neq y\}.$$

2. Two elements of $\mathcal{C}$ are always disjoint,

$$A \cap B = \varnothing \ \ \forall A, B \in \mathcal{C} : A \neq B.$$

3. The sum of the elements of a pair in $\mathcal{C}$ is always a power of two,
$$\{x, y\} \in \mathcal{C} \Longrightarrow \exists k \geq 0 : x + y = 2^k.$$
4. Any element of $\{0, \ldots, n\}$ that is not a power of two is contained in a pair that belongs to $\mathcal{C}$,
$$x \in \{0, \ldots, n\} \Longrightarrow x \in \bigcup_{C \in \mathcal{C}} C \text{ or } \exists k \geq 0 : x = 2^k.$$

Note that it is allowed for a power of two to belong to $\bigcup_{C \in \mathcal{C}} C$, we are just saying that if x is not a power of two then $x \in \bigcup_{C \in \mathcal{C}} C$.

If the previous statement is true, then the problem is solved. In order to see this, suppose that the statement is true and that given an integer $n \geq 1$ we have a set $S \subset \{0, \ldots, n\}$ such that S and $S + S$ do not contain powers of two. Take a collection $\mathcal{C}$ as before. We know that S does not contain powers of two, so using the fourth property we may assert that

$$S \subset \bigcup_{C \in \mathcal{C}} C.$$

We also know that $S + S$ does not contain powers of two. This means that if $C \in \mathcal{C}$, then at most one element of C may belong to S, otherwise the third property would imply that $S+S$ contains a power of two. This allows us to write

$$|S \cap C| \leq 1 = \frac{|C|}{2} \ \ \forall C \in \mathcal{C}.$$

Using our two observations and the second property we have

$$|S| = \sum_{C \in \mathcal{C}} |S \cap C| \leq \sum_{C \in \mathcal{C}} \frac{|C|}{2} = \frac{\left| \bigcup_{C \in \mathcal{C}} C \right|}{2} \leq \frac{n+1}{2} < \frac{n}{2} + 1.$$

Now it suffices to prove our initial statement, regarding the existence of collections $\mathcal{C}$ of pairs, in order to solve the problem. In

order to prove the statement we will prove by induction on $k \geq 0$ that for every $0 \leq m < 2^k$ the integer $n = 2^k + m$ admits a collection $\mathcal{C}$ of pairs with the desired properties.

For the base case take $k = 0$. The only possible value for m is $m = 0$, which implies $n = 1$. In this case we may take $\mathcal{C} = \{\{0, 1\}\}$.

In order to prove the inductive step we will consider two different cases, depending on the value of m.

First, suppose that $k \geq 1$ is any integer and $m = 2^k - 1$. In this case it is easy to check that we can take $\mathcal{C}$ to be the collection of pairs $\{2^k - i, 2^k + i\}$ with $1 \geq i \geq m$, and the pair $\{0, 2^k\}$. Note that in this case we do not need to use the induction hypothesis.

Now, suppose that the hypothesis is true for $r < k$ and fix $0 \leq m < 2^k - 1$. We want to find a collection $\mathcal{C}$ of pairs for $n = 2^k + m$. Note that $1 \leq 2^k - m - 1 < 2^k$, so there are integers $0 \leq r < k$ and $0 \leq t < 2^r$ such that

$$2^k - m - 1 = 2^r + t.$$

Using the induction hypothesis we may find a collection $\mathcal{A}$ satisfying our four properties for the integer $2^r + t$. Define $\mathcal{B}$ to be the collection of the pairs $\{2^k - i, 2^k + i\}$ with $1 \leq i \leq m$. We are going to see that $\mathcal{C} = \mathcal{A} \cup \mathcal{B}$ satisfies the four desired properties for n.

1. Both $\mathcal{A}$ and $\mathcal{B}$ are collections of pairs formed with two different elements. For $\mathcal{A}$ these elements belong to $\{1, \ldots, 2^r + t\}$, while for $\mathcal{B}$ they belong to $\{2^k - m, \ldots, 2^k + m\}$. Then we have that

$$\mathcal{C} \subset \{\{x, y\} : 1 \leq x, y \leq n \text{ and } x \neq y\}.$$

2. The induction hypothesis implies that two different elements of $\mathcal{A}$ are disjoint and this is also easy to check for $\mathcal{B}$. Moreover, the elements of $\mathcal{A}$ are subsets of $\{1, \ldots, 2^r + t\}$ and the elements of $\mathcal{B}$ are subsets of $\{2^k - m, \ldots, 2^k + m\}$ with

$2^r + t < 2^k - m$, which implies that an element of $\mathcal{A}$ and an element of $\mathcal{B}$ are always disjoint. Then we may state that

$$A \cap B = \varnothing \ \ \forall A, B \in \mathcal{C} : A \neq B.$$

3. The sum of a pair in $\mathcal{C}$ is always a power of two because this is true for both $\mathcal{A}$ and $\mathcal{B}$,

$$\{x, y\} \in \mathcal{C} \Longrightarrow \exists k \geq 0 : x + y = 2^k.$$

4. Given that $\mathcal{A}$ satisfies the desired properties for the integer $2^r + t$, we have that if $x \in \{1, \ldots, 2^r + t\}$ then $x \in \bigcup_{A \in \mathcal{A}} A$ or x is a power of two. On the other hand, if $x \in \{2^k - m, \ldots, 2^k + m\}$ and $x \notin \bigcup_{B \in \mathcal{B}} B$ then $x = 2^k$, which is a power of two. Using that $2^r + t = 2^k - m - 1$ we conclude that

$$x \in \{1, \ldots, n\} \Longrightarrow x \in \bigcup_{C \in \mathcal{C}} C \text{ or } \exists k \geq 0 : x = 2^k.$$

Also solved by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain, and Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.

A–19. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let α be a real number. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \alpha(1 + x^2)\left[1 + \int_0^x \frac{f(t)}{1 + t^2}\, dt\right].$$

Solution by the proposer. Since f is continuous, then the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \frac{f(t)}{1 + t^2}$$

is continuous and has a primitive, say G. Thus, the given condition can be written as $f(x) = \alpha(1 + x^2)[1 + G(x) - G(0)]$. Since

G is differentiable in $\mathbb{R}$ (by the fundamental theorem of calculus), then f is differentiable in $\mathbb{R}$ as well. Now, taking the derivative of both sides of the expression

$$\frac{f(x)}{1+x^2} = \alpha[1 + G(x) - G(0)]$$

we obtain

$$\frac{f'(x)(1+x^2) - 2xf(x)}{(1+x^2)^2} = \alpha G'(x) = \alpha \frac{f(x)}{1+x^2}$$

or, equivalently,

$$\frac{f'(x)}{f(x)} = \alpha + \frac{2x}{1+x^2}.$$

Integrating we get

$$\ln|f(x)| = \alpha x + \ln(1+x^2) + K \iff f(x) = C(1+x^2)e^{\alpha x}$$

for some constant C.

Substituting this expression of $f(x)$ into the statement, we have that

$$C(1+x^2)e^{\alpha x} = \alpha(1+x^2)\left[1 + \int_0^x \frac{C(1+t^2)e^{\alpha t}}{1+t^2}\,dt\right].$$

From the preceding, it immediately follows that $C = \alpha$. Therefore, the only solution is

$$f(x) = \alpha(1+x^2)e^{\alpha x}$$

and we are done.

Also solved by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and Alejandro Suárez Hernández, CFIS, BarcelonaTech, Barcelona, Spain

A–20. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain, and José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Calculate

$$\sum_{k=1}^{n} \sum_{1\le i_1<\ldots<i_k\le n} \frac{3^k}{(i_1+2)(i_2+2)\ldots(i_k+2)}.$$

Solution 1 by the proposers. For all function f for which $f(k)\neq 0$, $(1\le k\le n)$, we have

$$\prod_{k=1}^{n}\left(1+\frac{1}{f(k)}\right)=1+\sum_{k=1}^{n}\sum_{1\le i_1<\ldots<i_k\le n}\frac{1}{f(i_1)f(i_2)\ldots f(i_k)}.$$

Putting $f(x)=\dfrac{x+2}{3}$ into the preceding expression, we get

$$1+\sum_{k=1}^{n}\sum_{1\le i_1<\ldots<i_k\le n}\frac{3^k}{(i_1+2)(i_2+2)\ldots(i_k+2)}=\prod_{k=1}^{n}\left(1+\frac{3}{k+2}\right).$$

We claim that

$$\prod_{k=1}^{n}\left(1+\frac{3}{k+2}\right)=\frac{(n+5)(n+4)(n+3)}{60}.$$

To prove our claim, we argue by induction. The case when $n=1$ trivially holds. Assume that the identity holds for n. We have to prove that

$$\prod_{k=1}^{n+1}\left(1+\frac{3}{k+2}\right)=\frac{(n+6)(n+5)(n+4)}{60}.$$

Indeed,

$$\begin{aligned}\prod_{k=1}^{n+1}\left(1+\frac{3}{k+2}\right)&=\prod_{k=1}^{n}\left(1+\frac{3}{k+2}\right)\left(1+\frac{3}{n+3}\right)\\&=\frac{(n+5)(n+4)(n+3)}{60}\left(1+\frac{3}{n+3}\right)\\&=\frac{(n+6)(n+5)(n+4)}{60}.\end{aligned}$$

In conclusion,

$$\sum_{k=1}^{n} \sum_{1 \le i_1 < \ldots < i_k \le n} \frac{3^k}{(i_1+2)(i_2+2)\ldots(i_k+2)} = \frac{n(n^2+12n+47)}{60}$$

and we are done.

Solution 2 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. Let

$$S_n = \sum_{k=1}^{n} \sum_{1 \le i_1 < \ldots < i_k \le n} \frac{3^k}{(i_1+2)(i_2+2)\ldots(i_k+2)}\,.$$

We can compute S_n recursively. First of all, we compute $S_1 = 1$. For $n > 1$, we split the sum into two depending on whether the term $(n+2)$ appears in the denominator of each addend. On the one hand, each addend

$$\frac{3^k}{(i_1+2)(i_2+2)\ldots(i_k+2)}$$

with $i_k < n$ also appears in the sum which defines S_{n-1} (exactly once). On the other hand, the remaining addends are of the form

$$\begin{aligned}\frac{3^k}{(i_1+2)\ldots(i_{k-1}+2)(n+2)} &= \frac{3}{n+2}\frac{3^{k-1}}{(i_1+2)\ldots(i_{k-1}+2)} \\ &= \frac{3}{n+2}A,\end{aligned}$$

and the corresponding A's are again the addends appearing (exactly once) in the definition of S_{n-1} unless $k = 1$ (in which case $A = 1$). Therefore,

$$S_n = S_{n-1} + \frac{3}{n+2}(S_{n-1}+1) = \frac{n+5}{n+2}S_{n-1} + \frac{3}{n+2}\,.$$

We need to solve the previous recurrence, which is not homogeneous. If one is able to guess the answer, then it can be proved by

induction; or it can also be solved using generating functions. In this case, however, we examine the expression

$$S_n = S_{n-1} + \frac{3}{n+2}(S_{n-1}+1)$$

and define instead $T_n = S_n + 1$ so that $T_1 = 2$ and

$$T_n = T_{n-1} + \frac{3}{n+2}T_{n-1} = \frac{n+5}{n+2}T_{n-1}$$

(and this recurrence is homogeneous and much easier to solve). By expanding the recurrence, we obtain that

$$\begin{aligned} T_n &= \frac{n+5}{n+2}T_{n-1} = \frac{(n+5)(n+4)}{(n+2)(n+1)}T_{n-2} \\ &= \frac{(n+5)(n+4)(n+3)}{(n+2)(n+1)n}T_{n-3} = \frac{(n+5)(n+4)(n+3)}{(n+1)n(n-1)}T_{n-4} \\ &= \ldots = \frac{(n+5)(n+4)(n+3)}{6\cdot 5\cdot 4}T_1 = \frac{(n+5)(n+4)(n+3)}{60} \end{aligned}$$

and, consequently,

$$S_n = \frac{(n+5)(n+4)(n+3)}{60} - 1 = \frac{n(n^2+12n+47)}{60}\,.$$

Also solved by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.

A–21. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be distinct nonzero complex numbers. Determine

$$\frac{2a^2+b^2+c^2}{\left(1-\frac{a}{b}\right)\left(1-\frac{a}{c}\right)} + \frac{a^2+2b^2+c^2}{\left(1-\frac{b}{a}\right)\left(1-\frac{b}{c}\right)} + \frac{a^2+b^2+2c^2}{\left(1-\frac{c}{a}\right)\left(1-\frac{c}{b}\right)}\,.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. The result is $a^2+b^2+c^2$. In order to show this we

will prove that $\displaystyle\sum_{\text{cyclic}} \frac{1}{(1-\frac{a}{b})(1-\frac{a}{c})} = 1$, and $\displaystyle\sum_{\text{cyclic}} \frac{a^2}{(1-\frac{a}{b})(1-\frac{a}{c})} = 0$, from where the solution follows.

$$\begin{aligned}\sum_{\text{cyclic}} \frac{1}{(1-\frac{a}{b})(1-\frac{a}{c})} &= \frac{bc}{(b-a)(c-a)} + \frac{ca}{(a-b)(c-b)} \\ &\quad + \frac{ab}{(a-c)(b-c)} \\ &= \frac{bc(c-b) - ca(c-a) + ab(b-a)}{(b-a)(c-a)(c-b)} = 1, \\ \sum_{\text{cyclic}} \frac{a^2}{(1-\frac{a}{b})(1-\frac{a}{c})} &= \frac{a^2bc}{(b-a)(c-a)} + \frac{b^2ca}{(a-b)(c-b)} \\ &\quad + \frac{c^2ab}{(a-c)(b-c)} \\ &= abc\left(\frac{a(c-b) - b(c-a) + c(b-a)}{(b-a)(c-a)(c-b)}\right) = 0.\end{aligned}$$

Solution 2 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. The value which we want to compute can be written as

$$X = \frac{2a^2bc + b^3c + c^3b}{(b-a)(c-a)} + \frac{2b^2ca + c^3a + a^3c}{(a-b)(c-b)} + \frac{2c^2ab + a^3b + b^3a}{(a-c)(b-c)}.$$

We reduce X to common denominator $D = (a-b)(b-c)(c-a)$ and let N be the corresponding numerator.

On the one hand, the denominator is

$$\begin{aligned}D &= -a^2b + ca^2 - b^2c + ab^2 - c^2a + bc^2 \\ &= a^2(c-b) + b^2(a-c) + c^2(b-a).\end{aligned}$$

On the other hand, the numerator can be expressed as

$$\begin{aligned}
N &= c^4b - b^4c - c^3b^2 + b^3c^2 + a^4c - c^4a - a^3c^2 + c^3a^2 \\
&\quad + b^4a - a^4b - b^3a^2 + a^3b^2 \\
&= a^2(c^3 - b^3) + b^2(a^3 - c^3) + c^2(b^3 - a^3) \\
&\quad + a^4(c - b) + b^4(a - c) + c^4(b - a) \\
&= a^2(c - b)(a^2 + b^2 + c^2 + bc) + b^2(a - c)(a^2 + b^2 + c^2 + ca) \\
&\quad + c^2(b - a)(a^2 + b^2 + c^2 + ab) \\
&= (a^2 + b^2 + c^2)[a^2(c - b) + b^2(a - c) + c^2(b - a)] \\
&\quad + abc[a(c - b) + b(a - c) + c(b - a)] \\
&= (a^2 + b^2 + c^2)D + abc \cdot 0\,.
\end{aligned}$$

In conclusion,

$$X = \frac{N}{D} = a^2 + b^2 + c^2\,.$$

Solution 3 by the proposer. Let S be the above sum. We claim that $S = a^2 + b^2 + c^2$. Indeed, reducing to common denominator the parentheses and multiplying by $\dfrac{1}{abc}$ our claim is equivalent to proving the identity

$$\frac{2a^2 + b^2 + c^2}{a(a - b)(a - c)} + \frac{a^2 + 2b^2 + c^2}{b(b - a)(b - c)} + \frac{a^2 + b^2 + 2c^2}{c(c - a)(c - b)} = \frac{a^2 + b^2 + c^2}{abc}.$$

It is well known from the theory of divided differences that

$$f(z_0, z_1, \ldots, z_n) = \sum_{j=0}^{n} f(z_j) \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{1}{z_j - z_k}$$

(that is to say, the interpolation polynomials of f in the Lagrange form and in the Newton form coincide and, in particular, their leading coefficient is the same).

Applying the above result to the function $f(z) = z + \dfrac{a^2 + b^2 + c^2}{z}$,

the left side of the previous identity equals

$$\begin{aligned} f(a,b,c) &= \frac{f(b,c)-f(a,b)}{c-a} \\ &= \frac{1}{c-a}\Big(\frac{bc-a^2-b^2-c^2}{bc} - \frac{ab-a^2-b^2-c^2}{ab}\Big) \\ &= \frac{a^2+b^2+c^2}{abc}. \end{aligned}$$

On the other hand, its right side is

$$\begin{aligned} &\frac{f(a)}{(a-b)(a-c)} + \frac{f(b)}{(b-a)(b-c)} + \frac{f(c)}{(c-a)(c-b)} \\ &\quad = \frac{2a^2+b^2+c^2}{a(a-b)(a-c)} + \frac{a^2+2b^2+c^2}{b(b-a)(b-c)} + \frac{a^2+b^2+2c^2}{c(c-a)(c-b)}. \end{aligned}$$

Hence, the claimed identity is proven and we are done.

Also solved by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and Óscar Rivero Salgado, CFIS, BarcelonaTech, Barcelona, Spain.

A–22. *Proposed by Mihály Bencze, Braşov, Romania.* Consider $f\colon [0,1] \to \mathbb{R}$ a continuous function such that for all $x \in [0,1]$

$$\int_0^x f(t)\,dt \le \frac{1}{\pi(1+x^2)}\,.$$

Prove that

$$\int_0^1 (f(x)-1)^2\,dx \ge \frac{1}{6}\,.$$

Solution 1 by Diego Goldsztajn, Universidad de la República Uruguay, Montevideo, Uruguay. Setting $x = 1$, we obtain the following inequality,

$$\int_0^1 f(t)\,\mathrm{d}t \le \frac{1}{2\pi} \implies -2\int_0^1 f(t)\,\mathrm{d}t \ge -\frac{1}{\pi}.$$

Then we have

$$\int_0^1 [f(t)-1]^2\,\mathrm{d}t = \int_0^1 [f(t)]^2\,\mathrm{d}t - 2\int_0^1 f(t)\,\mathrm{d}t + 1$$
$$\geq 1 - \frac{1}{\pi} > \frac{1}{6},$$

where we obtain a better bound than the one given in the statement.

Solution 2 by Alejandro Suárez Hernández, CFIS, Barcelona-Tech, Barcelona, Spain. Let us recall the Cauchy-Schwarz inequality for integrable real functions:

$$\left(\int_a^b f(x)g(x)\mathrm{d}x\right)^2 \leq \int_a^b f(x)^2\mathrm{d}x \cdot \int_a^b g(x)^2\mathrm{d}x.$$

In our case, we can use it to obtain

$$\int_0^1 (f(x)-1)^2\mathrm{d}x \geq \left(\int_0^1 (f(x)-1)\mathrm{d}x\right)^2. \tag{4}$$

By the statement's inequality we can see that:

$$\int_0^1 (f(x)-1)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x - 1 \leq \frac{1}{2\pi} - 1 = -\frac{2\pi-1}{2\pi} < 0.$$

It is clear that $|\int_0^1 (f(x)-1)\mathrm{d}x| \geq \frac{2\pi-1}{2\pi}$. Thus:

$$\left(\int_0^1 (f(x)-1)\mathrm{d}x\right)^2 \geq \left(\frac{2\pi-1}{2\pi}\right)^2 > \frac{1}{6}.$$

And finally, by (4),

$$\int_0^1 (f(x)-1)^2\mathrm{d}x \geq \left(\frac{2\pi-1}{2\pi}\right)^2 > \frac{1}{6}.$$

This gives us the desired proof and an even stricter lower bound.

Solution 3 by the proposer. Let F denote a primitive of f. Then

$$\int_0^x f(t)\,dt \leq \frac{1}{\pi(1+x^2)} \iff F(x) - F(0) \leq \frac{1}{\pi(1+x^2)}\,.$$

We consider $g\colon [0,1] \to \mathbb{R}$ defined as

$$g(x) = \int_0^x F(t)dt - xF(0) - \frac{1}{\pi}\arctan x,$$

so that

$$g'(x) = F(x) - F(0) - \frac{1}{\pi(1+x^2)} \leq 0.$$

In particular,

$$g(1) \leq g(0) \Longrightarrow \int_0^1 F(t)\,dt \leq F(0) + \frac{1}{4}. \tag{5}$$

On the other hand,

$$\begin{aligned} \int_0^1 (xF(x))'\,dx &= \int_0^1 (F(x) + xf(x))\,dx \\ \Longrightarrow\ F(1) &= \int_0^1 F(x)\,dx + \int_0^1 xf(x)\,dx, \end{aligned}$$

so from (5) we get that

$$F(1) = \int_0^1 F(x)\,dx + \int_0^1 xf(x)\,dx \leq F(0) + \frac{1}{4} + \int_0^1 xf(x)\,dx,$$

and thus

$$\begin{aligned} F(1) - F(0) - \frac{1}{4} &\leq \int_0^1 xf(x)\,dx \\ \Longrightarrow\ \int_0^1 f(x)\,dx - \frac{1}{4} &\leq \int_0^1 xf(x)\,dx. \end{aligned} \tag{6}$$

Finally, from the fact that

$$\int_0^1 (f(x) - x)^2\,dx \geq 0$$

we deduce that

$$\int_0^1 f^2(x)\,dx \geq 2\int_0^1 xf(x)\,dx - \frac{1}{3} \underset{(6)}{\geq} 2\left(\int_0^1 f(x)\,dx - \frac{1}{4}\right) - \frac{1}{3}$$

and, therefore,

$$\int_0^1 (f(x) - 1)^2\,dx = \int_0^1 f^2(x)\,dx - 2\int_0^1 f(x)\,dx + 1 \geq \frac{1}{6}.$$

Also solved by Francesc Gispert Sánchez, CFIS, Barcelona-Tech, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and Jesús Ocáriz Gallego, Universidad Autónoma de Madrid, Madrid, Spain.

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Editors

Aim and Scope

The goal of *Arhimede Mathematical Journal* is to provide a means of publication of useful materials to train students for Mathematical Contests at all levels. Potential contributions include any work involving fresh ideas and techniques, problems and lessons helpful to train contestants, all written in a clear and elegant mathematical style. All areas of mathematics, including algebra, combinatorics, geometry, number theory and real and complex analysis, are considered appropriate for the journal.

Information for Authors

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