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Articles

Arhimede Mathematical Journal aims to publish interesting and attractive papers with elegant mathematical exposition. Articles should include examples, applications and illustrations, whenever possible. Manuscripts submitted should not be currently submitted to or accepted for publication in another journal.

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Hölder's inequality revisited

J. L. Díaz-Barrero

Abstract

In this short note, the classical Hölder's inequality for sums is revisited and some of its applications are given.

1 On Hölder's inequality

In 1888 Rogers published in [4] an inequality discovered independently by Hölder in 1889, who published it in [2], and known since then in the literature as Hölder's inequality. A discrete version of it for the n-dimensional Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ is stated in the following theorem.

Theorem 1. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real or complex numbers. If $p > 1$ and $q > 1$ are real numbers such that*

$$\frac{1}{p} + \frac{1}{q} = 1\,,$$

then

$$\sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} |b_k|^q \right)^{\frac{1}{q}}$$

with equality when $|b_k| = c|a_k|^{p-1}$ for $1 \leq k \leq n$.

Proof. First, we write the inequality in the more convenient form

$$\frac{\sum_{i=1}^{n} |a_i b_i|}{\left(\sum_{i=1}^{n} |a_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |b_i|^q\right)^{\frac{1}{q}}} \leq 1$$

or, equivalently,

$$\sum_{i=1}^{n}\left(\frac{|a_i|^p}{\sum_{j=1}^{n}|a_j|^p}\right)^{\frac{1}{p}}\left(\frac{|b_i|^q}{\sum_{j=1}^{n}|b_j|^q}\right)^{\frac{1}{q}} \le 1\,.$$

Now, using the power means inequality,

$$x^{\frac{1}{\alpha}}y^{\frac{1}{\beta}} \le \frac{1}{\alpha}x + \frac{1}{\beta}y\,,$$

we have

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\frac{|a_i|^p}{\sum_{j=1}^{n}|a_j|^p}\right)^{\frac{1}{p}}\left(\frac{|b_i|^q}{\sum_{j=1}^{n}|b_j|^q}\right)^{\frac{1}{q}}\\
&\quad\le \sum_{i=1}^{n}\left[\frac{1}{p}\frac{|a_i|^p}{\sum_{j=1}^{n}|a_j|^p} + \frac{1}{q}\frac{|b_i|^q}{\sum_{j=1}^{n}|b_j|^q}\right]\\
&\quad= \frac{1}{p}\frac{\sum_{i=1}^{n}|a_i|^p}{\sum_{i=1}^{n}|a_i|^p} + \frac{1}{q}\frac{\sum_{i=1}^{n}|b_i|^q}{\sum_{i=1}^{n}|b_i|^q} = \frac{1}{p}+\frac{1}{q} = 1\,,
\end{aligned}$$

as desired. And it is clear that equality holds if and only if the two n-tuples $(|a_1|^p, |a_2|^p, \ldots, |a_n|^p)$ and $(|b_1|^q, |b_2|^q, \ldots, |b_n|^q)$ are proportional, that is, $|b_k| = c|a_k|^{p-1}$ for $1 \le k \le n$. □

Next, we will give a second proof of the preceding theorem. We need the following lemma.

Lemma 1. *Let $p, q > 1$ be real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, and let α and β be two nonnegative numbers, at least one of which being nonzero. Then, the function $f\colon [0,1] \to \mathbb{R}$ defined by*

$$f(t) = \alpha t + \beta(1-t^q)^{\frac{1}{q}}$$

has a unique maximum at $M = \left(\frac{\alpha^p}{\alpha^p+\beta^p}\right)^{\frac{1}{q}}$ and $f(M) = (\alpha^p+\beta^p)^{\frac{1}{p}}$.

Proof. If $\beta = 0$, then $f(t) = \alpha t$ and the lemma trivially holds. Likewise, if $\alpha = 0$, then $f(t) = \beta(1-t^q)^{\frac{1}{q}}$. Using the inequality $(1-t^q)^{\frac{1}{q}} < 1$, which is valid for $0 < t \le 1$, we get $f(t) < f(0)$ for $0 < t \le 1$ and the lemma follows.

Next, we assume that both α and β are positive. Under the assumption that $\alpha, \beta > 0$, we have $0 < M < 1$. Now we will see that M is the only solution of $f'(t) = 0$ in $(0, 1)$. Indeed, from

$$f'(t) = \alpha - \beta(1 - t^q)^{-\frac{1}{p}} t^{q-1} = 0$$

we obtain

$$\frac{t^q}{1 - t^q} = \left(\frac{\alpha}{\beta}\right)^p \Leftrightarrow t^q = \frac{\alpha^p}{\alpha^p + \beta^p}$$

and $t = \left(\frac{\alpha^p}{\alpha^p + \beta^p}\right)^{\frac{1}{q}} = M$.

Thus, the possible candidates for the maximum are 0, 1 and M. We claim that the function f attains its maximum at M. Indeed, f' is continuous in $(0, 1)$,

$$\lim_{t \to 0^+} = \alpha > 0 \text{ and } \lim_{t \to 1^-} = -\infty$$

and $f'(t) = 0$ has exactly one solution in $(0, 1)$ given by $t = M$. Thus, $f'(t) > 0$ in $(0, M)$ and $f'(t) < 0$ in $(M, 1)$. This means that f is increasing on $[0, M]$ and decreasing on $[M, 1]$, and we are done. The maximum value of f is then given by

$$\max_{t \in [0,1]} f(t) = f(M) = (\alpha^p + \beta^p)^{\frac{1}{p}},$$

as can be easily computed. □

Alternative proof of Theorem 1. We will argue by induction. The case $n = 1$ trivially holds. To prove the case $n = 2$, we may assume that at least one of b_1 and b_2 is nonzero; otherwise everything is trivial. Now we define the number

$$c = \frac{b_1}{(b_1^q + b_2^q)^{\frac{1}{q}}}$$

Notice that $c \in [0, 1]$, and we have

$$\frac{b_2}{(b_1^q + b_2^q)^{\frac{1}{q}}} = (1 - c^q)^{\frac{1}{q}}.$$

Dividing the inequality $a_1b_1 + a_2b_2 \leq (a_1^p + a_2^p)^{\frac{1}{p}}(b_1^q + b_2^q)^{\frac{1}{q}}$ by $(b_1^q + b_2^q)^{\frac{1}{q}}$ yields

$$a_1c + a_2(1 - c^q)^{\frac{1}{q}} \leq (a_1^p + a_2^p)^{\frac{1}{p}}\,.$$

This is a trivial equality when $a_1 = a_2 = 0$. Assume, thus, that $(a_1, a_2) \neq (0, 0)$, and consider the function defined by

$$f(t) = a_1t + a_2(1 - t^q)^{1/q}, \quad 0 \leq t \leq 1\,.$$

Then, by Lemma 1, Hölder's inequality for $n = 2$ is proven. Equality holds when $a_1 = a_2 = 0$, and in this case (a_1, a_2) is clearly proportional to (b_1, b_2). Assume $(a_1, a_2) \neq (0, 0)$. Again, on account of Lemma 1, equality holds when

$$c = \left(\frac{a_1^p}{a_1^p + a_2^p}\right)^{\frac{1}{q}}.$$

That is, when

$$\frac{b_1}{(b_1^q + b_2^q)^{\frac{1}{q}}} = \left(\frac{a_1^p}{a_1^p + a_2^p}\right)^{\frac{1}{q}} \Leftrightarrow \frac{b_1^q}{b_1^q + b_2^q} = \frac{a_1^p}{a_1^p + a_2^p}$$

from which it follows that

$$\frac{a_1^p}{b_1^q} = \frac{a_1^p + a_2^p}{b_1^q + b_2^q} = \frac{a_2^p}{b_2^q}$$

and (a_1^p, a_2^p) is again proportional to (b_1^q, b_2^q).

Assume that the inequality holds for n and we have to prove it for $n + 1$. Indeed,

$$\begin{aligned}\sum_{i=1}^{n+1} a_ib_i &= \sum_{i=1}^{n} a_ib_i + a_{n+1}b_{n+1} \leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}} + a_{n+1}b_{n+1} \\ &= AB + a_{n+1}b_{n+1}\,.\end{aligned}$$

Applying the case $n = 2$, we have

$$\begin{aligned} AB + a_{n+1}b_{n+1} &\le (A^p + a_{n+1}^p)^{\frac{1}{p}}(B^q + b_{n+1}^q)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^{n} a_i^p + a_{n+1}^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q + b_{n+1}^q\right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^{n+1} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n+1} b_i^q\right)^{\frac{1}{q}}. \end{aligned}$$

Since equality holds for the previous cases when the corresponding vectors are proportional, equality holds for the case $n + 1$ precisely when we have that $(a_1^p, a_2^p, \ldots, a_n^p)$ and $(b_1^q, b_2^q, \ldots, b_n^q)$ are proportional, and (A^p, a_{n+1}^p) and (B^q, b_{n+1}^q) are proportional too; in this case, $(a_1^p, a_2^p, \ldots, a_n^p, a_{n+1}^p)$ and $(b_1^q, b_2^q, \ldots, b_n^q, b_{n+1}^q)$ are also proportional. This proves the result for the case $n + 1$, and the proof is complete. □

Notice that for $p = q = 2$, we get the inequality

$$\left(\sum_{k=1}^{n} |a_k b_k|\right)^2 \le \left(\sum_{k=1}^{n} |a_k|^2\right)\left(\sum_{k=1}^{n} |b_k|^2\right).$$

This is the Cauchy–Bunyakowski–Schwarz inequality.

The natural generalization of Hölder's inequality is given in the following theorem.

Theorem 2. *Let $\{a_{ij}\}$ for $1 \le i \le k$ and $1 \le j \le n$ be nonnegative real numbers, and let $p_1, p_2, \ldots, p_k$ be positive real numbers such that*

$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} = 1.$$

Then

$$\sum_{j=1}^{n} \prod_{i=1}^{k} a_{ij} \le \prod_{i=1}^{k} \left(\sum_{j=1}^{n} a_{ij}^{p_i}\right)^{\frac{1}{p_i}}.$$

Proof. The inequality follows immediately by induction on k. □

2 Some applications

Hölder's inequality is important in function theory, functional analysis, Fourier analysis, analytic number theory, probability and statistics [1], and also has a lot of applications in information theory [5]. Hereafter, we will use it to prove the following inequality due to Minkowski [3].

Theorem 3. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real or complex numbers and let $p > 1$. Then*

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}} + \left(\sum_{k=1}^{n} |b_k|^p\right)^{\frac{1}{p}}.$$

Equality holds when $(|a_1|, |a_2|, \ldots, |a_n|)$ and $(|b_1|, |b_2|, \ldots, |b_n|)$ are proportional.

Proof. Define $q = \frac{p}{p-1}$ so that $\frac{1}{p} + \frac{1}{q} = 1$. Applying Hölder's inequality, we obtain

$$\begin{aligned}
\sum_{i=1}^{n}(a_i + b_i)^p &= \sum_{i=1}^{n} a_i(a_i + b_i)^{p-1} + \sum_{i=1}^{n} b_i(a_i + b_i)^{p-1} \\
&\leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n}(a_i + b_i)^{(p-1)q}\right)^{\frac{1}{q}} \\
&\quad + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n}(a_i + b_i)^{(p-1)q}\right)^{\frac{1}{q}} \\
&= \left[\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}\right] \left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{p-1}{p}}.
\end{aligned}$$

Multiplying the preceding inequality by $\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{-\frac{p-1}{p}}$, we get

$$\left(\sum_{i=1}^{n}(a_i + b_i)^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}$$

and the proof is complete. Equality holds if and only if the tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are proportional. □

Since the beginning, Hölder's inequality has been used efficiently for locating zeros of polynomials. Next, we state a result where a bound for the zeros of a polynomial with complex coefficients is obtained.

Theorem 4. *Let $A(z) = \sum_{k=0}^{n} a_k z^k$ be a nonconstant polynomial of degree n with complex coefficients. Then, all its zeros lie on the disk $\mathcal{D} = \{z \in \mathbb{C} : |z| < r\}$, where*

$$r = \left\{ 1 + \left(\sum_{k=0}^{n-1} \left| \frac{a_k}{a_n} \right|^3 \right)^{\frac{1}{2}} \right\}^{\frac{2}{3}}.$$

Proof. From

$$A(z) = \sum_{k=0}^{n} a_k z^k, \ (a_n \neq 0)$$

we have

$$|A(z)| = \left| \sum_{k=0}^{n} a_k z^k \right| \geq |a_n z^n| - \left| \sum_{k=0}^{n-1} a_k z^k \right| \geq |a_n||z|^n - \sum_{k=0}^{n-1} |a_k||z|^k.$$

On account of Hölder's inequality with $p = 3$ and $q = \frac{3}{2}$, the last term of the preceding expression becomes

$$\sum_{k=0}^{n-1} |a_k||z|^k \leq \left(\sum_{k=0}^{n-1} |a_k|^3 \right)^{\frac{1}{3}} \left(\sum_{k=0}^{n-1} |z|^{\frac{3k}{2}} \right)^{\frac{2}{3}}.$$

Therefore,

$$|A(z)| \geq |a_n||z|^n \left\{ 1 - K \left(\sum_{k=0}^{n-1} \frac{1}{|z|^{\frac{3(n-k)}{2}}} \right)^{\frac{2}{3}} \right\},$$

where

$$K = \left(\sum_{k=0}^{n-1} \left| \frac{a_k}{a_n} \right|^3 \right)^{\frac{1}{3}}.$$

We observe that $r > 1$. Therefore, if z is a zero such that $|z| \leq 1$, it is clear that $|z| < r$. In contrast, if we assume that $|z| > 1$, then

$$\sum_{k=0}^{n-1} \frac{1}{|z|^{\frac{3(n-k)}{2}}} < \sum_{j=1}^{\infty} \frac{1}{|z|^{\frac{3j}{2}}} = \frac{1}{|z|^{\frac{3}{2}} - 1}.$$

Substituting the above expression into the preceding inequality, we have that

$$|A(z)| > |a_n||z|^n \left\{ 1 - \frac{K}{(|z|^{\frac{3}{2}} - 1)^{\frac{2}{3}}} \right\} \geq 0$$

holds when

$$\frac{K}{(|z|^{\frac{3}{2}} - 1)^{\frac{2}{3}}} \leq 1.$$

That is, when

$$|z| \geq (1 + K^{\frac{3}{2}})^{\frac{2}{3}} = \left\{ 1 + \left[\sum_{k=0}^{n-1} \left| \frac{a_k}{a_n} \right|^3 \right]^{\frac{1}{2}} \right\}^{\frac{2}{3}} = r.$$

Hence, a bound for the moduli of the zeros is $|z| < r$ and the proof is complete. □

Also some numerical inequalities may be derived using Hölder's inequality.

Theorem 5. *Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 + a_2 + \ldots + a_n = 1$. If $p > 1$, then*

$$a_1 \cdot a_2^{1/p} + a_2 \cdot a_3^{1/p} + \ldots + a_{n-1} \cdot a_n^{1/p} < \left(\frac{1}{4} \right)^{\frac{1}{p}}$$

holds.

Proof. Let $S = a_1 \cdot a_2^{1/p} + a_2 \cdot a_3^{1/p} + \ldots + a_{n-1} \cdot a_n^{1/p}$. Define $q = \frac{p}{p-1}$ so that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Applying Hölder's inequality to the sequences $\left\{(a_i a_{i+1})^{1/p}\right\}$ and $\left\{a_i^{1/q}\right\}$ (for $1 \leq i \leq n-1$) yields

$$\begin{aligned} S &= \sum_{i=1}^{n-1}\left(a_i^{1/p} a_{i+1}^{1/p}\right) a_i^{1/q} \leq \left(\sum_{i=1}^{n-1}\left((a_i a_{i+1})^{\frac{1}{p}}\right)^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n-1}\left(a_i^{1/q}\right)^q\right)^{\frac{1}{q}} \\ &= \left(\sum_{i=1}^{n-1} a_i a_{i+1}\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n-1} a_i\right)^{\frac{1}{q}}. \end{aligned}$$

From

$$\sum_{i=1}^{n-1} a_i < \sum_{i=1}^{n} a_i = 1$$

and

$$\begin{aligned} \sum_{i=1}^{n-1} a_i a_{i+1} &\leq \left(\sum_{i\equiv 0 \pmod 2} a_i\right)\left(\sum_{j\equiv 1 \pmod 2} a_j\right) \\ &\leq \left[\frac{1}{2}\left(\sum_{i\equiv 0 \pmod 2} a_i + \sum_{j\equiv 1 \pmod 2} a_j\right)\right]^2 = \frac{1}{4}, \end{aligned}$$

we obtain

$$S < \left(\frac{1}{4}\right)^{\frac{1}{p}}$$

and the proof is complete. □

An immediate consequence of the preceding result is the following:

Corollary 1. *Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 + a_2 + \ldots + a_n = 1$. Then,*

$$a_1 \cdot a_2^{2/3} + a_2 \cdot a_3^{2/3} + \ldots + a_{n-1} \cdot a_n^{2/3} < \frac{3}{7}.$$

Proof. Set $p = \frac{3}{2}$ and it can be checked that

$$\left(\frac{1}{4}\right)^{\frac{2}{3}} < \frac{3}{7}.$$

□

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[1] Freimer, M. and Mudholkar, G. S. "A class of generalizations of Hölder's inequality". *Inequalities in Statistics and Probability*. Vol. 5. Lecture notes–monograph series. IMS. 1984, pp. 59–67.

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[3] Minkowski, H. *Geometrie der Zahlen*. New York, NY, USA: Chelsea Pub. Co., 1953.

[4] Rogers, L. J. "An extension of a certain theorem in inequalities". *Messenger of Mathematics* 17 (1888), pp. 145–150.

[5] Singh, R. P., Kumar, R., and Tuteja, R. K. "Application of Hölder's inequality in information theory". *Information Sciences* 152 (2003), pp. 145–154.

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Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.

2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to: **José Luis Díaz-Barrero**, Applied Mathematics III, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to: `jose.luis.diaz@upc.edu`.

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

November 30, 2015

Elementary Problems

E–17. *Proposed by Nicolae Papacu, Slobozia, Romania.* Solve the equation $[x \cdot [x]] = 2014$. Here, $[t]$ represents the integer part of $t \in \mathbb{R}$.

E–18. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be the side lengths of a triangle with area $\mathcal{A}$ and inradius r. Characterize all triangles for which

$$r\left(\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}\right) = \mathcal{A}$$

holds.

E–19. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be three different numbers such that

$$\begin{cases} a^3 + 7a + 11 = 0, \\ b^3 + 7b + 11 = 0, \\ c^3 + 7c + 11 = 0. \end{cases}$$

Find the value of $a + b + c$.

E–20. *Proposed by Alberto Espuny-Díaz, CFIS, BarcelonaTech, Barcelona, Spain.* Let p and q be two prime numbers. In which cases is $p + q$ a prime number?

E–21. *Proposed by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain.* Let ABC be a triangle and let D be the point where the median from A meets the circumcircle of the triangle. Prove that the triangles ABD and ACD have the same area.

E–22. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find the number of zeros at the end of the positive integer

$$1 + 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + 2014 \cdot 2014!\,.$$

Easy–Medium Problems

EM–17. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all positive solutions of the following system of equations:

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = x_5^2, \\ x_2 + x_3 + x_4 + x_5 = x_6^2, \\ x_3 + x_4 + x_5 + x_6 = x_7^2, \\ x_4 + x_5 + x_6 + x_7 = x_8^2, \\ x_5 + x_6 + x_7 + x_8 = x_1^2, \\ x_6 + x_7 + x_8 + x_1 = x_2^2, \\ x_7 + x_8 + x_1 + x_2 = x_3^2, \\ x_8 + x_1 + x_2 + x_3 = x_4^2. \end{cases}$$

EM–18. *Proposed by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain.* Let AB be a diameter of a circle Γ. Let us consider M and N two points on Γ different from A and B and such that AN and BM are not parallel. Let H be the point of intersection of AN and BM and let P be the foot of the perpendicular from H to AB. Prove that the lines AM, BN and PH are concurrent.

EM–19. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Show that

$$\left[\prod_{k=1}^{n}\left(k+\frac{1}{n}\right)\right]^{\frac{1}{n+1}} < \left[\prod_{k=1}^{n}\left(k+\frac{1}{n+1}\right)\right]^{\frac{1}{n}}.$$

EM–20. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let H be the orthocenter of an acute triangle ABC with altitudes of length h_a, h_b, and h_c respectively. Compute

$$\frac{HA}{h_a} + \frac{HB}{h_b} + \frac{HC}{h_c}.$$

EM–21. *Proposed by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain.* Find all real solutions of the system of equations

$$\begin{cases} x^4 + y^2 = 2z(x+y) - 4, \\ y^4 + z^2 = 2x(y+z) - 4, \\ z^4 + x^2 = 2y(z+x) - 4. \end{cases}$$

EM–22. *Proposed by Félix Gimeno Gil, CFIS, BarcelonaTech, Barcelona, Spain.* Find all strictly increasing functions $f\colon [0,1] \to [0,1]$ such that for all $x \in [0,1]$ and for $r > 0$

$$f(f(x)) = x^2 \quad \text{and} \quad f(x^r) = f(x)^r\,.$$

Medium–Hard Problems

MH–17. *Proposed by Ander Lamaison Vidarte, CFIS, BarcelonaTech, Barcelona, Spain.* Find all pairs of non-negative integers (a, b) satisfying

$$a^3b^2 = (a^2 + b)^3 .$$

MH–18. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In a triangle ABC, let a, b and c denote the lengths of the sides BC, CA and AB, respectively. If $abc \geq 1$, then show that

$$\sqrt{\frac{\sin A}{a^3 + b^6 + c^6}} + \sqrt{\frac{\sin B}{b^3 + c^6 + a^6}} + \sqrt{\frac{\sin C}{c^3 + a^6 + b^6}} \leq \sqrt[4]{\frac{27}{4}} .$$

MH–19. *Proposed by Andrés Sáez-Schwedt, Universidad de León, León, Spain.* Triangle ABC is rectangle in A, with B, C fixed and A variable. Let M, N be the midpoints of AB, AC, respectively, and construct M', N' such that A is the common midpoint of MM' and NN'. Find the locus of the point of intersection of lines BN' and CM'.

MH–20. *Proposed by Andrés Sáez-Schwedt, Universidad de León, León, Spain.* Let ABC be an acute triangle. The circles having the medians BN and CM as diameters intersect at two points P and Q. The line through the orthocenter of ABC parallel to BC meets AC at R and AB at S. If X and Y are the orthocenters of PQR and PQS, respectively, show that $XY = BC$.

MH–21. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer with prime factorization $n = p_1^{k_1} \dots p_r^{k_r}$. Prove that

$$\frac{n^2}{2} < \prod_{i=1}^{r} \left(p_i^{k_i} - p_i^{k_i - 1}\right)\left(1 + p_i + \dots + p_i^{k_i}\right) < n^2 .$$

MH–22. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Prove that

$$\sum_{n=1}^{\infty}\left(\frac{1}{5}\right)^{\binom{n+1}{2}}$$

is an irrational number.

Advanced Problems

A–17. *Proposed by Ovidiu T. Pop, Satu Mare, Romania, and Mihály Bencze, Braşov, Romania.* Show that there do not exist invertible matrices $A, B \in \mathcal{M}_n(\mathbb{C})$ such that $(A+B)^2 = A^2 + B^2$ and $(A+B)^3 = A^3 + B^3$.

A–18. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $n \geq 1$ be a positive integer, and let $S \subset \{0, 1, 2, \ldots, n\}$ such that

$$|S| \geq \frac{n}{2} + 1.$$

Show that some power of 2 is either an element of S or the sum of two distinct elements of S.

A–19. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let α be a real number. Find all continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \alpha(1+x^2)\left[1 + \int_0^x \frac{f(t)}{1+t^2}\,dt\right].$$

A–20. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain, and José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Calculate

$$\sum_{k=1}^{n} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \frac{3^k}{(i_1+2)(i_2+2)\ldots(i_k+2)}.$$

A–21. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be distinct nonzero complex numbers. Determine

$$\frac{2a^2+b^2+c^2}{\left(1-\frac{a}{b}\right)\left(1-\frac{a}{c}\right)} + \frac{a^2+2b^2+c^2}{\left(1-\frac{b}{a}\right)\left(1-\frac{b}{c}\right)} + \frac{a^2+b^2+2c^2}{\left(1-\frac{c}{a}\right)\left(1-\frac{c}{b}\right)}.$$

A–22. *Proposed by Mihály Bencze, Braşov, Romania.* Consider $f\colon [0,1] \to \mathbb{R}$ a continuous function such that for all $x \in [0,1]$

$$\int_0^x f(t)\,dt \leq \frac{1}{\pi(1+x^2)}\,.$$

Prove that

$$\int_0^1 (f(x)-1)^2\,dx \geq \frac{1}{6}\,.$$

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to: **José Luis Díaz-Barrero**, Applied Mathematics III, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to: jose.luis.diaz@upc.edu.

Some inequalities for the triangle

J. Gibergans-Báguena and J. L. Díaz-Barrero

1 Introduction

Geometric inequalities date from the ancient times and they are as old as geometry itself. The first book of Euclid's Elements contains several theorems on inequalities for the sides and the angles of a triangle, and there is no doubt that the most important of them is Proposition XX, which states that the sum of any two sides is greater than the third.

The *Triangle inequality* states that if a, b, c are the side lengths of a triangle, then

$$a + b > c,\ b + c > a,\ c + a > b \Leftrightarrow |a - b| < c < a + b.$$

Since the beginning, a lot of inequalities on the subject have been published [1]. One of the most well-known is that which relates the inradius and circumradius of a triangle, stated by Euler in 1716. It states that $R \geq 2r$ where R is the circumradius and r is the inradius of any triangle.

The goal of this paper is to use well-known results to derive new inequalities involving the elements of a triangle.

2 Basic results

Hereafter, for ease of reference, we recall some well-known results that will be used later on.

Theorem 1. *Let a, b, c be the lengths of the sides opposite to the angles A, B, C of a triangle ABC with semiperimeter s. Then, it is true that*

$$[ABC] = \frac{abc}{4R} = sr = \sqrt{s(s-a)(s-b)(s-c)},$$

where the notation $[\cdot]$ stands for the area of the figure. This expression is known as Heron's formula.

Proof. From Figure 1 and the sine law, we have

$$[ABC] = \frac{ah}{2} = \frac{ac\sin B}{2} = \frac{abc}{4R}.$$

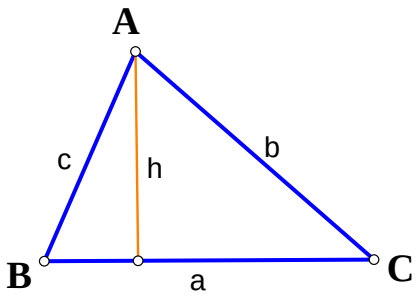

Figure 1: Scheme for Theorem 1.

To see that $[ABC] = sr$, it is enough to compute the area of $\triangle ABC$ as the sum of the areas of the three triangles of bases a, b, c and height r.

Now we will prove that $[ABC] = \sqrt{s(s-a)(s-b)(s-c)}$. Indeed, from Figure 1, we have $h = c\sin B$, and from the law of cosines

we get $b^2 = a^2 + c^2 - 2ac\cos B$. Then,

$$\begin{aligned}
[ABC] &= \frac{1}{2}ac\sin B \\
&= \frac{1}{2}ac\sqrt{1-\cos^2 B} \\
&= \frac{1}{2}ac\sqrt{(1-\cos B)(1+\cos B)} \\
&= \frac{1}{2}ac\sqrt{\left(1-\frac{a^2+c^2-b^2}{2ac}\right)\left(1+\frac{a^2+c^2-b^2}{2ac}\right)} \\
&= \frac{1}{4}\sqrt{(2ac-(a^2+c^2-b^2))(2ac+(a^2+c^2-b^2))} \\
&= \frac{1}{4}\sqrt{(b^2-(a-c)^2)((a+c)^2-b^2)} \\
&= \frac{1}{4}\sqrt{(b+a-c)(b+c-a)(a+c+b)(a+c-b)} \\
&= \sqrt{(s-c)(s-a)s(s-b)}\,.
\end{aligned}$$

□

From the foregoing expressions it immediately follows that

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \quad \text{and} \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}\,.$$

Useful formulae for the inradius and exradii of $\triangle ABC$ are

$$r = \frac{[ABC]}{s}, \quad r_a = \frac{[ABC]}{s-a}, \quad r_b = \frac{[ABC]}{s-b}, \quad r_c = \frac{[ABC]}{s-c}.$$

Let ABC be a triangle with side lengths a, b, c. An important substitution is the following: $a = x+y$, $b = y+z$, and $c = z+x$, where x, y, z are positive real numbers. From the preceding, it immediately follows that $x = \frac{1}{2}(c+a-b)$, $y = \frac{1}{2}(a+b-c)$ and

$z = \dfrac{1}{2}(b + c - a)$. Using this transformation, we have

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} = \frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}},$$
$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \frac{\sqrt{xyz(x+y+z)}}{x+y+z}.$$

From the above expressions, and taking into account the well-known Cesaro's inequality $(x+y)(y+z)(z+x) \geq 8xyz$ $(x > 0,\ y > 0,\ z > 0)$, it immediately follows that

$$\frac{R}{r} \geq 2 \quad \text{(Euler's inequality)}.$$

Theorem 2 (Stewart). *Let D be a point in the side BC of triangle ABC. Then, it is true that*

$$b^2 m + c^2 n = a(p^2 + mn),$$

where $m = BD$, $n = DC$ and $p = AD$, respectively.

Proof. Since $\angle ADB + \angle ADC = \pi$, then $\cos \angle ADB + \cos \angle ADC = 0$. Taking into account the law of cosines, we have

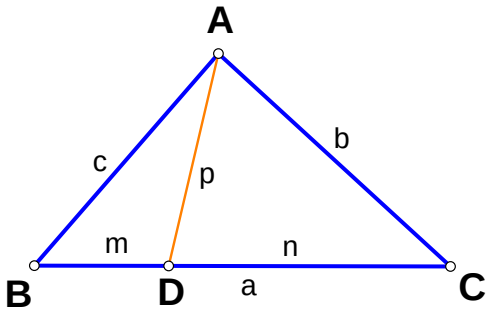

Figure 2: Scheme for Stewart's Theorem

$$\frac{m^2 + p^2 - c^2}{2mp} + \frac{n^2 + p^2 - b^2}{2np} = 0$$

or, equivalently,

$$n(m^2 + p^2 - c^2) + m(n^2 + p^2 - b^2) = 0,$$

from which we get

$$b^2m + c^2n = (m + n)(p^2 + mn) = a(p^2 + mn),$$

as claimed. □

Corollary 1 (length of the median). *In any triangle ABC, the length of the median is given by the expression*

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \quad \textit{(cyclic).}$$

Proof. Applying Stewart's theorem with $m = n = a/2$ and $p = m_a$, we have

$$\frac{b^2a}{2} + \frac{c^2a}{2} = a\left(m_a^2 + \frac{a^2}{4}\right)$$

from which we get $4m_a^2 = 2b^2+2c^2-a^2$, and the statement follows. □

The preceding expressions are also known in the literature as *Apollonius formulae*.

3 Main results

In the following we present some metric inequalities for the triangle.

Theorem 3. *Let a, b, c be the length of the sides of a triangle ABC and let k be a positive integer. Then it holds that*

$$\frac{a}{(b + c)^k} + \frac{b}{(c + a)^k} + \frac{c}{(a + b)^k} \geq \left(\frac{3}{2}\right)^k \left(\frac{1}{a + b + c}\right)^{k-1}.$$

Proof. We consider the function $f\colon [0, 2s] \to \mathbb{R}$ defined by

$$f(x) = \frac{x}{(2s - x)^k},$$

where s is the semiperimeter of $\triangle ABC$. Then,

$$f'(x) = \frac{2s + (k-1)x}{(2s - x)^{k+1}}$$

and

$$f''(x) = \frac{k(4s - x + kx)}{(2s - x)^{k+2}} > 0,$$

and consequently f is convex in $[0, 2s]$.

Applying Jensen's inequality yields

$$\begin{aligned} f(a) + f(b) + f(c) &= \frac{a}{(2s-a)^k} + \frac{b}{(2s-b)^k} + \frac{c}{(2s-c)^k} \\ &\geq 3\, f\left(\frac{a+b+c}{3}\right) = 3\, f\left(\frac{2s}{3}\right) \\ &= \left(\frac{3}{2}\right)^k \left(\frac{1}{2s}\right)^{k-1}. \end{aligned}$$

Putting $2s = a + b + c$ in the last inequality, we get

$$\frac{a}{(b+c)^k} + \frac{b}{(c+a)^k} + \frac{c}{(a+b)^k} \geq \left(\frac{3}{2}\right)^k \left(\frac{1}{a+b+c}\right)^{k-1}.$$

Equality holds when $a = b = c$, that is, when $\triangle ABC$ is equilateral. □

Theorem 4. *Let a, b, c be the length of the sides of a triangle ABC with heights h_a, h_b and h_c, respectively. Then it holds that*

$$h_a^2 + h_b^2 + h_c^2 \geq \frac{r(a+b+c)^2}{2R}.$$

Proof. WLOG we can assume that $a \geq b \geq c$. On account of rearrangement inequality applied to the positive numbers $\frac{1}{a} \leq \frac{1}{b} \leq \frac{1}{c}$, we have

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{a} \cdot \frac{1}{b} + \frac{1}{b} \cdot \frac{1}{c} + \frac{1}{c} \cdot \frac{1}{a} = \frac{a+b+c}{abc}.$$

Using Theorem 1, we may write

$$h_a = \frac{2rs}{a} \text{ (cyclic)} \Longrightarrow \frac{1}{a^2} = \frac{h_a^2}{4r^2s^2} \text{ (cyclic)}$$
$$\Longrightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{h_a^2 + h_b^2 + h_c^2}{4r^2s^2}.$$

On the other hand, also from Theorem 1,

$$[ABC] = \frac{abc}{4R} = \frac{(a+b+c)r}{2},$$

so we can write $a + b + c = \dfrac{2\,[ABC]}{r}$ and $abc = 4R\,[ABC]$, and we obtain

$$\frac{a+b+c}{abc} = \frac{1}{2rR}.$$

Finally, combining the above expressions, we get

$$\frac{1}{2rR} \leq \frac{h_a^2 + h_b^2 + h_c^2}{4r^2s^2},$$

from which it follows that

$$h_a^2 + h_b^2 + h_c^2 \geq \frac{r(a+b+c)^2}{2R}.$$

Equality holds when $a = b = c$, that is, when $\triangle ABC$ is equilateral. This completes the proof. □

Theorem 5. *Let d_a, d_b, d_c be the distances from the vertices of triangle ABC to its incenter. Then, we can prove that*

$$ad_a^2 + bd_b^2 + cd_c^2 \geq 8(s-a)(s-b)(s-c),$$

where s is the semiperimeter of $\triangle ABC$.

Proof. Let A, B, C be the angles of $\triangle ABC$ opposite to the sides a, b, c and let r be its inradius, as usual. Then,

$$\begin{aligned} ad_a^2 + bd_b^2 + cd_c^2 &= r^2\left[\frac{a}{\left(\sin\frac{A}{2}\right)^2} + \frac{b}{\left(\sin\frac{B}{2}\right)^2} + \frac{c}{\left(\sin\frac{C}{2}\right)^2}\right] \\ &= 2r^2\left(\frac{a}{\sin A}\cot\frac{A}{2} + \frac{b}{\sin B}\cot\frac{B}{2} + \frac{c}{\sin C}\cot\frac{C}{2}\right) \\ &= \frac{2ar^2}{\sin A}\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right) \end{aligned}$$

since by the law of sines we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Taking into account that $r\cot\frac{A}{2} = s - a$, $r\cot\frac{B}{2} = s - b$ and $r\cot\frac{C}{2} = s - c$, we have

$$s = \frac{a+b+c}{2} = r\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right)$$

and

$$ad_a^2 + bd_b^2 + cd_c^2 = \frac{2ars}{\sin A} = \frac{2a}{\sin A}[ABC] = \frac{2a}{\sin A}\left(\frac{bc}{2}\sin A\right) = abc.$$

Now, applying AM-GM inequality yields

$$\sqrt{(b+c-a)(a+c-b)} \leq \frac{b+c-a+a+c-b}{2} = c.$$

Likewise, we get

$$\begin{aligned} \sqrt{(b+c-a)(b+a-c)} &\leq b, \\ \sqrt{(b+a-c)(a+c-b)} &\leq a. \end{aligned}$$

Multiplying up the preceding inequalities, we obtain

$$8(s-a)(s-b)(s-c) = (b+c-a)(a+c-b)(a+b-c) \leq abc.$$

Notice that equality holds when $\triangle ABC$ is equilateral, and we are done. □

Theorem 6. *Let h_a, h_b, h_c be the heights of a triangle ABC and m_a, m_b, m_c the medians. It is true that*

$$\frac{h_a + h_b + h_c}{m_a^2 + m_b^2 + m_c^2} \leq \frac{2}{3R}.$$

Proof. Applying the well-known inequality $ab + bc + ac \leq a^2 + b^2 + c^2$ to the side lengths a, b, c of a triangle ABC, we have

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{a^2 + b^2 + c^2}{abc}.$$

Using the following formulae for the heights and medians of a triangle ABC,

$$h_a = \frac{r(a+b+c)}{a} \quad \text{(cyclic)}, \qquad m_a^2 = \frac{2(b^2+c^2) - a^2}{4} \quad \text{(cyclic)},$$

we have

$$\frac{h_a + h_b + h_c}{r(a+b+c)} \leq \frac{4(m_a^2 + m_b^2 + m_c^2)}{3abc}$$

or, equivalently,

$$\frac{h_a + h_b + h_c}{m_a^2 + m_b^2 + m_c^2} \leq \frac{4r(a+b+c)}{3abc}.$$

From Theorem 1, we have $abc = 2Rr(a+b+c)$ and

$$\frac{h_a + h_b + h_c}{m_a^2 + m_b^2 + m_c^2} \leq \frac{2}{3R}.$$

Equality holds when $\triangle ABC$ is equilateral, and we are done. □

Theorem 7. *Let D be a point in the side BC of $\triangle ABC$. Then, it holds that*

$$\frac{1}{p}\left(\frac{b^2}{n} + \frac{c^2}{m}\right) \geq 4,$$

where $m = BD$, $n = DC$, and $p = AD$, respectively.

Proof. Applying Theorem 2 to triangle ABC yields $b^2m + c^2n = a(p^2 + mn)$. On account of AM-GM-HM inequalities, we have

$$\frac{b^2}{a}\,m + \frac{c^2}{a}\,n = p^2 + mn \geq 2p\sqrt{mn} \geq 2p\left(\frac{2mn}{m+n}\right) = \frac{4pmn}{a},$$

and after simplification, we obtain

$$\frac{b^2}{n} + \frac{c^2}{m} \geq 4p,$$

from which the statement follows. □

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[1] Bottema, O. et al. *Geometric Inequalities*. Groningen, Netherlands: Walters-Noordhoff Publishing, 1968.

[2] Díaz-Barrero, J. L. and Bencze, M. *Mathematical Contest Notebook*. Iaşi, Romania: STUDIS Publishing House, 2015.

[3] Kazarinoff, N. D. *Geometric Inequalities*. USA: Yale University, 1961.

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Solutions

No problem is ever permanently closed. We will be very pleased to consider for publication new solutions or comments on the past problems.

Please, send submittals to: **José Luis Díaz-Barrero**, Applied Mathematics III, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to: jose.luis.diaz@upc.edu.

Elementary Problems

E–11. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* What is the value of

$$\sqrt[3]{45+29\sqrt{2}}+\sqrt[3]{45-29\sqrt{2}}$$

in its simplest form?

Solution 1 by Alberto Espuny-Díaz, CFIS, BarcelonaTech, Barcelona, Spain. We start by calculating the cube of the given num-

ber. We have that

$$\begin{aligned}
&\left(\sqrt[3]{45+29\sqrt{2}}+\sqrt[3]{45-29\sqrt{2}}\right)^3\\
&=45+29\sqrt{2}+3\sqrt[3]{\left(45+29\sqrt{2}\right)^2}\sqrt[3]{45-29\sqrt{2}}\\
&\quad+3\sqrt[3]{45+29\sqrt{2}}\sqrt[3]{\left(45-29\sqrt{2}\right)^2}+45-29\sqrt{2}\\
&=90+3\sqrt[3]{(45^2-2\cdot 29^2)\left(45+29\sqrt{2}\right)}\\
&\quad+3\sqrt[3]{(45^2-2\cdot 29^2)\left(45-29\sqrt{2}\right)}\\
&=90+3\sqrt[3]{45^2-2\cdot 29^2}\left(\sqrt[3]{45+29\sqrt{2}}+\sqrt[3]{45-29\sqrt{2}}\right).
\end{aligned}$$

We observe that the number we are trying to simplify appears again here. If we call this number x, then we have that it satisfies $x^3 = 90 + 21x$, so it is a root of the polynomial $x^3 - 21x - 90$. Using Ruffini's rule for the divisors of 90 we find that 6 is a root, so $x^3 - 21x - 90 = (x^2 + 6x + 15)(x - 6)$ and we can see that there are no more real roots (since the discriminant is negative). The solution for the problem, then, is 6.

Solution 2 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. We are going to compute explicitly the values of $\sqrt[3]{45+29\sqrt{2}}$ and $\sqrt[3]{45-29\sqrt{2}}$.

Suppose that $\sqrt[3]{45+29\sqrt{2}}$ is of the form $a+b\sqrt{2}$, with $a, b \in \mathbb{Q}$. In that case,

$$45+29\sqrt{2}=\left(a+b\sqrt{2}\right)^3=\left(a^3+3a2b^2\right)+\left(3a^2b+2b^3\right)\sqrt{2}$$

and we need to solve

$$\begin{cases} 45 = a\left(a^2+6b^2\right), \\ 29 = b\left(3a^2+2b^2\right). \end{cases}$$

We assume, moreover, that $a, b \in \mathbb{Z}$. As 29 is a prime number, necessarily $b = \pm 1$ and we find by inspection that the only solution is $(a, b) = (3, 1)$.

We have just proved that $\sqrt[3]{45+29\sqrt{2}} = 3+\sqrt{2}$. Similarly, we can obtain that $\sqrt[3]{45-29\sqrt{2}} = 3-\sqrt{2}$. Therefore,

$$\sqrt[3]{45+29\sqrt{2}} + \sqrt[3]{45-29\sqrt{2}} = 3+\sqrt{2}+3-\sqrt{2} = 6\,.$$

We observe, though, that a real number of the form $\sqrt[3]{x+y\sqrt{2}}$ with $x, y \in \mathbb{Z}$ cannot be expressed as $a+b\sqrt{2}$ with $a, b \in \mathbb{Q}$ in general. It worked only for these particular numbers.

Also solved by Guillermo Girona San Miguel, CFIS, BarcelonaTech, Barcelona, Spain; Ander Lamaison Vidarte, CFIS, BarcelonaTech, Barcelona, Spain, and the proposer.

E–12. *Proposed by Félix Gimeno Gil, CFIS, BarcelonaTech, Barcelona, Spain.* Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(a \cdot b) = f(a)f(b)$ and for all positive integer n we have $gcd(n, f(n)) = 1$.

Solution by the proposer. One such function is $f(n) = 1$ for all n. We will now prove that it is the only one. To do it, we will argue by contradiction. Indeed, suppose there exists another function f distinct from the constant function $f(n) = 1$ that satisfies the problem conditions. Let us calculate $f(nf(n))$. Since n and $f(n)$ are coprime and f is multiplicative ($f(a \cdot b) = f(a)f(b)$), then $f(nf(n)) = f(n)f(f(n))$, which needs to be coprime with $nf(n)$, but they both share the factor $f(n)$, so $f(n)$ must be 1 for all n, contradiction. This completes the proof.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain; Alberto Espuny-Díaz, CFIS, BarcelonaTech, Barcelona, Spain; Guillermo Girona San Miguel, CFIS, BarcelonaTech, Barcelona, Spain; Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain; Ander Lamaison Vidarte, CFIS, BarcelonaTech, Barcelona, Spain, and Isaac Sánchez Barrera, CFIS, BarcelonaTech, Barcelona, Spain.

E–13. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* The areas of the three distinct faces of an ortohedron are 182, 195 and 210, respectively. Find its volume.

Solution by the proposer. In general, if the sides of an ortohedron have lengths a, b, c, then the areas of its nonparallel faces are ab, bc, and ca, and its volume is

$$V = abc = \sqrt{(ab)\,(bc)\,(ca)}.$$

In our particular case

$$V = \sqrt{182 \cdot 195 \cdot 210} = 2730.$$

Also solved by Alberto Espuny-Díaz, CFIS, BarcelonaTech, Barcelona, Spain; Eduardo Espuny-Díaz, Universitat Pompeu Fabra, Barcelona, Spain; Alberto Espuny-Miró, Universidad de Murcia, Murcia, Spain; José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain; Guillermo Girona San Miguel, CFIS, BarcelonaTech, Barcelona, Spain; Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain; Ander Lamaison Vidarte, CFIS, BarcelonaTech, Barcelona, Spain; Marta Pérez-Larrañaga, CFIS, BarcelonaTech, Barcelona, Spain, and Isaac Sánchez Barrera, CFIS, BarcelonaTech, Barcelona, Spain.

E–14. *Proposed by José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.* Find the values of the positive integers n for which $n^2 - 27n + 181$ is a perfect square.

Solution 1 by Alberto Espuny-Díaz, CFIS, BarcelonaTech, Barcelona, Spain. We want to find all the values of n such that $n^2 - 27n + 181 = m^2$, where n, $m \in \mathbb{N}$. Using the solution for second order polynomials we may write

$$n = \frac{27 \pm \sqrt{27^2 - 4(181 - m^2)}}{2}.$$

We want n to be an integer, so we need $\sqrt{27^2 - 4(181 - m^2)}$ to be an odd integer, or $27^2 - 4(181 - m^2) = 4m^2 + 5$ to be an odd

perfect square. This is the same as writing $(2m)^2 + 5 = a^2$ for some a, so we have two perfect squares separated by 5 units. We know that the distance between perfect squares increases for every number, and that the only ones separated by five units are 4 and 9. Hence, $(2m)^2 = 4$, so $m = 1$. Substituting this we obtain the values $n = 12$ and $n = 15$, and we are done.

Solution 2 by Ander Lamaison Vidarte, CFIS, BarcelonaTech, Barcelona, Spain. If $n^2 - 27n + 181$ is a perfect square (say m^2 for a non-negative integer m), then multiplying by 4 we obtain $(2m)^2 = 4n^2 - 108n + 724 = (2n-27)^2 - 5$. Rearranging the terms we find $5 = (2n-27)^2 - (2m)^2 = (2n-27+2m)(2n-27-2m)$. This means that 5 is a product of two integers, the first greater than the second. Since 5 is a prime number, this can only be done as $5 \cdot 1$ or $(-1) \cdot (-5)$. Thus we produce two systems of equations

$$\begin{cases} 2n-27+2m = 5 \\ 2n-27-2m = 1 \end{cases} \quad \text{and} \quad \begin{cases} 2n-27+2m = -1 \\ 2n-27-2m = -5 \end{cases}$$

with solutions $(15, 1)$ and $(12, 1)$, respectively. We conclude that the only solutions are $n = 12$ and $n = 15$.

Solution 3 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. First of all, we observe that, for any $k \in \mathbb{N}$,

$$\begin{cases} k \equiv 0 \pmod 3 & \implies k^2 \equiv 0 \pmod 3, \\ k \not\equiv 0 \pmod 3 & \implies k^2 \equiv 1 \pmod 3. \end{cases}$$

As $n^2 - 27n + 181 \equiv n^2 + 1 \pmod 3$, this expression can be a perfect square only if n is a multiple of 3. Setting $n = 3m$, with $m \in \mathbb{N}$, we want to find the values of m such that $a_m = 9m^2 - 81m + 181$ is a perfect square. By inspection, we obtain $a_1 = 109$, $a_2 = 55$, $a_3 = 19$, $a_4 = 1$ and $a_5 = 1$. Therefore, $m = 4$ and $m = 5$ yield the only solutions with $m \leq 5$.

Furthermore, if $m \geq 6$, we can express

$$a_m = (3m-13)^2 - 3(m-4) < (3m-13)^2,$$
$$a_m = 9m^2 - 81m + 181 > 9m^2 - 84m + 196 = (3m-14)^2,$$

and, thus, a_m lies between two consecutive perfect squares.

In conclusion, the only solutions are $n = 12$ and $n = 15$.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain, and the proposer.

E–15. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be three odd positive integers. Show that the equation $ax^2 + bx + c = 0$ does not have rational roots.

Solution 1 by the proposer. Suppose that $x = \dfrac{m}{n}$ with $(m, n) = 1$ is a root of the equation. Then

$$a\left(\frac{m}{n}\right)^2 + b\left(\frac{m}{n}\right) + c = 0$$

or

$$am^2 + bmn + cn^2 = 0.$$

Now we distinguish the following cases:

- If m, n are odd, then odd + odd + odd = 0 (contradiction).
- If m is odd and n even, then odd + even + even = 0 (contradiction).
- If m even and n odd, then even + even + odd = 0 (contradiction).
- If m, n are even, then $(m, n) \geq 2$ (contradiction).

Solution 2 by Ander Lamaison Vidarte, CFIS, BarcelonaTech, Barcelona, Spain. The two roots of the equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Hence, if we show that $b^2 - 4ac$ is never the square of a rational number, then we will have proved the statement. $b^2 - 4ac$ is an integer, so if it is the square of a rational number then it is the square of an integer.

Taking modulo 8, we can check that $b^2 \equiv 1 \pmod 8$ (since b is odd) and $4ac \equiv 4 \pmod 8$ (since both a and c are odd), so $b^2 - 4ac \equiv 5 \pmod 8$. But the squares modulo 8 are 0, 1 and 4. We conclude that $b^2 - 4ac$ is not the square of an integer, and this proves the statement.

Solution 3 by Alberto Espuny-Díaz, CFIS, BarcelonaTech, Barcelona, Spain. Using the solution for degree two polynomials, we know that a root

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

is rational if, and only if, $\sqrt{b^2 - 4ac}$ is rational, that is, if its discriminant is a perfect square. Let us assume that it is a perfect square.

Since all three numbers we are considering are odd, we can say that $b = 2m + 1$ for some m, and then we have that the discriminant $(2m+1)^2 - 4ac$ is a perfect square, say d^2. We observe that the discriminant is odd, and so must be d. Let us say $d = 2n + 1$. We can now rewrite the previous as

$$\begin{aligned}(2m+1)^2 - (2n+1)^2 = 4ac &\implies 4m^2 - 4n^2 + 4m - 4n = 4ac \\ &\implies m^2 - n^2 + m - n = ac.\end{aligned}$$

However, we now observe that the RHS of this last expression is odd, while the LHS is even independently of the parity of m and n, and this is a contradiction. The conclusion is that the discriminant is not a perfect square and the roots of the polynomial are not rational.

Also solved by José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain, and Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain.

E–16. *Proposed by Alberto Espuny-Díaz and Ander Lamaison Vidarte, CFIS, BarcelonaTech, Barcelona, Spain.* Prove that the 2015-gon inscribed in a circle that has the maximum surface is the regular one.

Solution by the proposers. We are going to proceed by *reductio ad absurdum*. Let us assume that we have the inscribed 2015-gon of maximum area, and that it is not regular. Since it is not regular, there is at least one vertex which is not equidistant with respect to its two neighbors. Let us fix every vertex except this one. The area of the 2015-gon, which is maximal, can be computed as the sum of the area defined by the 2014 fixed points and the area of the triangle defined by our vertex and its neighbors.

Now let us consider this triangle. Let us say that the distance between the two neighbors is the base, and the area of the triangle is computed as half its base by its height. However, since the three points are on a circumference, the maximum height is achieved when the triangle is isosceles, that is, when our vertex is equidistant to its two neighbors. In this way, we can construct a 2015-gon with greater surface, which contradicts the hypothesis. Hence, the polygon must be regular.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain; José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain, and Francesc Gispert Sánchez, BarcelonaTech, Barcelona, Spain.

Easy–Medium Problems

EM–11. *Proposed by Alberto Espuny-Díaz, CFIS, BarcelonaTech, Barcelona, Spain.* How many palindrome integers with an even number of digits are also prime numbers?

Solution 1 by the proposer. We claim that there is only one: 11. Indeed, consider a palindrome with an even number of digits, say $2n$. It can be written as $a_1a_2\ldots a_na_{n+1}\ldots a_{2n-1}a_{2n}$, where $a_i = a_{2n-i+1}$ for all i between 1 and n. Suppose that it is different from 11. Now, considering the rule for divisibility by 11, we know that it is a multiple of 11 if

$$\sum_{i \text{ odd}} a_i - \sum_{i \text{ even}} a_i = 11\,k,$$

where $k \in \mathbb{Z}$. Because of the construction of our palindrome, we have that

$$\begin{aligned}\sum_{i \text{ odd}} a_i &= a_1 + a_3 + a_5 + \ldots + a_{2n-3} + a_{2n-1}\\ &= a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n\\ &= a_2 + a_4 + a_6 + \ldots + a_{2n-2} + a_{2n} = \sum_{i \text{ even}} a_i\,,\end{aligned}$$

and this means that our number is a multiple of 11 and, hence, not a prime number.

Solution 2 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. The only such number is 11.

In order to prove it, we observe that for any $k \in \mathbb{N} \cup \{0\}$

$$10^{2k+1} \equiv -1 \pmod{11} \qquad \text{and} \qquad 10^{2k} \equiv 1 \pmod{11}$$

(this can be shown by induction on k).

A palindrome number with an even number of digits has a decimal expression of the form $a_1a_2\ldots a_na_n\ldots a_2a_1$. That is to say, its

value is

$$\sum_{k=1}^{n} a_k 10^{k-1} + \sum_{k=1}^{n} a_k 10^{2n-k} = \sum_{k=1}^{n} a_k \left(10^{k-1} + 10^{2n-k}\right).$$

Moreover, $k-1$ and $2n-k$ have different parities for every value of k. Therefore,

$$\sum_{k=1}^{n} a_k \left(10^{k-1} + 10^{2n-k}\right) \equiv \sum_{k=1}^{n} a_k((\pm 1) + (\mp 1)) = 0 \pmod{11},$$

which means that $a_1 a_2 \ldots a_n a_n \ldots a_2 a_1$ is a multiple of 11.

Solution 3 by Isaac Sánchez Barrera, CFIS, BarcelonaTech, Barcelona, Spain. There is only one number with these properties: 11. In fact, we are going to prove by induction that every palindrome number with an even number of digits is a multiple of 11.

To start off, see that any palindrome number with $2n$ digits can be written as

$$a_1 a_2 \ldots a_n a_n a_{n-1} \ldots a_1 = \sum_{i=1}^{n} \left(10^{2\,(n-i)+1} + 1\right) 10^{i-1}\, a_i,$$

and we now only need to show that the terms of the sum are multiples of 11. In particular, numbers of the form $10^{2k+1} + 1$ for any non-negative integer k are multiples of 11.

The base case, when $k = 0$, is trivial since 11 is a multiple of itself.

Let's now take a look at the general case, when $k > 0$,

$$\begin{aligned} 10^{2k+1} + 1 &= 10^{2k-1} \cdot 100 + 1 \\ &= \left(10^{2k-1} + 1 - 1\right) 100 + 1 \\ &= \left(10^{2k-1} + 1\right) \cdot 100 - 100 + 1 \\ &= \left(10^{2k-1} + 1\right) \cdot 100 - 99 \end{aligned}$$

and since by induction $10^{2k-1} + 1$ is a multiple of 11, and $99 = 9 \cdot 11$, we conclude $10^{2k+1} + 1$ is a multiple of 11, too.

Also solved by José Luis Díaz-Barrero, Barcelona, Spain, and Sergi Rocamora-Ardèvol, BarcelonaTech, Barcelona, Spain.

EM–12. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let x, y, z be three non unitary real numbers such that $x+y+z+1=0$ and $x^2+y^2+z^2-1=0$. Show that

$$\frac{xy}{x+y}+\frac{yz}{y+z}+\frac{zx}{z+x}$$

is an integer and determine its value.

Solution by the proposer. Squaring both sides of the expression $x+y+z=-1$ yields

$$x^2+y^2+z^2+2(xy+yz+zx)=1.$$

Taking into account that $x^2+y^2+z^2=1$, we have $xy+yz+zx=0$. From the preceding equality it immediately follows that

$$x=-\frac{yz}{y+z},\quad y=-\frac{zx}{z+x},\quad z=-\frac{xy}{x+y}.$$

Since $|x|$, $|y|$, $|z|$ are distinct from 1, then $x+y$, $y+z$ and $z+x$ are nonzero and the expression to be computed is well defined. Furthermore, we have

$$x+y+z+1=-\left(\frac{yz}{y+z}+\frac{zx}{z+x}+\frac{xy}{x+y}\right)+1=0,$$

from which we obtain

$$\frac{xy}{x+y}+\frac{yz}{y+z}+\frac{zx}{z+x}=1,$$

and we are done.

Also solved by Alberto Espuny-Díaz, CFIS, BarcelonaTech, Barcelona, Spain; José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain; Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain, and Ander Lamaison Vidarte, CFIS, BarcelonaTech, Barcelona, Spain.

EM–13. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Show that in any triangle ABC the following holds:

$$\frac{1}{2}\left(\frac{\sin A+\sin B+\sin C}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}\right)\leq \mathcal{A},$$

where $\mathcal{A}$ represents the area of $\triangle ABC$.

Solution 1 by the proposer. First, we write the given inequality in the more convenient form

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\geq \frac{1}{2\mathcal{A}}(\sin A+\sin B+\sin C).$$

Multiplying both sides of the above inequality by abc yields

$$\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}\geq \frac{abc}{2\mathcal{A}}(\sin A+\sin B+\sin C).$$

From the law of sines, namely

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\frac{a+b+c}{\sin A+\sin B+\sin C}=2R,$$

and the fact that $2R=\dfrac{abc}{2\mathcal{A}}$, we get

$$\frac{abc}{2\mathcal{A}}(\sin A+\sin B+\sin C)=a+b+c.$$

Then the preceding inequality becomes

$$\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}\geq a+b+c.$$

The last inequality is proven by putting $x=\sqrt{\dfrac{bc}{a}}$, $y=\sqrt{\dfrac{ac}{b}}$, $z=\sqrt{\dfrac{ab}{c}}$ in the well-known inequality $x^2+y^2+z^2\geq xy+yz+zx$. Equality holds when $\triangle ABC$ is equilateral, and we are done.

Solution 2 by Scott H. Brown, Auburn University, Montgomery AL, USA (slightly modified by the editors). From the law of sines we find $\sin A + \sin B + \sin C = \frac{s}{R}$, and it is well-known that $\mathcal{A} = rs$, where R is the circumradius, r is the inradius and s is the semiperimeter, as usual. Now rewrite the inequality claimed as follows,

$$\frac{1}{2\mathcal{A}} \sum_{\text{cyclic}} \sin A \leq \sum_{\text{cyclic}} \frac{1}{a^2}\,.$$

Substitute the preceding equalities into the last inequality and simplifying we obtain

$$\frac{1}{2rR} \leq \sum_{\text{cyclic}} \frac{1}{a^2}\,.$$

From

$$\mathcal{A} = rs = \frac{abc}{4R}\,,$$

the last inequality becomes

$$\frac{a+b+c}{abc} = \frac{1}{bc} + \frac{1}{ca} + \frac{1}{ab} \leq \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

which is true by the rearrangement inequality.

Also solved by José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain; Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain, and Ander Lamaison Vidarte, CFIS, BarcelonaTech, Barcelona, Spain.

EM–14. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let α, β, γ be the roots of the polynomial equation $x^3 - 7x^2 + 5x + 3 = 0$. Find the value of $\alpha^4 + \beta^4 + \gamma^4$.

Solution by José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain. On account of Cardano–Viète formulae, we have $\alpha + \beta + \gamma = 7$, $\alpha\beta + \beta\gamma + \gamma\alpha = 5$, and $\alpha\beta\gamma = -3$, respectively. From the preceding and $(\alpha+\beta+\gamma)^2 = \alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha)$ it immediately follows that $\alpha^2 + \beta^2 + \gamma^2 = 39$.

Since α, β, γ are the roots of the equation $x^3 - 7x^2 + 5x + 3 = 0$, then

$$\left.\begin{array}{c} \alpha^3 - 7\alpha^2 + 5\alpha + 3 = 0 \\ \beta^3 - 7\beta^2 + 5\beta + 3 = 0 \\ \gamma^3 - 7\gamma^2 + 5\gamma + 3 = 0 \end{array}\right\} \tag{1}$$

Adding up the preceding equations and taking into account the above results, we obtain that $\alpha^3 + \beta^3 + \gamma^3 = 229$. Multiplying by α the first equation in (1), by β the second and by γ the third, we obtain

$$\left.\begin{array}{c} \alpha^4 - 7\alpha^3 + 5\alpha^2 + 3\alpha = 0 \\ \beta^4 - 7\beta^3 + 5\beta^2 + 3\beta = 0 \\ \gamma^4 - 7\gamma^3 + 5\gamma^2 + 3\gamma = 0 \end{array}\right\}$$

and adding up the above equations and taking into account the preceding, we get

$$\alpha^4 + \beta^4 + \gamma^4 = 1387,$$

and we are done.

Also solved by Francesc Gispert Sánchez, CFIS, Barcelona-Tech, Barcelona, Spain, and the proposer.

EM–15. *Proposed by Ander Lamaison Vidarte, CFIS, Barcelona-Tech, Barcelona, Spain.* Let Γ be a circumference of radius r, and A, B and C be three points inside Γ that lie on a circumference Ψ, with radius R. If $R > r$, prove that the triangle ABC has an obtuse angle.

Solution 1 by the proposer. Ψ is larger than Γ, so it cannot be entirely inside of Γ. But A lies on Ψ and is inside Γ. Hence Γ and Ψ intersect, and they do so in two points, P and Q.

The points A, B and C divide Ψ into three arcs AB, BC and CA. Since all three points are inside Γ, it follows that both intersection points are on the same arc. Without loss of generality, we can say that this arc is CA.

The arc PQ of Ψ containted inside Γ is less than half the circumference. Indeed, if it were not, then it would include two points S,

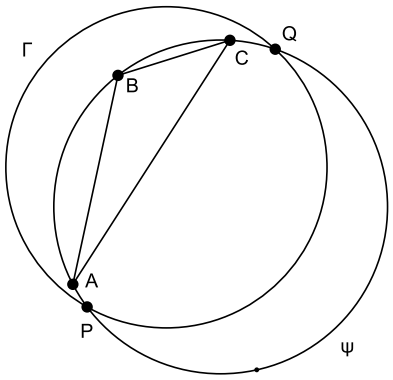

Figure 1: Scheme for the solution of Problem EM–15.

T of Ψ forming a diameter, and $2R = ST \leq 2r < 2R$, which is a contradiction. Hence, the arc CA is more than half the circumference and thus $\angle ABC > 90°$.

Solution 2 by the proposer. If ABC has a right angle, wlog in B, then $2R = AC < 2r$, which is impossible. Therefore, we need to prove that all three angles cannot be acute. Assume that they are, and we will find a contradiction. Let O be the circumcenter of ABC. Since triangle ABC is acute, O lies inside of ABC.

Extend OA, OB and OC past A, B and C, respectively, until they intersect Γ at A', B' and C'. It is clear that $OA' > OA = R$, and similarly $OB' > R$ and $OC' > R$. Therefore,

$$\begin{aligned} AB^2 &= OA^2 + OB^2 - 2OA \cdot OB \cos\angle AOB \\ &= R^2(2 - 2\cos\angle AOB) \\ &< OA' \cdot OB'(2 - 2\cos\angle A'OB') \\ &= 2OA' \cdot OB' - 2OA' \cdot OB' \cos\angle A'OB' \\ &\leq OA'^2 + OB'^2 - 2OA' \cdot OB' \cos\angle A'OB' \\ &= A'B'^2. \end{aligned}$$

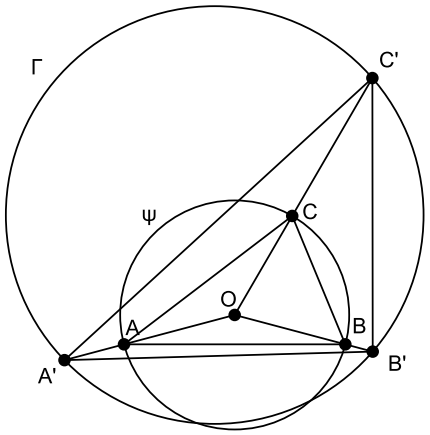

Figure 2: Scheme for the solution of Problem EM–15.

Similarly, $B'C' > BC$ and $C'A' > CA$.

Using the law of sines, $\sin\angle B'C'A' = \frac{A'B'}{2r} > \frac{AB}{2R} = \sin\angle BCA$. Since $\angle BCA$ is acute, this can only be the case if $\angle B'C'A' > \angle BCA$. Analogously we have $\angle A'B'C' > \angle ABC$ and $\angle C'AB' > CAB$. Adding all three we get a contradiction, since we obtain $180^\circ = \angle A'B'C' + \angle B'C'A' + \angle C'A'B' > \angle ABC + \angle BCA + \angle CAB = 180^\circ$.

Also solved by Francesc Gispert Sánchez, CFIS, Barcelona-Tech, Barcelona, Spain.

EM–16. *Proposed by Andrés Sáez-Schwedt, Universidad de León, León, Spain.* Find the smallest positive integer n such that 3^{2014} divides $(2014)^{(3^n)} - 1$.

Solution 1 by the proposer. If we define the sequence $b_{n+1} = b_n^3$, with $b_0 = 2014$, then $(2014)^{(3^n)}$ is clearly b_n, so that one has to investigate how many factors 3 the number $b_n - 1$ has, and see when it reaches 2014.

Note that $b_0 - 1 = 2013 = 3 \times 671$ has exactly one factor 3. If $b_n - 1 = a3^t$, for some $t \geq 1$ and 3 not dividing a, then

$$b_{n+1} - 1 = (b_n - 1)(b_n^2 + b_n + 1) = a3^t(b_n^2 + b_n + 1).$$

Looking at the term $B = b_n^2 + b_n + 1$ modulo 9, one has to consider only the cases where b_n is 1, 4 or 7 (mod 9), since $b_n \equiv 1 \pmod 3$ for all n. In all such cases it results that B is multiple of 3 but not of 9, so it adds only one extra factor 3. Consequently, the number $b_n - 1$ has exactly $n + 1$ factors 3, and hence the number of $3's$ reaches 2014 for the first time when $n = 2013$, which is the answer to the problem.

Solution 2 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. We are going to prove by induction that, for all $n \geq 0$, $2014^{3^n} = 3^{n+1}a_n + 1$ with $3 \nmid a_n$.

First of all, we observe that $2014 = 3a_0 + 1$, where $a_0 = 671$, and that $3 \nmid a_0$.

Now, for $n \geq 1$, we suppose that $2014^{3^{n-1}} = 3^n a_{n-1} + 1$ with $3 \nmid a_{n-1}$. Then, by the binomial theorem,

$$2014^{3^n} = (3^n a_{n-1} + 1)^3 = 3^{3n}a_{n-1}^3 + 3^{2n+1}a_{n-1}^2 + 3^{n+1}a_{n-1} + 1$$

and we can take $a_n = 3^{2n-1}a_{n-1}^3 + 3^{2n}a_{n-1}^2 + a_{n-1}$. Indeed, as $3 \mid 3^{2n-1}a_{n-1}^3$ and $3 \mid 3^{2n}a_{n-1}^2$ but $3 \nmid a_{n-1}$, necessarily $3 \nmid a_n$.

Therefore, the smallest positive integer which satisfies the requirement in the statement is $n = 2013$.

Also solved by Ander Lamaison Vidarte, CFIS, BarcelonaTech, Barcelona, Spain.

Medium–Hard Problems

MH–11. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $n \geq 2$ be a positive integer. Find all real solutions of the following system of equations:

$$\begin{cases} 3^{x_2} = 1 + \dfrac{1}{n}\left(3^{x_1} + 3^{-x_1} - 2\right), \\ 3^{x_3} = 1 + \dfrac{1}{n}\left(3^{x_2} + 3^{-x_2} - 2\right), \\ \qquad \vdots \\ 3^{x_n} = 1 + \dfrac{1}{n}\left(3^{x_{n-1}} + 3^{-x_{n-1}} - 2\right), \\ 3^{x_1} = 1 + \dfrac{1}{n}\left(3^{x_n} + 3^{-x_n} - 2\right). \end{cases}$$

Solution 1 by the proposer. Putting $3^{x_i} = a_i$, $1 \leq i \leq n$, the given system can be written as

$$\begin{cases} a_2 = 1 + \dfrac{1}{n}\left(a_1 + \dfrac{1}{a_1} - 2\right), \\ a_3 = 1 + \dfrac{1}{n}\left(a_2 + \dfrac{1}{a_2} - 2\right), \\ \qquad \vdots \\ a_n = 1 + \dfrac{1}{n}\left(a_{n-1} + \dfrac{1}{a_{n-1}} - 2\right), \\ a_1 = 1 + \dfrac{1}{n}\left(a_n + \dfrac{1}{a_n} - 2\right), \end{cases}$$

$$\iff \begin{cases} a_2 = \dfrac{1}{n}\left(a_1 + n - 2 + \dfrac{1}{a_1}\right), \\ a_3 = \dfrac{1}{n}\left(a_2 + n - 2 + \dfrac{1}{a_2}\right), \\ \quad \vdots \\ a_n = \dfrac{1}{n}\left(a_{n-1} + n - 2 + \dfrac{1}{a_{n-1}}\right), \\ a_1 = \dfrac{1}{n}\left(a_n + n - 2 + \dfrac{1}{a_n}\right). \end{cases}$$

Applying the AM-GM inequality, from the first equation we get

$$a_2 = \frac{1}{n}\left(a_1 + n - 2 + \frac{1}{a_1}\right) \geq \sqrt[n]{a_1 \cdot \underbrace{1 \cdot 1 \cdots 1}_{n-2} \cdot \frac{1}{a_1}} = 1.$$

Hence, $a_2 \geq 1$, and the same occurs with the remaining a_i. That is, for all $1 \leq i \leq n$, $a_i \geq 1$. Therefore,

$$\begin{aligned} a_2 &= \frac{1}{n}\left(a_1 + n - 2 + \frac{1}{a_1}\right) \\ &\leq \frac{1}{n}\left(a_1 + \underbrace{a_1 + a_1 + \ldots + a_1}_{n-2} + a_1\right) = a_1. \end{aligned}$$

Likewise, $a_3 \leq a_2, \ldots, a_n \leq a_{n-1}, a_1 \leq a_n$. Combining the preceding inequalities yields $a_1 = a_2 = \ldots = a_n$. Now the first equation becomes

$$a_1 = \frac{1}{n}\left(a_1 + n - 2 + \frac{1}{a_1}\right) \iff (n-1)a_1^2 - (n-2)a_1 - 1 = 0,$$

with solutions $\left\{1, -\dfrac{1}{n-1}\right\}$. Since the numbers a_i may not be negative, then $a_1 = a_2 = \ldots = a_n = 1$. Finally, we have that the unique solution of the given system is $x_1 = x_2 = \ldots = x_n = 0$, and we are done.

Solution 2 by Ander Lamaison Vidarte, CFIS, BarcelonaTech, Barcelona, Spain. Let $3^{x_k} = a_k$ for $1 \leq k \leq n$. Note that the a_k

are positive. Then the system becomes $a_{k+1} = 1+\frac{1}{n}\left(a_k + a_k^{-1} - 2\right)$ for $1 \leq k \leq n-1$, and $a_1 = 1+\frac{1}{n}\left(a_n + a_n^{-1} - 2\right)$. Since $x+x^{-1} \geq 2$ for positive x (by the arithmetic-geometric mean inequality), the RHS of each equation is greater than or equal to 1, so $a_k \geq 1$ for all k.

Let $f(x) = 1 + \frac{1}{n}(x + x^{-1} - 2)$. Note that $f(1) = 1$. Then the equations become $f(a_k) = a_{k+1}$ for $1 \leq k \leq n-1$ and $f(a_n) = a_1$. But for $x > 1$,

$$f(x) - x = \frac{1}{n}\left((n-2)(1-x) + \left(x^{-1} - x\right)\right) < 0.$$

This means that $a_1 \geq a_2 \geq \ldots \geq a_n \geq a_1$, with equality if, and only if, $a_k = 1$ for all k. We conclude that this is the only solution, $a_k = 1$ and $x_k = 0$ for all k.

Also solved by Francesc Gispert Sánchez, BarcelonaTech, Barcelona, Spain.

MH–12. *Proposed by Andrés Sáez-Schwedt, Universidad de León, León, Spain.* Let I be the incenter of triangle ABC. The line through I perpendicular to AI meets AC at E and AB at F. Let P and Q be the circumcenters of BEI and CFI, respectively. Prove that the midpoint of PQ belongs to AI.

Solution 1 by Ander Lamaison Vidarte, CFIS, BarcelonaTech, Barcelona, Spain. Let R and S be the midpoints of EI and FI, respectively. First we note that the triangles AEI and AFI both have a right angle at I, equal angles at A and a common side AI. Hence we have $EI = FI$, and consequently $RI = SI$.

The lines RP, SQ and AI are all perpendicular to EF, so they are parallel. If M is the intersection point of AI and PQ, then by Thales' theorem, $\frac{PM}{MQ} = \frac{RI}{IS} = 1$, so $PM = MQ$, which means that M is the midpoint of PQ. The midpoint of PQ lies on AI, just as we wanted to show.

Solution 2 by Candice Rowe, Auburn University–Montgomery, Eclectic, Alabama, USA. First, by the angle-side-angle theorem,

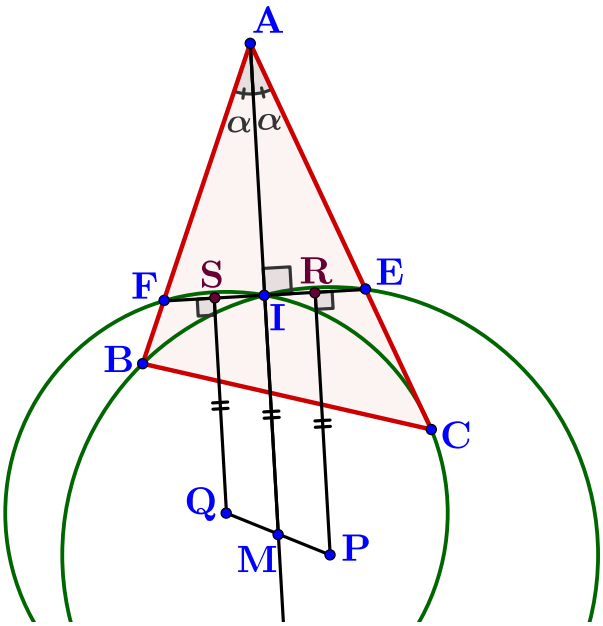

Figure 3: Construction for Solution 1 of Problem MH–12.

we can see that triangle AIE is congruent to triangle AIF. Therefore, $EI = IF$.

By definition, we know that the circumcenter is the intersection of the perpendicular side bisectors from each side of the triangle. Therefore, we know that the circumcenter of CFI, Q, must lie on the perpendicular bisector of IF. Similarly, the circumcenter of BEI, P, must lie on the perpendicular bisector of EI.

Since we know that $EI = IF$, it is clear that $ED = DI = IG = GF$, where D is the midpoint of EI and G is the midpoint of IF. Let M be the point of intersection between PQ and AI. As the angles between the line EF and the lines PD, MI and QG are right, these three lines are parallel. Then, if H and J are the orthogonal projections of P and Q onto AI, respectively, we have that $JQ = IG = ID = HP$, $\angle QJM = \angle PHM = 90°$ and $\angle QMJ = \angle PMH$. Thus, triangles QJM and PHM are congruent, by the angle-angle-side theorem. Hence, $PM = MQ$, which shows that the midpoint of PQ is M and belongs to AI.

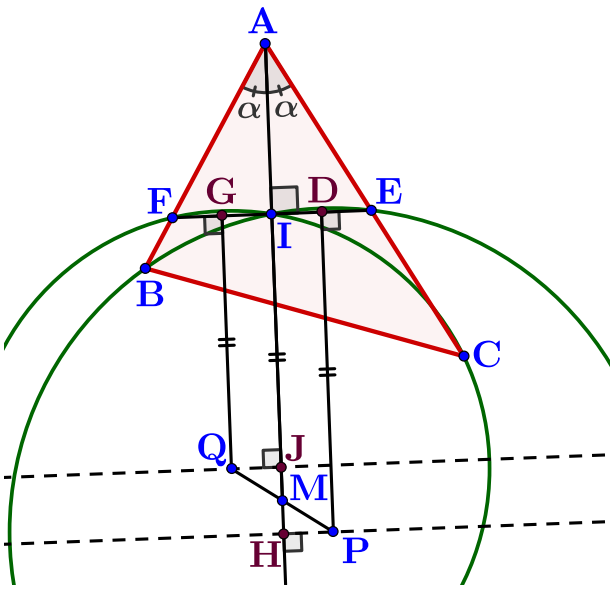

Figure 4: Construction for Solution 2 of Problem MH–12.

This also proves the general case for any two points that lie on the perpendicular side bisectors of EI and IF, not just for the circumcenters of BEI and CFI.

Solution 3 by the proposer. Let D be the intersection of PM and QN, being M, N the midpoints of BI and CI. As an intersection of two bisectors of triangle BCI, D is its circumcenter, and so it coincides with a known point of the triangle, namely the second point of intersection of AI with the circumcircle of ABC, i.e. the midpoint of the arc BC not containing A. We will use later the fact that D lies on AI.

On the other side, since central angles are equal to twice inscribed or half-inscribed angles, it follows that $\angle BPE = \gamma$ and $\angle CQF = \beta$. Therefore, the quadrilaterals $BECP$ and $BFCQ$ are cyclic. Now, P lies on the bisector of BE and on the circumcircle of BEC, so P must be a point of the internal or external bisector of C. From the relative positions of the points, it is clear that it must be the external bisector, so $\angle ICP = 90°$. By a similar reasoning, we deduce that $\angle IBQ = 90°$, hence CP and BQ meet at G, the excenter of ABC associated to the side BC. In particular, G

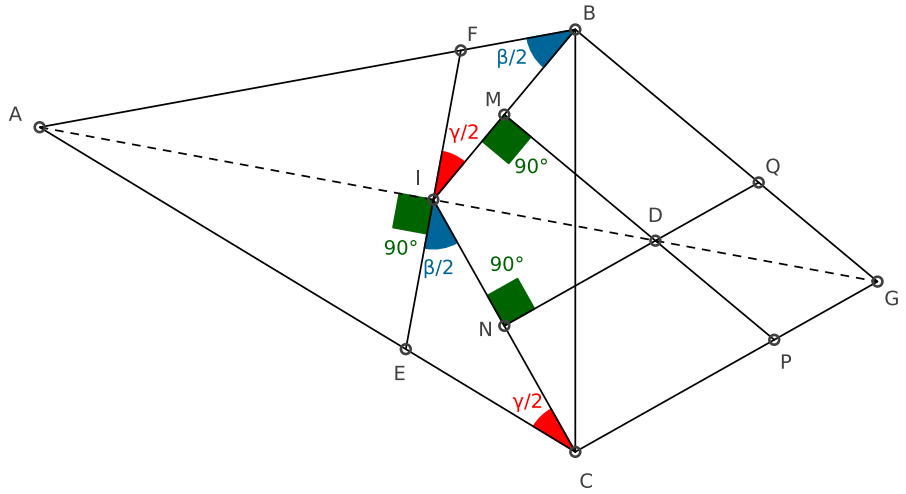

Figure 5: Construction for Solution 3 of Problem MH–12.

belongs to AD.

Finally, $DPGQ$ is a parallelogram (with sides perpendicular to BI, CI), so the midpoint of PQ is also the midpoint of DG, which is certainly a point of AI.

Also solved by Alberto Espuny-Díaz, CFIS, BarcelonaTech, Barcelona, Spain; Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain, and Jon Liberal Huarte, Pamplona, Spain.

MH–13. *Proposed by Ander Lamaison Vidarte, CFIS, BarcelonaTech, Barcelona, Spain.* Let a, b and c be three positive integers, each of them without repeated digits in their decimal expression. If $a + b + c = 58888$, prove that there is a digit, other than 0, which appears in at least two of the three numbers.

Solution by the proposer. We shall prove the following lemma:

Lemma. *The only way to express* 58888 *as the sum of no more than* 45 *powers of* 10 *is* $58888 = 5{\cdot}10^4{+}8{\cdot}10^3{+}8{\cdot}10^2{+}8{\cdot}10^1{+}8{\cdot}10^0$.

Proof. We notice that any way of expressing 58888 with the minimum number of powers of 10 does not contain 10 equal summands 10^k, otherwise we could turn them into a 10^{k+1}. But the sums of powers of 10, each summand not appearing more than nine times, correspond one-to-one to decimal expressions of numbers. Thus, the minimal way of expressing 58888 is $5 \cdot 10^4 + 8 \cdot 10^3 + 8 \cdot 10^2 + 8 \cdot 10^1 + 8 \cdot 10^0$, which has 37 summands. Since the number of summands must be congruent to $58888 \equiv 1$ modulo 9, any other expression has at least $37 + 9 = 46$ summands. □

Now, let D be the set of digits other than 0 appearing in a, b and c and assume that none of them appears twice. Then $58888 = \sum_{d \in D} d \cdot 10^{n_d}$ for some nonnegative integers n_d. This expresses 58888 as the sum of at most $1 + 2 + ... + 9 = 45$ powers of 10, and hence this expression is the one from the lemma.

Note that this expression does not include nine powers with the same exponent, so $9 \notin D$. But this reduces the possible number of powers to $1 + 2 + ... + 8 = 36$. Since we know that this number cannot be less than 37, we get a contradiction.

MH–14. *Proposed by Francesc Gispert Sánchez, CFIS, Barcelona-Tech, Barcelona, Spain.* Let I be the incentre of a triangle ABC and let D, E and F be the points of intersection of the circumcircle of the triangle and the extended angle bisectors AI, BI and CI, respectively. Then, the intersection of triangles ABC and DEF forms an hexagon with vertices K, L, M, N, P and Q, taken counterclockwise. Prove that KN, LP and MQ are concurrent at I.

Solution 1 by the proposer. Let $\angle CAB = 2\alpha$, $\angle ABC = 2\beta$ and $\angle BCA = 2\gamma$. Let G be the point of intersection of CA and BE.

We claim that NI is parallel to CA. Since angles subtended by the same arc are equal, we have that $\angle BED = \angle BAD = \alpha$ and also $\angle DEC = \angle DAC = \alpha$. Hence, EN is the angle bisector of $\angle BEC$ and we have that

$$\frac{BN}{NC} = \frac{BE}{EC}$$

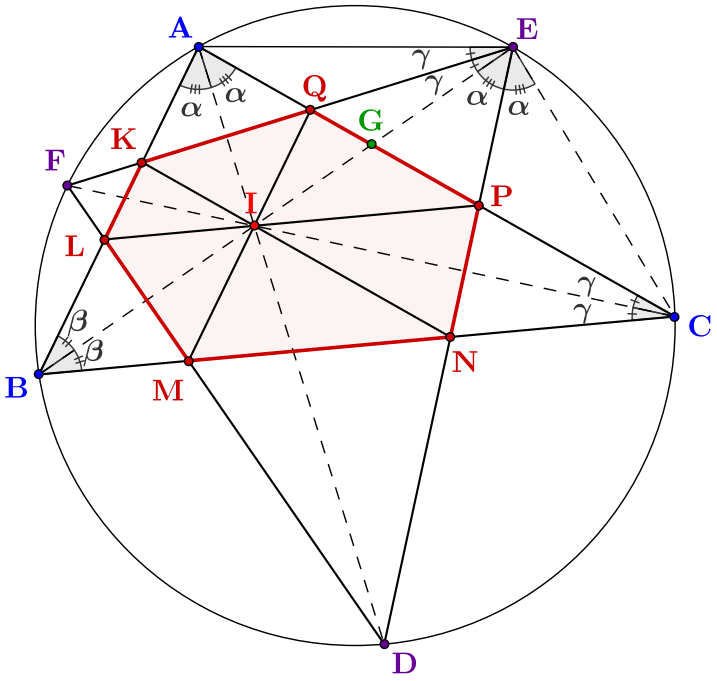

Figure 6: Construction for Solution 1 of Problem MH–14.

by the angle bisector theorem. Moreover, we observe that $\angle BAG = \angle BEC = 2\alpha$ and $\angle ABG = \angle EBC = \beta$. Then, triangles BEC and BAG are similar and

$$\frac{BE}{EC} = \frac{BA}{AG}.$$

Applying the angle bisector theorem to triangle BAG yields

$$\frac{BA}{AG} = \frac{BI}{IG}.$$

Finally, combining the previous expressions, we obtain that

$$\frac{BN}{NC} = \frac{BI}{IG},$$

and NI must be parallel to CA by the intercept theorem.

Similarly, IK is also parallel to CA. That is, N, I and K are collinear. Therefore, we have shown that I belongs to KN, and it can be proven analogously that I also belongs to LP and MQ.

Solution 2 by Ander Lamaison Vidarte, CFIS, BarcelonaTech, Barcelona, Spain. Let $\angle CAB = 2\alpha$, $\angle ABC = 2\beta$ and $\angle BCA = 2\gamma$. We claim that $CPIN$ is a rhombus.

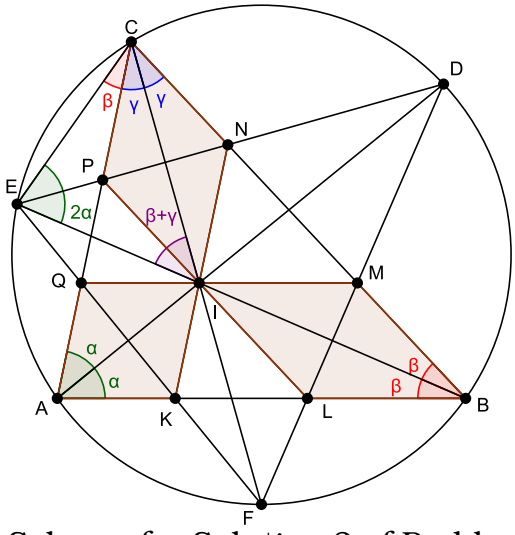

Figure 7: Scheme for Solution 2 of Problem MH–14.

From the definition of the angles, $2\alpha+2\beta+2\gamma = 180°$. We consider the triangle EIC, and look at its angles:

$$\angle IEC = \angle BEC = \angle BAC = 2\alpha,$$
$$\angle ECI = \angle ECA + \angle ACI = \angle EBA + \angle ACI = \beta + \gamma,$$
$$\angle EIC = 180° - \angle IEC - \angle ECI = 180° - 2\alpha - \beta - \gamma = \beta + \gamma.$$

We conclude that EIC is isosceles, $IE = CE$, and analogously $ID = CD$. The line DE is the perpendicular bisector of IC.

The line IC is perpendicular to PN, because PN is the bisector of IC. In addition, $IN = CN$ and $IP = CP$. Moreover, $\angle PCI = \gamma = \angle ICN$. IC is both the height and the bisector of vertex C in triangle PCN, so this triangle is isosceles, $IP = PC = NC = NI$, and hence $CPIN$ is a rhombus. $BLIM$ and $AKIQ$ are also rhombi, by the same argument.

As a consequence to the result above, $PI \parallel CN \parallel BM \parallel LI$, so I lies on the line LP. Analogously, I also lies on KN and MQ, so the statement follows.

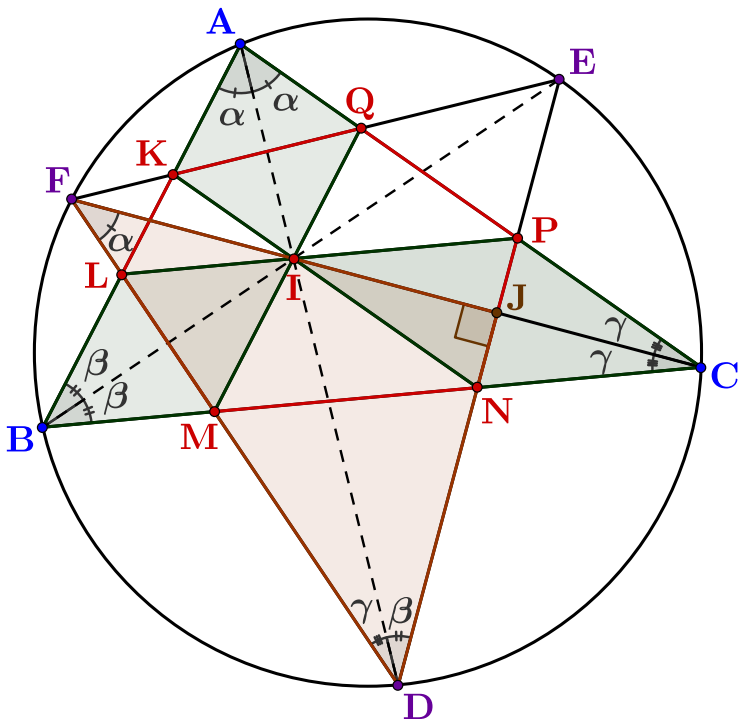

Figure 8: Construction for Solution 3 of Problem MH–14.

Solution 3 by Jon Liberal Huarte, Pamplona, Spain. First, we prove that I is the orthocenter of triangle $\triangle DEF$. Let $\widehat{A} = 2\alpha$, $\widehat{B} = 2\beta$ and $\widehat{C} = 2\gamma$. We have

$$\alpha + \beta + \gamma = 90°.$$

From Figure 8, it immediately follows that

$$\begin{aligned} \angle CAD &= \angle CFD = \alpha, \\ \angle ABE &= \angle ADE = \beta, \\ \angle FCA &= \angle FDA = \gamma. \end{aligned}$$

Let J be the intersection point of the angle bisector CI and the segment ED. $\triangle JFD$ is right-angled at J, as

$$\angle JFD + \angle FDJ = \alpha + \beta + \gamma = 90^\circ.$$

Thus, $FC \perp DE$ and, analogously, $AD \perp FE$ and $BE \perp FD$, which implies that I is the orthocenter of $\triangle DEF$.

As a property of the orthocenter, $\overline{IJ} = \overline{JC}$. Therefore, the quadrilateral $\Diamond CPIN$ is a rhombus and, likewise, so are $\Diamond BMIL$ and $\Diamond AKIQ$. From this, it follows that $PI \parallel NC$ and $IL \parallel BM$. That is, $PI, IL \parallel BC$, proving that P, I and L are collinear (and analogously, so are M, I and Q, and K, I and N).

MH–15. *Proposed by Marius Dragan, Bucharest, Romania.* Let A, B, C be the angles measured in radians of a triangle ABC with sides of lengths a, b, c and semiperimeter p. Prove that

$$aA^2 + bB^2 + cC^2 \geq \frac{2\pi^2}{9}\, p.$$

Solution 1 by the proposer. Applying Jensen's inequality to the convex function $f(x) = x^2$, we have

$$\begin{aligned}\frac{a}{\sum\limits_{\text{cyc}} a}A^2 + \frac{b}{\sum\limits_{\text{cyc}} a}B^2 + \frac{c}{\sum\limits_{\text{cyc}} a}C^2 &\geq \left(\frac{a}{\sum\limits_{\text{cyc}} a}A + \frac{b}{\sum\limits_{\text{cyc}} a}B + \frac{c}{\sum\limits_{\text{cyc}} a}C\right)^2 \\ &= \frac{(aA + bB + cC)^2}{\left(\sum\limits_{\text{cyc}} a\right)^2}.\end{aligned}$$

Since the largest side is opposite to the largest angle, we have

$$aA + bB + cC \geq \frac{1}{3}\left(\sum_{\text{cyc}} A\right)\left(\sum_{\text{cyc}} a\right) = \frac{\pi}{3}\left(\sum_{\text{cyc}} a\right)$$

on account of Chebyshev's inequality. Now, combining the preceding inequalities yields

$$a\,A^2 + b\,B^2 + c\,C^2 \geq \frac{(a\,A + b\,B + c\,C)^2}{\sum\limits_{\text{cyc}} a} \geq \frac{\pi^2}{9}\left(\sum_{\text{cyc}} a\right)$$

and the statement follows. Equality holds when $\triangle ABC$ is equilateral, and we are done.

Solution 2 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. Applying the Cauchy–Schwarz inequality with vectors $\left(\sqrt{a}A, \sqrt{b}B, \sqrt{c}C\right)$ and $\left(\sqrt{a}, \sqrt{b}, \sqrt{c}\right)$, we obtain that

$$(a+b+c)\left(aA^2 + bB^2 + cC^2\right) \geq (aA + bB + cC)^2.$$

Therefore, if we prove that

$$(aA + bB + cC)^2 \geq \frac{2\pi^2}{9}p(a+b+c)$$

or, equivalently,

$$3(aA + bB + cC) \geq \pi(a+b+c) = (A+B+C)(a+b+c),$$

we will obtain the desired result.

Now, suppose without loss of generality that $a \geq b \geq c$. Since A, B and C are the angles and a, b and c are the side lengths of the triangle ABC, this means that $A \geq B \geq C$. Finally, the last inequality is precisely Chebyshev's inequality.

Equality holds for equilateral triangles.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.

MH–16. *Proposed by Nicolae Papacu, Slobozia, Romania.* Let a, b, $c \in (0,\infty)$ such that $a \geq b \geq c$. Prove that

$$\begin{aligned}\frac{3}{2^{n-1}} &\leq \frac{a^n+b^n}{(a+b)^n}+\frac{b^n+c^n}{(b+c)^n}+\frac{c^n+a^n}{(c+a)^n}\\ &\leq \frac{3}{2^n}\left(\sqrt{\frac{a}{c}}+\sqrt{\frac{c}{a}}\right)^n+3\left(\frac{1}{2^{n-1}}-1\right),\end{aligned}$$

where $n \in \mathbb{N}^*$.

Solution by the proposer. We start with the known inequality

$$x^n+y^n \geq \frac{(x+y)^n}{2^{n-1}}, \quad \forall\, x,y>0,\ n \in \mathbb{N},$$

which can be proved by induction on n or with the arithmetic-n^{th} power inequality. We then have that

$$\frac{a^n+b^n}{(a+b)^n} \geq \frac{1}{2^{n-1}},$$

from which we directly obtain that

$$\sum_{\text{cyclic}} \frac{a^n+b^n}{(a+b)^n} \geq \frac{3}{2^{n-1}}.$$

Now we have to prove the right hand inequality. Since

$$(a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

and

$$2(a+b)^n=(a+b)^n+(b+a)^n,$$

then we have that

$$\begin{aligned}(a+b)^n+(b+a)^n &= \sum_{k=0}^{n}\binom{n}{k}\left(a^{n-k}b^k+b^{n-k}a^k\right)\\ &= 2(a^n+b^n)+\sum_{k=1}^{n-1}\binom{n}{k}\left(a^{n-k}b^k+b^{n-k}a^k\right).\end{aligned}$$

Using the AM-GM inequality we get

$$a^{n-k}b^k + b^{n-k}a^k \geq 2\sqrt{(ab)^n}.$$

Using this in the previous expression yields

$$2(a+b)^n \geq 2(a^n + b^n) + 2\sum_{k=1}^{n-1} \binom{n}{k} \sqrt{(ab)^n}$$

and then we obtain

$$(a+b)^n \geq a^n + b^n + (2^n - 2)\sqrt{(ab)^n},$$

since $\sum_{k=1}^{n-1} \binom{n}{k} = 2^n - 2$ (note that this inequality can also be proved using induction). Therefore,

$$\begin{aligned} \frac{a^n + b^n}{(a+b)^n} &\leq 1 - \frac{(2^n - 2)\sqrt{(ab)^n}}{(a+b)^n} \\ &= \frac{(a+b)^n - \left(2\sqrt{ab}\right)^n}{(a+b)^n} + \frac{2\sqrt{(ab)^n}}{(a+b)^n}. \end{aligned}$$

Using the AM-GM inequality $\left(\frac{1}{(a+b)^n} \leq \frac{1}{\left(2\sqrt{ab}\right)^n} \right)$ we obtain

$$\begin{aligned} \frac{a^n + b^n}{(a+b)^n} &\leq \frac{(a+b)^n - \left(2\sqrt{ab}\right)^n}{\left(2\sqrt{ab}\right)^n} + \frac{2\sqrt{(ab)^n}}{\left(2\sqrt{ab}\right)^n} \\ &= \frac{(a+b)^n}{\left(2\sqrt{ab}\right)^n} - 1 + \frac{2}{2^n} \\ &= \frac{1}{2^n}\left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}}\right)^n - 1 + \frac{1}{2^{n-1}}. \end{aligned}$$

We can do this for the three pairs of numbers, and adding them up we obtain

$$\sum_{\text{cyclic}} \frac{a^n + b^n}{(a+b)^n} \leq \frac{1}{2^n} \sum_{\text{cyclic}} \left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \right)^n + 3\left(-1 + \frac{1}{2^{n-1}} \right).$$

If we can prove that $\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \leq \sqrt{\frac{a}{c}} + \sqrt{\frac{c}{a}}$ and $\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}} \leq \sqrt{\frac{a}{c}} + \sqrt{\frac{c}{a}}$ we will solve the problem:

$$\begin{aligned} \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \leq \sqrt{\frac{a}{c}} + \sqrt{\frac{c}{a}} &\iff \frac{a}{b} + \frac{b}{a} + 2 \leq \frac{a}{c} + \frac{c}{a} + 2 \\ &\iff ac(a^2 + b^2) \leq ab(a^2 + c^2) \\ &\iff (c-b)(a^2 - bc) \leq 0, \end{aligned}$$

which is true since $0 \leq c \leq b \leq a$. In the same way, $\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}} \leq \sqrt{\frac{a}{c}} + \sqrt{\frac{c}{a}}$ is equivalent to the inequality $(a-b)(c^2 - ab) \leq 0$ which is true since $0 \leq c \leq b \leq a$, and we are done.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.

Advanced Problems

A–11. *Proposed by Francesc Gispert Sánchez, CFIS, Barcelona-Tech, Barcelona, Spain.* Let n be a positive integer such that $n \geq 5$ and $n \equiv 1 \pmod 4$ and consider $m = \left\lfloor \frac{n}{2} \right\rfloor$. We define the function

$$\begin{aligned} f(x_1, \ldots, x_m) &= \sum_{j=1}^{\frac{m}{2}} (2j-1) x_j^{8(2j)} + \sum_{j=\frac{m}{2}+1}^{m} (2j-1) x_j^{8[2(m-j)+1]} \\ &= x_1^{16} + 3x_2^{32} + 5x_3^{48} + \ldots + (m-3) x_{\frac{m}{2}-1}^{8(m-2)} \\ &\quad + (m-1) x_{\frac{m}{2}}^{8m} + (m+1) x_{\frac{m}{2}+1}^{8(m-1)} \\ &\quad + (m+3) x_{\frac{m}{2}+2}^{8(m-3)} + \ldots + (2m-3) x_{m-1}^{24} \\ &\quad + (2m-1) x_m^8, \end{aligned}$$

where $x_1, \ldots, x_m$ are positive real numbers.

Show that for any positive real numbers $a_1, a_2, \ldots, a_m$ the following inequality holds:

$$\prod_{\text{cyc}} [f(a_1, \ldots, a_m) + (n+m)(n-m)]^{\frac{1}{m}} \geq n^2 (a_1 \ldots a_m)^{\frac{n-1}{n}} .$$

Solution by the proposer. First, we observe that $n = 2m + 1$ and, since $n \equiv 1 \pmod 4$, n is odd and m is even. Apart from this, in the following proof we are going to use the well-known identity

$$\sum_{j=1}^{k} (2j-1) = k^2 \quad \forall\, k \in \mathbb{N} .$$

The stated inequality can be written in the equivalent form

$$\prod_{\text{cyc}} \frac{f(a_1, \ldots, a_m) + (n+m)(n-m)}{n^2} \geq (a_1 \ldots a_m)^{\frac{(n-1)m}{n}} .$$

By the definition of f, we can express $f(a_1,\dots,a_m)$ as the sum of exactly m^2 terms, each of which is a power of a_i for some $i \in \{1,\dots,m\}$. Moreover, $(n+m)(n-m) = n^2-m^2$. Therefore, the AM-GM inequality yields

$$\frac{f(a_1,\dots,a_m)+(n+m)(n-m)}{n^2} \geq \left[\left(\prod_{j=1}^{\frac{m}{2}} a_j^{(2j-1)8(2j)}\right)\left(\prod_{j=\frac{m}{2}+1}^{m} a_j^{(2j-1)8[2(m-j)+1]}\right)1^{n^2-m^2}\right]^{\frac{1}{n^2}},$$

and we can obtain analogous inequalities for the cyclic permutations of $(a_1,\dots,a_m)$. Multiplying all these inequalities, we obtain that

$$\prod_{\text{cyc}} \frac{f(a_1,\dots,a_m)+(n+m)(n-m)}{n^2} \geq (a_1\dots a_m)^{\frac{\alpha}{n^2}},$$

where

$$\alpha = \sum_{j=1}^{\frac{m}{2}}(2j-1)8(2j) + \sum_{j=\frac{m}{2}+1}^{m}(2j-1)8[2(m-j)+1].$$

In order to obtain the desired inequality, it remains to check that $\alpha = mn(n-1)$. To compute the value of α, we consider a fixed $r \in \left\{1,\dots,\frac{m}{2}\right\}$ and $s = m-r+1 \in \left\{\frac{m}{2}+1,\dots,m\right\}$ (that is, s is the r-th last index of the sum that we want to calculate). Then, adding the r-th and the s-th terms of the sum, we obtain

$$\begin{aligned}(2r-1)8(2r)+(2s-1)8[2(m-s)+1] \\ = 16r(2r-1)+8(2m-2r+1)(2r-1) = 8n(2r-1).\end{aligned}$$

Thus, taking into consideration all the possible values of r, we can express α in the form

$$\alpha = \sum_{r=1}^{\frac{m}{2}} 8n(2r-1) = 8n\left(\frac{m}{2}\right)^2 = n(n-1)m$$

and we are done.

A–12. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Compute

$$\lim_{n\to\infty}\left\{1+\sum_{k=1}^{n}\frac{1}{k}-(\ln n+\gamma)\right\}^{n},$$

where $\gamma = 0.577216\ldots$ is the Euler–Mascheroni constant.

Solution by the proposer. Since $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)$ by definition (it is not known whether γ is rational or irrational), then

$$\lim_{n\to\infty}\left\{1+\sum_{k=1}^{n}\frac{1}{k}-(\ln n+\gamma)\right\}=1$$

and the given limit is an indetermination of (1^{∞}) type. Therefore,

$$\lim_{n\to\infty}\left\{1+\sum_{k=1}^{n}\frac{1}{k}-\ln(n)-\gamma\right\}^{n}=e^{\alpha},$$

where

$$\begin{aligned}\alpha &= \lim_{n\to\infty} n\left(\sum_{k=1}^{n}\frac{1}{k}-\ln(n)-\gamma\right)\\ &= \lim_{n\to\infty}\left(\frac{\displaystyle\sum_{k=1}^{n}\frac{1}{k}-\ln n-\gamma}{\dfrac{1}{n}}\right)=\lim_{n\to\infty}\left[\frac{\dfrac{1}{n}+\ln\left(1-\dfrac{1}{n}\right)}{\dfrac{1}{n}-\dfrac{1}{n-1}}\right].\end{aligned}$$

Notice that in the preceding equality we have applied Stolz–Cesàro theorem. Setting $t = 1/n$ into the last limit and using l'Hôpital's rule, we get

$$\alpha=\lim_{t\to 0}\frac{t+\ln(1-t)}{t-\dfrac{t}{1-t}}=\lim_{t\to 0}\frac{t-1}{t-2}=\frac{1}{2},$$

from which it follows that

$$\lim_{n\to\infty}\left\{1+\sum_{k=1}^{n}\frac{1}{k}-(\ln n+\gamma)\right\}^{n}=\sqrt{e}.$$

A–13. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Prove that

$$1+\sum_{k=1}^{n} F_k \leq \frac{1}{2}\left(\frac{F_n^2}{F_{n+2}}+\frac{F_{n+1}^2}{F_n}+\frac{F_{n+2}^2}{F_{n+1}}\right),$$

where F_n is the n^{th} Fibonacci number, defined by $F_0=0$, $F_1=1$ and for all $n\geq 1$, $F_{n+1}=F_n+F_{n-1}$.

Solution 1 by Miguel Cidrás Senra, IES Valle Inclán, Pontevedra, Spain. First, we prove that

$$1+\sum_{k=1}^{n} F_k = F_{n+2}.$$

For $n=1$, $1+F_1=2=F_3$. Assume that the result holds for m. It must then be shown that it also holds for $m+1$, that is:

$$1+\sum_{k=1}^{m+1} F_k = 1+\sum_{k=1}^{m} F_k + F_{m+1} = F_{m+2}+F_{m+1} = F_{(m+1)+2}.$$

By mathematical induction, the statement holds for all positive integer n.

As $F_{n+2}^2 \geq F_{n+1}^2 \geq F_n^2$ and $\frac{1}{F_n} \geq \frac{1}{F_{n+1}} \geq \frac{1}{F_{n+2}}$, by the rearrangement inequality

$$\frac{F_{n+1}^2}{F_n}+\frac{F_{n+2}^2}{F_{n+1}}+\frac{F_n^2}{F_{n+2}} \geq \frac{F_n^2}{F_n}+\frac{F_{n+1}^2}{F_{n+1}}+\frac{F_{n+2}^2}{F_{n+2}}.$$

With these results and with the propierty that $F_{n+2}=F_n+F_{n+1}$, we have

$$\begin{aligned}\frac{1}{2}\left(\frac{F_{n+1}^2}{F_n}+\frac{F_{n+2}^2}{F_{n+1}}+\frac{F_n^2}{F_{n+2}}\right) &\geq \frac{1}{2}(F_n+F_{n+1}+F_{n+2})\\ &= F_{n+2} = 1+\sum_{k=1}^{n} F_k,\end{aligned}$$

as we wanted to prove.

Solution 2 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. It is well-known that

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1$$

(this identity can be easily proved by induction on n, for instance). Thus, the claimed inequality can be expressed as

$$2F_{n+2} \leq \frac{F_n^2}{F_{n+2}} + \frac{F_{n+1}^2}{F_n} + \frac{F_{n+2}^2}{F_n + 1}.$$

Applying the Cauchy–Schwarz inequality to the vectors

$$\left(\frac{F_n}{\sqrt{F_{n+2}}}, \frac{F_{n+1}}{\sqrt{F_n}}, \frac{F_{n+2}}{\sqrt{F_{n+1}}}\right) \quad \text{and} \quad \left(\sqrt{F_{n+2}}, \sqrt{F_n}, \sqrt{F_{n+1}}\right)$$

we obtain

$$\frac{F_n^2}{F_{n+2}} + \frac{F_{n+1}^2}{F_n} + \frac{F_{n+2}^2}{F_n + 1} \geq \frac{(F_n + F_{n+1} + F_{n+2})^2}{F_{n+2} + F_n + F_{n+1}}$$

(this is sometimes called the Cauchy–Schwarz inequality in Engel form). Finally, we observe that the RHS of the last inequality can be written as

$$\frac{(F_n + F_{n+1} + F_{n+2})^2}{F_{n+2} + F_n + F_{n+1}} = F_n + F_{n+1} + F_{n+2} = F_{n+2} + F_{n+2} = 2F_{n+2}$$

and we are done. Furthermore, the inequality is always strict (the vectors we used are never proportional).

Solution 3 by the proposer. By induction, it is easy to see that $\sum_{k=1}^{n} F_k = F_{n+2} - 1$. Then, the given equation is equivalent to prove that

$$2F_{n+2} \leq \frac{F_n^2}{F_{n+2}} + \frac{F_{n+1}^2}{F_n} + \frac{F_{n+2}^2}{F_{n+1}},$$

or

$$F_n + F_{n+1} + F_{n+2} \leq \frac{F_n^2}{F_{n+2}} + \frac{F_{n+1}^2}{F_n} + \frac{F_{n+2}^2}{F_{n+1}}. \tag{2}$$

Expanding out (2), we obtain that we have to prove that

$$F_n^3F_{n+1}+F_{n+1}^3F_{n+2}+F_{n+2}^3F_n-F_nF_{n+1}F_{n+2}(F_n+F_{n+1}+F_{n+2})\geq 0,$$

or

$$\begin{aligned}&F_nF_{n+1}(F_n^2-F_{n+2}^2)+F_{n+1}F_{n+2}(F_{n+1}^2-F_n^2)\\&\quad+F_{n+2}F_n(F_{n+2}^2-F_{n+1}^2)\geq 0.\end{aligned}\tag{3}$$

Adding and subtracting $F_nF_{n+1}^3$ to the left side of inequality (3) yields

$$\begin{aligned}&F_nF_{n+1}(F_n^2-F_{n+2}^2)+F_{n+1}F_{n+2}(F_{n+1}^2-F_n^2)\\&\quad+F_nF_{n+1}^3-F_nF_{n+1}^3+F_{n+2}F_n(F_{n+2}^2-F_{n+1}^2)\\&=F_n(F_{n+2}-F_{n+1})(F_{n+2}^2-F_{n+1}^2)+F_{n+1}(F_{n+2}-F_n)(F_{n+1}^2-F_n^2)\\&\geq 0\end{aligned}$$

and we are done.

A–14. *Proposed by Andrés Sáez-Schwedt, Universidad de León, León, Spain.* Let n be a positive integer. If $(1+\sqrt{3})^n$ is written as $a+b\sqrt{3}$, with a, b integers, find $\gcd(a,b)$ in terms of n.

Solution 1 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. We are going to prove by induction that, if we express $\left(1+\sqrt{3}\right)^n$ as $a_n+b_n\sqrt{3}$, then $\gcd(a_n,b_n)=2^{\left\lfloor\frac{n}{2}\right\rfloor}$.

First of all, we observe that $a_1=1$, $b_1=1$, $a_2=4$ and $b_2=2$. Therefore, we have the base cases $\gcd(a_1,b_1)=1$ and $\gcd(a_2,b_2)=2$.

Now, we compute

$$\left(1+\sqrt{3}\right)\left(a_n+b_n\sqrt{3}\right)=(a_n+3b_n)+(a_n+b_n)\sqrt{3}$$

and we get the recurrence relations

$$\begin{cases}a_{n+2}=a_{n+1}+3b_{n+1}=2(2a_n+3b_n),\\b_{n+2}=a_{n+1}+b_{n+1}=2(a_n+2b_n).\end{cases}$$

For the inductive step, we suppose that $a_n = 2^r A_n$ and $b_n = 2^r B_n$ with $\gcd(A_n, B_n) = 1$ and, from the previous expressions, we obtain that

$$\begin{cases} a_{n+2} = 2^{r+1}(2A_n + 3B_n), \\ b_{n+2} = 2^{r+1}(A_n + 2B_n), \end{cases}$$

and we claim that $2A_n + 3B_n$ and $A_n + 2B_n$ are coprime. To see it, assume for the sake of contradiction that there exists a prime number p which divides both $2A_n + 3B_n$ and $A_n + 2B_n$. Then, p must divide the linear combinations of these two numbers; in particular, it must divide $A_n + B_n = (2A_n + 3B_n) - (A_n + 2B_n)$ and $B_n = 2(A_n + 2B_n) - (2A_n + 3B_n)$. But, for the same reason, this means that p divides both $A_n = (A_n + B_n) - B_n$ and B_n, which contradicts the fact that A_n and B_n are coprime.

Solution 2 by the proposer. Let's take a look at the first values of n:

$$\begin{aligned} \left(1+\sqrt{3}\right)^2 &= 4 + 2\sqrt{3} \to \gcd(a,b) = 2, \\ \left(1+\sqrt{3}\right)^3 &= 10 + 6\sqrt{3} \to \gcd(a,b) = 2, \\ \left(1+\sqrt{3}\right)^4 &= 28 + 16\sqrt{3} \to \gcd(a,b) = 2^2, \\ \left(1+\sqrt{3}\right)^5 &= 76 + 44\sqrt{3} \to \gcd(a,b) = 2^2, \\ \left(1+\sqrt{3}\right)^6 &= 208 + 120\sqrt{3} \to \gcd(a,b) = 2^3, \\ &\dots \end{aligned}$$

This suggests that $\gcd(a,b) = 2^{\left\lfloor \frac{n}{2} \right\rfloor}$, where $\lfloor \cdot \rfloor$ denotes the integer part. If we denote by a_n, b_n the integer numbers such that $\left(1+\sqrt{3}\right)^n = a_n + b_n\sqrt{3}$, we will prove that $\gcd(a_n, b_n) = 2^k$ when $n = 2k$ or $n = 2k+1$.

The first obvious simplification comes from the factorization

$$\left(1+\sqrt{3}\right)^2 = 2\left(2+\sqrt{3}\right).$$

Therefore, if we denote $\left(2+\sqrt{3}\right)^n = c_n + d_n$, starting with $c_1 = 2$, $d_1 = 1$, then one has

$$\left(1+\sqrt{3}\right)^{2k} = 2^k\left(2+\sqrt{3}\right)^k = 2^k\left(c_k + d_k\sqrt{3}\right).$$

A recurrence relation for the sequences c_k, d_k is easily deduced:

$$\begin{aligned} c_{k+1} + d_{k+1}\sqrt{3} &= \left(2+\sqrt{3}\right)\left(c_k + d_k\sqrt{3}\right) \\ &= \underbrace{2c_k + 3d_k}_{c_{k+1}} + \underbrace{(c_k + 2d_k)}_{d_{k+1}}\sqrt{3}, \end{aligned}$$

and we use it to prove by induction that $\gcd(c_k, d_k) = 1$ for all k. Indeed, applying repeatedly the property $\gcd(x, y) = \gcd(x, x-y)$ one gets:

$$\begin{aligned} \gcd(c_{k+1}, d_{k+1}) &= \gcd(2c_k + 3d_k, c_k + 2d_k) \\ &= \gcd(c_k + d_k, c_k + 2d_k) \\ &= \gcd(c_k + d_k, d_k) = \gcd(c_k, d_k) = 1. \end{aligned}$$

This solves the case $n = 2k$. For the case $n = 2k+1$ we need an additional property of c_k, d_k, namely that their sum is always odd (this is immediate). Then,

$$\begin{aligned} \left(1+\sqrt{3}\right)^{2k+1} &= \left(1+\sqrt{3}\right)\left(2+\sqrt{3}\right)^k = \left(1+\sqrt{3}\right)\left(c_k + d_k\sqrt{3}\right) \\ &= \underbrace{c_k + 3d_k}_{a} + \underbrace{(c_k + d_k)}_{b}\sqrt{3} \end{aligned}$$

and

$$\begin{aligned} \gcd(c_k + 3d_k, c_k + d_k) &= \gcd(2d_k, c_k + d_k) \overset{*}{=} \gcd(d_k, c_k + d_k) \\ &= \gcd(c_k, d_k) = 1, \end{aligned}$$

where in the step $(*)$ it is used that 2 does not divide $c_k + d_k$. The solution is now complete.

A–15. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $A(z) = \sum_{k=0}^{n} a_k z^k$ be a monic polynomial with complex coefficients. If $\alpha = \max\limits_{1\le k\le n}\Big\{|z_k|\Big\}$, where z_k are the roots of $A(z)$, then show that

$$\frac{1}{\alpha}\sum_{k=1}^{n}\left|\frac{a_{n-k}}{n^k C(n,n-k)}\right|^{1/k} \le 1.$$

Solution by the proposer. The Cardano–Viète formulas for $A(z)$ are

$$\begin{aligned} z_1 + z_2 + \ldots + z_n &= (-1)^1 a_{n-1},\\ z_1z_2 + z_1z_3 + \ldots + z_{n-1}z_n &= (-1)^2 a_{n-2},\\ &\vdots\\ z_1z_2\ldots z_n &= (-1)^n a_0. \end{aligned}$$

Since $\alpha = \max\limits_{1\le k\le n}\Big\{|z_k|\Big\}$, then

$$\begin{aligned} \left|\frac{a_{n-1}}{nC(n,1)}\right| &= \left|\frac{(-1)^1 a_{n-1}}{nC(n,1)}\right| = \left|\frac{z_1+z_2+\ldots+z_n}{nC(n,1)}\right|\\ &\le \frac{|z_1|+|z_2|+\ldots+|z_n|}{nC(n,1)} \le \frac{\alpha}{n}, \end{aligned}$$

$$\begin{aligned} \left|\frac{a_{n-2}}{n^2C(n,2)}\right|^{1/2} &= \left|\frac{(-1)^2 a_{n-2}}{n^2C(n,2)}\right|^{1/2} = \left|\frac{z_1z_2+z_1z_3+\ldots+z_{n-1}z_n}{n^2C(n,2)}\right|^{1/2}\\ &\le \left[\frac{|z_1z_2|+|z_1z_3|+\ldots+|z_{n-1}z_n|}{n^2C(n,2)}\right]^{1/2} \le \frac{\alpha}{n},\\ &\vdots\\ \left|\frac{a_0}{n^nC(n,n)}\right|^{1/n} &= \left|\frac{(-1)^n a_0}{n^nC(n,n)}\right|^{1/n} = \left|\frac{z_1z_2\ldots z_n}{n^nC(n,n)}\right|^{1/n} \le \frac{\alpha}{n}. \end{aligned}$$

Therefore, we have

$$\left|\frac{a_{n-k}}{n^kC(n,k)}\right|^{1/k} = \left|\frac{a_{n-k}}{n^kC(n,n-k)}\right|^{1/k} \le \frac{\alpha}{n}, \quad k = 1,2,\ldots,n. \quad (4)$$

Adding up the inequalities in (4), we get

$$\sum_{k=1}^{n}\left|\frac{a_{n-k}}{n^k C(n,n-k)}\right|^{1/k} \leq \alpha$$

and we are done.

A–16. *Proposed by Ander Lamaison Vidarte, CFIS, Barcelona-Tech, Barcelona, Spain.* We say that a positive integer $n \geq 3$ is *isoscelable* if the regular n-gon can be divided into $n-2$ isosceles triangles using $n-3$ diagonals. Find all values of n such that n and $n+2013$ are both *isoscelable*.

Solution by the proposer. Let us start proving that n is *isoscelable* if and only if it can be written as $n = 2^a + 2^b$.

Given a diagonal in the polygon, we consider the number of edges of the polygon lying on each side of it. We will prove that, if we divide the polygon into isosceles triangles, the number of edges on the smaller side of each diagonal is a power of 2.

Indeed, we suppose that is not the case and consider the shortest diagonal AB leaving on the smaller side a number of edges $k \geq 3$ which is not a power of 2. AB is part of an isosceles triangle on the smaller side. Since no other side on this part can be as long as AB, this must be the different side. The third vertex C satisfies $AC = BC$. On the smaller side of AC and BC there is the same number of sides, $\frac{k}{2}$. But $\frac{k}{2}$ must be a power of 2 (because it is smaller than k), so k is also a power of 2, contradiction.

Now, we consider an isosceles triangulation of the polygon. If the center lies on a diagonal, this diagonal divides the polygon into two equal parts, and on each side there is a number of edges which is a power of 2 (both sides are the 'smaller side'). If the center is on the inside of a triangle, then the number of edges of the polygon equals the sum of the smaller sides of each of the diagonals forming the triangle. But two of the sides are equal, so there is the same number of edges in the smaller side of each and $n = 2^a + 2^a + 2^b = 2^{a+1} + 2^b$.

For the converse, we can easily check that, if $n = 2^a + 2^b$, then n admits an isosceles triangulation, by drawing a diagonal leaving 2^a edges on one side and 2^b on the other, and drawing isosceles triangles from there. That is, once we choose a diagonal, we look at the smaller side of it and, since there are 2^k edges, we can choose the vertex which leaves exactly 2^{k-1} edges at each side to draw a new isosceles triangle at every step until the triangulation is complete.

We want to find the values of $n = 2^c + 2^d$ such that $n + 2013 = 2^a + 2^b$. We assume that $a \geq b$ and $c \geq d$. That is, $2^a + 2^b - 2^c - 2^d = 2013$. Since 2013 is odd, either b or d is 0. If $d = 0$, $2^a + 2^b - 2^c = 2014$. But 4 does not divide 2014, so either $b = 1$ or $c = 1$. This means that either $2^a + 2^b = 2016$ or $2^a - 2^c = 2012$, none of which has a solution. Hence $b = 0$ and $2^a - 2^c - 2^d = 2012$. Thus, $2^a > 2012$, and therefore $a \geq 11$. If $a > 11$, then $2^{c+1} \geq 2^c + 2^d = 2^a - 2012 > 2^{a-1}$. We deduce that $c = a - 1$ and $2012 = 2^c - 2^d$, which has no solution. We conclude that $a = 11$, $n + 2013 = 2^{11} + 2^0 = 2049$, and $n = 36 = 2^5 + 2^2$. The only solution is $n = 36$.

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