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New generalizations and refinements of Ionescu-Weitzenböck inequality

by

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Abstract. In this paper we present some new generalizations and refinements of *Ionescu-Weitzenböck* inequality.

1. Introduction

In Romanian Mathematical Gazette, Vol. III (1897-1898), *Ion Ionescu* published [6]
Problem 273. *In any triangle ABC with sides of lengths a, b, c and area F, the following inequality holds:*

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}F \tag{I}$$

In 1919, *Roland Weitzenböck* published in Mathematische Zeitschrift, Vol. 5, No. 1-2, pp. 137-146 the paper [7], in which this statement appeared again:
In any triangle ABC with sides of lengths a, b, c and area F, the following inequality holds:

$$a^2+b^2+c^2\geq 4\sqrt{3}F \tag{W}$$

We note that the inequality of *Ion Ionescu* (I) is the same as the inequality of *Roland Weitzenböck* (W), and for this reason we named it *Ionescu-Weitzenböck* inequality (I-W); see for example [1] and [2]. In [5] 11 proofs for (I-W) are given, and 23 new proofs are presented in [2]. In [4], 10 generalizations for (I-W) are presented.

2. Preliminary results

In the following, we shall obtain several new algebraic inequalities.

Theorem 1. *If t, u, v, $w\in[0,\infty)$, $t+u=v+w\in(0,\infty)$ and x, $y\in(0,\infty)$, then*

$$(t+u)(x+y)\geq 2\sqrt{(vx+wy)(vy+wx)}\geq 2(t+u)\sqrt{xy} \tag{1}$$

Proof. We have

$$\begin{aligned}(t+u)(x+y)=&(v+w)(x+y)=vx+vy+wx+wy\\ =&(vx+wy)+(vy+wx)\\ \geq&2\sqrt{(vx+wy)(vy+wx)}\end{aligned} \tag{2}$$

We also have

$$\begin{aligned}(vx+wy)(vy+wx)&=v^2xy+vwx^2+vwy^2+w^2xy\\ &=vw(x^2+y^2)+(v^2+w^2)xy\\ &\geq 2vw\sqrt{x^2y^2}+(v^2+w^2)xy\\ &=xy(v^2+2vw+w^2)\\ &=xy(v+w)^2=(t+u)^2xy\end{aligned} \tag{3}$$

By using (2) and (3) we obtain (1). □

If t, $u \in [0, \infty)$, $t + u \in (0, \infty)$ and x, $y \in (0, \infty)$ we denote

$$D(t, u, x, y) = (tx + uy)(ty + ux)$$

Then, we have

$$\begin{aligned} D(t, u, x, y) =& tu(x^2 + y^2) + (t^2 + u^2)xy \\ =& tu(x + y)^2 + (t^2 + u^2)xy - 2tuxy \\ =& tu(x + y)^2 + (t - u)^2 xy \end{aligned} \tag{4}$$

As

$$(x + y)^2 - (x - y)^2 = 4xy, \quad \forall\, x, y \in \mathbb{R}$$

by (4) we deduce

$$\begin{aligned} D(t, u, x, y) &= tu(x + y)^2 + (t - u)^2 \frac{(x + y)^2 - (x - y)^2}{4} \\ &= \frac{1}{4}\left(4(x + y)^2 tu + (t - u)^2(x + y)^2 - (t - u)^2(x - y)^2\right) \\ &= \frac{1}{4}\left((t + u)^2(x + y)^2 - (t - u)^2(x - y)^2\right) \end{aligned}$$

and if we denote $q = t + u \in (0, \infty)$, the previous yields that

$$D(t, u, x, y) = \frac{1}{4}\left(q^2(x + y)^2 - (q - 2u)^2(x - y)^2\right) \tag{5}$$

Hence, the relations from theorem 1 become

$$\begin{aligned} &(t+u)(x + y) \geq 2\sqrt{D(t, u, x, y)} \geq 2(t + u)\sqrt{xy} \\ &\quad \iff q(x + y) \geq 2\sqrt{D(t, u, x, y)} \geq 2q\sqrt{xy} \end{aligned} \tag{6}$$

If we take $t + u = v + w = q = 1$, relation (6) becomes

$$\frac{x + y}{2} \geq \sqrt{D(t, u, x, y)} \geq \sqrt{xy}$$

If x, y, $z \in (0, \infty)$, then by (6) we obtain

$$\begin{aligned} &q \sum_{\text{cyclic}} (x + y) \geq 2 \sum_{\text{cyclic}} \sqrt{D(t, u, x, y)} \geq 2q \sum_{\text{cyclic}} \sqrt{xy} \\ &\iff q(x + y + z) \geq \sum_{\text{cyclic}} \sqrt{D(t, u, x, y)} \geq q \sum_{\text{cyclic}} \sqrt{xy} \end{aligned} \tag{7}$$

If we take $t+u=v+w=q=1$, $t=v=1-\lambda$, $u=w=\lambda$, $\lambda\in[0,1]$, then by theorem 1 we obtain lemma 2.1 from [1]. Therefore, we have

$$\begin{aligned}D(t,u,x,y)&=D(1-\lambda,\lambda,x,y)=D(\lambda,x,y)\\&=((1-\lambda)x+\lambda y)((1-\lambda)y+\lambda x)\\&=\frac{1}{4}(x+y)^2-\frac{1}{4}(1-2\lambda)^2(x-y)^2\\&=\left(\frac{x+y}{2}\right)^2-(1-2\lambda)^2\left(\frac{x-y}{2}\right)^2\end{aligned}$$

$$x+y\geq 2\sqrt{D(\lambda,x,y)}\geq 2\sqrt{xy}$$

$$x+y+z\geq\sum_{\text{cyclic}}\sqrt{D(\lambda,x,y)}\geq\sum_{\text{cyclic}}\sqrt{xy}$$

If $q\in(0,\infty)$ is a fixed constant, then (5) becomes

$$D(t,u,x,y)=D(t,x,y)=\frac{q^2}{4}(x+y)^2-\frac{(q-2t)^2}{4}(x-y)^2$$

where $t\in[0,q]$.

We observe that

$$\begin{aligned}D(q-t,x,y)=&\frac{1}{4}(x+y)^2q^2-\frac{1}{4}(q-2(q-t))^2(x-y)^2\\=&\frac{q^2}{4}(x+y)^2-\frac{(-q+2t)^2}{4}(x-y)^2\\=&\frac{q^2}{4}(x+y)^2-\frac{(q-2t)^2}{4}(x-y)^2=D(t,x,y)\end{aligned}$$

If we consider the function

$$D:[0,q]\to\mathbb{R},\quad D(t)=\frac{q^2}{4}(x+y)^2-\frac{(q-2t)^2}{4}(x-y)^2$$

we have

$$D'(t)=4(q-2t)\left(\frac{x-y}{2}\right)^2=(q-2t)(x-y)^2$$

As $D'(t)=0\iff t=\frac{1}{2}q$, the sign of D' is constant between 0 and $\frac{1}{2}q$, so we deduce that

$$D(0,x,y)\leq D(t,x,y)\leq D\left(\frac{q}{2},x,y\right),\quad\forall\,t\in[0,q]$$

Theorem 2. *If $t, u, v, w \in [0,\infty)$, $t+u=v+w \in (0,\infty)$ and $x, y, z \in (0,\infty)$, then*

$$\begin{aligned}(t+u)(x+y+z) &\geq \sum_{cyclic} \sqrt{(vx+wy)(vy+wx)} \\ &\geq (t+u)\sum_{cyclic}\sqrt{xy} \geq 3(t+u)\sqrt[3]{xyz} \end{aligned} \tag{8}$$

Proof. We have

$$\begin{aligned}(t+u)(x+y+z) =&(t+u)\left(\frac{x+y}{2}+\frac{y+z}{2}+\frac{z+x}{2}\right)\\ =&\frac{t+u}{2}(x+y)+\frac{t+u}{2}(y+z)+\frac{t+u}{2}(z+x)\end{aligned}$$

so by (1) we obtain

$$\begin{aligned}(t+u)(x+y+z) &\geq \sum_{\text{cyclic}} \sqrt{(vx+wy)(vy+wx)} \\ &\geq (t+u)\sum_{\text{cyclic}}\sqrt{xy} \\ &\geq 3(t+u)\sqrt[3]{\prod_{\text{cyclic}}\sqrt{xy}} \\ &= 3(t+u)\sqrt[3]{xyz}\end{aligned}$$

on account of AM-GM inequality. □

Theorem 3. *If $t, u, v, w \in [0,\infty)$, $t+u=v+w \in (0,\infty)$ and $x, y, z \in (0,\infty)$, then*

$$\begin{aligned}(t+u)(x+y+z) &\geq 3\sqrt[3]{(vx+wy)(vy+wz)(vz+wx)} \\ &\geq 3(t+u)\sqrt[3]{xyz}\end{aligned} \tag{9}$$

Proof. We have

$$\begin{aligned}(t+u)(x+y+z) &= (v+w)(x+y+z) \\ &= vx+vy+vz+wx+wy+wz \\ &= (vx+wy)+(vy+wz)+(vz+wx) \\ &\geq 3\sqrt[3]{(vx+wy)(vy+wz)(vz+wx)}\end{aligned} \tag{10}$$

$$
\begin{aligned}
&(vx+wy)(vy+wz)(vz+wx)\\
&\quad=v^3xyz+v^2wxz^2+v^2wy^2z+vw^2yz^2\\
&\quad+v^2wx^2y+vw^2x^2z+vw^2xy^2+w^3xyz\\
&\quad=\left(v^3+w^3\right)xyz+v^2w\left(xz^2+y^2z+x^2y\right) \qquad (11)\\
&\quad+vw^2\left(yz^2+x^2z+xy^2\right)\\
&\quad\geq\left(v^3+w^3\right)xyz+\left(v^2w+vw^2\right)3\sqrt[3]{xz^2\cdot y^2z\cdot x^2y}\\
&\quad=\left(v^3+w^3\right)xyz+3vw(v+w)\sqrt[3]{(xyz)^3}\\
&\quad=\left(v^3+w^3+3vw(v+w)\right)xyz\\
&\quad=(v+w)^3xyz=(t+u)^3xyz \qquad (12)
\end{aligned}
$$

By (10) and (11) we deduce (9) □

We denote

$$E(t,u,x,y,z)=(tx+uy)(ty+uz)(tz+ux)$$

Then (9) becomes

$$(t+u)(x+y+z)\geq 3\sqrt[3]{E(t,u,x,y,z)}\geq 3(t+u)\sqrt[3]{xyz} \qquad (13)$$

$\forall\ t,\ u\in[0,\infty),\ t>u,\ x,\ y,\ z\in(0,\infty)$.

Let $n,\ t,\ u$ be real numbers such that $n,\ t,\ u\in[0,\infty),\ t>u$ and $x,\ y\in(0,\infty)$. We denote

$$
\begin{aligned}
F(n,t,u,x,y)=&\left(\left(t+u+(t-u)^n\right)x+\left(t+u-(t-u)^n\right)y\right)\\
&\cdot\left(\left(t+u+(t-u)^n\right)y+\left(t+u-(t-u)^n\right)x\right)
\end{aligned} \qquad (14)
$$

Theorem 4. *If $n,\ t,\ u\in[0,\infty),\ t>u$ and $x,\ y\in(0,\infty)$, then*

$$F(n,t,u,x,y)=(t+u)^2(x+y)^2-(t-u)^{2n}(x-y)^2$$

Proof. By (14) we have

$$\begin{aligned}
F(n,t,u,x,y) =& \left(t+u+(t-u)^n\right)^2 xy + \left((t+u)^2-(t-u)^{2n}\right) y^2 \\
&+ \left((t+u)^2-(t-u)^{2n}\right) x^2 + \left(t+u-(t-u)^n\right)^2 xy \\
=& \left((t+u)^2-(t-u)^{2n}\right)\left(x^2+y^2\right) \\
&+ \left(\left(t+u+(t-u)^n\right)^2+\left(t+u-(t-u)^n\right)^2\right) xy \\
=& (t+u)^2(x^2+y^2) - (t-u)^{2n}(x^2+y^2) \\
&+2(t+u)^2xy + 2(t-u)^{2n}xy \\
=& (t+u)^2(x^2+2xy+y^2) - (t-u)^{2n}(x^2-2xy+y^2) \\
=& (t+u)^2(x+y)^2 - (t-u)^{2n}(x-y)^2
\end{aligned}$$

□

Theorem 5. *If n, p, t, $u \in [0,\infty)$, $n \geq p$, $t > u$ and x, $y \in (0,\infty)$, then*

$$\begin{aligned}
&F(n,t,u,x,y) - F(p,t,u,x,y) \\
&\qquad = (x-y)^2(t-u)^{2n}\left(1-(t-u)^{2(n-p)}\right) \qquad (15)
\end{aligned}$$

Proof. By theorem 4 we have

$$\begin{aligned}
&F(n,t,u,x,y) - F(p,t,u,x,y) \\
&\quad =(t+u)^2(x+y)^2 - (t-u)^{2n}(x-y)^2 \\
&\quad -(t+u)^2(x+y)^2 + (t-u)^{2p}(x-y)^2 \\
&\quad =(x-y)^2(t-u)^{2p}\left(1-(t-u)^{2(n-p)}\right)
\end{aligned}$$

□

Equation (15) yields that if $n \geq p$ and $t-u<1$, then

$$F(n,t,u,x,y) \geq F(p,t,u,x,y)$$

Hence, the function $F:[0,\infty) \to (0,\infty)$, $F(n) = F(n,t,u,x,y)$ is increasing if $t-u<1$.

If $t+u=q \in (0,\infty)$ we have

$$\begin{aligned}
F(n,t,u,x,y) =& \left((q+(t-u)^n)x + (q-(t-u)^n)y\right) \\
&\cdot \left((q+(t-u)^n)y + (q-(t-u)^n)x\right) \\
=& q^2(x+y)^2 - (q-2u)^{2n}(x-y)^2
\end{aligned}$$

Let n be a positive integer number, $t,\ u \in [0,\infty)$, $t+u=q \in (0,\infty)$, where q is fixed, and $x,\ y \in (0,\infty)$. We consider the function $G:[0,q]\to(-\infty,+\infty)$, $G(u)=F(n,t,u,x,y)$, namely

$$G(u)=q^2(x+y)^2-(q-2u)^{2n}(x-y)^2$$

The function G is derivable and

$$\begin{aligned}G'(u)=&-2n(q-2u)^{2n-1}(-2)(x-y)^2\\ =&4n(q-2u)^{2n-1}(x-y)^2\\ =&4n(q-2u)(q-2u)^{2(n-1)}(x-y)^2\end{aligned}$$

We have $G'(u)=0\Rightarrow u=\dfrac{1}{2}q$ and

$$G(q-u)=G(u),\ \ \forall\ u\in[0,q]$$

Since the sign of G' is constant between 0 and $\frac{1}{2}q$, we obtain that

$$G(0)=q^2(x+y)^2-q^{2n}(x-y)^2\le G(u)\le G\left(\frac{1}{2}q\right)=q^2(x+y)^2$$

$\forall\ u\in[0,q]$, which is equivalent to

$$q^2(x+y)^2-q^{2n}(x-y)^2\le F(n,t,u,x,y)\le q^2(x+y)^2,\ \ \forall\ u\in[0,q]$$

3. Main results

Theorem 6. *If $m,\ t,\ u\in[0,\infty)$, $t+u=q\in(0,\infty)$, then in any triangle ABC with usual notations and area S holds:*

$$\begin{aligned}q(a^{m+1}+b^{m+1}+c^{m+1})\ge&\sum_{cyclic}\sqrt{D(t,u,a^{m+1},b^{m+1})}\\ \ge&3\cdot q(abc)^{\frac{m+1}{3}}\\ \ge&2^{m+1}\cdot q\cdot 3^{\frac{3-m}{4}}\cdot S^{\frac{m+1}{2}}\end{aligned}\tag{16}$$

Proof. If we take $x = a^{m+1}$, $y = b^{m+1}$, $z = c^{m+1}$, then by (7) we obtain

$$\begin{aligned} q(a^{m+1} + b^{m+1} + c^{m+1}) &\geq \sum_{\text{cyclic}} \sqrt{D(t, u, a^{m+1}, b^{m+1})} \\ &\geq q \sum_{\text{cyclic}} (ab)^{\frac{m+1}{2}} \geq 3q \sqrt[3]{(ab)^{\frac{m+1}{2}} (bc)^{\frac{m+1}{2}} (ca)^{\frac{m+1}{2}}} \\ &= 3q \sqrt[3]{(abc)^{m+1}} = 3q \left(\sqrt[3]{(abc)^2} \right)^{\frac{m+1}{2}} \end{aligned} \tag{17}$$

By *Pólya-Szegő* inequality we have

$$\sqrt[3]{(abc)^2} \geq \frac{4S}{\sqrt{3}} \tag{P-S}$$

Then, (17) becomes

$$\begin{aligned} q(a^{m+1} + b^{m+1} + c^{m+1}) &\geq \sum_{\text{cyclic}} \sqrt{D(t, u, a^{m+1}, b^{m+1})} \\ &\geq q \sum_{\text{cyclic}} \sqrt{(ab)^{m+1}} q \geq 3q \left(\sqrt[3]{(abc)^2} \right)^{\frac{m+1}{2}} \\ &\geq 3q \left(\frac{4S}{\sqrt{3}} \right)^{\frac{m+1}{2}} = 2^{m+1} \cdot q \cdot 3^{\frac{3-m}{4}} \cdot S^{\frac{m+1}{2}} \end{aligned}$$

□

If $m = 1$, then by (16) we obtain generalizations and refinements of *Ionescu-Weitzenböck* inequality. That is,

$$\begin{aligned} q(a^2 + b^2 + c^2) &\geq \sum_{\text{cyclic}} \sqrt{D(t, u, a^2, b^2)} \\ &\geq q \sum_{\text{cyclic}} ab \geq 3q(abc)^{\frac{2}{3}} \geq 4q\sqrt{3}S \end{aligned} \tag{18}$$

If we take $q = 1$, then by (18) we deduce the following refinements of *Ionescu-Weitzenböck* inequality

$$\begin{aligned} a^2 + b^2 + c^2 &\geq \sum_{\text{cyclic}} \sqrt{D(t, u, a^2, b^2)} \\ &\geq \sum_{\text{cyclic}} ab \geq 3(abc)^{\frac{2}{3}} \geq 4\sqrt{3}S \end{aligned}$$

Theorem 7. *If $m, t, u, v, w \in [0,\infty)$, $t+u=v+w\in(0,\infty)$, then in any triangle ABC with usual notations and area S the following holds*

$$\begin{aligned}(t+u)(a^{m+1}+b^{m+1}+c^{m+1})&\\ \geq \sum_{cyclic}\sqrt{(va^{m+1}+wb^{m+1})\,(vb^{m+1}+wa^{m+1})}&\\ \geq (t+u)\sum_{cyclic}(ab)^{\frac{m+1}{2}}\geq 3(t+u)(abc)^{\frac{m+1}{3}}&\\ \geq 2^{m+1}\cdot 3^{\frac{3-m}{4}}(t+u)S^{\frac{m+1}{2}}&\end{aligned}\tag{19}$$

Proof. If we take $x=a^{m+1}$, $y=b^{m+1}$, $z=c^{m+1}$, then by (8) we deduce that

$$\begin{aligned}(t+u)(a^{m+1}+b^{m+1}+c^{m+1})&\\ \geq \sum_{\text{cyclic}}\sqrt{(va^{m+1}+wb^{m+1})\,(vb^{m+1}+wa^{m+1})}&\\ \geq (t+u)\sum_{\text{cyclic}}(ab)^{\frac{m+1}{2}}\geq 3(t+u)(abc)^{\frac{m+1}{3}}&\\ =3(t+u)\left(\sqrt[3]{(abc)^2}\right)^{\frac{m+1}{2}}&\end{aligned}$$

and using (P-S) we obtain

$$\begin{aligned}(t+u)(a^{m+1}+b^{m+1}+c^{m+1})&\\ \geq \sum_{\text{cyclic}}\sqrt{(va^{m+1}+wb^{m+1})\,(vb^{m+1}+wa^{m+1})}&\\ \geq (t+u)\sum_{\text{cyclic}}(ab)^{\frac{m+1}{2}}\geq 3(t+u)\left(\sqrt[3]{(abc)^2}\right)^{\frac{m+1}{2}}&\\ \geq 3(t+u)\left(\frac{4S}{\sqrt{3}}\right)^{\frac{m+1}{2}}=2^{m+1}\cdot 3^{\frac{3-m}{4}}(t+u)S^{\frac{m+1}{2}}&\end{aligned}$$

□

If we take $m=1$, then by (19) we obtain new generalizations and refinements of *Ionescu-Weitzenböck* inequality.

Theorem 8. *If $m, t, u, v, w \in [0, \infty)$, $t + u = v + w \in (0, \infty)$, then in any triangle ABC with usual notations and area S the following holds*

$$\begin{aligned}(t+u)&(a^{m+1} + b^{m+1} + c^{m+1}) \\ &\geq 3\sqrt[3]{(va^{m+1} + wb^{m+1})\,(vb^{m+1} + wc^{m+1})\,(vc^{m+1} + wa^{m+1})} \\ &\geq 3(t+u)\left(\sqrt[3]{abc}\right)^{m+1} \geq 2^{m+1}(t+u)\cdot 3^{\frac{3-m}{4}}\cdot S^{\frac{m+1}{2}}\end{aligned}$$

Proof. If we take $x = a^{m+1}$, $y = b^{m+1}$, $z = c^{m+1}$, then equation (9) becomes

$$\begin{aligned}(t+u)&(a^{m+1} + b^{m+1} + c^{m+1}) \\ &\geq 3\sqrt[3]{(va^{m+1} + wb^{m+1})\,(vb^{m+1} + wc^{m+1})\,(vc^{m+1} + wa^{m+1})} \\ &\geq 3(t+u)\left(\sqrt[3]{abc}\right)^{m+1} = 3(t+u)\left(\sqrt[3]{(abc)^2}\right)^{\frac{m+1}{2}}\end{aligned}$$

whence by (P-S) we obtain the result □

Theorem 9. *If $m, t, u, v, w \in [0, \infty)$, $t + u = v + w \in (0, \infty)$, then in any triangle ABC with usual notations and area S the following holds*

$$\begin{aligned}(t+u)(a^{2m} + b^{2m} + c^{2m}) &\geq 3\sqrt[3]{E(v, w, a^{2m}, b^{2m}, c^{2m})} \\ &\geq 3(t+u)\sqrt[3]{(abc)^{2m}}q \\ &\geq 2^{2m}(t+u)\cdot 3^{\frac{2-m}{2}}\cdot S^m\end{aligned}$$

Proof. If we take $x = a^{2m}$, $y = b^{2m}$, $z = c^{2m}$, then by (13) we obtain

$$\begin{aligned}(t+u)(a^{2m} + b^{2m} + c^{2m}) &\geq 3\sqrt[3]{E(v, w, a^{2m}, b^{2m}, c^{2m})} \\ &\geq 3(t+u)\sqrt[3]{(abc)^{2m}} = 3(t+u)\left(\sqrt[3]{(abc)^2}\right)^m \\ &\geq 3(t+u)\left(\frac{4S}{\sqrt{3}}\right)^m = 2^{2m}(t+u)\cdot 3^{\frac{2-m}{2}}\cdot S^m\end{aligned}$$

on account of P-S inequality. □

Theorem 10. *If $m, t, u, v, w \in [0, \infty)$, $t+u = v+w \in (0, \infty)$, then in any triangle ABC with usual notations and area S the following holds*

$$\begin{aligned}(t+u)(a^{2m}+b^{2m}+c^{2m}) &\\ \geq 3\sqrt[3]{(va^{2m}+wb^{2m})\,(vb^{2m}+wc^{2m})\,(vc^{2m}+wa^{2m})} &\\ \geq 3(t+u)\left(\sqrt[3]{abc}\right)^{2m} \geq 2^{2m}(t+u)\cdot 3^{\frac{2-m}{2}}\cdot S^m &\end{aligned}$$

Proof. If we take $x = a^{2m}$, $y = b^{2m}$, $z = c^{2m}$, then by (9) we deduce

$$\begin{aligned}(t+u)(a^{2m}+b^{2m}+c^{2m}) &\\ \geq 3\sqrt[3]{(va^{2m}+wb^{2m})\,(vb^{2m}+wc^{2m})\,(vc^{2m}+wa^{2m})} &\\ \geq 3(t+u)\left(\sqrt[3]{abc}\right)^{2m} = 3(t+u)\left(\sqrt[3]{(abc)^2}\right)^{m} &\qquad (20)\end{aligned}$$

Applying in (20) the (P-S) inequality we obtain new generalizations and refinements of *Ionescu-Weitzenböck* inequality, namely

$$\begin{aligned}(t+u)(a^{2m}+b^{2m}+c^{2m}) &\\ \geq 3\sqrt[3]{(va^{2m}+wb^{2m})\,(vb^{2m}+wc^{2m})\,(vc^{2m}+wa^{2m})} &\\ \geq 3(t+u)\left(\sqrt[3]{abc}\right)^{2m} \geq 2^{2m}(t+u)\cdot 3^{\frac{2-m}{2}}\cdot S^m &\end{aligned}$$

□

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A Generalization Of Chebyshev's Inequality

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In this paper we present a generalization of the well-known Chebyshev's Inequaliy, and some interesting consequences.

Theorem 1 *Let $(a_n)_{n\in N}$, $(b_n)_{n\in N}$, $(c_n)_{n\in N}$ and $(d_n)_{n\in N}$ be real sequences such that $(a_n)_{n\in N}$ and $(b_n)_{n\in N}$ are simultaneously increasing or decreasing, and $c_nd_m \geq 0$ for all n, m. Then:*

$$\left(\sum_{i=1}^{n} a_ib_ic_i\right)\left(\sum_{i=1}^{n} d_i\right)+\left(\sum_{i=1}^{n} a_ib_id_i\right)\left(\sum_{i=1}^{n} c_i\right)$$
$$\geq \left(\sum_{i=1}^{n} a_ic_i\right)\left(\sum_{i=1}^{n} b_id_i\right)+\left(\sum_{i=1}^{n} b_ic_i\right)\left(\sum_{i=1}^{n} a_id_i\right)$$

Proof.

$$\left(\sum_{i=1}^{n} a_ib_ic_i\right)\left(\sum_{i=1}^{n} d_i\right)+\left(\sum_{i=1}^{n} a_ib_id_i\right)\left(\sum_{i=1}^{n} c_i\right)$$
$$-\left(\sum_{i=1}^{n} a_ic_i\right)\left(\sum_{i=1}^{n} b_id_i\right)-\left(\sum_{i=1}^{n} b_ic_i\right)\left(\sum_{i=1}^{n} a_id_i\right)$$
$$=\sum_{i=1}^{n}\sum_{j=1}^{n}(a_ib_ic_id_j+a_jb_jc_id_j-a_ib_jc_id_j-a_jb_ic_id_j)$$
$$=\sum_{i=1}^{n}\sum_{j=1}^{n}c_id_j(a_ib_i+a_jb_j-a_ib_j-a_jb_i)$$
$$=\sum_{i=1}^{n}\sum_{j=1}^{n}c_id_j(a_i-a_j)(b_i-b_j)\geq 0$$

since $c_id_j \geq 0$ $(i, j = 1, 2, ...)$ and $(a_n)_{n\in N}$, $(b_n)_{n\in N}$ are simultaneously increasing or decreasing sequences. The equality holds if at least $a_1 = a_2 = ... = a_n$ or $b_1 = b_2 = ... = b_n$ (it is also valid for $c_n = 0$ or $d_n = 0$ $(n = 1, 2, ...)$ but this case is not relevant). □

Note 1. If one of the real number sequences $(a_n)_{n\in N}$ and $(b_n)_{n\in N}$ is increasing and the other one is decreasing, and $(c_n)_{n\in N}$ and $(d_n)_{n\in N}$ are real number sequences for which $c_n d_m \leq 0$ for all n, m, then we find again the inequality from Theorem 1.

Note 2. If one of the real number sequences $(a_n)_{n\in N}$ and $(b_n)_{n\in N}$ is increasing and the other one is decreasing, and $(c_n)_{n\in N}$ and $(d_n)_{n\in N}$ are such that $c_n d_m \geq 0$ for all n, m, then we find an inequality that is the opposite of the one from Theorem 1.

Note 3. If the real number sequences $(a_n)_{n\in N}$ and $(b_n)_{n\in N}$ are simultaneously increasing or decreasing and $(c_n)_{n\in N}$ and $(d_n)_{n\in N}$ are such that $c_n d_m \leq 0$ for all n, m, then we find an inequality that is the opposite of the one from Theorem 1.

Theorem 2 *If $f, g : [a, b] \to \mathbb{R}$ are simultaneously increasing or decreasing functions that are integrable in the interval $[a, b]$ and $h, l : [a, b] \to \mathbb{R}$ are functions that are integrable in the interval $[a, b]$ for which $h(x)l(y) \geq 0 \quad \forall\, x, y \in [a, b]$, then:*

$$\int_a^b f(x)g(x)h(x)\,\mathrm{d}x \int_a^b l(x)\,\mathrm{d}x + \int_a^b f(x)g(x)l(x)\,\mathrm{d}x \int_a^b h(x)\,\mathrm{d}x$$
$$\geq \int_a^b f(x)h(x)\,\mathrm{d}x \int_a^b g(x)l(x)\,\mathrm{d}x + \int_a^b g(x)h(x)\,\mathrm{d}x \int_a^b f(x)l(x)\,\mathrm{d}x$$

Proof. We divide $[a, b]$ into n equal parts and introduce the notation $x = a + \frac{b-a}{n} i$ $(i = 1, 2, ..., n)$. We can replace the real numbers a_i, b_i, c_i, d_i from Theorem 1 with real numbers $f(x_i)$, $g(x_i)$, $h(x_i)$, $l(x_i)$ $(i = 1, 2, ..., n)$, as these fulfill the conditions of said theorem. Consequently, the proposition from Theorem 1 can be written as follows:

$$\sum_{i=1}^{n} f(x_i)g(x_i)h(x_i) \sum_{i=1}^{n} l(x_i) + \sum_{i=1}^{n} f(x_i)g(x_i)l(x_i) \sum_{i=1}^{n} h(x_i)$$
$$\geq \sum_{i=1}^{n} f(x_i)h(x_i) \sum_{i=1}^{n} g(x_i)l(x_i) + \sum_{i=1}^{n} g(x_i)h(x_i) \sum_{i=1}^{n} f(x_i)l(x_i)$$

If we multiply both sides of the inequality by $\left(\frac{b-a}{n}\right)^2$, the above inequality takes the following form:

$$\begin{aligned}
&\sum_{i=1}^{n} f(x_i)g(x_i)h(x_i)\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} l(x_i)\left(\frac{b-a}{n}\right)\\
&+\sum_{i=1}^{n} f(x_i)g(x_i)l(x_i)\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} h(x_i)\left(\frac{b-a}{n}\right)\\
&\geq\sum_{i=1}^{n} f(x_i)h(x_i)\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} g(x_i)l(x_i)\left(\frac{b-a}{n}\right)\\
&+\sum_{i=1}^{n} g(x_i)h(x_i)\left(\frac{b-a}{n}\right)\sum_{i=1}^{n} f(x_i)l(x_i)\left(\frac{b-a}{n}\right)
\end{aligned}$$

Taking into account the notation $x_i = a + \frac{b-a}{n}i$ $(i = 1, 2, ..., n)$, the inequality can be rewritten as:

$$\begin{aligned}
&\sum_{i=1}^{n} f(x_i)g(x_i)h(x_i)(x_i - x_{i-1})\sum_{i=1}^{n} l(x_i)(x_i - x_{i-1})\\
&+\sum_{i=1}^{n} f(x_i)g(x_i)l(x_i)(x_i - x_{i-1})\sum_{i=1}^{n} h(x_i)(x_i - x_{i-1})\\
&\geq\sum_{i=1}^{n} f(x_i)h(x_i)(x_i - x_{i-1})\sum_{i=1}^{n} g(x_i)l(x_i)(x_i - x_{i-1})\\
&+\sum_{i=1}^{n} g(x_i)h(x_i)(x_i - x_{i-1})\sum_{i=1}^{n} f(x_i)l(x_i)(x_i - x_{i-1})
\end{aligned}$$

We observe that the sums in the above inequality correspond to the Riemann's sums of some functions for the interval $[a, b]$. Since the inequality is preserved when taking limits on both sides, we take limits and write the previous expression as the inequality from Theorem 2. The equality holds if at least one of functions f or g is constant. □

If in Theorem 2 we consider $f = g$, then we find the following proposition:

Corollary 3 *If function $f : [a,b] \to \mathbb{R}$ is integrable in the interval $[a,b]$ and functions $h, l : [a,b] \to \mathbb{R}$ are integrable in the interval $[a,b]$ and satisfy $h(x)l(y) \geq 0 \ \forall \ x, y \in [a,b]$, then the following holds:*

$$\int_a^b f^2(x)h(x)\,\mathrm{d}x \int_a^b l(x)\,\mathrm{d}x + \int_a^b f^2(x)l(x)\,\mathrm{d}x \int_a^b h(x)\,\mathrm{d}x \geq 2\int_a^b f(x)h(x)\,\mathrm{d}x \int_a^b f(x)l(x)\,\mathrm{d}x$$

If in Theorem 2 we consider $l(x) = \frac{A}{h(x)}$, $h(x) \neq 0$ for all $x \in [a,b]$, and $A > 0$ is a real constant, then we find the following proposition:

Corollary 4 *If functions $f, g : [a,b] \to \mathbb{R}$ are integrable in the interval $[a,b]$ and are simultaneously increasing or decreasing, and $h : [a,b] \to \mathbb{R}_+ \backslash \{0\}$ is a function that is integrable in the interval $[a,b]$, then:*

$$\int_a^b f(x)g(x)h(x)\,\mathrm{d}x \int_a^b \frac{\mathrm{d}x}{h(x)} + \int_a^b \frac{f(x)g(x)}{h(x)}\,\mathrm{d}x \int_a^b h(x)\,\mathrm{d}x \geq \int_a^b f(x)h(x)\,\mathrm{d}x \int_a^b \frac{g(x)}{h(x)}\,\mathrm{d}x + \int_a^b g(x)h(x)\,\mathrm{d}x \int_a^b \frac{f(x)}{h(x)}\,\mathrm{d}x$$

Setting $g = f^{-1}$ in Theorem 2, where f^{-1} is the inverse of f, then we find the following proposition:

Corollary 5 *If function* $f : [a,b] \to \mathbb{R}$ *is continuous in the interval* $[a,b]$ *and strictly monotone and surjective, and* $h, l : [a,b] \to \mathbb{R}$ *are integrable in the interval* $[a,b]$ *and satisfy* $h(x)l(y) \geq 0 \ \forall \ x, y \in [a,b]$*, then:*

$$\int_a^b f(x)f^{-1}(x)h(x)\,\mathrm{d}x \int_a^b l(x)\,\mathrm{d}x + \int_a^b f(x)f^{-1}(x)l(x)\,\mathrm{d}x \int_a^b h(x)\,\mathrm{d}x$$
$$\geq \int_a^b f(x)h(x)\,\mathrm{d}x \int_a^b f^{-1}(x)l(x)\,\mathrm{d}x + \int_a^b f^{-1}(x)h(x)\,\mathrm{d}x \int_a^b f(x)l(x)\,\mathrm{d}x$$

Taking into account Note 2 and the notation $g = \frac{A}{f}$, where $A > 0$ is a constant number from Theorem 2, we find:

Corollary 6 *If function* $f : [a,b] \to \mathbb{R}\backslash\{0\}$ *integrable in the interval* $[a,b]$ *is monotone, and* $h, l : [a,b] \to \mathbb{R}$ *are integrable in the interval* $[a,b]$ *and satisfy* $h(x)l(y) \geq 0 \ \forall \ x, y \in [a,b]$*, then:*

$$2\int_a^b h(x)\,\mathrm{d}x \int_a^b l(x)\,\mathrm{d}x$$
$$\leq \int_a^b f(x)h(x)\,\mathrm{d}x \int_a^b \frac{l(x)}{f(x)}\,\mathrm{d}x + \int_a^b \frac{h(x)}{f(x)}\,\mathrm{d}x \int_a^b f(x)l(x)\,\mathrm{d}x$$

If in Theorem 2 we consider $h(x) = l(x) = f^{m-1}(x)$, then we find the following proposition:

Corollary 7 *If* $f, g : [a,b] \to \mathbb{R}$ *are simultaneously increasing or decreasing functions integrable in the interval* $[a,b]$*,* $(f \neq 0)$ *then, for any positive integer* m*:*

$$\int_a^b f^m(x)g(x)\,\mathrm{d}x \int_a^b f^{m-1}(x)\,\mathrm{d}x$$
$$\geq \int_a^b f^m(x)\,\mathrm{d}x \int_a^b f^{m-1}(x)g(x)\,\mathrm{d}x$$

If in Theorem 2 we consider $h(x) = l(x) = C > 0$ then we find the following proposition:

Corollary 8 If $f, g : [a, b] \to \mathbb{R}$ are simultaneously increasing or decreasing functions integrable in the interval $[a, b]$, then:

$$(b-a)\int_a^b f(x)g(x)dx \geq \int_a^b f(x)dx \int_a^b g(x)dx$$

which is no less than Chebyshev's inequality for integrals.

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Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: **José Luis Díaz-Barrero**, Applied Mathematics III, UPC BARCELONA TECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain or by e-mail to: *jose.luis.diaz@upc.edu*.

The section is divided into four subsections: Elementary Problems, Easy-Medium High School Problems, Medium-Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

March 30, 2015

Elementary Problems

E–11. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* What is the value of

$$\sqrt[3]{45+29\sqrt{2}}+\sqrt[3]{45-29\sqrt{2}}$$

in its simplest form?

E–12. *Proposed by Félix Gimeno Gil, CFIS, Barcelona Tech, Barcelona, Spain.* Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that $f(a\cdot b)=f(a)f(b)$ and for all positive integer n we have $gcd(n,f(n))=1$.

E–13. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* The areas of the three distinct faces of an ortohedron are $182, 195$, and 210, respectively. Find its volume.

E–14. *Proposed by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain.* Find the values of the positive integer n for which $n^2-27n+181$ is a perfect square.

E–15. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let a, b, c be three odd positive integers. Show that the equation $ax^2+bx+c=0$ does not have rational roots.

E–16. *Proposed by Ander Lamaison Vidarte and Alberto Espuny-Díaz, CFIS, Barcelona Tech, Barcelona, Spain.* Prove that the 2015-gon inscribed in a circle that has the maximum surface is the regular one.

Easy–Medium Problems

EM–11. *Proposed by Alberto Espuny-Díaz, CFIS, Barcelona Tech, Barcelona, Spain.* How many palindrome integers with an even number of digits are also prime numbers?

EM–12. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let x, y, z be three non unitary real numbers such that $x + y + z + 1 = 0$ and $x^2 + y^2 + z^2 - 1 = 0$. Show that

$$\frac{xy}{x+y} + \frac{yz}{y+z} + \frac{zx}{z+x}$$

is an integer and determine its value.

EM–13. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Show that in any triangle ABC the following holds:

$$\frac{1}{2}\left(\frac{\sin A + \sin B + \sin C}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}\right) \leq \mathcal{A},$$

where $\mathcal{A}$ represents the area of $\triangle ABC$.

EM–14. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let α, β, γ be the roots of the polynomial equation $x^3 - 7x^2 + 5x + 3 = 0$. Find the value of $\alpha^4 + \beta^4 + \gamma^4$.

EM–15. *Proposed by Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.* Let Γ be a circumference of radius r, and A, B and C be three points inside Γ that lie on a circumference Ψ, with radius R. If $R > r$, prove that the triangle ABC has an obtuse angle.

EM–16. *Proposed by Andrés Sáez-Schwedt, Universidad de León, León, Spain.* Find the smallest positive integer n such that 3^{2014} divides $(2014)^{(3^n)} - 1$.

Medium–Hard Problems

MH–11. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let $n \geq 2$ be a positive integer. Find all real solutions of the following system of equations:

$$\begin{cases} 3^{x_2} = 1 + \dfrac{1}{n}\left(3^{x_1} + 3^{-x_1} - 2\right) \\ 3^{x_3} = 1 + \dfrac{1}{n}\left(3^{x_2} + 3^{-x_2} - 2\right) \\ \qquad \vdots \\ 3^{x_n} = 1 + \dfrac{1}{n}\left(3^{x_{n-1}} + 3^{-x_{n-1}} - 2\right) \\ 3^{x_1} = 1 + \dfrac{1}{n}\left(3^{x_n} + 3^{-x_n} - 2\right) \end{cases}$$

MH–12. *Proposed by Andrés Sáez-Schwedt, Universidad de León, León, Spain.* Let I be the incenter of triangle ABC. The line through I perpendicular to AI meets AC at E and AB at F. Let P and Q be the circumcenters of BEI and CFI, respectively. Prove that the midpoint of PQ belongs to AI.

MH–13. *Proposed by Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.* Let a, b and c be three positive integers, each of them without repeated digits in their decimal expression. If $a+b+c = 58888$, prove that there is a digit, other than 0, which appears in at least two of the three numbers.

MH–14. *Proposed by Francesc Gispert-Sánchez, CFIS, Barcelona Tech, Barcelona, Spain.* Let I be the incenter of a triangle ABC and let D, E and F be the points of intersection of the circumcircle of the triangle and the extended angle bisectors AI, BI and CI, respectively. Then, the intersection of triangles ABC and DEF forms an hexagon with vertices K, L, M, N, P and Q, taken counterclockwise. Prove that KN, LP and MQ meet at I.

MH–15. *Proposed by Marius Dragan, Bucharest, Romania.* Let A, B, C be the angles measured in radians of a triangle ABC with sides of length a, b, c and semiperimeter p. Prove that

$$aA^2 + bB^2 + cC^2 \geq \frac{2\pi^2}{9}\, p$$

MH–16. *Proposed by Nicolae Papacu, Slobozia, Romania.* Let $a, b, c \in (0, \infty)$ such that $a \geq b \geq c$. Prove that

$$\begin{aligned}\frac{3}{2^{n-1}} &\geq \frac{a^n + b^n}{(a+b)^n} + \frac{b^n + c^n}{(b+c)^n} + \frac{c^n + a^n}{(c+a)^n} \\ &\geq \frac{3}{2^n}\left(\sqrt{\frac{a}{c}} + \sqrt{\frac{c}{a}}\right)^n + 3\left(\frac{1}{2^{n-1}} - 1\right)\end{aligned}$$

where $n \in \mathbb{N}^*$

Advanced Problems

A–11. *Proposed by Francesc Gispert-Sánchez, CFIS, Barcelona Tech, Barcelona, Spain.* Let n be a positive integer such that $n \geq 5$ and $n \equiv 1 \pmod 4$ and consider $m = \left\lfloor \frac{n}{2} \right\rfloor$. We define the function

$$f(x_1, \ldots, x_m) = \sum_{j=1}^{\frac{m}{2}} (2j-1)\, x_j^{8(2j)} + \sum_{j=\frac{m}{2}+1}^{m} (2j-1)\, x_j^{8[2(m-j)+1]}$$

where $x_1, \ldots, x_m$ are positive real numbers. Show that for any positive real numbers $a_1, a_2, \ldots, a_m$ the following inequality holds:

$$\prod_{\text{cyc}} \left[f(a_1, \ldots, a_m) + (n+m)(n-m) \right]^{\frac{1}{m}} \geq n^2 (a_1 \ldots a_m)^{\frac{n-1}{n}}$$

A–12. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let n be a positive integer. Compute

$$\lim_{n\to\infty} \left\{ 1 + \sum_{k=1}^{n} \frac{1}{k} - (\ln n + \gamma) \right\}^n,$$

where $\gamma = 0.577216\ldots$ is the Euler–Mascheroni constant.

A–13. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let n be a positive integer. Prove that

$$1 + \sum_{k=1}^{n} F_k \leq \frac{1}{2} \left(\frac{F_n^2}{F_{n+2}} + \frac{F_{n+1}^2}{F_n} + \frac{F_{n+2}^2}{F_{n+1}} \right),$$

where F_n is the n^{th} Fibonacci number defined by $F_0 = 0, F_1 = 1$ and for all $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$.

A–14. *Proposed by Andrés Sáez-Schwedt, Universidad de León, León, Spain.* Let n be a positive integer. If $(1+\sqrt{3})^n$ is written as $a + b\sqrt{3}$, with a, b integers, find $\gcd(a, b)$ in terms of n.

A–15. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let $A(z) = \sum_{k=0}^{n} a_k z^k$ be a monic polynomial with complex coefficients. If $\alpha = \max\limits_{1\le k\le n}\left\{|z_k|\right\}$, where z_k are the roots of $A(z)$, then show that

$$\frac{1}{\alpha}\sum_{k=1}^{n}\left|\frac{a_{n-k}}{n^k C(n,n-k)}\right|^{1/k} \le 1$$

A–16. *Proposed by Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.* We say that a positive integer $n \ge 3$ is *isoscelable* if the regular n-gon can be divided into $n-2$ isosceles triangles using $n-3$ diagonals. Find all values of n such that n and $n+2013$ are both *isoscelable*.

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Send submittals to: **José Luis Díaz-Barrero**, Applied Mathematics III, UPC BARCELONA TECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain or by e-mail to: *jose.luis.diaz@upc.edu*.

The Torricelli point and some applications in algebra

Camelia Oprea

1 Introduction

In real life we are often confronted with the problem of placing an object at the smallest distance from three given points, for example, in order to minimize the costs of transport, pipes, wires, etc. For instance, one might want to find the optimal position of a thermoelectric power plant which is going to serve three towns A, B and C not on a line such that the total length of the used pipes is minimized.

Pierre de Fermat (1601-1665) was the first to propose the problem of determining the point located at the minimum distance from the vertices of a triangle, while Evangelista Torricelli (1608-1647) was the first one to solve it. Due to this, the point with the property mentioned before is known as the "Torricelli point" or the "Fermat point", denoted by T.

2 Theoretical results

Theorem 1 (Torricelli)**.** *Let ABC be a triangle with all angles strictly smaller than $120°$ and consider an equilateral triangle constructed outwards on each of its sides, namely ABC_1, ACB_1 and BCA_1. Then, the circumscribed circles of these triangles have a point in common.*

Proof. Let T be the intersection point other than A of the circumscribed circles of the triangles ABC_1 and ACB_1. Thus, the quadrilaterals $ATBC_1$ and $ATCB_1$ are inscriptible and $\angle ATB = 180° - \angle AC_1B = 120°$, since $\triangle AC_1B$ is equilateral. Similarly, $\angle ATC = 120°$.

Therefore, $\angle BTC = 360° - \angle ATB - \angle ATC = 120°$, from which it follows that the quadrilateral $BTCA_1$ is also inscriptible and T must lie on the circle circumscribed about BCA_1. □

The proof above also shows that there exists a unique point T such that $\angle ATB = \angle ATC = \angle BTC = 120°$. This point T is the Torricelli point of the triangle ABC.

Theorem 2. *Let ABC be a triangle with all angles strictly smaller than $120°$ and consider an equilateral triangle constructed outwards on each of its sides, namely ABC_1, ACB_1 and BCA_1, as in the previous theorem. Then,*

(a) the lines AA_1, BB_1 and CC_1 are concurrent at the Torricelli point T;

(b) $AT + BT + CT = AA_1 = BB_1 = CC_1$

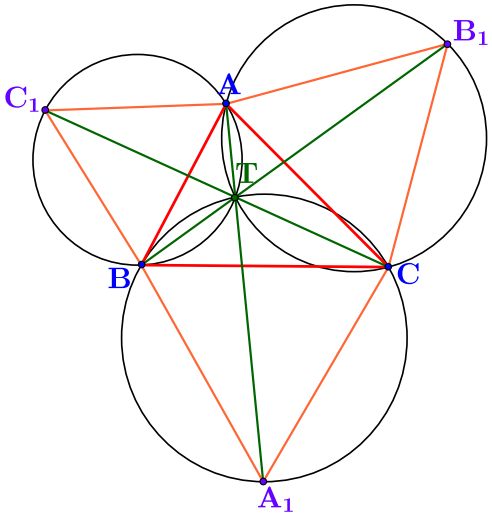

Figure 1: Possible construction to obtain the Torricelli point.

Proof. Let T be the intersection point of the circumscribed circles of $\triangle ABC_1$, $\triangle ACB_1$ and $\triangle BCA_1$. Considering the inscribed quadrilateral $BTCA_1$, we observe that $\angle A_1TC = \angle A_1BC = 60°$. Then, we have that $\angle A_1TA = \angle A_1TC + \angle CTA = 60° + 120° = 180°$ and it immediately follows that the point T lies on the line AA_1. Similar arguments yield that T belongs to the lines BB_1 and CC_1. In conclusion, $AA_1 \cap BB_1 \cap CC_1 = \{T\}$.

Applying Ptolemy's theorem to the convex inscribed quadrilateral $BTCA_1$ and knowing that $\triangle BCA_1$ is equilateral, we obtain that

$$BC \times TA_1 = BT \times A_1C + CT \times A_1B \Longrightarrow TA_1 = BT + CT$$

and then $AA_1 = AT + TA_1 = AT + BT + CT$. Finally, the same argument can be applied to the quadrilaterals $ATBC_1$ and $ATCB_1$. □

Theorem 3 (Fermat)**.** *Let ABC be a triangle with all angles strictly smaller than $120°$. The sum $S(M) = MA + MB + MC$ (where M is a point on the plane) attains its minimum value at the Torricelli point T of $\triangle ABC$.*

Proof. Let M be an arbitrary point on the plane.

We consider an equilateral triangle constructed outwards on each of the sides of $\triangle ABC$, namely ABC_1, ACB_1 and BCA_1. At least one of the quadrilaterals MAB_1C, MBA_1C and MAC_1B is convex. We can assume, without loss of generality, that MBA_1C is convex and, by Ptolemy's theorem, we have that

$$MA_1 \times BC \leq MB \times A_1C + MC \times A_1B \Longrightarrow MA_1 \leq MB + MC$$

and then $AA_1 \leq AM + MA_1 \leq MA + MB + MC$ and the equality holds if and only if $M = T$. □

From this, another geometric construction of the Torricelli point T follows. Let ABC be a triangle with all angles strictly smaller than 120° and let M be a point lying inside it. Determining the point T is equivalent to finding the point at which the sum $MA + MB + MC$ is minimized.

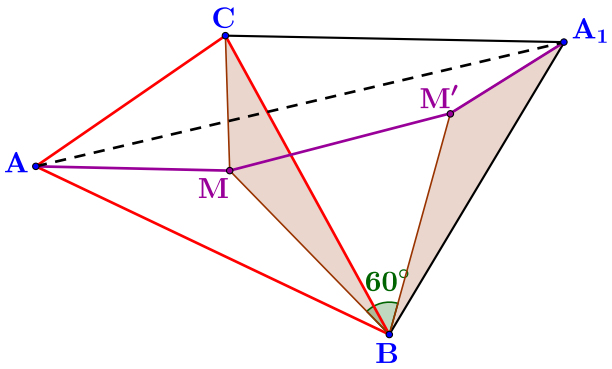

Figure 2: Another construction to obtain the Torricelli point.

Firstly, we rotate $\triangle BCM$ 60° about the point B towards the exterior of $\triangle ABC$. This rotation sends C to A_1 and M to M'. Moreover, the triangles BCA_1 and BMM' are equilateral, by construction. Therefore, the sum of the distances $MA + MB + MC = AM + MM' + M'A_1$ is in general the length of a polygonal chain from A to A_1. But the shortest polygonal chain from A to A_1 is a straight line and T must lie on the line AA_1. Similarly, we obtain that T lies on BB_1 and CC_1. That is, the lines AA_1, BB_1 and CC_1 meet at T.

In conclusion, we have seen that

$$\begin{aligned} MA + MB + MC \leq TA + TB + TC = AA_1 = BB_1 = CC_1 \ \forall \ M \\ \{T\} = AA_1 \cap BB_1 \cap CC_1 \\ 120^\circ = \angle ATB = \angle ATC = \angle BTC \end{aligned}$$

Hitherto, we have only considered the case in which the angles of the triangle ABC are strictly smaller than 120°. If $\triangle ABC$ has an angle equal to 120°, assume it is $\angle BCA$ and then the Torricelli point is C, as can be easily seen using the same construction as above. If $\angle BCA > 120^\circ$, then $T = C$.

Let a, b and c denote the lengths of the sides of the triangle ABC. Let $P = a + b + c$ be the perimeter of the triangle and $d_T = TA + TB + TC$ the sum of distances from the Torricelli point of $\triangle ABC$ to each of its vertices.

The ratio $\frac{d_T}{P}$ is maximized when $\triangle ABC$ is equilateral, in which case $d_T = \frac{P}{\sqrt{3}}$. On the other hand, the ratio is minimized when the points A, B and C are collinear, in which case the point T is the point of the triangle between the other two and $d_T = \frac{P}{2}$. Thus, the following double inequality holds:

$$\frac{1}{2} \leq \frac{d_T}{P} \leq \frac{1}{\sqrt{3}}$$

Let us determine an expression for d_T in terms of a, b, c and S, the area of $\triangle ABC$. We consider an equilateral triangle $\triangle BCA_1$ constructed outwards on the side BC. Then, applying the law of cosines to $\triangle ABA_1$, we obtain

$$\begin{aligned} d_T^2 = AA_1^2 &= a^2 + c^2 - 2ac\cos\left(\angle ABC + 60^\circ\right) \\ &= a^2 + c^2 - 2ac\left(\cos\angle ABC \times \cos 60^\circ - \sin\angle ABC \times \sin 60^\circ\right) \\ &= a^2 + c^2 - ac\cos\angle ABC + \sqrt{3}ac\sin\angle ABC, \end{aligned}$$

and applying the law of cosines to $\triangle ABC$, we obtain

$$b^2 = a^2 + c^2 - 2ac\cos\angle ABC \implies -ac\cos\angle ABC = \frac{b^2 - a^2 - c^2}{2}$$

Furthermore,

$$S = \frac{ac \sin \angle ABC}{2} \Longrightarrow ac \sin \angle ABC = 2S$$

Substituting into the first expression, we obtain

$$d_T^2 = a^2 + c^2 + \frac{b^2 - a^2 - c^2}{2} + 2\sqrt{3}S = \frac{a^2 + b^2 + c^2 + 4\sqrt{3}S}{2},$$

from which it immediately follows that

$$d_T = \sqrt{\frac{a^2 + b^2 + c^2 + 4\sqrt{3}S}{2}}$$

Let x, y and z denote the distances TA, TB and TC, respectively, from the Torricelli point T to each of the vertices of the triangle ABC. Applying the law of cosines to $\triangle BTC$, $\triangle ATB$ and $\triangle ATC$, we obtain the following expressions for the side lengths of $\triangle ABC$:

$$a = \sqrt{y^2 + yz + z^2}$$
$$b = \sqrt{z^2 + zx + x^2}$$
$$c = \sqrt{x^2 + xy + y^2}$$

3 Applications

Problem 1. *Let x, y and z be positive real numbers such that:*

(i) $x^2 + xy + y^2 = 25$
(ii) $y^2 + yz + z^2 = 169$
(iii) $z^2 + zx + x^2 = 144$

Compute the value of $xy + yz + zx$.

Solution. An algebraic solution for this kind of problems requires much effort and is hard to find.

We observe, taking into account the law of cosines, the following:

- (i) is equivalent to the existence of a triangle with side lengths x, y and 5 and with an angle of $120°$ between sides x and y;

- (ii) is equivalent to the existence of a triangle with side lengths y, z and 13 and with an angle of 120° between sides y and z;
- (iii) is equivalent to the existence of a triangle with side lengths z, x and 12 and with an angle of 120° between sides z and x.

We draw the three triangles with a common vertex T, where the sides of lengths x, y and z meet, forming the three angles of 120°, as in figure 3. We obtain a triangle ABC with side lengths 5, 13 and 12, which has a right angle (as it satisfies the Pythagorean theorem), and T is its Torricelli point because $\angle ATB = \angle ATC = \angle BTC = 120°$.

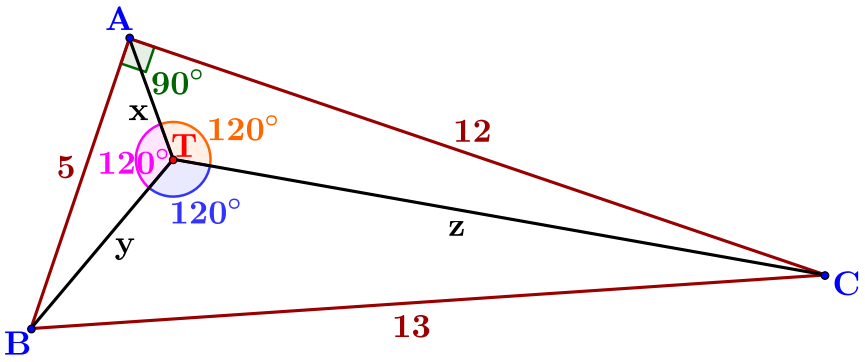

Figure 3: Scheme for the solution of problem 1.

Now, we compute the area of $\triangle ABC$ in two ways:

$$\begin{aligned} S_{ABC} &= S_{TAB} + S_{TBC} + S_{TAC} \\ &= \frac{xy \sin 120° + yz \sin 120° + zx \sin 120°}{2} \\ &= \frac{\sqrt{3}}{4}(xy + yz + zx) \end{aligned}$$

and

$$S_{ABC} = \frac{bc}{2} = 30$$

Setting equal the two right-hand sides yields

$$xy + yz + zx = 40\sqrt{3}$$

□

Problem 2. *Prove that*

$$3\left(x^2+xy+y^2\right)\left(y^2+yz+z^2\right)\left(z^2+zx+x^2\right) \geq (x+y+z)^2\,(xy+yz+zx)^2$$

for all positive real numbers x, y and z.

Solution. Since $x, y, z \in \mathbb{R}^+$, we can think of them as distances. Let T be a point on the plane. We draw segments TA, TB and TC of lengths x, y and z, respectively, and such that $\angle ATB = \angle ATC = \angle BTC = 120°$, as in figure 4. By construction, we have that T is the Torricelli point of $\triangle ABC$. Therefore, the side lengths of $\triangle ABC$ are $c = \sqrt{x^2+xy+y^2}$, $a = \sqrt{y^2+yz+z^2}$ and $b = \sqrt{z^2+zx+x^2}$.

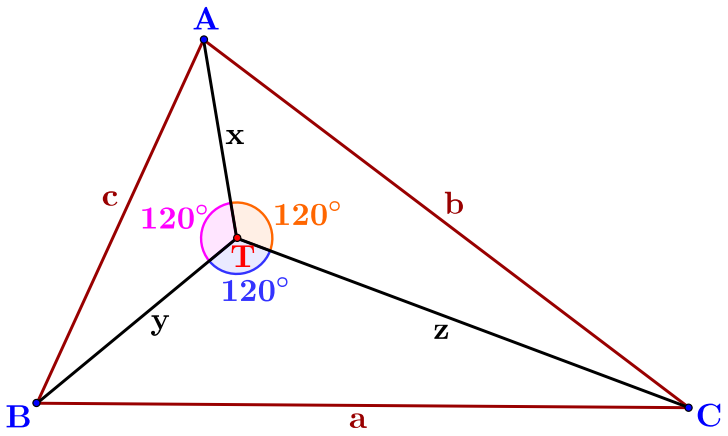

Figure 4: Scheme for the solution of problem 2.

We can compute the area S_{ABC} of $\triangle ABC$ as

$$\begin{aligned} S_{ABC} &= S_{TAB} + S_{TBC} + S_{TAC} \\ &= \frac{(xy+yz+zx)\sin 120°}{2} = \frac{\sqrt{3}}{4}(xy+yz+zx) \end{aligned}$$

and the initial inequality becomes

$$3a^2b^2c^2 \geq (x+y+z)^2\left(\frac{4S_{ABC}}{\sqrt{3}}\right)^2 \iff 3abc \geq 4S_{ABC}\,(x+y+z)$$

since all the terms are positive.

Furthermore, we have that

$$S_{ABC} = \frac{abc}{4R},$$

where R is the circumradius of $\triangle ABC$. Substituting in the previous expression, we obtain the equivalent inequality:

$$x + y + z \leq 3R$$

Finally, by Fermat's theorem, we have that $TA + TB + TC \leq MA + MB + MC$ for all points M on the plane. In particular, for the circumcenter O of $\triangle ABC$, we have that $x + y + z \leq 3R$, which is what we wanted to prove. □

Problem 3. *Prove that*

$$\sqrt{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\leq\frac{\sqrt{a^2+ab+b^2}}{ab}+\frac{\sqrt{b^2+bc+c^2}}{bc}+\frac{\sqrt{c^2+ca+a^2}}{ca}$$

for all positive real numbers a, b and c.

Solution. Let T be a point on the plane. We draw segments TX, TY and TZ of lengths a, b and c, respectively, and such that $\angle XTY = \angle YTZ = \angle ZTX = 120°$. T is the Torricelli point of $\triangle XYZ$ by construction. We drop the perpendicular TD to YZ from T, where D is the intersection point. Finally, we draw TE the bisector of $\angle YTZ$, with E on YZ.

On the one hand, we can express

$$S_{YTZ} = \frac{bc\sin 120°}{2} = \frac{YZ \times TD}{2},$$

from which it immediately follows that

$$TD = \frac{\sqrt{3}}{2}\frac{bc}{YZ} = \frac{\sqrt{3}}{2}\frac{bc}{\sqrt{b^2+bc+c^2}}$$

On the other hand, we can compute the length of the angle bisector

$$TE = \frac{\sqrt{bc}}{b+c}\sqrt{(b+c)^2-(b^2+bc+c^2)} = \frac{bc}{(b+c)}$$

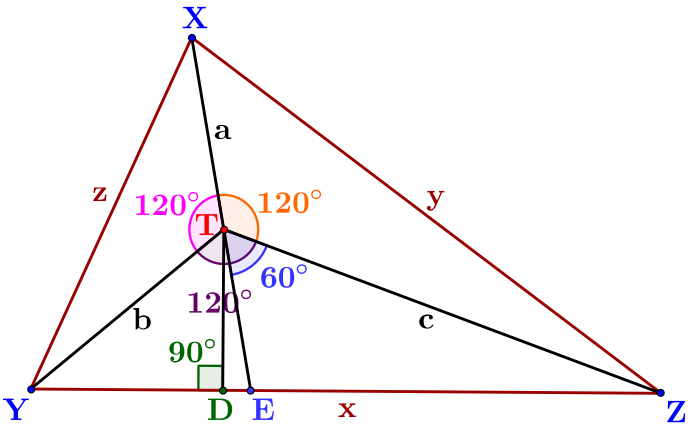

Figure 5: Scheme for the solution of problem 3.

And, by construction, we have that $TD \leq TE$. Therefore,

$$\frac{\sqrt{b^2+bc+c^2}}{bc} \geq \frac{\sqrt{3}}{2}\frac{b+c}{bc} = \frac{\sqrt{3}}{2}\left(\frac{1}{b}+\frac{1}{c}\right)$$

Similarly, we obtain that

$$\frac{\sqrt{c^2+ca+a^2}}{ca} \geq \frac{\sqrt{3}}{2}\left(\frac{1}{c}+\frac{1}{a}\right) \text{ and } \frac{\sqrt{a^2+ab+b^2}}{ab} \geq \frac{\sqrt{3}}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$$

and adding up these inequalities yields the desired inequality. □

Problem 4. *Show that for all positive real numbers x, y and z*

$$3\,(x+y+z) \leq \sum_{cyc} \sqrt{(x+y+z)^2+3\,(x^2-yz)}$$

Solution. Since x, y and z are positive, we can consider them distances. Let T be a point on the plane. We draw segments TA, TB and TC of lengths x, y and z, respectively, and such that $\angle ATB = \angle ATC = \angle BTC = 120°$. By construction, we have that T is the Torricelli point of $\triangle ABC$. Therefore, the side lengths of $\triangle ABC$ are $c = \sqrt{x^2+xy+y^2}$, $a = \sqrt{y^2+yz+z^2}$ and $b = \sqrt{z^2+zx+x^2}$.

The right-hand side of the inequality can then be written as

$$\sum_{cyc} \sqrt{(x+y+z)^2 + 3(x^2 - yz)}$$
$$= \sum_{cyc} \sqrt{2(x^2+xy+y^2) + 2(x^2+xz+z^2) - (y^2+yz+z^2)}$$
$$= 2\sum_{cyc} \sqrt{\frac{2(b^2+c^2)-a^2}{4}} = 2(m_a + m_b + m_c)$$

by Apollonius' theorem, where m_a, m_b and m_c are the lengths of the medians from A, B and C, respectively.

The inequality now becomes $x + y + z \leq \frac{2}{3}(m_a + m_b + m_c)$. Furthermore, we have that x, y and z are the distances TA, TB and TC from the Torricelli point of $\triangle ABC$ to each of its vertices, and $\frac{2}{3}m_a$, $\frac{2}{3}m_b$ and $\frac{2}{3}m_c$ are the distances GA, GB and GC from the centroid G of $\triangle ABC$ to each of its vertices. Therefore, the inequality immediately follows from Fermat's theorem, which asserts that $TA + TB + TC \leq GA + GB + GC$. □

Problem 5. *Let ABC be an acute triangle. Find the equilateral triangle with exactly one point of A, B and C on each of its sides which has the greatest possible area.*

Solution. We are going to construct an equilateral triangle MNP such that A lies on PM, B lies on MN and C lies on NP.

Since $\triangle MNP$ is equilateral, we have that $\angle CPA = \angle AMB = \angle BNC = 60°$. That is, if we consider the equilateral triangles ABC_1, CAB_1 and BCA_1 constructed outwards on each of the sides of $\triangle ABC$, then M must lie on the circumcircle of $\triangle ABC_1$, N must lie on the circumcircle of $\triangle BCA_1$ and P must lie on the circumcircle of $\triangle CAB_1$ (all three points in the exterior of $\triangle ABC$), as in figure 6.

We choose a point M on the circumcircle of $\triangle ABC_1$ (outside $\triangle ABC$). Let N be the point of intersection (different from B) between MB and the circumcircle of $\triangle BCA_1$ and let P be the point of intersection (different from A) between MA and the circumcircle of $\triangle CAB_1$.

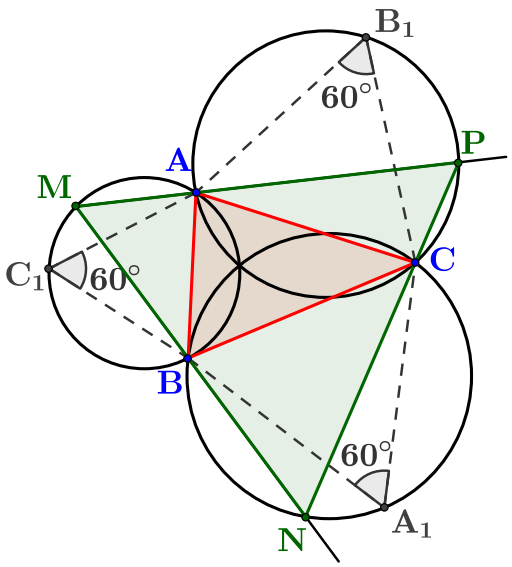

Figure 6: Possible equilateral triangles in problem 5.

We have to show that C lies on NP or, equivalently, that N, C and P are collinear. But it is easy to observe that

$$\begin{aligned}\angle ACP &= 120^\circ - \angle CAP = 120^\circ - (180^\circ - \angle BAC - \angle BAM)\\ &= \angle BAC + \angle BAM - 60^\circ\end{aligned}$$

and

$$\begin{aligned}\angle BCN &= 120^\circ - \angle CBN = 120^\circ - (180^\circ - \angle ABC - \angle ABM)\\ &= \angle ABC + \angle ABM - 60^\circ\end{aligned}$$

from which it immediately follows that

$$\begin{aligned}\angle NCP &= \angle NCB + \angle BCA + \angle ACP\\ &= \angle ABC + \angle BCA + \angle CAB + \angle BAM + \angle ABM - 120^\circ\\ &= 180^\circ\end{aligned}$$

Now, from all the triangles constructed this way, we want to obtain the one with the greatest area. As all these triangles are equilateral, the greatest area is attained with the largest side length. That is to say, we have to choose M in order to maximize the length of MN.

Let O_1 be the circumcenter of $\triangle ABC_1$ and O_2 be the circumcenter of $\triangle BCA_1$. We claim that the maximum length is attained when MN is parallel to O_1O_2. Let K be the perpendicular projection of O_1 on MB and let L be the perpendicular projection of O_2 on BN. Let S be the point of intersection between O_2L and a parallel to MN through O_1. On the one hand, it is clear that $O_1S = KL$. Furthermore, $KB = \frac{1}{2}MB$ and $BL = \frac{1}{2}BN$, so $KL = \frac{1}{2}MN$. On the other hand, we have that $O_1S \leq O_1O_2$ and the equality holds if and only if $S = O_2$ (because O_1S is the distance from O_1 to O_2L). In conclusion, $MN \leq 2O_1O_2$ with equality if and only if MN is parallel to O_1O_2.

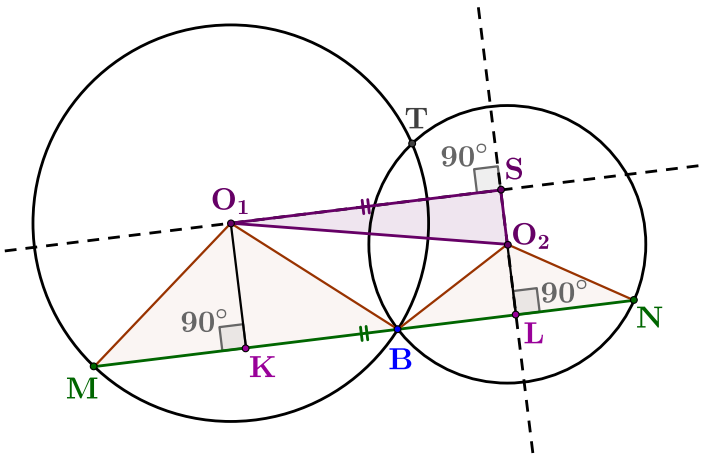

Figure 7: Sketch of the minimum side length in problem 5.

Equivalently, the triangle we are looking for has its sides perpendicular to TA, TB and TC (where T is the Torricelli point of $\triangle ABC$). □

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Rearrangement Inequality Revisited

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1 Introduction

There is no doubt that one of the most fundamental principles in the theory of inequalities is the systematic employment of order relationships. An elementary application arises in a natural way. Indeed, if $a_1 \leq a_2$ and $b_1 \leq b_2$ then $0 \leq (a_2 - a_1)(b_2 - b_1)$, from which $a_1b_2 + a_2b_1 \leq a_1b_1 + a_2b_2$ follows. This is precisely the rearrangement inequality for $n = 2$. In the following we will state and prove by induction the general case that appeared in 1934 in the famous book of Hardy-Littlewood and Pólya [1].

Theorem 1. *Let $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \leq b_2 \leq \ldots \leq b_n$ be sequences of real numbers and let $c_1, c_2, \ldots, c_n$ be a permutation of $b_1, b_2, \ldots, b_n$. Then, it is true that*

$$\sum_{k=1}^{n} a_k b_k \geq \sum_{k=1}^{n} a_k c_k \geq \sum_{k=1}^{n} a_k b_{n-k+1}$$

If the a_i are all different, equality holds if and only if $(c_1, c_2, \ldots, c_n) = (b_1, b_2, \ldots, b_n)$ or $(c_1, c_2, \ldots, c_n) = (b_n, b_{n-1}, \ldots, b_1)$, respectively. Inequalities reverse if the two sequences are sorted oppositely, one increasing and the other decreasing.

Proof. To prove the preceding result we argue by mathematical induction. First, we prove LHS inequality. Indeed, let

$$P(n) \ : \ \sum_{k=1}^{n} a_k b_k \geq \sum_{k=1}^{n} a_k c_k$$

The case when $n = 1$ trivially holds. Suppose that $P(k)$ holds for $1 \leq k \leq n$ and we have to see that $P(n+1)$ also holds. WLOG we can assume that

$$a_1 \leq a_2 \leq \ldots \leq a_i \leq \ldots \leq a_j \leq \ldots \leq a_n \leq a_{n+1}$$

$$b_1 \leq b_2 \leq \ldots \leq b_i \leq \ldots \leq b_j \leq \ldots \leq b_n \leq b_{n+1}$$

and suppose that $c_i = b_{n+1}$ and $c_{n+1} = b_j$ for some index i, j.

Since $(a_{n+1} - a_i)(b_{n+1} - b_j) \geq 0$, then $a_i b_j + a_{n+1} b_{n+1} \geq a_i c_i + a_{n+1} c_{n+1}$, or equivalently $a_i c_{n+1} + a_{n+1} c_i \geq a_i c_i + a_{n+1} c_{n+1}$. Taking into account $P(n)$ we can now prove $P(n+1)$:

$$\begin{aligned}
& a_1 c_1 + a_2 c_2 + \ldots + a_i c_i + \ldots + a_n c_n + a_{n+1} c_{n+1} \\
\leq\ & a_1 c_1 + a_2 c_2 + \ldots + a_i c_{n+1} + \ldots + a_n c_n + a_{n+1} c_i \\
=\ & a_1 c_1 + a_2 c_2 + \ldots + a_i b_j + \ldots + a_n c_n + a_{n+1} b_{n+1} \\
\overset{P(n)}{\leq}\ & a_1 b_1 + a_2 b_2 + \ldots + a_i b_i + \ldots + a_n b_n + a_{n+1} b_{n+1}
\end{aligned}$$

The RHS inequality is obtained from the preceding replacing the sequence $b_1 \leq b_2 \leq \ldots \leq b_n$ by $-b_n \leq -b_{n-1} \leq \ldots \leq -b_1$. This completes the proof.

□

Rearrangement Inequality can also be stated in the following form:

Let $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1, b_2, \ldots, b_n$ be sequences of real numbers and let $c_1, c_2, \ldots, c_n$ be a permutation of $b_1, b_2, \ldots, b_n$. The sum $S = a_1c_1 + a_2c_2 + \ldots + a_nc_n$ is maximal if the two sequences $a_1, a_2, \ldots, a_n$ and $c_1, c_2, \ldots, c_n$ are sorted in the same way and minimal if the two sequences are sorted oppositely, one increasing and the other decreasing.

Two special cases of the preceding are the following corollaries.

Corollary 2. *Let $a_1, a_2, \ldots, a_n$ be real numbers and let $c_1, c_2, \ldots, c_n$ be any permutation of them. Then, the following holds*

$$a_1^2 + a_2^2 + \ldots + a_n^2 \geq a_1c_1 + a_2c_2 + \ldots + a_nc_n$$

Corollary 3. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers and let $c_1, c_2, \ldots, c_n$ be any permutation of them. Then, the following holds*

$$\frac{c_1}{a_1} + \frac{c_2}{a_2} + \ldots + \frac{c_n}{a_n} \geq n$$

In 1935, participants in the famous Eötvös Competition in Hungary were asked to prove the above Corollary ([2]). From the above theorem and its corollaries it is possible to derive a lot of useful inequalities. For instance, all the classical inequalities used in this paper and derived in this way can be found in ([3],[4]).

2 Main Results

Hereafter, we present two applications of the rearrangement inequality. We begin with

Theorem 4. *Let a, b, and c be positive real numbers. Then,*

$$\left(\frac{(4n+1)a - b}{b+c}\right)^2 + \left(\frac{(4n+1)b - c}{c+a}\right)^2 + \left(\frac{(4n+1)c - a}{a+b}\right)^2 \geq 12n^2$$

holds for any positive integer $n \geq 1$.

Proof. Without loss of generality we can assume that $a \geq b \geq c$ from which it immediately follows that $a+b \geq a+c \geq b+c$ and $\dfrac{1}{b+c} \geq \dfrac{1}{c+a} \geq \dfrac{1}{a+b}$. Since the first and the last sequences are sorted in the same way, by applying rearrangement inequality, we get

$$\begin{aligned} 2n\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) &\geq 2n\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right) \\ 2n\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) &\geq 2n\left(\frac{b}{a+b}+\frac{c}{b+c}+\frac{a}{c+a}\right) \\ \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} &\geq \frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} \end{aligned}$$

Adding up the preceding inequalities yields

$$(4n+1)\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right) \geq 6n+\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}$$

from which we obtain

$$\frac{1}{3}\left(\frac{(4n+1)a-b}{b+c}+\frac{(4n+1)b-c}{c+a}+\frac{(4n+1)c-a}{a+b}\right) \geq 2n$$

Taking into account AM-QM inequality, we have

$$\begin{aligned} &\sqrt{\frac{1}{3}\left[\left(\frac{(4n+1)a-b}{b+c}\right)^2+\left(\frac{(4n+1)b-c}{c+a}\right)^2+\left(\frac{(4n+1)c-a}{a+b}\right)^2\right]} \\ &\geq \frac{1}{3}\left(\frac{(4n+1)a-b}{b+c}+\frac{(4n+1)b-c}{c+a}+\frac{(4n+1)c-a}{a+b}\right) \geq 2n \end{aligned}$$

from which the statement follows. Equality holds when $a=b=c$ and we are done.

□

The particular case when $n=1$ has appeared recently in [5].

Next we state a constrained inequality that will be proven using the rearrangement inequality.

Theorem 5. *Let a, b, c, d be four positive real numbers such that $abcd = 1$. Then, for all $n \geq 1$ it is true that*

$$\frac{a^{2n+1}+b^{2n-1}}{b+c+d}+\frac{b^{2n+1}+c^{2n-1}}{c+d+a}+\frac{c^{2n+1}+d^{2n-1}}{d+a+b}+\frac{d^{2n+1}+a^{2n-1}}{a+b+c} \geq \frac{8}{3}$$

Proof. Since the vectors $(a^{2n+1}, b^{2n+1}, c^{2n+1}, d^{2n+1})$ and $\left(\frac{1}{b+c+d}, \frac{1}{c+d+a}, \frac{1}{d+a+b}, \frac{1}{a+b+c}\right)$ have its components sorted in the same way, as can be easily checked, then we have

$$\begin{aligned}&\frac{a^{2n+1}}{b+c+d}+\frac{b^{2n+1}}{c+d+a}+\frac{c^{2n+1}}{d+a+b}+\frac{d^{2n+1}}{a+b+c}\\ &\geq \frac{b^{2n+1}}{b+c+d}+\frac{c^{2n+1}}{c+d+a}+\frac{d^{2n+1}}{d+a+b}+\frac{a^{2n+1}}{a+b+c},\end{aligned}$$

and

$$\begin{aligned}&\frac{a^{2n+1}+b^{2n-1}}{b+c+d}+\frac{b^{2n+1}+c^{2n-1}}{c+d+a}+\frac{c^{2n+1}+d^{2n-1}}{d+a+b}+\frac{d^{2n+1}+a^{2n-1}}{a+b+c}\\ \geq &\frac{b^{2n+1}+b^{2n-1}}{b+c+d}+\frac{c^{2n+1}+c^{2n-1}}{c+d+a}+\frac{d^{2n+1}+d^{2n-1}}{d+a+b}+\frac{a^{2n+1}+a^{2n-1}}{a+b+c}\end{aligned}$$

Applying AM-GM inequality, we have

$$\begin{aligned}&\frac{b^{2n+1}+b^{2n-1}}{b+c+d}+\frac{c^{2n+1}+c^{2n-1}}{c+d+a}+\frac{d^{2n+1}+d^{2n-1}}{d+a+b}+\frac{a^{2n+1}+a^{2n-1}}{a+b+c}\\ &\geq \frac{2b^{2n}}{b+c+d}+\frac{2c^{2n}}{c+d+a}+\frac{2d^{2n}}{d+a+b}+\frac{2a^{2n}}{a+b+c}\end{aligned}$$

Now, we apply Cauchy's inequality to the vectors

$$\vec{u} = \left(\frac{b^n}{\sqrt{b+c+d}}, \frac{c^n}{\sqrt{c+d+a}}, \frac{d^n}{\sqrt{d+a+b}}, \frac{a^n}{\sqrt{a+b+c}}\right)$$

and

$$\vec{v} = \left(\sqrt{b+c+d}, \sqrt{c+d+a}, \sqrt{d+a+b}, \sqrt{a+b+c}\right)$$

to obtain

$$\frac{2b^{2n}}{b+c+d}+\frac{2c^{2n}}{c+d+a}+\frac{2d^{2n}}{d+a+b}+\frac{2a^{2n}}{a+b+c}\geq\frac{2(a^n+b^n+c^n+d^n)^2}{3(a+b+c+d)}$$

On account of mean inequalities, we have $a^n+b^n+c^n+d^n\geq\frac{(a+b+c+d)^n}{4^{n-1}}$, and

$$\begin{aligned}&\frac{a^{2n+1}+b^{2n-1}}{b+c+d}+\frac{b^{2n+1}+c^{2n-1}}{c+d+a}+\frac{c^{2n+1}+d^{2n-1}}{d+a+b}+\frac{d^{2n+1}+a^{2n-1}}{a+b+c}\\&\geq\frac{2(a^n+b^n+c^n+d^n)^2}{3(a+b+c+d)}\geq\frac{2(a+b+c+d)^{2n}}{3\cdot4^{2n-2}(a+b+c+d)}\\&=\frac{2\,(a+b+c+d)^{2n-1}}{3\cdot4^{2n-2}}\geq\frac{2\left(4\sqrt[4]{abcd}\right)^{2n-1}}{3\cdot4^{2n-2}}=\frac{8}{3}\end{aligned}$$

Equality holds when $a=b=c=d=1$ and the proof is complete. □

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Solutions

No problem is ever permanently closed. We will be very pleased to consider for publication new solutions or comments on the past problems.

Elementary Problems

E–1. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Without the aid of a computer, calculate the following sum:

$$2014^2 - 2013^2 + 2012^2 - 2011^2 + \ldots + 4^2 - 3^2 + 2^2 - 1^2$$

Solution by the proposer. We have

$$2014^2 - 2013^2 + 2012^2 - 2011^2 + \ldots + 4^2 - 3^2 + 2^2 - 1^2$$

$$= (2-1)(2+1) + (4-3)(4+3) + \ldots + (2014-2013)(2014+2013)$$

$$= 1 + 2 + 3 + 4 + 5 + 6 + \ldots + 2013 + 2014 = \frac{(1+2014)\cdot 2014}{2}$$

$$= 2015 \cdot 1007 = 2029105$$

Solution by Alberto Espuny-Díaz, CFIS, Barcelona Tech, Barcelona, Spain. We have

$$2014^2 - 2013^2 + 2012^2 - 2011^2 + \ldots + 4^2 - 3^2 + 2^2 - 1^2$$

$$= \sum_{n=1}^{1007} (2n)^2 - (2n-1)^2 = \sum_{n=1}^{1007} (4n-1) = 4\sum_{n=1}^{1007} n - 1007$$

$$= 4\frac{1007}{2}1008 - 1007 = 1007 \cdot 2015 = 2029105$$

Also solved by Eduardo Espuny-Díaz, Universitat Pompeu Fabra, Barcelona, Spain; Guillermo Girona San Miguel, CFIS, Barcelona Tech, Barcelona, Spain, and Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.

E–2. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* On a table there are several notes of $5, 10, 20, 50, 100$, 200 and 500 Euro. If the total number of notes is 211, then show that there are at least 31 with the same value.

Solution by the proposer. We can consider the values of the notes as pigeonholes and the number of notes as pigeons. So, we have to place 211 pigeons in 7 pigeonholes. Since $211 = 7 \times 30 + 1$, then at least 31 pigeons must lie in the same pigeonhole. That is, there are at least 31 notes that have the same value.

Solution by Eduardo Espuny-Díaz, Universitat Pompeu Fabra, Barcelona, Spain. Let us suppose that we have at most 30 notes of each kind. Then, we have at most $30 \times 7 = 210$ notes, which is less than the number of notes on the table. Hence, there must be at least one more of one kind, that is, 31.

Also solved by Alberto Espuny-Díaz, CFIS, Barcelona Tech, Barcelona, Spain, and Guillermo Girona San Miguel, CFIS, Barcelona Tech, Barcelona, Spain.

E–3. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Suppose that in the following figure all the squares are of side length one. Compute the value of $A + B + C$.

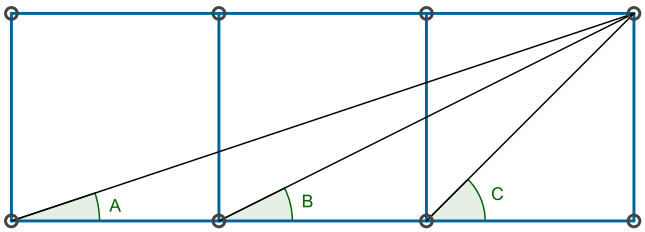

Figure 8: Statement of the problem

Solution 1 by Alberto Espuny-Díaz, CFIS, Barcelona Tech, Barcelona, Spain. Let us consider the following figure:

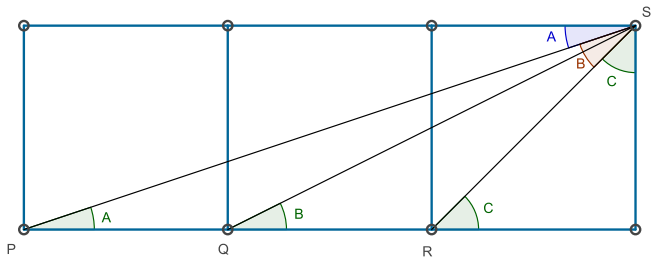

Figure 9: Scheme for the solution

For the angles on the top of the right square, we observe that C is the other angle of an isosceles triangle, and the angles A at P and S are equal because their sides are parallel. Finally, B appears because triangles PRS and QRS are similar. Indeed, we observe that they share one angle at point R, and that their side lengths incident on R are 1 and $\sqrt{2}$, and $\sqrt{2}$ and 2, respectively, so they are proportional. Hence, we observe that $A + B + C = 90°$

Solution 2 by Francesc Gispert Sánchez, CFIS, Barcelona Tech, Barcelona, Spain. We have that, if $0 \leq x, y < \frac{\pi}{4}$,

$$\tan(x+y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)}$$

Setting $x = \arctan(a)$ and $y = \arctan(b)$ in the previous expression and applying arctan to both sides of the equation, we obtain that

$$\arctan(a) + \arctan(b) = \arctan\left(\frac{a+b}{1-ab}\right)$$

if $0 \leq a, b < 1$.

Now, we can use this to compute

$$\begin{aligned} A + B + C &= \arctan\left(\frac{1}{3}\right) + \arctan\left(\frac{1}{2}\right) + \arctan(1) \\ &= \arctan\left(\frac{\frac{1}{3}+\frac{1}{2}}{1-\frac{1}{3}\cdot\frac{1}{2}}\right) + \frac{\pi}{4} = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \end{aligned}$$

Solution 3 by Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain. The right triangles with angles A, B and C all have altitude of length 1, and bases of length 3, 2 and 1, respectively. Therefore, the angles A, B and C correspond to the arguments of the complex numbers $3+i$, $2+i$ and $1+i$, respectively. The sum of the angles is the argument of the product of the three numbers. But

$$(3+i)(2+i)(1+i) = (5+5i)(1+i) = 10i$$

This number has argument $\dfrac{\pi}{2}$.

E–4. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Find the value of

$$\frac{1}{2+\sqrt{2}} + \frac{1}{3\sqrt{2}+2\sqrt{3}} + \cdots + \frac{1}{2025\sqrt{2024}+2024\sqrt{2025}}$$

Solution by the proposer. For each positive integer n, it is true that

$$\begin{aligned}\frac{1}{(n+1)\sqrt{n}+n\sqrt{n+1}} &= \frac{1}{\sqrt{n}\sqrt{n+1}\left(\sqrt{n+1}+\sqrt{n}\right)}\\ &= \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}\sqrt{n+1}}\\ &= \frac{1}{\sqrt{n}}-\frac{1}{\sqrt{n+1}}\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{2+\sqrt{2}}+\frac{1}{3\sqrt{2}+2\sqrt{3}}+\ldots+\frac{1}{2025\sqrt{2024}+2024\sqrt{2025}}\\ &=\left(1-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)+\ldots+\left(\frac{1}{\sqrt{2024}}-\frac{1}{\sqrt{2025}}\right)\\ &=1-\frac{1}{\sqrt{2025}}=1-\frac{1}{45}=\frac{44}{45}\end{aligned}$$

Also solved by Alberto Espuny-Díaz, CFIS, Barcelona Tech, Barcelona, Spain, and Guillermo Girona San Miguel, CFIS, Barcelona Tech, Barcelona, Spain.

E–5. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let I be the sum of the interior angles of a convex n-sided polygon and let E be the sum of its exterior angles taken counterclockwise. Compute

$$\frac{I-E}{I+E}$$

Solution by the proposer. Let $A_1, A_2, \ldots, A_n$ be the interior angles of the n-gon and let $B_1, B_2, \ldots, B_n$ be the exterior angles. To compute I we join the center of the polygon to each vertex and we obtain n triangles. The sum of the angles of all triangles is

$$n \cdot 180^\circ = A_1 + A_2 + \ldots + A_n + 360^\circ = I + 360^\circ$$

from which follows $I = (n-2)\,180°$. To compute E we observe that $A_1 + B_1 = A_2 + B_2 = \ldots = A_n + B_n = 180°$. on account of the fact that $I = (n-2)\,180°$, then

$$(n-2)\,180° + B_1 + B_2 + \ldots + B_n = n\,180° \Leftrightarrow (n-2)\,180° + E = n\,180°$$

from which we get that $E = 360°$. Finally,

$$\frac{I-E}{I+E} = \frac{(n-4)\,180°}{n\,180°} = \frac{n-4}{n}$$

Also solved by Alberto Espuny-Díaz, CFIS, Barcelona Tech, Barcelona, Spain, and Guillermo Girona San Miguel, CFIS, Barcelona Tech, Barcelona, Spain.

E–6. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Find the number of zeros at the end of

$$2014! = 1 \cdot 2 \cdot 3 \ldots 2012 \cdot 2013 \cdot 2014$$

Solution by the proposer. We begin with the following general question: *What is the maximum power of a prime p that divides $n!$?* To answer this question we consider, for instance, $n = 12$ and $p = 2$. We have

1	2	3	4	5	6	7	8	9	10	11	12
0	1	0	1	0	1	0	1	0	1	0	1
0	0	0	1	0	0	0	1	0	0	0	1
0	0	0	0	0	0	0	1	0	0	0	0

where we have written in each column indexed by a number from 1 to 12 as many $1's$ as the maximum exponent of 2 that divides it, otherwise we have written 0. So, the maximum exponent of 2 that divides 12! equals the number of ones in the previous table. That is, $e(2) = 10$. Now we count this number by rows and we get

$$e(2) = \left\lfloor \frac{12}{2} \right\rfloor + \left\lfloor \frac{12}{2^2} \right\rfloor + \left\lfloor \frac{12}{2^3} \right\rfloor = 6 + 3 + 1 = 10$$

In general for a prime p and any positive integer n, we have

$$e(p) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \ldots + \left\lfloor \frac{n}{p^k} \right\rfloor + \ldots$$

To find the number of zeros in which 2014! ends, it will suffice to know $e(5)$ because a zero is obtained multiplying 2×5 and $e(2) \geq e(5)$. That number is

$$e(5) = \left\lfloor \frac{2014}{5} \right\rfloor + \left\lfloor \frac{2014}{5^2} \right\rfloor + \left\lfloor \frac{2014}{5^3} \right\rfloor + \left\lfloor \frac{2014}{5^4} \right\rfloor = 402+80+16+3 = 501$$

Also solved by Guillermo Girona San Miguel, CFIS, Barcelona Tech, Barcelona, Spain.

E–7. *Proposed by Alberto Espuny-Díaz, CFIS, Barcelona Tech, Barcelona, Spain.* Compute the value of

$$\log \frac{1}{2} + \log \frac{2}{3} + \log \frac{3}{4} + \ldots + \log \frac{9999}{10000}$$

Solution by the proposer. Using the properties of the sum of logarithms, we have that

$$\begin{aligned} &\log \frac{1}{2} + \log \frac{2}{3} + \log \frac{3}{4} + \ldots + \log \frac{9999}{10000} \\ &= \log \left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \ldots \cdot \frac{9999}{10000} \right) \\ &= \log \frac{1}{10000} = -4 \end{aligned}$$

An alternative solution can be written using the properties of the logarithm of a fraction. Indeed, we have that

$$\begin{aligned} &\log \frac{1}{2} + \log \frac{2}{3} + \log \frac{3}{4} + \ldots + \log \frac{9999}{10000} \\ &= \log 1 - \log 2 + \log 2 - \log 3 + \log 3 - \log 4 + \ldots - \log 10000 \\ &= \log 1 - \log 10000 = 0 - 4 = -4 \end{aligned}$$

Also solved by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain; Eduardo Espuny-Díaz, Universitat Pompeu Fabra, Barcelona, Spain; Guillermo Girona San Miguel, CFIS, Barcelona Tech, Barcelona, Spain, and Isaac Sánchez Barrera, CFIS, Barcelona Tech, Barcelona, Spain.

E–8. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Find the value of

$$\left(1+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2014}+\frac{1}{2015}\right)\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2014}\right)$$
$$-\left(\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2014}+\frac{1}{2015}\right)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{2014}\right)$$

Solution by the proposer. Let $A=\dfrac{1}{3}+\dfrac{1}{4}+\ldots+\dfrac{1}{2014}+\dfrac{1}{2015}$ and $B=\dfrac{1}{2}+\dfrac{1}{3}+\dfrac{1}{4}+\ldots+\dfrac{1}{2014}$. Then, the given expression is

$$(1+A)B-A(1+B)=B-A=\frac{1}{2}-\frac{1}{2015}=\frac{2013}{4030}$$

Also solved by Alberto Espuny-Díaz, CFIS, Barcelona Tech, Barcelona, Spain; Guillermo Girona San Miguel, CFIS, Barcelona Tech, Barcelona, Spain, and Isaac Sánchez Barrera, CFIS, Barcelona Tech, Barcelona, Spain.

E–9. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* A set of 2016 points in general position (no three of them are collinear) are chosen in the plane. How many disjoint triangles can be built with them?

Solution by the proposer. Since no three of the points are collinear, we draw a line at the left of all of them and not parallel to any line joining two of the chosen points. Sweeping the plane from left to right, mark the first, second and third points found; after that, mark the fourth, fifth and sixth; and so on until we have swept all the points. We have $2016/3=672$ disjoint regions with three of the chosen points (non collinear) in each. Thus, we can construct 672 disjoint triangles.

Also solved by Alberto Espuny-Díaz, CFIS, Barcelona Tech, Barcelona, Spain, and Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.

E–10. *Proposed by José Gibergnas-Báguena, Barcelona Tech, Barcelona, Spain.* Find the last four digits of the number

$$9 + 99 + 999 + 9999 + \ldots + \underbrace{999\ldots9}_{2014}$$

Solution by the proposer. We have

$$N = 9 + 99 + 999 + 9999 + \ldots + \underbrace{999\ldots999}_{2014}$$

$$= (10 - 1) + (100 - 1) + (1000 - 1) + \ldots + (1\underbrace{00\ldots000}_{2014} - 1)$$

$$= 1\underbrace{11\ldots110}_{2014} - 2014 = 1\underbrace{11\ldots111}_{2010}09096$$

So, the number N ends in 9096 and we are done.

Also solved by Guillermo Girona San Miguel, CFIS, Barcelona Tech, Barcelona, Spain, and Isaac Sánchez Barrera, CFIS, Barcelona Tech, Barcelona, Spain.

Easy–Medium Problems

EM–1. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* A triangle is drawn in a sheet of paper. A blind man draws a line that does not pass through any vertex of the triangle. Show that the line does not cut the three sides of the triangle.

Solution by the proposer. The line divides the sheet of paper into two disjoint regions. We have two cases: (1) If the triangle lies in one of them, then the statement holds. (2) If not, then we have three points (the vertices) lying in two disjoint regions. By the pigeonhole principle, two of them lie in the same region and the side of the triangle joining them is not cut by the line. This completes the proof of the claim.

Also solved by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain.

EM–2. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let a, b, c be positive real numbers. Prove that

$$\left(a^{bc}\, b^{ca}\, c^{ab}\right)^{\frac{1}{ab+bc+ca}} \leq \frac{(a+b+c)^3}{9\,(ab+bc+ca)}$$

Solution by the proposer. First, we insert $\frac{3abc}{ab+bc+ca}$ and we claim that

$$\left(a^{bc}\, b^{ca}\, c^{ab}\right)^{\frac{1}{ab+bc+ca}} \leq \frac{3abc}{ab+bc+ca} \leq \frac{(a+b+c)^3}{9\,(ab+bc+ca)}$$

Indeed, the RHS inequality immediately follows from AM-GM inequality. To prove the LHS inequality, we will apply Jensen's inequality to the concave function $f : (0, +\infty) \to \mathbb{R}$ defined by $f(t) = \log t$ with $q_1 = \frac{bc}{ab+bc+ca}$, $q_2 = \frac{ca}{ab+bc+ca}$, $q_3 = \frac{ab}{ab+bc+ca}$, and $x_1 = a$, $x_2 = b$, $x_3 = c$, respectively. Thus, from

$$\sum_{k=1}^{3} q_k f(x_k) \leq f\left(\sum_{k=1}^{3} q_k x_k\right)$$

we get

$$\frac{bc\,\log a}{ab+bc+ca}+\frac{ca\,\log b}{ab+bc+ca}+\frac{ab\,\log c}{ab+bc+ca}\leq \log\left(\frac{3abc}{ab+bc+ca}\right)$$

or equivalently,

$$\log\left(a^{bc}\,b^{ca}\,c^{ab}\right)^{\frac{1}{ab+bc+ca}}\leq \log\left(\frac{3abc}{ab+bc+ca}\right)$$

from which the inequality follows. Equality holds when $a=b=c$, and we are done.

Also solved by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain.

EM–3. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Show that there exist infinitely many triples (x,y,z) of positive integers such that $x<y<z$ and $2y^2=x^2+z^2$.

Solution 1 by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain. Let a,b be positive integers such that $a>b$. Then, the numbers $x^2=(a^2+b^2)^2-4ab(a^2-b^2)$, $y^2=(a^2+b^2)^2$, $z^2=(a^2+b^2)^2+4ab(a^2-b^2)$ satisfy $2y^2=x^2+z^2$. Furthermore, $x^2=a^4+b^4+2a^2b^2-4a^3b+4ab^3=(a^2-b^2-2ab)^2$, $y^2=(a^2+b^2)^2$ and $z^2=a^4+b^4+2a^2b^2+4a^3b-4ab^3=(a^2-b^2+2ab)^2$. So, $x=a^2-b^2-2ab$, $y=a^2+b^2$ and $z=a^2-b^2+2ab$ satisfy $x<y<z$ and we are done.

Solution 2 by Jordi Bosch Bosch, IES Jaume Vicens Vives, Girona, Spain. The triple $(x,y,z)=(1,5,7)$ satisfies the hypothesis of the statement. Then, all triples of the form $(t,5t,7t)$ with $t>0$ also verify that $2y^2=50t^2=t^2+49t^2=x^2+z^2$ and $x<y<z$. Since $t\in(0,+\infty)$ then we have infinitely many solutions.

Also solved by the proposer.

EM–4. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let a, b, c be positive real numbers. Prove that

$$\frac{a^4+10a^2+5}{b^3+b^2+b+1}+\frac{b^4+10b^2+5}{c^3+c^2+c+1}+\frac{c^4+10c^2+5}{a^3+a^2+a+1}\geq 12$$

Solution by Jordi Bosch Bosch, IES Jaume Vicens Vives, Girona, Spain. We claim that $\dfrac{a^4+10a^2+5}{a^3+a^2+a+1}\geq 4$. Indeed, $a^4+10a^2+5-4\left(a^3+a^2+a+1\right)=a^4+10a^2+5-4a^3-4a^2-4a-4=a^4-4a^3+6a^2-4a+1=(a-1)^4\geq 0$. Equality holds when $a=1$.

Applying AM-GM inequality yields

$$\frac{a^4+10a^2+5}{b^3+b^2+b+1}+\frac{b^4+10b^2+5}{c^3+c^2+c+1}+\frac{c^4+10c^2+5}{a^3+a^2+a+1}$$

$$\geq 3\sqrt[3]{\frac{a^4+10a^2+5}{b^3+b^2+b+1}\cdot\frac{b^4+10b^2+5}{c^3+c^2+c+1}\cdot\frac{c^4+10c^2+5}{a^3+a^2+a+1}}$$

$$=3\sqrt[3]{4\cdot 4\cdot 4}=12$$

Equality holds when $a=b=c=1$, and we are done.

Also solved by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain, and the proposer.

EM–5. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* How many ways are there to weigh 31 grams with a balance if we have 7 weights of one gram, 5 of two grams, and 6 of five grams, respectively?

Solution by Alberto Espuny-Díaz, CFIS, Barcelona Tech, Barcelona, Spain. The required number is the number of solutions of $a+b+c=31$ with $a\in\{0,1,2,3,4,5,6,7\}$, $b\in\{0,2,4,6,8,10\}$, and $c\in\{0,5,10,15,20,25,30\}$. We claim that the number of solutions of this equation equals the coefficient of x^{31} in the product

$$\left(1+x+x^2+\ldots+x^7\right)\left(x+x^2+x^4+\ldots+x^{10}\right)\left(1+x^5+x^{10}+\ldots+x^{30}\right)$$

Indeed, a term with x^{31} is obtained by taking some term x^a from the first parentheses, some term x^b from the second, and x^c from the third, in such a way that $a+b+c=31$. Each such possible selection of a, b and c contributes 1 to the considered coefficient of x^{31} in the product. Since

$$(1+x+x^2+\ldots+x^7)\,(1+x^2+x^4+\ldots+x^{10})\,(1+x^5+x^{10}+\ldots+x^{30})$$
$$= 1+x+\ldots+10x^{30}+10x^{31}+10x^{32}+\ldots+x^{46}+x^{47},$$

then the number of ways to obtain 31 grams is 10, and we are done.

Also solved Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain, and the proposer.

EM–6. *Proposed by Mihály Bencze, Braşov, Romania.* Let a,k be positive integers and let n be a nonnegative integer. Show that $(ka^2+1)^{2n+1}$ can be expressed as a sum of $k+1$ squares and $(ka^2+1)^{2n+2}$ can be expressed as a sum of $(k+1)^2$ squares.

Solution by the proposer. First, we have

$$(ka^2+1)^{2n+1} = (ka^2+1)(ka^2+1)^{2n} = \Big(\underbrace{a^2+a^2+\ldots+a^2}_{k}+1^2\Big)\left(ka^2+1\right)^{2n}$$
$$= \underbrace{\left(a(ka^2+1)^n\right)^2+\left(a(ka^2+1)^n\right)^2+\ldots+\left(a(ka^2+1)^n\right)^2}_{k}+\left((ka^2+1)^n\right)^2$$

which is the sum of $k+1$ squares.

Likewise,

$$(ka^2+1)^{2n+2} = (ka^2+1)^2(ka^2+1)^{2n} = \left(k^2a^4+2ka^2+1\right)\left(ka^2+1\right)^{2n}$$
$$= \underbrace{\left(a^2(ka^2+1)^n\right)^2+\left(a^2(ka^2+1)^n\right)^2+\ldots+\left(a^2(ka^2+1)^n\right)^2}_{k^2}$$
$$+\underbrace{\left(a(ka^2+1)^n\right)^2+\left(a(ka^2+1)^n\right)^2+\ldots+\left(a(ka^2+1)^n\right)^2}_{2k}+\left((ka^2+1)^n\right)^2$$

is the sum of $(k+1)^2$ squares, and we are done.

Also solved by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain; José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain, and Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.

EM–7. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* If $P(a, b, c)$ is a common point of the plane $x + y + z + 3 = 0$ and the sphere $x^2 + y^2 + z^2 = 9$ not lying on the cartesian axes, then show that

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}$$

does not depend on the position of point P.

Solution by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain. Since P lies on the plane, then $a + b + c = -3$. Squaring the last expression, we obtain

$$9 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

and $ab + bc + ca = 0$ on account of P also lying on the sphere. From the preceding, it immediately follows that

$$a = -\frac{bc}{b+c}, \quad b = -\frac{ca}{c+a}, \quad c = -\frac{ab}{a+b}$$

Adding up the expressions above yields

$$-\left(\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b}\right) = a + b + c = -3$$

from which we get

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a} = 3$$

and we are done.

Also solved by the proposer.

EM–8. *Proposed by Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.* Find all pairs of non-negative integers (a, b) satisfying

$$a^2 2^a = 8b^3 3^b$$

Solution by the proposer. $a = 0 \iff a^2 2^a = 0 = 8b^3 3^b \iff b = 0$. Hence, $(0, 0)$ is a solution. Now assume a and b are positive. If $b \geq a$, $a^2 2^a \leq b^2 2^b < 8b^3 3^b = a^2 2^a$. This is impossible, so $a > b$. $2^a | 8b^3 3^b$, so $2^a | 8b^3$. Also, $3^b | a^2 2^a$, so $3^b | a^2$. Multiplying both expressions, $2^a 3^b | 8a^2 b^3$, $2^a 3^b \leq 8a^2 b^3$, and $\frac{2^a}{a^2} \frac{3^b}{b^3} \leq 8$.

Assume that $b \geq 6$ (and $a \geq 7$). $\frac{3^b}{b^3} \geq \frac{27}{8}$ (it is true for $b = 6$, and if it is true for b, then $\frac{3^{b+1}}{(b+1)^3} = 3\left(\frac{b}{b+1}\right)^3 \frac{3^b}{b^3} \geq 3\left(\frac{6}{7}\right)^3 \frac{3^b}{b^3} > \frac{3^b}{b^3} \geq \frac{27}{8}$). Similarly, $\frac{2^a}{a^2} \geq \frac{128}{49}$ (it is true for $a = 7$ and, if it is true for a, $\frac{2^{a+1}}{(a+1)^2} = 2\left(\frac{a}{a+1}\right)^2 \frac{2^a}{a^2} \geq 2\left(\frac{7}{8}\right)^2 \frac{2^a}{a^2} > \frac{2^a}{a^2} \geq \frac{128}{49}$). Therefore, $\frac{2^a}{a^2} \frac{3^b}{b^3} \geq \frac{27}{8} \frac{128}{49} > 8$ and there is no solution.

We have proved the following: $a > b$, $2^a | 8b^3$, $3^b | a^2$, $1 \leq b \leq 5$. We consider the following cases:

- b = 3 or b = 5: $2^a | 8b^3$, $2^a | 8$, $3 \geq a > b \geq 3$. There is no solution.
- b = 1: $2^a | 8b^3 = 8$, $a \leq 3$, $3 | a^2$, $a = 3$. $(3, 1)$ is a possible solution.
- b = 2: $2^a | 8b^3 = 2^6$, $a \leq 6$. $9 | a^2$, $3 | a$. $(3, 2)$ and $(6, 2)$ are possible solutions.
- b = 4: $2^a | 8b^3 = 2^9$, $a \leq 9$. $81 | a^2$, $9 | a$. $(9, 4)$ is a possible solution.

The possible solutions are $(0, 0), (3, 1), (3, 2), (6, 2)$ and $(9, 4)$. By checking in the original equation we find that the only solutions are $(0, 0)$ and $(9, 4)$.

Also solved by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.

EM–9. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let A, B, C, D be four points lying on the circle $\Gamma(O, R)$ such that the chords AC and BD are perpendicular and meet at point P. Prove that

$$AP^2 + BP^2 + CP^2 + DP^2 = 4R^2$$

Solution 1 by the proposer. We argue working backwards. Indeed, we observe that $4R^2$ is the square of a diameter of Γ. So, in the following figure we look at the diameter AE and we have that angles ABE and ACE are right. So, on account of Pythagoras theorem, we have

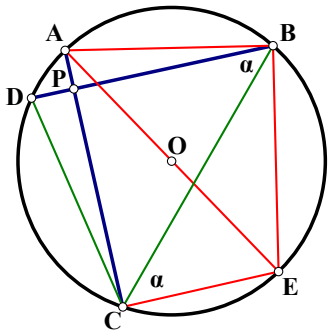

Figure 10: Scheme for Solution 1

$$4R^2 = AE^2 = AB^2 + BE^2 = AP^2 + BP^2 + BE^2$$

Now, we need to see that $BE^2 = CP^2 + DP^2$. Applying Pythagoras theorem to the right triangle CPD yields

$$DC^2 = CP^2 + DP^2$$

To finish the proof, it will suffice to see that $DC = BE$. Since $\angle PCE = \angle BPC = 90°$, then segments CE and DB are parallel. As a consequence, $\angle PBC = \angle BCE = \alpha$ and the chords CD and BE are equal. This completes the proof.

Solution 2 by by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain. In the following figure, on account of Pythagoras theorem, we have $AP^2+PC^2 = AC^2$ and $BP^2+PD^2 = BD^2$. Therefore, it will suffice to see that $AC^2+BD^2 = 4R^2$**.** To do it, let us consider the diameter MN parallel to AB. We then draw E the symmetric point of point B with respect to the diameter MN and the segment FE parallel to AB, where F belongs to CD. Now, we will see that $CE = BD$ and that $\triangle ACE$ is right. Indeed, tri-

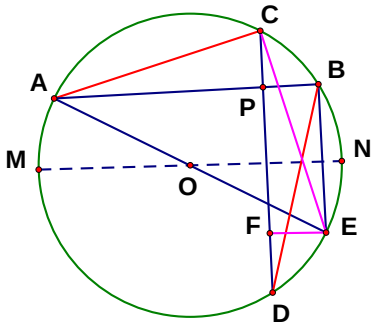

Figure 11: Scheme for Solution 2

angles BPD and EFC are congruent, as can be easily checked. And, by construction, $\angle ABE$ is a right angle, so $\angle ACE$, which is subtended by the same arc, is also right. Therefore, in triangle ACE we have the equality $4R^2 = AC^2 + CE^2 = AC^2 + BD^2$ on account of Pythagoras theorem, and we are done.

Also solved by Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.

EM–10. *Proposed by Nicolae Papacu, Slobozia, Romania.* Let $n \geq 1$ be a natural number and consider the equation

$$\left\lfloor \frac{x}{1\cdot 3} \right\rfloor + \left\lfloor \frac{x}{3\cdot 5} \right\rfloor + \left\lfloor \frac{x}{5\cdot 7} \right\rfloor + \ldots + \left\lfloor \frac{x}{(2n-1)(2n+1)} \right\rfloor = n,$$

where x is a natural number. Find all the values of n for which the equation has n solutions for x ($\lfloor\cdot\rfloor$ denotes the integer part).

Solution 1 by Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain. Assume that for an integer n the equation has at least four solutions $w < x < y < z$. Then it is clear that $z \geq w + 3$, and this implies that $\lfloor \frac{z}{3} \rfloor \geq \lfloor \frac{w+3}{3} \rfloor = \lfloor \frac{w}{3} + 1 \rfloor = \lfloor \frac{w}{3} \rfloor + 1 > \lfloor \frac{w}{3} \rfloor$. For any other positive integer, $\lfloor \frac{z}{k} \rfloor \geq \lfloor \frac{w}{k} \rfloor$. But then we get a contradiction since

$$\begin{aligned} 0 = n - n &= \sum_{k=1}^{n} \left\lfloor \frac{z}{(2k-1)(2k+1)} \right\rfloor - \sum_{k=1}^{n} \left\lfloor \frac{w}{(2k-1)(2k+1)} \right\rfloor \\ &\geq \left\lfloor \frac{z}{1 \cdot 3} \right\rfloor - \left\lfloor \frac{w}{1 \cdot 3} \right\rfloor > 0 \end{aligned}$$

This means that no integer $n \geq 4$ satisfies the condition from the statement. All that is left is to check the cases $n \leq 3$. For $n = 1$ the resulting equation is $\lfloor \frac{x}{3} \rfloor = 1$, which has solutions $x \in \{3, 4, 5\}$. For $n = 2$ the equation is $\lfloor \frac{x}{3} \rfloor + \lfloor \frac{x}{15} \rfloor = 2$, which has solutions $x \in \{6, 7, 8\}$. Finally, for $n = 3$ the equation is $\lfloor \frac{x}{3} \rfloor + \lfloor \frac{x}{15} \rfloor + \lfloor \frac{x}{35} \rfloor = 3$, with solutions $x \in \{9, 10, 11\}$.

The only integer n for which the equation has exactly n solutions is $n = 3$.

Solution 2 by the proposer. Taking into account the definition of the integer part of a number, we have that

$$\left\lfloor \frac{x}{(2k-1)(2k+1)} \right\rfloor \leq \frac{x}{(2k-1)(2k+1)} < \left\lfloor \frac{x}{(2k-1)(2k+1)} \right\rfloor + 1,$$

for $k = 1, \ldots, n$. Then,

$$\begin{aligned} \sum_{k=1}^{n} \left\lfloor \frac{x}{(2k-1)(2k+1)} \right\rfloor &\leq \sum_{k=1}^{n} \frac{x}{(2k-1)(2k+1)} \\ &< \sum_{k=1}^{n} \left\lfloor \frac{x}{(2k-1)(2k+1)} \right\rfloor + n \end{aligned}$$

Since

$$\sum_{k=1}^{n} \left\lfloor \frac{x}{(2k-1)(2k+1)} \right\rfloor = n$$

and

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{2k-1} - \frac{1}{2k+1}\right) = \frac{n}{2n+1},$$

we obtain

$$n \leq \frac{nx}{2n+1} < 2n,$$

which means that $x \in [2n+1, 4n+2)$. Since $x \in \mathbb{N}$, we have that $x \in \{2n+1, 2n+2, \ldots, 4n+1\}$, and we can consider different cases.

If $x \geq 2n+n+3$, then $\left\lfloor \frac{x}{1\cdot 3} \right\rfloor \geq n+1 > n$, and then

$$\sum_{k=1}^{n} \left\lfloor \frac{x}{(2k-1)(2k+1)} \right\rfloor > n$$

Thus, we have that $x \in \{2n+1, 2n+2, \ldots, 3n+1, 3n+2\}$.

If $n \geq 5$ we have three possibilities. If we set $x = 3n+2$, then $\left\lfloor \frac{x}{1\cdot 3} \right\rfloor = n$, $\left\lfloor \frac{x}{3\cdot 5} \right\rfloor = \left\lfloor \frac{n}{5} + \frac{2}{3\cdot 5} \right\rfloor \geq 1$; if we set $x = 3n+1$ we have that $\left\lfloor \frac{x}{1\cdot 3} \right\rfloor = n$, $\left\lfloor \frac{x}{3\cdot 5} \right\rfloor = \left\lfloor \frac{n}{5} + \frac{1}{3\cdot 5} \right\rfloor \geq 1$; finally, setting $x = 3n$ yields $\left\lfloor \frac{x}{1\cdot 3} \right\rfloor = n$, $\left\lfloor \frac{x}{3\cdot 5} \right\rfloor = \left\lfloor \frac{n}{5} \right\rfloor \geq 1$. In these three cases we obtain

$$\sum_{k=1}^{n} \left\lfloor \frac{x}{(2k-1)(2k+1)} \right\rfloor > n$$

Hence, for $n \geq 5$ we have that $x \in \{2n+1, 2n+2, \ldots, 3n-1 = 2n+n-1\}$, that is, the equation has less than $n-1$ solutions. Therefore, $n \leq 4$ and $x \in \{2n+1, 2n+2, \ldots, 3n+1, 3n+2\}$.

If $n = 4$ we have the equation $\left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{15} \right\rfloor + \left\lfloor \frac{x}{35} \right\rfloor + \left\lfloor \frac{x}{63} \right\rfloor = 4$ with $x \in \{9, 10, 11, 12, 13, 14\}$. By substituting x into the equation, we observe that the equality holds for $x \in \{12, 13, 14\}$, which means that there are 3 solutions.

If $n = 3$ we have the equation $\left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{15} \right\rfloor + \left\lfloor \frac{x}{35} \right\rfloor = 3$ with $x \in \{7, 8, 9, 10, 11\}$. Substitution into the equation yields that the solutions are $x \in \{9, 10, 11\}$, and we have 3 solutions.

If $n = 2$, the equation becomes $\left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{x}{15} \right\rfloor = 2$ with $x \in \{5, 6, 7, 8\}$. The solutions for this equation are $x \in \{6, 7, 8\}$, which means that we have 3 solutions again.

If $n = 1$ the equation becomes $\left\lfloor \frac{x}{3} \right\rfloor = 1$ with $x \in \{3, 4, 5\}$, the three of them being solutions. We have 3 solutions once again.

In conclusion, the only n for which the equation has n solutions is $n = 3$.

Also solved by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain and José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain.

Medium–Hard Problems

MH–1. *Proposed by Diana Alexandrescu, Bucharest, Romania and José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let p, n be positive integers such that p is prime and $p < n$. If p divides $n+1$ and $\left(\left[\frac{n}{p}\right], (p-1)!\right) = 1$, then prove that $p \cdot \left[\frac{n}{p}\right]^2$ divides $\binom{n}{p} - \left[\frac{n}{p}\right]$ (here $[x]$ is the integer part of the real number x).

Solution by the proposers. Since $p \mid n+1$, then $p \mid n+1-p$. So, there exists $k \in \mathbb{N}$ such that $n = kp + p - 1$ and $\left[\frac{n}{p}\right] = k$. Now, we have

$$\begin{aligned}
\binom{n}{p} - \left[\frac{n}{p}\right] &= \binom{kp+p-1}{p} - k \\
&= \frac{(kp+p-1)(kp+p-2)\ldots(kp+1)(kp)}{p!} - k \\
&= \frac{k(kp+1)(kp+2)\ldots(kp+p-1) - k(p-1)!}{(p-1)!} \\
&= \frac{k(k\cdot p\cdot r + (p-1)!) - k(p-1)!}{(p-1)!} \\
&= \frac{k^2\cdot p\cdot r}{(p-1)!} \in \mathbb{N}
\end{aligned}$$

where $r \in \mathbb{N}$. Since $\left(\left[\frac{n}{p}\right], (p-1)!\right) = 1$, then $(p-1)!$ divides r and therefore $\binom{n}{p} - \left[\frac{n}{p}\right]$ is divisible by $p \cdot \left[\frac{n}{p}\right]^2$ as we wanted to prove.

Also solved by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain.

MH–2. *Proposed by Nicolae Papacu, Slobozia, Romania.* For all positive integer n we consider the number $a_n = 4^{6^n} + 1943$. Prove that a_n is divisible by 2013 for all $n \geq 1$, and find all values of n for which $a_n - 207$ is the cube of a positive integer.

Solution by the proposer. To prove the first part, we begin observing that $2013 = 3 \cdot 11 \cdot 61 = 33 \cdot 61$. Since $9^5 = 81 \cdot 81 \cdot 9 = (61 + 20)(61 + 20) \cdot 9$ and $20 \cdot 20 \cdot 9 = 3600 = 61 \cdot 59 + 1$, then we have $9^5 \equiv 1 \pmod{61}$. Since $4^6 = 4096 = 61 \cdot 67 + 9 \equiv 9 \pmod{61}$ and $6^{n-1} = (5 + 1)^{n-1} = 5m + 1$, $m \in \mathbb{N}$, then for all $n \geq 1$, the following holds

$$4^{6^n} = \left(4^6\right)^{6^{n-1}} = \left(4^6\right)^{5m+1} \equiv 9^{5m} \cdot 9 \equiv 9 \pmod{61}$$

Therefore, $a_n = 4^{6^n} + 1943 \equiv 1952 \equiv 0 \pmod{61}$ and $61 | a_n$, for all $n \geq 1$.

On the other hand, $a_n = 4^{6^n} - 4 + 1947 = 4\left(4^{6^n - 1} - 1\right) + 33 \cdot 59$. Since $6^n - 1 \equiv 0 \pmod{5}$, then $6^n - 1 = 5p$, $p \in \mathbb{N}$. Then, we have

$$4^{6^n - 1} - 1 = 4^{5p} - 1 = \left(4^5\right)^p - 1 = 1024^p - 1$$

$$= (1024 - 1)(1024^{p-1} + \ldots + 1) = 1023 \cdot q = 33 \cdot 31 \cdot q$$

and $33 | a_n$, which jointly with the preceding yields $2013 = 33 \cdot 61 | a_n$ for all $n \geq 1$.

To solve the second part of the statement, we observe that $a_n - 207 = 4^{6^n} + 1736$ is an even integer, and therefore its cubic root is also an even integer, say $2x$ with $x \in \mathbb{N}$. From $4^{6^n} + 1736 = (2x)^3$ follows $2^{2 \cdot 6^n - 3} + 217 = x^3$ or $2^{3(4 \cdot 6^{n-1} - 1)} + 217 = x^3$. Setting $2^{4 \cdot 6^{n-1} - 1} = y$ in the last equation yields

$$x^3 - y^3 = 217 \Leftrightarrow (x - y)(x^2 + xy + y^2) = 217 = 7 \cdot 31$$

Since $x - y < x^2 + xy + y^2$, then we have two possibilities

$$\begin{cases} x - y = 1 \\ x^2 + xy + y^2 = 217 \end{cases} \quad \text{or} \quad \begin{cases} x - y = 7 \\ x^2 + xy + y^2 = 31 \end{cases}$$

The solutions of the first system are $(9, 8)$ and $(-8, -9)$, and the solutions of the second one are $(6, -1)$ and $(1, -6)$. Finally, since $y = 2^{4 \cdot 6^{n-1} - 1}$ is a positive integer, then $y = 2^{4 \cdot 6^{n-1} - 1} = 8 = 2^3$ from which follows $n = 1$, and we are done.

Also solved by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain; José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain, and Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.

MH–3. *Proposed by Radu Bairac, Chisinau, Republic of Moldova.* Let ABC be a triangle with $\angle ABC = 120°$ and angle bisectors AA_1, BB_1, CC_1, respectively. Let F be a point such that $B_1F \perp A_1C_1$, where $F \in A_1C_1$. If R, I and S are the incenters of the triangles C_1B_1F, $C_1B_1A_1$ and A_1B_1F, and $B_1S \cap A_1C_1 = \{Q\}$, then show that R, I, S and Q are concyclic.

Solution by Marc Felipe Alsina, CFIS, Barcelona Tech, Barcelona, Spain. In order to prove the statement, we will first prove that the triangle $A_1B_1C_1$ is right-angled at B_1. It is immediate to see that BA is the external bisector of $\widehat{B_1BC}$, and therefore we can state, due to the bisector theorem applied to the triangle B_1BC, the following result:

$$\frac{\overline{AB_1}}{\overline{AC}} = \frac{\overline{B_1B}}{\overline{CB}}$$

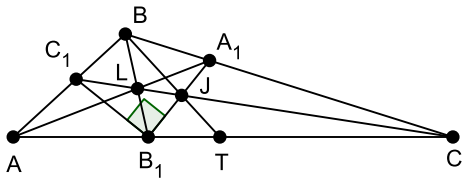

Figure 12: Scheme for first part of solution

Let L be the incentre of ABC. Let's call J the intersection point of the bisector CC_1 and the segment A_1B_1, and let T be the intersection point of BJ and AC. Then, by applying Menelaus' theorem to the triangle B_1BC we get:

$$\frac{\overline{B_1A}}{\overline{AC}} \cdot \frac{\overline{CA_1}}{\overline{A_1B}} \cdot \frac{\overline{BL}}{\overline{LB_1}} = -1$$

Similarly, applying Ceva's Theorem to the same triangle we get the relation:

$$\frac{\overline{B_1T}}{\overline{TC}} \cdot \frac{\overline{CA_1}}{\overline{A_1B}} \cdot \frac{\overline{BL}}{\overline{LB_1}} = 1$$

Dividing both expressions we obtain the following:

$$\frac{\overline{B_1T}}{\overline{TC}} = -\frac{\overline{B_1A}}{\overline{AC}} = \frac{\overline{AB_1}}{\overline{AC}} = \frac{\overline{B_1B}}{\overline{CB}}$$

That relation, by the converse of the angle bisector theorem, tells us that T is the foot of the internal bisector of $\widehat{CBB_1}$. That means J is the intersection of two angle bisectors of the triangle B_1BC, and therefore its incentre. Then B_1A_1 is the bisector of the angle $\widehat{BB_1C}$.

Analogously, B_1C_1 is the bisector of the angle $\widehat{AB_1B}$, and

$$\begin{aligned}\widehat{C_1B_1A_1} = \widehat{C_1B_1B} + \widehat{BB_1A_1} &= \frac{\widehat{AB_1B} + \widehat{BB_1C}}{2} \\ &= \frac{\widehat{AB_1C}}{2} = \frac{180^\circ}{2} = 90^\circ\end{aligned}$$

Now that we have proved that $\widehat{A_1B_1C_1} = 90^\circ$, we can proceed to prove the statement.

Let P be the intersection of B_1R and A_1C_1. Since FR and FS are bisectors of a right angle, we have

$$\widehat{QFS} = \widehat{SFB_1} = \widehat{B_1FR} = \widehat{RFP} = 45^\circ$$

On the other hand,

$$\begin{aligned}\widehat{RB_1S} = \widehat{RB_1F} + \widehat{FB_1S} &= \frac{\widehat{C_1B_1F} + \widehat{FB_1A_1}}{2} \\ &= \frac{\widehat{C_1B_1A_1}}{2} = \frac{90^\circ}{2} = 45^\circ\end{aligned}$$

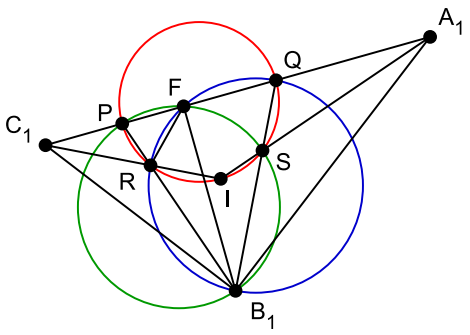

Figure 13: Scheme for second part of solution

We can observe that B_1RFQ is a cyclic quadrilateral because

$$\widehat{QFR} + \widehat{RB_1Q} = 135^\circ + 45^\circ = 180^\circ$$

Similarly, we obtain that $PFSB_1$ is also a cyclic quadrilateral. We can then trivially check that

$$\begin{aligned}\widehat{RIS} = \widehat{C_1IA_1} &= 180^\circ - \widehat{IA_1C_1} - \widehat{A_1C_1I} \\ &= 180^\circ - \frac{\widehat{A_1C_1B_1} + \widehat{B_1A_1C_1}}{2} = 180^\circ - \frac{90^\circ}{2} = 135^\circ\end{aligned}$$

and

$$\widehat{SQR} = \widehat{B_1QR} = \widehat{B_1FR} = 45^\circ$$

And, thus, we can affirm that $RISQ$ is a cyclic quadrilateral since

$$\widehat{RIS} + \widehat{SQR} = 135^\circ + 45^\circ = 180^\circ$$

Solution by the proposer. First, we will show that $\angle C_1B_1A_1 = 90°$. Let $K \in BC$ so that $B \in (KA_1)$, then $\angle ABK = 60°$. Point C_1 is on the bisector of $\angle ACB$ and this implies that $d(C_1, BC) = d(C_1, AC)$ or $C_1F_1 = C_1F_3$, where F_1 is the projection of C_1 on BC and F_3 is the projection of C_1 on AC. Segment BA is the bisector of $\angle KBB_1$, which implies that $d(C_1, KB) = d(C_1, BB_1)$ or $C_1F_1 = C_1F_2$, where F_2 is the projection of C_1 on BB_1. With this, $C_1F_2 = C_1F_3$ and C_1B_1 is the bisector of $\angle BB_1A$. Let us denote $\angle BB_1C_1 = \alpha$. Likewise, we prove that B_1A_1 is the bisector of $\angle BB_1C$. Let $\angle BB_1A_1 = \angle CB_1A_1 = \beta$. Then, from $\angle AB_1C = 180°$ we have $2\alpha + 2\beta = 180°$ and $\alpha + \beta = 90°$.

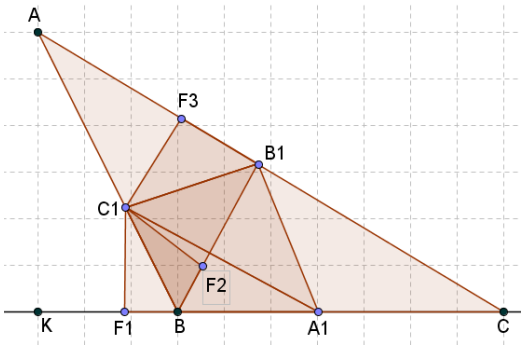

Figure 14: Scheme for first part of solution

Let r_1 be the radius of the inscribed circle of $\triangle A_1B_1C_1$, r_2 be the radius of the inscribed circle of $\triangle C_1B_1F$, and r_3 be the radius of the inscribed circle of $\triangle A_1B_1F$, respectively. Considering the properties of right triangles, we have

$$\triangle C_1FB_1 \sim \triangle C_1B_1A_1 \Rightarrow \frac{r_2}{r_1} = \frac{B_1C_1}{C_1A_1} = \cos C_1$$

from which $r_2 = r_1 \cos C_1 = r_1 \sin A_1$ follows. Likewise,

$$\frac{r_3}{r_1} = \frac{A_1B_1}{C_1A_1} = \cos A_1 \Rightarrow r_3 = r_1 \cos A_1 = r_1 \sin C_1$$

Now, we will see that $I_1R \parallel A_1B_1$ and $I_1S \parallel C_1B_1$, where I_1 is the projection of I on C_1A_1. Let R_2 and R^* be points such that $I_1R_2 \perp C_1B_1$, $R_2 \in (C_1B_1)$ and $I_1R_2 \cap C_1I = \{R^*\}$. On account of $\triangle C_1II_1 \sim \triangle C_1R^*R_2$, then

$$\frac{R^*R_2}{II_1} = \frac{C_1R_2}{C_1I_1} = \cos C_1 \Rightarrow R^*R_2 = r_1 \cos C_1 = r_2$$

from which we obtain $R^* = R, I_1R \perp C_1B_1 \Rightarrow I_1R \parallel A_1B_1$. Likewise, we get $I_1S \parallel C_1B_1$.

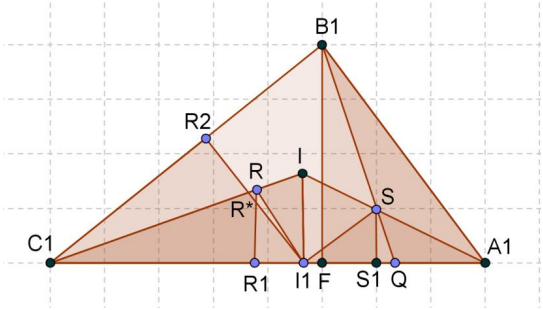

Figure 15: Scheme for second part of solution

In triangle I_1RR_1 we have $r_2 = I_1R \sin A_1 = r_1 \cos C_1 = r_1 \sin A_1$ and $I_1R = r_1$. In triangle I_1SS_1 we have $r_3 = I_1S \sin C_1 = r_1 \cos A_1 = r_1 \sin C_1$, from which $I_1S = r_1$. Finally, we get $I_1R = II_1 = I_1S = r_1$. Since $\triangle QSI_1 \sim \triangle QB_1C_1$, then

$$\frac{SI_1}{B_1C_1} = \frac{I_1Q}{C_1Q} \Longrightarrow SI_1 = I_1Q$$

on account of $B_1C_1 = C_1Q$. Now, we can conclude that points R, I, S, and Q lie on the same circle, and we are done.

MH–4. *Proposed by Iván Geffner Fuenmayor, Barcelona Tech, Barcelona, Spain.* Let Γ be the circumcircle of a triangle ABC and let E and F be the intersections of the bisectors of $\angle ABC$ and $\angle ACB$ with Γ. If EF is tangent to the incircle γ of $\triangle ABC$, then find the value of $\angle BAC$.

Solution by the proposer. Let us denote by I the incenter of $\triangle ABC$.

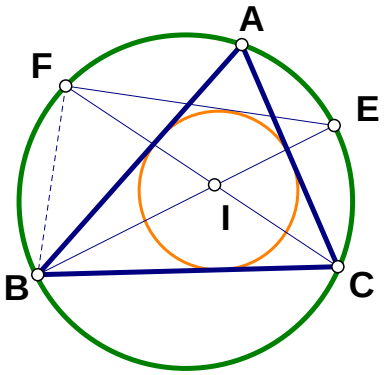

Figure 16: Scheme for the solution

From the figure it immediately follows that $\angle IBC = \angle IFE$ and $\angle ICB = \angle IEF$. Therefore, $\triangle IBC \sim \triangle IFE$. Since both have the same height (the radius of the incircle) because BC and EF are both tangent to γ, then $\triangle IBC = \triangle IFE$. Therefore, $IB = IF$. Now, if we denote $\angle ABC = 2\alpha$ and $\angle ACB = 2\beta$, then $\angle BIF = \alpha + \beta$. Now, since $\triangle IFB$ is isosceles, it follows that

$$\angle IBF = \angle IFB = \angle CFB = \angle BAC = 180^\circ - 2(\alpha + \beta)$$

Adding up the angles of $\triangle IFB$ yields

$$(\alpha + \beta) + 180^\circ - 2(\alpha + \beta) + 180^\circ - 2(\alpha + \beta) = 180^\circ$$

from which we conclude that $\alpha + \beta = 60^\circ$ and $\angle BAC = 60^\circ$.

Also solved by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain, and Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.

MH–5. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let x_i, $1 \leq i \leq 12$ be positive real numbers with product 1. Prove that

$$\sum_{cyclic} \frac{x_1^{13} + x_2^{11}}{x_2 + x_3 + \ldots + x_{12}} \geq \frac{24}{11}$$

Solution by the proposer. To prove the claimed inequality we need the following result: *Let* $x_k > 0, v_k > 0$, $1 \leq k \leq n$. *Then, for all* $m \geq 2$, *the following holds:*

$$\frac{x_1^m}{v_1} + \frac{x_2^m}{v_2} + \ldots + \frac{x_n^m}{v_n} \geq \frac{(x_1 + x_2 + \ldots + x_n)^m}{n^{m-2}(v_1 + v_2 + \ldots + v_n)}$$

Indeed, setting

$$\vec{u} = (\sqrt{v_1}, \sqrt{v_2}, \ldots, \sqrt{v_n}) \qquad \vec{v} = \left(\frac{u_1}{\sqrt{v_1}}, \frac{u_2}{\sqrt{v_1}}, \ldots, \frac{u_n}{\sqrt{v_n}}\right)$$

in Cauchy's inequality yields

$$(v_1 + v_2 + \ldots + v_n)\left(\frac{u_1^2}{v_1} + \frac{u_2^2}{v_2} + \ldots + \frac{u_n^2}{v_n}\right) \geq (u_1 + u_2 + \ldots + u_n)^2$$

or equivalently,

$$\frac{u_1^2}{v_1} + \frac{u_2^2}{v_2} + \ldots + \frac{u_n^2}{v_n} \geq \frac{(u_1 + u_2 + \ldots + u_n)^2}{v_1 + v_2 + \ldots + v_n}$$

Setting $u_k = \sqrt{x_k^m}$, $1 \leq k \leq n$, in the foregoing inequality, we obtain

$$\frac{x_1^m}{v_1} + \frac{x_2^m}{v_2} + \ldots + \frac{x_n^m}{v_n} \geq \frac{(\sqrt{x_1^m} + \sqrt{x_2^m} + \ldots + \sqrt{x_n^m})^2}{v_1 + v_2 + \ldots + v_n}$$

Since the function $f(t) = \sqrt{t^m}$ is convex for $m \geq 2$ and $t > 0$, then applying Jensen's inequality, we get

$$\frac{(\sqrt{x_1^m} + \sqrt{x_2^m} + \ldots + \sqrt{x_n^m})^2}{v_1 + v_2 + \ldots + v_n} \geq \frac{(x_1 + x_2 + \ldots + x_n)^m}{n^{m-2}(v_1 + v_2 + \ldots + v_n)}$$

that jointly with the preceding inequality proves the result.

Since $x_1 x_2 \cdots x_{12} = 1$, then applying AM-GM inequality, it immediately follows that $x_1 + x_2 + \ldots + x_{12} \geq 12$. Applying the preceding result twice, we obtain

$$\sum_{cyclic} \frac{x_1^{13}}{x_2 + x_3 + \ldots + x_{12}} \geq \frac{(x_1 + x_2 + \ldots + x_{12})^{13}}{11\,(x_1 + x_2 + \ldots + x_{12})12^{11}} \geq \frac{12}{11}$$

and

$$\sum_{cyclic} \frac{x_2^{11}}{x_2 + x_3 + \ldots + x_{12}} \geq \frac{(x_1 + x_2 + \ldots + x_{12})^{11}}{11\,(x_1 + x_2 + \ldots + x_{12})12^9} \geq \frac{12}{11}$$

Adding up the above inequalities yields

$$\sum_{cyclic} \frac{x_1^{13} + x_2^{11}}{x_2 + x_3 + \ldots + x_{12}} \geq \frac{24}{11}$$

from which the statement follows. Equality holds when $x_1 = x_2 = \cdots = x_{12} = 1$ and we are done.

Also solved by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain.

MH–6. *Proposed by Sorin Radulescu, Bucharest, Romania.* Let p be an odd positive integer. Find all natural numbers $n \geq 2$ for which the following holds:

$$\sum_{i=1}^{n} \prod_{j \neq i} (x_i - x_j)^p \geq 0 \quad \forall\, x_1, \ldots, x_n \in \mathbb{R}$$

Solution by the proposer. Let us denote

$$\begin{aligned} f_n(x_1, x_2, \ldots, x_n) &= \sum_{i=1}^{n} \prod_{j \neq i} (x_i - x_j)^p \\ &= (x_1 - x_2)^p (x_1 - x_3)^p \ldots (x_1 - x_n)^p \\ &+ (x_2 - x_1)^p (x_2 - x_3)^p \ldots (x_2 - x_n)^p + \ldots \\ &+ (x_n - x_1)^p (x_n - x_2)^p \ldots (x_n - x_{n-1})^p \end{aligned}$$

Since $f_2(x_1, x_2) = (x_1 - x_2)^p + (x_2 - x_1)^p = 0$ for all $x_1, x_2 \in \mathbb{R}$, then for $n = 2$ the statement holds and equality occurs. Let $n > 3$. Then

$$\begin{aligned} f_n(x_1, x_2, a, a, \ldots, a) &= (x_1 - x_2)^p (x_1 - a)^{p(n-2)} + (x_2 - x_1)^p (x_2 - a)^{p(n-2)} \\ &= (x_1 - x_2)^p \left[(x_1 - a)^{p(n-2)} - (x_2 - a)^{p(n-2)} \right] \end{aligned}$$

for all $x_1, x_2 \in \mathbb{R}$. Observe that $f_n(x_1, x_2, a, a, \ldots, a) \geq 0$ when the function $u_a(x) = (x-a)^{p(n-2)}$ is an increasing function. This occurs when $p(n-2)$ is an odd number, from which we deduce that n must be an odd number too.

Now we consider the case when $n \geq 7$ is an odd number. Then, it should be true that

$$f_n(x_1, a, a, a, b, \ldots, b) = (x_1 - a)^{3p}(x_1 - b)^{p(n-4)} \geq 0,$$

for all $x_1, a, b \in \mathbb{R}$. But the preceding inequality does not hold when $x_1 = \dfrac{a+b}{2}$ and $a \neq b$. Hence, we have to analyze the cases $n = 3$ and $n = 5$.

(1) For $n = 3$ we may suppose WLOG that $x_1 \geq x_2 \geq x_3$. Let $g(x_1, x_2, x_3) = (x_3 - x_1)^p (x_3 - x_2)^p$ and $u(x) = (x - x_3)^p$, $x \geq x_3$. Then, we have

(i) u is increasing.

(ii) $g(x_1, x_2, x_3) \geq 0$.
On account of (i) and (ii), we have that

$$\begin{aligned} f(x_1, x_2, x_3) &= (x_1 - x_2)^p \left[(x_1 - x_3)^p - (x_2 - x_3)^p\right] + (x_3 - x_1)^p (x_3 - x_2)^p \\ &= (x_1 - x_2)^p \left[u(x_1) - u(x_2)\right] + g(x_1, x_2, x_3) \geq 0 \end{aligned}$$

and the statement holds for $n = 3$.

(2) For $n = 5$ we also suppose that $x_1 \geq x_2 \geq x_3 \geq x_4 \geq x_5$. Let

$$h(x_1, x_2, x_3, x_4, x_5) = (x_3 - x_1)^p (x_3 - x_2)^p (x_3 - x_4)^p (x_3 - x_5)^p$$

and let $v(x) = (x - x_3)^p (x - x_4)^p (x - x_5)^p$, $x \geq x_3$ and $w(x) = (x - x_1)^p (x - x_2)^p (x - x_3)^p$, $x \leq x_3$. Observe that $h(x_1, x_2, x_3, x_4, x_5) \geq 0$ and v and w are increasing. Since

$$\begin{aligned} f(x_1, x_2, x_3, x_4, x_5) &= (x_1 - x_2)^p [v(x_1) - v(x_2)] \\ &+ (x_4 - x_5)^p [w(x_4) - w(x_5)] + h(x_1, x_2, x_3, x_4, x_5), \end{aligned}$$

then $f(x_1, x_2, x_3, x_4, x_5) \geq 0$. We conclude that the natural numbers for which the statement holds are $n = 2, n = 3$ and $n = 5$.

Also solved by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.

MH–7. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let u be the real root of the equation $x^3 - 3x^2 + 5x - 27 = 0$, and let v be the real root of the equation $x^3 - 3x^2 + 5x + 21 = 0$. Find $u + v$.

Solution by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^3 - 3x^2 + 5x$ and let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(t) = t^3 + 2t$. Then, $f(x) = (x-1)^3 + 2(x-1) + 3$ and g is an odd increasing and bijective function, as can be easily checked. Notice that $g(x-1) = f(x) - 3$. Furthermore,

$$g(u-1) = (u-1)^3 + 2(u-1) = f(u) - 3 = 27 - 3 = 24$$

and

$$g(v-1) = (v-1)^3 + 2(u-1) = f(v) - 3 = -21 - 3 = -24$$

Now, we have $g(u-1) = -g(v-1) = g(1-v)$ because g is odd, and taking into account that g is bijective, we get $u - 1 = 1 - v$, which means that $u + v = 2$ and we are done.

Also solved by the proposer.

MH–8. *Proposed by Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.* We have $2n$ positive integers, the sum of which is a multiple of n. A step consists of choosing n numbers and adding them a positive integer (the same to all of them). Prove that we can make all $2n$ numbers equal in at most $2n - 1$ steps.

Solution by the proposer. Denote $2n - 1$ of the numbers by $A_1, A_2, \ldots, A_{2n-1}$, and let B be the remaining one. From here on, we will use cyclic notation. That is, $A_{2n-1+k} = A_k$. We consider the sets $C_k = \{A_k, A_{k+1}, A_{k+2}, \ldots, A_{k+n-1}\}$. If the initial values of A_k, B and the sum of all numbers are a_k, b and s, respectively, we can make all numbers equal by adding $d_k = \frac{s}{n} - a_k - a_{k+n-1}$ to each number in C_k. If any d_k is negative, we add $-d_k$ to the complement of C_k.

Next we explain the way to obtain this. First, we notice that we only need to consider differences between numbers. If we add or subtract 1 to or from each number, the number of steps required to make all of them equal remains the same. We also notice that adding k to a set of n numbers is equivalent to adding $-k$ to the complement, so we can suppose we can add any nonzero number in a step. Finally, adding 0 is equivalent to skipping the step.

Let us first make all A_k equal. Notice that C_k and C_{k+n} contain every A_i once except for A_k, which appears twice. Thus, subtracting a_k from each number in C_k and C_{k+n} means subtracting it once from every A_i with $i \neq k$ and twice from A_k. If we do this for all k, we find that A_i ends up with the value $a_i - a_i - a_1 - a_2 - \ldots - a_{2n-1} = b - s$.

Finally, to make all A_i equal to b, we add $\frac{s}{n}$ to each number in C_k. Since each A_i belongs to exactly n of the sets C_k, the final value of A_i is $b - s + n\frac{s}{n} = b$. Now all $2n$ numbers are equal.

In conclusion, if we look at what we have added to each number in the set C_k, we see that we subtracted a_k and a_{k+n-1} in the first part and added $\frac{s}{n}$ in the second part. The procedure follows.

Also solved by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.

MH–9. *Proposed by Marius Dragan, Bucharest, Romania.* Let n be a positive integer. For $0 < x < 1$ prove that

$$\left\lfloor \sqrt{n-x} + \sqrt{n+x} + \sqrt{n} \right\rfloor = \left\lfloor \sqrt{9n-2} \right\rfloor$$

Solution by the proposer. First, we consider the case $n = 1$. We need to prove that

$$\left\lfloor \sqrt{1-x} + \sqrt{1+x} + 1 \right\rfloor = 2 \Leftrightarrow 1 < \sqrt{1-x} + \sqrt{1+x} < 2$$

and this follows from the QM-AM inequality.
We can assume thus that $n \geq 2$. We will prove the following inequalities:

$$\sqrt{9n-2} < \sqrt{n-1} + \sqrt{n+1} + \sqrt{n} < \sqrt{n-x} + \sqrt{n+x} + \sqrt{n} < \sqrt{9n}$$

On account of AM-GM inequality, we have

$$\sqrt{n-1}+\sqrt{n+1}+\sqrt{n} > 3\sqrt[3]{\sqrt{n^3-n}} = 3\sqrt[6]{n^3-n}$$

and

$$3\sqrt[6]{n^3-n} > \sqrt{9n-2} \Leftrightarrow 3\sqrt[6]{n^3-n} > 3\sqrt{n-\frac{2}{9}} \Leftrightarrow n^3-n > \left(n-\frac{2}{9}\right)^3$$

or equivalently $\frac{2}{3}n\left(n-\frac{31}{18}\right)+\frac{8}{9^3} > 0$ for all $n \geq 2$.

Next we see that $\sqrt{n-1}+\sqrt{n+1} < \sqrt{n+x}+\sqrt{n-x} \Leftrightarrow n^2-1 < n^2-x^2 \Leftrightarrow x^2 < 1$ which trivially holds from the statement.

Finally, on account of QM-AM inequality, we have

$$\sqrt{n+x}+\sqrt{n-x}+\sqrt{n} < \sqrt{3(n-x+n+x+n)} = \sqrt{9n}$$

Since $9n-1 \equiv 2 \pmod 3$, it is not a square. Then, there does not exist any positive integer between $\sqrt{9n-2}$ and $\sqrt{n-x}+\sqrt{n+x}+\sqrt{n}$, and

$$\left\lfloor \sqrt{n-x}+\sqrt{n+x}+\sqrt{n} \right\rfloor = \left\lfloor \sqrt{9n-2} \right\rfloor$$

as we wanted to prove.

Also solved by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain, and José Gibergans Báguena, Barcelona Tech, Barcelona, Spain.

MH–10. *Proposed by Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.* Let $S = \{1, 2, \ldots, n\}$, $n \geq 2$, and let $f : S \longrightarrow S$ be a bijective function distinct from the identity. Let $u = \sum_{k=1}^{n} |f(k)-k|$ and let v be the number of ordered pairs (a, b) of elements of S such that $a > b$ and $f(a) < f(b)$. Show that $v < u \leq 2v$, and that $u = 2v$ if and only if there do not exist positive integers $a > b > c$ such that $f(a) < f(b) < f(c)$.

Solution 1 by the proposer. We say a pair (a, b) is *a-good* if $a > b \geq f(a)$, and *b-good* if $b \leq f(a) < f(b)$. If $f(k) = k$, no pair (k, b) or (a, k) is *k-good*. If $f(k) > k$, there are no *k-good* pairs of the form (k, b) and there are exactly $f(k) - k$ of the form (a, k). Likewise, if $f(k) < k$ there are exactly $k - f(k)$ *k-good* pairs. In any case, there are exactly $|f(k) - k|$ *k-good* pairs.

For every pair (a, b) with $a > b$ and $f(a) < f(b)$, the pair is *a-good* if $f(a) \leq b$, and *b-good* if $f(a) \geq b$. The number of pairs counted in v is at most the number of *good* pairs, which is less than or equal to the sum of the $|f(k) - k|$. This means that $v \leq \sum_{k=1}^{n} |f(k) - k| = u$. Let k be the smallest integer with $f(k) \neq k$. Then $f(k) > k$, $f^{-1}(k) > k$. The pair $(f^{-1}(k), k)$ is both *$f^{-1}(k)$-good* and *k-good*. Equality does not hold, and $v < u$.

Let $A_k = \{f^{-1}(1), f^{-1}(2), ..., f^{-1}(f(k))\}$ and $B_k = \{1, 2, ..., k\}$. The element x is in B_k and not in A_k if and only if (k, x) is a pair counted in v. Similarly, x is in A_k and not in B_k if and only if (x, k) is a pair counted in v. Then,

$$\begin{aligned} 2v &= \sum_{k=1}^{n} |A_k \backslash B_k| + |B_k \backslash A_k| \\ &= \sum_{k=1}^{n} |A_k| + |B_k| - 2|A_k \cap B_k| \\ &\geq \sum_{k=1}^{n} |A_k| + |B_k| - 2\min\{|A_k|, |B_k\} \\ &= \sum_{k=1}^{n} f(k) + k - 2\min\{f(k), k\} = \sum_{k=1}^{n} |f(k) - k| = u \end{aligned}$$

We look at the equality case. $u = 2v \iff |A_b \cap B_b| = \min\{|A_b|, |B_b|\}$ for all $b \iff A_b \subseteq B_b$ or $B_b \subseteq A_b$ for all $b \iff$ There do not exist a, b, c such that $a \in A_b \backslash B_b$, $c \in B_b \backslash A_b \iff$ There do not exist a, b, c such that $a > b > c$, $f(a) < f(b) < f(c)$.

Solution 2 by the proposer. For bijective functions in general, we shall prove that $v \leq u \leq 2v$, with $u = v$ only for the identity and $u = 2v$ if and only if there do not exist $a > b > c$ such that $f(a) < f(b) < f(c)$. We argue by induction on u.

If $u = 0$, $\sum_{k=1}^{n} |f(k) - k| = 0$, $|f(k) - k| = 0$ for all k, $f(k) = k$ and f is the identity. In this case, $v = 0$, $v = u = 2v$ and there do not exist $a > b > c$ with $f(a) < f(b) < f(c)$. This satisfies the result.

Now, suppose that the statement is true for $0 \leq u < t$, and we have a function f with $u = t$. Then,

$$\begin{aligned}\sum_{k=1}^{n} f(k) - k &= \sum_{k=1}^{n} f(k) - \sum_{k=1}^{n} k \\ &= \sum_{k=1}^{n} k - \sum_{k=1}^{n} k = 0\end{aligned}$$

Since $u = \sum_{k=1}^{n} |f(k) - k| \neq 0$, there exist some k with $f(k) - k < 0$ (that is, $f(k) < k$), and some k with $f(k) - k > 0$. Let x be the smallest value of k with $f(k) < k$. Then, $f(f(x)) \neq f(x)$ because if they were equal, then $f(x) = x$. Since $f(x) < x$, and by definition of x, $f(f(x)) > f(x)$. Thus, there are values of $k < x$ with $f(k) > k$. Let y be the greatest one of these.

We define $\bar{k}$ as follows: $\bar{x} = y$, $\bar{y} = x$ and $\bar{k} = k$ for the rest of the values. We define $\tilde{f}$ as $\tilde{f}(k) = f(\bar{k})$, and define $\tilde{u}$ and $\tilde{v}$ analogously to u and v for f. Let $z = x - y$.

For all k with $y < k < x$, $f(k)$ is not less than k by definition of x, nor greater than k by definition of y. Therefore, $f(k) = k$. If $y < f(x) < x$, then $f(f(x)) = f(x)$, which is impossible. Hence, $f(x) \leq y$. In the same way, $f(y) \geq x$. Then,

$$\begin{aligned}u - \tilde{u} &= \sum_{k=1}^{n} |f(\bar{k}) - \bar{k}| - \sum_{k=1}^{n} |\tilde{f}(k) - k| \\ &= |f(x) - x| - |\tilde{f}(y) - y| + |f(y) - y| - |\tilde{f}(x) - x| \\ &= x - y - y + x = 2z\end{aligned}$$

Hence, $\tilde{u} < u = t$, so by induction hypothesis $\tilde{v} \leq \tilde{u} \leq 2\tilde{v}$.

Let us find the value of $v - \tilde{v}$, by finding the cases when $(\bar{a}, \bar{b})$ is included in v and (a, b) is not included in $\tilde{v}$, and vice versa. The condition for the first case can be written as $\bar{a} > \bar{b}$, $f(\bar{a}) < f(\bar{b})$ and $a < b$. It is clear that one of the terms must be x or y. For instance, $a = y$ and $b = x$ satisfies this. If only one of the terms appears and it is a, then $\bar{a} > \bar{b} = b > a$, $a = y$ and $x > b > y$. All the values of b in this interval satisfy the condition because $f(y) \geq x > b = f(b)$. Likewise, if the term that appears is b, then $b = x$ and $x > a > y$ satisfies the conditions. We can check that there are no pairs (a, b) counted in $\tilde{v}$ such that $(\bar{a}, \bar{b})$ is not counted in v. In conclusion, $v - \tilde{v} = 1 + 2(z - 1) = 2z - 1$.

Using the previous results,

$$\begin{aligned} v = \tilde{v} + 2z - 1 \leq \tilde{u} + 2z - 1 < \tilde{u} + 2z = u \\ \leq 2\tilde{v} + 2z \leq 2\tilde{v} + 2(2z - 1) = 2v \end{aligned}$$

This proves the inequality.

All that remains is to find when $u = 2v$ holds. This happens whenever $\tilde{u} = 2\tilde{v}$ and $z = 1$. If $z > 1$, $x > y + 1 > y$ and $f(x) \leq y < y + 1 = f(y + 1) < x \leq f(y)$. We can therefore suppose $z = 1$. We need to prove that there exist $a > b > c$ with $f(a) < f(b) < f(c)$ if and only if there exist $a > b > c$ with $\tilde{f}(a) < \tilde{f}(b) < \tilde{f}(c)$. $f(\bar{a}) < f(\bar{b}) < f(\bar{c}) \iff \tilde{f}(a) < \tilde{f}(b) < \tilde{f}(c)$. If two elements of $\{a, b, c\}$ are neither x nor y, for example a and c, then $\bar{a} = a$, $\bar{c} = c$ and since $x = y + 1$, $a > x > c \iff a > y > c$. If both x and y appear, they cannot be a and c because x and y are consecutive integers. Hence, one of them is b, for example x. $a > x > y$ and $f(a) < f(x) < f(y)$. As $f(x) < x$, there are more values of k with $k < x$ and $f(k) > f(x)$ than $k > x$ with $f(k) < f(x)$. Since a is an example of the latter, there is d not equal to y with $d < x$ and $f(d) > f(x)$, and therefore $a > x > d$ and $f(a) < f(x) < f(d)$, where y does not appear. We reduced the problem to the previous case. The same argument works if $\tilde{f}(a) < \tilde{f}(x) < \tilde{f}(y)$ or if $b = y$, and we are done.

Advanced Problems

A–1. *Proposed by Mihály Bencze, Braşov, Romania.* Find all real solutions of the system

$$\begin{aligned} \sin x + \sin y + \sin z &= 0, \\ \cos x + \cos y + \cos z &= 0, \\ \tan 3^k x + \tan 3^k y + \tan 3^k z &= 3(2 - \sqrt{3}), \end{aligned}$$

where k is a positive integer.

Solution by Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain. If $a = \cos x + i \sin x = e^{ix}$, $b = \cos y + i \sin y = e^{iy}$ and $c = \cos z + i \sin z = e^{iz}$, then $a + b + c = 0$ and $|a| = |b| = |c| = 1$. Thus, $A(a), B(b), C(c)$ are the affixes of the vertices of an equilateral triangle. Therefore, $b = a\epsilon, c = a\epsilon^2$, where $\epsilon^3 = 1$ and $\epsilon \neq 1$, which means that ϵ can either be $e^{i\frac{2\pi}{3}}$ or $e^{i\frac{4\pi}{3}}$. If we consider $\epsilon = e^{i\frac{2\pi}{3}}$, then we have

$$\begin{aligned} \cos y + i \sin y &= \left(\cos x + i \sin x\right) \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right) \\ &= \cos\left(x + \frac{2\pi}{3}\right) + i \sin\left(x + \frac{2\pi}{3}\right) \end{aligned}$$

from which we obtain that $y = x + \dfrac{2\pi}{3} + 2\pi n$, where $n \in \mathbb{Z}$. Likewise, from

$$\begin{aligned} \cos z + i \sin z &= \left(\cos x + i \sin x\right) \left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}\right) \\ &= \cos\left(x + \frac{4\pi}{3}\right) + i \sin\left(x + \frac{4\pi}{3}\right) \end{aligned}$$

we get $z = x + \dfrac{4\pi}{3} + 2\pi l$, where $l \in \mathbb{Z}$. Substituting the preceding values in the third equation, we obtain

$$\tan 3^k x + \tan 3^k \left(x + \frac{2\pi}{3}\right) + \tan 3^k \left(x + \frac{4\pi}{3}\right) = 3(2 - \sqrt{3})$$

or

$$\tan 3^k x + \tan\left(3^k x + 2\cdot 3^{k-1}\pi\right) + \tan\left(3^k x + 4\cdot 3^{k-1}\pi\right) = 3\,\tan 3^k x$$
$$= 3\,(2-\sqrt{3})$$

from which we get $\tan 3^k x = 2-\sqrt{3}$. This means that $3^k x = \dfrac{\pi}{12} + \pi m$, which implies that the solutions are

$$x = \frac{\pi}{4\cdot 3^{k+1}} + \frac{m\pi}{3^k}$$

$$y = \frac{\pi}{4\cdot 3^{k+1}} + \frac{m\pi}{3^k} + \frac{2\pi}{3} + 2\pi n$$

$$z = \frac{\pi}{4\cdot 3^{k+1}} + \frac{m\pi}{3^k} + \frac{4\pi}{3} + 2\pi l$$

where $l, m, n \in \mathbb{Z}$. Equivalently, if we consider $\epsilon = e^{i\frac{4\pi}{3}}$, the solutions are

$$x = \frac{\pi}{4\cdot 3^{k+1}} + \frac{m\pi}{3^k}$$

$$y = \frac{\pi}{4\cdot 3^{k+1}} + \frac{m\pi}{3^k} + \frac{4\pi}{3} + 2\pi n$$

$$z = \frac{\pi}{4\cdot 3^{k+1}} + \frac{m\pi}{3^k} + \frac{2\pi}{3} + 2\pi l$$

where $l, m, n \in \mathbb{Z}$.

Also solved by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain; Francesc Gispert Sánchez, CFIS, Barcelona Tech, Barcelona, Spain, and the proposer.

A–2. *Proposed by Dan Popescu, Suceava, Romania.* Let p be a prime number. Show that for all $x \in \mathbb{R}$ the following is true:

$$\sum_{k=0}^{p-1} (-1)^k \cos^p\left(x + \frac{k\pi}{p}\right) = \frac{p\cos px}{2^{p-1}}$$

Solution by the proposer. Let us denote $a = \cos x + i \sin x$ and $z = \cos \dfrac{\pi}{p} + i \sin \dfrac{\pi}{p}$. Then, for $0 \leq k \leq p-1$, we have

$$az^k = \cos\left(x + \frac{k\pi}{p}\right) + i \sin\left(x + \frac{k\pi}{p}\right)$$

from which we obtain

$$\cos\left(x + \frac{k\pi}{p}\right) = \frac{az^k + \overline{az^k}}{2} = \frac{a^2z^{2k} + 1}{2az^k}$$

and

$$\cos^p\left(x + \frac{k\pi}{p}\right) = \frac{(a^2z^{2k} + 1)^p}{2^p a^p z^{kp}} = \frac{(a^2z^{2k} + 1)^p}{(-1)^k 2^p a^p}$$

Then,

$$\begin{aligned}
\sum_{k=0}^{p-1}(-1)^k \cos^p\left(x + \frac{k\pi}{p}\right) &= \frac{1}{2^p a^p}\sum_{k=0}^{p-1}(a^2z^{2k} + 1)^p \\
&= \frac{1}{2^p a^p}\sum_{k=0}^{p-1}\sum_{j=0}^{p}\binom{p}{j}a^{2j}z^{2kj} \\
&= \frac{1}{2^p a^p}\sum_{j=0}^{p}\binom{p}{j}a^{2j}\sum_{k=0}^{p-1}z^{2kj} \\
&= \frac{p}{2^p a^p}(1 + a^{2p}) = \frac{p}{2^p}\left(a^p + a^{-p}\right) \\
&= \frac{p}{2^{p-1}}\cos px,
\end{aligned}$$

taking into account that $z^{2p} = 1$ (z is a primitive $2p$-th root of unity) and, for any $0 < j < p$,

$$\sum_{k=0}^{p-1} z^{2kj} = \sum_{k=0}^{p-1}(z^{2k})^j = \frac{1 - z^{2kp}}{1 - z^{2k}} = 0$$

Also solved by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain, and Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.

A–3. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain, and José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain.* Calculate

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i<j\le n}\left(\frac{(n^2+i^2)(n^2+j^2)}{n^4+i^2+j^2}\right)^{1/2}$$

Solution by the proposers. Let us denote by a_n the general term of the sequence the limit of which must be computed. Then, we have

$$\begin{aligned} a_n &= \frac{1}{n^2}\sum_{1\le i<j\le n}\left(\frac{(n^2+i^2)(n^2+j^2)}{n^4+i^2+j^2}\right)^{1/2} \\ &= \sum_{1\le i<j\le n}\frac{\sqrt{1+\left(\frac{i}{n}\right)^2}\sqrt{1+\left(\frac{j}{n}\right)^2}}{\sqrt{n^4+i^2+j^2}} \end{aligned}$$

Since

$$\sum_{1\le i<j\le n}\frac{\sqrt{1+\left(\frac{i}{n}\right)^2}\sqrt{1+\left(\frac{j}{n}\right)^2}}{\sqrt{n^4+2n^2}} \le a_n \le \sum_{1\le i<j\le n}\frac{\sqrt{1+\left(\frac{i}{n}\right)^2}\sqrt{1+\left(\frac{j}{n}\right)^2}}{\sqrt{n^4+2}},$$

then

$$\frac{n^2}{\sqrt{n^4+2n^2}}\sum_{1\le i<j\le n}\frac{\sqrt{1+\left(\frac{i}{n}\right)^2}\sqrt{1+\left(\frac{j}{n}\right)^2}}{n^2} \le a_n$$

$$\le \frac{n^2}{\sqrt{n^4+2}}\sum_{1\le i<j\le n}\frac{\sqrt{1+\left(\frac{i}{n}\right)^2}\sqrt{1+\left(\frac{j}{n}\right)^2}}{n^2}$$

Moreover,

$$\lim_{n\to\infty}\frac{n^2}{\sqrt{n^4+2n^2}} = \lim_{n\to\infty}\frac{n^2}{\sqrt{n^4+2}} = 1$$

Therefore,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sum_{1\le i<j\le n} \frac{\sqrt{1+\left(\frac{i}{n}\right)^2}\sqrt{1+\left(\frac{j}{n}\right)^2}}{n^2}$$

Let $f : [0,1] \to \mathbb{R}$ be a continuous function. We claim that

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right) = \frac{1}{2}\left(\int_0^1 f(x)\,dx\right)^2 \qquad (21)$$

Indeed, from the identity

$$\left(\sum_{k=1}^{n} x_k\right)^2 = \sum_{k=1}^{n} x_k^2 + 2\sum_{1\le i<j\le n} x_i x_j,$$

we get

$$\frac{1}{n^2}\left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right)^2 = \frac{1}{n^2}\sum_{k=1}^{n} f^2\left(\frac{k}{n}\right) + \frac{2}{n^2}\sum_{1\le i<j\le n} f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right)$$

Since

$$\lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} f^2\left(\frac{k}{n}\right)\right) = 0,$$

then

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right) = \frac{1}{2}\left[\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right]^2$$

$$= \frac{1}{2}\left(\int_0^1 f(x)\,dx\right)^2$$

as claimed. Finally, setting $f(x) = \sqrt{1+x^2}$ into (21), we have

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \frac{1}{2}\left(\int_0^1 \sqrt{1+x^2}\right)^2 \\
&= \frac{1}{2}\left(\left[\frac{x\sqrt{1+x^2}}{2} + \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)\right]_0^1\right)^2 \\
&= \frac{1}{8}\left(\sqrt{2}+\ln(1+\sqrt{2})\right)^2 \simeq 0.6587,
\end{aligned}$$

and we are done.

A–4. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let $\{x_n\}_{n\geq 1}$ be a sequence of real numbers. Compute

$$\sum_{n=1}^{\infty}\left[\sin^2 x_n \prod_{k=1}^{n-1}\cos^2 x_k\right] + \prod_{n=1}^{\infty}\cos^2 x_n$$

Solution by the proposer. We claim that the answer is 1. Setting $\sin^2 x_n = a_n$ we have that $0 \leq a_n \leq 1$ and the claimed identity becomes

$$\sum_{n=1}^{\infty}\left[a_n \prod_{k=1}^{n-1}(1-a_k)\right] + \prod_{n=1}^{\infty}(1-a_n) = 1$$

To prove it, suppose that an infinite number of coins are flipped. Let a_n be the probability that the n^{th} coin toss lands heads, and let us consider the first time heads comes up. Then $a_n \prod_{k=1}^{n-1}(1-a_k)$ is the probability that the first head appears in the n^{th} flip and $\prod_{n=1}^{\infty}(1-a_n)$ is the probability that all flips come up tails. With this,

$$\sum_{n=1}^{\infty}\left[a_n \prod_{k=1}^{n-1}(1-a_k)\right] + \prod_{n=1}^{\infty}(1-a_n) = 1$$

and we are done.

Also solved by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain, and Francesc Gispert Sánchez, CFIS, Barcelona Tech, Barcelona, Spain.

A–5. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let $a < b$ be real numbers and let $f : [a, b] \to \mathbb{R}$ be an increasing (decreasing) function. Show that the set of discontinuities of f is countable.

Solution 1 by the proposer. Suppose that $a < x_1 < x_2 < \ldots < x_n < b$. We claim that

$$\sum_{k=1}^{n} \left[f(x_k^+) - f(x_k^-)\right] \leq f(b^-) - f(a^+)$$

Indeed, since f is increasing then for all $x_k \in (a, b)$ there exists $f(x^+) = \lim_{t \to x^+} f(t)$ and $f(x^-) = \lim_{t \to x^-} f(t)$ verifying that $f(x^-) \leq f(x) \leq f(x^+)$. Thus, we have

$$f(b^-) - f(a^+) = f(b^-) + f(x_1^-) - f(x_1^-) + f(x_1^+) - f(x_1^+) + \ldots$$

$$+ f(x_n^-) - f(x_n^-) + f(x_n^+) - f(x_n^+) - f(a^+)$$

$$= f(x_1^+) - f(x_1^-) + f(x_2^+) - f(x_2^-) + \ldots + f(x_n^+) - f(x_n^-) + A,$$

where

$$A = \left[f(x_1^-) - f(a^+)\right] + \left[f(x_2^-) - f(x_1^+)\right] + \ldots + \left[f(b^-) - f(x_n^+)\right]$$

on account of f being increasing, we have $A \geq 0$ and

$$f(b^-) - f(a^+) \geq \left[f(x_1^+) - f(x_1^-)\right] + \ldots + \left[f(x_n^+) - f(x_n^-)\right]$$

$$= \sum_{k=1}^{n} \left[f(x_k^+) - f(x_k^-)\right] \geq 0$$

Now, for all $n \in \mathbb{N}$, we define

$$I_n = \left\{ x \in (a, b) \mid f(x^+) - f(x^-) > \frac{f(b^-) - f(a^+)}{n} \right\}$$

and we will see that I_n has at most $n - 1$ elements, say $y_1, y_2, \ldots, y_{n-1}$. In fact, if we assume that I_n has the n elements $y_1, y_2, \ldots, y_n$ then

$$\sum_{k=1}^{n} \left[f(y_k^+) - f(y_k^-)\right] > \sum_{k=1}^{n} \left[\frac{f(b^-) - f(a^+)}{n}\right] = f(b^-) - f(a^+)$$

This is a contradiction. Thus, for all $n \in \mathbb{N}$ the set I_n has a finite number of elements.

Since the set I of discontinuities of f is

$$I = \bigcup_{n \in \mathbb{N}} I_n$$

and it is a countable union of countable sets, then it is countable (the same occurs for decreasing functions).

Solution 2 by Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain. Assume that f is increasing. Let $\alpha \in (a,b)$. If we consider the set $S_\alpha = \{f(x)|x < \alpha\}$, this set is bounded by $f(\alpha)$ (since f is increasing), so there is a lowest bound $f(\alpha^-)$ for all the elements of S. Similarly, there is a greatest lower bound $f(\alpha^+)$ for the values of $f(x)$ with $x > \alpha$. These values satisfy $f(\alpha^-) \leq f(\alpha) \leq f(\alpha^+)$.

Let $x = \alpha$ be a discontinuity of f. Assume that $f(\alpha^-) = f(\alpha^+)$ (both equal to $f(\alpha)$). Then for every $\varepsilon > 0$ we have that there is $x_1 < \alpha$ with $f(\alpha) - f(x_1) < \varepsilon$, and $x_2 > \alpha$ with $f(x_2) - f(\alpha) < \varepsilon$. But then $|f(x) - f(\alpha)| < \varepsilon$ for all $x \in (x_1, x_2)$. Hence, f is continuous at $x = \alpha$, which is a contradiction. This means that $f(\alpha^-) < f(\alpha^+)$.

For every discontinuity α, we choose a rational number from the interval $(f(\alpha^-), f(\alpha^+))$. If $\alpha_1 < \alpha_2$ are two discontinuities, then $f(\alpha_1^+) < f\left(\frac{\alpha_1+\alpha_2}{2}\right) < f(\alpha_2^-)$, so the rational number assigned to each of them is different. The number of discontinuities is at most the number of rational numbers, which is countable.

Also solved by Alberto Espuny-Díaz, CFIS, Barcelona Tech, Barcelona, Spain.

A–6. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \ln\left(x + \sqrt{1+x^2}\right) + \left(1+x^2\right)^{-1/2} - x\left(1+x^2\right)^{-3/2}$$

and let $a < b$ be two real numbers such that $\ln\left(\dfrac{f(b)}{f(a)}\right) = b - a$. Show that there exists $c \in (a,b)$ for which it is true that

$$2c^2 = 1 + (1+c^2)^{5/2} \ln(c + \sqrt{1+c^2})$$

Solution by the proposer. Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x) = \ln(x + \sqrt{1+x^2})$. Since $g'(x) = \dfrac{1}{\sqrt{1+x^2}}$ and $g''(x) = \dfrac{-x}{\sqrt{(1+x^2)^3}}$, then $f(x) = g(x) + g'(x) + g''(x)$. Let $F(x) = (g(x) + g'(x) + g''(x))e^{-x} = f(x)e^{-x}$. Clearly, $F(x)$ is infinitely differentiable in $\mathbb{R}$. Furthermore, since

$$\ln\left(\frac{f(b)}{f(a)}\right) = b - a \Leftrightarrow \frac{f(b)}{f(a)} = \frac{e^b}{e^a},$$

then $F(b) = f(b)e^{-b} = f(a)e^{-a} = F(a)$. Applying Rolle's theorem, we have that there exists $c \in (a, b)$ such that $F'(c) = 0$. That is,

$$F'(c) = f'(c)e^{-c} - f(c)e^{-c} = 0 \Leftrightarrow g'''(c) - g(c) = 0$$

or

$$\frac{2c^2 - 1}{(1+c^2)^{5/2}} - \ln\left(c + \sqrt{1+c^2}\right) = 0$$

from which the statement immediately follows.

Also solved by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain.

A–7. *Proposed by Nicolae Papacu, Slobozia, Romania.* Let $\{I_n\}_{n\geq 1}$ be the sequence defined by

$$I_n = \int_0^n x^n \arctan x \, dx$$

Compute $\lim\limits_{n\to\infty} \dfrac{I_n}{n^n}$.

Solution by the proposer. Integrating by parts, we put $u = \arctan x$ and $dv = x^n\,dx$ and we have

$$I_n = \int_0^n x^n \arctan x\,dx = \frac{x^{n+1}}{n+1}\arctan x\Big|_0^n - \frac{1}{n+1}J_n,$$

where $J_n = \displaystyle\int_0^n \frac{x^{n+1}}{1+x^2}\,dx$. From the preceding, it immediately follows that

$$\frac{I_n}{n^n} = \frac{n}{n+1}\arctan n - \frac{1}{n^n(n+1)}J_n$$

Now we claim that for $0 \le x \le n$ it is true that $0 \le \dfrac{x^{n+1}}{x^2+1} \le \dfrac{n^{n+1}}{n^2+1}$. Indeed, the preceding inequality is equivalent to

$$n^2x^2(x^{n-1} - n^{n-1}) + x^{n+1} - n^{n+1} \le 0$$

which trivially holds.

Hence, $0 \le J_n \le \displaystyle\int_0^n \frac{n^{n+1}}{n^2+1}\,dx = \frac{n^{n+2}}{n^2+1}$, from which we obtain

$$0 \le \frac{J_n}{n^n(n+1)} \le \frac{n^2}{(n+1)(n^2+1)}$$

and $\displaystyle\lim_{n\to\infty} \frac{J_n}{n^n(n+1)} = 0$. Finally,

$$\lim_{n\to\infty}\frac{I_n}{n^n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\arctan n - \frac{J_n}{n^n(n+1)}\right) = \frac{\pi}{2},$$

and we are done.

Also solved by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain, and José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain.

A–8. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let a, b be real numbers such that $-b \leq a \leq b$. Compute the sum

$$\sum_{n=0}^{\infty}(-1)^n \frac{\sinh\left((2n+1)a\right)}{(2n+1)e^{(2n+1)b}}$$

Solution by the proposer. Let

$$S = \sum_{n=0}^{\infty}(-1)^n \frac{\cosh\left((2n+1)a\right)}{(2n+1)e^{(2n+1)b}}$$

and

$$T = \sum_{n=0}^{\infty}(-1)^n \frac{\sinh\left((2n+1)a\right)}{(2n+1)e^{(2n+1)b}}.$$

Taking into account that

$$\begin{aligned} e^{\alpha} &= \cosh\alpha + \sinh\alpha \\ e^{-\alpha} &= \cosh\alpha - \sinh\alpha \end{aligned}$$

we have

$$S + T = \sum_{n=0}^{\infty}(-1)^n \frac{e^{(2n+1)a}}{(2n+1)e^{(2n+1)b}}$$

and

$$S - T = \sum_{n=0}^{\infty}(-1)^n \frac{e^{-(2n+1)a}}{(2n+1)e^{(2n+1)b}}$$

Subtracting the second expression from the first one, we get

$$\begin{aligned} 2T &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(e^{a-b}\right)^{2n+1} - \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(e^{-(a+b)}\right)^{2n+1} \\ &= \arctan e^{a-b} - \arctan e^{-(a+b)} \end{aligned} \tag{22}$$

valid for $e^{a-b} \leq 1$ and $e^{-(a+b)} \leq 1$ or, equivalently, for $-b \leq a \leq b$. From (22), we get

$$T = \frac{1}{2}\left[\arctan e^{a-b} - \arctan e^{-(a+b)}\right]$$

and we are done.

Also solved by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain.

A–9. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let $A(x) \in \mathbb{Z}[x]$ be a polynomial of degree n and let $a_0 < a_1 < \ldots < a_n$ be integers. Show that $|A(a_k)| \geq 2^{-n}\, n!$ for some $k \in \{0, 1, 2, \ldots, n\}$.

Solution by José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain. Consider the polynomial

$$\begin{aligned} B(x) &= \left(\prod_{j=0}^{n}(x - a_j)\right) \sum_{i=0}^{n} \left(\frac{A(a_i)}{x - a_i} \prod_{j\neq i} \frac{1}{a_i - a_j}\right) \\ &= \sum_{i=0}^{n} A(a_i) \prod_{j\neq i} \frac{x - a_j}{a_i - a_j} \end{aligned}$$

It is easy to see that $B(a_k) = A(a_k)$ for $0 \leq k \leq n$ and $\deg B \leq n$. Hence $B(x) = A(x)$. The highest coefficient of $B(x)$ is equal to

$$\sum_{i=0}^{n} A(a_i) \prod_{j\neq i} \frac{1}{a_i - a_j}$$

Since by hypothesis this coefficient is a nonzero integer, then its absolute value is greater than or equal to one. That is,

$$\begin{aligned} 1 &\leq \left|\sum_{i=0}^{n} A(a_i) \prod_{j\neq i} \frac{1}{a_i - a_j}\right| \leq \sum_{i=0}^{n} |A(a_i)| \left|\prod_{j\neq i} \frac{1}{a_i - a_j}\right| \\ &\leq \sum_{i=0}^{n} \left(\max_{0\leq k\leq n} |A(a_k)|\right) \prod_{j\neq i} \frac{1}{|a_i - a_j|} \\ &\leq \left(\max_{0\leq k\leq n} |A(a_k)|\right) \sum_{i=0}^{n} \prod_{j\neq i} \frac{1}{|i - j|} \\ &= \left(\max_{0\leq k\leq n} |A(a_k)|\right) \sum_{i=0}^{n} \frac{1}{i!(n - i)!} \\ &= \left(\max_{0\leq k\leq n} |A(a_k)|\right) \sum_{i=0}^{n} \frac{1}{n!}\binom{n}{k} = \left(\max_{0\leq k\leq n} |A(a_k)|\right) \frac{2^n}{n!} \end{aligned}$$

Therefore, one of the numbers $|A(a_k)|$ is not less than $2^{-n}\, n!$ and we are done.

Also solved by the proposer.

A–10. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Compute

$$\int_0^\infty \frac{dx}{(1+x^2)(1+x^e)}$$

Solution by the proposer. Setting $t = \arctan(x)$, we have $dt = \dfrac{dx}{1+x^2}$ and

$$\int_0^\infty \frac{dx}{(1+x^2)(1+x^e)} = \int_0^{\pi/2} \frac{dt}{1+\tan^e t}$$

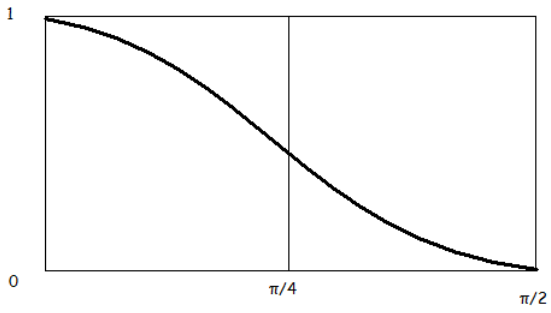

Figure 17: Graph of the function $f(x)$

Now we consider the function $f : [0, \pi/2] \to \mathbb{R}$ defined by $f(x) = \dfrac{1}{1+\tan^e x}$. Since $\tan\left(\dfrac{\pi}{2} - x\right) = \dfrac{1}{\tan x}$, then

$$\begin{aligned} f\left(\frac{\pi}{2} - x\right) &= \frac{1}{1+\tan^e(\pi/2 - x)} = \frac{\tan^e x}{1+\tan^e x} \\ &= 1 - \frac{1}{1+\tan^e x} = 1 - f(x) \end{aligned}$$

Therefore, $f(x) + f(\pi/2 - x) = 1$ for $0 \le x \le \pi/2$ and this means that the graph of $f(x)$ in $[0, \pi/2]$ has the symmetries presented in figure 17. Hence, the area under the curve is the area of the rectangle $[0, \pi/4] \times [0, 1]$ and

$$\int_0^{\pi/2} \frac{dt}{1+\tan^e t} = \frac{\pi}{4}$$

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