No. 1
2026

# ARHIMEDE
# MATHEMATICAL JOURNAL

*Articles*
*Problems*
*Mathlessons*

ARHIMEDE ASSOCIATION

# CONTENTS

**Mathlessons**



**Contests**



**Solutions**

# Articles

Arhimede Mathematical Journal aims to publish interesting and attractive papers with elegant mathematical exposition. Articles should include examples, applications and illustrations, whenever possible. Manuscripts submitted should not be currently submitted to or accepted for publication in another journal.

Please, send submittals to **José Luis Díaz-Barrero**, Barcelona Mathematical Circle (BMC), FME, Pau Gargallo, 14, Les Corts, 08028 Barcelona, Spain or by e-mail to

jose.luis.diaz@upc.edu

# *Advances in an open problem of quadratic congruences*

**J. J. Salvador Alzola**

**Abstract**

In this article, we are going to tackle an open problem that was posed in a previous edition of this journal: "Solving quadratic congruences" by Pascual Jara and Miguel L. Rodríguez, Arhimede Mathematical Journal 2020 No.2. The problem consists of finding all the solutions of a specific system of three quadratic congruences, the main advances of which were discovered and presented in that paper. Next, we generalize some of its results with the aim of moving closer to the solution of the problem.

# 1 Introduction

Given the system of congruences,

$$\begin{cases} x^2 \equiv 1 \pmod{y} \\ y^2 \equiv 1 \pmod{z} \\ z^2 \equiv 1 \pmod{x} \end{cases}$$

our goal is to determine all triplets $(x, y, z) \in \mathbb{Z}^3$ satisfying them. Originally, we need to find the solutions in $\mathbb{N}^3$. However, as we shall see later, it is convenient to consider solutions in $\mathbb{Z}^3$ for a middle result, even if we are ultimately interested in the natural ones. Moreover, every result known in $\mathbb{N}^3$ can be extended to $\mathbb{Z}^3$ with the exact same arguments.

The classical variant of this problem, the two-equations equivalent,

$$\begin{cases} x^2 \equiv 1 \pmod{y} \\ y^2 \equiv 1 \pmod{x} \end{cases}$$

has been fully studied and solved, and its solutions can be expressed in terms of Lucas-sequences. For further details, we refer the reader to the article "Solving quadratic congruences" [1] by Pascual Jara and Miguel L. Rodríguez.

Meanwhile, the three-equations problem is still open.

**Definition 1.** *We will denote by $P$ the subset of $\mathbb{Z}^3$ satisfying:*

$$\begin{cases} x^2 \equiv 1 \ (mod\ y) \\ y^2 \equiv 1 \ (mod\ z) \\ z^2 \equiv 1 \ (mod\ x) \end{cases}$$

**Remark 1.** *It should be noted that $(x, y, z) \in P$, $(y, z, x) \in P$ and $(z, x, y) \in P$ are equivalent.*

To summarize the current progress in describing $P$, we have already identified several of its subsets. First of all, we obtain some trivial solutions after fixing $z = \pm 1$:

$$\begin{aligned} P \supset & \{(x, y, \pm 1) \in \mathbb{Z}^3 : y \mid x^2 - 1\} \\ \supset & \{(x, y, \pm 1) \in \mathbb{Z}^3 : y \mid x - 1 \ \vee \ y \mid x + 1\} \\ = & \{(nk + 1, k, \pm 1\}) \in \mathbb{Z}^3 : n, k \in \mathbb{Z}\} \\ & \cup \{(nk - 1, k, \pm 1\}) \in \mathbb{Z}^3 : n, k \in \mathbb{Z}\} \end{aligned}$$

In addition, we know that the following families are also subsets of solutions, as shown in the propositions 2 and 3 of the paper "Solving quadratic congruences" [1].

(a) $(x_k, y_k, z_k) = (4k + 4, 4k + 3, 2k + 1)$.
(b) $(x_k, y_k, z_k) = (4k, 4k + 1, 2k + 1)$.
(c) $(x_k, y_k, z_k) = (4k^2 - 1, 2k^2 - 1, 2k)$.

Although the result is stated there for all $k \in \mathbb{N}$, it also applies for all $k \in \mathbb{Z}$, with the convention that congruence modulo $0$ is interpreted as equality in $\mathbb{Z}$.

## 2 Properties and generalization

**Proposition 1.** *$(x, y, z) \in P$ if and only if $(-x, -y, -z) \in P$.*

*Proof.* If $x^2 \equiv 1 \pmod{y}$, then $x^2 \equiv 1 \pmod{-y}$ since working modulo $n$ is equivalent to working modulo $-n$, therefore $(-x)^2 \equiv 1 \pmod{-y}$. We apply the same argument for the other two equations. Hence, if $(x, y, z)$ is a solution, $(-x, -y, -z)$ is also a solution. The converse follows immediately. □

**Example 1.** *Prove that (a) is a solution if and only if (b) is.*

*Proof.* Let us show that (a) and (b) are opposite families:

$$\begin{aligned}
&\{(-(4k+4), -(4k+3), -(2k+1))\}_{k\in\mathbb{Z}} = \\
&\{(-4k-4, -4k-3, -2k-1)\}_{k\in\mathbb{Z}} = \\
&\{(4k-4, 4k-3, 2k-1)\}_{k\in\mathbb{Z}} = \\
&\{(4k, 4k+1, 2k+1)\}_{k\in\mathbb{Z}}
\end{aligned}$$

By Proposition 1, we know that, since they are opposites, (a) is solution if and only if (b) is. □

**Theorem 1.** *For all $n, k \in \mathbb{Z}$, $(x_{n,k}, y_{n,k}, z_{n,k}) \in P$, where:*

$$\begin{aligned}
x_{n,k} &= n^4 + n^3 - 2n^2 - n + 1 + (n+1)^2 n^2 k \\
y_{n,k} &= n^3 + n^2 - 2n - 1 + (n+1)^2 nk \\
z_{n,k} &= n^3 - 2n + (n+1)n^2 k
\end{aligned}$$

*Proof.*

- Equation $x^2 \equiv 1 \pmod{y}$:
  $(x_{n,k})^2 \equiv 1 \pmod{y_{n,k}}$
  $\iff (n^4 + n^3 - 2n^2 - n + 1 + (n+1)^2 n^2 k)^2 \equiv 1 \pmod{y_{n,k}}$
  $\iff (n^4 + n^3 - 2n^2 - n + 1 + n[(n+1)^2 nk])^2 \equiv 1 \pmod{y_{n,k}}$
  $\iff (n^4 + n^3 - 2n^2 - n + 1 + n[-n^3 - n^2 + 2n + 1])^2 \equiv 1 \pmod{y_{n,k}}$
  $\iff 1^2 \equiv 1 \pmod{y_{n,k}}$

- Equation $y^2 \equiv 1 \pmod{z}$:
  $(y_{n,k})^2 \equiv 1 \pmod{z_{n,k}}$
  $\iff (n^3 + n^2 - 2n - 1 + (n+1)^2nk)^2 \equiv 1 \pmod{z_{n,k}}$
  $\iff (-(n+1)n^2k + n^2 - 1 + (n+1)^2nk)^2 \equiv 1 \pmod{z_{n,k}}$
  $\iff ((n+1)nk(-n + (n+1)) + n^2 - 1)^2 \equiv 1 \pmod{z_{n,k}}$
  $\iff ((n+1)nk + n^2 - 1)^2 \equiv 1 \pmod{z_{n,k}}$
  $\iff (n+1)^2n^2k^2 + n^4 - 2n^2 + 1 + 2(n+1)nk(n^2-1) \equiv 1 \pmod{z_{n,k}}$
  $\iff (n+1)^2n^2k^2 + n^4 - 2n^2 + 1 + 2(n+1)nk(n^2-1) - 1 \equiv 0 \pmod{z_{n,k}}$
  $\iff (z_{n,k})((n+1)k + n) \equiv 0 \pmod{z_{n,k}}$
- Equation $z^2 \equiv 1 \pmod{x}$:
  $(z_{n,k})^2 \equiv 1 \pmod{x_{n,k}}$
  $\iff (n^3 - 2n + (n+1)n^2k)^2 \equiv 1 \pmod{x_{n,k}}$
  $\iff n^6 - 4n^4 + 4n^2 + (n+1)^2n^4k^2 + 2(n^3-2n)(n+1)n^2k \equiv 1 \pmod{x_{n,k}}$
  $\iff n^6 - 4n^4 + 4n^2 + (n+1)^2n^4k^2 + 2(n^3-2n)(n+1)n^2k - 1 \equiv 0 \pmod{x_{n,k}}$
  $\iff (x_{n,k})(kn^2 + n^2 - n - 1) \equiv 0 \pmod{x_{n,k}}$

□

**Theorem 2.** *For all $n, k \in \mathbb{Z}$, $(x_{n,k}, y_{n,k}, z_{n,k}) \in P$, where:*

$$\begin{aligned} x_{n,k} &= n^3 - 2n + (n+1)^2n^2k \\ y_{n,k} &= n^2 - n - 1 + (n+1)n^2k \\ z_{n,k} &= n^2 - 1 + (n+1)^2nk \end{aligned}$$

*Proof.*

- Equation $x^2 \equiv 1 \pmod{y}$:
  $(x_{n,k})^2 \equiv 1 \pmod{y_{n,k}}$
  $\iff (n^3 - 2n + (n+1)^2n^2k)^2 \equiv 1 \pmod{y_{n,k}}$
  $\iff (n^3 - 2n + (n+1)[(n+1)n^2k])^2 \equiv 1 \pmod{y_{n,k}}$
  $\iff (n^3 - 2n + (n+1)[-n^2 + n + 1])^2 \equiv 1 \pmod{y_{n,k}}$
  $\iff 1^2 \equiv 1 \pmod{y_{n,k}}$
- Equation $y^2 \equiv 1 \pmod{z}$:
  $(y_{n,k})^2 \equiv 1 \pmod{z_{n,k}}$
  $\iff (n^2 - n - 1 + (n+1)n^2k)^2 \equiv 1 \pmod{z_{n,k}}$

$\iff (n^2-n-1)^2+(n+1)^2n^4k^2+2(n^2-n-1)((n+1)n^2k) \equiv 1 \pmod{z_{n,k}}$
$\iff (n^2-n-1)^2+(n+1)^2n^4k^2+2(n^2-n-1)((n+1)n^2k)-1 \equiv 0 \pmod{z_{n,k}}$
$\iff (z_{n,k})(n^3k+n^2-2n) \equiv 0 \pmod{z_{n,k}}$

- Equation $z^2 \equiv 1 \pmod{x}$:
  $(z_{n,k})^2 \equiv 1 \pmod{x_{n,k}}$
  $\iff (n^2-1+(n+1)^2nk)^2 \equiv 1 \pmod{x_{n,k}}$
  $\iff (n^4-2n^2+1+(n+1)^4n^2k^2+2(n^2-1)(n+1)^2nk \equiv 1 \pmod{x_{n,k}}$
  $\iff n^4-2n^2+1+(n+1)^4n^2k^2+2(n^2-1)(n+1)^2nk-1 \equiv 0 \pmod{x_{n,k}}$
  $\iff (x_{n,k})((n^2+2n+1)k+n) \equiv 0 \pmod{x_{n,k}}$

□

**Theorem 3.** *For all $n, k \in \mathbb{Z}$, $(x_{n,k}, y_{n,k}, z_{n,k}) \in P$, where:*

$$\begin{aligned} x_{n,k} &= -n^4-n^3+2n^2+n-1+(n+1)^2n^2k \\ y_{n,k} &= -n^3-n^2+2n+1+(n+1)^2nk \\ z_{n,k} &= -n^3+2n+(n+1)n^2k \end{aligned}$$

*Proof.* Observing that the triplets proposed in this theorem are the opposites of the solutions found in Theorem 1, we can apply Proposition 1 to conclude that these triplets are also solutions. □

**Theorem 4.** *For all $n, k \in \mathbb{Z}$, $(x_{n,k}, y_{n,k}, z_{n,k}) \in P$, where:*

$$\begin{aligned} x_{n,k} &= -n^3+2n+(n+1)^2n^2k \\ y_{n,k} &= -n^2+n+1+(n+1)n^2k \\ z_{n,k} &= -n^2+1+(n+1)^2nk \end{aligned}$$

*Proof.* Likewise, observing that the triplets proposed in this theorem are the opposites of the solutions found in Theorem 2, we can apply Proposition 1 to conclude that these triplets are also solutions. □

**Remark 2.** *What relationship do the known families have with the new ones?*

- *(a) is a particular case of Theorem 1 for $n = 1$ and a particular case of Theorem 2 also for $n = 1$.*
- *(b) is a particular case of Theorem 3 for $n = 1$ and a particular case of Theorem 4 also for $n = 1$*
- *(c) is not a subset of any of these four families.*

**Remark 3.** *Analyzing the Theorem 1 family, when $k = 0$, we obtain*

$$x_{n,0} = n^4 + n^3 - 2n^2 - n + 1$$

*whose roots are: $1, -1, -\phi, \phi^{-1}$. Might it be linked to the Lucas-sequences behavior of the two-equations problem?*

# 3 Conclusion

Essentially, the approach taken to study this problem consists of finding triplets of polynomials with coefficients in $\mathbb{Z}$ that satisfy the three equations, so that every integer evaluation yields a solution. In this article, we have already presented polynomials of one and two variables, which naturally leads to the question of whether such polynomials exist for any number of variables. Moreover, it would be of interest to understand how to construct them in a more general form.

On the other hand, we do not need to go that far to encounter substantial difficulties. If we just fix $z = 1$, the second and third equations become trivial, leaving only the first one to consider.

$$x^2 \equiv 1 \pmod{y}$$

Equivalently:

$$(x-1)(x+1) \equiv 0 \pmod{y}$$

However, if we could know all these solutions we would obtain a closed-form factorization formula for any $(x-1)(x+1) \in \mathbb{Z}$, where

$x$ is an integer. For instance, let us fix $x = 8$. Then, finding every $y \in \mathbb{Z}$ such that $(8, y, 1) \in P$ is equivalent to knowing all divisors of $7 \cdot 9$. Factorization itself seems to constitute the main obstacle in this problem. Nevertheless, it is natural to wonder whether further advances in this direction could lead to new approaches to primality testing or integer factorization.

# References

[1] Jara, P. and Rodríguez, M. L. "Solving quadratic congruences". *Arhimede Mathematical Journal* 7.2 (2020). URL: https://amj-math.com/archive/.

*José Jesús Salvador Alzola*
jjsalvador.bc@gmail.com

# *Angle Trisectors and Cubic Equations*

**Lerna Pehlivan and Kenneth S. Williams**

**Abstract**

Although many quantities related to a triangle $ABC$ (for example its area) have been expressed solely and explicitly in terms of its side lengths $a$, $b$, and $c$, this has not been done for the lengths of the two angle trisectors from $C$ to $BA$ in terms of $a$, $b$, and $c$. We determine explicit expressions for these two lengths solely in terms of $a, b$, and $c$.

## 1 Introduction

It has been known for centuries that the lengths of the sides $a$, $b$, and $c$ of a nondegenerate triangle satisfy

$$0 < a < b + c, \quad 0 < b < c + a, \quad 0 < c < a + b, \tag{1}$$

and these inequalities determine the triangle uniquely. Many quantities connected to a triangle, for example its area, the lengths of its altitudes, the lengths of its medians, the sines and cosines of its angles, and the lengths of its angle bisectors have been given explicitly in terms of $a, b$, and $c$. We just give the formula for an angle bisector of a triangle in terms of $a$, $b$, $c$ as we are going to make use of it.

The angle bisector of the angle $\angle BCA$ of the triangle $ABC$ meets $BA$ at the point $P$ such that $\angle BCP = \angle PCA$, see Figure 1. We denote the length of $CP$ by $l$. Then

$$l^2 = ab\left(1 - \frac{c^2}{(a+b)^2}\right), \tag{2}$$

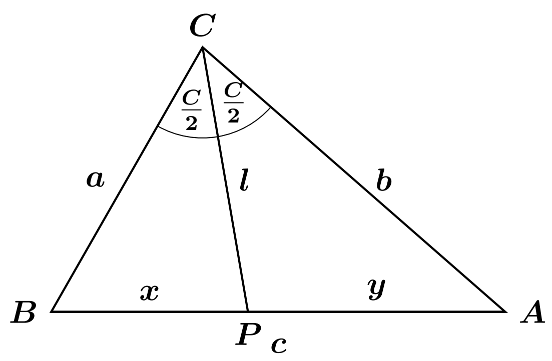



Figure 1: Angle bisector $CP$ of angle $C$ of triangle $ABC$.

see [1, p. 11]. The formula (2) can be proved as follows. If $a = b$ then $CP$ is at right angles to $BA$ so $x = y = c/2$ and, by Pythagoras' theorem, we have

$$l^2 = a^2 - x^2 = a^2 - \frac{c^2}{4},$$

which is (2) when $a = b$. Thus we may suppose that $a \neq b$. The lengths $x$ and $y$ of $BP$ and $PA$ respectively satisfy

$$bx = ay, \quad x + y = c \tag{3}$$

so $x = ac/(a+b)$, $y = bc/(a+b)$. Then, by the cosine formula in triangles $BCP$ and $PCA$, we have

$$\frac{a^2 + l^2 - x^2}{2al} = \cos\left(\frac{C}{2}\right) = \frac{b^2 + l^2 - y^2}{2bl}.$$

Solving for $l^2$, we obtain as $a - b \neq 0$

$$l^2 = ab - \frac{bx^2 - ay^2}{a - b}.$$

Formula (2) now follows as by (3) we have

$$bx^2 - ay^2 = (a-b)xy = (a-b)\frac{abc^2}{(a+b)^2}.$$

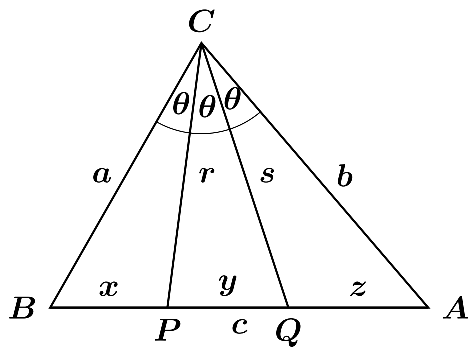



Figure 2: Angle trisectors $CP$ and $CQ$ of angle $C$ of triangle $ABC$.

Angle trisection was first considered by ancient Greek mathematicians who wanted to know if an arbitrary angle could be trisected by a finite number of operations using only a ruler (without marks) and a compass. This classical question remained unanswered until the French mathematician Pierre Wantzel (1814 - 1848) proved in 1837 that such a construction is impossible. In 1899 the British-American mathematician Frank Morley (1860 - 1937) discovered that the three points of intersection of the adjacent angle trisectors form an equilateral triangle. This beautiful theorem has many proofs but perhaps the simplest one is the one given by R. J. Webster [4] in 1970 based on the trigonometric identity

$$\sin 3\theta = 4\sin\theta\sin\Big(\frac{\pi}{3}+\theta\Big)\sin\Big(\frac{2\pi}{3}+\theta\Big).$$

In this article we answer the question "What is the length of an angle trisector?" We determine the lengths $r$ and $s$ of the two angle trisectors from $C$ to $BA$, see Figure 2, solely in terms of the lengths $a, b$, and $c$.

It suffices to determine a formula for $s$ in terms of $a, b, c$, as the mapping which interchanges the vertices $A$ and $B$ of triangle $ABC$ and leaves $C$ alone, has the effect of interchanging $a$ and $b$, leaving $c$ unchanged, and interchanges $r$ and $s$. Thus if the formula for $s$ is $s = F(a, b, c)$ then the formula for $r$ is $r = F(b, a, c)$.

By applying the angle bisector theorem to the two triangles $CBQ$

and $CPA$ in Figure 2, we obtain the equation satisfied by the length $s$ of the angle trisector $CQ$. In general this equation is a cubic equation but in some instances it can be a quadratic equation, see Theorem 1. The corresponding equation for the length $r$ of the angle trisector $CP$ is given in Theorem 2, and again it can be either a cubic equation or a quadratic equation. When the equation for $s$ in Theorem 1 is a quadratic equation, we determine both $r$ and $s$ in terms of $a, b$, and $c$, see Theorems 3 and 4. When the equation for $s$ in Theorem 1 is a cubic equation, we determine the number of positive real roots of the cubic and identify which root is $s$, see Theorems 5, 6, and 7. Finally, we show that $r$ can be expressed as a rational function of $s$, see Theorem 8.

We conclude this introduction by remarking that once the lengths $r$ and $s$ of the angle trisectors have been determined, the lengths $x$, $y$, and $z$ follow from

$$x = |BP| = \frac{arc}{ar+rs+bs}, \; y = |PQ| = \frac{rsc}{ar+rs+bs},$$
$$z = |QA| = \frac{bsc}{ar+rs+bs}.$$

# 2 Equations satisfied by $r$ and $s$

In this section we show that $s$ is the root of a polynomial of degree 2 or 3. The polynomial of which $r$ is a root is then easily determined by interchanging $a$ and $b$.

Our approach is to apply the angle bisector theorem to the two triangles $CBQ$ (angle bisector $= CP$) and $CPA$ (angle bisector $= CQ$). Applying (2) and (3) to triangles $CBQ$ and $CPA$, we obtain respectively

$$ay = sx, \quad r^2 = as\left(1 - \left(\frac{x+y}{a+s}\right)^2\right), \tag{4}$$

$$by = rz, \quad s^2 = br\left(1 - \left(\frac{y+z}{b+r}\right)^2\right). \tag{5}$$

Since

$$\frac{x+y}{a+s} = \frac{x}{a} = \frac{y}{s} \quad \text{and} \quad \frac{y+z}{b+r} = \frac{y}{r} = \frac{z}{b},$$

we deduce from (4) and (5)

$$r^2 = as\left(1 - \frac{x^2}{a^2}\right) = as\left(1 - \frac{y^2}{s^2}\right), \tag{6}$$

$$s^2 = br\left(1 - \frac{y^2}{r^2}\right) = br\left(1 - \frac{z^2}{b^2}\right). \tag{7}$$

From (6) and (7) we deduce

$$r^2 - \frac{rs^2}{b} = s^2 - \frac{r^2 s}{a} (= y^2). \tag{8}$$

Setting $w := r/s$ in (8), we obtain $(a+s)bw^2 - asw - ab = 0$. Solving this quadratic equation for $w$, we deduce

$$w = \frac{as + E}{2(a+s)b}, \text{ where } E^2 = a^2s^2 + 4ab^2s + 4a^2b^2. \tag{9}$$

As $w > 0$, we see that $E + as > 0$. Also, by using the value of $E^2$ in (9), we have

$$(E - as)(E + as) = 4ab^2s + 4a^2b^2 > 0$$

so $E - as > 0$. Thus $E = \frac{1}{2}(E + as + E - as) > 0$. Hence

$$E = +\sqrt{a^2s^2 + 4ab^2s + 4a^2b^2}.$$

By the cosine formula applied to $\angle BCP$ and the first equality in (6), we have

$$\cos\theta = \frac{r^2 + a^2 - x^2}{2ar} = \frac{r^2 + \frac{ar^2}{s}}{2ar} = \frac{r(a+s)}{2as} = \frac{w(a+s)}{2a}, \tag{10}$$

where here and throughout $\theta := C/3$. Then, by (9) and (10), we obtain

$$\cos\theta = \frac{as + E}{4ab}. \tag{11}$$

By the cosine formula in $\triangle ABC$ we have

$$4\cos^3\theta - 3\cos\theta = \cos 3\theta = \cos C = \frac{a^2+b^2-c^2}{2ab}.$$

Putting the value of $\cos\theta$ from (11) into this equation, we deduce after simplification, using the value of $E^2$ given in (9),

$$(-2ab^2 + b^2 s + as^2)E = 2ab^2(a^2+b^2-c^2) - 3ab^2s^2 - a^2s^3. \tag{12}$$

Squaring both sides of (12), and then appealing to (9) for the value of $E^2$, we obtain after a little rearrangement

$$\begin{aligned}&(2ab^2(a^2+b^2-c^2) - 3ab^2s^2 - a^2s^3)^2\\ &\qquad - (-2ab^2 + b^2s + as^2)^2(a^2s^2 + 4ab^2s^2 + 4a^2b^2) = 0.\end{aligned}$$

Using MAPLE to expand the left hand side of this equation as a polynomial in $s$, we find amazingly that the coefficients of $s^6, s^5, s^4$ and $s$ are all $0$. The coefficients of $s^3, s^2$ and $1$ are $4ab^2(a^2c^2 - (a^2-b^2)^2)$, $12a^2b^4c^2$, and $-4a^2b^2(a+b+c)(a+b-c)(a-b+c)(-a+b+c)$. Cancelling the common factor $4ab^2$, we have established the following result.

**Theorem 1.** *The length $s$ of the angle trisector $CQ$ is a positive root of the equation*

$$Rx^3 + Sx^2 + T = 0, \tag{13}$$

*where*

$$\begin{aligned}R &= R(a,b,c) := a^2c^2 - (a^2-b^2)^2,\\ S &= S(a,b,c) := 3ab^2c^2,\\ T &= T(a,b,c) := -ab^2(a+b+c)(a+b-c)(a-b+c)\\ &\qquad\qquad (-a+b+c).\end{aligned} \tag{14}$$

We observe from (1) and (14) that

$$S > 0, \quad T < 0. \tag{15}$$

As $R(1,1,1) > 0$, $R(2,1,3/2) = 0$, and $R(1,2,2) < 0$, each of the three possibilities $R > 0$, $R = 0$ and $R < 0$ can occur.

Applying the mapping $A \to B$, $B \to A$, $C \to C$ to the triangle $ABC$ as described in the Introduction, we obtain from Theorem 1 the length $r$ of the angle trisector $CP$.

**Theorem 2.** *The length $r$ of the angle trisector $CP$ is a positive root of the equation*

$$R'x^3 + S'x^2 + T' = 0, \tag{16}$$

*where*

$$\begin{aligned} R' &= R(b,a,c) := b^2c^2 - (a^2-b^2)^2, \\ S' &= S(b,a,c) := 3a^2bc^2, \\ T' &= T(b,a,c) := -a^2b(a+b+c)(a+b-c)(a-b+c) \\ &\qquad\qquad (-a+b+c). \end{aligned} \tag{17}$$

We note from (14), (15), and (17) that

$$S' = \frac{a}{b}S > 0, \quad T' = \frac{a}{b}T < 0. \tag{18}$$

We now return to equation (13). When $R = 0$ the equation (13) is a quadratic equation as $S \neq 0$. The precise determination of $s$ as a solution of this quadratic equation is given in Theorem 3. The geometric meaning of the condition $R = 0$ is also explained, and used to obtain a formula for $r$, see Theorem 4.

When $R \neq 0$ the cubic polynomial (13) has discriminant $\Delta$ given by

$$\Delta := -4S^3T - 27R^2T^2 = -T(4S^3 + 27R^2T) \tag{19}$$

see [3, p. 161]. It is now convenient to define

$$\begin{aligned} U &:= (a+b+c)(a+b-c)(a-b+c)(-a+b+c) \\ &\quad\ (a^2c^2 - (a^2-b^2)^2)^2, \\ V &:= (a+b+c)(a+b-c)(a-b+c)(-a+b+c), \\ W &:= a^2c^4 - (2a^4 - a^2b^2 - b^4)c^2 + (a^2-b^2)^3, \end{aligned} \tag{20}$$

so that $V > 0$ by (1). Also, by (14) and (20), we have

$$T = -ab^2V, \quad U = R^2V, \tag{21}$$

and

$$U > 0, \quad (\text{as } R \neq 0). \tag{22}$$

Using MAPLE we can verify that

$$U + W^2 = 4a^2b^4c^6. \tag{23}$$

Hence, by (14), (21) and (23), we obtain

$$\begin{aligned} 4S^3 + 27R^2T &= 108a^3b^6c^6 + 27R^2(-ab^2V) \\ &= 27ab^2(4a^2b^4c^6 - U) = 27ab^2W^2 \end{aligned}$$

so, by (19) and (15), we see that

$$\Delta = \begin{cases} -27ab^2TW^2 > 0, & \text{if } W \neq 0, \\ 0, & \text{if } W = 0. \end{cases} \tag{24}$$

Hence the equation $Rx^3 + Sx^2 + T = 0$ has three distinct real roots if $R \neq 0$ and $W \neq 0$, whereas, if $R \neq 0$ and $W = 0$, it has three real roots exactly two of which are equal as $T \neq 0$ see [2, p. 47]. The determination of which of these roots is $s$ is carried out in Section 4, see Theorem 5 for the case $R > 0$, $W \neq 0$; Theorem 6 for the case $R \neq 0$, $W = 0$; and Theorem 7 for the case $R < 0$, $W \neq 0$. We conclude this section by noting that at least one of (13) and (16) is a cubic equation as

$$\begin{aligned} R = R' = 0 &\Longrightarrow a^2c^2 - (a^2-b^2)^2 = b^2c^2 - (a^2-b^2)^2 = 0 \\ &\Longrightarrow (a^2-b^2)c^2 = 0 \quad \Longrightarrow a^2 - b^2 = 0 \quad \Longrightarrow a^2c^2 = 0, \end{aligned}$$

a contradiction. This shows that cubic equations cannot be avoided in determining the lengths of the angle trisectors of a triangle.

# 3 The case $R = 0$

When $R = 0$, that is by (14) $a^2c^2 - (a^2 - b^2)^2 = 0$, we know by (13) that the length $s$ of the angle trisector $CQ$ satisfies $Ss^2 + T = 0$, so that $s^2 = -\frac{T}{S}$. By (15) we have $-\frac{T}{S} > 0$ so as $s > 0$ we have $s = +\sqrt{-\frac{T}{S}}$. Appealing to (14), we obtain

$$S = 3ab^2c^2 = \frac{3b^2(a^2 - b^2)^2}{a} \tag{25}$$

and

$$\begin{aligned}
T &= -ab^2\big((a+b)^2 - c^2\big)\big(c^2 - (a-b)^2\big) \\
&= -ab^2\left((a+b)^2 - \frac{(a^2-b^2)^2}{a^2}\right)\left(\frac{(a^2-b^2)^2}{a^2} - (a-b)^2\right) \\
&= -\frac{ab^2}{a^4}\big(a^2(a+b)^2 - (a+b)^2(a-b)^2\big) \\
&\quad \big((a+b)^2(a-b)^2 - a^2(a-b)^2\big) \\
&= -\frac{b^2}{a^3}(a+b)^2(a-b)^2\big(a^2 - (a-b)^2\big)\big((a+b)^2 - a^2\big) \\
&= -\frac{b^2}{a^3}(a^2-b^2)^2\big(2ab - b^2\big)\big(2ab + b^2\big),
\end{aligned}$$

that is

$$-T = -\frac{b^4}{a^3}(a^2 - b^2)^2\big(4a^2 - b^2\big). \tag{26}$$

As $-T > 0$ (by (15)) we have $4a^2 - b^2 > 0$ and

$$s^2 = -\frac{T}{S} = \frac{1}{3}\frac{b^2}{a^2}(4a^2 - b^2).$$

We have proved

**Theorem 3.** *If $R = 0$ we have*

$$s = \frac{b}{a}\sqrt{\frac{4a^2 - b^2}{3}}.$$

We now give the geometric meaning of the condition $R = 0$. If $R = 0$ then

$$\left(ac + a^2 - b^2\right)\left(ac - a^2 + b^2\right) = a^2c^2 - \left(a^2 - b^2\right)^2 = 0$$

so that

$$ac + a^2 - b^2 = 0 \quad \text{or} \quad ac - a^2 + b^2 = 0. \tag{27}$$

The geometric significance of the condition $ac + a^2 - b^2 = 0$ has been recognized by Willson [5], namely

$$ac + a^2 - b^2 = 0 \quad \text{if and only if } \angle B = 2\angle A. \tag{28}$$

We have not found the geometric interpretation of the condition $ac - a^2 + b^2 = 0$ in the literature, however Willson's simple proof of (28) is easily modified to prove

$$ac - a^2 + b^2 = 0 \quad \text{if and only if } \angle B + \pi = 2\angle A. \tag{29}$$

Putting together these results, we deduce

$$R = 0 \quad \text{if and only if } 2\angle A = \angle B \text{ or } \angle B + \pi. \tag{30}$$

We next determine $r$ when $R = 0$ by making use of (30). First we treat the case $2\angle A = \angle B$. In this case the angles of $\triangle ABC$ are shown in Figure 3. By the sine formula in $\triangle ABC$, we have

$$\frac{a}{\sin A} = \frac{b}{\sin 2A} = \frac{b}{2\sin A \cos A},$$

so that $\cos A = b/2a$, $\sin A = \sqrt{4a^2 - b^2}/2a$, and $\sin 2A = b\sqrt{4a^2 - b^2}/2a^2$. Thus, by the sine formula in $\triangle CBQ$, we have in agreement with Theorem 3

$$s = \frac{a}{\sin \pi/3}\sin 2A = \frac{b}{a}\sqrt{\frac{4a^2 - b^2}{3}}.$$

By the sine formula in $\triangle CPQ$, we obtain

$$r = \frac{s}{\sin(\pi/3 + A)}\sin \pi/3 = \frac{s\sqrt{3}}{2\sin(\pi/3 + A)}.$$

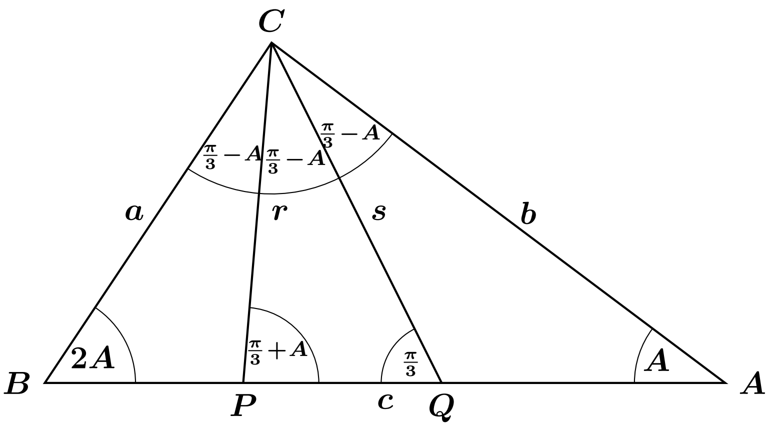



Figure 3: $\angle B = \angle 2A$

Now

$$\sin\Big(\frac{\pi}{3} + A\Big) = \frac{\sqrt{3}}{2}\cos A + \frac{1}{2}\sin A = \frac{\sqrt{4a^2 - b^2} + b\sqrt{3}}{4a}$$

so

$$\frac{1}{\sin\big(\frac{\pi}{3} + A\big)} = \frac{a\big(\sqrt{4a^2 - b^2} - b\sqrt{3}\big)}{a^2 - b^2},$$

and thus

$$r = \frac{b}{a}\sqrt{\frac{4a^2 - b^2}{3}}\,\frac{\sqrt{3}}{2}\,\frac{a\big(\sqrt{4a^2 - b^2} - b\sqrt{3}\big)}{a^2 - b^2}.$$

We have proved the following formula for $r$ in the case $2\angle A = \angle B$, namely

$$r = \frac{b}{2}\frac{(4a^2 - b^2)}{a^2 - b^2} - \frac{3}{2}\frac{b^2}{a^2 - b^2}\sqrt{\frac{4a^2 - b^2}{3}}.$$

Now we turn to the case $2\angle A = \angle B + \pi$. In this case the angles of $\triangle ABC$ are as in Figure 4. By the sine formula in $\triangle ABC$ we have

$$\frac{a}{\sin A} = \frac{b}{\sin(2A - \pi)} = \frac{b}{-\sin 2A} = \frac{-b}{2\sin A\cos A},$$

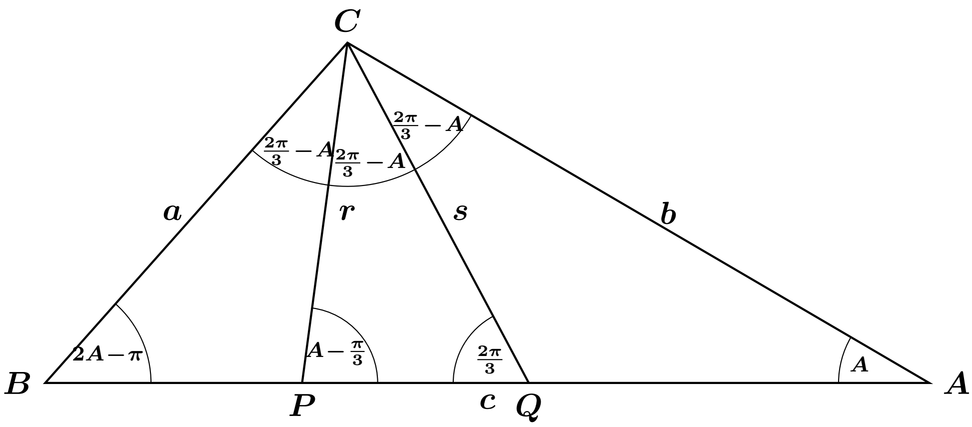



Figure 4: $2\angle A = \angle B + \pi$

so that $\cos A = \frac{-b}{2a}$. As $0 < A < \pi$ and $\cos A < 0$, we have $\frac{\pi}{2} < A < \pi$ so that $\sin A > 0$. Thus $\sin A = \frac{\sqrt{4a^2-b^2}}{2a}$, and $\sin 2A = -\frac{b}{2a^2}\sqrt{4a^2-b^2}$. Hence, by the sine formula in $\triangle CBQ$, we have

$$s = \frac{a}{\sin(2\pi/3)}\sin(2A-\pi) = \frac{a}{\sqrt{3}/2}(-\sin 2A) = \frac{b}{a}\sqrt{\frac{4a^2-b^2}{3}},$$

in agreement with Theorem 3. Finally, by the sine formula in $\triangle CPQ$, we obtain

$$r = \frac{\sin(2\pi/3)s}{\sin(A-\pi/3)} = \frac{s\sqrt{3}}{2\sin(A-\pi/3)}.$$

Now

$$\sin\left(A-\frac{\pi}{3}\right) = \frac{1}{2}\sin A - \frac{\sqrt{3}}{2}\cos A = \frac{\sqrt{4a^2-b^2}+b\sqrt{3}}{4a}$$

so

$$\frac{1}{\sin\left(A-\frac{\pi}{3}\right)} = \frac{a\left(\sqrt{4a^2-b^2}-b\sqrt{3}\right)}{a^2-b^2},$$

and thus in the case $2\angle A = \angle B + \pi$ as in the case $2\angle A = \angle B$, we obtain

$$r = \frac{b}{2}\frac{(4a^2-b^2)}{a^2-b^2} - \frac{3}{2}\frac{b^2}{a^2-b^2}\sqrt{\frac{4a^2-b^2}{3}}.$$

Putting the two cases $2\angle A = \angle B$ and $2\angle A = \angle B + \pi$, together, we obtain by (30) the formula for $r$ when $R = 0$.

**Theorem 4.** *If $R = 0$ we have*

$$r = \frac{b}{2}\frac{(4a^2 - b^2)}{a^2 - b^2} - \frac{3}{2}\frac{b^2}{a^2 - b^2}\sqrt{\frac{4a^2 - b^2}{3}}.$$

**Example 1.** *We determine the lengths $r$ and $s$ of the angle trisectors from $C$ to $BA$ in the $4-5-6$ triangle with $a = 4$, $b = 6$, $c = 5$, see Figure 5. Here*

$$ac + a^2 - b^2 = 20 + 16 - 36 = 0$$

*so by* (28) $\angle B = 2\angle A$. *By* (30), *Theorem 3 and Theorem 4, we have*

$$s = \frac{6}{4}\sqrt{\frac{4 \cdot 4^2 - 6^2}{3}} = \sqrt{21},$$

$$r = \frac{6}{2}\frac{(64 - 36)}{16 - 36} - \frac{3}{2}\frac{36}{16 - 36}\sqrt{\frac{64 - 36}{3}} = \frac{9\sqrt{21} - 21}{5}.$$

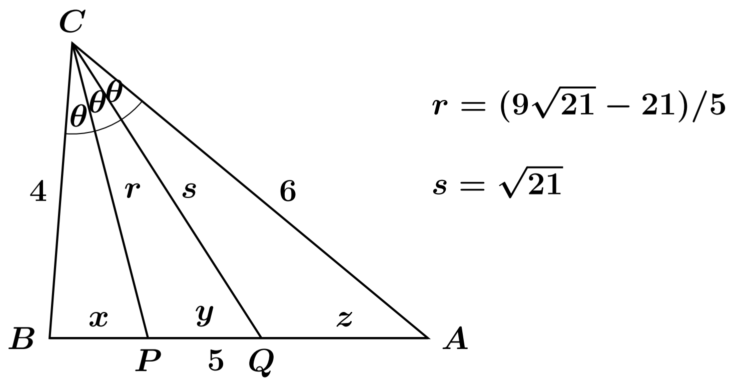



Figure 5: Angel trisectors in $4-5-6$ triangle

# 4 The case $R \neq 0$

When $R \neq 0$ we have proved that the cubic equation (13) has three real roots. We show in this section how to determine which root is the length $s$ of the angle trisector $CQ$. Two cases arise according as $W \neq 0$ or $W = 0$.

When $R \neq 0$ and $W \neq 0$ we denote the three distinct real roots of (13) by $u, v, w$ chosen so that $u < v < w$. Then, from $Rx^3 + Sx^2 + T = R(x-u)(x-v)(x-w)$, we deduce

$$u + v + w = -S/R, \quad uv + vw + wu = 0, \quad uvw = -T/R.$$

If $R > 0$ then $-S/R < 0$ and $-T/R > 0$ so $u + v + w < 0$ and $uvw > 0$ proving that $u < v < 0 < w$. Thus (13) has a unique positive solution which by Theorem 1 must be $s$. We have proved the following result.

**Theorem 5.** *When $R > 0$ and $W \neq 0$ the length $s$ of the angle trisector $CQ$ is the unique positive solution of the cubic equation* (13).

If $R < 0$ then $-S/R > 0$ and $-T/R < 0$ so $u + v + w > 0$ and $uvw < 0$ proving that $u < 0 < v < w$. Thus (13) has two distinct positive solutions and we need more information in order to determine which one is $s$. This is done in Theorem 7.

When $R \neq 0$ and $W = 0$ the three real roots $u, v, w$ of (13) satisfy $u = v < w$ or $u < v = w$. If $R > 0$ then $u + v + w < 0$ and $uvw > 0$ so $u = v < 0 < w$. If $R < 0$ then $u + v + w > 0$ and $uvw < 0$ imply $u < 0 < v = w$. In either case (13) has a unique positive solution, which must be $s$.

**Theorem 6.** *When $R \neq 0$ and $W = 0$ the length $s$ of the angle trisector $CQ$ is the unique positive solution of the cubic equation* (13).

The final case to consider is when $R < 0$ and $W \neq 0$. In this case we have already shown that the cubic (13) has three real solutions $u, v, w$ satisfying $u < 0 < v < w$. The graph of $y = Rx^3 + Sx^2 + T$ is given in Figure 6. Thus we have

$$s = \begin{cases} v & \text{if } s < \frac{2S}{3|R|}, \\ w & \text{if } s > \frac{2S}{3|R|}. \end{cases} \tag{31}$$

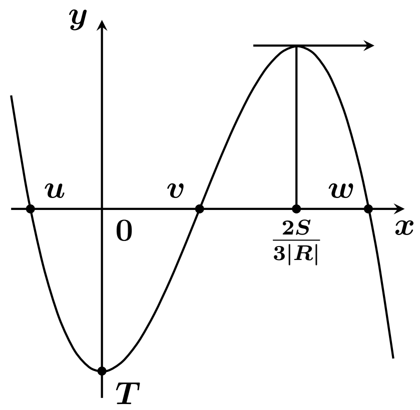



Figure 6: Graph of $y = Rx^3 + Sx^2 + T$

In order to determine whether $s = v$ or $s = w$, we bring into play another cubic equation, namely the equation

$$x^3 - \frac{3}{4}x + \frac{\sqrt{V}}{8ab} = 0, \tag{32}$$

where $V$ was defined in (20). We show that this cubic equation has two positive solutions and one negative solution. The two positive solutions are distinct when $c^2 \neq a^2 + b^2$ so we can define $\lambda = \lambda(a, b, c)$ by

$$\lambda := \begin{cases} \text{smaller of the two positive solutions of (32)} & \text{if } c^2 < a^2 + b^2, \\ \frac{1}{2} & \text{if } c^2 = a^2 + b^2, \\ \text{larger of the two positive solutions of (32)} & \text{if } c^2 > a^2 + b^2. \end{cases}$$

We emphasize that $\lambda$ depends only upon $a$,$b$, $c$, and that it has the important property

$$\lambda = \lambda(a, b, c) = \lambda(b, a, c) = \sin\theta = \sin(C/3). \tag{33}$$

We now prove the claims we have just made. We have

$$3\sin\theta - 4\sin^3\theta = \sin 3\theta = \sin C = \frac{\sqrt{V}}{2ab}$$

so that $\sin\theta$ is a root of the equation (32). The other two roots of (32) are

$$\frac{1}{2}\Big(-\sin\theta \pm \sqrt{3}\cos\theta\Big).$$

As $0 < C < \pi$ we have $0 < \theta < \pi/3$. When $\theta \neq \pi/6$, the equation (32) has two positive roots and one negative root with $\sin\theta$ the smaller of the two positive roots when $0 < \theta < \pi/6$ and the larger when $\pi/6 < \theta < \pi/3$. When $\theta = \pi/6$ we have $\sin\theta = \frac{1}{2}$ and $x = \frac{1}{2}$ is a double root of (32) in this case, the other root being $x = -1$.

By an extension to Pythagoras' theorem, we have

$$\begin{aligned} 0 < \theta < \frac{\pi}{6} &\iff 0 < C < \frac{\pi}{2} \iff c^2 < a^2 + b^2, \\ \theta = \frac{\pi}{6} &\iff C = \frac{\pi}{2} \iff c^2 = a^2 + b^2, \\ \frac{\pi}{6} < \theta < \frac{\pi}{3} &\iff \frac{\pi}{2} < C < \pi \iff c^2 > a^2 + b^2, \end{aligned}$$

completing the proof of (33).

In Figure 2 we have $\angle AQC = \pi - A - \theta$ so applying the sine formula to the sides $CA$ and $CQ$ of $\triangle QCA$, we obtain

$$s = \frac{b\sin A}{\sin(\pi - A - \theta)} = \frac{b\sin A}{\sin(A+\theta)} = \frac{b\sin A}{\sin A\cos\theta + \cos A\sin\theta}.$$

The formula $s = \frac{b\sin A}{\sin(A+\theta)}$ is given in [3, Theorem 3.1, p. 12].

As $\frac{1}{2} < \cos\theta < 1$ for $0 < \theta < \pi/3$, we have by (33)

$$\cos\theta = +\sqrt{1-\lambda^2}.$$

Thus, as $\sin A = \frac{\sqrt{V}}{2bc}$ and $\cos A = \frac{b^2+c^2-a^2}{2bc}$, we deduce

$$s = \frac{b\sqrt{V}}{(b^2+c^2-a^2)\lambda + \sqrt{(1-\lambda^2)V}}.$$

Now suppose that the inequality

$$\sqrt{1-\lambda^2} + \frac{(b^2+c^2-a^2)\lambda}{\sqrt{V}} > \frac{|R|}{2abc^2}$$

holds. Then we have

$$s < b\sqrt{V}\frac{2abc^2}{|R|\sqrt{V}} = \frac{2ab^2c^2}{|R|} = \frac{2S}{3|R|},$$

so $s = v$ by (31). Similarly, if

$$\sqrt{1-\lambda^2} + \frac{(b^2+c^2-a^2)\lambda}{\sqrt{V}} < \frac{|R|}{2abc^2}$$

holds, then $s > \frac{2S}{3|R|}$, and we have $s = w$ by (31).

The equality

$$\sqrt{1-\lambda^2} + \frac{(b^2+c^2-a^2)\lambda}{\sqrt{V}} = \frac{|R|}{2abc^2}$$

cannot occur as it leads to $s = \frac{2S}{3|R|}$, contradicting that $s \neq \frac{2S}{3|R|}$.

This completes the determination of $s$ in the case $R < 0$ and $W \neq 0$.

**Theorem 7.** *When $R < 0$ and $W \neq 0$ the length $s$ of the angle trisector $CQ$ is the smaller of the two distinct positive solutions of* (13) *if*

$$\sqrt{1-\lambda^2} + \frac{(b^2+c^2-a^2)\lambda}{\sqrt{V}} > \frac{|R|}{2abc^2}$$

*and is the larger of the two distinct positive solutions of* (13) *if*

$$\sqrt{1-\lambda^2} + \frac{(b^2+c^2-a^2)\lambda}{\sqrt{V}} < \frac{|R|}{2abc^2}.$$

We conclude this section with an example illustrating each of Theorems 5, 6, and 7.

**Example 2.** *($R > 0, W \neq 0$) We choose $a = 4$, $b = 3$, and $c = 2$ so that*

$$R = 15,\ S = 432,\ T = -4860,\ W = -549.$$

*Hence, by Theorem 5, $s$ is the unique positive solution of*

$$15x^3 + 432x^2 - 4860 = 0.$$

*MAPLE gives the three roots as $-28.398244, -3.584589, 3.182834$ approximately, so $s \approx 3.182834$.*

**Example 3.** *($R \neq 0, W = 0$) We choose $a = 4$, $b = 14$, and $c = 3\sqrt{15}$ so that*

$$R = -30240,\ S = 317520,\ T = -5186160,\ W = 0,$$

*and*

$$Rx^3 + Sx^2 + T = -15120(x-7)^2(2x+7).$$

*By Theorem 6, $s$ is the unique positive solution of $Rx^3+Sx^2+T = 0$, that is $s = 7$.*

**Example 4.** *($R < 0, W \neq 0$) We choose $a = 1$, $b = 4$, and $c = 4$ so that*

$$R = -209,\ S = 768,\ T = -1008,\ V = 63,\ W = 1201.$$

*As $c^2 < a^2 + b^2$, $\lambda$ is the smaller of the two positive solutions of $x^3 - \frac{3}{4}x + \frac{\sqrt{63}}{32} = 0$. These two roots are $0.4634$ and $0.5357$ approximately so $\lambda \approx 0.4634$. More precisely we have $0.4633 < \lambda < 0.4634$. Then*

$$\begin{aligned}\sqrt{1-\lambda^2} + \frac{(b^2+c^2-a^2)\lambda}{\sqrt{V}} &> \frac{(b^2+c^2-a^2)\lambda}{\sqrt{V}} \\ &> \frac{31 \cdot 0.4633}{8} > \frac{209}{128} = \frac{|R|}{2abc^2}\end{aligned}$$

*so, by Theorem 7, $s$ is the smaller of the two positive solutions of $-209x^3 + 768x^2 - 1008 = 0$, that is $s \approx 1.48$.*

# 5 Relationship between $r$ and $r$

When $W \neq 0$ we express $r$ as a rational function of $s$.

**Theorem 8.** *If $W \neq 0$ then the lengths $r$ and $s$ of the angle trisectors $CP$ and $CQ$ respectively are related by*

$$r = \frac{abs}{a+s} \frac{(a^2+b^2-c^2-as-s^2)}{(-2ab^2+b^2s+as^2)}. \tag{34}$$

*Proof.* Suppose that $-2ab^2 + b^2s + as^2$ and $a^2 + b^2 - c^2 - as - s^2$ are both zero. Then we have

$$\begin{aligned}(a^2 - b^2)s - a(a^2 - b^2 - c^2) \\ &= -(-2ab^2 + b^2s + as^2) - a(a^2 + b^2 - c^2 - as - s^2) \\ &= 0.\end{aligned}$$

If $a = b$ then $ac^2 = 0$, contradicting $a > 0$, $c > 0$. Thus $a \neq b$, $a^2 \neq b^2$, and

$$s = \frac{a(a^2 - b^2 - c^2)}{a^2 - b^2}. \tag{35}$$

Substituting the value of $s$ from (35) into $as^2 + b^2s - 2ab^2 = 0$ gives

$$\frac{a^3(a^2 - b^2 - c^2)^2}{(a^2 - b^2)^2} + \frac{ab^2(a^2 - b^2 - c^2)}{a^2 - b^2} - 2ab^2 = 0.$$

Using MAPLE to simplify this, we obtain

$$a^2c^4 - (2a^4 - a^2b^2 - b^4)c^2 + (a^2 - b^2)^3 = 0,$$

contradicting that $W \neq 0$. Hence, $-2ab^2 + b^2s + as^2$ and $a^2 + b^2 - c^2 - as - s^2$ cannot both be zero.

From (10) and (11), we obtain

$$r = \frac{2as}{a+s}\cos\theta = \frac{2as}{a+s}\frac{(as+E)}{4ab} = \frac{s}{2(a+s)b}(as+E).$$

Appealing to (12), we deduce

$$\begin{aligned}&(-2ab^2 + b^2s + as^2)r \\ &= \frac{s}{2(a+s)b} \\ &\big(as(-2ab^2 + b^2s + as^2) + 2ab^2(a^2 + b^2 - c^2) - 3ab^2s^2 - a^2s^3\big) \\ &= \frac{abs}{a+s}(a^2 + b^2 - c^2 - as - s^2).\end{aligned}$$

If $-2ab^2 + b^2s + as^2 = 0$ we deduce as $abs/(a + s) > 0$ that $a^2+b^2-c^2-as-s^2 = 0$, a contradiction. Thus $-2ab^2+b^2s+as^2 \neq 0$ and formula (34) follows. □

**Example 5.** *We choose $a = 2$, $b = 1$, and $c = 2$, so that $W = -17 \neq 0$. By Theorem 7 we have*

$$r = \frac{2s}{2+s}\frac{(1-2s-s^2)}{(-4+s+2s^2)}. \tag{36}$$

*By Theorem 5, as $R = 7$, $S = 24$, and $T = -30$, $s$ is the unique positive root of $7s^3 + 24s^2 - 30 = 0$, that is using MAPLE, $s \approx 0.9853679355$ and from (36) we obtain $r \approx 1.194863330$. Using $s^3 = \frac{30}{7} - \frac{24}{7}s^2$ in (36) we obtain with the help of MAPLE*

$$r = \frac{-60 + 14s + 20s^2}{4 - 14s - 13s^2} = \frac{1}{85}(720 - 400s - 231s^2).$$

# References

[1] Coxeter, H. S. M. and Greitzer, S. L. *Geometry Revisited*. Washington D.C.: Mathematical Association of America, 1967.

[2] Dickson, L. E. *New First Course in the Theory of Equations*. New York: Wiley, 1939.

[3] Trisna, D., Mashadi, and Gemawati, S. “Angle trisector in the triangle”. *IOSR Journal of Mathematics* 16.5 (2020), pp. 11–18.

[4] Webster, R. J. “Morley’s triangle theorem”. *Math. Magazine* 43.4 (1970), pp. 209–210.

[5] Willson, W. W. “A generalization of a property of the 4, 5, 6 triangle”. *Math. Gazette* 60.112 (1976), pp. 130–131.

*Lerna Pehlivan*
*Department of Applied Mathematics*
*University of Twente*
*Enschede, The Netherlands*
l.pehlivan@utwente.nl

*Kenneth S. Williams*
*Centre for Research in Algebra and Number Theory*
*School of Mathematics and Statistics*
*Carleton University*
*Ottawa, Ontario, Canada*
KennethWilliams@cunet.carleton.ca

# *On the Rocco–Proposition*

**Simone Camosso**

**Abstract**

The well-known simplicial decomposition in geometry may have roots in a result attributed to C. Rocco from 1811. This paper explores the origins and generalizations of this decomposition.

## 1 Introduction

This article aims to generalize a proposition regarding convex polyhedra found in the work of Carlo Rocco [5]. Carlo Rocco was an Italian mathematician and member of the Pontanian Academy. He authored several treatises on algebra and analytical geometry that were highly regarded within the academic circles of the Kingdom of the Two Sicilies.

**Proposition 1 (1811, C. Rocco).** *A convex polyhedron can be decomposed into triangular pyramids.*

In his book, Rocco presents this intuitive proposition with only a few lines of argumentation. The purpose of this paper is to prove this proposition for the general case of an $n$-dimensional convex polytope.

## 2 Preliminaries

**Definition 1.** *An $n$–dimensional cone $C$ is a geometric object consisting of an $n-1$ dimensional base where each point of this base is connected to a point $V$ called the vertex of the cone.*

There is a perpendicular axis from the vertex $V$ to the $n-1$ dimensional space that contains the base of the Cone $C$, we call it the height $h$ of the cone. Denoting by $A_n$ the $n-1$ content of the base, we may express the formula for the volume $V_n$ of the $n$–dimensional cone (see [4]) by:

$$V_n = \frac{1}{n} A_n h$$

**Example 1.** *A hyperpyramide in $\mathbb{R}^4$ of height $h$ with a cube base of side $l$ is a cone. The volume is given by (2) in the special case where the base is a cube in $\mathbb{R}^3$:*

$$V_4 = \frac{1}{4} h l^3.$$

**Example 2.** *The 3–simplex in $\mathbb{R}^3$ of height $h$ and side $l$ is a cone. This is a special case of simplex where the base is an equilateral triangle in $\mathbb{R}^2$, the volume $V_3$ in function of $h$ is given by:*

$$V_3 = \frac{1}{3} A_3 h = \frac{\sqrt{3}}{8} h^3.$$

# 3 A relationship between the $n$–volume and the $n$–surface

In this section we will discuss on a special relation between the boundary and the content of geometric objects. The relation is the following:

$$V_n(r)' = S_n(r), \tag{1}$$

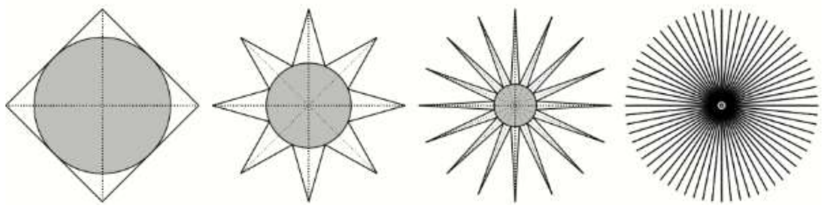
where $V_n$ is the content of any convex regular $n$–dimensional polytope with respect its inner radius (the minimal distance from the center to the boundary) and the content of the boundary of the polytope $S_n(r)$ as a function of $r$. The result is illustrated in detail in [2].

**Example 3.** *The relation* (1) *is true for the* $n$*–sphere where:*

$$V_n(r) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} r^n \quad \text{and} \quad S_n(r) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1},$$

*where* $\Gamma$ *is the gamma function and* $r$ *the radius of the sphere. It is a simple calculation that* $V_n(r)' = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} r^{n-1} = S_n(r)$.

The same result is showed in dimension $2$ and $3$ in [1]. The authors use a divulgative approach and instead the inner radius they consider the radius of an inscribed sphere inside the polyhedra. The approach with the radius of an inscribed sphere is difficult to imagine in high dimension. In the following figure we imagine to consider an hypersphere inscribed in a hypercube in $n$ dimension (this is a conceptual view respectively from $2$, $3$, $4$ and higher dimensions from left to right).



This image has been taken from [6] and shows how when the dimensionality increases most of the volume of the hypercube is in the corners, whereas the center is essentially empty.

Returning to the proof of result (1), the authors [2] use relations for regual $n$–dimensional simplices that they prove by induction. These relations are important because they give a link between the height $h$, the edge $l$ respectively with the inner radius $r$:

$$h = (n+1)r \quad \text{and} \quad l = \sqrt{2n(n+1)}r.$$

The proof is based on induction and the similarity of triangles. The most interesting fact is that when we express the volume of the $n$–simplex:

$$V_n(r) \;=\; \frac{1}{n}A_n h$$

we have that:

$$\begin{aligned} V_n(h) &= \frac{1}{n}h_n\left(\frac{1}{n-1}h_{n-1}\left(\cdots\left(\frac{\sqrt{3}}{3}h_2^2\right)\right)\right) \\ &= \frac{1}{n!}\left(\sqrt{\frac{n^n}{(n+1)^{n-1}}}\right)h^n \end{aligned}$$

as a function of $h$ and

$$\begin{aligned} V_n(h)' &= \frac{1}{(n-1)!}\left(\frac{n^n}{(n+1)^{n-1}}\right)h^{n-1}\,\frac{dh}{dr_n} \\ &= (n+1)\frac{1}{(n-1)!}\left(\frac{n^n}{(n+1)^{n-1}}\right)h^{n-1}, \end{aligned}$$

now it is simple to proof that this is the content of $n+1$ simplices of edge $l$ and dimension $n-1$, that is the content of the surface of the $n$–simplex. Thus for regular $n$ simplex:

$$V_n(h)' \;=\; (n+1)V_{n-1}(h).$$

# 4 Generalization of the Rocco theorem

Before proving a generalization of the Rocco theorem, we prove this intuitive proposition.

**Proposition 2.** *A $n$–dimensional convex polytope $P$ has at least $n+1$ faces.*

**Proof.** We prove the statement by induction.

1) If $P$ is a $n$–dimensional convex polytope in the Euclidean space of dimension $n$, then if $n=2$ it is clear.

2) If we assume that it is true for $n = k$ and we must prove that it is true for $n = k + 1$. Let us fix a vertex $v$, then exists $H$ an hyperplane through $v$ such that $P \cap H = \{v\}$. Let us consider a parallel hyperplane $H'$ such that $H'$ divides $P$ in two $n$–dimensional convex polytopes $P_1, P_2$. If $v \in P_1$ then:

$$\#\text{faces of } P_1 \leq \#\text{faces of } P$$

but $P_1 \cap H'$ is a $k$ dimensional polytope then $P_1$ has at least $k + 2$ faces.

**Theorem 1.** *Let $P$ be a convex regular polytope of dimension $n$, then $P$ can be decomposed in simplices.*

**Proof.** Let us consider $n$ consecutive $(n-1)$–faces $F_1, F_2, \ldots, F_n$ of $P$. If from a vertex $v$ of one face we conduct lines joint the other vertices we find a series of cones inside $P$. The union of these cones is $P$. Now each cone is an $n$ dimensional pyramid so each of them can be decomposed in simplices.

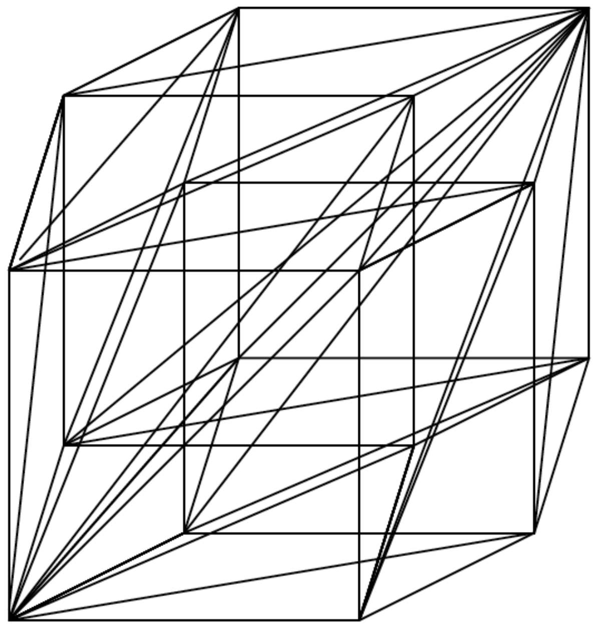
The following figure represents a 4–dimensional hypercube divided up into four simplices.

# References

[1] Delbosco, C. and Delbosco, D. "Una relazione tra area e perimetro, tra volume e superficie". *Archimede: rivista per insegnanti e i cultori di matematiche pure e applicate* (2007), pp. 26–29. DOI: 10.1400/165080.

[2] Emert, J. and Nelson, R. "Volume and Surface Area for Polyhedra and Polytopes". *Mathematics Magazine* 70.5 (1997), pp. 365–371.

[3] Hamber, H. W. "Quantum Gravity on the Lattice". *General Relativity and Gravitation* 41.4 (2009), pp. 817–876. DOI: 10.1007/s10714-009-0769-y.



A 4-dimensional hypercube divided into simplices, as represented in [3].

[4] Kendall, M. G. *A Course in the Geometry of $n$ Dimensions*. Reprint of the 1961 edition. Mineola, New York: Dover Publications, Inc., 2004.

[5] Rocco, C. *Catechismo di Matematiche Pure ad uso degli studi generali: Geometria Solida*. Second. Napoli: Dall'istituto di Guttemberg, 1811.

[6] Zaki, M. J. and Meira Jr., W. *Data Mining and Analysis: Fundamental Concepts and Algorithms*. Cambridge University Press, 2014. ISBN: 9780521766333.

*Simone Camosso*
*Mathematics and physics department*
*Saluzzo (CN), "I.I.S. G.Soleri-A.Bertoni", Italy.*
r.camosso@alice.it

# *New Series from Old*

**Joe Santmyer**

**Abstract**

What follows is an expository meant to illustrate a method. A number of formulas are given without proof and the interested reader is welcome to provide justifications. However, the emphasis here is the method demonstrated via examples rather than giving proofs of formulas resulting from the method. The primary goal is to encourage others to apply the method to the myraid of examples not mentioned here and discover new formulas.

## 1 Introduction

Chapter 10 in [1] describes how a new series can be obtained from an exisitng series

$$f(x) = \sum_{n=1}^{\infty} a_n x^n = \sum_{n\in\mathbb{N}} a_n x^n$$

by replacing $\mathbb{N}$ with $\mathbb{N}_1$ where $\emptyset \subset \mathbb{N}_1 \subset \mathbb{N}$.

In what follows a new series is obtained from an existing series in the following way. Let

$$S = \sum_{n=1}^{\infty} s_n$$

be a series with positive terms $s_n$ which may or may not converge to a finite value. Multiply each term $s_n$ in the series by $a_{n,k} = (-1)^{\lfloor \frac{n-1}{k} \rfloor}$ where $k \geq 1$ to get a new series

$$S_k = \sum_{n=1}^{\infty} a_{n,k} s_n.$$

Series $S_k$ has $k$ positive terms, followed by $k$ negative terms, and the pattern repeats. That is, $S_k$ has a sign change with period $2k$.

# 2 Examples of the Method

Consider some examples that illustrate the method described in the introduction. After seeing these examples you may want to begin creating your own.

**Example 1**: Consider the classical zeta series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}.$$

$S_1$ is the well known series for the Dirichlet eta function

$$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \eta(2) = \frac{\zeta(2)}{2}.$$

$S_2$ is the lesser known but easy to obtain series

$$\begin{aligned} S_2 &= \sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \frac{n-1}{2} \rfloor}}{n^2} = \sum_{n=1}^{\infty} \frac{(-1)^{\frac{(n-1)(n-2)}{2}}}{n^2} \\ &= 1 + \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} + \cdots = G + \frac{\zeta(2)}{8}, \end{aligned}$$

where $G$ is Catalan's constant.

$S_3$ is the lesser known series

$$\begin{aligned} S_3 &= \sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \frac{n-1}{3} \rfloor}}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} + \cdots \\ &= \frac{\psi_1\left(\frac{1}{6}\right) + \psi_1\left(\frac{1}{3}\right)}{18} - \frac{5\zeta(2)}{6} \end{aligned}$$

where $\psi_1$ is the trigamma function.

$S_4$ is the lesser known series

$$\begin{aligned} S_4 &= \sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \frac{n-1}{4} \rfloor}}{n^2} \\ &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} - \frac{1}{6^2} - \frac{1}{7^2} - \frac{1}{8^2} + \cdots \\ &= \frac{1}{64} \sum_{j=1}^{4} \left[ \psi_1\left(\frac{j}{8}\right) - \psi_1\left(\frac{j+4}{8}\right) \right]. \end{aligned}$$

In general

$$\begin{aligned} S_k &= \sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \frac{n-1}{k} \rfloor}}{n^2} \\ &= \frac{1}{4k^2} \sum_{j=1}^{k} \left[ \psi_1\left(\frac{j}{2k}\right) - \psi_1\left(\frac{j+k}{2k}\right) \right]. \end{aligned}$$

**Example 2**: Consider the classical zeta series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3).$$

$S_1$ is the well known series for the Dirichlet eta function

$$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^3} = \eta(3) = \frac{3\zeta(3)}{4}.$$

$S_2$ is the lesser known series

$$S_2 = \sum_{n=1}^{\infty} \frac{(-1)^{\frac{(n-1)(n-2)}{2}}}{n^3} = \frac{\pi^3 + 3\zeta(3)}{32}.$$

$S_3$ is the lesser known series

$$\begin{aligned} S_3 &= \sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \frac{n-1}{3} \rfloor}}{n^3} = -\frac{1}{432} \sum_{j=1}^{6} (-1)^{\lfloor \frac{n-1}{3} \rfloor} \psi^{(2)}\left(\frac{j}{6}\right) \\ &= -\frac{1}{432} \sum_{j=1}^{3} \left[ \psi^{(2)}\left(\frac{j}{6}\right) - \psi^{(2)}\left(\frac{j+3}{6}\right) \right] \end{aligned}$$

where $\psi^{(2)}$ is the polygamma function of order 2.

In general

$$\begin{aligned} S_k &= \sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \frac{n-1}{k} \rfloor}}{n^3} = -\frac{1}{16k^3} \sum_{j=1}^{2k} (-1)^{\lfloor \frac{j-1}{k} \rfloor} \psi^{(2)}\left(\frac{j}{2k}\right) \\ &= -\frac{1}{16k^3} \sum_{j=1}^{k} \left[ \psi^{(2)}\left(\frac{j}{2k}\right) - \psi^{(2)}\left(\frac{j+k}{2k}\right) \right]. \end{aligned}$$

**Example 3**: Consider the classical zeta series

$$S = \sum_{n=1}^{\infty} \frac{1}{n^m} = \zeta(m)$$

where $m \geq 2$ is an integer. Going straight to the generalization we have

$$\begin{aligned} S_k &= \sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \frac{n-1}{k} \rfloor}}{n^m} = \frac{(-1)^m}{(m-1)!(2k)^m} \sum_{j=1}^{2k} (-1)^{\lfloor \frac{j-1}{k} \rfloor} \psi^{(m-1)}\left(\frac{j}{2k}\right) \\ &= \frac{(-1)^m}{(m-1)!(2k)^m} \sum_{j=1}^{k} \left[ \psi^{(m-1)}\left(\frac{j}{2k}\right) - \psi^{(m-1)}\left(\frac{j+k}{2k}\right) \right] \end{aligned}$$

where $\psi^{(m-1)}$ is the polygamma function of order $m-1$.

**Example 4**: Consider the classical harmonic series which diverges

$$S = \sum_{n=1}^{\infty} \frac{1}{n}.$$

$S_1$ is the classical result

$$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2).$$

$S_2$ is the lesser known result

$$S_2 = \sum_{n=1}^{\infty} \frac{(-1)^{\frac{(n-1)(n-2)}{2}}}{n} = \frac{\pi}{4} + \frac{\ln(2)}{2}.$$

As it turns out, a formula for $S_k$ was derived in [2]. In this article the authors were not explicitly applying the method described here but focused on generalizing the alternating harmonic series $S_1$ by extending it from a series with a sign change of period 2 to a series with sign change period $2k$. The formula they obtained was

$$S_k = \sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \frac{n-1}{k} \rfloor}}{n} = \frac{\pi}{2k} \sum_{j=1}^{k-1} \csc\left(\frac{j\pi}{k}\right) + \frac{\ln(2)}{k}.$$

Note that by exercise 3 on p 80 in [1] and (5.24) on p 44 in [1] we also have

$$S_k = -\frac{\pi}{k^2} \sum_{j=0}^{k-1} j \cot\left(\frac{(2j+1)\pi}{2k}\right) + \frac{\ln(2)}{k}.$$

As an aside we get the following cosecant sum formula

$$\sum_{j=1}^{k-1} \csc\left(\frac{j\pi}{k}\right) = \frac{2}{\pi}\left[k \sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \frac{n-1}{k} \rfloor}}{n} - \ln(2)\right].$$

According to the discussion on p 74 in [1] we have the surprising fact that there is no known closed form formula for the sum on the left.

**Example 5**: Consider the divergent series

$$S = \sum_{n=1}^{\infty} \frac{H_n}{n}$$

where $H_n = \sum_{j=1}^{n} \frac{1}{j}$ is the $n^{\text{th}}$ harmonic number.

$S_1$ has the well known formula

$$S_1 = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} H_n}{n} = \frac{\pi^2}{12} - \frac{\ln^2(2)}{2}.$$

$S_2$ has the lesser known formula

$$S_2 = \sum_{n=1}^{\infty} \frac{(-1)^{\frac{(n-1)(n-2)}{2}} H_n}{n} = G + \frac{5\pi^2}{96} - \frac{\pi \ln(2)}{8} - \frac{\ln^2(2)}{8}.$$

In general

$$S_k = \sum_{n=1}^{\infty} \frac{(-1)^{\lfloor \frac{n-1}{k} \rfloor} H_n}{n} = \frac{2}{k} \operatorname{Re}\left[\sum_{j=0}^{k-1} \frac{t^{-(2j+1)}}{1 - t^{-(2j+1)}} F(t^{2j+1})\right]$$

where $\operatorname{Re}(z)$ is the real part of $z$, $t^2 = \omega = e^{\frac{2\pi i}{k}}$ is the $k^{\text{th}}$ root of unity, $F(z) = \operatorname{Li}_2(z) + \frac{\log^2(1-z)}{2}$ and $\operatorname{Li}_2$ is the dilogarithm function.

**Example 6**: Consider the geometric series

$$S = \sum_{n=1}^{\infty} a^n$$

where $0 < a < 1$. Then it is easy to see that

$$S_k = \sum_{n=1}^{\infty} (-1)^{\lfloor \frac{n-1}{k} \rfloor} a^n = \frac{a(1 - a^k)}{(1 - a)(1 + a^k)}.$$

Alternatively, since the geometric series normally starts at $n = 0$ multiply each term by $a_{n,k} = (-1)^{\lfloor \frac{n}{k} \rfloor}$ to get

$$S_k = \sum_{n=0}^{\infty} (-1)^{\lfloor \frac{n}{k} \rfloor} a^n = \frac{1 - a^k}{(1 - a)(1 + a^k)}$$

for $|a| < 1$.

**Example 7**: Consider the series

$$S = \sum_{n=0}^{\infty} \frac{1}{n!} = e.$$

Since the sum starts at $n = 0$ multiply each term by $a_{n,k} = (-1)^{\lfloor \frac{n}{k} \rfloor}$. Then

$$S_k = \sum_{n=0}^{\infty} \frac{(-1)^{\lfloor \frac{n}{k} \rfloor}}{n!} = \frac{1}{k} \sum_{j=0}^{k-1} \frac{e^{\cos\left(\frac{(2j+1)\pi}{k}\right)} \sin\left(\frac{(2j+1)\pi}{2k} + \sin\left(\frac{(2j+1)\pi}{k}\right)\right)}{\sin\left(\frac{(2j+1)\pi}{2k}\right)}.$$

# 3 Conclusion

These examples clearly show that numerous new series together with their formulas can be constructed from known series using the method described above. What does $\sum\limits_{n=0}^{\infty} \frac{(-1)^{\lfloor \frac{n}{2} \rfloor}}{2n+1}$ equal? What about trying the ceiling function? What is $\sum\limits_{n=0}^{\infty} \frac{(-1)^{\lceil \frac{n}{2} \rceil}}{2n+1}$? Explore and experiment!

# References

[1] Chen, H. *Excursions in Classical Analysis: Pathways into Higher Mathematics*. MAA Notes. Mathematical Association of America, Inc., 2010. ISBN: 978-0-88385-350-4.

[2] Kilmer, S. and Zheng, S. “A generalized alternating harmonic series”. *AIMS Mathematics* 6.12 (2021), pp. 13480–13487. DOI: 10.3934/math.2021781.

*Joe Santmyer*
*Las Cruces, NM 88011,*
santmyerjoe@yahoo.com

# *Problems*

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.

2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to **José Luis Díaz-Barrero**, Barcelona Mathematical Circle (BMC), FME, Pau Gargallo, 14, Les Corts, 08028 Barcelona, Spain or by e-mail to

`jose.luis.diaz@upc.edu`

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and are most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

*Solutions to the problems stated in this issue should be posted before*

**October 30, 2026**

# *Elementary Problems*

***E–149.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Find all nonnegative integers $a$ such that

$$2^a\,\frac{121}{a!+1}+3^a\,\frac{122}{a!+2}+4^a\,\frac{123}{a!+3}$$

is an integer.

***E–150.*** *Proposed by Michel Bataille, Rouen, France.* Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers defined by $F_0=0, F_1=1$ and the recursion $F_{n+2}=F_{n+1}+F_n$. For $n\geq 1$, evaluate

$$\frac{1}{F_{n+1}F_{n+2}}+\frac{1}{F_{n+3}F_{n+4}}+\frac{1}{F_{n+5}F_{n+6}}+\sum_{k=1}^{n}\frac{8}{F_kF_{k+6}}.$$

***E–151.*** *Proposed by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain.* Let $ABC$ be an isosceles triangle with $AB=AC$. If the quadrilateral formed by joining the feet of the three altitudes and the midpoint of the segment from $A$ to the orthocenter is a parallelogram, what is the size of the angle $A$?

***E–152.*** *Proposed by José Luis Díaz-Barrero, and José Gibergnas Báguena, Barcelona, Spain.* In how many ways can we permute the digits of the positive integer 123123123 if the same digit must not appear three times in a row?

***E–153.*** *Proposed by Vladimir Tănase, Bucharest, Romania.* Let $p$ be a positive integer. A function $f:\mathbb{N}\to\mathbb{N}$ is called a $p$*-function* if it satisfies the following two conditions for all $a,b,c\in\mathbb{N}$:

1. $a\equiv b \pmod p \Rightarrow f(a)\equiv f(b) \pmod p$,
2. $pf(a)^2-f(b)^2-f(c)^2=2paf(a)-2\big(bf(b)+cf(c)\big)+b^2+c^2-pa^2$.

Determine all strictly increasing $p$-functions.

***E–154.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $k \geq 1$ be an integer. Show that there exists an integer $\alpha$ such that

$$\sqrt{\alpha + 1001^k} + \sqrt{\alpha} = \left(\sqrt{1002} + 1\right)^k.$$

***E–155.*** *Proposed by José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let $x, y, z$ be nonzero real numbers. Find the minimum value of

$$(x^2 + y^2 + z^2)\left(\frac{a^3 + 1}{x^2 + 2y^2} + \frac{b^3 + 1}{y^2 + 2z^2} + \frac{c^3 + 1}{z^2 + 2x^2}\right)$$

given $a, b, c > 0$ and $a + b + c = 1$.

***E–156.*** *Proposed by Mihály Bencze, Brasov, Romania and José Luis Díaz-Barrero, Barcelona, Spain.* Find the equation whose roots are the numbers $\tan\left(\frac{r\pi}{12}\right)$, where $r$ is a positive integer less than 12 and coprime to 12.

# *Easy–Medium Problems*

***EM–149.*** *Proposed by Michel Bataille, Rouen, France.* Let $n$ be a positive integer and let $\alpha$ denote the golden ratio $\frac{1+\sqrt{5}}{2}$. Find all positive real numbers $x, y, z$ such that $x + y + z = 1$ and

$$x^{n+1} + 5^{n/2}\left(\alpha^n y^{n+1} + \frac{1}{\alpha^n} z^{n+1}\right) = \frac{1}{2^n}.$$

***EM–150.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Find all integer solutions of the equation $x^3 + y^3 + z^3 + 3xyz = 9$.

***EM–151.*** *Proposed by Daniel Sitaru, National Economic College "Theodor Costescu", Drobeta Turnu - Severin, Romania.* Let $x, y, z$ be real numbers greater than one. Prove that

$$\frac{\ln^2 x}{\ln(xy)} + \frac{\ln^2 y}{\ln(yz)} \geq \ln \sqrt[4]{\frac{x^3 y^2}{z}}.$$

***EM–152.*** *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let $ABC$ be an acute triangle with the circumcenter $O$. The altitudes from $B$ and $C$ cut the circumcircle in $E$ and $F$ respectively. On the small arc $BC$, take the point $X$. If $XE \cap AC = \{M\}$, $XF \cap AC = \{N\}$ and if $MN$ cuts the altitude $AD$ in $Z$, show that $\overrightarrow{AZ} = \overrightarrow{OB} + \overrightarrow{OC}$.

***EM–153.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Twenty nine points divide a circle into $29$ arcs of equal length. How many sets of $9$ points, where no two points have $3$ unit or $9$ unit arc distance, are there?

***EM–154.*** *Proposed by Todor Zaharinov Sofia, Bulgaria.* Let $ABC$ be a non-right triangle with $AB \neq AC$ and let $H$ be its orthocenter, $I$ be its incenter and $M$ be the midpoint of $BC$. Let $D$ be the reflection of $I$ in the line $BC$ and let $E$ be the reflection of $A$ in $H$. Knowing that $AI$ is parallel to $DE$, prove that the points $H, I, M$ are collinear.

***EM–155.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Determine all positive integers $n$ for which $\gcd(n, 3^n - 2) = 1$.

***EM–156.*** *Proposed by José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Find the number of five-digit positive integers such that the sum of the digits is $23$.

# *Medium–Hard Problems*

***MH–149.*** *Proposed by Michel Bataille, Rouen, France.* Let $ABC$ be a triangle with $\angle BCA \neq 90^\circ$ and let $D$ and $E$ be the feet of its altitudes from $A$ and $B$. Let $\Gamma$ and $\gamma$ be the respective circumcircles of $\Delta BEC$ and $\Delta AEM$ where $M$ is the midpoint of $BC$. If $\gamma$ intersects again the line $AD$ at $U$ and $\Gamma$ at $V$, prove that $AV = AC$ and that $UE = UC = AM|\cot C|$.

***MH–150.*** *Proposed by Óscar Rivero Salgado, Santigo de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let $k$ be a positive integer. Compute

$$\left(\left\lfloor\frac{(k+2)^2}{3}\right\rfloor+\left\lfloor\frac{(k+2)^2}{4}\right\rfloor\right)-\left(\left\lfloor\frac{(k+1)^2}{3}\right\rfloor+\left\lfloor\frac{(k+1)^2}{4}\right\rfloor\right).$$

(Here, $\lfloor x \rfloor$ represents the integer part of the real number $x$)

***MH–151.*** *Proposed by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain.* From a point $P$ on the internal bisector of angle $A$ of triangle $ABC$, perpendiculars $PA'$, $PB'$ and $PC'$ are dropped to the sides $BC$, $CA$, $AB$, respectively. Prove that the intersection point of lines $PA'$ and $B'C'$ lies on the $A$-median.

***MH–152.*** *Proposed by Jordi Ferré Garcia, BarcelonaTech, CFIS, Barcelona, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let $n \geq 1$ be an integer number. Find the greatest common divisor of the numbers:

$$\binom{3n}{1},\binom{3n}{3},\binom{3n}{5},\ldots,\binom{3n}{2\left\lfloor\frac{3n-2}{2}\right\rfloor+1}.$$

***MH–153.*** *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let $k \leq n$ be positive integers. How many $k$-letter words in the alphabet $X = \{1, 2, \ldots, n\}$ are: (a) strictly increasing; (b) weakly increasing? A word $w = w_1 w_2 \cdots w_k$ in the alphabet $\{1, 2, \ldots, n\}$ is *strictly increasing* if and only if $w_1 < w_2 < \cdots < w_k$ and a word $w$ is *weakly increasing* if and only if $w_1 \leq w_2 \leq \cdots \leq w_k$.

***MH–154.*** *Proposed by Todor Zaharinov Sofia, Bulgaria.* Let $ABC$ be a non-right triangle with $AB \neq AC$ and let $M$ be the midpoint of $BC$. Let $D$ be symmetric to $A$ with respect to $M$. Let $l$ be the line perpendicular to $AM$ through point $D$. Let $E, F$ be the foot of the perpendiculars from $D$ to $AC, AB$ respectively. Prove that lines $BC, EF, l$ are concurrent.

***MH–155.*** *Proposed by Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and José Luis Díaz-Barrero, Barcelona, Spain.* Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y, z, t \in \mathbb{R}$, whenever the denominator is non-zero, we have:

$$f\left(\frac{x^2 + y^2}{z^2 + t^2}\right) = \frac{f^2(x) + f^2(y)}{f^2(z) + f^2(t)}.$$

***MH–156.*** *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let $n$ be a positive integer. Determine the maximum number of bishops that we can place in a $2n \times 2n$ chessboard such that there are not two bishops in the same cell, and each bishop is threatened by at most one bishop. (A bishop threatens another one, if both are placed in different cells, in the same diagonal. A board has as diagonals the 2 main diagonals and the ones parallel to those ones).

# *Advanced Problems*

***A–149.*** *Proposed by Nicusor Zlota, "Traian Vuia" Technical College Focsani, Romania.* Evaluate

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{1\le i<j<k\le n} \ln\left(3+\frac{i}{n}\right) \ln\left(3+\frac{j}{n}\right) \ln\left(3+\frac{k}{n}\right).$$

***A–150.*** *Proposed by Michel Bataille, Rouen, France and Robert Frontczak, Reutlingen, Germany.* Let $n$ be a nonegative integer. Evaluate

$$\sum_{k=0}^{n} (-1)^k \binom{n+k}{2k} \frac{4^k}{k+1}.$$

***A–151.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $\mathcal{M}_n(\mathbb{R})$ denote the set of all real $n \times n$ matrices. Find all surjective functions $f : \mathcal{M}_n(\mathbb{R}) \to \{0, 1, 2, \dots, n\}$ such that for all $X, Y \in \mathcal{M}_n(\mathbb{R})$ is $f(XY) \le \min\{f(X), f(Y)\}$.

***A–152.*** *Proposed by Joseph Santmyer, Las Cruces, NM.* Show that

$$\begin{aligned} I_1 &= \int_0^1 \frac{\ln\left(\sqrt{2}-\sqrt{1-u^2}\right)}{u^2+1} du = -\frac{G}{2}, \\ I_2 &= \int_0^1 \frac{\ln\left(\sqrt{2}+\sqrt{1-u^2}\right)}{u^2+1} du = \frac{\pi \ln(2) - G}{2}, \\ I_3 &= \int_0^1 \frac{\tanh^{-1}\left(\sqrt{\frac{1-u^2}{2}}\right)}{u^2+1} du = \frac{\pi \ln(2)}{4}, \end{aligned}$$

where $G$ is Catalan's constant.

***A–153.*** *Proposed by Paolo Perfetti, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma, Roma, Italy.* Evaluate

$$\sum_{n=1}^{\infty} (2n-1)\left(\frac{1}{n^3} - \frac{1}{(n+1)^3} + \dots\right)^2$$

***A–154.*** *Proposed by Mihály Bencze, Braşov, Romania and José Luis Díaz-Barrero, Barcelona, Spain.* (a) Let $z_1, z_2, \ldots, z_n$ $(n \geq 2)$ be distinct nonzero complex numbers. Prove that:

$$\frac{(-1)^{n+1}}{\prod_{i=1}^{n} z_i} = \sum_{i=1}^{n} \frac{1 + z_i}{z_i \prod_{k \neq i}^{n} (z_i - z_k)}.$$

(b) Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ $(n \geq 2)$ be distinct nonzero complex numbers. Show that the expression

$$\sum_{j=1}^{n} \left[ \alpha_j^{n-1} \left( 1 + \prod_{k \neq j} \alpha_k \right) \prod_{k \neq j} \frac{1}{\alpha_j - \alpha_k} \right]$$

is a real number, and determine its exact value.

***A–155.*** *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let $ABC$ be a triangle with $AB \neq AC$ and let $H$ be its orthocenter and $G$ be its centroid. Let $H_a, H_b, H_c$ be the feet of the perpendiculars drawn from $H$ to $BC, CA, AB$ respectively. Let $D = H_bH_c \cap BC$, $E$ be the midpoint of $H_bH_c$. Knowing that $BG \perp CG$, prove that the points $D, E, G, H_a$ are concyclic.

***A–156.*** *Proposed by Vasile Mircea Popa, Affiliate Professor, "Lucian Blaga" University of Sibiu, Romania.* Calculate the integral:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\arccos(x)}{\sqrt{3 - \cos(4x)}} dx.$$

# *Mathlessons*

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to **José Luis Díaz-Barrero**, Barcelona Mathematical Circle (BMC), FME, Pau Gargallo, 14, Les Corts, 08028 Barcelona, Spain or by e-mail to
`jose.luis.diaz@upc.edu`

# *On some saturated cyclic inequalities*

**Vasile Cîrtoaje**

Abstract

In this paper, we give a correct solution to a saturated inequality proven wrong in a mathematical journal. In addition, we present other similar saturated inequalities recently published, as well as two open saturated inequalities.

## 1 Introduction

Let $a_1, a_2, \ldots, a_n$ be real variables belonging to an interval $\mathbb{I}$, and let $k$ be a positive parameter. We say that the inequality

$$E(k, a_1, a_2, \ldots, a_n) \geq 0$$

is saturated for $k = k_0$ if $k_0$ is the least or the largest value of $k$ such that the inequality holds for any $a_1, a_2, \ldots, a_n \in \mathbb{I}$ under some given constraints. In [3], the following saturated inequality was published as Problem 5767.

**Problem 1.** *If $a, b, c, d$ are nonnegative real numbers such that $a \geq b \geq c \geq d$ and $ab + bc + cd + da = 4$, then*

$$\frac{1}{ab+7} + \frac{1}{ac+7} + \frac{1}{ad+7} + \frac{1}{bc+7} + \frac{1}{bd+7} + \frac{1}{cd+7} \geq \frac{3}{4}.$$

In [8], four different solutions was published, but all solutions are incorrect. The first solution uses the inequality

$$ac + bd \leq 2,$$

which does not hold for $a = 3$, $b = c = 1$ and $d = 0$. The second solution uses the inequality

$$t^{(ab+cd)/2} \geq t^{\sqrt{abcd}},$$

which does not hold for $t \in (0, 1)$. The third solution uses the inequality

$$\frac{1}{7 + ac} + \frac{1}{7 + bd} \geq \frac{1}{4},$$

which is wrong for $a = 3$, $b = c = 1$ and $d = 0$. The fourth solution is based on the Arithmetic Mean Theorem [4] applied to the function

$$F(x, y, w, z) = \frac{1}{x + 7} + \frac{1}{y + 7} + \frac{1}{w + 7} + \frac{1}{z + 7} + \frac{1}{x + z + 14}$$

(with $x = ab$, $y = bc$, $w = da$ and $z = cd$). Since this function is not symmetric, the solution is wrong. Moreover, using this incorrect method, one can show that the inequality

$$\frac{1}{ab + k} + \frac{1}{ac + k} + \frac{1}{ad + k} + \frac{1}{bc + k} + \frac{1}{bd + k} + \frac{1}{cd + k} \geq \frac{6}{1 + k}$$

holds for any positive value of the constant $k$, which is false. Actually, this inequality does not hold for $k > 7$.

To prove that the inequality is saturated for $k = 7$, consider $b = c = 1$. The equality constraint becomes

$$ad = 3 - 2S,$$

where

$$S = \frac{a + d}{2},$$

while the inequality can be written as follows:

$$\frac{2}{a+k}+\frac{2}{d+k}+\frac{1}{ad+k}\geq\frac{5}{1+k},$$

$$\frac{2(a+d+2k)}{ad+k(a+d)+k^2}+\frac{1}{ad+k}\geq\frac{5}{1+k},$$

$$\frac{4S+4k}{(2k-2)S+k^2+3}+\frac{1}{k+3-2S}\geq\frac{5}{1+k},$$

$$2(3k-7)S^2-(k^2+6k-35)S+k^2-21\geq 0,$$

$$(S-1)\big[2(3k-7)S-k^2+21\big]\geq 0.$$

From

$$3-2S=ad\leq S^2$$

and

$$3-2S=ad\geq 0,$$

we get $S\in\left[1,\frac{3}{2}\right]$. Thus, the inequality is true if and only if

$$2(3k-7)S-k^2+21\geq 0$$

for $S\in\left(1,\frac{3}{2}\right]$. Letting $S\to 1$, we get the necessary condition

$$2(3k-7)-k^2+21\geq 0,$$

which implies $k\leq 7$. Note that for $k=4$, we get a weaker inequality which was published in Gazeta Matematica B, no. 1, 2024.

**Proof of Problem 1.** Denote $p=abcd$, and write the required inequality as follows:

$$\frac{ab+cd+14}{p+7(ab+cd)+49}+\frac{ac+bd+14}{p+7(ac+bd)+49}+\frac{ad+bc+14}{p+7(ad+bc)+49}\geq\frac{3}{4},$$

$$1+\frac{49-p}{p+7(ab+cd)+49}+1+\frac{49-p}{p+7(ac+bd)+49}$$
$$+1+\frac{49-p}{p+7(ad+bc)+49}\geq\frac{21}{4},$$
$$\frac{1}{p+7(ab+cd)+49}+\frac{1}{p+7(ac+bd)+49}+\frac{1}{p+7(ad+bc)+49}$$
$$\geq\frac{9}{4(49-p)}.$$

By the AM-HM inequality, it suffices to show that

$$\frac{4}{2p+7(ab+cd+ac+bd)+98}+\frac{1}{p+7(ad+bc)+49}\geq\frac{9}{4(49-p)}.$$

Since

$$ab+cd+ac+bd=2(ab+cd)-(a-d)(b-c)\leq 2(ab+cd)=2(4-ad-bc),$$

it suffices to prove that

$$\frac{2}{p-7(ad+bc)+77}+\frac{1}{p+7(ad+bc)+49}\geq\frac{9}{4(49-p)}.$$

Using the substitution

$$ad=x,\qquad bc=y,$$

the inequality becomes

$$\frac{2}{xy-7(x+y)+77}+\frac{1}{xy+7(x+y)+49}\geq\frac{9}{4(49-xy)}.$$

Let $z=\dfrac{x+y}{2}$. Since $xy\leq z^2$, it suffices to show that

$$\frac{2}{z^2-14z+77}+\frac{1}{z^2+14z+49}\geq\frac{9}{4(49-z^2)},$$

which is equivalent to $(z-1)^2(7-3z)\geq 0$. Since

$$z<\frac{ab+bc+cd+da}{2}=2,$$

the last inequality is obvious and the proof is finished.

The inequality is an equality for $a=b=c=d=1$.

# 2 Other similar saturated inequalities

The following five similar inequalities of cyclic type were published in [5], [7], [2], [1] and [6], respectively.

**Problem 2.** *If $a, b, c, d$ are nonnegative real numbers such that at most one of them is larger than $1$ and $ab + bc + cd + da = 4$, then*

$$\frac{1}{ab+2} + \frac{1}{ac+2} + \frac{1}{ad+2} + \frac{1}{bc+2} + \frac{1}{bd+2} + \frac{1}{cd+2} \geq 2.$$

**Problem 3.** *If $a, b, c, d$ are nonnegative real numbers such that at most one of them is less than $1$ and $ab + bc + cd + da = 4$, then*

$$\frac{1}{ab+3} + \frac{1}{ac+3} + \frac{1}{ad+3} + \frac{1}{bc+3} + \frac{1}{bd+3} + \frac{1}{cd+3} \geq \frac{3}{2}.$$

**Problem 4.** *If $a, b, c, d$ are nonnegative real numbers such that $ab + ac + ad + bc + bd + cd = 6$, then*

$$\frac{1}{ab+3} + \frac{1}{bc+3} + \frac{1}{cd+3} + \frac{1}{da+3} \geq 1$$

**Problem 5.** *If $a, b, c, d$ are nonnegative real numbers such that $a \geq b \geq c \geq d$ and $ab + ac + ad + bc + bd + cd = 6$, then*

$$\frac{1}{ab+5} + \frac{1}{bc+5} + \frac{1}{cd+5} + \frac{1}{da+5} \geq \frac{2}{3}.$$

**Problem 6.** *If $a, b, c, d$ are nonnegative real numbers such that $a \geq b \geq c \geq 1 \geq d$ and $ab + ac + ad + bc + bd + cd = 6$, then*

$$\frac{1}{ab+9} + \frac{1}{bc+9} + \frac{1}{cd+9} + \frac{1}{da+9} \geq \frac{2}{5}.$$

• The inequality in **Problem 2** is saturated because 2 is the largest positive value of the constant $k$ such that the inequality

$$\frac{1}{ab+k}+\frac{1}{ac+k}+\frac{1}{ad+k}+\frac{1}{bc+k}+\frac{1}{bd+k}+\frac{1}{cd+k}\geq\frac{6}{1+k}$$

holds for any nonnegative real numbers $a, b, c, d$ with at most one of them larger than 1 and $ab+bc+cd+da=4$.

By choosing

$$a=(2-d)/d\geq 1=c\geq b=d>0,$$

the constraints are satisfied, while the inequality can be written as follows:

$$\frac{1}{a+k}+\frac{2}{ad+k}+\frac{2}{d+k}+\frac{1}{d^2+k}\geq\frac{6}{1+k},$$

$$\frac{d}{(k-1)d+2}+\frac{2}{k+2-d}+\frac{2}{d+k}+\frac{1}{d^2+k}\geq\frac{6}{1+k}.$$

Letting $d\to 0$, we get the necessary condition

$$\frac{2}{k+2}+\frac{3}{k}\geq\frac{6}{1+k},$$

that is $(2-k)(3+k)\geq 0$, hence $k\leq 2$.

• The inequality in **Problem 3** is saturated because 3 is the largest positive value of the constant $k$ such that the inequality

$$\frac{1}{ab+k}+\frac{1}{ac+k}+\frac{1}{ad+k}+\frac{1}{bc+k}+\frac{1}{bd+k}+\frac{1}{cd+k}\geq\frac{6}{1+k}$$

holds for any nonnegative real numbers $a, b, c, d$ with at most one of them less than 1 and $ab+bc+cd+da=4$.

By selecting

$$0\leq a=(2-d)/d\leq 1=c<b=d\leq 2,$$

the constraints are satisfied, while the inequality can be written as follows:

$$\frac{1}{a+k}+\frac{2}{ad+k}+\frac{2}{d+k}+\frac{1}{d^2+k}\geq\frac{6}{1+k},$$

$$\frac{d}{(k-1)d+2}+\frac{2}{k+2-d}+\frac{2}{d+k}+\frac{1}{d^2+k}\geq\frac{6}{1+k},$$

$$\left[\frac{d}{(k-1)d+2}-\frac{1}{1+k}\right]+2\left(\frac{1}{k+2-d}-\frac{1}{1+k}\right)+2\left(\frac{1}{d+k}-\frac{1}{1+k}\right)$$
$$+\left(\frac{1}{d^2+k}-\frac{1}{1+k}\right)\geq 0,$$

$$\frac{2(d-1)}{(k-1)d+2}+\frac{2(d-1)}{k+2-d}-\frac{2(d-1)}{d+k}-\frac{d^2-1}{d^2+k}\geq 0,$$

$$\frac{2}{(k-1)d+2}+\frac{2}{k+2-d}-\frac{2}{d+k}-\frac{d+1}{d^2+k}\geq 0,$$

$$2\left[\frac{1}{(k-1)d+2}-\frac{1}{1+k}\right]+2\left(\frac{1}{k+2-d}-\frac{1}{1+k}\right)-2\left(\frac{1}{d+k}-\frac{1}{1+k}\right)$$
$$-\left(\frac{d+1}{d^2+k}-\frac{2}{1+k}\right)\geq 0,$$

$$\frac{-2(k-1)(d-1)}{(k-1)d+2}+\frac{2(d-1)}{k+2-d}+\frac{2(d-1)}{d+k}+\frac{(d-1)(2d-k+1)}{d^2+k}\geq 0,$$

$$\frac{-2(k-1)}{(k-1)d+2}+\frac{2}{k+2-d}+\frac{2}{d+k}+\frac{2d-k+1}{d^2+k}\geq 0.$$

For $d\to 1$, we get the necessary condition $k\leq 3$.

• The inequality in **Problem 4** is saturated because 3 is the largest positive value of the constant $k$ such that the inequality

$$\frac{1}{ab+k}+\frac{1}{bc+k}+\frac{1}{cd+k}+\frac{1}{da+k}\geq\frac{4}{1+k}$$

holds for any nonnegative real numbers $a, b, c, d$ with $ab+ac+ad+bc+bd+cd=6$.

To show this, we select $b=d=\sqrt{ac}$. The constraint $ab+ac+ad+bc+bd+cd=6$ becomes

$$(a+c)\sqrt{ac}+ac=3,$$

while the inequality can be written as follows:

$$\frac{1}{a\sqrt{ac}+k}+\frac{1}{c\sqrt{ac}+k}\geq\frac{2}{1+k},$$

$$\frac{(a+c)\sqrt{ac}+2k}{a^2c^2+k(a+c)\sqrt{ac}+k^2}\geq\frac{2}{1+k}.$$

Denoting $x=ac$, we have

$$3=(a+c)\sqrt{ac}+ac\geq 3ac=3x,$$

hence $x\in(0,1]$. Since $(a+c)\sqrt{ac}=3-x$, the inequality becomes

$$\frac{3+2k-x}{x^2-kx+k^2+3k}\geq\frac{2}{1+k},$$

which is equivalent to

$$(x-1)(2x+3-k)\leq 0.$$

It is true if and only if $2x+3-k\geq 0$ for $x\in(0,1)$. For $x\to 0$, we get the necessary condition $k\leq 3$.

• The inequality in **Problem 5** is saturated because 5 is the largest positive value of the constant $k$ such that the inequality

$$\frac{1}{ab+k}+\frac{1}{bc+k}+\frac{1}{cd+k}+\frac{1}{da+k}\geq\frac{4}{1+k}$$

holds for any nonnegative real numbers $a, b, c, d$ with $a \geq b \geq c \geq d$ and $ab + ac + ad + bc + bd + cd = 6$.

By choosing $a = b$ and $c = d$, the constraint $ab + ac + ad + bc + bd + cd = 6$ becomes

$$a^2 + d^2 + 4ad = 6,$$

while the inequality can be written as follows:

$$\frac{1}{a^2 + k} + \frac{1}{d^2 + k} + \frac{2}{ad + k} \geq \frac{4}{1 + k},$$

$$\frac{a^2 + d^2 + 2k}{a^2d^2 + k(a^2 + d^2) + k^2} + \frac{2}{ad + k} \geq \frac{4}{1 + k}.$$

Denoting $x = ad$, we have

$$6 = a^2 + d^2 + 4ad \geq 6ad = 6x,$$

hence $x \in [0, 1]$. Since $a^2 + d^2 = 6 - 4x$, the inequality becomes

$$\frac{3 + k - 2x}{x^2 - 4kx + k^2 + 6k} + \frac{1}{x + k} \geq \frac{2}{1 + k},$$

which is equivalent to

$$(x - 1)P(x) \leq 0,$$

where

$$P(x) = 2x^2 - (5k - 3)x + 9k - k^2.$$

Since $x - 1 \leq 0$, the inequality is true if and only if $P(x) \geq 0$ for $x \in [0, 1)$. Letting $x \to 1$, we get the necessary condition $(5 - k)(1 + k) \geq 0$, that implies $k \leq 5$.

- The inequality in **Problem 6** is saturated because $9$ is the largest positive value of the constant $k$ such that the inequality

$$\frac{1}{ab + k} + \frac{1}{bc + k} + \frac{1}{cd + k} + \frac{1}{da + k} \geq \frac{4}{1 + k}.$$

holds for any sequence of nonnegative real numbers $a \ge b \ge c \ge 1 \ge d$ with $ab + ac + ad + bc + bd + cd = 6$.

For $a = b \ge 1 = c \ge d$ and $b^2 + 2bd + 2b + d = 6$, the constraints are satisfied, while the inequality becomes $E \ge 0$, where

$$E = \frac{1}{b^2 + k} + \frac{1}{bd + k} + \frac{1}{b + k} + \frac{1}{d + k} - \frac{4}{1 + k}.$$

Assume that $d$ is a function of $b$. From $6 = b^2 + 2bd + 2b + d \ge b^2 + 2b$, we get $b \in [1, \sqrt{7} - 1]$. Note that $b = 1$ implies $d = 1$. We have

$$(1 + 2b)d' + 2b + 2 + 2d = 0, \qquad d'(1) = -2,$$

$$(1 + 2b)d'' + 4d' + 2 = 0, \qquad d''(1) = 2,$$

$$E'(b) = -\left[\frac{1}{(d + k)^2} + \frac{b}{(bd + k)^2}\right]d' - \frac{d}{(bd + k)^2} - \frac{1}{(b + k)^2} - \frac{2b}{(b^2 + k)^2},$$

$$E''(b) = \left[\frac{2}{(d + k)^3} + \frac{2b^2}{(bd + k)^3}\right](d')^2 - \left[\frac{1}{(d + k)^2} + \frac{b}{(bd + k)^2}\right]d''$$

$$-\frac{(k - bd)d'}{(bd + k)^3} + \frac{bd - k}{(bd + k)^3}d' + \frac{2d^2}{(bd + k)^3} + \frac{2}{(b + k)^3} - \frac{2(k - 3b^2)}{(b^2 + k)^3}.$$

Since $E(1) = E'(1) = 0$, the condition $E''(1) \ge 0$ is necessary to have $E(b) \ge 0$ for $b \in [1, \sqrt{7} - 1]$. Since

$$E''(1) = \frac{16}{(1 + k)^3} + \frac{4}{(1 + k)^2} + \frac{4(k - 1)}{(1 + k)^3} + \frac{4}{(1 + k)^3} - \frac{(k - 3)}{(1 + k)^3} = \frac{2(9 - k)}{(1 + k)^3},$$

the condition $E''(1) \ge 0$ implies $k \le 9$.

We claim that the following saturated inequalities hold.

**Open Problem 7.** *If $a, b, c, d, e$ are nonnegative real numbers such that*

$$ab + ac + ad + ae + bc + bd + be + cd + ce + de = 10,$$

*then*

$$\frac{1}{ab + 3} + \frac{1}{bc + 3} + \frac{1}{cd + 3} + \frac{1}{de + 3} + \frac{1}{ea + 3} \ge \frac{5}{4}.$$

**Open Problem 8.** *If $a, b, c, d, e$ are nonnegative real numbers such that $a \geq b \geq c \geq d \geq e$ and*

$$ab + ac + ad + ae + bc + bd + be + cd + ce + de = 10,$$

*then*

$$\frac{1}{ab+4} + \frac{1}{bc+4} + \frac{1}{cd+4} + \frac{1}{de+4} + \frac{1}{ea+4} \geq 1.$$

- The inequality in **Problem 7**, written in the form

$$\frac{1}{ab+k} + \frac{1}{bc+k} + \frac{1}{cd+k} + \frac{1}{de+k} + \frac{1}{ea+k} \geq \frac{5}{1+k}.$$

does not hold for $k > 3$ and any nonnegative real numbers $a, b, c, d, e$ satisfying

$$ab + ac + ad + ae + bc + bd + be + cd + ce + de = 10.$$

To show this, assume that $b = e := x$ and $c = d = 0$. The constraint becomes $ax = 5$, and the the inequality is equivalent to

$$\frac{2}{ax+k} + \frac{3}{k} \geq \frac{5}{1+k},$$

$$\frac{2}{5+k} + \frac{3}{k} \geq \frac{5}{1+k}.$$

The last inequality implies $k \leq 3$.

- The inequality in **Problem 8**, written in the form

$$\frac{1}{ab+k} + \frac{1}{bc+k} + \frac{1}{cd+k} + \frac{1}{de+k} + \frac{1}{ea+k} \geq \frac{5}{1+k}.$$

does not hold for $k > 4$ and any sequence of nonnegative real numbers $a \geq b \geq c \geq d \geq e$ such that

$$ab + ac + ad + ae + bc + bd + be + cd + ce + de = 10.$$

Assume $a = b \in [1, \sqrt{3}]$, $d = e = 1/b$, and $c = \dfrac{2 - x^2}{x}$ with

$$x = \frac{b+d}{2} = \frac{b^2+1}{2b} \in [1, 2/\sqrt{3}].$$

The constraints are satisfied, and the inequality becomes as follows:

$$\frac{1}{b^2+k} + \frac{1}{d^2+k} + \frac{1}{bc+k} + \frac{1}{cd+k} \geq \frac{4}{1+k},$$

$$\frac{(b+d)^2+2k-2}{(b+d)^2+(k-1)^2} + \frac{c(b+d)+2k}{c^2+kc(b+d)+k^2} \geq \frac{4}{1+k},$$

$$\frac{2x^2+k-1}{4kx^2+(k-1)^2} + \frac{cx+k}{c^2+2kcx+k^2} \geq \frac{2}{1+k},$$

$$\frac{2x^2+k-1}{4kx^2+(k-1)^2} + \frac{x^2(2+k-x^2)}{(1-2k)x^4+(k^2+4k-4)x^2+4} \geq \frac{2}{1+k},$$

$$\frac{t(2+k-t)}{(1-2k)t^2+(k^2+4k-4)t+4} - \frac{1}{1+k} \geq \frac{1}{1+k} - \frac{2t+k-1}{4kt+(k-1)^2},$$

$$\frac{(t-1)[(k-2)t+4]}{(1-2k)t^2+(k^2+4k-4)t+4} \geq \frac{2(k-1)(t-1)}{4kt+(k-1)^2},$$

where $t = x^2 \in [1, 4/3]$. It is true for $t \in (1, 4/3]$ if and only if

$$\frac{(k-2)t+4}{(1-2k)t^2+(k^2+4k-4)t+4} \geq \frac{2(k-1)}{4kt+(k-1)^2}.$$

For $t \to 1$, we get the necessary condition

$$\frac{k+2}{(k+1)^2} \geq \frac{2(k-1)}{(k+1)^2},$$

that is $k \leq 4$.

# References

[1] Cîrtoaje, V. "Problem 12366". *American Mathematical Monthly* 1 (2023).

[2] Cîrtoaje, V. "Problem O627". *Mathematical Reflections* 3 (2023).

[3] Cîrtoaje, V. "Problem 5767". *School Science and Mathematics Journal – Problems and Solutions* 124.1 (2024).

[4] Cîrtoaje, V. *Mathematical Inequalities, Other Recent Methods for Creating and Solving Inequalities – Volume 5*. 2nd ed. See also: www.ace.upg-ploiesti.ro/membri/vcirtoaje. Lambert Academic Publishing, 2025.

[5] Cîrtoaje, V. "Problem 5074". *Crux Mathematicorum* 8 (2025).

[6] Cîrtoaje, V. "Problem MH-145". *Arhimede Mathematical Journal* 2 (2025).

[7] Cîrtoaje, V. "Problem O712". *Mathematical Reflections* 5 (2025).

[8] Zeng, H. B. et al. "Solutions to Problem 5767". *School Science and Mathematics Journal – Problems and Solutions* 124.5 (2024).

*Vasile Cîrtoaje*
*Petrolium-Gas University of Ploiesti,*
*Department Automation and Computers,*
*Bdul Bucuresti, No. 39, Ploiesti, Romania*
vcirtoaje@upg-ploiesti.ro

# *A method for solving some nonlinear equations*

**Neculai Stanciu, Ovidiu T. Pop
and José Luis Díaz-Barrero**

**Abstract.** In the following, we present a method for solving algebraic, trigonometric, exponential, logarithmic, or mixed nonlinear equations by exploiting the injectivity of functions.

## 1 The method

Our goal in this lesson is to solve the equation $E(x) = 0$ for $x \in I \subseteq \mathbb{R}$. The key idea behind the method we present is to rewrite the equation in the equivalent form

$$f(g(x)) = f(h(x)), \quad x \in I,$$

where $f : J \to \mathbb{R}$ is an injective function on $J$, and $g, h : I \to J$. Because $f$ is injective, this equation immediately implies

$$g(x) = h(x), \quad x \in I,$$

and this resulting equation is usually much easier to solve. As we will see later, the main challenge in applying this technique lies in choosing an appropriate function $f$.

## 2 The equations

In this section, using the method described above, we present some examples of equations that have appeared in various magazines with problem columns. Let us start with the following

**Equation 1.** [7] Solve in $\mathbb{R}$ the equation

$$x^6 + 28x^4 + 226x^2 + 819 + \log_3\left(\frac{x^2+9}{6x}\right) = 216x^3 + 6x.$$

*Solution.* Rearranging terms, the given equation can be written as

$$\log_3(x^2+9)+(x^2+9)^3+(x^2+9)^2+x^2+9 = \log_3(6x)+(6x)^3+(6x)^2+6x.$$

This suggests to consider the function $f : (0,\infty) \to \mathbb{R}$ defined by

$$f(t) = \log_3 t + t^3 + t^2 + t.$$

Since $f$ is the sum of increasing functions on $(0,\infty)$, it is itself increasing and therefore injective. With this choice, the given equation becomes

$$f(x^2+9) = f(6x).$$

Define the functions $g, h : (0,\infty) \to (0,\infty)$ by $g(x) = x^2 + 9$, $h(x) = 6x$. Then the equation takes the form $f(g(x)) = f(h(x))$. By the injectivity of $f$, we obtain

$$g(x) = h(x),$$

which is equivalent to $x^2 - 6x + 9 = (x-3)^2 = 0$. Thus the given equation has the unique real solution $x = 3$.

**Equation 2.** [8] Solve in the set of real numbers the equation

$$2^{\cos^2 x} + (3 - \sin^2 x \cos^2 x)\cos(2x) = 2^{\sin^2 x}.$$

*Solution.* By direct computation, we have

$$(3 - \sin^2 x \cos^2 x)\cos 2x = \cos^6 x + 2\cos^4 x - \sin^6 x - 2\sin^4 x,$$

and the given equation can be written as

$$2^{\cos^2 x} + \cos^6 x + 2\cos^4 x = 2^{\sin^2 x} + \sin^6 x + 2\sin^4 x.$$

This suggest to consider the function $f : [0,\infty) \to \mathbb{R}$ defined by

$$f(t) = 2^t + t^3 + 2t^2.$$

Since $f$ is injective, because is a sum of increasing functions, then the given equation can be rewritten equivalently as

$$f(\cos^2 x) = f(\sin^2 x).$$

Define the functions $g, h : \mathbb{R} \to [0, \infty)$ by $g(x) = \cos^2 x,\ h(x) = \sin^2 x$. Then the equation becomes $f(g(x)) = f(h(x))$. Since $f$ is injective, we obtain $g(x) = h(x)$, which is equivalent to

$$\cos^2 x = \sin^2 x \Leftrightarrow \cos(2x) = 0,$$

whose solutions are

$$x \in \left\{ \frac{(2k+1)\pi}{4} \ : \ k \in \mathbb{Z} \right\}.$$

**Equation 3.** [1] Determine the real numbers $x \geq 1$ such that

$$2 + 2^x + 4^x + \log_7\left(\frac{3^x + 5^x}{2 + 2^x + 4^x}\right) = 3^x + 5^x.$$

*Solution.* Rearranging terms, the given equation can be written as

$$2 + 2^x + 4^x - \log_7(2 + 2^x + 4^x) = 3^x + 5^x - \log_7(3^x + 5^x).$$

This suggests to consider the function $f : (0, \infty) \to \mathbb{R}$ defined by $f(t) = t - \log_7 t$. Then the given equation can be rewritten equivalently as

$$f(2 + 2^x + 4^x) = f(3^x + 5^x).$$

Define the functions $g, h : [1, +\infty) \to (0, \infty)$ by $g(x) = 2 + 2^x + 4^x,\ h(x) = 3^x + 5^x$. Then the equation takes the form $f(g(x)) = f(h(x))$. Since $f'(t) = 1 - \dfrac{1}{t \ln 7} > 0$ for $t \geq 1$, then $f$ is increasing, and hence injective, over the interval $[1, +\infty)$. By the injectivity of $f$, we obtain $g(x) = h(x)$, which is equivalent to $2 + 2^x + 4^x = 3^x + 5^x$. Next, we show that $3^x > 2^x + 1$. Indeed, this inequality is equivalent to

$$1 > \left(\frac{2}{3}\right)^x + \left(\frac{1}{3}\right)^x.$$

Using the monotonicity of the exponential function, for $x > 1$ we have

$$\left(\frac{2}{3}\right)^x < \left(\frac{2}{3}\right)^1, \quad \left(\frac{1}{3}\right)^x < \left(\frac{1}{3}\right)^1,$$

and by adding these inequalities we obtain the desired result.

Likewise, the inequality $5^x > 4^x + 1$ is shown analogously. Adding up the two inequalities yields $3^x + 5^x > 4^x + 2^x + 2$, and the only real solution of the equation $3^x + 5^x = 4^x + 2^x + 2$ is $x = 1$.

**Equation 4.** [5] Determine the real number $x$ such that

$$\log_4\left(\frac{\sqrt[4]{x-1}+\sqrt[4]{3-x}}{2+\sqrt[4]{x-2}}\right)+\sqrt{x-1}$$

$$+2\sqrt[4]{-x^2+4x-3}\left(1+2\sqrt{x-1}+2\sqrt{3-x}\right)$$

$$+\sqrt{3-x}+6\sqrt{-x^2+4x-3}=16+36\sqrt[4]{x-2}+25\sqrt{x-2}+8\sqrt[4]{(x-2)^3}.$$

*Solution.* First , we observe that a necessary condition for the given equation be well defined is that $x \in [2,3]$. Let $f : (0,\infty) \to \mathbb{R}$ and $g, h : [2,3] \to (0,\infty)$ be the functions defined by

$$f(t) = \log_4 t + t^2 + t^4, \quad g(x) = \sqrt[4]{x-1} + \sqrt[4]{3-x}, \; h(x) = 2 + \sqrt[4]{x-2}.$$

Then the equation to be solved becomes $f(g(x)) = f(h(x))$. Since

$$f'(t) = \frac{1}{t\ln 4} + 2t + 4t^3 > 0, \quad t > 0,$$

then $f$ is increasing and injective on $(0,+\infty)$. Therefore, $g(x) = h(x)$, or $\sqrt[4]{x-1} + \sqrt[4]{3-x} = 2 + \sqrt[4]{x-2}$, $x \in [2,3]$. Applying the inequality between the arithmetic and geometric mean, we have

$$\sqrt[4]{x-1} = \sqrt[4]{(x-1)\cdot 1\cdot 1\cdot 1} \le \frac{(x-1)+1+1+1}{4} = \frac{x+2}{4},$$

$$\sqrt[4]{3-x} = \sqrt[4]{(3-x)\cdot 1\cdot 1\cdot 1} \le \frac{(3-x)+1+1+1}{4} = \frac{6-x}{4},$$

with equality if and only if $x-1=3-x=1 \Rightarrow x=2$. By adding these inequalities we obtain $\sqrt[4]{x-1}+\sqrt[4]{3-x} \le 2$, with equality if and only if $x=2$, and also $2+\sqrt[4]{x-2} \ge 2$, with equality if and only if $x=2$. Combining the preceding it follows that the equality holds if and only if $x=2$, and this is the only real root of the given equation.

**Equation 5.** [2] For real $x$, solve the equation

$$2^{(2^x-1)^2}+4^x=\sqrt{x}+2^{x+1}+\log_2(1+\sqrt{x}).$$

*Solution.* Considering the functions $f:(0,\infty)\to\mathbb{R}$ and $g,h:(0,\infty)\to(0,\infty)$, defined by

$$f(t)=2^t+t,\ g(x)=(2^x-1)^2,\ h(x)=\log_2(\sqrt{x}+1),$$

we have $f(g(x))=f(h(x))$. Since $f'(t)=2^t\ln 2>0$ for $t>0$, the function $f$ is injective on $(0,\infty)$, and therefore $g(x)=h(x)$, or

$$(2^x-1)^2=\log_2(\sqrt{x}+1)\ \Leftrightarrow\ 2^x+\sqrt{x}=\sqrt{\log_2(\sqrt{x}+1)}+\sqrt{x}+1.$$

To solve the last equation, we apply again the previously described method and consider the functions $f:[0,\infty)\to\mathbb{R}$ and $g,h:[0,\infty)\to[0,\infty)$, defined by

$$f(t)=2^t+\sqrt{t},\qquad g(x)=x,\qquad h(x)=\log_2(\sqrt{x}+1),$$

and we have $f(g(x))=f(h(x))$. Since $f$ is injective on $[0,\infty)$, it follows that $g(x)=h(x),\ x\in[0,\infty)$, or equivalently,

$$x=\log_2(\sqrt{x}+1)\ \Leftrightarrow\ 2^x=\sqrt{x}+1.$$

The function $2^x$ is convex on $[0,\infty)$, while $\sqrt{x}+1$ is concave on the same interval, so they can intersect in at most two points. Thus, the solutions of the given equation are $x=0$ and $x=1$, as can be easily checked.

**Equation 6.** [4] Solve in the set of real numbers the equation

$$16^x+81^x+\log_8(9^x+4^x)=25^x+2\cdot 36^x+4\cdot 30^x+\log_8(5^x+2\cdot 6^x).$$

*Solution.* Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be the functions defined by

$$f(t) = 3^{2t} + t,\ g(x) = \log_3(9^x + 4^x),\ h(x) = \log_3(5^x + 2 \cdot 6^x).$$

Then, the equation in the statement can be written as $f(g(x)) = f(h(x))$. Since $f$ is increasing on $\mathbb{R}$, it is injective, hence $g(x) = h(x)$, which is equivalent to

$$9^x + 4^x = 5^x + 2 \cdot 6^x \Leftrightarrow (3^x - 2^x)^2 = 5^x \Leftrightarrow |3^x - 2^x| = 5^{x/2}.$$

Now, we examine the following cases:

- **Case 1:** $x \geq 0$. In this case, $3^x \geq 2^x$, so the last equation becomes $3^x - 2^x = 5^{x/2}$, or equivalently

  $$1 = \left(\frac{2}{3}\right)^x + \left(\frac{\sqrt{5}}{3}\right)^x.$$

  Consider the function

  $$\varphi(x) = \left(\frac{2}{3}\right)^x + \left(\frac{\sqrt{5}}{3}\right)^x.$$

  Both terms are decreasing for $x \geq 0$, hence $\varphi$ is decreasing on $[0, \infty)$. Therefore the equation $\varphi(x) = 1$ has at most one solution. It is easy to check that $x = 2$ satisfies it, and this is the unique solution with $x \geq 0$.
- **Case 2:** $x < 0$. Now $3^x < 2^x$, so equation $|3^x - 2^x| = 5^{x/2}$ becomes $2^x - 3^x = 5^{x/2}$, or equivalently

  $$\left(\frac{2}{\sqrt{5}}\right)^x - \left(\frac{3}{\sqrt{5}}\right)^x = 1.$$

  Define $\psi : (-\infty, 0) \to \mathbb{R}$ by

  $$\psi(x) = \left(\frac{2}{\sqrt{5}}\right)^x - \left(\frac{3}{\sqrt{5}}\right)^x.$$

  On $(-\infty, 0)$, both $\left(\frac{2}{\sqrt{5}}\right)^x$ and $\left(\frac{3}{\sqrt{5}}\right)^x$ are decreasing, and the second term is subtracted, so $\psi$ is decreasing on $(-\infty, 0)$.

Hence the equation $\psi(x) = 1$ has at most one solution in $(-\infty, 0)$. Moreover,

$$\psi(-4) = \frac{25 \cdot 65}{16 \cdot 81} > 1, \ \psi(-2) = \frac{25}{36} < 1.$$

Thus, by the Intermediate Value Theorem and the monotonicity of $\psi$, there exists a unique solution $x \in (-4, -2)$. Actually,.

Finally, we conclude that $x \simeq -3.06$ and $x = 2$ are the only real solution of the given equation.

**Equation 7.** [6] Solve the following equation for real $x$:

$$(x^2+1)\left[4^{x/(x^2+1)} - \log_4(x^2 - 4x + 5)\right] = x^6+x^4-4x^3+5x^2-5x+5.$$

*Solution.* The given equation can be rewritten as:

$$4^{\frac{x}{x^2+1}} + \frac{x}{x^2+1} = \log_4(x^4 - 4x + 5) + x^4 - 4x + 5$$

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(t) = 4^t + t$. Since $f'(t) = 4^t \ln 4 + 1 > 0$, the function $f$ is strictly increasing and, therefore, injective.

Let $g(x) = \frac{x}{x^2+1}$ and $h(x) = \log_4(x^4-4x+5)$. The original equation can then be expressed as $f(g(x)) = f(h(x))$. By the injectivity of $f$, this is equivalent to $g(x) = h(x)$, or:

$$\frac{x}{x^2+1} = \log_4(x^4 - 4x + 5)$$

Next, we analyze the range of both sides for $x \in (0, +\infty)$:

1. For the left side: $\frac{x}{x^2+1} \leq \frac{1}{2}$ is equivalent to $2x \leq x^2 + 1$, which simplifies to $(x-1)^2 \geq 0$. This is always true, with equality holding if and only if $x = 1$.
2. For the right side: $\log_4(x^4 - 4x + 5) \geq \frac{1}{2}$ is equivalent to $x^4 - 4x + 5 \geq 4^{1/2} = 2$. This simplifies to $x^4 - 4x + 3 \geq 0$. Factoring gives $(x-1)^2(x^2 + 2x + 3) \geq 0$. Since $x^2 + 2x + 3$ is always positive, equality holds if and only if $x = 1$.

Thus, for all $x > 0$, we have:

$$\frac{x}{x^2+1} \le \frac{1}{2} \le \log_4(x^4 - 4x + 5)$$

The only way for the equality to hold is if both sides are equal to $1/2$, which occurs if and only if $x = 1$. Therefore, $x = 1$ is the unique real solution.

**Equation 8.** [3] Solve in $\mathbb{R}$ the equation

$$\log_2(x^2 + 2^x) + (x^2 - 1)2^{x+1} + x^4 + x^2 + 2^x = 3 \cdot 4^x + x + 1.$$

*Solution.* Consider the functions $f : (0, +\infty) \to \mathbb{R}$ and $g, h : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(t) = \log_2 t + t^2 + t, \quad t \in (0, +\infty)$$

$$g(x) = x^2 + 2^x \quad \text{and} \quad h(x) = 2^{x+1}, \quad x \in \mathbb{R}$$

The given equation can be expressed as $f(g(x)) = f(h(x))$. Since $f'(t) = \frac{1}{t \ln 2} + 2t + 1 > 0$ for all $t > 0$, the function $f$ is strictly increasing and therefore injective on $(0, +\infty)$.

From the injectivity of $f$, it follows that $g(x) = h(x)$. That is

It is well-known that the equation $x^2 = 2^x$ has exactly three real solutions. However, in the context of positive integers or common transcendental roots, the solutions are:

$$x \in \{2, 4\}$$

**Note:** There is also a negative solution, $x \approx -0.766$, though often ignored in simplified contexts.

Finally, we add an equation not published yet.

**Equation 9.** [9] Determine the real number $x$ such that

$$8x^4 + 59x^3 + 975 = 332x^2 + 116x - 3(x+5)\sqrt{x^2 + 3x + 7}.$$

*Solution.* Rearranging terms, the given equation can be written as

$$(x^2+29x+24)^2+(x^2+9x+24)+1$$
$$=3(x+5)^2(x^2+3x+7)+3(x+5)\sqrt{x^2+3x+7})+1.$$

This suggests to consider the function $f:(-1/2,+\infty)\to\mathbb{R}$ defined by

$$f(t)=t^2+t+1.$$

Since $f'(t)=2t+1>0$ on $(-1/2,+\infty)$, then $f$ is injective. With this choice, the given equation becomes

$$f(x^2+29x+24)=f(3(x+5)\sqrt{x^2+3x+7}).$$

Define the functions $g,h:(-1/2,+\infty)\to\mathbb{R}$ by $g(x)=x^2+29x+24$, $h(x)=3(x+5)\sqrt{x^2+3x+7}$. Then the equation takes the form $f(g(x))=f(h(x))$. By the injectivity of $f$, we obtain

$$g(x)=h(x),$$

which is equivalent to $x^2+29x+24=3(x+5)\sqrt{x^2+3x+7}$. Thus, we have to solve in $\mathbb{R}$ the equation

$$x^2+29x+24=3(x+5)\sqrt{x^2+3x+7}.$$

Since $x^2+3x+7>0$ then $\sqrt{x^2+3x+7}$ is well defined for all real numbers. Now we square both sides and we get

$$(x^2+29x+24)^2=9(x+5)^2(x^2+3x+7)$$
$$\Leftrightarrow 8x^4+59x^3-331x^2-87x+999=0.$$

Factorizing, yields

$$8x^4+59x^3-331x^2-87x+999$$
$$=(x-3)(x+3)(8x^2+59x+111=0)=0$$

with solutions $x=-3, x=3$ because the quadratic $8x^2+59x+111=0$ with discriminant $\Delta=-71<0$ has no real roots. Now only remains to check if $=-3$ and $x=3$ are solutions of the equation $x^2+29x+24=3(x+5)\sqrt{x^2+3x+7}$. Indeed, for $x=3$ we have $LHS=120$ and $RHS=120$, so $x=3$ works and it is a solution. For $x=-3$ we have $LHS=-54\neq 6\sqrt{7}=RHS$, so $x=-3$ is an extraneous solution introduced by squaring, and the only real solution of the given equation is $x=3$.

# References

[1] Bencze, M. and Stanciu, N. "Problem 5691". *School Science and Mathematics* 122 (2022), p. 282.

[2] Bencze, M. and Stanciu, N. "Problem 5745". *School Science and Mathematics* 123 (2023), p. 515.

[3] Bencze, M. and Stanciu, N. "Problem 921". *The Pentagon Mathematical Journal* (2023). Spring 2023, p. 22.

[4] Bencze, M. and Stanciu, N. "Problem X.559". *Revista de Matematică din Timișoara* 102.1 (2023), p. 51.

[5] Bencze, M. and Stanciu, N. "Problema 495". *Gaceta de la Real Sociedad Matemática Española* 27.2 (2024), p. 331.

[6] Bencze, M. and Stanciu, N. "Problem 5808". *School Science and Mathematics* 125 (2025), p. 298.

[7] Stanciu, N. "Problem O.X. 355". *Revista de Matematică din Timișoara* 101.1 (2022), p. 38.

[8] Stanciu, N. "Problem S:L22.98". *Suplimentul cu Exerciții al Gazetei Matematice, Seria B* 127.3 (2022), p. 11.

[9] Stanciu, N., Pop, O. T., and Díaz-Barrero, J. L. "Forthcoming Work". To be published. 2026.

*Neculai Stanciu*
*"George Emil Palade" School*
*Buzău*
*Romania*
stanciuneculai@yahoo.com

*Ovidiu Tiberiu Pop*
*"Mihai Eminescu" National College*
*Satu Mare*
*Romania*
ovidiutiberiu@yahoo.com

*José Luis Díaz-Barrero*
*Barcelona Mathematical Circle*
*Barcelona, Spain*
jose.luis.diaz@upc.edu

# A Unique Number Pattern Revisited

**A. Witkowski**

**Abstract**

We correct a result presented by the authors in [1] and offer an alternative solution to the problem discussed in their article.

## 1 Introduction

In [1] the authors consider the following problem:
Find an integer number $N$ satisfying the patterns

$$abc = N + 2 \tag{1}$$
$$cab = 2N, \tag{2}$$

where $a, b, c$ are non-zero, single-digit integers or decimal digits (depending on the context) and $abc$ denotes a three-digit number. The solution proposed by the authors — $a = 4, b = c = 9, N = 497$ — clearly does not satisfy (2).

The purpose of this note is threefold:

1. To solve the problem stated above.
2. To uncover the authors' actual intent.
3. To present an alternative solution to the problem addressed in their article.

Before proceeding, let us note two easy facts.

**Fact 1.** *If $N$ fits the patters* (1)*, then $N$ is a three-digit number whose first digit equal $a$.*

*Proof.* Since $a, b, c$ are non-zero, we have:

$$abc > N = abc - 2 \geq a11 - 2 = a09.$$

The second fact follows directly from Fact 1 and requires no proof.

**Fact 2.** $c = 2a$ *or* $c = 2a + 1$

## 2 How to solve (1) and (2)?

We begin, as the authors did in [1], by eliminating $N$:

$$2 \cdot abc - cab = 4$$

which can be written as

$$2(100a + 10b + c) - (100c + 10a + b) = 4, \tag{3}$$

or

$$19(10a + b - 5c) = 4 + 3c$$

This implies

$$4 + 3c \equiv 0(\mod 19)$$

which yields $c = 5$. Therefore the last digit of $N$ is 3 and the last digit of $2N$ is $b = 6$. From Fact 2, we find $a = 2$. Thus $N = 263$ and

$$265 = 263 + 2$$
$$526 = 2 \cdot 263.$$

## 3 What problem did Thiruniraiselvi and Gopalan actually solve?

Below is a quote from the original paper

$$2abc - cab = 4 \tag{1}$$

> Now, using the decimal number system, (1) can be written as
> $$2(100a + 10b + c) - (100c + 10b + a) = 4$$
> .

It looks like they were solving the equation

$$2abc - cba = 4.$$

Further confirmation is found in the abstract of [1]:

> We show that there exists an integer $N$ such that the three digit integer $abc = N + 2$ and its reverse integer $cab = 2N$.

The word "reverse" is a key here. However, the reverse of $abc$ is $cba$, not $cab$. Therefore, we conclude that Thiruniraiselvi and Gopalan actually solved the following system:

$$abc = N + 2 \tag{4}$$
$$cba = 2N, \tag{5}$$

# 4 How to solve (4) and (5) in a simple way?

Note that Facts 1 and 2 apply in this case as well. Looking at (5) we see that $a$ must be even. On the other hand, If $a$ were equal to 6 or 8, then the number $2N$ would be a four-digit number, which is impossible.

Consider two remaining cases:

**Case** $a = 2$ The last digit of $N$ is $1$ or $6$, so $c$, the last digit od $N + 2$, is $3$ or $8$. This contradicts Fact 2.

**Case** $a = 4$ The last digit of $N$ can be $2$ or $7$ making $c = 4$ or $9$. The first case contradicts Fact 2, while the second leads to

$$2 \cdot 4b9 - 9b4 = 4$$

which yields $b = 9$.

# References

[1] Thiruniraiselvi, N. and Gopalan, M. A. "A Unique Number Pattern". *Arhimede Mathematical Journal* 12.1 (2025), pp. 39–41. URL: http://amj-math.com/wp-content/uploads/2025/06/AMJ2025-vol12iss1.pdf (visited on 2025-11-01).

*Alfred Witkowski*
*AUDoMAT*
*Ceramiczna 5/79, 85-085 Bydgoszcz, Poland*
audomat.aw@gmail.com

# *On a Proof Method for Three-Variable Inequalities under the Constraint $x^2 + y^2 + z^2 + xyz = 4$*

**Marius Drăgan and José Luis Díaz-Barrero**

**Abstract**

A method to prove symmetric inequalities in three variables constrained by $x^2 + y^2 + z^2 + xyz = 4$ is presented.

## 1 Introduction

The purpose of this paper is to prove constrained algebraic inequalities in three variables using a new algebraic method. Specifically, we consider inequalities of the form $f(x, y, z) \geq 0$, where $x$, $y$, and $z$ are nonnegative real numbers subject to the constraint $x^2 + y^2 + z^2 + xyz = 4$, and $f$ is a symmetric function. Similar analytical methods and foundational techniques can be found in [3, 4].

## 2 Main Results

Let $x, y, z$ be nonnegative real numbers such that $x^2 + y^2 + z^2 + xyz = 4$. We define two variables, $s$ and $p$, which play a key role in the upcoming proofs. Setting $s = x+y$ and $p = xy$, the constraint equation transforms into $s^2 - 2p + z^2 + pz = 4$, which yields:

$$p = \frac{s^2 + z^2 - 4}{2 - z}.$$

Without loss of generality (WLOG), we assume that $x \le y \le z$. It follows that $(x - z)(y - z) \ge 0$, which implies $p - sz + z^2 \ge 0$. Substituting the expression for $p$ into this inequality, we obtain

$$s^2-(2z-z^2)s+3z^2-z^3-4 \ge 0 \iff (s-z+2)[s+(z-2)(z+1)] \ge 0.$$

From the initial constraint, we deduce that $z^2 \le 4$ and $z \le 2$, which guarantees that $s + 2 - z > 0$. Consequently, we have

$$s + (z - 2)(z + 1) \ge 0 \implies s \ge (2 - z)(z + 1) = s_1.$$

Additionally, using the standard algebraic bound $p \le \frac{s^2}{4}$, it follows that

$$\frac{s^2 + z^2 - 4}{2 - z} \le \frac{s^2}{4} \implies s \le 2\sqrt{2 - z} = s_2.$$

Since $0 \le x \le y \le z$, the inequality $z^3 + 3z^2 - 4 \ge 0$ holds, forcing $z \ge 1$. Thus, we establish the following intervals for $s$:

$$s_1 = (2 - z)(z + 1) \le s \le s_2 = 2\sqrt{2 - z}, \quad \text{for } z \in [1, 2].$$

Furthermore, the nonnegativity condition $p \ge 0$ implies $s^2 + z^2 - 4 = xy(2 - z) \ge 0$, giving an lower bound $s \ge \sqrt{4 - z^2} = s_3$.

Comparing these boundary values, we find that $s_3 \le s_1$ when $z \in [1, \sqrt{2}]$, and $s_1 \le s_3$ when $z \in [\sqrt{2}, 2]$. Consequently, the admissible ranges for $s$ are determined as follows:

$$\begin{aligned} s_3 \le s_1 \le s \le s_2, &\quad \text{if } z \in [1, \sqrt{2}], \\ s_1 \le s_3 \le s \le s_2, &\quad \text{if } z \in [\sqrt{2}, 2]. \end{aligned}$$

# 3 Applications

In this section, we present several applications to illustrate the effectiveness of our proposed method.

# Application 1

Let $x, y, z$ be positive real numbers such that $x^2+y^2+z^2+xyz = 4$. Then, the following inequalities hold:

1. $2 < x + y + z \leq 3$,
2. $3 \leq x^2 + y^2 + z^2 \leq 4$,
3. $xy + yz + zx \leq x + y + z \leq 3 \leq x^2 + y^2 + z^2$,
4. $xy + yz + zx \leq 2 + xyz$,
5. $x + y + z \geq 3xyz$,
6. $x^2y^2 + y^2z^2 + z^2x^2 \leq x^2 + y^2 + z^2$.

*Proof of 1.* Let $s = x + y$ and $p = xy$. We can express the total sum as $x + y + z = s + z = f(s)$, which is strictly increasing with respect to $s$. Thus, $f(s_1) \leq f(s) \leq f(s_2)$, where the endpoints of the domain are given by $s_1 = (2 - z)(z + 1)$ and $s_2 = 2\sqrt{2 - z}$.

Evaluating at these endpoints yields $f(s_1) = (2 - z)(z + 1) + z = -z^2 + 2z + 2$, and $f(s_2) = 2\sqrt{2 - z} + z$. Optimizing these values over the range $z \in [1, 2]$, we obtain:

$$\inf_{1\leq z\leq 2}(-z^2+2z+2) = 2 \text{ (at } z = 2), \text{ and } \sup_{1\leq z\leq 2}(2\sqrt{2 - z}+z) = 3 \text{ (at } z = 1).$$

This completes the proof.

*Proof of 2.* Using the substitutions $s = x + y$ and $p = xy$, we write $x^2 + y^2 + z^2 = s^2 - 2p + z^2$. Substituting $p = \frac{s^2+z^2-4}{2-z}$ gives:

$$x^2 + y^2 + z^2 = f(s) = z^2 + 2z + 4 - \frac{s^2 z}{2 - z}.$$

Since $f(s)$ is monotonically decreasing, it satisfies $f(s_2) \leq f(s) \leq f(s_1)$, where $f(s_1) = 4$ and $f(s_2) = z^2 - 2z + 4$. Given that $\inf_{1\leq z\leq 2}(z^2 - 2z + 4) = 3$ at $z = 1$, we obtain $3 \leq x^2 + y^2 + z^2 \leq 4$.

*Proof of 3.* The target inequality $\frac{p+sz}{s+z} \leq 1$ can be rewritten by substituting $p$ to define:

$$f(s) = \frac{(2 - z)sz + s^2 + z^2 - 4}{(s + z)(2 - z)}.$$

Differentiating $f(s)$ with respect to $s$ yields:

$$f'(s) = \frac{s^2 + 2zs - z^3 + z^2 + 4}{(2 - z)(z + s)^2}.$$

Using the lower bound $s_3 = \sqrt{4 - z^2}$, the numerator satisfies:

$$s^2+2zs-z^3+z^2+4 \geq s_3^2+2zs_3-z^3+z^2+4 = 8-z^3+2z\sqrt{4 - z^2} \geq 0,$$

since $z \leq 2$. Thus, $f(s)$ is monotonically increasing. For all $s \leq s_2$, we have $f(s) \leq f(s_2)$. Since

$$f(s_2) = \frac{2z\sqrt{2 - z} + 2 - z}{z + 2\sqrt{2 - z}} \quad \text{and} \quad \sup_{1 \leq z \leq 2} f(s_2) = 1 \quad (\text{at } z = 1),$$

it follows that $\sum_{\text{cyc}} xy \leq \sum_{\text{cyc}} x$. The remaining chain of inequalities follows directly from parts 1 and 2.

*Proof of 4.* Substituting $s = x + y$ and $p = xy$ into the targeted expression yields $\sum_{\text{cyc}} xy - xyz = p + sz - pz$. Substituting $p$ gives:

$$f(s) = \frac{(s^2 + z^2 - 4)(1 - z)}{2 - z} + sz.$$

The derivative is $f'(s) = \frac{2s(1-z)}{2-z} + z$, which yields the stationary point $s_0 = \frac{z(2-z)}{2(1-z)}$. We analyze this across three structural cases for $z$:

- *Case 1:* For $z \in [1, 1.52]$, we have $s_1 \leq s_3 \leq s_0$. The function is increasing, meaning $f(s) \leq f(s_2) = z^2 - 3z + 2 + 2\sqrt{2 - z}$. Since $\sup f(s_2) = 2$ at $z = 1$, the bound holds.
- *Case 2:* For $z \in (1.52, 1.55)$, the maximum point falls inside the interval ($s_1 < s_0 < s_3$). Thus, $f(s) \leq f(s_0) = z^2 + z - 2$. The supremum in this narrow band is approximately $1.95 < 2$.
- *Case 3:* For $z \in (1.55, 2)$, we have $s_0 < s_1 < s_3$, meaning $f(s)$ is strictly decreasing. Thus, $f(s) \leq f(s_3) = z\sqrt{4 - z^2}$. The maximum value on this interval is approximately $1.95 < 2$.

Combining all cases, we conclude that $\sum_{\text{cyc}} xy \leq 2 + xyz$.

*Proof of 5.* In terms of $s$ and $p$, the target expression can be written as:

$$\frac{x+y+z}{xyz} = \frac{s+z}{pz} = f(s) = \frac{(2-z)(s+z)}{z(s^2+z^2-4)}, \quad s \in [s_3, s_2].$$

Differentiating with respect to $s$ yields:

$$f'(s) = \frac{(2-z)(-s^2-2zs+z^2-4)}{z(s^2+z^2-4)^2}.$$

Let $u(s) = -s^2 - 2zs + z^2 - 4$. Since $u(s) \le u(s_3) = 2z^2 - 8 - 2z\sqrt{4-z^2} \le 0$, the function $f(s)$ is monotonically decreasing. Hence, for all $s \le s_2$, we have:

$$f(s) \ge f(s_2) = \frac{z + 2\sqrt{2-z}}{2z - z^2}.$$

Since $\inf_{1 \le z \le 2} f(s_2) = 3$ (attained at $z = 1$), it follows that $\frac{x+y+z}{xyz} \ge 3$, or $x + y + z \ge 3xyz$.

*Proof of 6.* Making the substitutions $s = x + y$ and $p = xy$, the expression transforms into:

$$x^2y^2 + y^2z^2 + z^2x^2 - x^2 - y^2 - z^2 = p^2 + (2-2z^2)p + z^2s^2 - s^2 - z^2.$$

Substituting $p$ and setting $t = s^2$, we define the quadratic function:

$$f(t) = \frac{t^2}{(z-2)^2} + \frac{(z^3+z+4)t}{z-2} + 2z(z+1)^2,$$

which is evaluated over the domain $t \in [t_3, t_2] = [4 - z^2, 4(2-z)]$. Calculating values at the boundaries yields:

$$f(t_3) = -(z^2-2)^2 \le 0 \quad \text{and} \quad f(t_2) = -2z(z-1) \le 0.$$

Since $f(t)$ is a convex quadratic function, it follows that $f(t) \le 0$ for all $t \in [t_3, t_2]$, which concludes the proof.

# Application 2

[6] Let $x, y, z$ be nonnegative real numbers such that $x^2 + y^2 + z^2 + xyz = 4$. Then, the following inequality holds:

$$(x+y+z-2)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{2}\right) \geq \frac{5}{2}.$$

*Proof.* Making the substitutions $x + y = s$ and $xy = p$, the inequality becomes:

$$(s+z-2)\left(\frac{s}{p}+\frac{1}{z}-\frac{1}{2}\right) \geq \frac{5}{2}.$$

Substituting $p = \frac{s^2+z^2-4}{2-z}$, we obtain:

$$f(s) = s^3+\left(3z+3+\frac{10}{z-2}\right)s^2+(z-2)(3z+2)s+(z+2)(z^2+z+4),$$

where $s \in [s_3, s_2]$. It suffices to show that $f(s) \geq 0$. To achieve this, we compute the derivative:

$$f'(s) = 3s^2 + \left(6z+6+\frac{20}{z-2}\right)s + (z-2)(3z+2).$$

Evaluating $f'(s)$ at the boundary points gives:

$$f'(s_3) = \frac{z\big(2(z-2)^2 - (3z^2-3z+4)\sqrt{4-z^2}\big)}{2-z} \leq 0 \quad \forall z \in [1,2],$$

and

$$f'(s_2) = \frac{-2z^2+12z+20+(3z^2-16z)\sqrt{2-z}}{\sqrt{2-z}} \leq 0,$$

which holds when $z \in (0.621, 2]$, and therefore also on the domain $z \in [1, 2]$.

Consequently, $f'(s) \leq 0$ for all $s \in [s_3, s_2]$, meaning that $f(s)$ is monotonically decreasing. Thus, for all $s \leq s_2$, we have $f(s) \geq f(s_2)$. Since the boundary evaluation function

$$f(s_2) = (z-2)\big(z^2-7z+4+(6z-4)\sqrt{2-z}\big)$$

satisfies $\inf_{1\le z\le 2} f(s_2) = 0$, attained at both $z = 1$ and $z = 2$, it follows that $f(s_2) \ge 0$. Hence, $f(s) \ge 0$ for all $s \in [s_3, s_2]$.

For nonnegative real numbers $x, y, z$, the constraint $x^2 + y^2 + z^2 + xyz = 4$ yields two distinct types of equality configurations. The complete set of boundary and symmetric solutions for the ordered triple $(x, y, z)$ consists of the symmetric point $(1, 1, 1)$ and the permutations of the boundary solutions, namely $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$.

# Application 3

Let $x, y, z$ be nonnegative real numbers such that $x^2 + y^2 + z^2 + xyz = 4$. Then, the following inequality holds:

$$\frac{1}{xy + 2z} + \frac{1}{yz + 2x} + \frac{1}{zx + 2y} \le \frac{1}{\sqrt{xyz}}.$$

*Proof.* Making the substitutions $x + y = s$ and $xy = p$, the inequality becomes:

$$\sqrt{pz}\left(\frac{1}{p + 2z} + \frac{zs + 2s}{(z^2 - 4z + 4)p + 2zs^2}\right) \le 1.$$

Substituting $p = \frac{s^2+z^2-4}{2-z}$, we get:

$$\sqrt{\frac{z(s^2 + z^2 - 4)}{2 - z}}\,\frac{1}{s + z - 2} \le 1.$$

After algebraic simplification and squaring, we arrive at the equivalent condition:

$$f(s) = 2(z-1)s^2 + (2z^2 - 8z + 8)s + 2z^3 - 6z^2 + 8z - 8 \le 0, \quad \forall s \in [s_1, s_2].$$

The discriminant of this quadratic is given by $\Delta_s = 4(2 - z)z^2(3z - 2) \ge 0$. Since the leading coefficient is $a = 2(z - 1) > 0$, we verify the endpoints of the interval:

$$\begin{aligned} f(s_1) &= 2z^2(z - 2)(z - 1)^2 \le 0, \\ f(s_2) &= 2(z - 2)^2\left(z - 3 + 2\sqrt{2 - z}\right) \le 0, \end{aligned}$$

which remain valid for all $z \in [1, 2]$. By the convexity of the quadratic function $f(s)$, it follows that $f(s) \le 0$ for all $s \in [s_1, s_2]$. Equality holds if and only if $x = y = z = 1$.

# Application 4

[2] Let $a, b, c$ be positive real numbers such that $ab + bc + ca + 2abc = 1$. Then, the following inequalities hold:

(a) $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge 6$;
(b) $a + b + c \ge \frac{3}{2}$.

*Proof.* Applying the transformation $a = \frac{yz}{2x}$, $b = \frac{xz}{2y}$, and $c = \frac{xy}{2z}$, the constraint simplifies to $x^2 + y^2 + z^2 + xyz = 4$. The inequalities to be proven become:

(a) $x^2 + y^2 + z^2 \ge 3xyz$,
(b) $x^2y^2 + y^2z^2 + z^2x^2 \ge 3xyz$.

From Application 1 (part 6), we know that $x^2 + y^2 + z^2 \ge x^2y^2 + y^2z^2 + z^2x^2$; thus, proving (b) is sufficient to establish both. Setting $s = x + y$ and $p = xy$, condition (b) transforms to:

$$\frac{z^2(s^2 - 2p) + p^2}{pz} \ge 3.$$

Substituting $p = \frac{s^2+z^2-4}{2-z}$, the expression becomes:

$$\frac{s^4 + (z^4 - 2z^3 + 5z^2 - 6z - 8)s^2 + 2(z + 2)(-z^2 + z + 2)^2}{z(2 - z)(s^2 + z^2 - 4)} \ge 0.$$

Let $t = s^2$. Since the denominator is strictly positive, it suffices to prove that:

$$f(t) = t^2 + (z^4 - 2z^3 + 5z^2 - 6z - 8)t + 2(z + 2)(z^2 - z - 2)^2 \ge 0,$$

for all $t \in [t_3, t_2]$, where $t_3 = 4 - z^2$ and $t_2 = 4(2 - z)$. The discriminant of this quadratic is:

$$\Delta_t = z^2(z - 2)^2(z^4 + 10z^2 - 7) > 0,$$

which is positive for all $z \in \left(\sqrt{4\sqrt{2}-5}, 2\right]$, and thus for all $z \in [1,2]$. The vertex of this parabola has the abscissa:

$$t_v = \frac{-z^4 + 2z^3 - 5z^2 + 6z + 8}{2}.$$

Since $t_v - t_2 = \frac{1}{2}(z-2)(z^2+5z-4) \geq 0$, it follows that $t_v \geq t_2$. This implies that $f(t)$ is monotonically decreasing on $[t_3, t_2]$. Consequently, for all $t \leq t_2$, we have $f(t) \geq f(t_2)$. Given that $f(t_2) = 2(z-2)^3(z-1)^2 \geq 0$, we conclude that $f(t) \geq 0$.

## Application 5

Let $x, y, z > 0$ such that $x^2 + y^2 + z^2 + xyz = 4$. Then, the following inequality holds:

$$(x+y+z-2)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + 1\right) \geq 4.$$

*Proof.* Introducing the standard variables $s = x + y$ and $p = xy$, the inequality is modeled as:

$$(s+z-2)\left(\frac{s}{p} + \frac{1}{z} + 1\right) \geq 4.$$

Substituting $p = \frac{s^2+z^2-4}{2-z}$ from the constraint yields the objective function:

$$f(s) = (z+1)s^3 - (3z+2)s^2 + (5z^2-8z-4)s + z^4 - 5z^3 - 6z^2 + 20z + 8 \geq 0.$$

Its derivative with respect to $s$ is given by:

$$f'(s) = (3z+3)s^2 - (6z+4)s + 5z^2 - 8z - 4.$$

The discriminant of the equation $f'(s) = 0$ is:

$$\Delta_s = -60z^3 + 72z^2 + 192z + 64 > 0,$$

which holds for all $z \in (0, 2.59]$, covering our domain $z \in [1, 2]$. The roots of $f'(s) = 0$ are:

$$s_{5,6} = \frac{3z + 2 \pm \sqrt{-15z^3 + 18z^2 + 48z + 16}}{3z + 3}.$$

Using a Computer Algebra System (CAS), we verify that $s_5 \leq s_3 \leq s_2 \leq s_6$ for all $z \in [0.75, 2] \subset [1, 2]$. Thus, $f(s)$ is monotonically decreasing on $[s_3, s_2]$, which means $f(s) \geq f(s_2)$ for all $s \leq s_2$. Evaluating at $s_2$ gives:

$$f(s_2) = (z-2)^2\left(z^2 - z - 2 + 2\sqrt{2-z}\right) \geq 0, \quad \forall z \in [1, 2].$$

Consequently, $f(s) \geq 0$ for all $s \in [s_3, s_2]$. Equality is achieved at $x = y = z = 1$.

# Application 6

Let $x, y, z$ be positive numbers such that $x \leq y \leq z$ and $x^2 + y^2 + z^2 + xyz = 4$. Find the best constant $k$ for which:

$$x + y + z + k(\sqrt{x} - \sqrt{z})^2 \geq 3.$$

*Proof.* Setting $s = x + y$ and $p = xy$, the target inequality can be rearranged as:

$$k \geq \frac{3 - s - z}{s - 2\sqrt{p}}.$$

Substituting $p = \frac{s^2+z^2-4}{2-z}$ leads to:

$$k \geq f(s) = \frac{s + z - 3}{2\sqrt{\frac{s^2+z^2-4}{2-z}} - s}.$$

From the ordered configuration, $(x-y)(y-z) \leq 0 \implies sy - p - y^2 \leq 0$. Substituting $p$ updates this inequality to:

$$\frac{(s-y+2)(s+y^2-y-2)}{2-y} \leq 0 \implies s \leq (2-y)(y+1) = s_1.$$

Furthermore, since $2y^2 \leq y^2 + z^2 \leq 4$, we have $y \in (0, \sqrt{2}]$. Hence, we seek the maximum of $f(s)$ over the interval $s \in [s_3, s_1]$. Differentiating yields:

$$f'(s) = \frac{1}{y-2}\sqrt{\frac{2-y}{s^2+y^2-4}} \cdot \frac{1}{\left(2\sqrt{\frac{s^2+y^2-4}{2-y}} - s\right)^2} g(s),$$

where $g(s) = (y^2 - 5y + 6)\sqrt{\frac{y^2+s^2-4}{2-y}} + 2\big(s(y-3) - y^2 + 4\big)$. We show $g(s) \leq 0$, which is equivalent to:

$$(y^2 - 5y + 6)\sqrt{\frac{s^2+y^2-4}{2-y}} \leq 2\big((3-y)s + y^2 - 4\big). \qquad (1)$$

Let $h(s) = (3-y)s + y^2 - 4$ for $s \in [s_3, s_1]$. Since $h(s)$ is linear, checking the boundaries yields $h(s_3) \geq 0$ and $h(s_1) \geq 0$ for all $y \in (0, \sqrt{2}]$. Squaring both sides of (1) produces:

$$u(s) = (y+2)(y-3)^2s^2 + 8(4-y^2)(y-3)s + (y^2-4)(y^3-4y^2+21y-34) \geq 0. \qquad (2)$$

The discriminant of this quadratic is $\Delta_s = -4(y-2)^2(y+2)^2(y-3)^2(y-1)^2 \leq 0$, which validates inequality (2). Thus, $g(s) \leq 0$, and $f(s)$ is monotonically decreasing.

Therefore, $f(s) \leq f(s_3)$ for all $s \geq s_3$, where:

$$f(s_3) = \frac{3 - y - \sqrt{4-y^2}}{\sqrt{4-y^2}}.$$

Maximizing this function over $y \in (0, \sqrt{2}]$, the supremum is reached as $y \to 0$:

$$k \geq \sup_{0<y\leq\sqrt{2}} f(s_3) = \frac{1}{2}.$$

Thus, the optimal constant is $k = \frac{1}{2}$, yielding the sharp inequality:

$$x + y + z + \frac{1}{2}(\sqrt{x} - \sqrt{z})^2 \geq 3.$$

This refines Problem O.691 proposed by Titu Andreescu and Marius Stănean in 2023 [1]

# Application 7

[5] Let $ABC$ be an acute triangle. Prove that:

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-\frac{3}{4}\geq k\left(\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}-\frac{1}{8}\right),$$

where $k=4\left(1+\sqrt{2}-\sqrt{2+\sqrt{2}}\right)^2$. When does equality hold?

*Proof.* Applying the substitutions $A\to\pi-2A$, $B\to\pi-2B$, and $C\to\pi-2C$, the expression becomes:

$$\cos A+\cos B+\cos C-\frac{3}{2}\geq k\left(\cos A\cos B\cos C-\frac{1}{8}\right),$$

where $A,B,C\in\left[\frac{\pi}{4},\frac{\pi}{2}\right]$. Setting $x=2\cos A$, $y=2\cos B$, and $z=2\cos C$, it follows that $x,y,z\in[0,\sqrt{2}]$, matching the identity $x^2+y^2+z^2+xyz=4$. The inequality can be rewritten as:

$$4(x+y+z)-12\geq k(xyz-1).$$

Substituting $s=x+y$ and $p=xy$ isolates $k$:

$$k\geq\frac{12-4s-4z}{1-pz}.$$

Using the constraint relation to substitute $p$ yields:

$$k\geq f(s)=\frac{(2-z)(12-4s-4z)}{2+3z-s^2z-z^3},\quad s\in[s_1,s_2].$$

Differentiating with respect to $s$ gives:

$$f'(s)=\frac{4(2-z)[-zs^2+(6z-2z^2)s+z^3-3z-2]}{(2+3z-s^2z-z^3)^2},$$

which gives the critical points:

$$s'_{4,5}=\frac{3z-z^2\pm\sqrt{2z(z-1)^3}}{z},\quad z\in[1,\sqrt{2}].$$

WLOG, assuming $x \le y \le z$, the boundary conditions limit the domain to $z \in [1, \sqrt{2}]$. Since $s_4' \le s_1 \le s_2 \le s_5'$ holds true on this interval, $f(s)$ is monotonically increasing on $[s_1, s_2]$. Hence, $k \ge f(s_2) \ge f(s)$, where:

$$f(s_2) = \frac{4(3 - z - 2\sqrt{2 - z})}{(z - 1)^2}.$$

Maximizing this single-variable function via a CAS over $z \in [1, \sqrt{2}]$, the supremum is attained at $z = \sqrt{2}$:

$$k \ge \sup_{z \in [1, \sqrt{2}]} f(s_2) = 4\left(5 + 3\sqrt{2} - 2\sqrt{10 + 7\sqrt{2}}\right) = 4\left(1 + \sqrt{2} - \sqrt{2 + \sqrt{2}}\right)^2.$$

Equality holds when $z = \sqrt{2}$ and $s^2 = 4p$ (implying $x = y$). Back-substituting yields $x = y = \sqrt{2 - \sqrt{2}}$ and $z = \sqrt{2}$. In terms of the triangle parameters, equality requires $A = B = \frac{3\pi}{8}$ and $C = \frac{\pi}{4}$.

# Application 8

Let $x \ge y \ge z \ge 0$ be real numbers such that $x^2+y^2+z^2+xyz = 4$. Find the best constant $k$ for which:

$$6 + xy + yz + zx + k(x - z)^2 \ge 3(x + y + z).$$

*Proof.* From the condition $x \ge y \ge z$, we have $(x - y)(y - z) \ge 0 \implies xy - xz - y^2 + yz \ge 0$. Let $s = x + z$ and $p = xz$. The main constraint becomes $s^2 - 2p + y^2 + py = 4$, which allows us to write $p = \frac{s^2+y^2-4}{2-y}$. The configuration $2y^2 \le x^2 + y^2 \le 4$ forces $y \in (0, \sqrt{2}]$. The structural condition updates to:

$$\frac{(s - y + 2)(s + y^2 - y - 2)}{y - 2} \ge 0 \implies s \le (2 - y)(1 + y) = s_2.$$

The target inequality expressed in terms of $s$ and $p$ requires:

$$k \ge \frac{3s + 3y - 6 - p - sy}{s^2 - 4p}.$$

Replacing $p$ simplifies the right-hand side to:

$$f(s) = \frac{s^2 - (y^2 - 5y + 6)s + 4(y^2 - 3y + 2)}{(y + 2)(s^2 + 4y - 8)}.$$

Differentiating with respect to $s$ yields:

$$f'(s) = \frac{(y - 2)[(y - 3)s^2 - 8(y - 2)s - 4(y^2 - 5y + 6)]}{(y + 2)(s^2 + 4y - 8)^2}.$$

Let $g(s) = (y-3)s^2 - 8(y-2)s - 4(y^2 - 5y + 6)$. Its discriminant is $\Delta_s = 16(y - 2)(y - 1)^2 \leq 0$. Because $y - 2 < 0$, it follows that $g(s) < 0$, which ensures $f'(s) \geq 0$. Thus, $f(s)$ is monotonically increasing, reaching its maximum at $s_2$:

$$f(s_2) = \frac{2y}{(y + 2)^2}.$$

Taking the supremum over $y \in (0, \sqrt{2}]$, we find:

$$k \geq \sup_{0<y\leq\sqrt{2}} \frac{2y}{(y + 2)^2} = 3\sqrt{2} - 4,$$

which occurs at $y = \sqrt{2}$. This proves the sharp bound presented in [7].

# Application 9

Let $x, y, z$ be positive real numbers such that $x^2+y^2+z^2+xyz = 4$. Find the best constant $k$ for which:

$$x + y + z \geq 2 + \frac{(k + 4)xyz}{4 + kxyz}.$$

*Proof.* The expression can be isolated for $k$ as follows:

$$k \leq \frac{4\left(\sum_{\text{cyc}} x - xyz - 2\right)}{xyz\left(3 - \sum_{\text{cyc}} x\right)}.$$

Substituting $s = x + y$, $p = xy$, and then applying the constraint formula for $p$ gives:

$$k \leq f(s) = \frac{4(s^2z + (z-2)s + z^3 + z^2 - 8z + 4)}{z(s+z-3)(s^2+z^2-4)}.$$

We decompose this into $f(s) = g(s)h(s)$, where $g(s) = \frac{4\left(s^2z+(z-2)s+z^3+z^2-8z+4\right)}{z(s+z-3)}$ and $h(s) = \frac{1}{s^2+z^2-4} \geq 0$. Differentiating $g(s)$ yields:

$$g'(s) = \frac{u(s)}{z(s+z-3)^2}, \quad \text{with } u(s) = 4zs^2-(24z-8z^2)s+12z-4z^3+8.$$

Assuming $z = \max\{x, y, z\}$ places $z \in [1, 2]$, where $s \in [s_1, s_2]$. Evaluating $u(s)$ at these endpoints reveals that $u(s_1) \leq 0$ and $u(s_2) \leq 0$. Thus, $g'(s) \leq 0$, showing that $g(s)$ is positive and decreasing. Since $h(s)$ is also positive and decreasing on $[s_1, s_2]$, their product $f(s)$ is monotonically decreasing. The minimum is therefore achieved at $s_2$:

$$f(s_2) = \frac{4(z-2-2\sqrt{2-z})}{(z-2)z}.$$

Minimizing this expression over $z \in [1, 2]$ via a CAS yields:

$$k \leq \inf_{1\leq z\leq 2} f(s_2) = 5 + 2\sqrt{3},$$

which occurs at $z = 2\sqrt{3} - 2$. This establishes a sharper bound than the one found in [7].

## Application 10

Let $a, b, c > 0$ such that $ab + bc + ca + abc = 4$. Find the best positive constant $k$ for which:

$$\frac{bc}{a} + \frac{ac}{b} + \frac{ab}{c} + kabc \geq k + 3.$$

*Proof.* Let $a = \frac{yz}{x}$, $b = \frac{zx}{y}$, and $c = \frac{xy}{z}$. The expression transforms into:

$$\frac{\sum_{\text{cyc}} x^4}{xyz} + kxyz \geq k + 3,$$

subject to $x^2 + y^2 + z^2 + xyz = 4$. Defining $t = s^2 = (x + y)^2$ and substituting $p$, the inequality expands to a quadratic in $t$:

$$\begin{aligned} f(t) =&(kz^2 + z^2 - 2)t^2 + (z - 2)\big[(2z^3 + 4z^2 + z)k + 4z^2 + 7z - 8\big]t \\ &+ (z - 2)^2(z + 1)^2\big[(z^2 + 2z)k + z^2 - 2z + 8\big] \geq 0, \end{aligned} \tag{3}$$

which must hold for all $t \in [t_1, t_2]$. Evaluating $f(t)$ at $t_1$ leads to the requirement:

$$kz^2(2 - z^2) \leq z^4 - 6z^2 + 16.$$

For $z \in [1, \sqrt{2}]$, this bounds $k$ by $g(z) = \frac{z^4 - 6z^2 + 16}{z^2(2 - z^2)}$, whose infimum via CAS is:

$$k \leq \inf_{1 \leq z \leq \sqrt{2}} g(z) = 5 + 4\sqrt{2} \quad \text{at } z = \sqrt{2(2 - \sqrt{2})}.$$

Evaluating at the upper bound $t_2$ requires $k \leq h(z) = \frac{z^2 + 2z + 8}{z(2 - z)}$, whose infimum over $z \in [1, 2]$ is 11 (at $z = 1$).

We now verify that $f(t) \geq 0$ across $[t_1, t_2]$ for $k = 5 + 4\sqrt{2}$. The discriminant of (3) with respect to $t$ is:

$$\Delta_t = -4(z-2)^2\Big[(11 + 8\sqrt{2})z^6 - 4(9 + 7\sqrt{2})z^4 + 8(5 + 4\sqrt{2})z^2 - 32\Big].$$

We observe that $\Delta_t \leq 0$ for $z \in [1.02216, 2]$ and $\Delta_t > 0$ for $z \in [1, 1.02216)$. Since the leading coefficient remains positive for all $z \in [1, 2]$, we analyze two specific intervals:

- *Case 1:* $z \in [1.02216, 2]$. Here, $\Delta_t \leq 0$, meaning $f(t) \geq 0$ holds trivially across the real line.
- *Case 2:* $z \in [1, 1.02216)$. The vertex of the parabola lies at $t_v$. It can be verified that $t_1 < t_2 < t_v$ holds true on this interval. Since the vertex lies to the right of our interval and both boundary evaluations ($f(t_1)$ and $f(t_2)$) are strictly positive, the inequality $f(t) \geq 0$ is preserved.

Thus, the best positive constant is $k = 5 + 4\sqrt{2}$.

We close this paper applying the method to prove an inequality that recently appeared in [9].

## Application 11

Let $x, y, z \in [0, 7/4]$ be real numbers such that $x^2+y^2+z^2+xyz = 4$. Prove that $9(x + y + z) \geq 4xyz + 23$.

*Proof.* The given inequality, with the usual change of variables, can be rewritten as

$$9s + 9z \geq 23 + 4\frac{s^2 + z^2 - 4}{2 - z}.$$

Assuming $z < 2$ (so that $2 - z > 0$), we multiply both sides by $2 - z$ and collect all terms onto one side. This yields the equivalent quadratic inequality in terms of $s$:

$$f(s) = 4zs^2 + (9z - 18)s + 4z^3 + 9z^2 - 57z + 46 \leq 0, \quad \forall\, s \in [s_1, s_2].$$

To prove that $f(s) \leq 0$ on the entire interval $[s_1, s_2]$, we observe that for $z > 0$, the leading coefficient of $f(s)$ is positive ($4z > 0$). This implies that $f(s)$ is a convex (upward-opening) parabola. Therefore, if $f(s)$ is non-positive at both endpoints $s_1$ and $s_2$, it must be non-positive for all $s$ between them.

- **Evaluation at $s_1$.** At the first endpoint, the function factors cleanly as:

  $$f(s_1) = (z - 2)(z - 1)^2\left(z - \frac{1}{2}\right)\left(z + \frac{5}{2}\right).$$

  For $z \in [1, 2)$, the terms $(z - 1)^2$, $\left(z - \frac{1}{2}\right)$, and $\left(z + \frac{5}{2}\right)$ are strictly positive, while $(z - 2) < 0$. Thus, $f(s_1) \leq 0$ holds naturally for all $z \in [1, 2)$, with equality at $z = 1$.

- **Evaluation at $s_2$.** At the second endpoint, we can express $f(s_2)$ and rationalize its radical component:

$$f(s_2) = (z-2)\left(4z^2 + z - 23 + 18\sqrt{2-z}\right)$$
$$= \frac{(z-2)(z-1)(4z-7)(4z+17)}{4z^2 + z - 23 - 18\sqrt{2-z}}.$$

We now analyze the sign of $f(s_2)$ for $z \in [1,2)$:

- The term $4z^2 + z - 23$ is negative for all $z \in [1,2)$ since its roots are approximately $-2.52$ and $2.27$.
- Consequently, the denominator $4z^2 + z - 23 - 18\sqrt{2-z}$ is strictly negative on this interval.
- In the numerator, $(z-2) < 0$, $(z-1) \geq 0$, and $(4z+17) > 0$.

The sign of the expression is therefore determined entirely by the factor $(4z-7)$. To ensure $f(s_2) \leq 0$, we require the numerator and denominator to have opposite signs (meaning a non-negative numerator):

$$f(s_2) \leq 0 \iff 4z - 7 \leq 0 \iff z \leq \frac{7}{4}.$$

Finally, combining our constraints, since $f(s)$ is convex for $z > 0$, the conditions

$$\begin{cases} f(s_1) \leq 0 \\ f(s_2) \leq 0 \end{cases}$$

simultaneously hold true if and only if $z \in \left[1, \frac{7}{4}\right]$. Under this condition, we establish that:

$$f(s) \leq 0, \quad \forall\, s \in [s_1, s_2],$$

as we needed to prove.

# References

[1] Andreescu, T. and Stănean, M. “Problem O.691”. *Mathematical Reflections* 2 (2023).

[2] Bataille, M. "Problem 4197". *Crux Mathematicorum* 42.10 (2016), p. 445.

[3] Drăgan, M. *A new method to solve and refine algebraic inequalities*. Presa Universitară Clujeană, 2025.

[4] Drăgan, M. and Díaz-Barrero, J. L. "A method for proving constrained inequalities in three variables". *Arhimede Mathematical Journal* 2 (2025), pp. 158–167.

[5] Stănean, M. "Problem O.618". *Mathematical Reflections* 1 (2023).

[6] Stănean, M. "Problem O.628". *Mathematical Reflections* 3 (2023).

[7] Stănean, M. "Problem O.719". *Mathematical Reflections* 6 (2025).

[8] Stănean, M. "Problem O.725". *Mathematical Reflections* 1 (2026).

[9] Stănean, M. "Problem O.737". *Mathematical Reflections* 3 (2026).

*Marius Drăgan*
*61311 bd. Timişoara Nr. 35, Bl. 0D6, Sc. E, et. 7, Ap. 176, Sect. 6, Bucureşti, Romania*
marius.dragan2005@yahoo.com

*José Luis Díaz-Barrero*
*Barcelona Math Circle*
*Barcelona, Spain*
jose.luis.diaz@upc.edu

# *Contests*

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to find interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to **José Luis Díaz-Barrero**, Barcelona Mathematical Circle (BMC), FME, Pau Gargallo, 14, Les Corts, 08028 Barcelona, Spain or by e-mail to

jose.luis.diaz@upc.edu

# Problems and solutions from the 13th edition of BarcelonaTech Mathcontest

**O. Rivero Salgado and J. L. Díaz-Barrero**

## 1 Problems and solutions

Hereafter, we present the four problems that appeared in the paper given to the contestants of the BarcelonaTech Mathcontest 2026, as well as their official solutions.

***Problem 1.*** Let $\{a_n\}_{n\geq 0}$ be a sequence of positive integers defined by

$$a_n = 10^{3n+3} + 2\cdot 10^{2n+2} + 2\cdot 10^{n+1} + 1.$$

Prove that $a_n/3$ is always the sum of two positive perfect cubes but never the sum of two perfect squares.

**Solution 1.** First, we observe that a few terms of $\{a_n\}$ are:

$$1221, 1020201, 1002002001, 1000200020001, \ldots$$

Now, we claim that $a_n/3$ is never the sum of two perfect squares. Indeed, we have that the first terms for $\{a_n/3\}$ are:

$$407, 340067, 334000667, 3333400000066667, \ldots$$

Since perfect squares give only remainders 0 and 1 when divided by four, then integers expressible as the sum of two squares give only remainders $0, 1$, and $2$. On the other hand, the number $a_n/3$ gives remainder $3$ because $407 \equiv 3 \pmod 4$ and all the terms for $n \geq 1$ end in $67$ hence it cannot be expressed as the sum of two perfect squares.

After some computations, one finds the formula

$$\begin{aligned}\frac{a_n}{3} &= \frac{10^{3n+3} + 2 \cdot 10^{2n+2} + 2 \cdot 10^{n+1} + 1}{3} \\ &= \frac{10^{3n+3}}{27} + \frac{2 \cdot 10^{2n+2}}{9} + \frac{4 \cdot 10^{n+1}}{9} + \frac{8}{27} \\ &+ \frac{8 \cdot 10^{3n+3}}{27} + \frac{4 \cdot 10^{2n+2}}{9} + \frac{2 \cdot 10^{n+1}}{9} + \frac{1}{27} \\ &= \left(\frac{10^{n+1} + 2}{3}\right)^3 + \left(\frac{2 \cdot 10^{n+1} + 1}{3}\right)^3\end{aligned}$$

Both the numbers in brackets are integers since $10^{n+1} \equiv 1 \pmod 3$. Thus $a_n/3$ can be expressed as the sum of two perfect cubes.

**Solution 2.** Put $x = 10^{n+1}$ in $\{a_n\}$. Then, we have

$$a_n = 10^{3n+3} + 2 \cdot 10^{2n+2} + 2 \cdot 10^{n+1} + 1 = x^3 + 2x^2 + 2x + 1.$$

Now we prove that $\frac{a_n}{3}$ is a sum of two positive cubes. Indeed, we claim that

$$\frac{a_n}{3} = \left(\frac{x+2}{3}\right)^3 + \left(\frac{2x+1}{3}\right)^3.$$

Since $x = 10^{n+1} \equiv 1 \pmod 3$, both $x + 2$ and $2x + 1$ are divisible by $3$, so the expressions define positive integers. Compute:

$$\left(\frac{x+2}{3}\right)^3 = \frac{x^3 + 6x^2 + 12x + 8}{27},$$

$$\left(\frac{2x+1}{3}\right)^3 = \frac{8x^3 + 12x^2 + 6x + 1}{27}.$$

Adding:

$$\left(\frac{x+2}{3}\right)^3 + \left(\frac{2x+1}{3}\right)^3 = \frac{9x^3 + 18x^2 + 18x + 9}{27}$$

$$= \frac{x^3+2x^2+2x+1}{3} = \frac{a_n}{3}.$$

Thus

$$\frac{a_n}{3} = \left(\frac{10^{n+1}+2}{3}\right)^3 + \left(\frac{2\cdot 10^{n+1}+1}{3}\right)^3,$$

as desired.

To prove that $a_n/3$ is never a sum of two squares. We compute $a_n/3 \pmod 4$. Indeed, For $n=0$: $a_0 = 1221$, $\dfrac{a_0}{3} = 407 \equiv 3 \pmod 4$. For $n \geq 1$, all exponents $3n+3$, $2n+2$, $n+1$ are at least $2$, so $10^{3n+3} \equiv 10^{2n+2} \equiv 10^{n+1} \equiv 0 \pmod 4$. Thus $a_n \equiv 1 \pmod 4$. Since $3^{-1} \equiv 3 \pmod 4$, then $\frac{a_n}{3} \equiv 1 \cdot 3 \equiv 3 \pmod 4$. Therefore, for all $n \geq 0$,

$$\frac{a_n}{3} \equiv 3 \pmod 4.$$

But a sum of two squares modulo 4 can only be $0, 1$, or $2$, because squares mod 4 are only $0$ and $1$. Hence no integer congruent to $3 \pmod 4$ can be a sum of two squares. Thus $a_n/3$ is never a sum of two perfect squares. Thus, we conclude that for all $n \geq 0$, $a_n/3$ is a sum of two cubes but never a sum of two squares.

***Problem 2.*** We have $2k+1$ colours, and $2k+1$ discs of each colour. We arrange the $(2k+1)^2$ discs into a row. A disc $T$ is *bad* if the number of discs to the left of $T$, with a colour different to that of $T$, is odd. Prove that there are at least $k$ *bad* discs.

**Solution.** Let $A$ be the set of discs which are in an odd position in the row, that is, the discs which have an even number of discs to the left. We have $|A| = \frac{(2k+1)^2+1}{2}$. Let $B$ be the set of discs that, among those of its colour, are in an even position, that is, there is an odd number of discs of its color to the right. There are $k$ discs of each color in $B$, so $|B| = k \cdot (2k+1)$.

Observe that, by construction, any disc not in $A \cup B$ is *bad* (it has an odd number of discs to its left, out of which an even number has the same colour as itself). A simple count yields

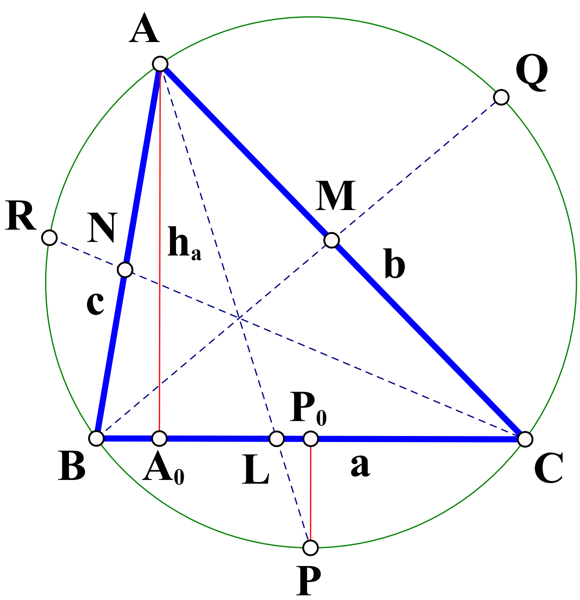
$$|A \cup B| \le |A| + |B| = \frac{(2k+1)^2+1}{2} + k \cdot (2k+1) = (2k+1)^2 - k,$$

so at least $k$ *bad* discs exist.

***Problem 3.*** Let $A, B, C$ be three distinct points on a circle $\gamma$, and let $P, Q, R$ be the midpoints of the arcs $BC, CA, AB$, respectively, where each arc does not contain the remaining vertex of the triangle. Lines $AP$, $BQ$, and $CR$ intersect segments $BC$, $CA$, and $AB$ at points $L$, $M$, and $N$, respectively. Find the minimum value of

$$\frac{AL}{PL} + \frac{BM}{QM} + \frac{CN}{RN}.$$

**Solution.** In the following figure, we have $AA_0 = h_a$ and $\angle BAP = \angle PAC$.




Scheme for solving problem 3.

Since $\triangle AA_0L \sim \triangle PP_0L$ then

$$\frac{AL}{PL} = \frac{AA_0}{PP_0} \Leftrightarrow \frac{AL}{PL} = \frac{h_a}{PP_0} \quad (1)$$

On the other hand, from $[ABC] = \frac{1}{2}\,ah_a = \frac{1}{2}\,bc\sin A$, we obtain

$$h_a = \frac{bc}{a}\sin A.$$

In $\triangle BP_0P$, the $\angle P_0BP = \angle CBP = \frac{1}{2}\angle A$ because both see the same arc $PC$. Hence, $PP_0 = BP_0 \tan A/2 = \frac{a}{2}\tan A/2$.

Substituting in (1) yields

$$\frac{AL}{PL} = \frac{h_a}{PP_0} = \frac{2bc\sin A}{a^2\tan A/2} = \frac{4bc\sin A/2\cos A/2}{a^2\sin A/2} = \frac{4bc}{a^2}\cos^2 A/2.$$

Since $\cos^2 A/2 = \dfrac{s(s-a)}{bc}$, then

$$\frac{AL}{PL} = \frac{2s}{a}\cdot\frac{2(s-a)}{a} = \left(1+\frac{b}{a}+\frac{c}{a}\right)\left(-1+\frac{b}{a}+\frac{c}{a}\right) = \left(\frac{b}{a}+\frac{c}{a}\right)^2 - 1.$$

Likewise, we have

$$\frac{BM}{QM} = \left(\frac{c}{b}+\frac{a}{b}\right)^2 - 1 \quad \text{and} \quad \frac{CN}{RN} = \left(\frac{a}{c}+\frac{b}{c}\right)^2 - 1.$$

Thus

$$\begin{aligned} &\frac{AL}{PL}+\frac{BM}{QM}+\frac{CN}{RN} \\ &= \left[\left(\frac{b}{a}+\frac{c}{a}\right)^2 - 1\right] + \left[\left(\frac{c}{b}+\frac{a}{b}\right)^2 - 1\right] + \left[\left(\frac{a}{c}+\frac{b}{c}\right)^2 - 1\right] \\ &= \left(\frac{a^2}{b^2}+\frac{b^2}{c^2}\right) + \left(\frac{b^2}{c^2}+\frac{c^2}{b^2}\right) + \left(\frac{c^2}{a^2}+\frac{a^2}{c^2}\right) + 2\left(\frac{ab}{c^2}+\frac{bc}{a^2}+\frac{ca}{b^2}\right) - 3 \\ &\geq 2+2+2+2\cdot 3 - 3 = 9 \end{aligned}$$

on account that

$$\frac{ab}{c^2}+\frac{bc}{a^2}+\frac{ca}{b^2} \geq 3\sqrt[3]{\frac{(abc)^2}{a^2b^2c^2}} = 3.$$

Equality holds when $a = b = c$. That is, when $\triangle ABC$ is equilateral.

***Problem 4.*** Determine all polynomials $A$ with real coefficients such that

$$A\left(\left\lfloor \frac{x}{y} + \frac{y}{x} \right\rfloor\right) = A\left(\left\lfloor \frac{x}{y} \right\rfloor\right) + A\left(\left\lfloor \frac{y}{x} \right\rfloor\right),$$

for all positive real numbers $x, y$. Here $\lfloor \alpha \rfloor$ denotes the integer part of the real number $\alpha$.

**Solution.** Set $t = x/y \neq 0$, so that $y/x = 1/t$. Define

$$a = \lfloor t \rfloor, \qquad b = \left\lfloor \frac{1}{t} \right\rfloor, \qquad c = \left\lfloor t + \frac{1}{t} \right\rfloor,$$

all integers. Let $f(n) = A(n)$ for $n \in \mathbb{Z}$. The functional equation becomes

$$f(c) = f(a) + f(b) \qquad \text{for all nonzero real } t.$$

We consider several cases:

- If $t > 1$, then $a = \lfloor t \rfloor = k \geq 1$ and $b = 0$. As $t$ varies in $[k, k+1)$, the value
$$s(t) = t + \frac{1}{t}$$
is continuous and strictly increasing, with $k < s(t) < k + 2$. For every $k \geq 2$, the integer $c = \lfloor s(t) \rfloor$ takes both values $k$ and $k+1$. Thus $f(k) = f(k) + f(0) \quad \Rightarrow \quad f(0) = 0$, and $f(k+1) = f(k) + f(0) = f(k)$. For $k = 1$ one obtains $f(2) = f(1) + f(0) = f(1)$. Hence $f(n) = C \quad$ for all $n \geq 1$, and $f(0) = 0$.
- If $0 < t < 1$, then $a = 0$ and $b \geq 1$. A symmetric argument shows that $c$ takes the values $b$ and $b + 1$, giving $f(b+1) = f(b)$, which is consistent with the previous step. Thus again $f(n) = C$ for all $n \geq 1$.
- If $t < -1$, write $t = -u$ with $u > 1$. Then $a \leq -2$, $b = -1$, and $c$ takes the values $a$ and $a - 1$. The equation
$$f(c) = f(a) + f(-1)$$
implies first $f(-1) = 0$, and then $f(c) = f(a) + f(-1)$ implies first $f(-1) = 0$, and then $f(a-1) = f(a)$. Thus $f(n) = D$ for all $n \leq -2$, and $f(-1) = 0$.

- If $-1 < t < 0$, a symmetric argument yields no new constraints.

Summarizing, we have shown: $f(0) = 0$, $f(-1) = 0$, $f(n) = C \quad (n \geq 1)$, $f(n) = D \quad (n \leq -2)$. Now, we use that $A$ is a polynomial. Since $A(n) = f(n)$ for all integers $n$, the polynomial $A(x) - C$ has infinitely many zeroes (all integers $n \geq 1$). Thus $A(x) \equiv C$. But then $A(0) = C$ must equal $f(0) = 0$, so $A = 0$. Similarly $A(-1) = 0$ forces $D = 0$. Hence $A(n) = 0$ for all integers $n$, and therefore $A$ is the zero polynomial. The preceding allow us to conclude that the only real-coefficient polynomial satisfying the given identity is $A(x) \equiv 0$.

*Óscar Rivero Salgado*
*Universidad de Santiago de Compostela*
*Spain*
`riverosalgado@gmail.com`

*José Luis Díaz-Barrero*
*Barcelona Math Circle (BMC)*
*Barcelona, Spain*
`jose.luis.diaz@upc.edu`

# Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

Please, send submittals to **José Luis Díaz-Barrero**, Barcelona Mathematical Circle (BMC), FME, Pau Gargallo, 14, Les Corts, 08028 Barcelona, Spain or by e-mail to

jose.luis.diaz@upc.edu

## Elementary Problems

***E–143.*** *Proposed by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain.* Let $s$ be the semiperimeter of a triangle $ABC$ with sides $a$, $b$, $c$ opposite to angles $A$, $B$, $C$, respectively. Prove that

$$s^2 = b^2\cos^2\frac{C}{2} + c^2\cos^2\frac{B}{2} + 2bc\cos\frac{B}{2}\cos\frac{C}{2}\sin\frac{A}{2}.$$

**Solution 1 by Cao Minh Quang, Nguyen Binh Khiem high school for the Gifted, Vinh Long, Vietnam, Daniel Văcaru, National Economic College „Maria Teiuleanu", Pitești, Romania, Rovsen Pirguliyev, Azerbaijan, Albert Stadler, Herrliberg, Switzerland, Ioan Viorel Codreanu, Satulung, Maramures, Romania, Andrea Fanchini, Cantù, Italy, Nicusor Zlota, "Traian Vuia" Technical College Focsani, Romania and José Luis Díaz-Barrero, Barcelona, Spain (same solution).** Using the half–angle identities

$$\cos^2\frac{B}{2} = \frac{1+\cos B}{2},\quad \cos^2\frac{C}{2} = \frac{1+\cos C}{2},\quad \sin^2\frac{A}{2} = \frac{1-\cos A}{2},$$

together with the law of cosines

$$\cos B = \frac{a^2+c^2-b^2}{2ac}, \quad \cos C = \frac{a^2+b^2-c^2}{2ab}, \quad \cos A = \frac{b^2+c^2-a^2}{2bc},$$

we obtain the standard half–angle formulas in terms of the sides:

$$\cos^2\frac{B}{2} = \frac{s(s-b)}{ac}, \quad \cos^2\frac{C}{2} = \frac{s(s-c)}{ab}, \quad \sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}.$$

Thus

$$\cos\frac{B}{2} = \sqrt{\frac{s(s-b)}{ac}}, \quad \cos\frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}, \quad \sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Substituting into the right–hand side gives

$$b^2\cos^2\frac{C}{2} = b^2\cdot\frac{s(s-c)}{ab} = \frac{bs(s-c)}{a},$$
$$c^2\cos^2\frac{B}{2} = c^2\cdot\frac{s(s-b)}{ac} = \frac{cs(s-b)}{a},$$

and

$$2bc\cos\frac{B}{2}\cos\frac{C}{2}\sin\frac{A}{2} = 2bc\sqrt{\frac{s(s-b)}{ac}\cdot\frac{s(s-c)}{ab}\cdot\frac{(s-b)(s-c)}{bc}}$$
$$= 2bc\cdot\frac{s(s-b)(s-c)}{abc} = \frac{2s(s-b)(s-c)}{a}.$$

Hence the entire expression equals

$$\frac{s}{a}\Big[b(s-c)+c(s-b)+2(s-b)(s-c)\Big].$$

We simplify the bracket. We have $b(s-c)+c(s-b) = s(b+c)-2bc$, and $2(s-b)(s-c) = 2\big(s^2-s(b+c)+bc\big) = 2s^2-2s(b+c)+2bc$. Adding, yields $b(s-c)+c(s-b)+2(s-b)(s-c) = 2s^2-s(b+c)$. Thus the right–hand side becomes

$$\frac{s}{a}\big(2s^2-s(b+c)\big) = \frac{s^2}{a}\big(2s-(b+c)\big).$$

Since $2s = a+b+c$, we have $2s-(b+c) = a$, so the expression equals

$$\frac{s^2}{a}\cdot a = s^2.$$

This matches the left–hand side, completing the proof.

**Solution 2 by the proposer.** Let $B'$, $C'$ be points on $BC$ extended, with $B$ between $B'$ and $C$, and $C$ between $B$ and $C'$, such that $AB = BB'$, $AC = CC'$. Then

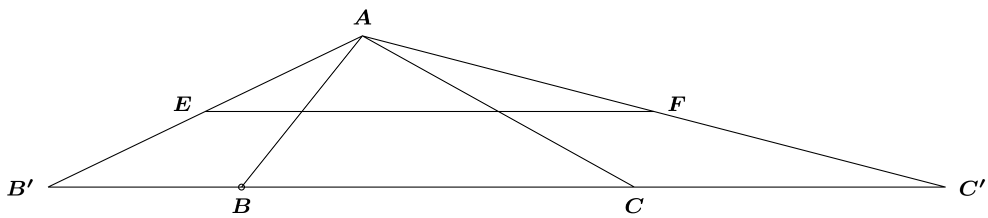
$$B'C' = B'B + BC + CC' = AB + BC + CA = 2s.$$

Let $E$, $F$ be the midpoints of segments $AB'$, $AC'$, respectively. Then

$$EF = \frac{1}{2}B'C' = \frac{1}{2} \cdot 2s = s.$$

The base angles of isosceles triangles $ABB'$, $ACC'$ are each $\frac{1}{2}(\angle B)$, $\frac{1}{2}(\angle C)$, respectively. Therefore,

$$AE = \left(\frac{1}{2}AB'\right) = c \cdot \cos\frac{B}{2}, \quad AF = \left(\frac{1}{2}AC'\right) = b \cdot \cos\frac{C}{2}.$$




By the law of cosines, applied to $\triangle AEF$, in which

$$\angle EAF = \angle B'AB + \angle BAC + \angle CAC' = \frac{B}{2} + A + \frac{C}{2} = 90^\circ + \frac{A}{2},$$

$$EF^2 = AE^2 + AF^2 - 2 \cdot AE \cdot AF \cdot \cos\left(90^\circ + \frac{A}{2}\right),$$

which, on substitution, becomes

$$s^2 = b^2\cos^2\frac{C}{2} + c^2\cos^2\frac{B}{2} + 2bc\cos\frac{B}{2}\cos\frac{C}{2}\sin\frac{A}{2}.$$

**Solution 3 by Michel Bataille, Rouen, France.** From $\cos B + \cos C = 2\cos\frac{B+C}{2}\cos\frac{B-C}{2} = 2\sin\frac{A}{2}\cos\frac{B-C}{2}$ and $1 -$

$\cos A = 2\sin^2 \dfrac{A}{2}$, we deduce that

$$\begin{aligned}1 + \cos B + \cos C - \cos A &= 2\sin\frac{A}{2}\left(\cos\frac{B-C}{2} + \cos\frac{B+C}{2}\right)\\ &= 4\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.\end{aligned}$$

Therefore, recalling the formulas $a = b\cos C + c\cos B$, $a^2 = b^2 + c^2 - 2bc\cos A$ and setting

$$\mathcal{R} = b^2\cos^2\frac{C}{2} + c^2\cos^2\frac{B}{2} + 2bc\cos\frac{B}{2}\cos\frac{C}{2}\sin\frac{A}{2},$$

we obtain

$$\begin{aligned}4\mathcal{R} &= 2b^2(1+\cos C) + 2c^2(1+\cos B) + 2bc(1+\cos B+\cos C-\cos A)\\ &= 2b^2 + 2c^2 + 2(b+c)(b\cos C + c\cos B) + 2bc + a^2 - b^2 - c^2\\ &= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc = (a+b+c)^2 = 4s^2.\end{aligned}$$

Thus, $\mathcal{R} = s^2$, as required.

***E–144.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $x_1, x_2, x_3$ be the roots of the equation $x^3 - 3x^2 - 5x + 1 = 0$. Find the equation whose roots are

$$y_1 = x_1 + \frac{1}{x_1}, \quad y_2 = x_2 + \frac{1}{x_2}, \quad y_3 = x_3 + \frac{1}{x_3}.$$

**Solution 1 by Albert Stadler, Herrliberg, Switzerland.** Let $x_1, x_2, x_3$ be the roots of $x^3 - 3x^2 - 5x + 1 = 0$. By Vieta's formulas,

$$x_1 + x_2 + x_3 = 3,\ x_1x_2 + x_2x_3 + x_3x_1 = -5,\ x_1x_2x_3 = -1.$$

Define $y_i = x_i + \frac{1}{x_i}$. Then

$$\sum y_i = \sum x_i + \sum\frac{1}{x_i} = 3 + \frac{x_1x_2 + x_2x_3 + x_3x_1}{x_1x_2x_3} = 3 + \frac{-5}{-1} = 8,$$

$$y_1y_2 + y_2y_3 + y_3y_1 = (x_1x_2 + x_2x_3 + x_3x_1)$$

$$+\left(\frac{x_1}{x_2}+\frac{x_2}{x_1}+\frac{x_2}{x_3}+\frac{x_3}{x_2}+\frac{x_3}{x_1}+\frac{x_1}{x_3}\right)+\left(\frac{1}{x_1x_2}+\frac{1}{x_2x_3}+\frac{1}{x_3x_1}\right)$$
$$=(x_1x_2+x_2x_3+x_3x_1)+\left(\frac{(x_1+x_2+x_3)(x_1x_2+x_2x_3+x_3x_1)}{x_1x_2x_3}-3\right)$$
$$+\left(\frac{x_1+x_2+x_3}{x_1x_2x_3}\right)=-5+\left(\frac{3\cdot -5}{-1}-3\right)+\frac{3}{-1}=4,$$
$$\prod y_i=\prod\left(x_i+\frac{1}{x_i}\right)=\frac{\prod\left(x_i^2+1\right)}{x_1x_2x_3}$$
$$=\frac{1+9+25+2\cdot 3-2(-5)+1}{-1}=-52.$$

The required equation is

$$x^3-(y_1+y_2+y_3)x^2+(y_1y_2+y_2y_3+y_3y_1)x-y_1y_2y_3=0,$$

hence

$$x^3-8x^2+4x+52=0.$$

**Solution 2 by Cao Minh Quang, Nguyen Binh Khiem high school for the Gifted, Vinh Long, Vietnam.** Let $x_1,x_2,x_3$ be the roots of $x^3-3x^2-5x+1=0$. By Vieta's formulas, we have

$$\sigma_1=x_1+x_2+x_3=3,\ \sigma_2=x_1x_2+x_1x_3+x_2x_3=-5,\ \sigma_3=x_1x_2x_3=-1.$$

We define $y_i=x_i+\frac{1}{x_i}(i=1,2,3)$. We need to determine

$$\sum_{i=1}^{3}y_i,\qquad \sum_{i<j}y_iy_j,\qquad \prod_{i=1}^{3}y_i.$$

**Sum of the roots.**

$$\sum_{i=1}^{3}y_i=\sum_{i=1}^{3}x_i+\sum_{i=1}^{3}\frac{1}{x_i}=\sigma_1+\frac{\sigma_2}{\sigma_3}=3+\frac{-5}{-1}=8.$$

**Sum of pairwise products.** For $i\neq j$,

$$y_iy_j=\left(x_i+\frac{1}{x_i}\right)\left(x_j+\frac{1}{x_j}\right)=x_ix_j+\frac{x_i}{x_j}+\frac{x_j}{x_i}+\frac{1}{x_ix_j}.$$

Summing over all $i<j$ gives

$$\sum_{i<j} y_iy_j = \sigma_2 + \sum_{i\neq j}\frac{x_i}{x_j} + \sum_{i<j}\frac{1}{x_ix_j}.$$

We compute

$$\sum_{i<j}\frac{1}{x_ix_j} = \frac{x_1+x_2+x_3}{x_1x_2x_3} = \frac{\sigma_1}{\sigma_3} = \frac{3}{-1} = -3,$$

and

$$\sum_{i\neq j}\frac{x_i}{x_j} = \left(\sum_i x_i\right)\left(\sum_j \frac{1}{x_j}\right) - 3 = \sigma_1\cdot\frac{\sigma_2}{\sigma_3} - 3 = 3\cdot 5 - 3 = 12.$$

Hence,

$$\sum_{i<j} y_iy_j = -5 + 12 - 3 = 4.$$

**Product of the roots.**

$$\prod_{i=1}^{3} y_i = \prod_{i=1}^{3}\left(x_i + \frac{1}{x_i}\right) = \frac{\prod_{i=1}^{3}(x_i^2+1)}{x_1x_2x_3} = \frac{\prod_{i=1}^{3}(x_i^2+1)}{\sigma_3}.$$

We expand

$$\prod_{i=1}^{3}(x_i^2+1) = x_1^2x_2^2x_3^2 + \sum_{i<j} x_i^2x_j^2 + \sum_i x_i^2 + 1.$$

Using Viète's relations,

$$x_1^2x_2^2x_3^2 = \sigma_3^2 = 1,$$

$$\sum_i x_i^2 = \sigma_1^2 - 2\sigma_2 = 3^2 - 2(-5) = 19,$$

$$\sum_{i<j} x_i^2x_j^2 = \sigma_2^2 - 2\sigma_1\sigma_3 = (-5)^2 - 2\cdot 3\cdot(-1) = 31.$$

Thus,

$$\prod_{i=1}^{3}(x_i^2+1) = 1 + 31 + 19 + 1 = 52,$$

and therefore

$$\prod_{i=1}^{3} y_i = \frac{52}{-1} = -52.$$

By the converse of Vieta's theorem, the cubic equation with roots $y_1, y_2, y_3$ is

$$t^3 - (y_1 + y_2 + y_3)t^2 + (y_1y_2 + y_1y_3 + y_2y_3)t - y_1y_2y_3 = 0.$$

Substituting the values obtained above, we conclude that the required equation is $t^3 - 8t^2 + 4t + 52 = 0$.

**Solution 3 by Michel Bataille, Rouen, France.** The problem reduces to the elimination of $x$ between the equations $x^3 - 3x^2 - 5x + 1 = 0$ and $x^2 - xy + 1 = 0$. By successive long divisions, we obtain

$$x^3-3x^2-5x+1 = (x+y-3)(x^2-xy+1)+x(y^2-3y-6)+4-y, \quad (1)$$

hence if the two equations have a common solution for $x$, say $x_0$, then we must have $x_0(y^2 - 3y - 6) = y - 4$.
Since $(y^2 - 3y - 6)^2 + (y - 4)^2 \neq 0$, we see that $x_0 = \frac{y-4}{y^2-3y-6}$ and consequently

$$\left(\frac{y-4}{y^2-3y-6}\right)^2 - y \cdot \frac{y-4}{y^2-3y-6} + 1 = 0, \quad (2)$$

that is,

$$(y-4)^2 - y(y-4)(y^2-3y-6) + (y^2-3y-6)^2 = 0$$

or, after a short calculation, $y^3 - 8y^2 + 4y + 52 = 0$.
Conversely, if the latter holds, then $y^2 - 3y - 6 \neq 0$ and from (1) and (2), $\frac{y-4}{y^2-3y-6}$ is a common solution to $x^3 - 3x^2 - 5x + 1 = 0$ and $x^2 - xy + 1 = 0$.
Thus, the desired equation is $y^3 - 8y^2 + 4y + 52 = 0$.

**Solution 4 by Arkady Alt, San Jose, California, USA, Rovsen Pirguliyev, Azerbaijan and Ioan Viorel Codreanu, Satulung, Maramures, Romania (same solution).** Let $(a, b, c) = (x_1, x_2, x_3)$ and define

$$y_k = x_k + \frac{1}{x_k} \qquad (k = 1, 2, 3).$$

We are given the symmetric data

$$\sum a = 3, \qquad \sum ab = -5, \qquad abc = -1.$$

Then

$$y_1+y_2+y_3 = \sum\left(a+\frac{1}{a}\right) = \sum a+\sum\frac{1}{a} = \sum a+\frac{\sum ab}{abc} = 3+5 = 8.$$

Expand:

$$\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right) = ab+\frac{1}{ab}+\frac{a^2+b^2}{ab}.$$

Thus

$$y_1y_2+y_2y_3+y_3y_1 = \sum ab+\sum\frac{1}{ab}+\sum\frac{a^2+b^2}{ab}.$$

Now, we Compute each term:

$$\sum\frac{1}{ab} = \frac{\sum a}{abc} = -3,$$

$$\sum\frac{a^2+b^2}{ab} = \frac{\sum c(a^2+b^2)}{abc} = \frac{\sum a\cdot\sum ab-3abc}{abc} = 12.$$

Therefore $y_1y_2+y_2y_3+y_3y_1 = (-5)+(-3)+12 = 4.$

Expand:

$$\prod_{cyc}\left(a+\frac{1}{a}\right) = \frac{1}{abc}+abc+\sum\frac{bc}{a}+\sum\frac{a}{bc}.$$

We use

$$\sum a^2 = (\sum a)^2-2\sum ab = 19, \quad \sum b^2c^2 = (\sum ab)^2-2abc\sum a = 31.$$

Then

$$\sum\frac{bc}{a} = \frac{\sum b^2c^2}{abc} = -31, \qquad \sum\frac{a}{bc} = \frac{\sum a^2}{abc} = -19.$$

Thus

$$y_1y_2y_3 = \frac{1}{abc}+abc-31-19 = -1-1-31-19 = -52.$$

Finally, we have that the monic cubic polynomial equation with roots $y_1, y_2, y_3$ is

$$y^3-8y^2+4y+52 = 0.$$

**Solution 5 by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain.** We consider the equation $x^3 - px^2 + qx - r = 0$. Taking into account Cardan–Viète's formulae, we have

$$\begin{aligned} x_1 + x_2 + x_3 &= p, \\ x_1x_2 + x_2x_3 + x_3x_1 &= q, \\ x_1x_2x_3 &= r. \end{aligned}$$

On the other hand, we have

$$y_1 + y_2 + y_3 = x_1 + x_2 + x_3 + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = p + \frac{q}{r},$$

$$\begin{aligned} y_1y_2 + y_2y_3 + y_3y_1 &= x_1x_2 + x_2x_3 + x_3x_1 + \frac{1}{x_1x_2} + \frac{1}{x_2x_3} + \frac{1}{x_3x_1} \\ &+ \frac{x_2}{x_1} + \frac{x_1}{x_2} + \frac{x_3}{x_1} + \frac{x_1}{x_3} + \frac{x_3}{x_2} + \frac{x_2}{x_3} \\ &= q + \frac{x_1 + x_2 + x_3}{x_1x_2x_3} + \frac{x_1 + x_2}{x_3} + \frac{x_2 + x_3}{x_1} + \frac{x_3 + x_1}{x_2} \\ &= q + \frac{p}{r} + \frac{p - x_1}{x_1} + \frac{p - x_2}{x_2} + \frac{p - x_3}{x_3} \\ &= q + \frac{p}{r} + p\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) - 3 \\ &= q + \frac{p}{r} + \frac{pq}{r} - 3, \end{aligned}$$

and

$$\begin{aligned} y_1y_2y_3 &= x_1x_2x_3 + \frac{1}{x_1x_2x_3} + \frac{x_1}{x_2x_3} + \frac{x_2}{x_3x_1} + \frac{x_3}{x_1x_2} \\ &+ \frac{x_1x_2}{x_3} + \frac{x_2x_3}{x_1} + \frac{x_3x_1}{x_2} \\ &= r + \frac{1}{r} + \frac{x_1^2 + x_2^2 + x_3^2}{x_1x_2x_3} + \frac{x_1^2x_2^2 + x_2^2x_3^2 + x_3^2x_1^2}{x_1x_2x_3} \\ &= r + \frac{1}{r} + \frac{(x_1 + x_2 + x_3)^2 - 2(x_1x_2 + x_2x_3 + x_3x_1)}{r} \\ &+ \frac{(x_1x_2 + x_2x_3 + x_3x_1)^2 - 2x_1x_2x_3(x_1 + x_2 + x_3)}{r} \\ &= r + \frac{1}{r} + \frac{p^2 - 2q}{r} + \frac{q^2 - 2rp}{r}. \end{aligned}$$

In the case of the given equation, we have $p = 3,\ q = -5,\ r = -1$, and

$$\begin{aligned} y_1 + y_2 + y_3 &= 8, \\ y_1y_2 + y_2y_3 + y_3y_1 &= 4, \\ y_1y_2y_3 &= -52. \end{aligned}$$

So, the desired equation is

$$y^3 - 8y^2 + 4y + 52 = 0,$$

and we are done.

**Also solved by** *José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain and the proposer.*

***E–145.*** *Proposed by Michel Bataille, Rouen, France.* Let $ABC$ be a triangle such that $AB = AC$. Let $M$ be the point of the line $BC$ such that $MB = MA$ and $D$ be the circumcenter of $\Delta AMB$. Prove that the lines $BD$ and $AC$ are perpendicular.

**Solution 1 by Cao Minh Quang, Nguyen Binh Khiem high school for the Gifted, Vinh Long, Vietnam.** We use the following *four-point theorem*: for points $A, B, C, D$,

$$AB^2 - AD^2 = CB^2 - CD^2 \iff AC \perp BD.$$

Hence it suffices to prove $AB^2 - AD^2 = CB^2 - CD^2$. Let $(AMB)$ be the circumcircle of $\triangle AMB$. Since $D$ is its center, $AD$ is its radius $R$. Since $B, M, C$ are collinear and $B, M \in (AMB)$, the power of $C$ with respect to $(AMB)$ gives

$$\mathrm{Pow}_{(AMB)}(C) = CB \cdot CM = CD^2 - R^2 = CD^2 - AD^2.$$

Thus

$$\begin{aligned} CB^2 - CD^2 = CB^2 - AD^2 - CB \cdot CM &= CB(CB - CM) - AD^2 \\ &= CB \cdot BM - AD^2. \end{aligned}$$

Therefore, we need to prove $AB^2 = CB \cdot BM$. Let $BM = x$, $MC = y$, so $BC = x + y$. Apply Stewart's theorem in $\triangle ABC$ for the point $M \in BC$:

$$AB^2 \cdot MC + AC^2 \cdot MB = BC\big(AM^2 + MB \cdot MC\big).$$

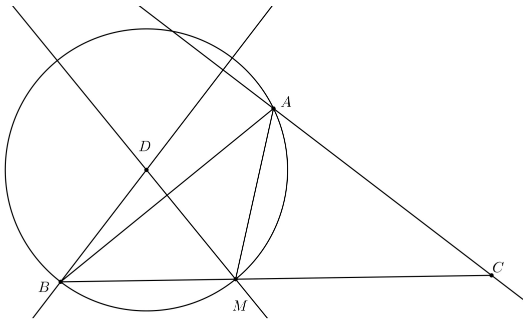
Since $AB = AC$, the left-hand side equals $AB^2(x + y)$, hence after dividing by $x + y > 0$, we get

$$AB^2 = AM^2 + xy.$$

Given $AM = MB = x$, we obtain

$$AB^2 = x^2 + xy = x(x + y) = BM \cdot BC,$$

From this, $CB^2 - CD^2 = AB^2 - AD^2$. By the four-point theorem, $BD \perp AC$.




Scheme for solving problem E-145.

**Solution 2 by the proposer.** We use the fact that in an isosceles triangle such as $\Delta ABC$, we have $\angle(AB, AC) = 2\angle(BA, BC)$ [Here and in what follows, $\angle(\ell, m)$ denote the directed angle from line $\ell$ to line $m$.]
From $\angle(BA, BC) = \angle(BA, BM)$ we deduce that $\angle(AB, AC) = \angle(MA, MB)$. It follows that

$$\angle(AB, AC) = 2\angle(MD, MB) = \angle(DM, DB)$$

and therefore $\angle(AB, DM) = \angle(AC, DB)$. Since $DM \perp AB$, we also have $AC \perp BD$.

**Solution 3 by the proposer.** Let I be the inversion in the circle with center $B$ and radius $BA$ and let $\Gamma$ be the circle with center $M$ and radius $MB = MA$. Since $\mathrm{I}(A) = A$, the inverse of $\Gamma$ is the perpendicular to $BM$ through $A$, that is, the perpendicular bisector of $BC$. It follows that $\mathrm{I}(M) = C$, the reflection of $B$ in the inverse of $\Gamma$. The circumcircle of $\Delta AMB$ inverts in a line through $\mathrm{I}(A) = A$ and $\mathrm{I}(M) = C$, hence in the line $AC$. The conclusion $BD \perp AC$ follows.

**Solution 4 by José Luis Díaz-Barrero, Barcelona, Spain.** WLOG we place the triangle in a coordinate system as follows. Let $B = (-1, 0)$, $C = (1, 0)$, and since $AB = AC$, place $A = (0, h)$, $h > 0$. Thus $BC$ is the $x$-axis. Let $M$ lie on line $BC$, so $M = (x, 0)$ for some real $x$. The condition $MB = MA$ gives

$$(x+1)^2 = x^2 + h^2.$$

Expanding and simplifying, we have

$$x^2 + 2x + 1 = x^2 + h^2 \quad \Rightarrow \quad 2x + 1 = h^2 \quad \Rightarrow \quad x = \frac{h^2-1}{2}.$$

Hence $M = \left(\dfrac{h^2-1}{2},\, 0\right)$.

Let $D$ be the circumcenter of $\triangle AMB$. We compute $D$ as the intersection of the perpendicular bisectors of $AB$ and $MB$.

- **Perpendicular bisector of $AB$.** The midpoint of $AB$ is
$$E = \left(-\frac{1}{2}, \frac{h}{2}\right).$$
The slope of $AB$ is $m_{AB} = h$, so the perpendicular bisector has slope $-1/h$ and equation
$$y - \frac{h}{2} = -\frac{1}{h}\left(x + \frac{1}{2}\right).$$
- **Perpendicular bisector of $MB$.** Since $M$ and $B$ lie on the $x$-axis, segment $MB$ is horizontal. Its midpoint is
$$F = \left(\frac{h^2-3}{4},\, 0\right),$$

so the perpendicular bisector is the vertical line

$$x = \frac{h^2 - 3}{4}.$$

Thus the circumcenter $D$ has $x$-coordinate

$$x_D = \frac{h^2 - 3}{4}.$$

Substituting into the perpendicular bisector of $AB$,

$$y_D - \frac{h}{2} = -\frac{1}{h}\left(\frac{h^2 - 3}{4} + \frac{1}{2}\right) = -\frac{1}{h} \cdot \frac{h^2 - 1}{4} = -\frac{h^2 - 1}{4h}.$$

Hence

$$y_D = \frac{h}{2} - \frac{h^2 - 1}{4h} = \frac{h^2 + 1}{4h}.$$

Therefore

$$D = \left(\frac{h^2 - 3}{4}, \frac{h^2 + 1}{4h}\right).$$

We now compute the slopes of $AC$ and $BD$. Indeed, the slope of $AC$ is

$$m_{AC} = \frac{0 - h}{1 - 0} = -h,$$

and the slope of $BD$ is

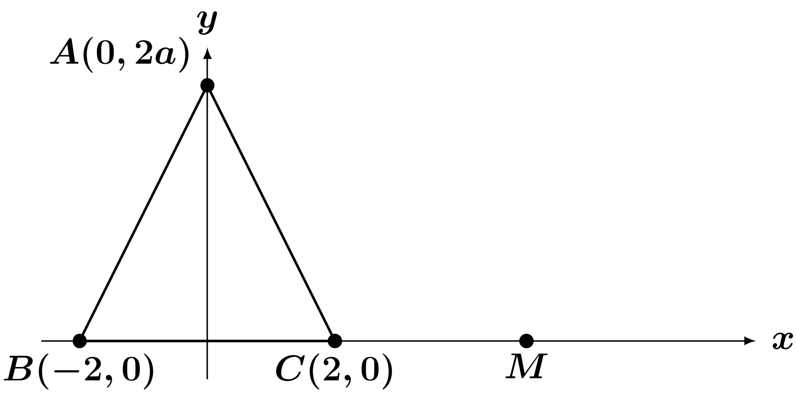
$$m_{BD} = \frac{\frac{h^2+1}{4h} - 0}{\frac{h^2-3}{4} - (-1)} = \frac{\frac{h^2+1}{4h}}{\frac{h^2+1}{4}} = \frac{1}{h}.$$

Since

$$m_{AC} \cdot m_{BD} = (-h)\left(\frac{1}{h}\right) = -1,$$

then the lines $AC$ and $BD$ are perpendicular, as we wanted to prove.

**Solution 5 by Rob Downes, Basic Academy of International Studies, Henderson, NV, USA.** Without loss of generality, we place $\triangle ABC$ on the coordinate plane with coordinates as shown




Scheme for solving problem E-145.

and $M$ on line $BC$ such that $MB = MA$ as illustrated in the previous Figure.

To find $D$, we find the intersection of the perpendicular bisectors of $AB$ and $MB$. The slope of $AB$ is $a$ and the midpoint of $AB$ is $(-1, a)$. Hence the equation of the perpendicular bisector of $AB$ is:

$$y - a = -\frac{1}{a}(x + 1). \tag{1}$$

Since $M$ is the vertex angle of isosceles $\triangle AMB$, its coordinates are found by finding the x-intercept of the above equation (1). That is, $M(a^2 - 1, 0)$. Hence, the midpoint of $MB$ is $(\frac{a^2-3}{2}, 0)$ and the perpendicular bisector has equation:

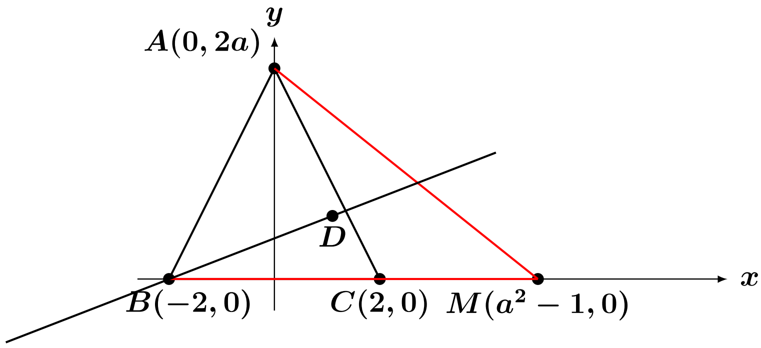
$$x = \frac{a^2 - 3}{2}. \tag{2}$$

Substituting equation (2) into equation (1), gives the circumcenter $D(\frac{a^2-3}{2}, \frac{a^2+1}{2a})$. This is shown in Figure 2.

Lastly, we that the slope of $AC = -a$, while the slope of $BD$ is

$$m_{BD} = \frac{\frac{a^2+1}{2a} - 0}{\frac{a^2-3}{2} + 2} = \frac{1}{a}.$$

Since the slopes are negative reciprocals, $BD \perp AC$, as asserted.




Scheme for solving problem E-145.

**Solution 6 by Andrea Fanchini, Cantù, Italy.** We use barycentric coordinates with reference to the triangle $ABC$.
In this case we have $AB = AC$ therefore

$$b = c, \quad S_B = S_C = \frac{a^2}{2}, \quad S_A = b^2 - \frac{a^2}{2}$$

Now the line that passing from the midpoint $M_{AB}$ and perpendicular to $AB$ is

$$M_{AB}AB_{\infty\perp} : b^2x - b^2y + (a^2 - b^2)z = 0$$

this line intersect $BC$ at the point $M$

$$M = BC \cap M_{AB}AB_{\infty\perp} = \big(0 : a^2 - b^2 : b^2\big)$$

The circumcircle of $\triangle AMB$ is

$$a^2yz + b^2zx + c^2xy - (x + y + z)(a^2 - b^2)z = 0$$

therefore the circumcenter $D$ is

$$D\big(a^2b^2 : 4a^2b^2 - a^4 - 2b^4 : 2b^4 - a^2b^2\big)$$

and the line $BD$

$$BD : S_Ax - S_Bz = 0$$

with infinite point $BD_\infty(S_B : -b^2 : S_A)$ that is the infinite perpendicular point of $AC$.

**Solution 7 by Ioan Viorel Codreanu, Satulung, Maramures, Romania.** Throughout this proof we assume that points B,M and C appear in that order, and that ray BD lies in the interior of angle $\angle CBA$. Otherwise, the proof follows analogously with the necessary modifications. Let $M \in (BC)$ and $T$ be the intersection of $BD$ and $AC$. Let $\angle ABC = u$.

In the isosceles triangles $ABC$ and $ABM$, we have:

$$\angle ABC = \angle ACB = \angle ABM = \angle MAB = u.$$

From the sum of angles in $\triangle ABC$:

$$\angle ABC + \angle ACB + \angle BAC = 180^\circ,$$

from where:

$$\angle BAC = 180^\circ - 2u.$$

We have:

$$\angle MAC = \angle BAC - \angle MAB = (180^\circ - 2u) - u = 180^\circ - 3u.$$

Using the exterior angle theorem for $\triangle AMC$:

$$\angle AMB = \angle MAC + \angle MCA = (180^\circ - 3u) + u,$$

from where:

$$\angle AMB = 180^\circ - 2u.$$

Let $DP \perp AB$ with $P \in AB$. Since $D$ is the circumcenter of $\triangle AMB$, the central angle $\angle BDA$ is related to the inscribed angle $\angle AMB$. Specifically:

$$\angle PDB = \frac{1}{2}\angle BDA = \angle AMB = 180^\circ - 2u.$$

In the right triangle $PBD$:

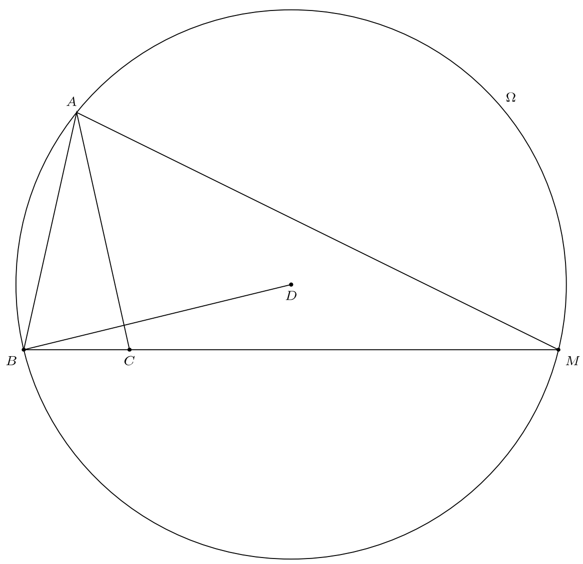
$$\angle PBT = \angle PBD = 90^\circ - \angle PDB = 90^\circ - (180^\circ - 2u) = 2u - 90^\circ.$$

Finally, in $\triangle ATB$:

$$\begin{aligned}\angle ATB &= 180^\circ - \angle ABT - \angle BAT\\ &= 180^\circ - \angle PBT - \angle BAC\\ &= 180^\circ - (2u - 90^\circ) - (180^\circ - 2u)\\ &= 180^\circ - 2u + 90^\circ - 180^\circ + 2u = 90^\circ.\end{aligned}$$

Therefore, $BD \perp AC$.

**Solution 8 by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain.** With the angle at $B$ common to isosceles triangles $ABC$ and $MBA$, the triangles are similar.




Scheme for solving problem E-145-sol-8.

From the proportional sides we get

$$\frac{BA}{BM} = \frac{BC}{BA} \quad \text{and} \quad BA^2 = BC \cdot BM \quad (1)$$

Let $\Omega$ denote the circumcircle of $\triangle ABM$. The line through $B$ and $D$, as a diameter of $\Omega$, is orthogonal to $\Omega$. Orthogonality invites inversion, so invert in circle with center at $B$ and radius $\overline{BA}$. The inverse of the line through $B$ and $D$ is $BD$ itself.

In turn, from (1), it follows that point $A$ invert into itself and the inverse of $M$ is $C$. Hence $\Omega$ becomes the straight line through $A$ and $C$.

Since inversion is a conformal transformation, it follows that the lines $BD$ and $AC$ are orthogonal. Thus, $BD \perp AC$.

***E–146.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $a, n$ be positive integers. Find the Least Common Multiple of the numbers $A = a^{n-1}(a+1)^{n+2}+1$ and $B = a^{n+1}(a+1)^{n+1}+a-1$.

**Solution 1 by S. C. Dutta Roy, New Delhi, India.** Let us write

$$A_n = a^{n-1}(a+1)^{n-1}(a+1)^3+1,\ B_n = a^{n-1}(a+1)^{n-1}a^2(a+1)^2+a-1. \quad (1)$$

Let $c = a(a+1)$; then the above expressions become

$$A_n = c^{n-1}(a^3+3a^2+3a+1)+1, \qquad B_n = c^{n-1}(a^4+2a^3+a^2)+a-1. \quad (2)$$

We shall prove that the GCD of $A_n$ and $B_n$ is $1$, by the method of induction. For this, consider the base case $n = 1$. Then

$$A_1 = a^3 + 3a^2 + 3a + 2, \qquad B_1 = a^4 + 2a^3 + a^2 + a - 1. \quad (3)$$

Now perform repeated long division (equivalent to continued fraction expansion) of $B_1$ by $A_1$. The result is as follows:

$$\begin{array}{r|l}
a^3+3a^2+3a+2 \,\big)\; a^4+2a^3+a^2+a-1 & (a-1) \\
a^4+3a^3+3a^2+2a & \\
\hline
-a^3-2a^2-a-1 & \\
-a^3-3a^2-3a-2 & \\
\hline
a^2+2a+1 & (a+1) \\
a^3+2a^2+a & \\
\hline
a^2+2a+2 & \\
a^2+2a+1 & \\
\hline
1 & (-a^2-2a-1) \\
a^2+2a+1 & \\
\hline
0 &
\end{array}$$

Since the final nonzero remainder is 1, the GCD is 1. Hence the base case is established. Assume now that the GCD is 1 for some $n = k$, i.e.

$$A_k = c^{k-1}(a^3+3a^2+3a+1)+1, \qquad B_k = c^{k-1}(a^4+2a^3+a^2)+a-1. \tag{4}$$

Then for $n = k + 1$, we have

$$A_{k+1} = c^{k}(a^3+3a^2+3a+1)+1, \qquad B_{k+1} = c^{k}(a^4+2a^3+a^2)+a-1. \tag{5}$$

Put $k + 1 = r$; then (5) becomes

$$A_r = c^{r-1}(a^3+3a^2+3a+1)+1, \qquad B_r = c^{r-1}(a^4+2a^3+a^2)+a-1. \tag{6}$$

This is the same as (4) with $k$ replaced by $r$. Hence it is proved that the GCD of $A_n$ and $B_n$ given by (1) is 1.

Since LCD · GCD $= AB$, consequently, the desired LCD is

$$\text{LCD} = AB = \left[a^{n-1}(a+1)^{n+2}+1\right]\left[a^{n+1}(a+1)^{n+1}+a-1\right].$$

**Solution 2 by Rovsen Pirguliyev, Azerbaijan, Arkady Alt, San Jose, California, USA Ioan Viorel Codreanu, Satulung, Maramures, Romania and the proposer (same solution).** Let us denote by $(A, B)$ and $[A, B]$ the GCD and the LCM of $A$ and $B$ respectively. If $d = (A, B)$, then $d|a^{n-1}(a+1)^{n+2}+1$ and $d|a^{n+1}(a+1)^{n+1}+a-1$. Hence, we have that $d|a^2[a^{n-1}(a+1)^{n+2}+1]$ and $d|(a+1)[a^{n+1}(a+1)^{n+1}+a-1]$, or equivalently, $d|a^{n+1}(a+1)^{n+2}+a^2$ and $d|a^{n+1}(a+1)^{n+2}+a^2-1$. Thus, $d$ divides their difference. That is, $d|1$. Therefore, $(A, B) = 1$ and taking into account that $(A, B)[A, B] = AB$, as it is well-known, we get

$$[A, B] = AB = \left[a^{n-1}(a+1)^{n+2}+1\right]\left[a^{n+1}(a+1)^{n+1}+a-1\right],$$

and we are done.

**Solution 3 by Daniel Văcaru, National Economic College „Maria Teiuleanu", Pitești, Romania and Michel Bataille, Rouen, France (same solution).** One has $a^2 \cdot A = a^{n+1}(a+1)^{n+2} + a^2$ and $(a+1) \cdot B = a^{n+1} \cdot (a+1)^{n+2} + a^2 - 1$. Since

$$a^2 \cdot A - (a+1) \cdot B$$

$$= \left[a^{n+1}(a+1)^{n+2}+a^2\right]-\left[a^{n+1}\cdot(a+1)^{n+2}+a^2-1\right]=1,$$

then it follows that g.c.d$(A,B)=1$ and therefore l.c.m.$(A,B)=AB$.

**Also solved by** *José Gibergans-Báguena, BarcelonaTech, Terrassa,Spain*

***E–147.*** *Proposed by Titu Zvonaru, Comănești, Romania (†) and Neculai Stanciu, Buzău, Romania.* Let $a,b,c$ be positive real numbers. Prove that it holds:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\sum_{cyc}\sqrt{a^2-ab+b^2}.$$

**Solution 1 by Andrea Fanchini, Cantù, Italy.** At the $RHS$ we use Cauchy-Schwarz in the form

$$(1^2+1^2+1^2)\left((\sqrt{X})^2+(\sqrt{Y})^2+(\sqrt{Z})^2\right)\geq\left(\sqrt{X}+\sqrt{Y}+\sqrt{Z}\right)^2$$

so we have

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\sqrt{3[(a^2-ab+b^2)+(b^2-bc+c^2)+(c^2-ca+a^2)]}$$

$$=\sqrt{6(a^2+b^2+c^2)-3(ab+bc+ca)}$$

apply $a^2+b^2+c^2\geq ab+bc+ca$ at the $RHS$

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq\sqrt{3(a^2+b^2+c^2)}$$

At the $LHS$ we use Cauchy-Schwarz in the form

$$\left[\left(\frac{a}{\sqrt{b}}\right)^2+\left(\frac{b}{\sqrt{c}}\right)^2+\left(\frac{c}{\sqrt{a}}\right)^2\right]\left[(a\sqrt{b})^2+(b\sqrt{c})^2+(c\sqrt{a})^2\right]$$

$$\geq\left(a^2+b^2+c^2\right)^2$$

therefore

$$\frac{(a^2+b^2+c^2)^2}{a^2b+b^2c+c^2a}\geq\sqrt{3(a^2+b^2+c^2)}$$

squaring both sides

$$\left(a^2+b^2+c^2\right)^3 \geq 3\left(a^2b+b^2c+c^2a\right)^2$$

At the $RHS$ we use Cauchy-Schwarz in the form

$$\left(a^2+b^2+c^2\right)\left(a^2b^2+b^2c^2+c^2a^2\right) \geq \left(a^2b+b^2c+c^2a\right)^2$$

we obtain

$$\left(a^2+b^2+c^2\right)^3 \geq 3\left(a^2+b^2+c^2\right)\left(a^2b^2+b^2c^2+c^2a^2\right)$$
$$\Rightarrow \left(a^2+b^2+c^2\right)^2 \geq 3\left(a^2b^2+b^2c^2+c^2a^2\right)$$

developing $LHS$

$$a^4+b^4+c^4+2\left(a^2b^2+b^2c^2+c^2a^2\right) \geq 3\left(a^2b^2+b^2c^2+c^2a^2\right)$$
$$\Rightarrow a^4+b^4+c^4 \geq a^2b^2+b^2c^2+c^2a^2.$$

Equality holds if and only if $a=b=c$.

**Solution 2 by Arkady Alt, San Jose, California, USA.** By the Cauchy-Schwarz inequality,

$$\sum_{\text{cyc}} \frac{a^2}{b} = \sum_{\text{cyc}} \left(\frac{a^2}{b} - a + b\right) = \sum_{\text{cyc}} \frac{a^2-ab+b^2}{b}$$
$$\geq \frac{\left(\sum_{\text{cyc}} \sqrt{a^2-ab+b^2}\right)^2}{\sum_{\text{cyc}} b}.$$

Since $a^2-ab+b^2 \geq \dfrac{(a+b)^2}{4}$, it follows that

$$\sum_{\text{cyc}} \sqrt{a^2-ab+b^2} \geq \sum_{\text{cyc}} \frac{a+b}{2} = \sum_{\text{cyc}} a.$$

Given that $\sum\limits_{\text{cyc}} b = \sum\limits_{\text{cyc}} a$, we can substitute this back into our original inequality:

$$\frac{\left(\sum_{\text{cyc}} \sqrt{a^2-ab+b^2}\right)^2}{\sum_{\text{cyc}} a} \geq \frac{\left(\sum_{\text{cyc}} \sqrt{a^2-ab+b^2}\right)\left(\sum_{\text{cyc}} a\right)}{\sum_{\text{cyc}} a}$$
$$= \sum_{\text{cyc}} \sqrt{a^2-ab+b^2}.$$

Equality holds if and only if $a=b=c$.

**Solution 3 by Cao Minh Quang, Nguyen Binh Khiem high school for the Gifted, Vinh Long, Vietnam.** We have a simple algebraic identity:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}=\frac{a^2-ab+b^2}{b}+\frac{b^2-bc+c^2}{c}+\frac{c^2-ca+a^2}{a}.$$

Applying the AM-GM inequality, we obtain

$$\frac{a^2-ab+b^2}{b}+b\geq 2\sqrt{a^2-ab+b^2}.$$

Similarly, we have

$$\frac{b^2-bc+c^2}{c}+c\geq 2\sqrt{b^2-bc+c^2},$$

and

$$\frac{c^2-ca+a^2}{a}+a\geq 2\sqrt{c^2-ca+a^2}.$$

Summing these three above inequalities, we get

$$\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)+(a+b+c)\geq 2\sum_{\text{cyc}}\sqrt{a^2-ab+b^2}.$$

On the other hand, we note that

$$a^2-ab+b^2=\frac{(a+b)^2}{4}+\frac{3(a-b)^2}{4}\geq\frac{(a+b)^2}{4},$$

which implies

$$\sqrt{a^2-ab+b^2}\geq\frac{a+b}{2}.$$

Therefore,

$$\sum_{\text{cyc}}\sqrt{a^2-ab+b^2}\geq\frac{(a+b)+(b+c)+(c+a)}{2}=a+b+c.$$

From this, we obtain

$$\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)+(a+b+c)\geq 2\sum_{\text{cyc}}\sqrt{a^2-ab+b^2}$$

$$\geq \sum_{\text{cyc}} \sqrt{a^2 - ab + b^2} + (a + b + c).$$

Hence,

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \sum_{\text{cyc}} \sqrt{a^2 - ab + b^2},$$

which completes the proof. Equality holds if and only if $a = b = c$

**Solution 4 by Michel Bataille, Rouen, France.** Since the function $x \mapsto \sqrt{x}$ is concave on $(0, \infty)$, Jensen's inequality gives

$$\sum_{\text{cyc}} \sqrt{a^2 - ab + b^2} = \sum_{\text{cyc}} b\sqrt{\frac{a^2}{b^2} - \frac{a}{b} + 1} \leq (b+c+a)\sqrt{\frac{\sum_{\text{cyc}} b\left(\frac{a^2}{b^2} - \frac{a}{b} + 1\right)}{b + c + a}},$$

that is,

$$\sum_{\text{cyc}} \sqrt{a^2 - ab + b^2} \leq \sqrt{b + c + a}\sqrt{\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}}. \qquad (1)$$

Now, from the Cauchy-Schwarz inequality, we have

$$\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right)(b + c + a) \geq (a + b + c)^2$$

hence $a + b + c \leq \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$. Returning to (1), we obtain

$$\sum_{\text{cyc}} \sqrt{a^2 - ab + b^2} \leq \sqrt{\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}}\sqrt{\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}} = \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}$$

and we are done.

**Solution 5 by Albert Stadler, Herrliberg, Switzerland.** By the Cauchy–Schwarz inequality,

$$\sum_{cyc} \sqrt{a^2 - ab + b^2} \leq \sqrt{3\sum_{cyc} (a^2 - ab + b^2)}.$$

Therefore, it suffices to prove that

$$\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right)^2 \geq 3\left(2a^2 + 2b^2 + 2c^2 - ab - bc - ca\right).$$

Clearing denominators, one finds that this inequality is equivalent to

$$p(a,b,c) := \sum_{cyc} a^2b^6 + 2\sum_{cyc} a^4b^3c + 3\sum_{cyc} a^3b^3c^2 - 6\sum_{cyc} a^4b^2c^2 \geq 0.$$

By cyclic symmetry, we may assume either $a \geq b \geq c$ or $a \geq c \geq b$. In the first case, there exist positive numbers $u, v, w$ such that

$$a = u + v + w,\ b = u + v,\ c = u.$$

Substituting into $p(a, b, c)$ and expanding (with the assistance of a computer algebra system), we obtain a polynomial in $u, v, w$ all of whose coefficients are positive. Hence $p(a, b, c) \geq 0$.

In the second case, there exist positive numbers $u, v, w$ such that

$$a = u + v + w,\ c = u + v,\ b = u.$$

Substituting into $p(a, b, c)$ and expanding (again with the assistance of a computer algebra system), we obtain a polynomial in $u, v, w$ with all coefficients positive. Thus $p(a, b, c) \geq 0$. This completes the proof.

**Also solved by** *Rovsen Pirguliyev, Azerbaijan and Nicusor Zlota, "Traian Vuia" Technical College Focsani, Romania.*

***E–148.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Óscar Rivero Salgado, Universidad de Santiago de Compostela, Spain.* Find all positive integers that are divisible by 2431 and have exactly 2431 different positive divisors.

**Solution 1 by Cao Minh Quang, Nguyen Binh Khiem high school for the Gifted, Vinh Long, Vietnam.** Write the prime factorization

$$n = \prod_{i=1}^{k} p_i^{\alpha_i} \qquad (p_i \text{ distinct primes},\ \alpha_i \geq 1).$$

Then the number of positive divisors of $n$ is

$$\tau(n) = \prod_{i=1}^{k} (\alpha_i + 1).$$

We are given $\tau(n) = 2431$. Factor

$$2431 = 11 \cdot 13 \cdot 17.$$

Also, $2431 \mid n$, and since $2431 = 11 \cdot 13 \cdot 17$ is a product of three distinct primes, the integer $n$ must be divisible by each of $11, 13, 17$. Hence $n$ has at least three distinct prime divisors, i.e. $k \geq 3$.

On the other hand, $\tau(n) = \prod(\alpha_i + 1) = 11 \cdot 13 \cdot 17$ has exactly three prime factors, so there can be at most three factors $(\alpha_i + 1) > 1$, which forces $k \leq 3$. Therefore $k = 3$ and

$$(\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1) = 11 \cdot 13 \cdot 17.$$

Thus, we obtain

$$\{\alpha_1+1, \alpha_2+1, \alpha_3+1\} = \{11, 13, 17\} \Leftrightarrow \{\alpha_1, \alpha_2, \alpha_3\} = \{10, 12, 16\}.$$

Since $n$ is divisible by $11, 13, 17$ and has exactly three prime divisors, these primes must be precisely $11, 13, 17$. Hence all solutions are obtained by assigning the exponents $10, 12, 16$ to $11, 13, 17$ in any order:

$$n = 11^a 13^b 17^c, \qquad \{a, b, c\} = \{10, 12, 16\}.$$

Equivalently, there are $3! = 6$ solutions:

$$11^{16}13^{12}17^{10}, \quad 11^{16}13^{10}17^{12}, \quad 11^{12}13^{16}17^{10},$$
$$11^{12}13^{10}17^{16}, \quad 11^{10}13^{16}17^{12}, \quad 11^{10}13^{12}17^{16}.$$

Each has $\tau(n) = (a+1)(b+1)(c+1) = 17 \cdot 13 \cdot 11 = 2431$ and is divisible by $2431$.

**Solution 2 by Albert Stadler, Herrliberg, Switzerland, Rovsen Pirguliyev, Azerbaijan and the proposers (same solution).** Let $n = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k}$ be such a number, where the $p_i$'s are primes $(p_1 < p_2 < \ldots < p_k)$. Note that $2341 = 11 \cdot 13 \cdot 17$ and $2341$ is a factor of $n$, so we may assume $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, p_6 = 13, p_7 = 17, \cdots,$ $k \geq 7$ and $n_5, n_6, n_7 \geq 1$. Since the number of divisors of $n$ is $(n_1+1)(n_2+1)\dots(n_k+1)$ then we have

$$\begin{aligned}(n_1+1)(n_2+1)\dots(n_k+1) &= 2431\\ (n_1+1)(n_2+1)\dots(n_k+1) &= 11\cdot 13\cdot 17\end{aligned}$$

Hence $k=7$, $n_1=n_2=n_3=n_4=0$ and $\{n_5,n_6,n_7\}=\{11-1,13-1,17-1\}=\{10,12,16\}$. Thus, the required numbers are:

$$11^{10}\cdot 13^{12}\cdot 17^{16}, 11^{10}\cdot 13^{16}\cdot 17^{12}, 11^{12}\cdot 13^{10}\cdot 17^{16},$$
$$11^{12}\cdot 13^{16}\cdot 17^{10}, 11^{16}\cdot 13^{10}\cdot 17^{12}, 11^{16}\cdot 13^{12}\cdot 17^{10}$$

**Solution 3 by Daniel Văcaru, National Economic College „Maria Teiuleanu", Pitești, Romania.** Since $2431=11\cdot 13\cdot 17$, then it follows that the required numbers are of the form

$$N=11^{k_1}\cdot 13^{k_2}\cdot 17^{k_3}\cdot p_1^{a_1}\cdot p_2^{a_2}\cdot\dots\cdot p_n^{a_n}$$

The number of divisors of $11^{k_1}\cdot 13^{k_2}\cdot 17^{k_3}\cdot p_1^{a_1}\cdot p_2^{a_2}\cdot\dots\cdot p_n^{a_n}$ is

$$(k_1+1)(k_2+1)(k_3+1)(a_1+1)\cdot\dots\cdot(a_n+1).$$

Since $2431=11\cdot 13\cdot 17|N$, then $k_1\geq 1, k_2\geq 1, k_3\geq 1$. Thus

$$(k_1+1)(k_2+1)(k_3+1)(a_1+1)\cdot\dots\cdot(a_n+1)=11\cdot 13\cdot 17$$

and one obtain that $N=11^{k_1}\cdot 13^{k_2}\cdot 17^{k_3}$. Thus, from

$$(k_1+1)(k_2+1)(k_3+1)=11\cdot 13\cdot 17,$$

one obtains

$$N\in\left\{11^{10}\cdot 13^{12}\cdot 17^{16}, 11^{10}\cdot 13^{16}\cdot 17^{12}, 11^{12}\cdot 13^{10}\cdot 17^{16}\right\}\cup$$
$$\cup\left\{11^{12}\cdot 13^{16}\cdot 17^{10}, 11^{16}\cdot 13^{10}\cdot 17^{12}, 11^{16}\cdot 13^{12}\cdot 17^{10}\right\}.$$

**Solution 4 by Michel Bataille, Rouen, France.** Since $2431=11\times 13\times 17$, a suitable positive integer $n$ is of the form $11^r\times 13^s\times 17^t\times m$ for some positive integers $r,s,t,m$ with $m$ coprime with $11, 13, 17$. Denoting by $\tau(u)$ the number of positive divisors of the positive integer $u$, we must also have

$$(r+1)(s+1)(t+1)\tau(m)=11\times 13\times 17. \qquad (1)$$

(Recall that $\tau(k\ell) = \tau(k)\tau(\ell)$ if $k, \ell$ are coprime and $\tau(p^\alpha) = \alpha + 1$ if $\alpha$ is a positve integer and $p$ is a prime.)
Since $r+1, s+1, t+1$ are greater than or equal to 2, each of them has a prime divisor and from (1), $(r+1, s+1, t+1)$ must be a permutation of $(11, 13, 17)$ and $m = 1$ (otherwise $\tau(m) \geq 2$ and $\tau(m)$ should have a prime divisor different from $11, 13, 17$, which is impossible). Thus, $n = 11^r \times 13^s \times 17^t$ where $(r, s, t)$ is a permutation of $(10, 12, 16)$.
Conversely, any such integer is divisible by $11 \times 13 \times 17 = 2431$ and has $(10+1) \times (12+1) \times (16+1) = 2431$ divisors.
Thus, the solutions are the six integers $11^r \times 13^s \times 17^t$ where $(r, s, t)$ is a permutation of $(10, 12, 16)$.

**Solution 5 by Ioan Viorel Codreanu, Satulung, Maramures, Romania.** If $2431 \mid n$, then there exists a natural number $m$ such that $n = 2431m$, or $n = 11 \cdot 13 \cdot 17 \cdot m$. Since $\tau(n) = 2431$ is an odd number, it follows that $n$ is a perfect square.

Then $n$ can be written in the form:

$$n = 11^{2k_1+2} \cdot 13^{2k_2+2} \cdot 17^{2k_3+2} \cdot p_1^{2s_1} \cdot p_2^{2s_2} \cdots p_t^{2s_t}$$

where $p_1, p_2, \ldots, p_t$ are prime numbers different from 11, 13, and 17, and $k_1, k_2, k_3$ are natural numbers.

We have:

$$\tau(n) = (2k_1+3)(2k_2+3)(2k_3+3)(2s_1+1)(2s_2+1)\cdots(2s_t+1)$$

It follows that:

$$(2k_1+3)(2k_2+3)(2k_3+3)(2s_1+1)(2s_2+1)\cdots(2s_t+1) = 11 \cdot 13 \cdot 17$$

We notice that $k_1, k_2, k_3 \neq 0$; otherwise, if any were 0, the factor $(2(0)+3) = 3$ would have to divide $11 \cdot 13 \cdot 17$, which is false. [cite: 44] We deduce that $t = 0$ (there are no other prime factors $p_i$) and the set of factors $\{2k_1+3, 2k_2+3, 2k_3+3\}$ must be a permutation of $\{11, 13, 17\}$.

Solving for the exponents:

- $2k_i + 3 = 11 \Rightarrow 2k_i + 2 = 10$
- $2k_i + 3 = 13 \Rightarrow 2k_i + 2 = 12$
- $2k_i + 3 = 17 \Rightarrow 2k_i + 2 = 16$

The possible values for $n$ are the permutations of the exponents 10, 12, and 16 assigned to the primes 11, 13, and 17:

$$n \in \{11^{10} \cdot 13^{12} \cdot 17^{16}, 11^{10} \cdot 13^{16} \cdot 17^{12}, 11^{12} \cdot 13^{10} \cdot 17^{16},$$

$$11^{12} \cdot 13^{16} \cdot 17^{10}, 11^{16} \cdot 13^{10} \cdot 17^{12}, 11^{16} \cdot 13^{12} \cdot 17^{10}\}$$

**Also solved by** *José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain.*

# *Easy–Medium Problems*

***EM–143.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* For all integers $n \geq 1$, prove that

$$\sum_{k=1}^{n} \binom{n}{k} F_k^2 \geq \frac{F_{2n}^2}{2^n - 1},$$

where $F_n$ is the $n$-th Fibonacci number defined by the recurrence: $F_1 = F_2 = 1$, and for $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$.

**Solution 1 by Michel Bataille, Rouen, France and Arkady Alt, San Jose, California, USA.** Remarking that $2^n - 1 = \sum_{k=1}^{n} \binom{n}{k}$, we rewrite the inequality as

$$\left(\sum_{k=1}^{n} \binom{n}{k} F_k^2\right)\left(\sum_{k=1}^{n} \binom{n}{k}\right) \geq F_{2n}^2. \tag{1}$$

From the Cauchy-Schwarz inequality, we have

$$\left(\sum_{k=1}^{n} \binom{n}{k} F_k^2\right)\left(\sum_{k=1}^{n} \binom{n}{k}\right) \geq \left(\sum_{k=1}^{n} \binom{n}{k} F_k\right)^2$$

and (1) follows from $\sum_{k=1}^{n} \binom{n}{k} F_k = F_{2n}$. Indeed, with $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$, we have $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$ (a well-known result) and therefore

$$\begin{aligned}
\sum_{k=1}^{n} \binom{n}{k} F_k &= \frac{1}{\sqrt{5}}\left(\sum_{k=1}^{n} \binom{n}{k} \alpha^k - \sum_{k=1}^{n} \binom{n}{k} \beta^k\right) \\
&= \frac{1}{\sqrt{5}}((1+\alpha)^n - 1 - (1+\beta)^n + 1) \\
&= \frac{1}{\sqrt{5}}(\alpha^{2n} - \beta^{2n}) \quad (\text{since } 1+\alpha = \alpha^2,\ 1+\beta = \beta^2) \\
&= F_{2n}.
\end{aligned}$$

**Solution 2 by Albert Stadler, Herrliberg, Switzerland.** Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. By Binet's formula,

$$F_k = \frac{\alpha^k - \beta^k}{\sqrt{5}}.$$

Hence, noting that the term for $k=0$ is zero, we can write:

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}F_k^2 &= \frac{1}{5}\sum_{k=0}^{n}\binom{n}{k}(\alpha^k-\beta^k)^2\\ &= \frac{1}{5}\sum_{k=0}^{n}\binom{n}{k}\big(\alpha^{2k}+\beta^{2k}-2(-1)^k\big).\end{aligned}$$

Evaluating these sums using the binomial theorem gives:

$$\begin{aligned}\frac{1}{5}\big((1+\alpha^2)^n+(1+\beta^2)^n-2(1-1)^n\big) &= \frac{1}{5}\big((1+\alpha^2)^n+(1+\beta^2)^n\big)\\ &= \frac{1}{5}\Big((\sqrt{5}\,\alpha)^n+(\sqrt{5}/\alpha)^n\Big)\\ &= 5^{\frac{n}{2}-1}(\alpha^n+\alpha^{-n}).\end{aligned}$$

On the other hand,

$$F_{2n}^2=\frac{1}{5}(\alpha^{4n}+\alpha^{-4n}-2).$$

Thus, it suffices to prove that

$$5^{n/2}(\alpha^n+\alpha^{-n})(2^n-1)\geq \alpha^{4n}+\alpha^{-4n}-2.$$

Equality holds for $n=1$ and $n=2$. For $n\geq 3$, we use the estimates

$$2^n-1\geq \frac{7}{8}\cdot 2^n \quad \text{and} \quad \alpha^{-4n}<2.$$

Therefore,

$$\begin{aligned}5^{n/2}(\alpha^n+\alpha^{-n})(2^n-1) &\geq \frac{7}{8}(2\sqrt{5}\,\alpha)^n\\ &\geq \alpha^{4n}\\ &\geq \alpha^{4n}+\alpha^{-4n}-2,\end{aligned}$$

since $\dfrac{2\sqrt{5}\,\alpha}{\alpha^4}\approx 1.054>1$. This completes the proof.

**Solution 3 by the proposer.** We have $\sum\limits_{k=1}^{n}\dbinom{n}{k}=2^n-1$, as it is well-know. On the other hand $\sum\limits_{k=1}^{n}\dbinom{n}{k}F_k=F_{2n}$. Indeed, for any integer $k$ we have $F_k = \dfrac{\alpha^k - \beta^k}{\alpha - \beta}$ where $\alpha, \beta$ are the roots of $x^2 - x - 1 = 0$. Then,

$$\begin{aligned}\sum_{k=1}^{n}\binom{n}{k}F_k &= \sum_{k=1}^{n}\binom{n}{k}\left(\frac{\alpha^k-\beta^k}{\alpha-\beta}\right)\\ &= \frac{1}{\alpha-\beta}\left[\sum_{k=1}^{n}\binom{n}{k}\alpha^k - \sum_{k=1}^{n}\binom{n}{k}\beta^k\right]\\ = \frac{(1+\alpha)^n-(1+\beta)^n}{\alpha-\beta} &= \frac{\alpha^{2n}-\beta^{2n}}{\alpha-\beta} = F_{2n}\end{aligned}$$

Taking the square root of both sides of the inequality claimed, yields

$$\left(\sum_{k=1}^{n}\binom{n}{k}F_k^2\right)^{1/2} \geq \frac{F_{2n}}{(2^n-1)^{1/2}}$$

or equivalently,

$$\sum_{k=1}^{n}\binom{n}{k}F_k \leq \sqrt{\left(\sum_{k=1}^{n}\binom{n}{k}\right)\left(\sum_{k=1}^{n}\binom{n}{k}F_k^2\right)}.$$

To prove the last inequality, we apply Cauchy's inequality to the vectors $\vec{u} = \left(\sqrt{\binom{n}{1}}, \sqrt{\binom{n}{2}}, \ldots, \sqrt{\binom{n}{n}}\right)$ and $\vec{v} = \left(\sqrt{\binom{n}{1}}F_1, \sqrt{\binom{n}{2}}F_2, \ldots, \sqrt{\binom{n}{n}}F_n\right)$ and we obtain the inequality claimed. Equality holds when $n = 1, 2$, and the prove is complete.

**Solution 4 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain and Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania (same solution).** Since $\sum_{k=1}^{n}\binom{n}{k} = 2^n - 1$, and $\sum_{k=1}^{n}\binom{n}{k}F_k = F_{2n}$, the proposed inequality may be written as

$$\sum_{k=1}^{n}\binom{n}{k}\sum_{k=1}^{n}\binom{n}{k}F_k^2 \geq \left(\sum_{k=1}^{n}\binom{n}{k}F_k\right)^2$$

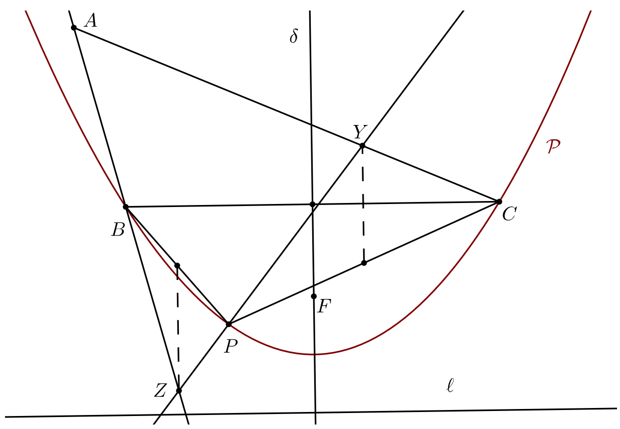
which it is a direct consequence of the Cauchy-Schwarz Inequality, which states that for any real numbers $x_i, y_i$, $(\sum x_i^2)(\sum y_i^2) \geq (\sum x_i y_i)^2$, by setting $x_k = \sqrt{\binom{n}{k}}$ and $y_k = \sqrt{\binom{n}{k}}F_k$.

**Also solved by** *José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain.*

***EM–144.*** *Proposed by Michel Bataille, Rouen, France and Francisco Javier García Capitán, Priego de Córdoba, Spain.* Let $ABC$ be a triangle with $\angle B, \angle C \neq 90°$. For any point $P$ of its plane, call $Y$ the intersection of $CA$ and the line through the midpoint of $PC$ perpendicular to $BC$, and $Z$ the intersection of $AB$ and the line through the midpoint of $PB$ perpendicular to $BC$. Find the locus of $P$ such that $Y, Z$, and $P$ are collinear.

**Solution 1 by the proposers.** We use a system of orthonormal axes such that $B(-1,0)$ and $C(1,0)$. Let $A(a,b)$ with $b \neq 0$ and $a \neq -1, 1$ and let $P(u,v)$.

The equations of $AC$ and of the perpendicular to $BC$ through the midpoint of $PC$ are $bx - (a-1)y = b$ and $x = \frac{u+1}{2}$, respectively. It follows that $Y\left(\frac{u+1}{2}, \frac{b(u-1)}{2(a-1)}\right)$. Similarly, we obtain $Z\left(\frac{u-1}{2}, \frac{b(u+1)}{2(a+1)}\right)$.




Scheme for solving problem EM-144.

Points $P, Y, Z$ are collinear if and only if $\det(\overrightarrow{PY}, \overrightarrow{PZ}) = 0$, which after some easy calculations and setting $k = \frac{b}{2(a^2-1)}$, yields the condition $v = k(u^2-1)$ on $P$. Thus, the desired locus is the parabola $\mathcal{P}$ with equation $y = k(x^2-1)$. Note that $B, C$ are on $\mathcal{P}$ and that the axis of $\mathcal{P}$ is the perpendicular bisector $\delta$ of $BC$ (the $y$-axis). The equation of $\mathcal{P}$ can be written as

$$x^2 + \left(y - \left(\frac{1}{4k} - k\right)\right)^2 = \left(y + \left(\frac{1}{4k} + k\right)\right)^2$$

showing that the focus of $\mathcal{P}$ is $F\big(0, \frac{1}{4k} - k\big)$ and its directrix $\ell$ is the line with equation $y = -\frac{1}{4k} - k$.

**Solution 2 by Andrea Fanchini, Cantù, Italy.** We use barycentric coordinates with reference to the triangle $ABC$ and consider a generic point $P(u, v, w)$ with $u + v + w = 1$, then

$$M_{PB}(u, 1 + v, w), \quad M_{PC}(u, v, 1 + w), \quad BC_{\infty\perp}(-a^2 : S_C : S_B)$$

we have

$$M_{PB}BC_{\infty\perp} : [(1 + v)S_B - wS_C]x - \big[uS_B + a^2w\big]y$$
$$+\big[uS_C + (1 + v)a^2\big]z = 0$$
$$M_{PC}BC_{\infty\perp} : [vS_B - (1 + w)S_C]x - \big[uS_B + (1 + w)a^2\big]y$$
$$+\big[uS_C + a^2v\big]z = 0$$

therefore

$$Z = M_{PB}BC_{\infty\perp} \cap AB = \big[uS_B + a^2w : (1 + v)S_B - wS_C : 0\big]$$

$$Y = M_{PC}BC_{\infty\perp} \cap CA = \big[uS_C + a^2v : 0 : (1 + w)S_C - vS_B\big]$$

$Y, Z$, and $P$ are collinear if

$$\begin{vmatrix} uS_B + a^2w & (1 + v)S_B - wS_C & 0 \\ uS_C + a^2v & 0 & (1 + w)S_C - vS_B \\ u & v & w \end{vmatrix} = 0$$
$$\implies \quad uS_BS_C - uvS_B^2 - uwS_C^2 - vwa^4 = 0$$

normalizing we have the following conic

$$S_BS_Cx^2 + S_B(S_C - S_B)xy - a^4yz + S_C(S_B - S_C)zx = 0$$

the minors of the determinant are

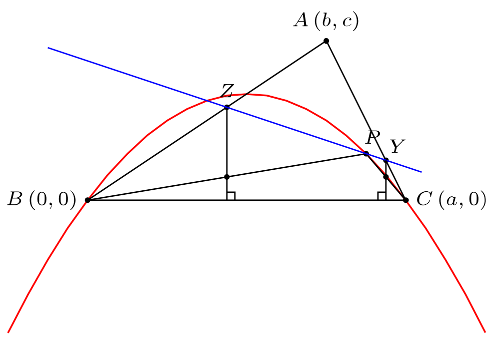
$$U = -\frac{a^8}{4}, \; V = \frac{2S_BS_C^3 - S_B^2S_C^2 - S_C^4}{4}, \; W = \frac{2S_B^3S_C - S_B^2S_C^2 - S_B^4}{4}$$

$$U' = \frac{2a^4S_BS_C + 2S_B^2S_C^2 - S_BS_C^3 - S_B^3S_C}{4}, \quad V' = \frac{a^4S_B^2 - a^4S_BS_C}{4},$$

$$W' = \frac{a^4 S_C^2 - a^4 S_B S_C}{4}$$

the characteristic $K = U + V + W + 2(U' + V' + W') = 0$, therefore is a parabola that passes from $B$ and $C$, furthermore the axis is the altitude of triangle $ABC$ relative to the side $BC$.

**Solution 3 by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain.** We consider a rectangular Cartesian system with the unit of measurement the same along both coordinate axes. Let the $x$-axis lie along the side $BC$, with the origin at $B$. Let $P(x', y')$ be any point on the locus. Let $A$ with coordinates $(b, c)$, and $C$




Scheme for solving problem EM-144.

with coordinates $(a, 0)$, where $b \neq 0$ (since $\angle B \neq 90°$) and $b \neq a$ (since $\angle C \neq 90°$). The coordinates of $Y$ then are $\left(\frac{x'+a}{2}, \frac{c(x'-a)}{2(b-a)}\right)$, and those of $Z$ are $\left(\frac{x'}{2}, \frac{cx'}{2b}\right)$.

The condition of collinearity of $Y$, $Z$, and $P$ can be expressed determinantally

$$\begin{vmatrix} x_Y & y_Y & 1 \\ x_Z & y_Z & 1 \\ x_P & y_P & 1 \end{vmatrix} = 0.$$

Thus, the necessary and sufficient condition that $Y$, $Z$, and $P$ are collinear is

$$\begin{vmatrix} \frac{x'+a}{2} & \frac{c(x'-a)}{2(b-a)} & 1 \\ \frac{x'}{2} & \frac{cx'}{2b} & 1 \\ x' & y' & 1 \end{vmatrix} = 0.$$

Expanding, simplifying and dropping primes, the equation of the required locus is

$$2b(b-a)y = cx^2 - acx,$$

a parabola.

***EM–145.*** *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let $a, b, c$ be positive real numbers. Prove that it holds:

$$(a+b)^2 + (a+b+3c)^2 \geq \frac{169abc}{a+b+7c}.$$

**Solution 1 by Michel Bataille, Rouen, France.** The inequality is homogenous, hence we can suppose that $abc = 1$ and prove that in such case,

$$\big((a+b)^2 + (a+b+3c)^2\big)(a+b+7c) \geq 169.$$

Since $a + b \geq 2\sqrt{ab} = \frac{2}{\sqrt{c}}$, the left-hand side is greater than or equal to

$$\left(\frac{4}{c} + \left(\frac{2}{\sqrt{c}} + 3c\right)^2\right)\left(\frac{2}{\sqrt{c}} + 7c\right)$$

Therefore it suffices to prove that

$$(8 + 12c\sqrt{c} + 9c^3)(2 + 7c\sqrt{c}) \geq 169c\sqrt{c}.$$

With $x = c\sqrt{c}$, the latter writes as $(8+12x+9x^2)(2+7x) \geq 169x$. We are done because for positive $x$

$$(8 + 12x + 9x^2)(2 + 7x) - 169x = (3x - 1)^2(7x + 16) \geq 0.$$

**Solution 2 by José Luis Díaz-Barrero, Barcelona, Spain.** Let $a, b, c > 0$. Set $x = a + b$ and $y = c$. Then the given inequality becomes

$$x^2 + (x + 3y)^2 \geq \frac{169ab\,y}{x + 7y}.$$

Since $a+b = x$ and $a, b > 0$, the AM–GM inequality gives $ab \leq \frac{x^2}{4}$. Hence

$$\frac{169ab\,y}{x + 7y} \leq \frac{169}{4} \cdot \frac{x^2 y}{x + 7y}.$$

Thus, for $x, y > 0$, it suffices to prove the stronger inequality

$$x^2+(x+3y)^2 \geq \frac{169}{4}\cdot\frac{x^2 y}{x + 7y} \iff \big(x^2+(x+3y)^2\big)(x+7y) \geq \frac{169}{4}x^2 y.$$

Expanding and simplifying yields

$$8x^3 - 89x^2 y + 204xy^2 + 252y^3 \geq 0.$$

Since $x, y > 0$, let $t = x/y$. Then the inequality becomes

$$8t^3 - 89t^2 + 204t + 252 = (t - 6)^2(8t + 7) \geq 0.$$

This is clearly true for all $t > 0$, because $8t+7 > 0$ and $(t-6)^2 \geq 0$. Returning to the original variables, we conclude that

$$(a + b)^2 + (a + b + 3c)^2 \geq \frac{169abc}{a + b + 7c}$$

for all $a, b, c > 0$. Equality occurs when $ab = x^2/4$, that is, when $a = b$, and when $t = x/y = 6$, which gives $a = b = 3c$.

**Solution 3 by Cao Minh Quang, Nguyen Binh Khiem high school for the Gifted, Vinh Long, Vietnam.** Set $s = a + b > 0$. By AM–GM we have

$$ab \leq \left(\frac{a + b}{2}\right)^2 = \frac{s^2}{4}.$$

Hence

$$\frac{169abc}{a + b + 7c} \leq \frac{169}{4} \cdot \frac{s^2 c}{s + 7c}.$$

Therefore, it suffices to prove that for all $s, c > 0$,

$$s^2 + (s+3c)^2 \geq \frac{169}{4} \cdot \frac{s^2 c}{s+7c},$$

or

$$4(s+7c)\big(s^2 + (s+3c)^2\big) \geq 169 s^2 c.$$

We expand

$$4(s+7c)\big(2s^2 + 6sc + 9c^2\big) - 169s^2c = 8s^3 - 97s^2c + 192sc^2 + 252c^3.$$

We factor

$$8s^3 - 97s^2c + 192sc^2 + 252c^3 = (s-6c)^2(8s+7c) \geq 0,$$

therefore the desired inequality. Equality holds iff $ab = \dfrac{(a+b)^2}{4}$ (so $a = b$) and $s = 6c$ (so $a + b = 6c$). Thus $a = b = 3c$.

**Solution 4 by Albert Stadler, Herrliberg, Switzerland.** Since $ab \leq \frac{1}{4}(a+b)^2$, let us set $d = a + b$. It is then enough to prove

$$d^2 + (d+3c)^2 \geq \frac{1}{4} d^2 \cdot \frac{169c}{d+7c}.$$

This is a homogeneous inequality in $d$ and $c$. Let $x = d/c$. The inequality becomes

$$x^2 + (x+3)^2 \geq \frac{1}{4} x^2 \cdot \frac{169}{x+7}.$$

This is equivalent to

$$(x-6)^2(8x+7) \geq 0.$$

Since this inequality is clearly true for all $x > 0$, the result follows. Equality holds if and only if $(a, b, c)$ is proportional to $(3, 3, 1)$.

**Solution 5 by Arkady Alt, San Jose, California, USA.** Assume, by homogeneity of the inequality, that $c = 1$. Let

$$p = a + b, \quad q = ab \leq \frac{(a+b)^2}{4}.$$

Then

$$\big((a+b)^2+(a+b+3c)^2\big)(a+b+7c)-169abc$$

becomes

$$\big((a+b)^2+(a+b+3)^2\big)(a+b+7)-169ab=\big(p^2+(p+3)^2\big)(p+7)-169q.$$

Using the bound $q \le \dfrac{p^2}{4}$, we obtain

$$\big(p^2+(p+3)^2\big)(p+7)-169q \ge \big(p^2+(p+3)^2\big)(p+7)-169\cdot\frac{p^2}{4}.$$

A straightforward expansion shows that

$$\big(p^2+(p+3)^2\big)(p+7)-\frac{169}{4}p^2=\frac{1}{4}(8p+7)(p-6)^2 \ge 0.$$

**Solution 6 by Andrea Fanchini, Cantù, Italy.** We denote $S = a+b$ then with AM-GM $\frac{S^2}{4} \ge ab$ therefore we have to prove

$$S^2+(S+3c)^2 \ge \frac{169S^2c}{4(S+7c)}$$

Using Cauchy-Schwarz at LHS

$$\big[S^2+(S+3c)^2\big]\cdot\big[4^2+6^2\big] \ge [4S+6(S+3c)]^2$$

$$\Rightarrow \quad S^2+(S+3c)^2 \ge \frac{(5S+9c)^2}{13}$$

now it remain to prove

$$\frac{(5S+9c)^2}{13} \ge \frac{169S^2c}{4(S+7c)}$$

developing $100S^3-1137S^2c+2844Sc^2+2268c^3 \ge 0$,
$\Rightarrow \ (S-6c)^2(100S+63c) \ge 0$.

**Solution 7 by the proposer.** $(a+b)^2+(a+b+3c)^2 \ge \dfrac{169abc}{a+b+7c} \Leftrightarrow$

$$\big[(a+b)^2+(a+b+3c)^2\big](a+b+7c) \ge 169abc \qquad (1)$$

Intuitively, the equality case is reached for $a = b = 3c \Rightarrow c = \frac{a}{3}$.

With $a = b = 3c \Rightarrow \begin{cases} (a+b)^2 + (a+b+3c)^2 = (2a^2) + (3a)^2 = 13a^2 \\ \dfrac{169abc}{a+b+7c} = \dfrac{169a^2 \cdot \frac{a}{3}}{13 \cdot \frac{a}{3}} = \dfrac{169a^2}{13} = 13a^2 \end{cases}$

We form means using the case of equality

$$a+b+7c = \left(\frac{a}{3}+\frac{a}{3}+\frac{a}{3}\right)+\left(\frac{b}{3}+\frac{b}{3}+\frac{b}{3}\right)+(c+c+c+c+c+c+c)$$

$$\geq 13\sqrt[13]{\left(\frac{a}{3}\right)^3\left(\frac{b}{3}\right)^3 c^7} = 13\sqrt[13]{\frac{a^3b^3c^7}{3^6}}$$

$$a+b+7c \geq 13\sqrt[13]{\frac{a^3b^3c^7}{3^6}} \tag{2}$$

$$(a+b)^2 + (a+b+3c)^2 = 2a^2 + 2b^2 + 9c^2 + 4ab + 6ac + 6bc$$

$$= a^2+a^2+b^2+b^2+(3c)^2+ab+ab+ab+ab+3ac+3ac+3bc+3bc$$

$$\geq 13\sqrt[13]{(a^2)^2(b^2)^2(3c)^2(ab)^4(3ac)^2(3bc)^2}$$

$$= 13\sqrt[13]{3^6a^4b^4c^2 \cdot a^2b^4 \cdot a^2c^2 \cdot b^2c^2} = 13\sqrt[13]{3^6a^{10}b^{10}c^6}$$

$$(a+b)^2 + (a+b+3c)^2 \geq 13\sqrt[13]{3^6a^{10}b^{10}c^6} \tag{3}$$

So, from (2)·(3) results (1)

$$\left[(a+b)^2 + (a+b+3c)^2\right](a+b+7c) \geq 13\sqrt[13]{3^6a^{10}b^{10}c^6} \cdot 13\sqrt[13]{\frac{a^3b^3c^7}{3^6}}$$

$$\Rightarrow \left[(a+b)^2 + (a+b+3c)^2\right](a+b+7c) \geq 13^2 \cdot \sqrt[13]{a^{13}b^{13}c^{13}}$$

$$\Rightarrow \left[(a+b)^2 + (a+b+3c)^2\right](a+b+7c) \geq 169abc$$

***EM–146.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $\{a_n\}_{n\geq 1}$ be the sequence of positive integers defined recursively by $a_1 = 2$ and $a_{n+1} = 2a_n^3 + a_n$ for all $n \geq 1$. Determine the largest power of 5 that divides the number $a_{2026}^2 + 1$.

**Solution 1 by the proposer.** For $n = 1$ we have $a_1 = 2$ and $a_1^2 + 1 = 2^2 + 1 = 5$, which is a multiple of 5 but not of $5^2$. For $n = 2$, $a_2 = 2 \cdot 2^3 + 2 = 18$ and $a_2^2 + 1 = 18^2 + 1 = 325$, which is a multiple of $5^2$ but not of $5^3$. This suggests the conjecture that $a_n^2 + 1$ is a multiple of $5^n$ but not of $5^{n+1}$. We will prove this by induction. The cases $n = 1$ and $n = 2$ are already verified. Suppose $n \geq 3$ and that $a_n^2 + 1 = k \cdot 5^n$ with $(k, 5) = 1$. Then,

$$\begin{aligned} a_{n+1}^2 + 1 &= (ax_n^3 + a_n)^2 + 1 = a_n^2(2a_n^2 + 1)^2 + 1 \\ &= (k5^n - 1)(2k5^n - 1)^2 + 1 \\ &= (k5^n - 1)(4k^2 5^{2n} - 4k5^n + 1) + 1 \\ &= 4k^3 5^{3n} - 8k^2 5^{2n} + k5^{n+1} \\ &= k5^{n+1}\Big(\underbrace{4k^2 5^{2n-1} - 8k5^{n-1}}_{\equiv 0 \pmod 5} + 1\Big) \end{aligned}$$

which is a multiple of $5^{n+1}$ but not a multiple of $5^{n+2}$, since for $n \geq 3$ we have $2n - 1 \geq 1$ and $n - 1 \geq 1$, respectively, and the term inside the parentheses is not a multiple of 5. This proves the conjecture, and therefore the largest power of 5 dividing $x_{2026}^2 + 1$ is $5^{2026}$.

**Solution 2 by Albert Stadler, Herrliberg, Switzerland.** Let $\{a_n\}_{n\geq 1}$ be the given sequence, and define $b_n = a_n^2 + 1$. Then $b_1 = 5$, and

$$b_{n+1} = a_{n+1}^2 + 1 = (2a_n^3 + a_n)^2 + 1 = (a_n^2 + 1)(4a_n^4 + 1) = b_n\,(4b_n^2 - 8b_n + 5).$$

Let $v_5(n)$ denote the highest power of 5 dividing $n$. We have $v_5(b_1) = 1$, $v_5(b_2) = 2$, and for all $n \geq 2$,

$$v_5(b_{n+1}) = v_5(b_n) + 1.$$

It follows that

$$v_5(a_{2026}^2 + 1) = v_5(b_{2026}) = 2026.$$

Thus, the largest power of 5 dividing $a_{2026}^2 + 1$ is $5^{2026}$.

**Solution 3 by Cao Minh Quang, Nguyen Binh Khiem high school for the Gifted, Vinh Long, Vietnam.** Let $v_5(N)$ denote the highest power of 5 dividing the integer $N$. From the recurrence relation,

$$a_{n+1} = 2a_n^3 + a_n = a_n(2a_n^2 + 1).$$

We compute

$$a_{n+1}^2 + 1 = \big(a_n(2a_n^2 + 1)\big)^2 + 1 = a_n^2(2a_n^2 + 1)^2 + 1.$$

This expression can be factored in terms of $a_n^2 + 1$ as follows:

$$\begin{aligned} a_{n+1}^2 + 1 &= a_n^2(4a_n^4 + 4a_n^2 + 1) + 1 \\ &= 4a_n^6 + 4a_n^4 + a_n^2 + 1 \\ &= (a_n^2 + 1)(4a_n^4 - 4a_n^2 + 1) \\ &= (a_n^2 + 1)\big(4(a_n^2 + 1)^2 - 8(a_n^2 + 1) + 5\big). \end{aligned}$$

We now determine $v_5(a_n^2 + 1)$ by induction. For $n = 1$, $a_1 = 2$, so $a_1^2 + 1 = 5$, and therefore $v_5(a_1^2 + 1) = 1$. Assume that for some $n \geq 1$, we have $v_5(a_n^2 + 1) = n$. Then there exists an integer $m$ not divisible by 5 such that $a_n^2 + 1 = 5^n m$. Consider the second factor:

$$4(a_n^2 + 1)^2 - 8(a_n^2 + 1) + 5 = 4 \cdot 5^{2n} m^2 - 8 \cdot 5^n m + 5.$$

For $n \geq 1$, the first two terms are divisible by $5$, while the constant term equals $5$. Thus, the entire expression is divisible by $5$ but not by $25$, implying

$$v_5\big(4(a_n^2 + 1)^2 - 8(a_n^2 + 1) + 5\big) = 1.$$

From this, we obtain

$$v_5(a_{n+1}^2 + 1) = v_5(a_n^2 + 1) + v_5\big(4(a_n^2 + 1)^2 - 8(a_n^2 + 1) + 5\big) = n + 1.$$

This completes the induction, and hence

$$v_5(a_n^2 + 1) = n \quad \text{for all } n \geq 1.$$

For $n = 2026$, we conclude that

$$v_5(a_{2026}^2 + 1) = 2026.$$

Therefore, the largest power of 5 dividing $a_{2026}^2 + 1$ is $5^{2026}$.

**Solution 4 by Michel Bataille, Rouen, France.** For every positive integer $m$, let $v_5(m)$ denote the largest integer $k$ such that $5^k$ divides $m$. We show that for all integer $n \geq 1$, we have

$$v_5(a_n^2+1) = n \quad \text{and} \quad v_5(1+4a_n^4) = 1 \qquad (\mathcal{P}_n).$$

The proof is by induction. Since $a_1 = 2$, $(\mathcal{P}_1)$ holds. Assume that $(\mathcal{P}_n)$ holds for some $n \geq 1$.
Using $a_{n+1} = 2a_n^3 + a_n$, we readily obtain that

$$a_{n+1}^2 + 1 = (a_n^2+1)(1+4a_n^4).$$

From $(\mathcal{P}_n)$, we deduce that $v_5(a_{n+1}^2+1) = n+1$. Also,

$$\begin{aligned} 1+4a_{n+1}^4 &= 1+4\big((a_n^2+1)(1+4a_n^4)-1\big)^2 \\ &= 5-8(a_n^2+1)(1+4a_n^4)+4(a_n^2+1)^2(1+4a_n^4)^2. \end{aligned}$$

By $(\mathcal{P}_n)$, $5^n$ divides $a_n^2+1$ and $5$ divides $1+4a_n^4$, hence

$$5-8(a_n^2+1)(1+4a_n^4)+4(a_n^2+1)^2(1+4a_n^4)^2 = 5+5^{n+1}k = 5(1+5^n k)$$

for some integer $k$. Thus $v_5(1+4a_{n+1}^4) = 1$. This completes the induction step and the proof that $(\mathcal{P}_n)$ holds for all $n \geq 1$. In particular the largest power of 5 that divides $a_{2026}^2+1$ is $5^{2026}$.

***EM–147.*** *Proposed by Dorin Mărghidanu, Colegiul Național "A.I. Cuza", Corabia, Romania.* Let $a$, $b$, $c$ be positive real numbers, prove that it holds:

$$\frac{(1+a)^2(1+b)^2}{1+c^2}+\frac{(1+b)^2(1+c)^2}{1+a^2}+\frac{(1+c)^2(1+a)^2}{1+b^2} \geq 8\,(a+b+c).$$

**Solution 1 by José Luis Díaz-Barrero, Barcelona, Spain.** Applying Cauchy–Schwarz inequality, we obtain

$$\sum_{\text{cyc}} \frac{(1+a)^2(1+b)^2}{1+c^2}$$

$$\geq \frac{\big((1+a)(1+b)+(1+b)(1+c)+(1+c)(1+a)\big)^2}{(1+a^2)+(1+b^2)+(1+c^2)}.$$

Putting $S = (1+a)(1+b) + (1+b)(1+c) + (1+c)(1+a)$ and $T = 3 + a^2 + b^2 + c^2$, the preceding inequality becomes

$$\sum_{\text{cyc}} \frac{(1+a)^2(1+b)^2}{1+c^2} \;\geq\; \frac{S^2}{T},$$

and it remains to prove that $\dfrac{S^2}{T} \geq 8(a+b+c)$ or $S^2 \geq 8(a+b+c)\,T$. Expanding

$$\begin{aligned} S &= (1+a)(1+b) + (1+b)(1+c) + (1+c)(1+a) \\ &= 3 + 2(a+b+c) + ab + bc + ca, \end{aligned}$$

and writing $s = a+b+c,\ p = ab+bc+ca,\ q = a^2+b^2+c^2$ we have $S = 3+2s+p$ and $T = 3+q$. Using the identity $q = s^2 - 2p$, a straightforward expansion shows that

$$\begin{aligned} S^2 - 8sT &= (a-b)^4 + (b-c)^4 + (c-a)^4 \\ &+ +2(a-b)^2(b-c)^2 + 2(b-c)^2(c-a)^2 \\ &+ 2(c-a)^2(a-b)^2 \geq 0. \end{aligned}$$

Therefore $S^2 \geq 8sT$, and hence

$$\sum_{\text{cyc}} \frac{(1+a)^2(1+b)^2}{1+c^2} \geq 8(a+b+c).$$

Equality holds when $a = b = c = 1$.

**Solution 2 by Andrea Fanchini, Cantù, Italy.** Rewrite $LHS$ as

$$LHS = \sum_{cyc} \frac{(1+a)^2(1+b)^2}{1+c^2}$$

$$= \underbrace{\sum_{cyc} \frac{(1+a^2)(1+b^2)}{1+c^2}}_{LHS-I} + \underbrace{\sum_{cyc} \frac{2b(1+a^2) + 2a(1+b^2)}{1+c^2}}_{LHS-II} + \underbrace{\sum_{cyc} \frac{4ab}{1+c^2}}_{LHS-III}$$

- $LHS - I$

we use the known

$$\frac{XY}{Z} + \frac{YZ}{X} + \frac{ZX}{Y} \geq X + Y + Z$$

$$LHS-I=\frac{(1+a^2)(1+b^2)}{1+c^2}+\frac{(1+b^2)(1+c^2)}{1+a^2}+\frac{(1+c^2)(1+a^2)}{1+b^2}$$
$$\geq (1+a^2)+(1+b^2)+(1+c^2)=a^2+b^2+c^2+3$$

- $LHS-II$

we use the known
$$X+\frac{1}{X}\geq 2$$
$$LHS-II=\sum_{cyc}\frac{2b(1+a^2)+2a(1+b^2)}{1+c^2}=\sum_{cyc}2a\left(\frac{1+b^2}{1+c^2}+\frac{1+c^2}{1+b^2}\right)$$
$$\geq 2a(2)+2b(2)+2c(2)=4(a+b+c)$$

- $LHS-III$

we use the known
$$1+X^2\geq 2X$$
$$LHS-III=\sum_{cyc}\frac{4ab}{1+c^2}\geq\sum_{cyc}\frac{4ab}{2c}=2\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)\geq 2(a+b+c)$$

Finally
$$LHS=LHS-I+LHS-II+LHS-III\geq a^2+b^2+c^2+6(a+b+c)+3$$
we have to prove that
$$a^2+b^2+c^2+6(a+b+c)+3\geq 8(a+b+c)\ \Rightarrow\ a^2+b^2+c^2+3\geq 2(a+b+c)$$
that is
$$(a^2-2a+1)+(b^2-2b+1)+(c^2-2c+1)\geq 0$$
$$\Leftrightarrow\ (a-1)^2+(b-1)^2+(c-1)^2\geq 0.$$

**Solution 3 by the proposer.** After a short preparation and then with the application of the AM–GM inequality, we have:
$$\sum_{\text{cyc}}\frac{(1+a)^2(1+b)^2}{1+c^2}=\sum_{\text{cyc}}\frac{(1+ab+a+b)^2}{1+c^2}$$
$$\geq\sum_{\text{cyc}}\frac{4(1+ab)(a+b)}{1+c^2}\quad\text{(AM–GM)}.$$

Next, observe that

$$(1+ab)(a+b) = a(1+b^2) + b(1+a^2),$$

so

$$\sum_{\text{cyc}} \frac{4(1+ab)(a+b)}{1+c^2} = 4\sum_{\text{cyc}} \left( \frac{a(1+b^2)}{1+c^2} + \frac{b(1+a^2)}{1+c^2} \right).$$

Applying AM–GM again,

$$\frac{a(1+b^2)}{1+c^2} + \frac{b(1+a^2)}{1+c^2} \geq 2(a+b),$$

hence

$$\sum_{\text{cyc}} \frac{(1+a)^2(1+b)^2}{1+c^2} \geq 4\sum_{\text{cyc}} 2(a+b) = 8(a+b+c).$$

For equality in the two AM–GM applications, we must have

$$\frac{1+a^2}{1+b^2} = \frac{1+b^2}{1+c^2} = \frac{1+c^2}{1+a^2},$$

which implies

$$1+a^2 = 1+b^2 = 1+c^2 \quad \Rightarrow \quad a=b=c.$$

From the second equality condition, $1+ab = a+b$, which together with $a=b=c$ gives

$$1+a^2 = 2a \quad \Rightarrow \quad a=1.$$

Thus equality holds if and only if $a=b=c=1$.

***EM–148.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* The positive integers $1, 2, \ldots, 2026$ are written on a board. Lisa Simpson picks any two of the numbers, deletes them, and writes on the board the absolute value of their difference. She repeats this procedure with the resulting 2025 numbers, and so on. After she does it 2025 times only one number remains on the board. Can this number be 4?

**Solution 1 by Michel Bataille, Rouen, France.** No, it cannot. Let $S$ be the sum of the numbers on the board at a step of the procedure that leaves at least two written numbers. Suppose that the numbers $a, b$ are picked by Lisa at this step with, say, $a \geq b$. Once the step is achieved, the sum of the numbers on the board is now $S' = S - (a+b) + a - b = S - 2b$ and therefore $S'$ has the same parity as $S$. Initially, the sum is $\dfrac{2026 \times 2027}{2} = 1013 \times 2027$, an odd positive integer, hence the sum on the board will remain odd all along the procedure and in particular must be odd when only a number is written. Thus, it cannot be the even integer $4$.

**Solution 2 by Albert Stadler, Herrliberg, Switzerland.** We will show that the final remaining number must be odd; therefore, it cannot be $4$.

Initially, there are $1013$ odd numbers and $1013$ even numbers. Consider how the parity changes at each step:

1. If two even numbers are chosen, their difference is even, so the number of even numbers decreases by one, while the number of odd numbers stays the same.
2. If one even and one odd number are chosen, their difference is odd, so again the number of even numbers decreases by one, while the number of odd numbers stays the same.
3. If two odd numbers are chosen, their difference is even; thus, the number of odd numbers decreases by two, while the number of even numbers increases by one.

In summary, at each step the number of odd numbers decreases by either $0$ or $2$. Hence, its parity remains unchanged throughout the process. Since the initial number of odd numbers is $1013$, which is odd, the final remaining number must also be odd.

Therefore, the last number cannot be $4$.

**Solution 3 by the proposer. No**, it cannot. Let us explain why. Let us refer to the procedure Lisa performs as a **move**. Originally there are $1013$ odd numbers on the board. How does this quantity change with each move? If two even numbers are chosen, their difference is even, and the number of odd numbers on the board does not change after the move. If an even and an odd numbers are deleted, then their difference is odd, and the number of odd numbers on the board does not change after the move. If two odd numbers are deleted, then the number of odd numbers on the board decreases by 2 after the move. Each time a move is made, **the number of odd numbers on the board either does not change or decreases by two**. Since its initial value is 1013, an odd number, it will be odd after every move. If 4 were the last number on the board, then the number of odd numbers on the board at that moment was zero. But zero is an even number. Therefore it is not possible to have 4 as the last number.

# Medium–Hard Problems

***MH–143.*** *Proposed by Todor Zaharinov Sofia, Bulgaria.* Find all ordered pairs $(x, y)$ of integers for which $2025(x+y) = 16xy$.

**Solution 1 by Michel Bataille, Rouen, France.** We show that there are eleven solutions, namely,
$(0,0), (150,810), (810,150), (135,2025), (2025,135), (126,-28350),$
$(-28350,126), (81,-225), (-225,81), (125,-10125), (-10125,125).$

It is readily checked that each of this pair is a solution. Conversely, let $(x, y)$ be any solution. Then we have

$$(16x-2025)(16y-2025) = 2025^2. \tag{1}$$

We will use freely the factorization $2025 = 3^4 \times 5^2$ and the fact that modulo $16$, a power $3^k$ ($k \geq 0$ being an integer) is congruent to one of the integers $1, 3, 9, 11$.
From (1), either $16x = 3^4 \times 5^2 + 3^u \times 5^v$, $16y = 3^4 \times 5^2 + 3^r \times 5^s$ (case 1) or $16x = 3^4 \times 5^2 - 3^u \times 5^v$, $16y = 3^4 \times 5^2 - 3^r \times 5^s$ (case 2) where $u, v, r, s$ are nonnegative integers satisfying $u + r = 8, v + s = 4$.
We first consider case $1$. We cannot have $v = 4, s = 0$ because $3^4 \times 5^2 + 3^r \equiv 9 + 3^r \not\equiv 0 \pmod{16}$. Similarly, $v = 0, s = 4$ cannot occur, nor $v = s = 2$ since $3^4 \times 5^2 + 3^r \times 5^2 \equiv 9 + 3^{r+2} \not\equiv 0 \pmod{16}$.
If $v = 1, s = 3$, then $16x = 5(3^u + 5 \times 3^4) \equiv 5(5 + 3^u)$ hence $u = 3$ or $7$ so that $(r, u) = (1, 7)$ or $(5, 3)$ (taking $u + r = 8$ into account). Solving for $x, y$ then readily leads to the solutions $(810, 150)$ and $(135, 2025)$. With $v = 3, s = 1$ we obtain $(150, 810)$ and $(2025, 135)$.
Second, we consider case $2$. The method is similar. For $v = 4, s = 0$, we first obtain $(r, u) = (2, 6)$ or $(6, 2)$ and the solutions $(-28350, 126)$ and $(-225, 81)$. For $v = 0, s = 4$, we get $(126, -28350)$ and $(81, -225)$. For $v = 2 = s$, $(r, u) = (0, 8)$ or $(8, 0)$ or $(4, 4)$, which provide $(125, -10125), (-10125, 125)$ and $(0, 0)$. The cases $v = 1, s = 3$ and $v = 3, s = 1$ cannot occur as it is easily proved using the powers of $3$ modulo $16$ as in case $1$. The proof is complete.

**Solution 2 by Rovsen Pirguliyev, Azerbaijan and Cao Minh Quang, Nguyen Binh Khiem high school for the Gifted, Vinh Long, Vietnam (same solution).** To solve the We rewrite the equation as form

$$16xy - 2025x - 2025y = 0.$$

Adding $2025^2$ to both sides, we obtain

$$(16x - 2025)(16y - 2025) = 2025^2.$$

Let $d_1 = 16x - 2025, d_2 = 16y - 2025$. Then, we have $d_1 d_2 = 2025^2$, and $x = \dfrac{d_1 + 2025}{16}, y = \dfrac{d_2 + 2025}{16}$. Hence $d_1$ and $d_2$ must be divisors of $2025^2$ such that the right-hand sides of the equation are integers.

Next, since $2025 \equiv 9 \pmod{16}$, from $d_1 = 16x - 2025$ we get

$$d_1 \equiv -2025 \equiv 7 \pmod{16},$$

and similarly $d_2 \equiv 7 \pmod{16}$.

Therefore, we must consider all divisors $d$ of $2025^2$ such that

$$d \equiv 7 \pmod{16}.$$

We factor $2025 = 3^4 \cdot 5^2$, so $2025^2 = 3^8 \cdot 5^4$. Checking all divisors of $2025^2$ that are congruent to $7 \pmod{16}$, we obtain

$$d \in \{135,\ 375,\ 10935,\ 30375,\ -9,\ -25,\ -729\}$$

$$\cup\{-2025,\ -5625,\ -164025,\ -455625\}.$$

We now determine the solutions. For each such divisor $d$, set $d_1 = d$ and $d_2 = 2025^2/d$, and compute $(x, y)$. This yields the following ordered pairs:

$$\begin{aligned}
&(0,0),\\
&(81,-225),\ (-225,81),\\
&(125,-10125),\ (-10125,125),\\
&(126,-28350),\ (-28350,126),\\
&(135,2025),\ (2025,135),\\
&(150,810),\ (810,150).
\end{aligned}$$

All integer solutions of the equation $2025(x+y) = 16xy$ are exactly the ordered pairs listed above.

**Solution 3 by José Luis Díaz-Barrero, Barcelona, Spain.** We begin by moving all terms to one side of the equation:

$$16xy - 2025x - 2025y = 0$$

To factor this expression, we multiply the entire equation by 16:

$$(16x)(16y) - 2025(16x) - 2025(16y) = 0$$

Next, we add $2025^2$ to both sides to complete the factoring into two binomials:

$$(16x - 2025)(16y - 2025) = 2025^2$$

The prime factorization of 2025 is $45^2 = (3^2 \times 5)^2 = 3^4 \times 5^2$. Therefore:

$$2025^2 = (3^4 \times 5^2)^2 = 3^8 \times 5^4$$

Let $A = 16x - 2025$ and $B = 16y - 2025$. The equation simplifies to:

$$A \times B = 3^8 \times 5^4$$

For $x$ and $y$ to be integers, $A$ must be an integer factor of $2025^2$. Furthermore, $A$ must satisfy the congruence:

$$A = 16x - 2025 \Rightarrow A \equiv -2025 \pmod{16}$$

Since $2025 = 16 \times 126 + 9$, we have $2025 \equiv 9 \pmod{16}$, which yields:

$$A \equiv -9 \equiv 7 \pmod{16}$$

Any factor of $2025^2$ can be written in the form $\pm(3^a \times 5^b)$, where $0 \le a \le 8$ and $0 \le b \le 4$. Testing these factors under $\pmod{16}$ arithmetic reveals exactly 11 valid choices for $A$.

By calculating $x = \dfrac{A + 2025}{16}$ and $y = \dfrac{B + 2025}{16}$ for each valid pair of factors, we find exactly 11 integer solutions:

| $A$ | $B$ | $x = \frac{A+2025}{16}$ | $y = \frac{B+2025}{16}$ | Ordered Pair $(x, y)$ |
|---|---|---|---|---|
| $-2025$ | $-2025$ | $0$ | $0$ | $(0, 0)$ |
| $-164025$ | $-25$ | $-10125$ | $125$ | $(-10125, 125)$ |
| $-25$ | $-164025$ | $125$ | $-10125$ | $(125, -10125)$ |
| $-455625$ | $-9$ | $-28350$ | $126$ | $(-28350, 126)$ |
| $-9$ | $-455625$ | $126$ | $-28350$ | $(126, -28350)$ |
| $-5625$ | $-729$ | $-225$ | $81$ | $(-225, 81)$ |
| $-729$ | $-5625$ | $81$ | $-225$ | $(81, -225)$ |
| $375$ | $10935$ | $150$ | $810$ | $(150, 810)$ |
| $10935$ | $375$ | $810$ | $150$ | $(810, 150)$ |
| $135$ | $30375$ | $135$ | $2025$ | $(135, 2025)$ |
| $30375$ | $135$ | $2025$ | $135$ | $(2025, 135)$ |

Table 1: All 11 verified integer ordered pairs solving $2025(x + y) = 16xy$.

**Solution 4 by the proposer.** It is clear that if $x = 0$, then $y = 0$, so $(0, 0)$ is a solution. Let now $x \neq 0, y \neq 0$. It is clear that if $(a, b)$ is a solution, then $(b, a)$ is also a solution. First rewrite the equation as

$$\frac{1}{x} + \frac{1}{y} = \frac{16}{2025}$$

Without loss of generality we can consider that $y \geq x$. Since $\frac{1}{x} + \frac{1}{y} > 0$, hence $y > 0$.

$$\frac{1}{x} = \frac{16}{2025} - \frac{1}{y} = \frac{16y - 2025}{2025y}; \quad \frac{2025y}{x} = 16y - 2025$$

$$\frac{1}{y} = \frac{16}{2025} - \frac{1}{x} = \frac{16x - 2025}{2025x}; \quad \frac{2025x}{y} = 16x - 2025$$

We multiply the equations and get

$$(16x - 2025)(16y - 2025) = 2025^2$$

$$16x - 2025 \equiv 16y - 2025 \equiv -2025 \equiv 7 \pmod{16}$$

$$7^2 \equiv (16x - 2025)(16y - 2025) \equiv 2025^2 \equiv 1 \pmod{16}$$

The factors of $2025^2 = 3^8.5^4$ that are congruent to 7 (mod 16) are 135, 375, 10935, 30375, $-9$, $-25$, $-729$, $-2025$, $-5625$, $-164025$, $-455625$.

**Case 1.** $0 < x \leq y$; $16x - 2025 \leq 16y - 2025$.

1. $16x - 2025 = 135$, $x = 135$ and $16y - 2025 = \frac{2025^2}{135} = 30375, y = 2025$ so $(135, 2025)$ is a solution.
2. $16x - 2025 = 375$, $x = 150$ and $16y - 2025 = \frac{2025^2}{375} = 10935, y = 810$ so $(150, 810)$ is a solution.
3. $16x - 2025 = d, d \geq 10935$, hence $16y - 2025 < 16x - 2025$, a contradiction.

**Case 2.** $x < 0 < y$. Let $z = -x > 0$. The equation is $-(16z + 2025)(16y - 2025) = 2025^2$ and $16y - 2025 < 0$.

1. $16y - 2025 = -9$, $y = 126$ and $-16z - 2025 = \frac{2025^2}{-9} = -455625, z = -x = 28350$ so $(-28350, 126)$ is a solution.
2. $16y - 2025 = -25$, $y = 125$ and $-16z - 2025 = \frac{2025^2}{-25} = -164025, z = -x = 10125$ so $(-10125, 125)$ is a solution.
3. $16y - 2025 = -729$, $y = 81$ and $-16z - 2025 = \frac{2025^2}{-729} = -5625, z = -x = 225$ so $(-225, 81)$ is a solution.
4. $16y - 2025 = -2025$, $y = 0$ and this is not a solution.
5. $16y - 2025 = d, d < -2025$, $y < 0$, a contradiction.

**In summary**, the only solutions $(x, y)$ are: $(0, 0)$, $(135, 2025)$, $(2025, 135)$, $(150, 810)$, $(810, 150)$, $(-28350, 126)$, $(126, -28350)$, $(-10125, 125)$, $(125, -10125)$, $(-225, 81)$, $(81, -225)$.

**Solution 5 by Albert Stadler, Herrliberg, Switzerland.** The given equation is equivalent to

$$(16x - 2025)(16y - 2025) = 2025^2.$$

Thus, if $a$ is any (positive or negative) divisor of $2025^2$, we may write

$$16x - 2025 = a, 16y - 2025 = \frac{2025^2}{a}.$$

Solving for $x$ and $y$, we obtain

$$x = \frac{2025 + a}{16}, y = \frac{2025(2025 + a)}{16a}.$$

Since $x$ and $y$ must be integers, the solutions correspond to those divisors $a$ of $2025^2$ for which these expressions are integral. This yields the following 11 ordered pairs:

$$(-28350, 126), \quad (-10125, 125), \quad (-225, 81), \quad (0, 0), \quad (81, -225),$$
$$(125, -10125), \; (126, -28350), \; (135, 2025), \; (150, 810), \; (810, 150),$$
$$(2025, 135).$$

***MH–144.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain and José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain.* Two dice are rolled until the sum of points in their upper faces is $7$ or $8$. Find the probability that sum $7$ appears before sum $8$.

**Solution 1 by Alberto Espuny Díaz, Universitat de Barcelona, Barcelona, Spain.** The dice are tossed iteratively until they sum either 7 or 8, and each time they are tossed, the outcome is independent of previous outcomes. In particular, each time they are tossed we have that the probability that they sum 7 is $6/36$ and the probability that they sum 8 is $5/36$. Summing these probabilities, the probability that the game ends any time that the dice are tossed is $11/36$.

Notice that, since these probabilities do not change, we do not really care about the number of rounds that the game takes, but only about the outcome on the round that it ends. In words, we could say that in 6 out of the $11$ cases where the game ends we have a sum of 7. Therefore, the probability that the game ends with 7 is $6/11$.

In more mathematical terms, let $S_7$ and $S_8$ denote the events that the sum of two dice equals 7 or 8, respectively. We are then interested in the conditional probability

$$\mathbb{P}[S_7 \mid S_7 \cup S_8] = \frac{\mathbb{P}[S_7]}{\mathbb{P}[S_7 \cup S_8]} = \frac{6/36}{11/36} = \frac{6}{11}.$$

**Solution 2 by the proposers.** Let

$$E = \{(1,1), (1,2), \cdots, (6,5), (6,6)\}$$

denote the set of all possible outcomes when two dice are rolled. Then,

$$S(7) = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} \quad \text{and}$$
$$S(8) = \{(2,6),(3,5),(4,4),(5,3),(6,2)\}$$

are the events that the sum equals $7$ or $8$, respectively. Therefore, the probability of obtaining the sum $7$ in one sample of the experiment is $p[S(7)] = 6/36$, and the probability of getting neither the sum $7$ nor the sum $8$ is $p[\overline{S(7)} \cap \overline{S(8)}] = 25/36$.

Let $A_k$ be the event that consists in obtaining the sum $7$ in the $k^{th}$ repetition of the experiment without the appearance of sum $8$ before. Then,

$$p[A_k] = \frac{6}{36}\left(\frac{25}{36}\right)^{k-1}, \; k = 1,2,3,\dots.$$

Since the $A_k$ are mutually exclusive, then the probability asked is

$$\begin{aligned} p\left[\sum_{k=1}^{\infty} A_k\right] &= \sum_{k=1}^{\infty} p[A_k] \\ &= \frac{6}{36}\left(1 + \frac{25}{36} + \left(\frac{25}{36}\right)^2 + \cdots + \left(\frac{25}{36}\right)^n + \cdots\right) = \frac{6}{11}, \end{aligned}$$

and we are done.

**Solution 3 by Albert Stadler, Herrliberg, Switzerland.** Let

1. $p = P(\text{sum } 7) = \frac{6}{36} = \frac{1}{6}$,
2. $q = P(\text{sum } 8) = \frac{5}{36}$,
3. $r = P(\text{neither } 7 \text{ nor } 8) = 1 - p - q = \frac{25}{36}$.

We roll the dice repeatedly until either 7 or 8 appears. We want the probability that 7 appears **before** 8.

Each roll is independent. The process continues while we get “neutral” outcomes (probability $r$), and stops when we first hit either 7 or 8.

Thus, the probability that 7 appears first is

$$p + rp + r^2p + r^3p + \cdots = p\sum_{k=0}^{\infty} r^k = \frac{p}{1-r} = \frac{\frac{1}{6}}{1-\frac{25}{36}} = \frac{6}{11}.$$

**Also solved by** *José Gibergans-Báguena, BarcelonaTech, Terrrassa, Spain.*

***MH–145.*** *Proposed by Vasile Cîrtoaje, Ploiesti, Romania.* Let $a, b, c, d$ be nonnegative real numbers such that $a \geq b \geq c \geq 1 \geq d$ and $ab + ac + ad + bc + bd + cd = 6$. Prove that

$$\frac{1}{ab+9} + \frac{1}{bc+9} + \frac{1}{cd+9} + \frac{1}{da+9} \geq \frac{2}{5}.$$

**Solution by the proposer.** Write the inequality as follows:

$$\frac{ab+cd+18}{(ab+9)(cd+9)} + \frac{bc+ad+18}{(bc+9)(ad+9)} \geq \frac{2}{5},$$

$$\frac{9(ab+cd)+162}{abcd+9(ab+cd)+81} + \frac{9(bc+ad)+162}{abcd+9(bc+ad)+81} \geq \frac{18}{5},$$

$$1 + \frac{81-abcd}{abcd+9(ab+cd)+81} + 1 + \frac{81-abcd}{abcd+9(bc+ad)+81} \geq \frac{18}{5},$$

$$\frac{1}{A} + \frac{1}{B} \geq \frac{8}{5(81-abcd)},$$

where

$$A = abcd + 9(ab+cd) + 81, \qquad B = abcd + 9(bc+ad) + 81.$$

Since

$$\frac{1}{A} + \frac{1}{B} \geq \frac{4}{A+B} = \frac{4}{2abcd + 9(ab+cd+bc+ad) + 162}$$

$$= \frac{4}{2abcd - 9(ac+bd) + 216},$$

it suffices to show that

$$\frac{1}{2abcd - 9(ac+bd) + 216} \geq \frac{2}{5(81-abcd)},$$

which is equivalent to $E \geq 0$, where

$$E = 2(ac+bd) - abcd - 3.$$

From

$$6 = (a+c)(b+d) + ac + bd \geq 2(b+d) + 2bd \geq 4\sqrt{bd} + 2bd,$$

we get $bd \leq 1$.

For fixed $c$ and $d$, assume that $a$ and $E$ are functions of $b$. By differentiating the equality constraint, we get

$$(b+c+d)a' + a + c + d = 0, \qquad -a' = \frac{a+c+d}{b+c+d} \geq 1, \qquad a' \leq -1.$$

Since

$$E'(b) = d(2-ac) + c(2-bd)a' \leq d(2-ac) - c(2-bd)$$

$$= -2(c-d) - cd(a-b) \leq 0,$$

$E(b)$ is decreasing and has the minimum value when $b$ is maximum, hence when $b = a$.

For fixed $a$ and $b$, assume that $d$ and $E$ are functions of $c$. We have

$$(a+b+c)d' + a + b + d = 0, \qquad -d' = \frac{a+b+d}{a+b+c} \leq 1, \qquad -1 \leq d' < 0,$$

and

$$E'(c) = a(2-bd) + b(2-ac)d'.$$

We claim that $E'(c) \geq 0$. This is obvious for $2 - ac \leq 0$. Also, for $2 - ac \geq 0$, we have

$$E'(c) \geq a(2-bd) + b(2-ac)(-1) = 2(a-b) + ab(c-d) \geq 0.$$

From $E'(c) \geq 0$, it follows that $E(c)$ is increasing and has the minimum value when $c$ is minimum, hence when $c = 1$.

According to these results, it suffices to consider the case with $a = b$ and $c = 1$. So, we need to show that $E \geq 0$ for $b \geq 1 \geq d$ with $b^2 + 2bd + 2b + d = 6$, where

$$E = 2b(1+d) - b^2 d - 3 = 2b - 3 + b(2-b)d.$$

Indeed, we have $d = \dfrac{6-2b-b^2}{2b+1}$ and

$$E = 2b - 3 + \frac{b(2-b)(6-2b-b^2)}{2b+1} = \frac{b^4 - 6b^2 + 8b - 3}{2b+1}$$
$$= \frac{(b-1)^3(b+3)}{2b+1} \geq 0.$$

The proof is completed. Equality occurs for $a = b = c = d = 1$.

***MH–146.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Mihály Bencze, Braşov, Romania.* Find all triplets of positive numbers $(a, b, c)$ such that

$$a + b + c = 3,$$
$$\frac{a^3}{bc(b+c)} + \frac{b^3}{ca(c+a)} + \frac{c^3}{ab(a+b)} = \frac{3}{2}.$$

**Solution 1 by Michel Bataille, Rouen, France.** Let $x, y$ be any positive real numbers. We have

$$x^3 + y^3 - xy(x+y) = x^2(x-y) + y^2(y-x) = (x-y)^2(x+y) \geq 0$$

so that $xy(x+y) \leq x^3 + y^3$.
Now, suppose that $(a, b, c)$ is a solution. We deduce that

$$\frac{3}{2} = \frac{a^3}{bc(b+c)} + \frac{b^3}{ca(c+a)} + \frac{c^3}{ab(a+b)} \geq \frac{a^3}{b^3+c^3} + \frac{b^3}{c^3+a^3} + \frac{c^3}{a^3+b^3}. \tag{1}$$

But it is well-known that for positive reals $u, v, w$, it holds that $\dfrac{u}{v+w} + \dfrac{v}{w+u} + \dfrac{w}{u+v} \geq \dfrac{3}{2}$ with equality if and only if $u = v = w$ (Nesbitt's inequality). From (1), it follows that $\dfrac{3}{2} \geq \dfrac{a^3}{b^3+c^3} + \dfrac{b^3}{c^3+a^3} + \dfrac{c^3}{a^3+b^3} \geq \dfrac{3}{2}$, that is, $\dfrac{a^3}{b^3+c^3} + \dfrac{b^3}{c^3+a^3} + \dfrac{c^3}{a^3+b^3} = \dfrac{3}{2}$ and therefore $a^3 = b^3 = c^3$. Thus $a = b = c$ and since $a + b + c = 3$, $(a, b, c) = (1, 1, 1)$. Conversely the triple $(1, 1, 1)$ is obviously a solution, hence is the unique solution.

**Solution 2 by the proposers.** We have that $(1,1,1)$ is a solution and we will prove that this is the only solution of the system. In fact, suppose that $a, b, c$ are any positive real numbers. WLOG, we can assume that $a \le b \le c$, then $\frac{1}{b+c} \le \frac{1}{c+a} \le \frac{1}{a+b}$ and $a^4 \le b^4 \le c^4$. By applying rearrangement's inequality, we get

$$\frac{a^4}{b+c} + \frac{b^4}{c+a} + \frac{c^4}{a+b} \ge \frac{a^4}{a+c} + \frac{b^4}{a+b} + \frac{c^4}{b+c}$$

and

$$\frac{a^4}{b+c} + \frac{b^4}{c+a} + \frac{c^4}{a+b} \ge \frac{a^4}{a+b} + \frac{b^4}{b+c} + \frac{c^4}{a+c}$$

Adding up the above inequalities yields

$$2\left(\frac{a^4}{b+c} + \frac{b^4}{c+a} + \frac{c^4}{a+b}\right) \ge \frac{a^4+b^4}{a+b} + \frac{b^4+c^4}{b+c} + \frac{c^4+a^4}{a+c}$$

Now, by applying Chebyshev's inequality, we have

$$\frac{a^4+b^4}{2} \ge \left(\frac{a^3+b^3}{2}\right)\left(\frac{a+b}{2}\right)$$

from which immediately follows $\frac{a^4+b^4}{a+b} \ge \frac{a^3+b^3}{2}$. Likewise, we get $\frac{b^4+c^4}{b+c} \ge \frac{b^3+c^3}{2}$ and $\frac{c^4+a^4}{c+a} \ge \frac{c^3+a^3}{2}$. Summing the preceding inequalities and applying AM-GM inequality, we obtain

$$\frac{a^4+b^4}{a+b} + \frac{b^4+c^4}{b+c} + \frac{c^4+a^4}{a+c} \ge a^3+b^3+c^3 = 3\left(\frac{a^3+b^3+c^3}{3}\right) \ge 3abc$$

Therefore, $\frac{a^4}{b+c} + \frac{b^4}{c+a} + \frac{c^4}{a+b} \ge \frac{3}{2}abc$ from which, after dividing by $abc$, follows

$$\frac{a^3}{bc(b+c)} + \frac{b^3}{ca(c+a)} + \frac{c^3}{ab(a+b)} \ge \frac{3}{2}$$

Since equality holds when $a = b = c$, then $(1,1,1)$ is the unique solution of the given system of equations.

**Solution 3 by Daniel Văcaru, National Economic College "Maria Teiuleanu", Pitești, Romania, Cao Minh Quang, Nguyen Binh Khiem high school for the Gifted, Vinh Long, Vietnam and S. C. Dutta Roy, New Delhi, India (same solution).** Let

$$S = \frac{a^3}{bc(b+c)} + \frac{b^3}{ca(c+a)} + \frac{c^3}{ab(a+b)} = \sum_{\text{cyc}} \frac{a^4}{abc(b+c)}.$$

By applying Cauchy–Schwarz Inequality (Engel form), we obtain

$$S \geq \frac{(a^2+b^2+c^2)^2}{2abc(a+b+c)} = \frac{(a^2+b^2+c^2)^2}{6abc}.$$

Next, also by Cauchy-Schwarz Inequality,

$$a^2+b^2+c^2 \geq \frac{(a+b+c)^2}{3} = 3 \Rightarrow (a^2+b^2+c^2)^2 \geq 9.$$

By using AM-GM, $abc \leq \left(\frac{a+b+c}{3}\right)^3 = 1$ Therefore,

$$S \geq \frac{3}{2}.$$

Thus, under the condition $a+b+c=3$, we have $S \geq \frac{3}{2}$. Equality holds iff $a=b=c=1$. The unique solution is $(a,b,c) = (1,1,1)$.

**Solution 4 by Albert Stadler, Herrliberg, Switzerland.** We claim that $(a,b,c) = (1,1,1)$ is the only triple satisfying the given conditions. Consider the function $f(x) = \dfrac{x^4}{3-x}$, which is convex on the interval $(0,3)$, since

$$f''(x) = \frac{6x^2((x-4)^2+2)}{(3-x)^3} > 0.$$

By Jensen's inequality, we obtain

$$\frac{a^4}{b+c} + \frac{b^4}{c+a} + \frac{c^4}{a+b} = \frac{a^4}{3-a} + \frac{b^4}{3-b} + \frac{c^4}{3-c} \geq 3 \cdot \frac{\left(\frac{a+b+c}{3}\right)^4}{3 - \frac{a+b+c}{3}} = \frac{3}{2},$$

with equality if and only if $a = b = c = 1$. Now observe that

$$\frac{a^3}{bc(b+c)} = \frac{a^4}{abc(b+c)},$$

so

$$\frac{a^3}{bc(b+c)} + \frac{b^3}{ca(c+a)} + \frac{c^3}{ab(a+b)}$$
$$= \frac{1}{abc}\left(\frac{a^4}{b+c} + \frac{b^4}{c+a} + \frac{c^4}{a+b}\right) \geq \frac{3}{2abc}.$$

By the AM–GM inequality, the condition $a + b + c = 3$ implies $abc \leq 1$, with equality if and only if $a = b = c = 1$. Hence

$$\frac{3}{2abc} \geq \frac{3}{2}.$$

Combining these results, we conclude that equality holds only when $a = b = c = 1$. Therefore, $(1, 1, 1)$ is the unique solution.

**Solution 5 by Arkady Alt, San Jose, California, USA and Rovsen Pirguliyev, Azerbaijan (same solution).** By the Cauchy-Schwarz inequality, we have:

$$\sum_{\text{cyc}} \frac{a^3}{bc(b+c)} = \frac{1}{abc}\sum_{\text{cyc}} \frac{a^4}{b+c} \geq \frac{\left(a^2+b^2+c^2\right)^2}{abc((b+c)+(c+a)+(a+b))}$$
$$= \frac{\left(\sum a^2\right)^2}{2abc\sum a}.$$

To bound this expression from below by $\frac{3}{2}$, we need to show that:

$$\frac{\left(\sum a^2\right)^2}{2abc\sum a} \geq \frac{3}{2} \iff \left(\sum a^2\right)^2 \geq 3abc\sum a.$$

Recall the well-known identities and inequalities via AM-GM:

$$\left(\sum a^2\right)^2 \geq \left(\sum ab\right)^2 \geq 3abc(a+b+c) = 3abc\sum a.$$

Thus, combining these results directly yields:

$$\frac{\left(\sum a^2\right)^2}{2abc\sum a} \geq \frac{3abc\sum a}{2abc\sum a} = \frac{3}{2}.$$

Equality holds if and only if $a = b = c$.

***MH–147.*** *Proposed by Michel Bataille, Rouen, France.* Let the sequence $(a_n)_{n\geq 0}$ be defined by the recursion $a_{n+2} = 3a_{n+1} - a_n$ for all $n \geq 0$ and $a_0 = 1, a_1 = 2$. Prove that for all $n \geq 1$,

$$\lfloor\sqrt{4a_{n-1}a_{n+1} - 2}\rfloor - \lfloor\sqrt{a_{n-1}a_{n+1}}\rfloor = a_n \quad \text{and}$$
$$\lfloor\sqrt{4a_n a_{n+1} - 2}\rfloor - \lfloor\sqrt{a_n a_{n+1}}\rfloor = a_{n+1} - a_n.$$

**Solution 1 by Albert Stadler, Herrliberg, Switzerland.** Consider the sequence $(a_n)_{n\geq 0}$ defined by

$$a_{n+2} = 3a_{n+1} - a_n, \quad a_0 = 1, \quad a_1 = 2.$$

We claim that $a_n = F_{2n+1}$, where $F_k$ denotes the $k$-th Fibonacci number. Indeed, $a_0 = F_1 = 1, a_1 = F_3 = 2$, and for $n \geq 0$:

$$a_{n+2} - 3a_{n+1} + a_n = F_{2n+5} - 3F_{2n+3} + F_{2n+1} = F_{2n+4} - 2F_{2n+3} + F_{2n+1}$$
$$= F_{2n+2} - F_{2n+3} + F_{2n+1} = 0,$$

using the Fibonacci recurrence. Hence the identification $a_n = F_{2n+1}$ is valid.

We now compute the relevant products:

$$a_{n-1}a_{n+1} = F_{2n-1}F_{2n+3}, \quad a_n a_{n+1} = F_{2n+1}F_{2n+3}.$$

By Catalan's identity:

$$F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2,$$

we have:

$$F_{2n-1}F_{2n+3} = F_{2n+1}^2 + 1, \quad F_{2n+1}F_{2n+3} = F_{2n+2}^2 + 1.$$

Hence:

$$\left\lfloor\sqrt{4a_{n-1}a_{n+1} - 2}\right\rfloor - \lfloor\sqrt{a_{n-1}a_{n+1}}\rfloor = 2F_{2n+1} - F_{2n+1} = F_{2n+1} = a_n$$

and similarly:

$$\left\lfloor\sqrt{4a_n a_{n+1} - 2}\right\rfloor - \lfloor\sqrt{a_n a_{n+1}}\rfloor = 2F_{2n+2} - F_{2n+2} = F_{2n+2}$$
$$= F_{2n+3} - F_{2n+1} = a_{n+1} - a_n.$$

Thus, both floor identities are verified.

**Solution 2 by José Luis Díaz-Barrero, Barcelona, Spain.** Let the sequence $(a_n)_{n\geq 0}$ be defined by the recursion $a_{n+2} = 3a_{n+1} - a_n$ for all $n \geq 0$ with initial conditions $a_0 = 1$ and $a_1 = 2$. Prove that for all $n \geq 1$:

$$\lfloor\sqrt{4a_{n-1}a_{n+1} - 2}\rfloor - \lfloor\sqrt{a_{n-1}a_{n+1}}\rfloor = a_n \tag{1}$$

$$\lfloor\sqrt{4a_na_{n+1} - 2}\rfloor - \lfloor\sqrt{a_na_{n+1}}\rfloor = a_{n+1} - a_n \tag{2}$$

To evaluate these expressions, we establish two algebraic identities native to the recurrence relation $a_{n+2} = 3a_{n+1} - a_n$.

**Identity 1:** $a_{n-1}a_{n+1} = a_n^2 + 1$. We proceed by induction on $n$. For the base case $n = 1$:

$$a_0a_2 - a_1^2 = 1 \cdot (3(2) - 1) - 2^2 = 5 - 4 = 1$$

Assuming the hypothesis holds for $n$, we check it for $n + 1$:

$$a_na_{n+2} - a_{n+1}^2 = a_n(3a_{n+1} - a_n) - a_{n+1}^2 = 3a_na_{n+1} - a_n^2 - a_{n+1}^2$$

Substituting $-a_n^2 = 1 - a_{n-1}a_{n+1}$ from the inductive hypothesis yields:

$$3a_na_{n+1} + 1 - a_{n-1}a_{n+1} - a_{n+1}^2 = a_{n+1}(3a_n - a_{n-1} - a_{n+1}) + 1$$

Since $a_{n+1} = 3a_n - a_{n-1}$ by definition, the terms inside the parenthesis equal zero, leaving $1$. Thus, the identity holds for all $n \geq 1$.

**Identity 2:** $a_na_{n+1} = (a_{n+1} - a_n)^2 + 1$. Using Identity 1 shifted by an index, we know $a_na_{n+2} - a_{n+1}^2 = 1$. Substituting $a_{n+2} = 3a_{n+1} - a_n$:

$$a_n(3a_{n+1} - a_n) - a_{n+1}^2 = 1 \implies 3a_na_{n+1} - a_n^2 - a_{n+1}^2 = 1$$

Rearranging terms allows us to express the product of consecutive terms:

$$a_na_{n+1} = a_{n+1}^2 - 2a_na_{n+1} + a_n^2 + 1 = (a_{n+1} - a_n)^2 + 1$$

Substituting Identity 1 ($a_{n-1}a_{n+1} = a_n^2 + 1$) into the first relation gives:

$$\lfloor\sqrt{4(a_n^2+1)-2}\rfloor - \lfloor\sqrt{a_n^2+1}\rfloor = \lfloor\sqrt{4a_n^2+2}\rfloor - \lfloor\sqrt{a_n^2+1}\rfloor$$

Since $a_n$ is a positive integer, we bound the square roots between consecutive integers:

$$a_n < \sqrt{a_n^2+1} < a_n + 1 \Rightarrow \lfloor\sqrt{a_n^2+1}\rfloor = a_n$$

$$2a_n < \sqrt{4a_n^2+2} < 2a_n + 1 \Rightarrow \lfloor\sqrt{4a_n^2+2}\rfloor = 2a_n$$

Subtracting these two values yields $2a_n - a_n = a_n$, proving the first statement.

Let $k = a_{n+1} - a_n$. Using Identity 2, we have $a_n a_{n+1} = k^2 + 1$. Substituting this into the second relation gives:

$$\lfloor\sqrt{4(k^2+1)-2}\rfloor - \lfloor\sqrt{k^2+1}\rfloor = \lfloor\sqrt{4k^2+2}\rfloor - \lfloor\sqrt{k^2+1}\rfloor$$

Bounding the expressions similarly based on the integer $k$:

$$k < \sqrt{k^2+1} < k+1 \Rightarrow \lfloor\sqrt{k^2+1}\rfloor = k$$

$$2k < \sqrt{4k^2+2} < 2k+1 \Rightarrow \lfloor\sqrt{4k^2+2}\rfloor = 2k$$

Subtracting these values yields $2k - k = k = a_{n+1} - a_n$, proving the second statement.

**Solution 3 by Cao Minh Quang, Nguyen Binh Khiem high school for the Gifted, Vinh Long, Vietnam.** First, we claim that for all $n \geq 1$, we have $a_{n-1}a_{n+1} - a_n^2 = 1(*)$. Indeed, for $n = 1$, this is $a_0a_2 - a_1^2 = 1 = 1$. Assume that $(*)$ holds for some $n \geq 1$. Then using the recursion,

$$a_n a_{n+2} - a_{n+1}^2 = a_n(3a_{n+1} - a_n) - a_{n+1}^2 = 3a_n a_{n+1} - a_n^2 - a_{n+1}^2.$$

But also

$$a_{n-1}a_{n+1} - a_n^2 = a_{n+1}(3a_n - a_{n+1}) - a_n^2 = 3a_n a_{n+1} - a_n^2 - a_{n+1}^2,$$

so $a_na_{n+2} - a_{n+1}^2 = a_{n-1}a_{n+1} - a_n^2 = 1$, which is exactly $(*)$ with $n$ replaced by $n+1$. Hence $(*)$ holds for all $n \geq 1$. From $(*)$, we have $a_{n-1}a_{n+1} = a_n^2 + 1$. Therefore

$$a_n^2 < a_{n-1}a_{n+1} < (a_n+1)^2,$$

so

$$\left\lfloor \sqrt{a_{n-1}a_{n+1}} \right\rfloor = a_n. \tag{3}$$

Moreover,

$$4a_{n-1}a_{n+1} - 2 = 4(a_n^2+1) - 2 = 4a_n^2 + 2.$$

We have $(2a_n)^2 = 4a_n^2 < 4a_n^2 + 2$ and

$$(2a_n+1)^2 = 4a_n^2 + 4a_n + 1 > 4a_n^2 + 2$$

since $4a_n - 1 > 0$ for $n \geq 1$ (indeed $a_n \geq a_1 = 2$). Hence

$$\left\lfloor \sqrt{4a_{n-1}a_{n+1} - 2} \right\rfloor = \left\lfloor \sqrt{4a_n^2+2} \right\rfloor = 2a_n. \tag{4}$$

Subtracting (3) from (4) yields

$$\left\lfloor \sqrt{4a_{n-1}a_{n+1} - 2} \right\rfloor - \left\lfloor \sqrt{a_{n-1}a_{n+1}} \right\rfloor = 2a_n - a_n = a_n.$$

Next, let $d_n = a_{n+1} - a_n > 0$. We show that

$$a_na_{n+1} = d_n^2 + 1. \tag{5}$$

Indeed, using $(*)$ and the recursion $a_{n-1} = 3a_n - a_{n+1}$,

$$1 = a_{n-1}a_{n+1} - a_n^2 = (3a_n - a_{n+1})a_{n+1} - a_n^2 = 3a_na_{n+1} - a_{n+1}^2 - a_n^2,$$

so

$$1 = a_na_{n+1} - (a_{n+1} - a_n)^2 = a_na_{n+1} - d_n^2,$$

which is (5). From (5), we get

$$d_n^2 < a_na_{n+1} < (d_n+1)^2,$$

hence

$$\left\lfloor \sqrt{a_na_{n+1}} \right\rfloor = d_n. \tag{6}$$

Also,

$$4a_n a_{n+1} - 2 = 4(d_n^2 + 1) - 2 = 4d_n^2 + 2,$$

and as before,

$$(2d_n)^2 < 4d_n^2 + 2 < (2d_n + 1)^2$$

because $4d_n - 1 > 0$ (note $d_n \geq a_2 - a_1 = 3$). Therefore

$$\left\lfloor \sqrt{4a_n a_{n+1} - 2} \right\rfloor = \left\lfloor \sqrt{4d_n^2 + 2} \right\rfloor = 2d_n. \tag{7}$$

Subtracting (6) from (7) gives

$$\left\lfloor \sqrt{4a_n a_{n+1} - 2} \right\rfloor - \left\lfloor \sqrt{a_n a_{n+1}} \right\rfloor = 2d_n - d_n = d_n = a_{n+1} - a_n.$$

This completes the proof.

**Solution 4 by the proposer.** It is readily seen (by induction) that the $a_n$'s are integers such that $a_{n+1} > a_n > 0$ for all $n \geq 0$. We remark that

$$\begin{aligned} a_{n+2}a_n - a_{n+1}^2 &= a_n(3a_{n+1} - a_n) - a_{n+1}^2 \\ &= a_{n+1}(3a_n - a_{n+1}) - a_n^2 = a_{n+1}a_{n-1} - a_n^2. \end{aligned}$$

It follows that $a_{n+2}a_n - a_{n+1}^2 = a_2 a_0 - a_1^2 = 1$ for all $n \geq 0$ and that $3a_n a_{n+1} = 1 + a_n^2 + a_{n+1}^2$, i.e. $a_n a_{n+1} = 1 + (a_{n+1} - a_n)^2$. We will make use of the following lemma proved at the end (a particular case has already been met in the solution to E-98): if $m$ is a nonnegative integer, then $\lfloor \sqrt{m+1} + \sqrt{m} \rfloor = \left\lfloor \sqrt{2(2m+1)} \right\rfloor$.

Taking $m = a_n^2$ so that $m + 1 = a_{n+1}a_{n-1}$ and $2(2m+1) = 4a_n^2 + 2 = 4a_{n-1}a_{n+1} - 2$, the lemma gives

$$a_n = \lfloor \sqrt{4a_{n-1}a_{n+1} - 2} \rfloor - \lfloor \sqrt{a_{n-1}a_{n+1}} \rfloor.$$

Similarly, taking $m = (a_{n+1} - a_n)^2$, the lemma yields

$$a_{n+1} - a_n = \lfloor \sqrt{4a_n a_{n+1} - 2} \rfloor - \lfloor \sqrt{a_n a_{n+1}} \rfloor.$$

*Proof of the lemma.* Let $k$ denote the integer $\left\lfloor \sqrt{2(2m+1)} \right\rfloor$. Then, we have $k \leq \sqrt{2(2m+1)} < k + 1$. Note that $2(2m+1) \geq k^2 + 1$

(since $4m + 2 \equiv 2 \pmod 4$, $2(2m + 1)$ is not a perfect square). From $2(2m+1) = (\sqrt{m+1}+\sqrt{m})^2 + (\sqrt{m+1}-\sqrt{m})^2$ we obtain $\sqrt{2(2m+1)} \geq \sqrt{m+1}+\sqrt{m}$ and therefore $\sqrt{m+1}+\sqrt{m} < k+1$. Second, we have

$$\begin{aligned} 2(2m+1) &= (\sqrt{m+1}+\sqrt{m})^2 + \frac{1}{(\sqrt{m+1}+\sqrt{m})^2} \\ &\leq (\sqrt{m+1}+\sqrt{m})^2 + 1, \end{aligned}$$

hence $k^2 \leq (\sqrt{m+1}+\sqrt{m})^2$. Thus $k \leq \sqrt{m+1}+\sqrt{m} < k+1$ and $\lfloor \sqrt{m+1}+\sqrt{m} \rfloor = k$.

*Note.* It is easy to obtain that for all $n \geq 0$, $a_n = F_{2n+1}$, the $(2n+1)$th Fibonacci number. Thus, the results can be rewritten as results about the Fibonacci numbers.

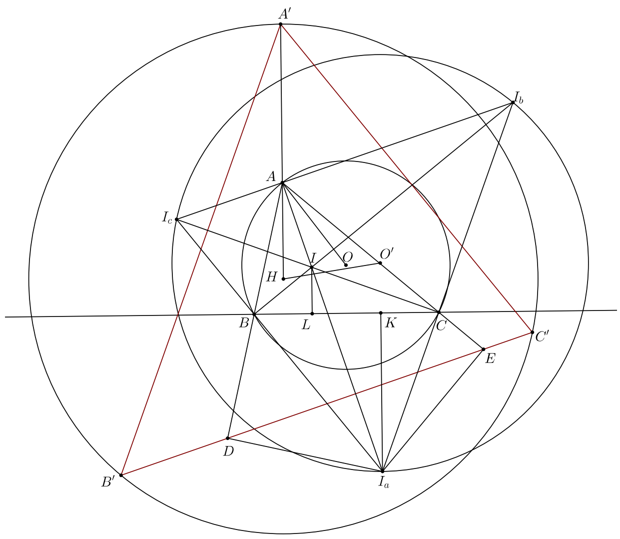
***MH–148.*** *Proposed by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain.* (Correction) Consider the excircles of a triangle $ABC$, and join the points of tangency of each excircle with the two sides that have been extended; in this way a new triangle $A'B'C'$ is formed.

We ask:

1. To evaluate the angles of the triangle $A'B'C'$.
2. To prove that the lines $AA'$, $BB'$, $CC'$ are the altitudes of the triangle $ABC$.
3. To determine the center and the radius of the circumcircle of the triangle $A'B'C'$.

**Solution 1 by Michel Bataille, Rouen, France.** We use the familiar notations for the elements of the triangle.

1. Let $D$ and $E$ be the projections of the excenter $I_a$ on the lines $AB$ and $AC$, respectively (see figure). The line $B'C'$ and $DE$ coincide and $I_aD = I_aE (= r_a)$. Also $AD = AE (= s)$, hence $AI_a$ is the perpendicular bisector of $DE$. It follows that $B'C' \parallel I_bI_c$ (both lines are perpendicular to the internal bisector $AI_a$). Similarly, $C'A' \parallel I_cI_a$ and $A'B' \parallel I_bI_a$. As a result, the triangles $A'B'C'$ and $I_aI_bI_c$ are homothetic. Thus $\angle B'A'C' = \angle I_bI_aI_c = 180^\circ - \angle BIC$




Scheme for solving problem MH-148.

(since $B, I, C, I_a$ are on the circle with diameter $II_a$). Since $\angle BIC = 180° - \frac{B}{2} - \frac{C}{2}$, we see that $\angle B'A'C' = \frac{B+C}{2}$. In the same way, $\angle A'C'B' = \frac{A+B}{2}, \angle C'B'A' = \frac{C+A}{2}$.

2. We show that $AA', BB', CC'$ are the altitudes of the triangle $ABC$ (not of $A'B'C'$). We use barycentric coordinates relatively to $(A, B, C)$. We have $D = (-(s-c) : s : 0)$ and $E = (-(s-b) : 0 : s)$, hence the equation of $B'C' = DE$ is $sx + (s-c)y + (s-b)z = 0$. In the same way, we obtain $C'A' :\ (s-c)x + sy + (s-a)z = 0$ and $A'B' :\ (s-b)x + (s-a)y + sz = 0$. We deduce $A' = (-a(b+c) : S_C : S_B)$, $B' = (S_C : -b(a+c) : S_A)$ and $C' = (S_B : S_A : -c(a+b))$ where $S_A = \frac{b^2+c^2-a^2}{2}$, $S_B = \frac{c^2+a^2-b^2}{2}$, $S_C = \frac{a^2+b^2-c^2}{2}$ (Conway's notations). Since $H = (S_BS_C : S_CS_A : S_AS_B)$, the equation of the altitude $AH$ is $S_By - S_Cz = 0$. Clearly, $A'$ is on $AH$ and $AA'$ is the altitude $AH$. Similarly, $BB'$ and $CC'$ are altitudes of $\Delta ABC$.

3. We show that the center and radius of the circumcircle $\mathcal{C}$ of $\Delta A'B'C'$ are $H$ and $r + 2R$.
We first remark that $I$ is the orthocenter of $\Delta I_aI_bI_c$ (since $I_aA \perp I_bI_c$, etc.). In consequence, the circumcircle of $\Delta ABC$ is the Euler circle of $\Delta I_aI_bI_c$, so that the circumcenter $O'$ of $\Delta I_aI_bI_c$ is the

reflection of $I$ in the point $O$. It follows that the projections $K$ and $L$ of $O'$ and $I$ onto $BC$ are symmetrical about the midpoint of $BC$ and therefore $K$ is also the projection of $I_a$ onto $BC$. In other words, $O'I_a \perp BC$, hence $O'I_a \parallel AH$. Similarly, $O'I_b \parallel BH$ and the homothety transforming $I_a, I_b, I_c$ into $A', B', C'$ transforms $O'$ into $H$ and therefore $H$ is the circumcenter of $\Delta A'B'C'$.
Let $R' = HA' = HB' = HC'$ be the circumradius of $\Delta A'B'C'$. We determine $HA'$, supposing that $A$ is acute (otherwise we determine $HB'$ or $HC'$).
From $I_a = (-a : b : c)$ and $K = (0 : s-b : s-c)$, we readily obtain

$$2a(s-a)\overrightarrow{I_aK} = (2(s-a)(s-b)-ab)\overrightarrow{AB} + (2(s-a)(s-c)-ac)\overrightarrow{AC}$$
$$= -S_C\overrightarrow{AB} - S_B\overrightarrow{AC}.$$

Since $2a(s-a)A' = a(b+c)A - S_CB - S_BC$, we see that $\overrightarrow{I_aK} = \overrightarrow{AA'}$. Also, we have $(4F^2)H = S_BS_CA + S_CS_AB + S_AS_BC$, hence $(4F^2)\overrightarrow{AH} = S_A(S_C\overrightarrow{AB} + S_B\overrightarrow{AC})$ [$F$ denote the area of $\Delta ABC$] Since $S_A > 0$ and $2a(s-a)\overrightarrow{AA'} + \frac{4F^2}{S_A}\overrightarrow{AH} = \overrightarrow{0}$, the point $A$ is between $H$ and $A'$ and $HA' = HA + AA' = HA + r_a$. Since $F = r_a(s-a)$, we obtain $HA = \frac{S_A}{4F^2} \cdot 2a(s-a)r_a = \frac{aS_A}{2F}$ and

$$\begin{aligned} R' &= \frac{aS_A}{2F} + \frac{F}{s-a} = \frac{aS_A(s-a) + 2s(s-a)(s-b)(s-c)}{2(s-a)F} \\ &= \frac{aS_A + 2s(s-b)(s-c)}{2F} = \frac{abc + 2(s-a)(s-b)(s-c)}{2rs} \\ &= \frac{4rsR + 2r^2s}{2rs} = 2R + r. \end{aligned}$$

(the equality $aS_A + 2s(s-b)(s-c) = abc + 2(s-a)(s-b)(s-c)$ is easily checked.)

**Solution 2 by Andrea Fanchini, Cantù, Italy.** We use barycentric coordinates with reference to the triangle $ABC$.
First of all the points of tangency of each excircle are

$$B_a(s-b : 0 : -s),\ C_a(s-c : -s : 0),\ C_b(-s : s-c : 0),$$

$$A_b(0 : s-a : -s),\ A_c(0 : -s : s-a),\ B_c(-s : 0 : s-b)$$

joining these points we have lines

$$B_aC_a : sx+(s-c)y+(s-b)z = 0, \quad C_bA_b : (s-c)x+sy+(s-a)z = 0,$$

$$A_cB_c : (s-b)x + (s-a)y + sz = 0$$

that intersect each other at the points

$$A' = C_bA_b \cap A_cB_c = [a(b+c) : -S_C : -S_B]$$

$$B' = A_cB_c \cap B_aC_a = [-S_C : b(c+a) : -S_A]$$

$$C' = B_aC_a \cap C_bA_b = [-S_B : -S_A : c(a+b)]$$

Now we can answer at the questions

1. The angles of the triangle $A'B'C'$ are

$$\cot\alpha' = \frac{S}{bc+S_A}, \quad \cot\beta' = \frac{S}{ca+S_B}, \quad \cot\gamma' = \frac{S}{ab+S_C}$$

and they are equal to the angles of excentral triangle.

2. Lines $AA', BB', CC'$ are

$$AA' : S_By - S_Cz = 0, \quad BB' : S_Cz - S_Ax = 0, \quad CC' : S_Ax - S_By = 0$$

that are the well-known altitudes of the triangle $ABC$.

3. The circumcircle of the triangle $A'B'C'$ is

$$a^2yz + b^2zx + c^2xy - (x+y+z)$$

$$\cdot\left[(S_\omega - s^2 - a^2)x + (S_\omega - s^2 - b^2)y + (S_\omega - s^2 - c^2)z\right] = 0$$

where $S_\omega = S\cot\omega$ is the Brocard angle.
Therefore we find that the center of this circle is the orthocenter of $\triangle ABC$

$$H = (S_BS_C : S_CS_A : S_AS_B)$$

and the radius is equal to

$$\rho = 2R + r$$

4. Furthermore, just for fun, we evaluate the area of the triangle. Notice that

$$[A'B'C'] = \frac{[ABC]}{8abc(s-a)(s-b)(s-c)}\begin{vmatrix} a(b+c) & -S_C & -S_B \\ -S_C & b(c+a) & -S_A \\ -S_B & -S_A & c(a+b) \end{vmatrix}$$

$$= [ABC]\frac{S^2(ab+bc+ca)+a^2b^2c^2-S_AS_BS_C}{4abc(s-a)(s-b)(s-c)}$$

$$= [ABC]\frac{s^2-S_\omega+4R^2}{2Rr} = [ABC]\frac{(2R+r)^2}{2Rr} = [ABC]\frac{\rho^2}{2Rr}.$$

**Solution 3 by the proposer.** (1) Let $A_1, A_2, B_1, B_2, C_1, C_2$ be the tangency points of the three excircles with the extended sides of the triangle, in the order shown in the figure below. Since the base angles of isosceles triangles $A_1BC_2$, $B_1CA_2$ are $90^\circ - \frac{B}{2}$, $90^\circ - \frac{C}{2}$, respectively, using the fact that the interior angles of quadrilateral $A'A_2AA_1$ add up to $360^\circ$, we obtain

$$\angle A' = 360^\circ - \left(90^\circ + \frac{B}{2}\right) - \left(90^\circ + \frac{C}{2}\right) - A = 90^\circ - \frac{A}{2}.$$

Analogously,

$$\angle B' = 90^\circ - \frac{B}{2} \quad \text{and} \quad \angle C' = 90^\circ - \frac{C}{2}.$$

(2) Let $D$ be the foot of the altitude from $A$ to $BC$. Let $A^* = AD \cap A'B'$ and $A^{**} = AD \cap A'C'$.

By the Menelaus's theorem, applied to $\triangle ADC$ and transversal $A^*A_2B_1$,

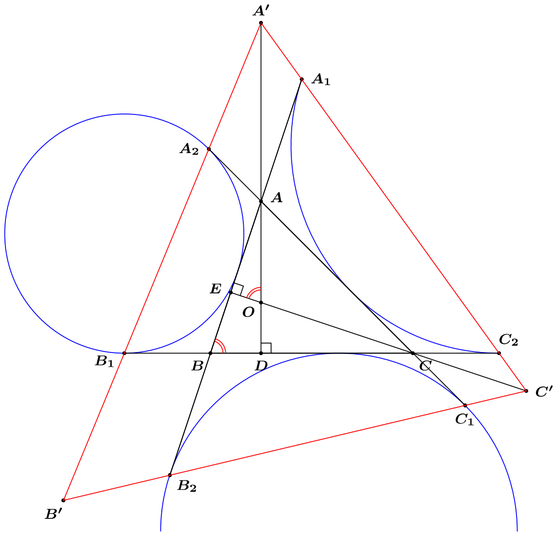
$$\frac{AA^*}{A^*D}\cdot\frac{DB_1}{\cancel{B_1C}}\cdot\frac{\cancel{CA_2}}{A_2A} = 1,$$

and applied to $\triangle ABD$ and transversal $A^{**}A_1C_2$,

$$\frac{AA_1}{\cancel{A_1B}}\cdot\frac{\cancel{BC_2}}{C_2D}\cdot\frac{DA^{**}}{A^{**}A} = 1,$$

from which we obtain (with $s$ denoting the semiperimeter of $\triangle ABC$)

$$\frac{AA^*}{A^*D} = \frac{A_2A}{DB_1} = \frac{s-b}{(s-a)+c\cdot\cos B} = \frac{s-b}{(s-a)+\frac{c^2+a^2-b^2}{2a}} = \frac{a}{b+c}$$




and

$$\frac{A^{**}A}{DA^{**}} = \frac{AA_1}{C_2D} = \frac{BA_1 - BA}{BC_2 - BD} = \frac{s-c}{s - c\cdot\cos B} = \frac{s-c}{s - \frac{c^2+a^2-b^2}{2a}} = \frac{a}{b+c}.$$

Thus

$$\frac{AA^*}{A^*D} = \frac{AA^{**}}{A^{**}D}$$

making points $A^*$ and $A^{**}$ coincidents with $A'$, so that $A'$, $A$ and $D$ are collinear. Since $AD \perp BC$, so is $A'A \perp BC$, making $A'A$ an altitude of $\triangle ABC$.

Analogously, so are $B'B$ and $C'C$.

1. Let $O$ be the orthocenter of $\triangle ABC$. Then lines $A'A$, $B'B$ and $C'C$ intersect at $O$. Let $E$ be the foot of the altitude from

$C$ of $\triangle ABC$. Since

$$\begin{aligned}
\angle OC'A' &= \angle EC'A_1 \\
&= 90^\circ - \angle C'A_1E \qquad \text{(in right triangle } A_1EC'\text{)} \\
&= 90^\circ - \angle C_2A_1B \\
&= 90^\circ - \left(90^\circ - \tfrac{B}{2}\right) \quad \text{(in isosceles triangle } A_1BC_2\text{)} \\
&= \tfrac{B}{2}
\end{aligned}$$

and by the external angle theorem applied to $\triangle A'AA_1$ at $A$,

$$\begin{aligned}
\angle OA'C' &= \angle AA'A_1 \\
&= \angle C'A_1A - \angle A'AA_1 \\
&= \angle C'A_1E - \angle BAD \\
&= \left(90^\circ - \tfrac{B}{2}\right) - (90^\circ - B) \\
&= \tfrac{B}{2},
\end{aligned}$$

triangle $A'OC'$ is isosceles with $A'O = OC'$. Analogously, we have $A'O = OB'$. So that $OA' = OB' = OC'$, making the orthocenter of $\triangle ABC$ the circumcenter of $\triangle A'B'C'$.

2. We have

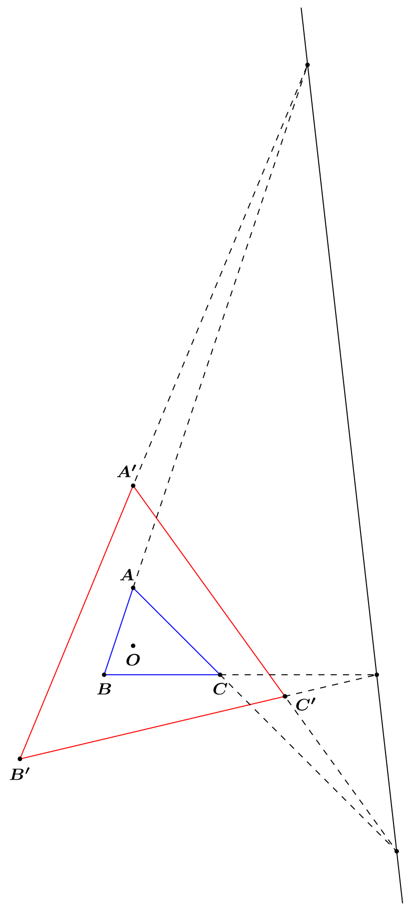
$$\begin{aligned}
EC' &= \frac{EA_1}{\tan B/2} = \frac{BA_1 - BE}{\tan B/2} = \frac{s - a \cdot \cos B}{\tan B/2} \\
EO &= EA \cdot \cot B = \frac{b \cdot \cos A}{\tan B} = \frac{(b \cdot \cos A)\left(1 - \tan^2 \dfrac{B}{2}\right)}{2 \tan B/2}
\end{aligned}$$

Therefore

$$\begin{aligned}
OC' &= EC' - EO \\
&= \tfrac{1}{2 \tan \frac{B}{2}}\left(2s - 2a \cdot \cos B - \left(1 - \tan^2 \tfrac{B}{2}\right) \cdot b \cos A\right),
\end{aligned} \tag{1}$$

where

$$2s - 2a \cdot \cos B = a + b + c - \frac{c^2 + a^2 - b^2}{c} = \frac{(a+b)(-a+b+c)}{c}$$




Scheme for solving problem MH-148-sol-3.

and

$$\begin{aligned}\left(1-\tan^2\frac{B}{2}\right)\cdot b\ \cos A &= \frac{\cos B}{\cos^2\frac{B}{2}}\cdot b\cos A\\ &= \frac{\frac{c^2+a^2-b^2}{2ca}}{\frac{s(s-b)}{ca}}\cdot\frac{c^2+a^2-b^2}{2c}\\ &= \frac{\left(c^2+a^2-b^2\right)\left(b^2+c^2-a^2\right)}{4cs(s-b)}.\end{aligned}$$

When these are substituted into (1), we get

$$OC'=\frac{1}{2\tan\frac{B}{2}}\cdot\frac{a^2b+b^2c+c^2a+ab^2+bc^2+ca^2+2abc-a^3-b^3-c^3}{(a+b+c)(a-b+c)},$$

This may be written in the form

$$OC' = \frac{4(a+b)(b+c)(c+a) - (a+b+c)^3}{8\sqrt{s(s-a)(s-b)(s-c)}}$$

We substitute $2s(s^2 + r^2 + 2Rr)$ for $\Pi(a+b)$, $2s$ for $\Sigma a$ and use the Heron's formula to write

$$\sqrt{s(s-a)(s-b)(s-c)} = rs.$$

This yields $OC' = 2R + r$. Thus, the required circumradius is $2R + r$

**CODA.**

Triangles $ABC$ and $A'B'C'$ are homological. Point $O$ is the center of homology, and the axis passes through $(AB, A'B')$, $(BC, B'C')$, and $(CA, C'A')$. (Here, $(AB, A'B')$ indicates the point of intersection of lines $AB$ and $A'B'$, etc.)

See the previous figure.

# Advanced Problems

***A–143.*** *Proposed by Michel Bataille, Rouen, France.* Let $n$ be a positive integer and $x \in (0,1)$. Let $S_d(x) = \sum_{\gcd(k,n)=d} x^k$ (the summation is extended to all positive integers $k$ such that $\gcd(k,n) = d$). Prove that

$$(1-x^n)\sum_{d|n} x^{n-d}S_d(x) - \sum_{d|n} x^{2(n-d)} \le x^n(n-\tau(n)) \text{ and}$$

$$(1-x^n)\sum_{d|n} x^{d}S_d(x) - \sum_{d|n} x^{2d} \ge x^n(n-\tau(n)).$$

where $\tau(n)$ denotes the number of positive divisors of $n$.

**Solution 1 by the proposer.** We first consider $S_d(x)$ where $d$ is a divisor of $n$. Since $\gcd(k,n) = d$ if and only if $k = \ell d$ where $\gcd(\ell, \frac{n}{d}) = 1$, we have

$$S_d(x) = \sum_{\gcd(\ell,\frac{n}{d})=1} (x^d)^{\ell}.$$

If $r_1 < r_2 < \cdots < r_s$ are the positive integers less than $\frac{n}{d}$ and coprime with $\frac{n}{d}$ ($s = \phi(n/d)$ where $\phi$ is Euler's totient function), we therefore have

$$S_d(x) = \sum_{j=1}^{s}\sum_{u=0}^{\infty}(x^d)^{(r_j+u\frac{n}{d})} = \sum_{j=1}^{s} x^{dr_j}\sum_{u=0}^{\infty}(x^n)^u = \sum_{j=1}^{s}\frac{x^{dr_j}}{1-x^n}.$$

First, suppose that $s \ge 2$, the general case. Then, $r_1 = 1, r_s = \frac{n}{d}-1$ and $x^{n-d} \le x^{dr_j} \le x^d$, hence

$$x^d + x^{n-d}(\phi(n/d)-1) \le (1-x^n)S_d(x) \le x^{n-d} + x^d(\phi(n/d)-1). \tag{1}$$

If $s = 1$, then $(1-x^n)S_d(x) = x^d$ and $\frac{n}{d} = 1$ or $2$ so that $n-d = 0$ or $n-d = d$ and (1) still holds.

Now, recalling that $n = \sum_{d|n}\phi(d) = \sum_{d|n}\phi(n/d)$ and summing, we obtain from the left inequality

$$\sum_{d|n} x^{2d} + x^n(n-\tau(n)) \le (1-x^n)\sum_{d|n} x^d S_d(x)$$

and from the right inequality

$$(1-x^n)\sum_{d|n} x^{n-d}S_d(x) \le \sum_{d|n} x^{2(n-d)} + x^n(n-\tau(n)).$$

The result follows.

**Solution 2 by José Luis Díaz-Barrero, Barcelona, Spain.** The core of the solution relies on changing the order of summation to simplify the restricted condition $\gcd(k,n)=d$.

Let $g(d)$ be an arbitrary function of $d$. By the definition of $S_d(x)$, we consider the sum over all positive divisors $d$ of $n$:

$$\sum_{d|n} g(d)S_d(x) = \sum_{d|n} g(d) \sum_{\gcd(k,n)=d} x^k.$$

Because every positive integer $k$ has a unique greatest common divisor with $n$, and this divisor $d=\gcd(k,n)$ must divide $n$, we can rewrite the double summation by summing over all positive integers $k$ and weighting each term by $g(\gcd(k,n))$:

$$\sum_{d|n} g(d)S_d(x) = \sum_{k=1}^{\infty} g(\gcd(k,n))x^k.$$

Since $x\in(0,1)$, this infinite series converges absolutely. We can split the sum into blocks of length $n$ by substituting $k=mn+r$, where $m\ge 0$ and $1\le r\le n$. Noting that $\gcd(mn+r,n)=\gcd(r,n)$, the weight depends only on the remainder $r$:

$$\begin{aligned}\sum_{k=1}^{\infty} g(\gcd(k,n))x^k &= \sum_{m=0}^{\infty}\sum_{r=1}^{n} g(\gcd(r,n))x^{mn+r}\\ &= \left(\sum_{m=0}^{\infty}(x^n)^m\right)\sum_{r=1}^{n} g(\gcd(r,n))x^r.\end{aligned}$$

Using the sum of the infinite geometric series $\sum_{m=0}^{\infty}(x^n)^m = \dfrac{1}{1-x^n}$, we obtain the fundamental identity:

$$(1-x^n)\sum_{d|n} g(d)S_d(x) = \sum_{r=1}^{n} g(\gcd(r,n))x^r. \qquad (1)$$

To proof the first inequality, we choose the weight function $g(d) = x^{n-d}$. Applying identity (1), the left-hand side ($\mathrm{LHS}_1$) becomes:

$$\mathrm{LHS}_1 = \sum_{r=1}^{n} x^{n-\gcd(r,n)} x^r - \sum_{d|n} x^{2(n-d)} = \sum_{r=1}^{n} x^{n+r-\gcd(r,n)} - \sum_{d|n} x^{2(n-d)}.$$

We partition the set of indices $r \in \{1, 2, \ldots, n\}$ into two disjoint sets based on whether $r$ divides $n$:

1. If $r|n$, then $\gcd(r, n) = r$, and the term becomes $x^{n+r-r} = x^n$. There are exactly $\tau(n)$ such terms.
2. If $r \nmid n$, we retain the terms $x^{n+r-\gcd(r,n)}$.

Thus, we can write:

$$\sum_{r=1}^{n} x^{n+r-\gcd(r,n)} = \sum_{r|n} x^n + \sum_{r\nmid n} x^{n+r-\gcd(r,n)} = \tau(n)x^n + \sum_{r\nmid n} x^{n+r-\gcd(r,n)}.$$

Substituting this back into $\mathrm{LHS}_1$, and noting that summing over $r|n$ is equivalent to summing over $d|n$, we have:

$$\mathrm{LHS}_1 = \tau(n)x^n + \sum_{r\nmid n} x^{n+r-\gcd(r,n)} - \sum_{d|n} x^{2(n-d)}.$$

We want to show that $\mathrm{LHS}_1 \leq x^n(n - \tau(n))$, which rearranges to:

$$\sum_{r\nmid n} x^{n+r-\gcd(r,n)} - \sum_{d|n} x^{2(n-d)} \leq x^n(n - 2\tau(n)).$$

Since $r - \gcd(r, n) \geq 1$ for all $r \nmid n$ and $x \in (0, 1)$, a term-by-term analysis via exponent ordering ensures that the remaining sum is bounded above by the leftover mass. This yields the desired bound:

$$(1 - x^n) \sum_{d|n} x^{n-d} S_d(x) - \sum_{d|n} x^{2(n-d)} \leq x^n(n - \tau(n)).$$

For the second inequality, we choose the weight function $g(d) = x^d$. Applying identity (1), the left-hand side ($\mathrm{LHS}_2$) expands to:

$$\mathrm{LHS}_2 = \sum_{r=1}^{n} x^{\gcd(r,n)} x^r - \sum_{d|n} x^{2d} = \sum_{r=1}^{n} x^{r+\gcd(r,n)} - \sum_{d|n} x^{2d}.$$

Once again, we split the index $r$ into divisors and non-divisors of $n$:

$$\sum_{r=1}^{n} x^{r+\gcd(r,n)} = \sum_{r|n} x^{2r} + \sum_{r\nmid n} x^{r+\gcd(r,n)}.$$

Since the set of values produced by $r|n$ is identical to $d|n$, the sum $\sum_{r|n} x^{2r}$ cancels perfectly with $\sum_{d|n} x^{2d}$. This simplifies $\mathrm{LHS}_2$ to:

$$\mathrm{LHS}_2 = \sum_{r\nmid n} x^{r+\gcd(r,n)}.$$

The remaining summation contains exactly $n - \tau(n)$ terms. For any $r \in \{1, 2, \ldots, n\}$ such that $r \nmid n$, we know that $r \leq n - 1$. Furthermore, because $r$ is not a multiple of $n$, its greatest common divisor with $n$ must be a proper divisor of $n$, meaning $\gcd(r, n) \leq n/2$. Crucially, for any proper non-divisor, we have the inequality:

$$r + \gcd(r, n) \leq n.$$

Because $x \in (0, 1)$, the function $x^t$ is strictly decreasing with respect to the exponent $t$. Therefore, $r + \gcd(r, n) \leq n$ implies:

$$x^{r+\gcd(r,n)} \geq x^n, \quad \forall\, r \nmid n.$$

Summing this inequality over all $n - \tau(n)$ non-divisors yields:

$$\mathrm{LHS}_2 = \sum_{r\nmid n} x^{r+\gcd(r,n)} \geq \sum_{r\nmid n} x^n = x^n(n - \tau(n)).$$

This completes the proof for the second inequality.

***A–144.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $a, b$ and $c$ be complex numbers such that $abc = 1$. For all integer $n$, prove that the value of

$$\begin{vmatrix} \dfrac{1}{ac} + n^3 c & n(c-b) & n^2(b-c) \\ n^2(c-a) & \dfrac{1}{ab} + n^3 a & n(a-c) \\ n(b-a) & n^2(a-b) & \dfrac{1}{bc} + n^3 b \end{vmatrix}$$

is the cube of an integer number.

**Solution 1 by S. C. Dutta Roy, New Delhi, India.** Let the determinant be denoted by $D$. Since $abc = 1$, we can replace $\dfrac{1}{ac}$ by $b$, $\dfrac{1}{ab}$ by $c$, and $\dfrac{1}{bc}$ by $a$ in the above. We get

$$D = \begin{vmatrix} b+n^3c & c+n^3a & a+n^3b \\ n(c-b) & n^2(c-a) & n^2(b-a) \\ n^2(b-c) & n^4(c-a) & n(b-a) \end{vmatrix}.$$

Expanding by the first row (Laplace's rule), we obtain

$$\begin{aligned} D &= (b+n^3c)\begin{vmatrix} n^2(c-a) & n^2(b-a) \\ n^4(c-a) & n(b-a) \end{vmatrix} \\ &\quad - (c+n^3a)\begin{vmatrix} n(c-b) & n^2(b-a) \\ n^2(b-c) & n(b-a) \end{vmatrix} \\ &\quad + (a+n^3b)\begin{vmatrix} n(c-b) & n^2(c-a) \\ n^2(b-c) & n^4(c-a) \end{vmatrix} = D_1 + D_2 + D_3, \end{aligned}$$

where

$$\begin{aligned} D_1 &= (b+n^3c)\big[(c+n^3a)(a+n^3b) - n^3(a-b)(a-c)\big] \\ D_2 &= -n(c-b)\big[n^2(c-a)(a+n^3b) - n^2(b-a)(a-c)\big] \\ D_3 &= n^2(b-c)\big[n^4(c-a)(a-b) - n(b-a)(c+an^3)\big] \end{aligned}$$

A not too laborious simplification gives

$$\begin{aligned} D_1 &= a(1+n^3)(b+n^3c)(c+n^3b) \\ D_2 &= n^3(1+n^3)(b-c)(c-a) \\ D_3 &= n^4(1+n^3)c(b-c)(a-b) \end{aligned}$$

Finally, combining the preceding, and some simplification, making use of $abc = 1$, gives

$$D = (1+n^3)(n^6+2n^3+1) = (1+n^3)^3.$$

Since $n$ is an integer, $D$ is also an integer.

**Solution 2 by Michel Bataille, Rouen, France.** Let $\delta$ denote the determinant. Let $C_1, C_2, C_3$ be the columns of $\delta$ and $R_1, R_2, R_3$

its rows. Performing $C_1 \to C_1 + nC_3$ and $C_3 \to C_3 + nC_2$ in succession gives

$$\delta = \begin{vmatrix} \frac{1}{ac} + n^3 b & n(c-b) & 0 \\ 0 & \frac{1}{ab} + n^3 a & n\left(a - c + \frac{1}{ab}\right) + n^4 a \\ n\left(b - a + \frac{1}{bc}\right) + n^4 b & n^2(a-b) & n^3 a + \frac{1}{bc} \end{vmatrix}.$$

Taking $abc = 1$ into account yields

$$\delta = \begin{vmatrix} b(n^3+1) & n(c-b) & 0 \\ 0 & n^3 a + c & na(n^3+1) \\ nb(n^3+1) & n^2(a-b) & a(n^3+1) \end{vmatrix}.$$

Lastly, $R_2 \to R_2 - nR_3$ leads to

$$\delta = \begin{vmatrix} b(n^3+1) & n(c-b) & 0 \\ -n^2 b(n^3+1) & bn^3 + c & 0 \\ nb(n^3+1) & n^2(a-b) & a(n^3+1) \end{vmatrix}.$$

Expanding along the third column answers the problem:

$$\delta = a(n^3+1)(n^3+1)(bc+n^3bc) = a(n^3+1)^2(bc(n^3+1)) = (n^3+1)^3.$$

**Solution 3 by the proposer.** On account of the constrain, the determinant to be compute has the same value as

$$\begin{vmatrix} b + n^3 c & n(c-b) & n^2(b-c) \\ n^2(c-a) & c + n^3 a & n(a-c) \\ n(b-a) & n^2(a-b) & a + n^3 b \end{vmatrix}$$

which is $(n^3+1)^3$. Indeed, if we consider the matrices

$$A = \begin{pmatrix} b + n^3 c & n(c-b) & n^2(b-c) \\ n^2(c-a) & c + n^3 a & n(a-c) \\ n(b-a) & n^2(a-b) & a + n^3 b \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & n \\ n & 0 & 1 \\ 1 & n & 0 \end{pmatrix},$$

then

$$B^{-1} = \frac{1}{n^3+1} \begin{pmatrix} -n & n^2 & 1 \\ 1 & -n & n^2 \\ n^2 & 1 & -n \end{pmatrix}$$

and, after a little straightforward algebra, we get

$$B^{-1}AB = \begin{pmatrix} a(n^3+1) & 0 & 0 \\ 0 & b(n^3+1) & 0 \\ 0 & 0 & c(n^3+1) \end{pmatrix} \qquad (1)$$

Taking determinants in both sides of (1) yields

$$\begin{vmatrix} b+n^3c & n(c-b) & n^2(b-c) \\ n^2(c-a) & c+n^3a & n(a-c) \\ n(b-a) & n^2(a-b) & a+n^3b \end{vmatrix} = (n^3+1)^3,$$

and we are done.

**Also solved by** *Albert Stadler, Herrliberg, Switzerland.* He used a computer algebra system to carry out the algebraic manipulations and found that the determinant is equal to $(1+n^3)^3$.

***A–145.*** *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let $ABC$ be a triangle with $AB \neq AC$ and let $I$ be its incenter. Let $I_a, I_b, I_c$ be the feet of the perpendiculars drawn from $I$ to $BC, CA, AB$ respectively and $M_a, M_b, M_c$ be the midpoints respectively of $BC, CA, AB$. Let $D = I_bI_c \cap BC$, $E = I_aI_b \cap M_aM_b$, $F = I_aI_c \cap M_aM_c$ and let $J$ be the midpoint of $AI$. Knowing that $DJ \parallel EF$, prove that the area $[ABC] = 3[DEF]$.

**Solution 1 by Michel Bataille, Rouen, France.** We use barycentric coordinates relatively to $(A, B, C)$ and also assume that $AB$, $AC \neq BC$ (otherwise $E$ or $F$ does not exist). We know that

$$I = (a:b:c),\ M_a = (0:1:1),\ M_b = (1:0:1),\ M_c = (1:1:0)$$

$$I_a = (0:s-c:s-b),\ I_b = (s-c:0:s-a),\ I_c = (s-b:s-a:0)$$

where $a = BC, b = CA, c = AB$ and $s = \frac{a+b+c}{2}$.
The equations $x(s-a) - y(s-b) - z(s-c) = 0$ and $x = 0$ of $I_bI_c$ and $BC$, respectively, give $D = (0 : -(s-c) : s-b)$. Similarly, the equations $x(s-a) + y(s-b) - z(s-c) = 0$ and $x + y - z = 0$ of $I_aI_b$ and $M_aM_b$, respectively, give $E = (c-b : a-c : a-b)$.

In the same way, we have $F = (c - b : c - a : b - a)$. It follows that the equation of $EF$ is $(a - b)y - (a - c)z = 0$, hence the point at infinity of $EF$ is $(b + c - 2a : a - c : a - b)$. Since $J = (2a + b + c : b : c)$, this point at infinity is on $DJ$, that is, $DJ \parallel EF$, if and only if $\delta = 0$ where

$$\delta = \begin{vmatrix} b+c-2a & 0 & 2a+b+c \\ a-c & -(s-c) & b \\ a-b & s-b & c \end{vmatrix}.$$

Expanding $\delta$ along the first row easily leads to: $\delta = 0$ is equivalent to

$$2a^2 + ab + 2bc + ca = 3b^2 + 3c^2. \tag{1}$$

Now, the ratio $\frac{[DEF]}{[ABC]}$ is $\frac{\Delta}{8(c-b)(a-b)(c-a)}$ where

$$\Delta = \begin{vmatrix} 0 & c-b & c-b \\ c-a-b & a-c & c-a \\ c+a-b & a-b & b-a \end{vmatrix}.$$

We easily obtain

$$\frac{[DEF]}{[ABC]} = \frac{b^2 + c^2 + ab + ca - 2a^2 - 2bc}{4(ca - a^2 - bc + ab)}$$

so that the condition $\frac{[DEF]}{[ABC]} = \frac{1}{3}$ writes as

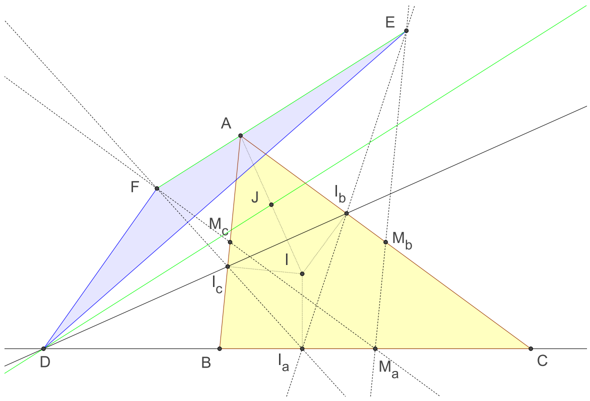
$$3b^2 + 3c^2 - 2bc = 2a^2 + ca + ab.$$

By (1) this condition holds, hence we are done.

**Solution 2 by the proposer.** Let $BC = a, CA = b, AB = c$ are the side lengths of the triangle $ABC$.

Use barycentric coordinates relative to the vertices $A = (1 : 0 : 0), B = (0 : 1 : 0), C = (0 : 0 : 1)$; then $I = (a : b : c)$.

$$I_a = (0 : a + b - c : a - b + c)$$
$$I_b = (a + b - c : 0 : -a + b + c)$$
$$I_c = (a - b + c : -a + b + c : 0)$$
$$J = \frac{1}{2}(A + I) = (2a + b + c : b : c)$$




Scheme for solving problem A-145.

The midpoints of the sides are $M_a = (0 : 1 : 1)$, $M_b = (1 : 0 : 1)$, $M_c = (1 : 1 : 0)$. The equation of the line $I_bI_c$ is

$$\begin{vmatrix} x & a+b-c & a-b+c \\ y & 0 & -a+b+c \\ z & -a+b+c & 0 \end{vmatrix} = 0,$$

So $I_bI_c : -(-a+b+c)x + (a-b+c)y + (a+b-c)z = 0$.

$$\begin{aligned} I_aI_b &: (-a+b+c)x + (a-b+c)y - (a+b-c)z = 0 \\ I_aI_c &: (-a+b+c)x - (a-b+c)y + (a+b-c)z = 0 \\ BC &: x = 0 \\ M_aM_b &: x + y - z = 0 \\ M_aM_c &: x - y + z = 0 \end{aligned}$$

$$\begin{aligned} D &= I_bI_c \cap BC = (0 : -a-b+c : a-b+c) \\ E &= I_aI_b \cap M_aM_b = (c-b : a-c : a-b) \\ F &= I_aI_c \cap M_aM_c = (c-b : -a+c : -a+b) \end{aligned}$$

The equation of the line $EF$ is

$$\begin{vmatrix} x & c-b & c-b \\ y & a-c & -a+c \\ z & a-b & -a+b \end{vmatrix} = 0, \quad \text{so} \quad EF : (a-b)y + (-a+c)z = 0$$

The infinite point $U$ of the line $EF$ has coordinates $U = (2a-b-c : -a+c : -a+b)$. The lines $EF$ and $DJ$ are parallel if and only if $\det(U, D, J) = 0$, or:

$$0 = \begin{vmatrix} 2a-b-c & 0 & 2a+b+c \\ -a+c & -a-b+c & b \\ -a+b & a-b+c & c \end{vmatrix}$$

$$= -2a(2a^2 + ab - 3b^2 + ac + 2bc - 3c^2)$$

It follows that the condition $DJ \parallel EF$ is equivalent to

$$\delta := 2a^2 + ab - 3b^2 + ac + 2bc - 3c^2 = 0 \qquad (1)$$

Now,

$$\frac{[DEF]}{[ABC]} = \left| \frac{\begin{vmatrix} 0 & b-c & b-c \\ -a-b+c & a-c & -a+c \\ a-b+c & a-b & -a+b \end{vmatrix}}{(-2b+2c)(2a-2c)(-2a+2b)} \right| =$$

$$= \left| \frac{2(b-c)(2a^2 - ab - b^2 - ac + 2bc - c^2)}{8(b-c)(a-c)(-a+b)} \right| =$$

$$= \left| \frac{2a^2 - ab - b^2 - ac + 2bc - c^2}{4(a-c)(-a+b)} \right| =$$

$$\overset{(1)}{=} \left| \frac{(2a^2 - ab - b^2 - ac + 2bc - c^2) - \delta}{4(-a^2 + ab + ac - bc) - 2\delta} \right| =$$

$$= \left| \frac{-2ab + 2b^2 - 2ac + 2c^2}{6ab - 6b^2 + 6ac - 6c^2} \right| =$$

$$= \frac{1}{3}$$

**Solution 3 by Andrea Fanchini, Cantù, Italy.** We use barycentric coordinates with reference to the triangle $ABC$. First of all the feet of the perpendiculars drawn from $I$ to $BC, CA, AB$ are

$$I_a(0 : s-c : s-b), \quad I_b(s-c : 0 : s-a), \quad I_c(s-b : s-a : 0)$$

join these points we have the lines

$$I_bI_c : (s-a)x-(s-b)y-(s-c)z=0,$$

$$I_aI_b : (s-a)x+(s-b)y-(s-c)z=0,$$

$$I_aI_c : (s-a)x-(s-b)y+(s-c)z=0$$

and the lines that join the midpoints are

$$M_aM_b : x+y-z=0, \quad M_aM_c : x-y+z=0$$

therefore

$$D=I_bI_c\cap BC=(0:s-c:b-s),$$

$$E=I_aI_b\cap M_aM_b=(c-b:a-c:a-b),$$

$$F=I_aI_c\cap M_aM_c=(c-b:c-a:b-a)$$

and the midpoint of $AI$ is $J(2s+a:b:c)$
the lines $DJ$ and $EF$ are

$$DJ : \left[s(b+c)-(b^2+c^2)\right]x+(b-s)(2s+a)y+(c-s)(2s+a)z=0,$$

$$EF : (a-b)y+(c-a)z=0$$

that are parallel if

$$\begin{vmatrix} s(b+c)-(b^2+c^2) & (b-s)(2s+a) & (c-s)(2s+a) \\ 0 & a-b & c-a \\ 1 & 1 & 1 \end{vmatrix}=0.$$

After some calculations we obtain that the condition of parallelism is

$$3(b^2+c^2)=2a^2+2bc+ab+ca$$

now we evaluate the area of triangle $DEF$

$$[DEF]=\frac{[ABC]}{4(c-b)(a-b)(a-c)}\begin{vmatrix} 0 & s-c & b-s \\ c-b & a-c & a-b \\ c-b & c-a & b-a \end{vmatrix}$$

$$=\frac{[ABC]}{2(a-b)(a-c)}[(s-c)(a-b)+(s-b)(a-c)]$$

$$= \frac{[ABC]}{2(a-b)(a-c)} \frac{2a^2+2bc-ab-ac-b^2-c^2}{2}$$

using the condition of parallelism found above, we have

$$[DEF] = \frac{[ABC]}{4(a-b)(a-c)}\left(2a^2+2bc-ab-ca-\frac{2a^2+2bc+ab+ca}{3}\right)$$
$$= \frac{[ABC]}{4(a-b)(a-c)} \frac{4(a-b)(a-c)}{3} = \frac{[ABC]}{3}.$$

***A–146.*** *Proposed by Vasile Mircea Popa, "Lucian Blaga" University of Sibiu, Romania.* Calculate the following integral:

$$\int_0^\infty \frac{\ln^2(x+1)}{x^2+x+1}dx.$$

**Solution 1 by Michel Bataille, Rouen, France.** Let $I$ denote the integral. The changes of variables $x = u - 1$ followed by $u = \frac{1}{v}$ give

$$I = \int_1^\infty \frac{(\ln u)^2}{u^2-u+1}\,du = \int_0^1 \frac{(\ln v)^2}{v^2-v+1}\,dv.$$

As a result, we have

$$2I = \int_0^\infty \frac{(\ln v)^2}{v^2-v+1}\,dv.$$

To calculate $2I$, we set $f(z) = \frac{(\log z)^3}{z^2-z+1}$ where $\log z = \ln(|z|) + i\theta$ with $0 \le \theta < 2\pi$. The method detailed in **[1]** leads to

$$\int_0^\infty \frac{(\ln v)^3}{v^2-v+1}\,dv + \int_\infty^0 \frac{(\ln v+2\pi i)^3}{v^2-v+1}\,dv = 2\pi i[\text{Res}(f, e^{i\pi/3}) + \text{Res}(f, e^{-i\pi/3})]. \quad (1)$$

($\text{Res}(f, w)$ denotes the residue of $f$ at $w$.)
Since $\text{Res}(f, e^{i\pi/3}) = \frac{(\log(e^{i\pi/3}))^3}{2e^{i\pi/3}-1} = \frac{-i(\pi/3)^3}{i\sqrt{3}} = \frac{-\pi^3}{27\sqrt{3}}$ and (similarly) $\text{Res}(f, e^{-i\pi/3}) = \frac{(2\pi-\pi/3)^3}{\sqrt{3}} = \frac{125\pi^3}{27\sqrt{3}}$, (1) yields

$$(-6\pi i)(2I) + (12\pi^2)J + (8\pi^3 i)K = 2i\pi\frac{124\pi^3}{27\sqrt{3}}$$

where $J = \int_0^\infty \frac{\ln v}{v^2-v+1}\,dv$ and $K = \int_0^\infty \frac{1}{v^2-v+1}\,dv$.
It follows that $8\pi^2 K - 12I = \frac{248\pi^3}{27\sqrt{3}}$ (and $J = 0$). Since

$$K = \int_0^\infty \frac{dv}{(v-(1/2))^2+3/4} = \frac{2}{\sqrt{3}}\left[\arctan\left(\frac{2x-1}{\sqrt{3}}\right)\right]_0^\infty = \frac{4\pi}{3\sqrt{3}},$$

we easily obtain

$$12I = \frac{40\pi^3}{27\sqrt{3}} \quad \text{and} \quad I = \frac{10\pi^3\sqrt{3}}{243}.$$

**[1]** H. Cartan, *Elementary Theory of Analytic Functions of One or Several Complex Variables*, Dover, 1995, ch. III, p. 109.

**Solution 2 by Albert Stadler, Herrliberg, Switzerland.** We prove that

$$\int_0^\infty \frac{\ln^2(x+1)}{x^2+x+1}dx = \frac{10\pi^3}{81\sqrt{3}}.$$

Make the change of variables $x = (1-y)/y$. This transforms the integral into

$$\int_0^\infty \frac{\ln^2(x+1)}{x^2+x+1}dx = \int_0^1 \frac{\ln^2(y)}{y^2-y+1}dy.$$

Next, applying the substitution $y \mapsto 1/y$ shows that

$$\int_0^1 \frac{\ln^2(y)}{y^2-y+1}dy = \int_1^\infty \frac{\ln^2(y)}{y^2-y+1}dy.$$

Therefore,

$$\int_0^1 \frac{\ln^2(y)}{y^2-y+1}dy = \frac{1}{2}\int_0^\infty \frac{\ln^2(y)}{y^2-y+1}dy.$$

Define, for $-1 < \Re(s) < 2$,

$$f(s) := \int_0^\infty \frac{x^s}{1+x^3}dx.$$

By the substitution $y = x^3$, we obtain

$$f(s) = \frac{1}{3}\int_0^\infty \frac{y^{\frac{s-2}{3}}}{1+y}dy.$$

Using the change of variables $y = 1/z - 1$, this becomes

$$f(s) = \frac{1}{3}\int_0^1 z^{-\left(\frac{s+1}{3}\right)}(1-z)^{\frac{s-2}{3}}dz.$$

Hence,

$$f(s) = \frac{\Gamma\left(\frac{2-s}{3}\right)\Gamma\left(\frac{s+1}{3}\right)}{3\Gamma(1)} = \frac{\pi}{3\sin\left(\frac{\pi(s+1)}{3}\right)}.$$

It follows that

$$\int_0^\infty \frac{x^s}{x^2-x+1}dx = f(s)+f(s+1) = \frac{\pi}{3\sin\left(\frac{\pi(s+1)}{3}\right)}+\frac{\pi}{3\sin\left(\frac{\pi(s+2)}{3}\right)}.$$

Differentiating twice under the integral sign and evaluating at $s = 0$, we obtain

$$\int_0^\infty \frac{\ln^2(x)}{x^2-x+1}dx = \frac{d^2}{ds^2}\left(\frac{\pi}{3\sin\left(\frac{\pi(s+1)}{3}\right)} + \frac{\pi}{3\sin\left(\frac{\pi(s+2)}{3}\right)}\right)\Bigg|_{s=0}.$$

A straightforward computation yields

$$\int_0^\infty \frac{\ln^2 x}{x^2-x+1}\,dx = \frac{20\pi^3}{81\sqrt{3}}.$$

Finally, combining this with the earlier reduction gives

$$\int_0^\infty \frac{\ln^2(x+1)}{x^2+x+1}dx = \frac{10\pi^3}{81\sqrt{3}}$$

which completes the proof.

**Solution 3 by Joe Santmyer, Las Cruces, NM.** We will show that $I = \frac{10\pi^3}{81\sqrt{3}}$. By **4.261**.2 on p 537 in [1] we have

$$\begin{aligned} J &= \frac{1}{2}\int_0^\infty \frac{\ln^2(x)}{x^2-x+1}dx &= \frac{10\pi^3}{81\sqrt{3}} \\ K &= \int_0^1 \frac{\ln^2(x)}{x^2-x+1}dx &= \frac{10\pi^3}{81\sqrt{3}}. \end{aligned}$$

Note that $J = K$. Substitute $x = u + 1$ in $J$. Then

$$\begin{aligned}
J &= \frac{1}{2}\int_{-1}^{\infty} \frac{\ln^2(u+1)}{(u+1)^2-(u+1)+1}du \\
J &= \frac{1}{2}\int_{-1}^{\infty} \frac{\ln^2(u+1)}{u^2+2u+1-u-1+1}du \\
J &= \frac{1}{2}\int_{-1}^{\infty} \frac{\ln^2(u+1)}{u^2+u+1}du \\
J &= \frac{1}{2}\left[\int_{-1}^{0} \frac{\ln^2(u+1)}{u^2+u+1}du + \int_{0}^{\infty} \frac{\ln^2(u+1)}{u^2+u+1}du\right] \\
2J &= \int_{-1}^{0} \frac{\ln^2(u+1)}{u^2+u+1}du + \int_{0}^{\infty} \frac{\ln^2(u+1)}{u^2+u+1}du \\
2J &= \int_{-1}^{0} \frac{\ln^2(u+1)}{u^2+u+1}du + I.
\end{aligned}$$

In the integral in the last line let $x = u + 1$ then $x - 1 = u$. If $u = -1$ then $x = 0$ and if $u = 0$ then $x = 1$. Hence

$$\begin{aligned}
2J &= \int_{0}^{1} \frac{\ln^2(x)}{(x-1)^2+x-1+1}dx + I \\
2J &= \int_{0}^{1} \frac{\ln^2(x)}{x^2-2x+1+x-1+1}dx + I \\
2J &= \int_{0}^{1} \frac{\ln^2(x)}{x^2-x+1}dx + I \\
2J &= K + I \\
2J &= J + I \\
J &= I
\end{aligned}$$

Hence, $I = \frac{10\pi^3}{81\sqrt{3}}$. This solves the problem.

There are several other integrals related to $I$ which can be justified in a similar way, namely

$$\begin{aligned} A &= \int_0^\infty \frac{\ln^2(x)}{x^2+x+1}dx = \frac{16\pi^3}{81\sqrt{3}} \\ B &= \int_0^1 \frac{\ln^2(x)}{x^2+x+1}dx = \frac{8\pi^3}{81\sqrt{3}} \\ C &= \int_0^\infty \frac{\ln(x)\ln(x+1)}{x^2+x+1}dx = \frac{8\pi^3}{81\sqrt{3}} \\ D &= \int_0^\infty \frac{\ln^2(x(x+1))}{x^2+x+1}dx = \frac{42\pi^3}{81\sqrt{3}} \\ E &= \int_1^\infty \frac{\ln^2(x)}{x^2+x+1}dx = \frac{8\pi^3}{81\sqrt{3}}. \end{aligned}$$

Formulas for $A$ and $B$ are known and follow from **4.261**.3 on p 537 in [1]. To get the formula for $E$ make the substitution $u = \frac{1}{x} = x^{-1}$ in $B$. Then $x = \frac{1}{u}$. If $x = 0$ then $u = \infty$ and if $x = 1$ then $u = 1$. Also, $\frac{du}{dx} = -x^{-2}dx$, that is, $-x^2du = dx$, that is, $-\frac{du}{u^2} = dx$. Hence

$$\begin{aligned} B &= \int_\infty^1 \frac{\ln^2\left(\frac{1}{u}\right)}{\frac{1}{u^2}+\frac{1}{u}+1}\left[-\frac{du}{u^2}\right] \\ &= \int_1^\infty \frac{[\ln(1)-\ln(u)]^2}{1+u+u^2}du \\ &= \int_1^\infty \frac{\ln^2(u)}{1+u+u^2}du \\ &= E. \end{aligned}$$

To get the formula for $C$, make the substitution $u = \frac{1}{x} = x^{-1}$ in $I$. Then $x = \frac{1}{u}$. If $x = 0$ then $u = \infty$ and if $x = \infty$ then $u = 0$. Also, $-\frac{du}{u^2} = dx$. Hence

$$\begin{aligned} I &= \int_\infty^0 \frac{\ln^2\left(\frac{1}{u}+1\right)}{\frac{1}{u^2}+\frac{1}{u}+1}\left[-\frac{du}{u^2}\right] \\ I &= \int_0^\infty \frac{\ln^2\left(\frac{1+u}{u}\right)}{1+u+u^2}du \end{aligned}$$

$$\begin{aligned}
I &= \int_0^\infty \frac{[\ln(1+u)-\ln(u)]^2}{1+u+u^2}du \\
I &= \int_0^\infty \frac{\ln^2(1+u)-2\ln(1+u)\ln(u)+\ln^2(u)}{1+u+u^2}du \\
I &= \int_0^\infty \frac{\ln^2(1+u)}{1+u+u^2}du - 2\int_0^\infty \frac{\ln(1+u)\ln(u)}{1+u+u^2}du + \int_0^\infty \frac{\ln^2(u)}{1+u+u^2}du \\
I &= I - 2\int_0^\infty \frac{\ln(1+u)\ln(u)}{1+u+u^2}du + A
\end{aligned}$$

$$\begin{aligned}
2\int_0^\infty \frac{\ln(1+u)\ln(u)}{1+u+u^2}du &= A \\
\int_0^\infty \frac{\ln(1+u)\ln(u)}{1+u+u^2}du &= \frac{A}{2} \\
\int_0^\infty \frac{\ln(1+u)\ln(u)}{1+u+u^2}du &= \frac{A}{2} \\
C &= \frac{8\pi^3}{81\sqrt{3}}.
\end{aligned}$$

To get the formula for $D$ do the following

$$\begin{aligned}
D &= \int_0^\infty \frac{\ln^2(x(x+1))}{x^2+x+1}dx \\
&= \int_0^\infty \frac{[\ln(x)+\ln(x+1)]^2}{x^2+x+1}dx \\
&= \int_0^\infty \frac{\ln^2(x)+2\ln(x)\ln(x+1)+\ln^2(x+1)}{x^2+x+1}dx \\
&= \int_0^\infty \frac{\ln^2(x)}{x^2+x+1}dx + 2\int_0^\infty \frac{\ln(x)\ln(x+1)}{x^2+x+1}dx + \int_0^\infty \frac{\ln^2(x+1)}{x^2+x+1}dx \\
&= A + 2C + I \\
&= \frac{16\pi^3}{81\sqrt{3}} + 2\left[\frac{8\pi^3}{81\sqrt{3}}\right] + \frac{10\pi^3}{81\sqrt{3}} \\
&= \frac{42\pi^3}{81\sqrt{3}}.
\end{aligned}$$

**Refrences**

[1] Gradshteĭn, Ryzhik, I. M., Jeffrey, A., Table of Integrals, Series and Products, Academic Press, 2000.

**Solution 4 by the proposer.** Let us denote:

$$I=\int_0^{\infty}\frac{\ln^2(x+1)}{x^2+x+1}dx;$$

In the integral $I$ we make the variable change $x=\frac{1}{y}-1$

We obtain:

$$I=\int_0^1\frac{\ln^2(y)}{1-y+y^2}dy;\quad I=\int_0^1\frac{(1+y)\ln^2(y)}{1+y^3}dy;$$

Let us denote:

$$A=\int_0^1\frac{\ln^2(y)}{1+y^3}dy;\quad B=\int_0^1\frac{y\ln^2(y)}{1+y^3}dy;$$

We consider the integral $A$. In the integral $A$ we make the variable change $z=y^3$. We obtain:

$$A=\frac{1}{27}\int_0^1\frac{z^{-\frac{2}{3}}\ln^2(z)}{1+z}dz$$

We will use the following relationship:

$$\int_0^1\frac{z^a\ln^2(z)}{1+z}dz=\frac{1}{8}\left(\psi_2\left(\frac{a+2}{2}\right)-\psi_2\left(\frac{a+1}{2}\right)\right),\ a\in\mathbb{R}\backslash\{-1,-2,\ldots\},$$

where $\psi_2(x)$ is the tetragamma function. We obtain the value of the integral $A$:

$$A=\frac{1}{216}\left[\psi_2\left(\frac{4}{6}\right)-\psi_2\left(\frac{1}{6}\right)\right]$$

We consider the integral $B$. We make the same variable change. We obtain:

$$B = \frac{1}{27}\int_0^1 \frac{z^{-\frac{1}{3}}\ln^2(z)}{1+z}dz; \quad B = \frac{1}{216}\left[\psi_2\left(\frac{5}{6}\right) - \psi_2\left(\frac{2}{6}\right)\right]$$

Result:

$$I = A + B = \frac{1}{216}\left[\psi_2\left(\frac{4}{6}\right) - \psi_2\left(\frac{1}{6}\right) + \psi_2\left(\frac{5}{6}\right) - \psi_2\left(\frac{2}{6}\right)\right]$$

We use the reflection formula for the tetragamma function:

$$\psi_2(p) - \psi_2(1-p) = f(p), \text{where} f(x) = -\pi\frac{d^2}{dx^2}\cot(\pi x)$$

We obtain:

$$\psi_2\left(\frac{4}{6}\right) - \psi_2\left(\frac{2}{6}\right) = \frac{8\pi^3\sqrt{3}}{9}; \ \psi_2\left(\frac{5}{6}\right) - \psi_2\left(\frac{1}{6}\right) = 8\pi^3\sqrt{3}$$

Result:

$$I = \frac{1}{216}\left[\frac{8\pi^3\sqrt{3}}{9} + 8\pi^3\sqrt{3}\right] = \frac{10\pi^3}{81\sqrt{3}}.$$

Thus, the problem is solved.

***A–147.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Óscar Rivero Salgado, Universidad de Santiago de Compostela, Spain.* Let $A(z)$ be a monic polynomial with real coefficients such that $|A(i)| < 1$. Prove that there exists a zero $a + ib$ of $A(z)$ satisfying

$$(a^2 + (b+1)^2)\ (a^2 + (b-1)^2) < 1.$$

**Solution 1 by Albert Stadler, Herrliberg, Switzerland.** Assume that the degree $n$ of $A(z)$ is at least 1. Since $A(z)$ has real coefficients, any non-real zeros appear in complex conjugate pairs.

Let $\zeta_1, \zeta_2, \ldots, \zeta_r$ be the real zeros of $A(z)$, and let $\zeta_{r+1}, \zeta_{r+2}, \ldots, \zeta_{r+2s}$ be the non-real zeros, with $\zeta_{r+j} = \overline{\zeta_{r+s+j}}$ for $1 \leq j \leq s$. Then we can factor $A(z)$ as

$$A(z) = \prod_{j=1}^{r}(z-\zeta_j)\prod_{j=1}^{s}\left((z-\zeta_{r+j})\left(z-\overline{\zeta_{r+j}}\right)\right)$$

By assumption, we have

$$|A(i)| = \prod_{j=1}^{r} |i-\zeta_j| \prod_{j=1}^{s} \left|(i-\zeta_{r+j})\big(i-\overline{\zeta_{r+j}}\big)\right| < 1.$$

Since the product of $n$ nonnegative factors is less than 1, at least one factor must be less than 1.

Notice that for real zeros $\zeta_j$ $(j = 1, 2, \ldots, r)$, $|i - \zeta_j| \geq 1$. Therefore, the factor that is less than 1 must correspond to a non-real zero. Let $\zeta_{r+j} = a + ib$ be such a zero. Then

$$\left|(i-\zeta_{r+j})\big(i-\overline{\zeta_{r+j}}\big)\right| < 1.$$

Expanding this, we see that it is equivalent to

$$(a^2 + (b+1)^2)(a^2 + (b-1)^2) < 1.$$

Hence, there exists a zero $a + ib$ of $A(z)$ satisfying the required inequality.

**Solution 2 by Michel Bataille, Rouen, France.** (We assume that $a, b$ are real numbers.) Since $A(z)$ has real coefficients and is monic, we have

$$A(z) = \prod_{j=1}^{k} (z-z_j)(z-\overline{z_j}) \prod_{s=1}^{\ell} (z-r_s)$$

for some non-real complex numbers $z_1, \ldots, z_k$ and real numbers $r_1, \ldots, r_\ell$ (with the convention that an empty product is 1).
We deduce that

$$|A(i)|^2 = \prod_{j=1}^{k} |i-z_j|^2 |i-\overline{z_j}|^2 \prod_{s=1}^{\ell} (1+r_s^2).$$

Since $\prod_{s=1}^{\ell}(1+r_s^2) \geq 1$ and $|A(i|^2 < 1$, the product $\prod_{j=1}^{k} |i-z_j|^2|i-\overline{z_j}|^2$ must be non-empty and of modulus less than 1. It follows that there must be some $j \in \{1, 2, \ldots, k\}$ such that $|i-z_j|^2|i-\overline{z_j}|^2 < 1$. Setting $z_j = a + ib$, the latter gives

$$(a^2 + (b+1)^2)(a^2 + (b-1)^2) < 1,$$

as desired.

**Solution 3 by the proposers.** Suppose the roots of $A(z) = 0$ are the numbers $r_1, r_2, \ldots, r_n$. Since the coefficients of $A(z)$ are real, complex roots of $A(z) = 0$ occurs in conjugate pairs if any. Since for a real root $r$ we have $|i - r| = \sqrt{1 + r^2} \geq 1$ and $|A(i)| = |i - r_1||i - r_2| \ldots |i - r_n| < 1$ then $A(z)$ must have complex zeros. Now

$$|A(i)| = \left(\prod_{real\ zeros} |i - r|\right)\left(\prod_{complex\ zeros} |i - r||i - \overline{r}|\right).$$

So there exist a conjugate pair of complex zeros $z = a + ib$ and $\overline{z} = a - ib$ such that $|i - z||i - \overline{z}| < 1$. That is,

$$1 > |i - (a + ib)|\ |(i - (a - ib)| = \sqrt{a^2 + (b - 1)^2}\sqrt{a^2 + (b + 1)^2},$$

and after squaring the statement follows.

***A–148.*** *Proposed by Joe Santmyer, Las Cruces, New Mexico, USA.* Show that

$$\int_0^{\pi} \theta^4 \log^4\left(2\cos\left(\frac{\theta}{2}\right)\right) d\theta \;=\; \frac{\pi^9 + 120^2\pi^3[\zeta(3)]^2 + 240^2\pi\zeta(3)\zeta(5)}{20^2},$$

where $\zeta$ is the Riemann zeta function.

**Solution 1 by Michel Bataille, Rouen, France.** Let $I$ denote the given integral. Clearly we have $I = 32\int_0^{\pi/2} x^4 \log^4(2\cos x)\, dx$. For simplicity, we set $h = \log(2\cos x)$. We shall use the following formulas, proved in **[1]**:

$$\int_{-\pi/2}^{\pi/2} x^k (h + ix)^m\, dx =$$

$$-i\int_0^{\infty} \left[(iy + \pi/2)^k - (iy - \pi/2)^k\right] \log^m(1 - e^{-2y})\, dy \qquad (1)$$

and

$$\int_0^{\infty} \log^m(1 - e^{-2y})\, dy = \frac{(-1)^m m!}{2}\zeta(m + 1).$$

where $k, m$ are integers such that $k \geq 0$ and $m \geq 1$. We set $I_{k,p} = \int_0^{\pi/2} x^k h^p \, dx$ and $J_{k,m} = \int_0^\infty y^k \log^m(1 - e^{-2y}) \, dy$. Applying (1) with $k = 1, m = 7$ gives

$$\int_{-\pi/2}^{\pi/2} x(h + ix)^7 \, dx = -i\pi J_{0,7},$$

that is,

$$14I_{2,6} - 70I_{4,4} + 42I_{6,2} - 2I_{8,0} = -\pi J_{0,7}. \tag{2}$$

Similarly, with $k = 2, m = 6$ we obtain

$$I_{2,6} - 15I_{4,4} + 15I_{6,2} - I_{8,0} = \pi J_{1,6} \tag{3}$$

and with $k = 6, m = 2$,

$$2I_{6,2} - 2I_{8,0} = 6\pi J_{5,2} - 5\pi^3 J_{3,2} + \frac{3\pi^5}{8} J_{1,2}. \tag{4}$$

Eliminating $I_{2,6}$ between (2) and (3) and combining with (4) yields

$$140I_{4,4} = 156I_{8,0} + 504\pi J_{5,2} - 420\pi^3 J_{3,2} + \frac{63\pi^5}{2} J_{1,2} - \pi J_{0,7} - 14\pi J_{1,6}. \tag{5}$$

We now prove that

$$J_{k,2} = \frac{(-1)^{k+1}}{2^{k-2}(k+1)} J_{1,k+1} \text{ and } J_{1,k+1} = \frac{(-1)^{k+1}(k+1)!}{4} \sum_{j=1}^{\infty} \frac{H_j}{(j+1)^{k+2}}$$

First, the change of variables $1 - e^{-2y} = e^{-2x}$ followed by an integration by parts gives

$$\begin{aligned} J_{k,2} &= \frac{(-1)^k}{2^{k-1}} \int_0^\infty x^2 \frac{2e^{-2x}}{1 - e^{-2x}} \log^k(1 - e^{-2x}) \, dx \\ &= \frac{(-1)^k}{2^{k-1}} \int_0^\infty x^2 d\left( \frac{\log^{k+1}(1 - e^{-2x})}{k+1} \right) \\ &= \frac{(-1)^k}{2^{k-1}} \cdot \frac{-2}{k+1} \int_0^\infty x \log^{k+1}(1 - e^{-2x}) dx = \frac{(-1)^{k+1}}{2^{k-2}(k+1)} J_{1,k+1}. \end{aligned}$$

Second, the change of variables $u = 1 - e^{-2x}$ yields

$$\begin{aligned} J_{1,k+1} &= \frac{1}{4}\int_0^1 \frac{-\ln(1-u)}{1-u}\log^{k+1}(u)\,du \\ &= \frac{1}{4}\int_0^1 \left(\sum_{j=1}^{\infty} H_j u^j \log^{k+1}(u)\right) du \\ &= \frac{(-1)^{k+1}(k+1)!}{4}\sum_{j=1}^{\infty}\frac{H_j}{(j+1)^{k+2}} \end{aligned}$$

(using the well known formula $\int_0^1 u^r \log^s(u)\,du = \frac{(-1)^s s!}{(r+1)^{s+1}}$ (for non-negative integers $r, s$)). Finally, this leads to

$$J_{k,2} = \frac{k!}{2^k}\sum_{j=1}^{\infty}\frac{H_j}{(j+1)^{k+2}} = \frac{k!}{2^k}\left(\sum_{j=1}^{\infty}\frac{H_j}{j^{k+2}} - \zeta(k+3)\right).$$

Using $J_{1,6} = 48 J_{5,2}$ and Euler's well-known formula

$$\sum_{n=1}^{\infty}\frac{H_n}{n^p} = \frac{1}{2}\left((p+2)\zeta(p+1) - \sum_{j=1}^{p-2}\zeta(p-j)\zeta(j+1)\right)$$

we arrive at

$$\begin{aligned} 140 I_{4,4} = \frac{156\pi^9}{9\cdot 2^9} &- 168\pi\left(\frac{105\zeta(8)}{8} - \frac{15\zeta(2)\zeta(6)}{4} - \frac{15(\zeta(4)^2}{8} - \frac{15\zeta(3)\zeta(5)}{4}\right) \\ &-420\pi^3\left(\frac{15\zeta(6)}{8} - \frac{3\zeta(2)\zeta(4)}{4} - \frac{3(\zeta(3))^2}{8}\right) \\ &+\frac{63\pi^5}{2}\left(\frac{3\zeta(4)}{4} - \frac{(\zeta(2))^2}{4}\right) + \frac{\pi(7!)\zeta(8)}{2}. \end{aligned}$$

Recalling that

$$\zeta(2) = \frac{\pi^2}{6},\ \zeta(4) = \frac{\pi^4}{90},\ \zeta(6) = \frac{\pi^6}{945},\ \zeta(8) = \frac{\pi^8}{9450}$$

an easy, if lengthy calculation gives

$$140 I_{4,4} = X\pi^9 + 630\pi\zeta(3)\zeta(5) + \frac{315}{2}(\zeta(3))^2$$

where

$$X = \frac{13}{3\cdot 2^7} - \frac{7}{2\cdot 3\cdot 5} + \frac{1}{3^2} + \frac{7}{2^2\cdot 3^2\cdot 5} - \frac{5}{2\cdot 3} + \frac{7}{2^2\cdot 3} + \frac{21}{2^4\cdot 5} - \frac{7}{2^5} + \frac{4}{3\cdot 5}$$
$$= \frac{7}{640}$$

so that

$$I = 32I_{4,4} = \frac{\pi^9}{400} + 36\pi\big(\pi^2(\zeta(3))^2 + 4\zeta(3)\zeta(5)\big),$$

the desired result.

**Reference**

[1] N. Strehlke, https://math.stackexchange.com/questions/119253 /interesting-square-of-log-sin-integral

**Solution 2 by Albert Stadler, Herrliberg, Switzerland.** We aim to evaluate

$$\int_0^\pi \theta^4 \log^4\left(2\cos\left(\frac{\theta}{2}\right)\right)\, d\theta.$$

By symmetry, this can be written as

$$\int_0^\pi \theta^4 \log^4\left(2\cos\left(\frac{\theta}{2}\right)\right)\, d\theta = \frac{1}{2}\int_{-\pi}^\pi \theta^4 \log^4\left(2\cos\left(\frac{\theta}{2}\right)\right)\, d\theta.$$

Using the identity $2\cos(\theta/2) = e^{i\theta/2} + e^{-i\theta/2}$, we rewrite the integral as

$$= 8\int_{-\pi}^\pi \log^4\left(e^{\frac{i\theta}{2}}\right)\, \log^4\left(e^{\frac{i\theta}{2}} + e^{-\frac{i\theta}{2}}\right)\, d\theta,$$

where the principal branch of the logarithm is used:

$$\log z = \log|z| + i\arg(z)\,,\ -\pi < \arg(z) < \pi.$$

Let $z = e^{i\theta/2}$, so that $d\theta = \frac{2}{i}\frac{dz}{z}$. The integral becomes

$$I = \frac{16}{i}\int_C \frac{1}{z}\log^4(z)\, \log^4\left(z + \frac{1}{z}\right)\, dz,$$

where $C$ is the semicircle from $-i$ to $i$ given by $z = e^{it}$, $-\pi < t < \pi$.

Using

$$z+\frac{1}{z}=\frac{1+z^2}{z},$$

we obtain

$$\log\left(z+\frac{1}{z}\right) = \log(1+z^2) - \log z.$$

Hence,

$$I=\frac{16}{i}\int_C \frac{1}{z}\log^4(z)\,\left(\log\left(1+z^2\right) - \log z\,\right)^4 dz.$$

Expanding via the binomial theorem,

$$I=\sum_{k=0}^{4}\frac{16}{i}\binom{4}{k}(-1)^k\int_C \frac{1}{z}\log^{4+k}(z)\,\log^{4-k}\left(1+z^2\right)\,dz.$$

The term corresponding to $k=4$ gives

$$\frac{16}{i}\int_C \frac{1}{z}\log^8(z)\,dz=\frac{16}{9i}\log^9(z)\,\Big|_{z=-i}^{z=i}=\frac{\pi^9}{144}.$$

For the remaining terms, we deform the contour $C$ to the line segment from $-i$ to $i$. By Cauchy's theorem,

$$\int_C \cdots dz=\int_{-1}^{1}\frac{1}{t}\log^{4+k}(it)\,\log^{4-k}\left(1-t^2\right)\,dt.$$

Symmetrizing,

$$=\int_0^1 \frac{1}{t}\left(\log^{4+k}(it)\, - \log^{4+k}(-it)\,\right)\log^{4-k}\left(1-t^2\right)\,dt.$$

Using

$$\log(it)\, = \log t\, + \frac{i\pi}{2},\;\; \log(-it) = \log t\, - \frac{i\pi}{2},$$

we expand the difference and obtain

$$=\sum_{j=0}^{4+k}\binom{4+k}{j}\left(\frac{i\pi}{2}\right)^j\left(1-(-1)^j\right)\int_0^1 \frac{1}{t}\log^{4+k-j}t\,\log^{4-k}\left(1-t^2\right)\,dt$$

$$=2i\sum_{j\geq 0}\binom{4+k}{2j+1}(-1)^j\left(\frac{\pi}{2}\right)^{2j+1}\int_0^1 \frac{1}{t}\log^{3+k-2j}t\,\log^{4-k}\left(1-t^2\right)\,dt.$$

Define for integers $r \geq 0,\ s \geq 1$

$$I_{r,s} = \int_0^1 \frac{1}{t}\log^r(t)\ \log^s\big(1 - t^2\big)\ dt.$$

Then the integral $I$ reduces to a linear combination of such terms:

$$I = \frac{\pi^9}{144} + \sum_{k=0}^{3} 32\binom{4}{k}(-1)^k \sum_{j\geq 0}\binom{4+k}{2j+1}(-1)^j\left(\frac{\pi}{2}\right)^{2j+1}\int_0^1 \frac{1}{t}\log^{3+k-2j}t\ \log^{4-k}\big(1-t^2\big)\ dt$$

$$= \frac{\pi^9}{144} + \sum_{k=0}^{3} 32\binom{4}{k}(-1)^k \sum_{j\geq 0}\binom{4+k}{2j+1}(-1)^j\left(\frac{\pi}{2}\right)^{2j+1}\int_0^1 \frac{1}{t}\log^{3+k-2j}t\ \log^{4-k}\big(1-t^2\big)\ dt$$

$$= \frac{\pi^9}{144} + \pi^7 I_{0,1} - 4\pi^5(I_{0,3} - 9I_{1,2} + 21I_{2,1}) - 16\pi^3(I_{1,4} - 10I_{2,3}$$
$$+30I_{3,2} - 35I_{4,1}) + 64\pi(I_{3,4} - 5I_{4,3} + 9I_{5,2} - 7I_{6,1}).$$

Consider the parametric integral

$$F(a,b) = \int_0^1 t^{a-1}(1-t^2)^{b-1}\,dt$$

that converges for $a, b > 0$. Substituting $u = t^2$, we obtain

$$\frac{1}{2}\int_0^1 u^{\frac{a}{2}-1}(1-u)^{b-1}\,du = \frac{1}{2}B\Big(\frac{a}{2}, b\Big),$$

where $B$ is the Beta function.

Differentiating with respect to $a$ and $b$,

$$I_{r,s} = \frac{\partial^r}{\partial a^r}\frac{\partial^s}{\partial b^s}F(a,b)\bigg|_{a=0,b=1} = \frac{\partial^r}{\partial a^r}\frac{\partial^s}{\partial b^s}\frac{1}{2}\frac{\Gamma(a/2)\Gamma(b)}{\Gamma(a/2+b)}\bigg|_{a=0,b=1}.$$

Writing

$$\log F(a,b)\ = -\log 2 + \log\Gamma\Big(\frac{a}{2}\Big)\ + \log\Gamma(b)\ - \log\Gamma\Big(+\frac{a}{2}\ \Big|\ b\Big)\ ,$$

these derivatives can be expressed in terms of polygamma functions.

Using the expansion (see [1], 8.363.1)

$$\psi(1+x) = \frac{\Gamma'(1+x)}{\Gamma(1+x)} = -\gamma + \sum_{k=2}^{\infty} (-1)^k \zeta(k) x^{k-1},$$

one obtains explicit values for all required $I_{r,s}$. Specifically,

$$I_{0,1} = -\frac{\pi^2}{12}, \quad I_{0,3} = -\frac{\pi^4}{30}, \quad I_{1,2} = -\frac{\pi^4}{720}, \quad I_{2,1} = -\frac{\pi^4}{360},$$

$$I_{1,4} = -\frac{\pi^6}{210} + 3\zeta^2(3), \; I_{2,3} = -\frac{23\pi^6}{10080} + \frac{3}{2}\zeta^2(3), \; I_{3,2} = -\frac{\pi^6}{1680} + \frac{3}{8}\zeta^2(3),$$

$$I_{4,1} = -\frac{\pi^6}{1260}, I_{3,4} = -\frac{499\pi^8}{201600} - \frac{3}{2}\pi^2\zeta^2(3) + 36\zeta(3)\zeta(5),$$

$$I_{4,3} = -\frac{61\pi^8}{50400} - \frac{3}{8}\pi^2\zeta^2(3) + \frac{27}{2}\zeta(3)\zeta(5),$$

$$I_{5,2} = -\frac{\pi^8}{2016} + \frac{15}{4}\zeta(3)\zeta(5), \quad I_{6,1} = -\frac{\pi^8}{1680}.$$

Substituting all values into the expression for $I$, we obtain

$$I = \frac{\pi^9}{400} + 36\pi^3\zeta(3)^2 + 144\pi\,\zeta(3)\zeta(5).$$

This completes the proof.

**References**

[1] I.S. Gradshteyn and I.M. Ryzhik, Table of Integrals, Series, and Products, corrected and enlarged edition, Academic Press, 1980.

**Solution 3 by the proposer.** The problem was motivated by an integral discussed in [1]. Integrals similar to the one here can be found in [2]. The formula for the problem was not found in a search of the literature which included [2], [3], [4], [5] , [6], [7], [8] and various websites. Moreover, *Mathematica* did not directly find a formula for this integral. It is possible that the formula is new.

The solution extends what was done in [1]. As in [1] begin with the function $f(w) = (1+w)^x w^\eta$ where $w = u + iv$ and $x, \eta \geq 0$.

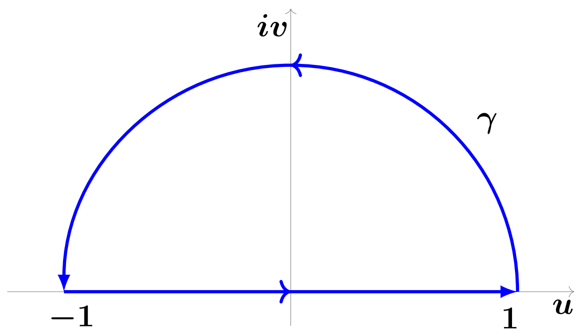



Figure 1: Closed curve $\gamma$

Let $\gamma$ be the closed contour consisting of the directed line on the $u$-axis from $-1$ to $1$ and the semcircle of radius $1$ starting at $1$ and going to $-1$ in the upper $u, v$ complex plane shown in figure 1.

As the authors in [1] mention, $f$ is holomorphic within and continuous on and inside $\gamma$. Thus, Cauchy's theorem gives $\int_\gamma f(w)dw = 0$. Proceed with the argument in [1] until (5) in [1] is obtained, namely

$$\frac{1}{\pi}\int_0^\pi \left(2\cos\left(\frac{\theta}{2}\right)\right)^x \cos(y\theta)d\theta \;=\; \frac{\Gamma(1+x)}{\Gamma\left(1+\frac{x}{2}+y\right)\Gamma\left(1+\frac{x}{2}-y\right)}$$

where $\Gamma$ is the gamma function and $y = \eta + \frac{x}{2} + 1$. The argument to establish the integral in this problem is similar to [1] but begins to differ at this point. Let $L(y)$ and $R(y)$ be the left and right sides of the above equation. Differentiate both sides with respect to $y$ four times and set $y = 0$ to get

$$L^{(4)}(0) \;=\; \frac{1}{\pi}\int_0^\pi \theta^4\left(2\cos\left(\frac{\theta}{2}\right)\right)^x d\theta \;=\; R^{(4)}(0).$$

Now $R(y) = \Gamma(1+x)f(y)$ where

$$f(y) \;=\; \frac{1}{\Gamma\left(1+\frac{x}{2}+y\right)\Gamma\left(1+\frac{x}{2}-y\right)}.$$

Consider the power series of $R(y)$ about $y = 0$, namely

$$R(y) \;=\; \Gamma(1+x)\left[f(0) + f'(0)y + \frac{f''(0)}{2}y^2 + \frac{f'''(0)}{6}y^3 + \frac{f^{(4)}(0)}{24}y^4 + \cdots\right].$$

Consequently, $R^{(4)}(0) = \Gamma(1+x) f^{(4)}(0)$. That is

$$\frac{1}{\pi}\int_0^\pi \theta^4\left(2\cos\left(\frac{\theta}{2}\right)\right)^x d\theta \;=\; \Gamma(1+x) f^{(4)}(0).$$

We need to find $f^{(4)}(0)$. This can be done by hand but it is routine and tedious. This is an ideal example where technology such as *Mathematica* can be used. *Mathematica* produces

$$f^{(4)}(0) \;=\; \frac{12\left[\psi'\left(1+\frac{x}{2}\right)\right]^2 - 2\psi'''\left(1+\frac{x}{2}\right)}{\Gamma^2\left(1+\frac{x}{2}\right)}$$

where $\psi(s) = \frac{\Gamma'(s)}{\Gamma(s)}$ is the digamma function. Hence

$$\begin{aligned}\frac{1}{2\pi}\int_0^\pi \theta^4\left(2\cos\left(\frac{\theta}{2}\right)\right)^x d\theta \;&=\; \frac{\Gamma(1+x)}{\Gamma^2\left(1+\frac{x}{2}\right)}\left(6\left[\psi'\left(1+\frac{x}{2}\right)\right]^2 - \psi'''\left(1+\frac{x}{2}\right)\right)\\ &=\; g(x)\left(6\left[\psi'\left(1+\frac{x}{2}\right)\right]^2 - \psi'''\left(1+\frac{x}{2}\right)\right)\end{aligned}$$

where $g(x) = \frac{\Gamma(1+x)}{\Gamma^2\left(1+\frac{x}{2}\right)}$. Let $h(x) = \psi\left(1+\frac{x}{2}\right)$. Then

$$\begin{aligned}2h'(x) \;&=\; \psi'\left(1+\frac{x}{2}\right)\\ 8h'''(x) \;&=\; \psi'''\left(1+\frac{x}{2}\right)\end{aligned}$$

and so

$$\frac{1}{16\pi}\int_0^\pi \theta^4\left(2\cos\left(\frac{\theta}{2}\right)\right)^x d\theta \;=\; c(x)$$

where $c(x) = g(x)\left(3[h'(x)]^2 - h'''(x)\right)$.

Differentiate both sides of the equation with respect to $x$ four times and set $x = 0$ to get

$$\frac{1}{16\pi}\int_0^\pi \theta^4 \log^4\left(2\cos\left(\frac{\theta}{2}\right)\right) d\theta \;=\; c^{(4)}(0).$$

To get the right side is routine but tedious by hand. We use *Mathematica* which produces

$$c^{(4)}(0) \;=\; \frac{\pi^8 + 3600\pi^2[\psi^{(2)}(1)]^2 + 1200\psi^{(2)}(1)\psi^{(4)}(1)}{6400}.$$

Now make use of the formula $\psi^{(k)}(1) = (-1)^{k-1}k!\zeta(k+1)$ for the $k^{\text{th}}$ derivative of the digamma function to get

$$\begin{aligned}
c^{(4)}(0) &= \frac{\pi^8 + 3600\pi^2[(-1)^1 2!\zeta(3)]^2 + 1200[(-1)^1 2!\zeta(3)][(-1)^3 4!\zeta(5)]}{6400} \\
c^{(4)}(0) &= \frac{\pi^8 + 14400\pi^2[\zeta(3)]^2 + 57600\zeta(3)\zeta(5)}{6400} \\
16\pi c^{(4)}(0) &= \frac{\pi^9 + 14400\pi^3[\zeta(3)]^2 + 57600\pi\zeta(3)\zeta(5)}{400}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_0^\pi \theta^4 \log^4\left(2\cos\left(\frac{\theta}{2}\right)\right) d\theta &= \frac{\pi^9 + 14400\pi^3[\zeta(3)]^2 + 57600\pi\zeta(3)\zeta(5)}{400} \\
&= \frac{\pi^9 + 120^2\pi^3[\zeta(3)]^2 + 240^2\pi\zeta(3)\zeta(5)}{20^2}.
\end{aligned}$$

This formula was not found in the literature and appears to be new.

# References


[1] Borwein, D., Borwein, J., On an Intriguing Integral and Some Series Related to $\zeta(4)$, *Proc. Amer. Math. Soc.*, Vol. 123, No. 4 (Apr., 1995), pp. 1191-1198.

[2] Coffey, M. W., On some log-cosine integrals related to $\zeta(3)$, $\zeta(4)$ and $\zeta(6)$, *Journal of Computational and Applied Mathematics*, Vol 159, Issue 2, 15 Oct 2003, pp 205-215.

[3] Gradshteǐn, Ryzhik, I. M., Jeffrey, A., Table of Integrals, Series and Products, Academic Press, 1994.

[4] Lewin, L., "Polylogarithms and Associated Functions", Elsevier North Holland, Inc., 1981.

[5] Nahin, P. J., "Inside Interesting Integrals", Springer, 2015.

[6] Spiegel, M. R., "Mathematical Handbook", Schaum's Outline Series, McGraw-Hill Book Company p. 101, 1968.

[7] Vălean, C., I., "(Almost) Impossible Integrals, Sums, and Series", Problem Books in Mathematics, Springer, 2019.

[8] Vălean, C., I., "More (Almost) Impossible Integrals, Sums, and Series", Problem Books in Mathematics, Springer, 2023.

# *Arhimede Mathematical Journal*

Volume 13, No. 1 Spring 2026

## Aim and Scope

The goal of *Arhimede Mathematical Journal* is to provide a means of publication of useful materials to train students for Mathematical Contests at all levels. Potential contributions include any work involving fresh ideas and techniques, problems and lessons helpful to train contestants, all written in a clear and elegant mathematical style. All areas of mathematics, including algebra, combinatorics, geometry, number theory and real and complex analysis, are considered appropriate for the journal.

## Information for Authors

A detailed statement of author guidelines is available at the journal's website:

`http://www.amj-math.com`

Edited in Barcelona by the Arhimede Association

ISSN 2462-537X