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Beyond Quadratic Gauss Sums

Joe Santmyer

Abstract

The name Gauss alone should attract your interest. In this article a known recursive formula, see (2) below, for the quadratic Gauss sum is extended to cubic, quartic, quintic and beyond. Formula (2) can be found in [4], as an exercise in [1] and at the website [8]. For an entirely different approach to a quadratic Gauss sum using elementary methods in linear algebra see [7]. Formula (4) below is the main result. It will be established by following the same method used in [4] to prove (2). A different way to establish (4) is the proof of proposition 3.2 on p 40 in [2] which is referred to as the *Reduction Property*.

1 Introduction

The quadratic Gauss sum $G_2(k;n)$ is defined as

$$G_2(k;n) = \sum_{r=1}^{n} e^{\frac{2\pi i k r^2}{n}}, \tag{1}$$

where n and k are integers with $n > 0$. What makes (1) quadratic is r^2 in the exponent of e. Replace r^2 with r^3 and you have a cubic Gauss sum, replace r^2 with r^4 and you have a quartic Gauss sum, and so on.

If p is a prime with $p \nmid k$, $p > 2$ and if $\alpha \geq 2$ then

$$G_2(k;p^\alpha) \;=\; pG_2(k;p^{\alpha-2}). \tag{2}$$

A generalization of (2) that holds for quadratic, cubic, quartic, *etc.*, Gauss sums will be the main result developed here. The proof of the generalization uses only elementary results and material comparable to what can be found in [1].

The interest in a Gauss sum with prime powers is that

$$G_2(k; n_1 n_2) \;=\; G_2(kn_1; n_2)G_2(kn_2; n_1) \tag{3}$$

if $\gcd(n_1, n_2) = 1$. This reduces the study of Gauss sums to the case $G_2(k; p^\alpha)$. Recursion (2) will be generalized as

$$G_m(k; p^\alpha) \;=\; p^{m-1}G_m(k; p^{\alpha-m}) \tag{4}$$

where $m \geq 2$, $\alpha \geq m$ and p is a prime with $p > m$ and $p \nmid k$. If $m = 2, 3, 4$ then formula (4) holds for quadratic, cubic and quartic Gauss sums, respectively. $G_m(k; n)$ can be regarded as a m^{th} order Gauss sum and looks as follows

$$G_m(k; n) = \sum_{r=1}^{n} e^{\frac{2\pi i k r^m}{n}}. \tag{5}$$

Sums of the form $G_m(k; n)$ are discussed in [6] but in the context of solving a congruence of the form $r^m \equiv x (\bmod\, p)$ and in [3] associated with a Dirichlet character. The sum of interest here is

$$G_m(k; p^\alpha) = \sum_{r=1}^{p^\alpha} e^{\frac{2\pi i k r^m}{p^\alpha}}. \tag{6}$$

The analogue of (3) associated with (4) is

$$G_m(k; n_1 n_2) \;=\; G_m(kn_1^{m-1}; n_2)G_m(kn_2^{m-1}; n_1), \tag{7}$$

where, again, $\gcd(n_1, n_2) = 1$. Formula (7) is justified after the recursion formula (4) is established. As in [2] formula (4) will be referred to as the *Reduction Property*.

2 Proof of the Reduction Property

Formula (4) is the main result. It will be justified by following the same method used in [4] to prove (2). Two key ideas in the method are

a. translate the summation from the (integer) interval $[1, p^\alpha]$ to the (integer) interval $[p^{\alpha-1}+1, p^{\alpha-1}+p^\alpha]$.
b. use the division algorithm to express $r \in [p^{\alpha-1}+1, p^{\alpha-1}+p^\alpha]$ as $r = p^{\alpha-1}s + t$ where s and t are unique integers satisfying $1 \le s \le p$ and $1 \le t \le p^{\alpha-1}$.

The term *integer* interval means just integers in the interval are considered. After the two key ideas are used, the binomial formula will be applied to finish the proof of (4).

To get started, note that

$$\begin{aligned} G_m(k; p^\alpha) &= \sum_{r\in[1,p^\alpha]} e^{\frac{2\pi i k r^m}{p^\alpha}} \\ &= \sum_{r=1}^{p^\alpha} e^{\frac{2\pi i k r^m}{p^\alpha}} \end{aligned}$$

is equal to the sum

$$\begin{aligned} S_{p^{\alpha-1}} &= \sum_{r\in[p^{\alpha-1}+1,p^{\alpha-1}+p^\alpha]} e^{\frac{2\pi i k r^m}{p^\alpha}} \\ &= \sum_{r=p^{\alpha-1}+1}^{p^{\alpha-1}+p^\alpha} e^{\frac{2\pi i k r^m}{p^\alpha}} \end{aligned}$$

since the terms in both sums range over exactly one complete system of residues mod p^α.

By introducing integers s and t associated with r via the division algorithm, the sum $S_{p^{\alpha-1}}$ can be expressed as two sums, one ranging over s and the other ranging over t. The two integers s and t separate terms involving p so that the sum $g(n)$ in theorem 8.1 on p 158 in [1] can be applied to the sum ranging over s. To see this consider

$$S_{p^{\alpha-1}} = \sum_{t=1}^{p^{\alpha-1}} \sum_{s=1}^{p} e^{\frac{2\pi i k (p^{\alpha-1}s+t)^m}{p^\alpha}}.$$

Lets focus on the terms $e^{\frac{2\pi i k (p^{\alpha-1}s+t)^m}{p^\alpha}}$. In the numerator of the fraction in the power of e apply the binomial theorem to get

$$\begin{aligned}(p^{\alpha-1}s+t)^m &= \sum_{a=0}^{m}\binom{m}{a}(p^{\alpha-1}s)^{m-a}t^a\\ &= \sum_{a=0}^{m-2}\binom{m}{a}(p^{\alpha-1}s)^{m-a}t^a + \binom{m}{m-1}p^{\alpha-1}st^{m-1}+t^m\\ &= \sum_{a=0}^{m-2}\binom{m}{a}(p^{\alpha-1}s)^{m-a}t^a + mp^{\alpha-1}st^{m-1}+t^m\\ &= \sum_{a=0}^{m-2}\binom{m}{a}p^{(\alpha-1)(m-a)}s^{m-a}t^a + mp^{\alpha-1}st^{m-1}+t^m.\end{aligned}$$

And so

$$\begin{aligned}\frac{(p^{\alpha-1}s+t)^m}{p^\alpha} &= \frac{1}{p^\alpha}\sum_{a=0}^{m-2}\binom{m}{a}p^{(\alpha-1)(m-a)}s^{m-a}t^a + mp^{\alpha-1}st^{m-1}+t^m\\ &= \sum_{a=0}^{m-2}\binom{m}{a}p^{(\alpha-1)(m-a)-\alpha}s^{m-a}t^a + \frac{mp^{\alpha-1}st^{m-1}}{p^\alpha}+\frac{t^m}{p^\alpha}\\ &= S+\frac{mst^{m-1}}{p}+\frac{t^m}{p^\alpha}\end{aligned}$$

where $S = \sum_{a=0}^{m-2}\binom{m}{a}p^{(\alpha-1)(m-a)-\alpha}s^{m-a}t^a$. The term in S having the <u>least</u> power of p is $a = m-2$ and this term is $\binom{m}{m-2}p^{\alpha-2}s^2t^{m-2}$. Since $\alpha \geq m \geq 2$ this term is an integer. Consequently, all terms in S are integers. This means that S itself is an integer. So, the terms $e^{\frac{2\pi ik(p^{\alpha-1}s+t)^m}{p^\alpha}}$ satisfy

$$\begin{aligned}e^{\frac{2\pi ik(p^{\alpha-1}s+t)^m}{p^\alpha}} &= e^{2\pi ikS}e^{\frac{2\pi ikmst^{m-1}}{p}}e^{\frac{2\pi ikt^m}{p^\alpha}}\\ &= 1\cdot e^{\frac{2\pi ikmst^{m-1}}{p}}e^{\frac{2\pi ikt^m}{p^\alpha}}\\ &= e^{\frac{2\pi ikmst^{m-1}}{p}}e^{\frac{2\pi ikt^m}{p^\alpha}}.\end{aligned}$$

Consequently

$$\begin{aligned}
S_{p^{\alpha-1}} &= \sum_{t=1}^{p^{\alpha-1}} \sum_{s=1}^{p} e^{\frac{2\pi ik(p^m s+t)^m}{p^\alpha}} \\
&= \sum_{t=1}^{p^{\alpha-1}} \sum_{s=1}^{p} e^{\frac{2\pi ikmst^{m-1}}{p}} e^{\frac{2\pi ikt^m}{p^\alpha}} \\
&= \sum_{t=1}^{p^{\alpha-1}} e^{\frac{2\pi ikt^m}{p^\alpha}} \sum_{s=1}^{p} e^{\frac{2\pi ik(mt^{m-1})s}{p}} \\
&= \sum_{t=1}^{p^{\alpha-1}} e^{\frac{2\pi ikt^m}{p^\alpha}} g(mt^{m-1}),
\end{aligned}$$

where

$$g(mt^{m-1}) = \sum_{s=1}^{p} e^{\frac{2\pi ik(mt^{m-1})s}{p}}.$$

Since $s \in [1, p]$ ranges over one complete system of residues mod p, $g(mt^{m-1})$ is equal to the function defined in theorem 8.1 on p 158 in [1]. From that theorem we can conclude that

$$g(mt^{m-1}) = \begin{cases} 0 & \text{if } p \nmid mt^{m-1} \\ p & \text{if } p \mid mt^{m-1}. \end{cases}$$

Since p is a prime and $p > m$

$$g(mt^{m-1}) = \begin{cases} 0 & \text{if } p \nmid t \\ p & \text{if } p \mid t. \end{cases}$$

Consequently, if $p|t$ then

$$\begin{aligned}
S_{p^{\alpha-1}} &= \sum_{t=1}^{p^{\alpha-1}} e^{\frac{2\pi ikt^m}{p^\alpha}} \cdot p \\
&= p \sum_{t=1}^{p^{\alpha-1}} e^{\frac{2\pi ikt^m}{p^\alpha}}.
\end{aligned}$$

If $p|t$ then $xp = t$ for some integer x. Since $t \in [1, p^{\alpha-1}]$ and $t = xp$ then as $t = p, 2p, 3p, \ldots, p^{\alpha-1}$ we have $x = 1, 2, 3, \ldots, p^{\alpha-2}$.

So the sum $S_{p^{\alpha-1}}$ can be expressed as

$$\begin{aligned}
S_{p^{\alpha-1}} &= p \sum_{t\in[1,p^{\alpha-1}]} e^{\frac{2\pi ikt^m}{p^\alpha}} \\
&= p \sum_{t\in\{p,2p,3p,\ldots,p^{\alpha-1}\}} e^{\frac{2\pi ikt^m}{p^\alpha}} \\
&= p \sum_{x\in\{1,2,3,\ldots,p^{\alpha-2}\}} e^{\frac{2\pi ik(xp)^m}{p^\alpha}} \\
&= p \sum_{x=1}^{p^{\alpha-2}} e^{\frac{2\pi ikx^m p^m}{p^\alpha}} \\
&= p \sum_{x=1}^{p^{\alpha-2}} e^{\frac{2\pi ikx^m}{p^{\alpha-m}}} \\
&= p \Bigg[\sum_{x=1}^{p^{\alpha-m}} e^{\frac{2\pi ikx^m}{p^{\alpha-m}}} + \sum_{x=1+p^{\alpha-m}}^{2p^{\alpha-m}} e^{\frac{2\pi ikx^m}{p^{\alpha-m}}} \\
&\quad + \sum_{x=1+2p^{\alpha-m}}^{3p^{\alpha-m}} e^{\frac{2\pi ikx^m}{p^{\alpha-m}}} + \cdots + \sum_{x=1+(p^{m-2}-1)p^{\alpha-m}}^{p^{\alpha-2}} e^{\frac{2\pi ikx^m}{p^{\alpha-m}}} \Bigg] \\
&= p \sum_{a=0}^{p^{m-2}-1} \sum_{x=1+ap^{\alpha-m}}^{(a+1)p^{\alpha-m}} e^{\frac{2\pi ikx^m}{p^{\alpha-m}}} \\
&= p \sum_{m=0}^{p^{m-2}-1} S_a,
\end{aligned}$$

where

$$S_a = \sum_{x=1+ap^{\alpha-m}}^{(a+1)p^{\alpha-m}} e^{\frac{2\pi ikx^m}{p^{\alpha-m}}}.$$

Since each sum S_a ranges over a complete residue system mod $p^{\alpha-m}$, each sum has the same value. That is, $S_a = S_0$ for $a =$

$0, 1, \ldots, p^{m-2} - 1$. Consequently

$$\begin{aligned}
S_{p^{\alpha-1}} &= p \sum_{a=0}^{p^{m-2}-1} S_a \\
&= p \sum_{a=0}^{p^{m-2}-1} S_0 \\
&= p(p^{m-2} S_0) \\
&= p^{m-1} S_0 \\
&= p^{m-1} \sum_{x=1}^{p^{\alpha-m}} e^{\frac{2\pi i k x^m}{p^{\alpha-m}}} \\
&= p^{m-1} G_m(k; p^{\alpha-m}).
\end{aligned}$$

Since $S_{p^{\alpha-1}} = G_m(k; p^{\alpha})$ we have $G_m(k; p^{\alpha}) = p^{m-1} G_m(k; p^{\alpha-m})$ which establishes the recursive formula (4).

Repeated applications of (4) easily yield

$$G_m(k; p^{\alpha}) = \begin{cases} p^{(m-1)\lfloor \frac{\alpha}{m} \rfloor} G_m(k; p^{\alpha - m\lfloor \frac{\alpha}{m} \rfloor}) & \text{if } m \nmid \alpha \\ p^{\frac{(m-1)\alpha}{m}} & \text{if } m \mid \alpha \end{cases}$$

where $\lfloor x \rfloor$ is the floor function of x, namely, the largest integer at most x.

3 Justifying Formula (7)

It remains to justify (7). Consider

$$G_m(k; n_1 n_2) = \sum_{t=1}^{n_1 n_2} e^{\frac{2\pi i k t^m}{n_1 n_2}}$$

and

$$\begin{aligned}
G_m(k n_1^{m-1}; n_2) G_m(k n_2^{m-1}; n_1) &= \left(\sum_{r=1}^{n_2} e^{\frac{2\pi i k n_1^{m-1} r^m}{n_2}} \right) \left(\sum_{s=1}^{n_1} e^{\frac{2\pi i k n_2^{m-1} s^m}{n_1}} \right) \\
&= \sum_{r=1}^{n_2} \sum_{s=1}^{n_1} e^{\frac{2\pi i k (n_1^m r^m + n_2^m s^m)}{n_1 n_2}}.
\end{aligned}$$

If it can be shown that for each term $e^{\frac{2\pi ikt^m}{n_1n_2}}$ in $G_m(k;n_1n_2)$ there is a corresponding term $e^{\frac{2\pi ik(n_1^m r^m + n_2^m s^m)}{n_1n_2}}$ in $G_m(kn_1^{m-1};n_2)\cdot G_m(kn_2^{m-1};n_1)$ such that the terms are equal, then we can conclude that $G_m(k;n_1n_2) = G_m(kn_1^{m-1};n_2)G_m(kn_2^{m-1};n_1)$. The only difference in the terms $e^{\frac{2\pi ikt^m}{n_1n_2}}$ and $e^{\frac{2\pi ik(n_1^m r^m + n_2^m s^m)}{n_1n_2}}$ is in the exponent of e that is different, namely, t^m in $e^{\frac{2\pi ikt^m}{n_1n_2}}$ and $n_1^m r^m + n_2^m s^m$ in $e^{\frac{2\pi ik(n_1^m r^m + n_2^m s^m)}{n_1n_2}}$.

If it can be shown that for every pair (r,s) where $1 \leq r \leq n_2$ and $1 \leq s \leq n_1$ there is exactly one value t where $1 \leq t \leq n_1n_2$ such that $(n_1^m r^m + n_2^m s^m) \equiv t^m \bmod n_1n_2$ we can conclude that $e^{\frac{2\pi ik(n_1^m r^m + n_2^m s^m)}{n_1n_2}} = e^{\frac{2\pi ikt^m}{n_1n_2}}$. If this is true for each of the n_1n_2 terms on the left and right, we can conclude that $G_m(k;n_1n_2) = G_m(kn_1^{m-1};n_2)G_m(kn_2^{m-1};n_1)$.

Let $R = \{1,2,3,\ldots,n_2\}$, $S = \{1,2,3,\ldots,n_1\}$ and $T = \{1,2,3,\ldots,n_1n_2\}$. Define the function $f : R\times S \to T$ as $f(r,s) = n_1r + n_2s$. Note that if $t = f(r,s)$ then

$$\begin{aligned}
t^m &= (f(r,s))^m \\
&= (n_1r+n_2s)^m \\
&= \sum_{a=0}^{m}\binom{m}{a}(n_1r)^a(n_2s)^{m-a} \\
&= n_2^m s^m + \sum_{a=1}^{m-1}\binom{m}{a}(n_1r)^a(n_2s)^{m-a} + n_1^m r^m \\
&= n_2^m s^m + n_1n_2\sum_{a=1}^{m-1}\binom{m}{a}n_1^{a-1}r^a n_2^{m-a-1}s^{m-a} + n_1^m r^m \\
&= n_1^m r^m + n_1n_2S + n_2^m s^m,
\end{aligned}$$

where $S = \sum_{a=1}^{m-1}\binom{m}{a}n_1^{a-1}r^a n_2^{m-a-1}s^{m-a}$. Consequently,

$$(n_1^m r^m + n_2^m s^m) \equiv t^m \bmod n_1n_2.$$

To show that exactly one $t \in T$ corresponds to a unique pair $(r,s) \in R\times S$, it suffices to show that f is injective. Since the

$|R \times S| = |T| = n_1n_2$ we can conclude that f is also surjective and hence a bijection.

So far, the assumption $(n_1, n_2) = 1$ has not been used. This is needed in order for f to be injective. Assume $f(r_1, s_1) = f(r_2, s_2)$. Then $n_1r_1+n_2s_1 = n_1r_2+n_2s_2$. This gives $n_1(r_1-r_2) = n_2(s_1-s_2)$. Since $(n_1, n_2) = 1$ then $n_1|(s_1 - s_2)$. So $n_1a = s_1 - s_2$ for some integer $a \geq 0$. So $s_1 = s_2 + n_1a$. Since $1 \leq s_1 \leq n_1$ then $1 \leq s_2 + n_1a \leq n_1$. If $a \neq 0$ then $s_2 + n_1a > n_1$, a contradiction. So $a = 0$ and $s_1 = s_2$.

A similar argument shows that $r_1 = r_2$. Consequently, $(r_1, s_1) = (r_2, s_2)$ implying thst f is injective. Since f is injective and surjective it is a bijection. This shows that for every $t \in T$ there is a unique pair $(r, s) \in R \times S$ such that $(n_1^m r^m + n_2^m s^m) \equiv t^m \bmod n_1n_2$. From this, it follows that for every term $e^{\frac{2\pi i k t^m}{n_1n_2}}$ in $G_m(k; n_1n_2)$ there is a unique term $e^{\frac{2\pi i k(n_1^m r^m + n_2^m s^m)}{n_1n_2}}$ in $G_m(kn_1^{m-1}; n_2)G_m(kn_2^{m-1}; n_1)$ such that $e^{\frac{2\pi i k t^m}{n_1n_2}} = e^{\frac{2\pi i k(n_1^m r^m + n_2^m s^m)}{n_1n_2}}$. Consequently,

$$G_m(k; n_1n_2) = G_m(kn_1^{m-1}; n)G_m(kn_2^{m-1}; n_1).$$

4 Closing Remarks

Interest in cubic, quartic and octic Gauss sums and/or reciprocity laws is discussed in more detail in [5] where evidence is cited that Gauss himself had invested time studying these sums and laws. Among the people mentioned in [5], from a historical perspective, Eisenstein stands out as being first to give complete proofs of cubic and quartic reciprocity laws.

The student reader is encouraged to take an exercise such as the one in [1] and explore ways to extend a formula like (2). It can be a starting point for undergraduate research.

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A generalization of Ji Chen's Inequality

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Abstract

In this paper we prove a generalization of Ji Chen's Inequality and give some applications.

1 Introduction

In this section, we recall some known inequalities which we will use in the following.

Theorem 1 (J. Radon's Inequality). (see [1] or [4]) *If* $n \in \mathbb{N}$, $x_k \geq 0$, $y_k > 0$, $k \in \{1, 2, \ldots, n\}$ *and* $p \geq 0$, *then*

$$\frac{x_1^{p+1}}{y_1^p} + \frac{x_2^{p+1}}{y_2^p} + \cdots + \frac{x_n^{p+1}}{y_n^p} \geq \frac{(x_1 + x_2 + \cdots + x_n)^{p+1}}{(y_1 + y_2 + \cdots + y_n)^p}, \tag{1}$$

with equality if and only if $\dfrac{x_1}{y_1} = \dfrac{x_2}{y_2} = \cdots = \dfrac{x_n}{y_n}$.

Theorem 2 (Ji Chen's Inequality). (see [3]) *Prove that*

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \geq \frac{9}{4} \tag{2}$$

for any $x, y, z \in \mathbb{R}$, $x, y, z > 0$, *with equality if and only if* $x = y = z$.

In this paper, we will write a, b, c the lengths of the sides and F the area of a given triangle.

Theorem 3 (V. O. Gordon's Inequality). (see [2]) *In any triangle the inequality*

$$ab + bc + ca \geq 4F\sqrt{3} \tag{3}$$

holds, with equality if and only if the triangle is equilateral.

2 Main results and applications

In this section we will give a generalization of Ji Chen's Inequality and then some applications.

Theorem 4. *If $m, n \geq 0$, then the inequality*

$$\begin{aligned}&\left((xy)^{m+n+1} + (yz)^{m+n+1} + (zx)^{m+n+1}\right)\cdot\\ &\cdot\left(\frac{1}{(x+y)^{2m+2}} + \frac{1}{(y+z)^{2m+2}} + \frac{1}{(z+x)^{2m+2}}\right) \geq\\ &\geq \frac{1}{3^{n-2} \cdot 4^{m+1}} (xy + yz + zx)^n\end{aligned} \tag{4}$$

holds for any $x, y, z > 0$, with equality if and only if $x = y = z$.

Proof. By using Radon's Inequality we have

$$(xy)^{m+n+1} + (yz)^{m+n+1} + (zx)^{m+n+1} \geq \frac{(xy + yz + zx)^{m+n+1}}{3^{m+n}} \tag{5}$$

with equality if and only if $xy = yz = zx$, equivalent to $x = y = z$ and

$$\begin{aligned}&\frac{1}{(x+y)^{2m+2}} + \frac{1}{(y+z)^{2m+2}} + \frac{1}{(z+x)^{2m+2}} =\\ &= \left(\frac{1}{(x+y)^2}\right)^{m+1} + \left(\frac{1}{(y+z)^2}\right)^{m+1} + \left(\frac{1}{(z+x)^2}\right)^{m+1} \geq\\ &\geq \frac{\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right)^{m+1}}{3^m}, \quad \text{from where}\end{aligned}$$

$$\frac{1}{(x+y)^{2m+2}}+\frac{1}{(y+z)^{2m+2}}+\frac{1}{(z+x)^{2m+2}}\geq$$
$$\geq\frac{1}{3^m}\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)^{m+1},\tag{6}$$

with equality if and only if $\dfrac{1}{(x+y)^2}=\dfrac{1}{(y+z)^2}=\dfrac{1}{(z+x)^2}$, equivalent to $x=y=z$.
From (5) and (6) we have

$$\left((xy)^{m+n+1}+(yz)^{m+n+1}+(zx)^{m+n+1}\right)\cdot$$
$$\cdot\left(\frac{1}{(x+y)^{2m+2}}+\frac{1}{(y+z)^{2m+2}}+\frac{1}{(z+x)^{2m+2}}\right)\geq$$
$$\geq\frac{1}{3^{2m+n}}(xy+yz+zx)^n\left((xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+x)^2}+\frac{1}{(z+x)^2}\right)\right)^{m+1}$$

and by using Ji Chen's Inequality, (4) follows.
Equality in (4) holds if and only if $x=y=z$. $\square$

Corollary 1. *If $m\geq 0$, the inequality*

$$\left((xy)^{m+1}+(yz)^{m+1}+(zx)^{m+1}\right)\cdot$$
$$\cdot\left(\frac{1}{(x+y)^{2m+2}}+\frac{1}{(y+z)^{2m+2}}+\frac{1}{(z+x)^{2m+2}}\right)\geq\frac{9}{4^{m+1}}\tag{7}$$

holds for any $x,y,z>0$, with equality if and only if $x=y=z$.

Proof. The inequality (7) is obtained from inequality (4) for $n=0$. $\square$

Remark 1. If we considering $m=0$ in inequality (7), then we obtain Ji Chen's Inequality.

Application 2.1 The inequality

$$\left((xy)^2+(yz)^2+(zx)^2\right)\left(\frac{1}{(x+y)^3}+\frac{1}{(y+z)^3}+\frac{1}{(z+x)^3}\right)\geq$$
$$\geq\frac{3\sqrt{3}}{8}\sqrt{xy+yz+zx}\tag{8}$$

holds for any $x, y, z > 0$.

Proof. The inequality (8) is obtained from inequality (4) for $m = n = \frac{1}{2}$. □

Application 2.2 For any $a, b, c > 0$, the following inequality

$$\left(a^2+b^2+c^2+3(ab+bc+ca)\right)\left(\frac{1}{a+2b+c}+\frac{1}{a+b+2c}+\frac{1}{2a+b+c}\right) \geq \frac{9}{4} \tag{9}$$

is true.

Proof. In the inequality (4) we consider $x = a + b$, $y = b + c$, $z = c + a$, $m = n = 0$ and then the inequality (9) follows. □

Application 2.3 If $x, y, z > 0$ with $xy + yz + zx = 256$, then the inequality

$$\left(x^{10} + y^{10} + z^{10}\right)\left(\frac{1}{(x+y)^8} + \frac{1}{(y+z)^8} + \frac{1}{(z+x)^8}\right) \geq 3 \tag{10}$$

holds.

Proof. We have $x^{10} + y^{10} + z^{10} \geq x^5y^5 + y^5z^5 + z^5x^5$ and then

$$\left(x^{10} + y^{10} + z^{10}\right)\left(\frac{1}{(x+y)^8} + \frac{1}{(y+z)^8} + \frac{1}{(z+x)^8}\right) \geq$$
$$\geq \left((xy)^5 + (yz)^5 + (zx)^5\right)\left(\frac{1}{(x+y)^8} + \frac{1}{(y+z)^8} + \frac{1}{(z+x)^8}\right).$$

But, in the last inequality using inequality (4) for $m = 3$, $n = 1$ we have that

$$\left((xy)^5 + (yz)^5 + (zx)^5\right)\left(\frac{1}{(x+y)^8} + \frac{1}{(y+z)^8} + \frac{1}{(z+x)^8}\right) \geq$$
$$\geq \frac{3}{4^4}\,(xy + yz + zx).$$

Taking that $xy+yz+zx = 256$ into account, from the inequalities above, the inequality (10) follows. □

Application 2.4 In any triangle, the following inequality

$$(a^8+b^8+c^8)\left(\frac{1}{(a+b)^4}+\frac{1}{(b+c)^4}+\frac{1}{(c+a)^4}\right)\geq 3F^2, \qquad (11)$$

holds, with equality if and only if $a=b=c$.

Proof. By using the inequality $x^2+y^2+z^2\geq xy+yz+zx$, we have
$a^8+b^8+c^8\geq a^4b^4+b^4c^4+c^4a^4$, with equality if and only if $a^4=b^4=c^4$, equivalent to $a=b=c$. Then

$$(a^8+b^8+c^8)\left(\frac{1}{(a+b)^4}+\frac{1}{(b+c)^4}+\frac{1}{(c+a)^4}\right)\geq$$
$$\geq\left((ab)^4+(bc)^4+(ca)^4\right)\left(\frac{1}{(a+b)^4}+\frac{1}{(b+c)^4}+\frac{1}{(c+a)^4}\right)$$

and applying Theorem 2.1 for $m=1$ and $n=2$ we obtain

$$(a^8+b^8+c^8)\left(\frac{1}{(a+b)^4}+\frac{1}{(b+c)^4}+\frac{1}{(c+a)^4}\right)\geq\frac{1}{16}(ab+bc+ca)^2, \qquad (12)$$

with equality if and only if $a=b=c$.
Taking in (12) into account by Gordon's Inequality, inequality from (11) follows. □

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An elementary approach for the best constant in some inequalities

Neculai Stanciu

A tribute to Titu Zvonaru (1953–2025)

1 Introduction

There are two situations where determining the best k is of interest:

- We consider the inequalities $A \geq p$, $B \leq q$. An inequality of the form $A + kB > p + kq$ is stronger when the parameter k is larger.
- An inequality of the form $f(a, b, c) \leq k$ is stronger when k is smaller (here $f(a, b, c)$ is a symmetric and homogeneous function).

The purpose of this note is to present elementary solutions for some of the problems from [2].

2 The results

Hereafter, we present the main results of this work.

Problem 1. *Determine the smallest constant k such that*

$$(a-b)^4+(b-c)^4+(c-a)^4 \;\le\; k\big(a^4+b^4+c^4-abc(a+b+c)\big)$$

*holds for any positive real numbers a, b, c (*see Problem 1 from [2]*).*

Solution. For $b=c=0$ we obtain $k \ge 2$. It remains to prove the inequality

$$(a-b)^4+(b-c)^4+(c-a)^4 \;\le\; 2\big(a^4+b^4+c^4-abc(a+b+c)\big),$$

which is successively equivalent to

$$a^2b^2+b^2c^2+c^2a^2 \;\ge\; abc(a+b+c)$$

and

$$a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2 > 0.$$

Problem 2. *Determine the largest constant k such that*

$$\big[(x-y)^2+(y-z)^2+(z-x)^2\big]^3 \;\ge\; k\,(x-y)^2(y-z)^2(z-x)^2$$

*holds for any positive real numbers x, y, z (*see Problem 3 from [2]*).*

Solution. We can assume that $x \ge y \ge z$. Denoting $a = x-y$, $b = y-z$, the inequality can be written as

$$\big(a^2+b^2+(a+b)^2\big)^3 \ge k\,a^2b^2(a+b)^2.$$

For $a=b$ we obtain $54 \ge k$. It remains to prove the inequality

$$\big(a^2+b^2+(a+b)^2\big)^3 \ge 54\,a^2b^2(a+b)^2.$$

Using the inequalities $a^2+b^2 \ge \dfrac{(a+b)^2}{2}$ and $(a+b)^2 \ge 4ab$, we have

$$\big(a^2+b^2+(a+b)^2\big)^3 \;\ge\; \left(\frac{(a+b)^2}{2}+(a+b)^2\right)^3 = 27(a+b)^6$$

$$\ge 27(a+b)^2\cdot 16a^2b^2 = 54\,a^2b^2(a+b)^2.$$

The equality occurs if and only if $a=b$. That is,

$$x+z=2y, \qquad x+y=2z, \qquad \text{or} \qquad y+z=2x.$$

Problem 3. *Determine the largest value of constant k such that*

$$(4-3k)\,\frac{(x+y)(y+z)(z+x)}{32xyz}$$

$$+\,k\left(\frac{x^2}{x+y+z}+\frac{y^2}{x+2y+z}+\frac{z^2}{x+y+2z}\right)\geq 1$$

*holds for any positive real numbers x, y, z (*see Problem 10 from [1]*).*

Solution. We have

$$(x+y)(y+z)(z+x)-8xyz=z(x-y)^2+x(y-z)^2+y(z-x)^2$$

and

$$\frac{3}{4}-\sum_{\text{cyc}}\frac{x^2}{x+y+z}=\sum_{\text{cyc}}\frac{(x-y)^2}{4(2x+y+z)(x+2y+z)}.$$

The inequality is written in the form

$$(4-3k)\left(\frac{(x+y)(y+z)(z+x)}{32xyz}-\frac{1}{4}\right)$$

$$\geq\ k\left(\frac{3}{4}-\frac{x^2}{x+y+z}-\frac{y^2}{x+2y+z}-\frac{z^2}{x+y+2z}\right),$$

that is,

$$(4-3k)\sum_{\text{cyc}}\frac{z(x-y)^2}{32xyz}\ \geq\ k\sum_{\text{cyc}}\frac{(x-y)^2}{4(2x+y+z)(x+2y+z)}.$$

For $y=z$ we obtain

$$(4-3k)\frac{(x-y)^2}{8xy}\ \geq\ k\,\frac{(x-y)^2}{2}(x+y)(x+3y).$$

We divide by $(x-y)^2$ and obtain

$$k\leq\frac{4(x+y)(x+3y)}{4xy+3(x+y)(x+3y)}.$$

Denoting $t = xy$, we need to determine the minimum of the expression

$$\frac{t^2+4t+3}{3t^2+16t+9}.$$

We have

$$\frac{t^2+4t+3}{3t^2+16t+9} = \frac{1}{3}\cdot\frac{3t^2+12t+9}{3t^2+16t+9} = \frac{1}{3}\left(1-\frac{4t}{3t^2+16t+9}\right)$$
$$\geq \frac{1}{3}\left(1-\frac{4t}{6\sqrt{3}t+16t}\right).$$

Thus,

$$\frac{t^2+4t+3}{3t^2+16t+9} = \frac{(6+3\sqrt{3})(8-3\sqrt{3})}{3(8+3\sqrt{3})(8-3\sqrt{3})} = \frac{7+2\sqrt{3}}{37},$$

so

$$k \leq \frac{4(7+2\sqrt{3})}{37}.$$

Problem 4. *Determine the smallest constant k such that*

$$k\cdot\frac{a^3+b^3+c^3}{abc} \;+\; (8-24k)\,\frac{abc\,(a+b)(b+c)(c+a)}{} \;\geq\; 1$$

*holds for any positive real numbers a, b, c (*see Problem 5 from [2]*).*

Solution. The inequality is written in the form

$$k\left(\frac{a^3+b^3+c^3}{abc}-3\right) \;\geq\; (1-3k)(1-8abc(a+b)(b+c)(c+a)),$$

$$k\,\frac{(a+b+c)\big((a-b)^2+(b-c)^2+(c-a)^2\big)}{2abc}$$
$$\geq\; (1-3k)\,\frac{c(a-b)^2+a(b-c)^2+b(c-a)^2}{(a+b)(b+c)(c+a)}.$$

For $b = c$ we obtain

$$k\,\frac{(a+2b)(a-b)^2}{2ab^2} \;\geq\; (1-3k)\,\frac{b(a-b)^2}{2b(a+b)^2}.$$

We divide by $(a-b)^2$ and obtain

$$k \geq \frac{ab^2}{(a+2b)(a+b)^2+3ab^2}.$$

Denoting $t = ab$, we must determine the maximum of the function

$$f(t) = \frac{t}{t^3+4t^2+8t+2}.$$

Since

$$f'(t) = 0 \iff t^3+2t^2-1 = 0 \iff (t+1)(t^2+t-1) = 0,$$

it follows that the maximum is reached for

$$t = \frac{\sqrt{5}-1}{2}.$$

That is,

$$k \geq \frac{17-5\sqrt{5}}{82}.$$

Problem 5. *Determine the smallest $k \leq \frac{1}{8}$ such that*

$$(3-24k)\,\frac{abc}{a^3+b^3+c^3} \;+\; k\,\frac{(a+b)(b+c)(c+a)}{abc} \;\geq\; 1$$

*holds for any positive real numbers a, b, c (*see Problem 6 from [2]*).*

Solution. The inequality is written in the form

$$k\left(\frac{(a+b)(b+c)(c+a)}{abc}-8\right) \;\geq\; (1-8k)\left(1-3\,\frac{abc}{a^3+b^3+c^3}\right),$$

$$k\,\frac{c(a-b)^2+a(b-c)^2+b(c-a)^2}{abc}$$

$$\geq\; (1-8k)\,\frac{(a+b+c)\big((a-b)^2+(b-c)^2+(c-a)^2\big)}{2(a^3+b^3+c^3)}.$$

For $b = c$ we obtain

$$k\,\frac{b(a-b)^2}{ab^2} \;\geq\; (1-8k)\,\frac{(a+2b)(a-b)^2}{2(a^3+2b^3)}.$$

We divide by $(a-b)^2$ and obtain

$$k \geq \frac{a^2b+2ab^2}{2a^3+8a^2b+16ab^2+4b^3}.$$

Denoting $t = ab$, we need to determine the maximum of the function

$$f(t) = \frac{t^2+2t}{2t^3+8t^2+16t+8}.$$

This maximum can be found numerically. For aesthetic reasons, we prefer a rational value slightly larger than the maximum of f. We have

$$\frac{t^2+2t}{2t^3+8t^2+16t+8} < \frac{2}{19} \iff 4t^3-3t^2-6t+16>0$$
$$\iff (t^3+t^3+1-3t^2)+2(t^3+1+1-3t)+11>0.$$

The expressions in parentheses are positive by the inequality of means. Thus we may take

$$k = \frac{2}{19}.$$

The method used in [2] to solve the problems above is the method presented in [1].

References

[1] Bencze, M. and Drăgan, M. "A New Method to Solve and Refine Algebraic Inequalities". *Mathematical Reflections* 5 (2023).

[2] Bencze, M. and Drăgan, M. "The best constant for some algebraic inequalities". *Mathproblems* 7.1 (2024), pp. 681–693.

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Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.
2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and are most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

April 30, 2026

Elementary Problems

E–143. *Proposed by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain.* Let s be the semiperimeter of a triangle ABC with sides a, b, c opposite to angles A, B, C, respectively. Prove that

$$s^2 = b^2 \cos^2 \frac{C}{2} + c^2 \cos^2 \frac{B}{2} + 2bc \cos \frac{B}{2} \cos \frac{C}{2} \sin \frac{A}{2}.$$

E–144. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let x_1, x_2, x_3 be the roots of the equation $x^3 - 3x^2 - 5x + 1 = 0$. Find the equation whose roots are

$$y_1 = x_1 + \frac{1}{x_1}, \quad y_2 = x_2 + \frac{1}{x_2}, \quad y_3 = x_3 + \frac{1}{x_3}.$$

E–145. *Proposed by Michel Bataille, Rouen, France.* Let ABC be a triangle such that $AB = AC$. Let M be the point of the line BC such that $MB = MA$ and D be the circumcenter of ΔAMB. Prove that the lines BD and AC are perpendicular.

E–146. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let a, n be positive integers. Find the Least Common Multiple of the numbers $A = a^{n-1}(a+1)^{n+2}+1$ and $B = a^{n+1}(a+1)^{n+1}+a-1$.

E–147. *Proposed by Titu Zvonaru, Comăneşti, Romania* (†) *and Neculai Stanciu, Buzău, Romania.* Let a, b, c be positive real numbers. Prove that it holds:

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq \sum_{cyc} \sqrt{a^2 - ab + b^2}.$$

E–148. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Óscar Rivero Salgado, Universidad de Santiago de Compostela, Spain.* Find all positive integers that are divisible by 2431 and have exactly 2431 different positive divisors.

Easy–Medium Problems

EM–143. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* For all integers $n \geq 1$, prove that

$$\sum_{k=1}^{n} \binom{n}{k} F_k^2 \geq \frac{F_{2n}^2}{2^n - 1},$$

where F_n is the n-th Fibonacci number defined by the recurrence: $F_1 = F_2 = 1$, and for $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$.

EM–144. *Proposed by Michel Bataille, Rouen, France and Francisco Javier García Capitán, Priego de Córdoba, Spain.* Let ABC be a triangle with $\angle B, \angle C \neq 90°$. For any point P of its plane, call Y the intersection of CA and the line through the midpoint of PC perpendicular to BC, and Z the intersection of AB and the line through the midpoint of PB perpendicular to BC. Find the locus of P such that Y, Z, and P are collinear.

EM–145. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let a, b, c be positive real numbers. Prove that it holds:

$$(a+b)^2 + (a+b+3c)^2 \geq \frac{169abc}{a+b+7c}.$$

EM–146. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $\{a_n\}_{n\geq 1}$ be the sequence of positive integers defined recursively by $a_1 = 2$ and $a_{n+1} = 2a_n^3 + a_n$ for all $n \geq 1$. Determine the largest power of 5 that divides the number $a_{2026}^2 + 1$.

EM–147. *Proposed by Dorin Mărghidanu, Colegiul Național "A.I. Cuza", Corabia, Romania.* Let a, b, c be positive real numbers, prove that it holds:

$$\frac{(1+a)^2(1+b)^2}{1+c^2} + \frac{(1+b)^2(1+c)^2}{1+a^2} + \frac{(1+c)^2(1+a)^2}{1+b^2} \geq 8\,(a+b+c).$$

EM–148. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* The positive integers $1, 2, \ldots, 2026$ are written on a board. Lisa Simpson picks any two of the numbers, deletes them, and writes on the board the absolute value of their difference. She repeats this procedure with the resulting 2025 numbers, and so on. After she does it 2025 times only one number remains on the board. Can this number be 4?

Medium–Hard Problems

MH–143. *Proposed by Todor Zaharinov Sofia, Bulgaria.* Find all ordered pairs (x, y) of integers for which $2025(x+y) = 16xy$.

MH–144. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain and José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain.* Two dice are rolled until the sum of points in their upper faces is 7 or 8. Find the probability that sum 7 appears before than sum 8.

MH–145. *Proposed by Vasile Cîrtoaje, Ploiesti, Romania.* Let a, b, c, d be nonnegative real numbers such that $a \geq b \geq c \geq 1 \geq d$ and $ab + ac + ad + bc + bd + cd = 6$. Prove that

$$\frac{1}{ab+9} + \frac{1}{bc+9} + \frac{1}{cd+9} + \frac{1}{da+9} \geq \frac{2}{5}.$$

MH–146. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Mihály Bencze, Braşov, Romania.* Find all triplets of positive numbers (a, b, c) such that

$$\begin{gathered} a+b+c = 3, \\ \frac{a^3}{bc(b+c)} + \frac{b^3}{ca(c+a)} + \frac{c^3}{ab(a+b)} = \frac{3}{2}. \end{gathered}$$

MH–147. *Proposed by Michel Bataille, Rouen, France.* Let the sequence $(a_n)_{n\geq 0}$ be defined by the recursion $a_{n+2} = 3a_{n+1} - a_n$ for all $n \geq 0$ and $a_0 = 1, a_1 = 2$. Prove that for all $n \geq 1$,

$$\lfloor\sqrt{4a_{n-1}a_{n+1} - 2}\rfloor - \lfloor\sqrt{a_{n-1}a_{n+1}}\rfloor = a_n \quad \text{and}$$
$$\lfloor\sqrt{4a_n a_{n+1} - 2}\rfloor - \lfloor\sqrt{a_n a_{n+1}}\rfloor = a_{n+1} - a_n.$$

MH–148. *Proposed by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain.* Consider the excircles of a triangle ABC, and join the points of tangency of each excircle with the two sides that have been extended; in this way a new triangle $A'B'C'$ is formed.

We ask:

1. To evaluate the angles of the triangle $A'B'C'$.
2. To prove that the lines AA', BB', CC' are the altitudes of the triangle $A'B'C'$.
3. To determine the center and the radius of the circumcircle of the triangle $A'B'C'$.

Advanced Problems

A–143. *Proposed by Michel Bataille, Rouen, France.* Let n be a positive integer and $x \in (0,1)$. Let $S_d(x) = \sum\limits_{\gcd(k,n)=d} x^k$ (the summation is extended to all positive integers k such that $\gcd(k,n) = d$). Prove that

$$(1 - x^n)\sum_{d|n} x^{n-d} S_d(x) - \sum_{d|n} x^{2(n-d)} \leq x^n(n - \tau(n)) \text{ and}$$

$$(1 - x^n)\sum_{d|n} x^{d} S_d(x) - \sum_{d|n} x^{2d} \geq x^n(n - \tau(n)).$$

where $\tau(n)$ denotes the number of positive divisors of n.

A–144. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let a, b and c be complex numbers such that $abc = 1$. For all integer n, prove that the value of

$$\begin{vmatrix} \dfrac{1}{ac} + n^3 c & n(c-b) & n^2(b-c) \\ n^2(c-a) & \dfrac{1}{ab} + n^3 a & n(a-c) \\ n(b-a) & n^2(a-b) & \dfrac{1}{bc} + n^3 b \end{vmatrix}$$

is the cube of an integer number.

A–145. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a triangle with $AB \neq AC$ and let I be its incenter. Let I_a, I_b, I_c be the feet of the perpendiculars drawn from I to BC, CA, AB respectively and M_a, M_b, M_c be the midpoints respectively of BC, CA, AB. Let $D = I_bI_c \cap BC$, $E = I_aI_b \cap M_aM_b$, $F = I_aI_c \cap M_aM_c$ and let J be the midpoint of AI. Knowing that $DJ \parallel EF$, prove that the area $[ABC] = 3[DEF]$.

A–146. *Proposed by Vasile Mircea Popa, "Lucian Blaga" University of Sibiu, Romania.* Calculate the following integral:

$$\int_0^{\infty} \frac{\ln^2(x+1)}{x^2+x+1} dx.$$

A–147. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Óscar Rivero Salgado, Universidad de Santiago de Compostela, Spain.* Let $A(z)$ be a monic polynomial with real coefficients such that $|A(i)| < 1$. Prove that there exists a zero $a + ib$ of $A(z)$ satisfying

$$(a^2 + (b+1)^2)\,(a^2 + (b-1)^2) < 1.$$

A–148. *Proposed by Joe Santmyer, Las Cruces, New Mexico, USA.* Show that

$$\int_0^\pi \theta^4 \log^4\left(2\cos\left(\frac{\theta}{2}\right)\right) d\theta \;=\; \frac{\pi^9 + 120^2\pi^3[\zeta(3)]^2 + 240^2\pi\zeta(3)\zeta(5)}{20^2},$$

where ζ is the Riemann zeta function.

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to
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An Enchanting Number Pattern

S. Devibala and M. A. Gopalan

Abstract

Let a, b, c, d, e and f be six non-zero single-digit integers. Some cases in which $(ab)^2 + (cd)^2 + (ef)^2 = (ba)^2 + (dc)^2 + (fe)^2$ are presented.

1 Introduction

It is worth observing that, if a, b, c are three non-zero single-digit integers, then in some cases

$$(ab)^2 + (bc)^2 + (ca)^2 = (ba)^2 + (cb)^2 + (ac)^2.$$

For example,

$$(68)^2 + (89)^2 + (96)^2 = (86)^2 + (98)^2 + (69)^2 = 21761.$$

In [**1**], the authors have shown that there exists an integer N such that the three-digit integer $abc = N + 2$ and its reverse integer $cba = 2N$, where a, b, c are three non-zero single-digit integers. In this short note, another enchanting number pattern is illustrated.

2 Methodology

In what follows we describe a procedure to determine six non-zero single-digit integers a, b, c, d, e and f that satisfy the equation

$$(ab)^2 + (cd)^2 + (ef)^2 = (ba)^2 + (dc)^2 + (fe)^2. \tag{1}$$

Indeed, note that ab, cd, ef, ba, dc, fe represent two-digit numbers respectively. Utilizing the decimal representation system, equation (1) can be written as

$$(10a+b)^2+(10c+d)^2+(10e+f)^2 = (10b+a)^2+(10d+c)^2+(10f+e)^2.$$

On simplification, the above equation is written as

$$a^2 + c^2 + e^2 = b^2 + d^2 + f^2. \tag{2}$$

Two options to solve (2) are presented below:

- **Option 1.** Assuming $a = d - f, b = c - e$ in (2), one has $d \cdot f = c \cdot e$. It is possible to choose c, d, e, f suitably so that the last expression is satisfied and from the preceding the values of a and b are obtained. It is also seen that (1) is satisfied. For the sake of clear understanding, a few illustrations are exhibited in Table 1 below:

Table 1: Numerical examples

d	f	c	e	a	b	Observation
6	1	3	2	5	1	$51^2 + 36^2 + 21^2 = 15^2 + 63^2 + 12^2 = 4338$
8	1	4	2	7	2	$72^2 + 48^2 + 21^2 = 27^2 + 84^2 + 12^2 = 7929$
8	3	6	4	5	2	$52^2 + 68^2 + 43^2 = 25^2 + 86^2 + 34^2 = 9177$
4	3	6	2	1	4	$14^2 + 64^2 + 23^2 = 41^2 + 46^2 + 32^2 = 4821$

Likewise, one may have $a = d + f, b = c + e$. For this choice, the numerical illustrations are presented in Table 2 below:

Table 2: Numerical examples

d	f	c	e	a	b	Observation
6	1	3	2	7	5	$75^2 + 36^2 + 21^2 = 57^2 + 63^2 + 12^2 = 7362$
8	1	4	2	9	6	$96^2 + 48^2 + 21^2 = 69^2 + 84^2 + 12^2 = 11961$
4	3	6	2	7	8	$78^2 + 64^2 + 23^2 = 87^2 + 46^2 + 32^2 = 10709$

- **Option 2.** Assuming $a = b - d, f = c - e$ in (2), one has $b \cdot d = c \cdot e$. It is possible to choose b, d, c, e suitably so that the last equation is satisfied. From the preceding, the values of a and f are obtained. It is also seen that (1) is satisfied. For the sake of clear understanding, a few illustrations are exhibited in Table 3 below: Likewise, one may have $a = b + d$, $f = c + e$.

Table 3: Numerical examples

b	d	c	e	a	f	Observation
6	1	3	2	5	1	$56^2 + 31^2 + 21^2 = 65^2 + 13^2 + 12^2 = 4538$
8	3	6	4	5	2	$58^2 + 63^2 + 42^2 = 85^2 + 36^2 + 24^2 = 9097$
8	1	4	2	7	2	$78^2 + 41^2 + 22^2 = 87^2 + 14^2 + 22^2 = 8249$
4	3	6	2	1	4	$14^2 + 63^2 + 24^2 = 41^2 + 36^2 + 42^2 = 4741$

For this choice, the numerical illustrations are presented in Table 4 below:

Table 4: Numerical examples

b	d	c	e	a	f	Observation
6	1	3	2	7	5	$76^2 + 31^2 + 25^2 = 67^2 + 13^2 + 52^2 = 7362$
8	1	4	2	9	6	$98^2 + 41^2 + 26^2 = 89^2 + 14^2 + 62^2 = 11961$
4	3	6	2	7	8	$74^2 + 63^2 + 28^2 = 47^2 + 36^2 + 82^2 = 10229$

3 Conclusion

To conclude, the readers and researchers in this area may examine for options to solve (1) and other number patterns.

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A method for proving constrained inequalities in three variables

Marius Drăgan and José Luis Díaz-Barrero

Abstract

A method to prove symmetric inequalities with homogeneous constrain is presented.

1 Introduction

The purpose of this paper is to prove constrained inequalities in three variables using a new method. Specifically, we consider inequalities of the form $f(x, y, z) \geq 0$, where x, y, and z are positive numbers, subject to the constraint $g(x, y, z) = 0$, with f and g being symmetric and homogeneous functions in x, y, and z. A related work can be found in [1]. In the following section, we will develop a corresponding method for each type of constraint.

2 Main results

Hereafter, we present several types of homogeneous constraints involving three positive real numbers, along with a method to prove the corresponding symmetric inequality. The types are:

- **Constrain 1:** $xyz = 1$. We make the substitution $x + y = s$, $xy = p$, so that $pz = 1$, or equivalently $p = \frac{1}{z}$. Since $(x - y)^2 \geq 0$ then we have $p \leq \frac{s^2}{4}$, which implies $\frac{1}{z} \leq \frac{s^2}{4}$, or $s \geq \frac{2}{\sqrt{z}} = s_1$. Without loss of generality, assume $x \leq y \leq z$. Note that among the variables x, y, z, at least one must be greater than 1. Without loss of generality, let $z \geq 1$. Similarly, at least one must be less than 1, so we take $x \leq 1$.

Application 1.1. Let x, y, z be positive real numbers such that $xyz = 1$. Then the following inequality holds:

$$\frac{1}{(1+x)^2} + \frac{1}{(1+y)^2} + \frac{1}{(1+z)^2} + \frac{2}{(1+x)(1+y)(1+z)} \geq 1.$$

Indeed, the given inequality can be rewritten as

$$\frac{(1+x)^2 + (1+y)^2}{(1+x+y+xy)^2} + \frac{1}{(1+z)^2} + \frac{1}{(1+z)(1+x+y+xy)} - 1 \geq 0.$$

Making the substitutions $s = x + y$ and $p = xy$, the preceding inequality becomes

$$\frac{s^2 - 2p + 2s + 2}{(1+s+p)^2} + \frac{1}{(1+z)(1+s+p)} - 1 + \frac{1}{(1+z)^2} \geq 0.$$

Substituting $p = 1/z$ yields

$$\frac{z[(s^2-3)z + z^3 - 2]}{(z+1)^2(sz+z+1)^2} \geq 0.$$

To prove the last inequality, it suffices to show that

$$f(s) = (s^2 - 3)z + z^3 - 2 = zs^2 + z^3 - 3z - 2 \geq 0.$$

Since $f(s)$ is increasing, because $f'(s) = 2sz \geq 0$ for all $z \geq 0$, we have $f(s) \geq f(s_1)$, which completes the proof because

$$f(s_1) = (z-1)^2(z+2) \geq 0.$$

Equality holds when $x = y = z = 1$.

Application 1.2. Let x, y, z be positive real numbers such that $xyz = 1$. Then the following inequality holds:

$$x^2 + y^2 + z^2 + 15(xy + yz + zx) \geq 16(x + y + z).$$

Indeed, making the substitutions $s = x + y$ and $p = xy$, the preceding inequality becomes

$$s^2 - 2p + z^2 + 15(p + sz) - 16(s + z) \geq 0.$$

Substituting $p = 1/z$ the inequality to be proven is

$$f(s) = zs^2 + (15z^2 - 16z)s + z^3 - 16z^2 + 13 \geq 0.$$

Since $f'(s) = 2zs + z(15z - 16)$, and $f'(s) = 0$ when $s = \frac{16-15z}{2} = s_v$, then $f(s)$ is increasing for all $s > s_v$. Now we set $s_1 > s_v$, or

$$\frac{2}{\sqrt{z}} > \frac{16 - 15z}{2},$$

which holds when $16 - z(16 - 15z)^2 > 0$, and this occurs when $z \in (0.761, 1.301)$. Then for all $s \geq s_1$, we have $f(s) \geq f(s_1) \geq 0$. Since

$$f(s_1) = (\sqrt{z} - 1)^2(2z^{3/2} + z^2 - 13z + 2\sqrt{z} + 17) \geq 0,$$

as can be easily checked, then

$$zs^2 + (15z^2 - 16z)s + z^3 - 16z^2 + 13$$

$$\geq (\sqrt{z} - 1)^2(2z^{3/2} + z^2 - 13z + 2\sqrt{z} + 17) \geq 0,$$

and the inequality is proven. Equality holds when $x = y = z = 1$, and the proof is complete.

Application 1.3. Let x, y, z be positive real numbers such that $xyz = 1$. Then the following inequality holds:

$$x^2 + y^2 + z^2 + x + y + z \geq 2(xy + yz + zx).$$

Indeed, suppose $z \leq 1$ and making the substitutions $s = x+y$ and $p = xy$, the preceding inequality becomes $s^2 - 2p + z^2 +$

$s + z - 2(p + sz) \geq 0$. Substituting $p = 1/z$ the inequality to be proven is

$$f(s) = sz^2 - z(2z - 1)s + z^3 + z^2 - 4 \geq 0.$$

Since $f'(s) = 2sz - z(2z - 1)$ then the minimum value of $f(s)$ is attained at $s_v = \frac{2z-1}{2}$ and $f(s)$ is increasing for all $s > s_v$. Now we set $s_1 > s_v$, or

$$\frac{2}{\sqrt{z}} > \frac{2z - 1}{2},$$

which holds when $16 - z(2z - 1)^2 > 0$, and this occurs when $z \in (0, 1.9371)$ and therefore $z \in (0, 1]$. Then for all $s \geq s_1$, we have $f(s) \geq f(s_1)$. Since

$$f(s_1) = \sqrt{z}(\sqrt{z} - 1)^2(z^{3/2} + 2z + 4\sqrt{z} + 2) \geq 0,$$

because $z > 0$, then

$$sz^2 - z(2z - 1)s + z^3 + z^2 - 4 \geq \sqrt{z}(\sqrt{z} - 1)^2(z^{3/2} + 2z + 4\sqrt{z} + 2) \geq 0,$$

and the inequality is proven. Equality holds when $x = y = z = 1$, and the proof is complete.

Application 1.4. Let x, y, z be positive real numbers such that $xyz = 1$. Then the following inequality holds:

$$(x + y)(y + z)(z + x) \geq 4(x + y + z - 1).$$

Indeed, using the same change of variables as in the preceding applications, the inequality claimed becomes $s(z^2 + sz + p) - 4(s + z - 1) \geq 0$ and putting $p = 1/z$ we obtain that we have to establish that

$$f(s) = z^2 s^2 + (z^3 - 4z + 1)s - 4z(z - 1) \geq 0.$$

Since $f'(s) = 2z^2 s + z^3 - 4z + 1$ then the minimum value of $f(s)$ is attained at $s_v = \frac{-z^3+4z+1}{2z^2}$ and $f(s)$ is increasing for $s \geq s_1$. The values of z for which $s_v \leq s_1$ are those for which

$$\frac{-z^3 + 4z + 1}{2z^2} \leq \frac{2}{\sqrt{z}}$$

or $z \in (0.193, 2.913)$. Then for all $s \geq s_1$, we have $f(s) \geq f(s_1)$. But,

$$f(s_1) = \frac{4}{z} + \frac{2(z^3 - 4z + 1)}{\sqrt{z}} - 4z(z-1) > 0$$

when $z \in (0, 1.265) \cup (14.383, +\infty)$, then

$$z^2s^2+(z^3-4z+1)s-4z(z-1) \geq \frac{4}{z}+\frac{2(z^3 - 4z + 1)}{\sqrt{z}}-4z(z-1) > 0,$$

and is these cases the inequality is proven. Equality holds when $x = y = z = 1$.

- **Constrain 2:** Let x, y, z be nonnegative real numbers such that $xy + yz + zx = 3$. Now, we define two variables s_1 and s_2 that will be play a key role in what follows. Putting $s = x + y, p = xy$ the constrain becomes $p = 3 - sz$. Since $p \geq 0$ then $3-sz \geq 0$ and $s \leq \frac{3}{z} = s_2$. Also $p = 3-sz \leq \frac{s^2}{4}$ or $s^2 + 4zs - 12 \geq 0$ from which we get $s \geq 2\sqrt{z^2+3} - 2z = s_1$. So $s_1 \leq 2\sqrt{z^2+3} - 2z \leq s \leq 3/z = s_2$. WLOG we may assume $z \leq 1$ or $z \geq 1$.

Application 2.1. Let x, y, z be positive real numbers such that $xy + yz + zx = 3$. Prove that

$$\frac{1}{x^2+1} + \frac{1}{y^2+1} + \frac{1}{z^2+1} \geq \frac{3}{2}.$$

Indeed, making the substitutions $s = x + y$ and $p = xy$ the inequality claimed is equivalent to

$$\frac{s^2 - 2p + 2}{p^3 + s^2 - 2p + 1} + \frac{1}{z^2+1} - \frac{3}{2} \geq 0.$$

Putting $p = 3 - sz$ yields

$$-\frac{(z^2+1)(3z^2-1)s^2 - 8z(2z^2+1) + 20z^2 + 12}{2(z^2+1)[s^2(z^2+1) - 4zs + 4]} \geq 0.$$

Since the discriminant of $s^2(z^2 + 1) - 4zs + 4 = 0$ is $\Delta_s = -16 < 0$ then $s^2(z^2+1) - 4zs + 4 > 0$. So, it will be suffice to prove that $f(s) = (z^2+1)(3z^2-1)s^2 - 8z(2z^2+1)s + 4(5z^2 + 3) \leq 0$. To do that we assume $z \leq 1$. The discriminant of $f(s)$, say $\delta_s = 16z^6 + 48(1-z^2)z^2 + 48 \geq 0$ because $z \leq 1$, then $f(s)$ has two roots. On the other hand, we have

$$f(s_1) = 4z[6z^5 + 21z^3 + 13z - (6z^4 + 12z^2 + 2)\sqrt{z^2+3}]$$

and $f(s_2) = \dfrac{(z^2-3)^2}{z^2}$. Trivially holds that $f(s_2) \leq 0$ for all $z \geq 0$. To prove that $f(s_1) \leq 0$ it will be suffice to see that $(6z^5 + 21z^3 + 13z^2)^2 \leq (6z^4 + 12z^2 + 2)^2(z^2+3)$ which is equivalent to $3(z-1)^2(z+1)^2(z^2+4) \geq 0$ for all $z \geq 0$. Next, we have $f'(s) = 2(z^2+1)(3z^2-1)s - 8z(2z^2+1) = 0$ for $s_v = \frac{4z(2z^2+1)}{(z^2+1)(3z^2-1)}$ which is well defined for all $z \neq 1/\sqrt{3}$. Thus, for $z \in (0, 1/\sqrt{3})$ setting $s_1 \geq s_v$ we have $f(s) \leq f(s_1) \leq 0$ for all $s \geq s_1$. Likewise, for $z \in (1/\sqrt{3}, 1]$ we have $f(s) \leq \max\{f(s_1), f(s_2)\} \leq 0$. Equality holds when $x = y = z = 1$. This completes the proof.

Application 2.2. Let x, y, z be positive real numbers such that $xy + yz + zx = 3$. Prove that

$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \geq \frac{x+y+z}{6} + \frac{3}{x+y+z}.$$

Indeed, carrying out the usual substitutions $s = x + y$ and $p = xy$ the inequality claimed becomes

$$\frac{1}{s} + \frac{s+2z}{z^2+sz+p} - \frac{s+z}{6} - \frac{3}{s+z} \geq 0.$$

Assuming $z \leq 1$ and putting $p = 3 - sz$ the inequality to be proven is

$$\frac{(z^2-3)s^3 + 2z(z^2-6)s^2 + (z^4+3z^2+36)s - 6z(z^2+3)}{6s(z^2+3)(z+s)} \leq 0.$$

To do that is suffice to prove that it holds:

$$f(s) = (z^2-3)s^3 + (2z^3-12z)s^2 + (z^4+3z^2+36)s - 6z(z^2+3) \leq 0.$$

Next, we claim that $f'(s) = (3z^2 - 9)s^2 + (4z^3 - 24z)s + z^4 + 3z^2 + 36$ is decreasing. In fact, we have that its discriminant $\delta_s = 4z^6 - 192z^4 + 252z^2 + 1296 > 0$ for all $z < 1.87$ and therefore for $z \leq 1$. On the other hand, $f'(s_1) = 17z^4 + 15z^2 - 72 + (24z - 16z^3)\sqrt{z^2+3} < 0$ for $z \in (0, 1.4036)$ and therefore for $z \in (0,1]$ and $f'(s_2) = \dfrac{z^6 + 15z^4 - 9z^2 - 81}{z^2} < 0$ $z \in (0, 1.55)$ and therefore for $z \in (0,1]$. Now, we consider the linear function $f''(s) = (6z^2 - 18)s + 4z^3 - 24z$. We have, $f''(s_1) = 4(-2z^3 + 3z + 3(z^2-3)\sqrt{z^2+3}) < 0$ when $z \in (0, 2.149)$ and $f''(s_2) = \frac{2(z^2+3)(2z^2-9)}{z} < 0$ when $z \in (0, 3/\sqrt{2})$ and therefore for $z \in (0,1]$. Thus, $f''(s) \leq 0$ because $f''(s)$ is linear and $f''(s_1) \leq 0$ and $f''(s_2) \leq 0$ and this implies that $f'(s)$ is decreasing. Since $f'(s_1) < 0$ then $f'(s) \leq f'(s_1) < 0$ for all $s \geq s_1$ and $f(s)$ is decreasing. That is, for all $s \geq s_1$ it holds $f(s) \leq f(s_1) < 0$, as we wanted to prove.

- **Constrain 3:** Let x, y, z be positive real numbers such that $x + y + z = 1$. Making the substitution $x + y = s$ and $xy = p$ from the constraint we get $s = 1 - z$. WLOG we assume $z \leq x \leq y$ or $(z-x)(z-y) \geq 0 \quad \Leftrightarrow \quad z^2 - sz + p \geq 0$. Putting $s = 1 - z$ yields $2z^2 - z + p \geq 0$ from which $p \geq z - 2z^2 = p_1$. Since $p \leq \dfrac{s^2}{4}$, then $p \leq \frac{(1-z)^2}{4} = p_2$. Thus, we have

$$p_1 = z - 2z^2 \ \leq \ p \ \leq \ \frac{(1-z)^2}{4} = p_2$$

for all $z \in \left[0, \frac{1}{3}\right]$.

Application 3.1. Let x, y, z be nonnegative real numbers such that $x + y + z = 1$. Then, it holds:

$$\frac{1}{8} \leq x^4 + y^4 + z^4 + 26xyz \leq 1.$$

Indeed, making the substitutions $s = x + y$ and $p = xy$ the LHS of the inequality claimed becomes

$$2p^2 + s^4 - 4ps^2 + z^4 - 26pz \geq \frac{1}{8}.$$

Putting $s = 1 - z$ we have to prove that $f(p) \geq 0$ where

$$f(p) = 2p^2 + [26z - 4(1-z)^2]p + (1-z)^4 + z^4 - \frac{1}{8}.$$

To do that we assume $z \leq x \leq y$ that jointly with the constrain gives $z \in [0, \frac{1}{3}]$.

Now we analyze the discriminant Δ_p of $f(p) = 0$. That is, $\Delta_p = -3(80z^3 - 380z^2 + 80z - 3)$. For $z \in (0.048, 0.1702)$ is $\Delta_p < 0$ and therefore $f(p) > 0$. For $z \in (0, 0.048)$ is $\Delta_p \geq 0$. In this case we compute $f'(p) = 4p + 26z - 4(1-z)^2$ and solving $f'(p) = 0$ we obtain $p_v = z^2 - \frac{17}{z} + 1$. Now we set $p_2 < p_v$ which occurs when $z \in [0, \frac{16-\sqrt{2}}{3}]$ and therefore for $z \in (0, 0.048)$. Then for all $p \leq p_2$ we have $f(p) \geq f(p_2)$. But for all $z \geq 0$ is

$$f(p_2) = \frac{1}{8}\left(9z^3 + 48z^2 - 98z + 48\right) \geq 0.$$

Now set $p_v < p_1$ which occurs when $0.109 \simeq \frac{1}{12}(19 - \sqrt{313}) < z < \frac{1}{12}(19 + \sqrt{313}) \simeq 3.057$ and also when $z \in (0.1702, 1/3)$. Then for all $p \geq p_1$ we have $f(p) > f(p_1)$. Since $f(p_1) = 18z^4 - 84z^3 + 50z^2 - 8z + 7/8 \geq 0$ for $z < 1/2$ and therefore for $z \in (0.1702, 1/3)$ then $f(p) \geq 0$, as desired.

The RHS inequality after making the usual substitutions $s = x + y$ and $p = xy$ becomes $2p^2 + s^4 - 4ps^2 + z^4 - 26pz \geq 1$. Putting $s = 1 - z$ yields

$$f(p) = 2p^2 + [26z - 4(1-z)^2]p + (1-z)^4 + z^4 - 1 \leq 0.$$

To prove the preceding inequality, we assume that $z \leq x \leq y$ or $z \in [0, \frac{1}{3}]$. The discriminant of $f(p) = 0$ is $\Delta_p = -240z^3 + 1140z^2 - 240z + 16$. We have $\Delta_p > 0$ for all $z < 4.5326$ and therefore for $z \in [0, 1/3]$. Furthermore, for these values of z, it holds $f(p_1) = 2z(z-4)(1-3z)^2 \leq 0$ and $f(p_2) = \frac{1}{8}(1-3z)^2(z-1)(z+7) \leq 0$. So by convexity $f(p) \leq \max\{f(p_1), f(p_2)\} \leq 0$, as we wanted to prove.

Application 3.2. Let x, y, z be positive real numbers such that $x + y + z = 1$. Prove that

$$5(x^2 + y^2 + z^2) \leq 6(x^3 + y^3 + z^3) + 1.$$

Indeed, making the usual substitutions $s = x + y$ and $p = xy$ the inequality claimed becomes

$$-6s^3 + 5s^2 + 18sp - 10p - 6z^3 + 5z^2 - 1 \leq 0.$$

Putting $s = 1 - z$ we obtain

$$f(p) = (8 - 18z)p - 2(1 - 2z)^2 \leq 0.$$

To prove it, we assume that $z \leq x \leq y$ or $0 \leq z \leq 1/3$. Furthermore, for these values of z, it holds $f(p_1) = 2(1 - 3z)^2(2z - 1) \leq 0$ and $f(p_2) = -\frac{1}{2}z(1 - 3z)^2 \leq 0$. Since $f(p)$ is a linear function, then $f(p) \leq 0$, and the proof in complete.

Application 3.3. Find the greatest constant λ such that the inequality

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} + \lambda\left(\frac{a^2 + b^2 + c^2}{(a + b + c)^2} - \frac{1}{3}\right)$$

holds for all nonnegative real numbers x, y, z, no two of which are zero [2].

Letting $t = a + b + c$ and $x = \frac{a}{t}, y = \frac{b}{t}$, and $z = \frac{c}{t}$ the inequality given in the statement becomes

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{3}{2} + \lambda\left(x^2 + y^2 + z^2 - \frac{1}{3}\right).$$

Making the usual change of variables $s = x + y, p = xy$ the preceding inequality becomes

$$\lambda \leq \frac{3[2s(s^2 - 2p + sz) + (2z - 3s)(z^2 + sz + p)]}{2s(z^2 + sz + p)(3s^2 - 6p + 3z^2 - 1)}$$

Putting $s = 1 - z$, yields

$$\lambda \leq \frac{3[(9z - 7)p + 7z^2 - 7z + 2]}{4(1 - z)(p + z)(-3p + 3z^2 - 3z + 1)}.$$

Let
$$f(p)=\frac{3[(9z-7)p+7z^2-7z+2]}{4(1-z)(p+z)(-3p+3z^2-3z+1)}.$$
We will see that $f(p)$ is decreasing or $f'(p)\le 0$. Indeed,
$$f'(p)=\frac{3[3(9z-7)p^2+6(7z^2-7z+2)p+6z^4+15z^3-25z^2+12z-2]}{4(1-z)(p+z)^2(-3p+3z^2-3z+1)^2}.$$
Since $x+y+z=1$ the we may assume $z\le x\le y$ or $z\in[0,1/3]$. Then, $f'(p)\le 0$ when
$$g(p)=3[3(9z-7)p^2+6(7z^2-7z+2)p+6z^4+15z^3-25z^2+12z-2]\le 0$$
which is true because its discriminant $\Delta_p=24(3z-1)^3(1-z)(z+1)\le 0$ and the leading coefficient of the trinomial $g(p)$ is $3(9z-7)<0$. Thus, $f'(p)\le 0$ and $f(p)$ is decreasing which implies that for all $p\le p_2$ is
$$\lambda\le f(p_2)\le f(p).$$
Since $f(p_2)=\dfrac{3}{1-z^2}$ then $\lambda\le\inf\limits_{0\le z\le 1/3}\left\{\dfrac{3}{1-z^2}\right\}=3$ taken at $z=0$. So, $\lambda\le 3$.

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On the heights of a triangle

Jordi Ferré García and José Luis Díaz-Barrero

1 Introduction

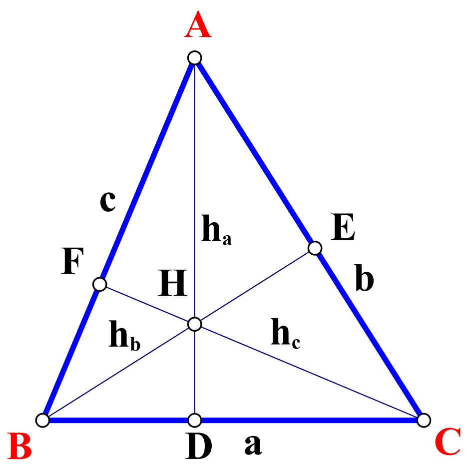
This math lesson is a continuation of [1]. In it, various relationships involving the lengths of the altitudes of a triangle, or heights, are presented. The altitude plays an important role in many theorems in the geometry of a triangle and their proofs. For example, the altitude plays a central role in proofs of both the Law of Sines and the Law of Cosines, as is well-known.

2 Main results

Hereafter, we present some results involving the heights of a triangle. We begin with

Theorem 1 (Length of the altitude). *In any triangle ABC the length of the altitude is given by the expressions*

$$\begin{aligned} h_a &= \frac{2}{a}\sqrt{s(s-a)(s-b)(s-c)} \\ &= 2R\sin B\sin C \\ &= \frac{2s\sin\dfrac{B}{2}\sin\dfrac{C}{2}}{\cos\dfrac{A}{2}} \quad \textit{(cyclic)} \end{aligned}$$



Scheme for proving the theorems.

Before proving the above theorem we need the following results.

Lemma 1. *In any triangle ABC, with the usual notations, it holds*

$$\sin A + \sin B + \sin C = 4\,\sin\frac{A+B}{2}\,\sin\frac{B+C}{2}\,\sin\frac{C+A}{2}.$$

Proof. On account of the well-known formulae $\sin A + \sin B = 2\,\sin\dfrac{A+B}{2}\,\cos\dfrac{A-B}{2}$, and $\sin(A+B+C) = \sin(A+B)\cos C + \cos(A+B)\sin C$, we have $\sin A + \sin B + \sin C = \sin A + \sin B + \sin C - \sin(A+B+C) = 2\,\sin\dfrac{A+B}{2}\,\cos\dfrac{A-B}{2}$

$$-\sin(A+B)\cos C + \sin C\left[1 - \cos(A+B)\right] \quad (1)$$

Since $\sin(A+B) = 2\sin\dfrac{A+B}{2}\cos\dfrac{A+B}{2}$ and $1 - \cos(A+B) = 2\,\sin^2\dfrac{A+B}{2}$ then (1) becomes

$$2\,\sin\frac{A+B}{2}\,\cos\frac{A-B}{2} - 2\sin\frac{A+B}{2}\left[\cos\frac{A+B}{2}\cos C - \sin\frac{A+B}{2}\sin C\right]$$

$$= 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} - 2\sin\frac{A+B}{2}\cos\left(\frac{A+B}{2}+C\right)$$
$$= 2\sin\frac{A+B}{2}\left[\cos\frac{A-B}{2} - \cos\left(\frac{A+B}{2}+C\right)\right]$$
$$= 4\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}$$

on account that $\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$.

Lemma 2. *In any triangle ABC, with the usual notations, it holds*

$$S = \frac{a^2 \sin B \sin C}{2\sin A}.$$

Proof. Let D be the foot of altitude h_a then $S = \frac{aAD}{2}$ but in the right triangle ACD we have $AD = b\sin C$. On account of the Sine Law we have $b = \frac{a\sin B}{\sin A}$ and $AD = \frac{a\sin B\sin C}{\sin A}$. Substituting in the preceding yields $S = \frac{a^2\sin B\sin C}{2\sin A}$, as we wanted to prove.

Lemma 3. *In any triangle ABC, with the usual notations, it holds*

$$\sin A + \sin B + \sin C = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Proof. Since $A+B+C = 180^\circ$ then $\frac{A+B}{2} = \frac{180^\circ - C}{2}$ and $\sin\frac{A+B}{2} = \cos\frac{C}{2}(cyclic)$. On account of Lemma 1, we have

$$\sin A + \sin B + \sin C = 4\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}$$
$$= 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

Proof of Theorem 1. In right triangle ADC we have $AD^2 = AC^2 - DC^2$ and $DC = b\cos C$. Cosine Law tells us that $c^2 = a^2 + b^2 - 2bc\cos C$ and $CD = \dfrac{b^2+a^2-c^2}{2a}$. Substituting in the preceding yields

$$h_a^2 = b^2 - \left(\frac{b^2+a^2-c^2}{2a}\right)^2 = \frac{4a^2b^2-(a^2+b^2-c^2)^2}{4a^2}$$
$$= \frac{[2ab+(a^2+b^2-c^2)][2ab-(a^2+b^2-c^2)]}{4a^2}$$
$$= \frac{[(a+b)^2-c^2][c^2-(a-b)^2)]}{4a^2}$$
$$= \frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{4a^2}$$
$$= \frac{4s(s-a)(s-b)(s-c)}{a^2}$$

from which $h_a = \dfrac{2}{a}\sqrt{s(s-a)(s-b)(s-c)}$ (cyclic) follows.

Lemma 2 tells us that $ah_a = \dfrac{a^2 \sin B \sin C}{\sin A}$ and using the Sine Law we get

$$h_a = \frac{a \sin B \sin C}{\sin A} = 2R \sin B \sin C \quad \text{(cyclic)}$$

From the preceding, we obtain $a = \dfrac{h_a \sin A}{\sin B \sin C}$ but right triangles ADB and ADC give $b = \dfrac{h_a}{\sin C}$ and $c = \dfrac{h_a}{\sin B}$ therefore

$$\frac{h_a}{\sin B} + \frac{h_a}{\sin C} + \frac{h_a \sin A}{\sin B \sin C} = 2s$$

and

$$h_a\left(\sin A + \sin B + \sin C\right) = 2s \sin B \sin C.$$

Using Lemma 3, we have

$$h_a(4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}) = 2s\sin B\sin C$$

from which

$$h_a = \frac{2s\sin\dfrac{B}{2}\sin\dfrac{C}{2}}{\cos\dfrac{A}{2}} \quad \text{(cyclic)}$$

follows and the theorem is proven.

Next, we give an expression for the distances from the orthocenter to the vertices of a triangle.

Theorem 2. *In any triangle ABC, the distances from the orthocenter to the vertices are given by*

$$\begin{aligned} d_a = d(H, A) &= 2R\cos A \\ &= \frac{a}{\tan A} \\ &= \sqrt{(2R-a)(2R+a)} \quad \textit{(cyclic)} \end{aligned}$$

Proof. Using the figure we have $AH = \dfrac{AF}{\sin B}$ and $AF = b\cos A$. Since $\frac{a}{\sin A} = \frac{b}{\sin B} = 2R$ by the Sine Law, we get

$$AH = \frac{b\cos A}{\sin B} = 2R\cos A = \frac{a\cos A}{\sin A} = \frac{a}{\tan A},$$

which shows the first two expressions. Then

$$AH^2 = \left(\frac{a\cos A}{\sin A}\right)^2 = \frac{a^2(1-\sin^2 A)}{\sin^2 A} = \frac{a^2}{\sin^2 A} - a^2 = 4R^2 - a^2$$

from which $AH = \sqrt{(2R-a)(2R+a)}$ (cyclic) follows.

Now we compute the distance from the orthocenter to the circumcenter of the triangle.

Theorem 3. *In any triangle ABC, the distance from the orthocenter to the circumcenter is given by the expressions*

$$\begin{aligned} HO &= \sqrt{9R^2 - (a^2 + b^2 + c^2)} \\ &= R\sqrt{1 - 8\cos A \cos B \cos C}. \end{aligned}$$

To show the theorem we will compute the power of H with respect to the circumcircle of triangle ABC. To do this comfortably, we will be assuming that all three angles of the triangle are at most $90°$, so that H lies inside the circumcircle. Otherwise, all calculations will still hold, up to a negative sign that cancels out.

Before proving the theorem we need the following well-known result.

Lemma 4. In any triangle ABC, the reflection of H with respect to any side of the triangle lies on the circumcircle of the triangle.

Proof. Call H' the reflection of H with respect to side BC. Then triangles $\triangle HBC \sim \triangle H'BC$ are similar (in fact congruent), so $\angle CH'B = \angle BHC = 180 - \angle A$, which shows that H' lies on the circumcircle of ABC.

Proof of Theorem 3. As H lies inside the circumcircle of ABC, the power of H with respect to this circumference is $R^2 - HO^2$. But this power can also be computed as $HA \cdot HH'$. So, using the lemma, if D is the projection of A over BC, we get that

$$R^2 - HO^2 = HA \cdot HH' = HA \cdot (2HD) = 2 \cdot HA \cdot (AD - HA)$$

$$= 2 \cdot (AH \cdot AD) - 2 \cdot AH^2.$$

We know by Theorem 2 that $AH^2 = 4R^2 - a^2$. Now to compute $AH \cdot AD$ we call E the projection of B over AC, and observe that cuadrilateral $CDHE$ is cyclic. So, by power of a point, we obtain that

$$AH \cdot AD = AE \cdot AC = (c\cos A) \cdot b = bc\cos A = \frac{b^2 + c^2 - a^2}{2}.$$

Putting everything together we get that

$$R^2 - HO^2 = (b^2 + c^2 - a^2) - 2(4R^2 - a^2)$$
$$\implies HO^2 = 9R^2 - (a^2 + b^2 + c^2).$$

This shows the first expression. For the second one, we just need to see that

$$a^2 + b^2 + c^2 = 8R^2(1 + \cos A \cos B \cos C).$$

Check how after substituting $a = 2R \sin A$, and cyclically for b and c, the condition turns into

$$\sin^2 A + \sin^2 B + \sin^2 C = 2 + 2\cos A \cos B \cos C$$
$$\iff \frac{1-\cos 2A}{2} + \frac{1-\cos 2B}{2} + \frac{1-\cos 2C}{2} = 2 + 2\cos A \cos B \cos C$$
$$\iff 4\cos A \cos B \cos C = -1 - (\cos 2A + \cos 2B + \cos 2C).$$

This is not difficult to show, it follows from applying identity $\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$ twice. Indeed,

$$4\cos A \cos B \cos C = 4\left[\frac{1}{2}(\cos(A+B) + \cos(A-B))\right]\cos C$$
$$= 2\cos(A+B)\cos C + 2\cos(A-B)\cos C$$
$$= \cos(180^\circ) + \cos(180^\circ - 2C) + \cos(180^\circ - 2B) + \cos(180^\circ - 2A)$$
$$= -1 - (\cos 2A + \cos 2B + \cos 2C),$$

which ends the proof of the theorem.

We'll now end the lecture by using the obtained formulaes to show two inequalities .

Theorem 4. *In any triangle ABC we get that*

$$\cos A \cos B \cos C \leq \frac{1}{8},$$

and equality holds if and only if $\angle A = \angle B = \angle C = 60^\circ$.

Proof. From last Theorem we know that $HO^2 = R^2(1-8\cos A\cos B\cos C)$, so

$$1 - 8\cos A\cos B\cos C \geq 0 \implies \cos A\cos B\cos C \leq \frac{1}{8}.$$

Also, equality holds if and only if $HO = 0$, that is, if the orthocenter and the circumcenter of the triangle coincide. Clearly this can only happen if ABC is an equilateral triangle, that is, if $\angle A = \angle B = \angle C = 60°$.

Theorem 5. *In any triangle ABC, with the usual notations, it holds*

$$\frac{h_a}{h_b+h_c} + \frac{h_b}{h_c+h_a} + \frac{h_c}{h_a+h_b} \leq \frac{3}{2} + \frac{a^2+b^2+c^2-ab-bc-ca}{2rs}.$$

Proof. To prove the inequality stated, we need the following claims.

- **Claim 1.** In any triangle ABC it holds:

$$\frac{1}{h_a+h_b} + \frac{1}{h_b+h_c} + \frac{1}{h_c+h_a} \leq \frac{1}{2r} \quad (1)$$

Indeed, since $ah_a = 2S$ (cyclic), then

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{a+b+c}{2S} = \frac{s}{S} = \frac{1}{r}.$$

On the other hand, we have

$$(h_a+h_b)\left(\frac{1}{h_a} + \frac{1}{h_b}\right) \geq 4 \quad \text{(cyclic)},$$

which yields

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \geq 2\left(\frac{1}{h_b+h_c} + \frac{1}{h_c+h_a} + \frac{1}{h_a+h_b}\right),$$

from which the claim follows.

- **Claim 2.** In any triangle ABC it holds:

$$h_a + h_b + h_c \leq 9r + \frac{a^2 + b^2 + c^2 - ab - bc - ca}{s} \qquad (2)$$

First, we see that if $a \leq b$ then $a + h_a \leq b + h_b$. Indeed,

$$a+h_a-b-h_b = a+b\sin C-b-a\sin C = (a-b)(1-\sin C) \leq 0.$$

This tells us that the triples (a, b, c) and $(a+h_a, b+h_b, c+h_c)$ are sorted in the same way. Using Chebyshev's inequality we get

$$a(a+h_a)+b(b+h_b)+c(c+h_c) \geq \frac{1}{3}(a+b+c)(a+b+c+h_a+h_b+h_c),$$

which implies

$$3(a^2+b^2+c^2)+18S \geq (a+b+c)^2+(a+b+c)(h_a+h_b+h_c),$$

or equivalently,

$$2(a^2 + b^2 + c^2 - ab - bc - ca) + 18S \geq 2s(h_a + h_b + h_c),$$

from which the claim follows.

Multiplying inequalities (1) and (2) yields

$$3+\frac{h_a}{h_b+h_c}+\frac{h_b}{h_c+h_a}+\frac{h_c}{h_a+h_b} \leq \frac{9}{2}+\frac{a^2+b^2+c^2-ab-bc-ca}{2rs},$$

which simplifies to

$$\frac{h_a}{h_b+h_c}+\frac{h_b}{h_c+h_a}+\frac{h_c}{h_a+h_b} \leq \frac{3}{2}+\frac{a^2+b^2+c^2-ab-bc-ca}{2rs}.$$

Equality holds when $h_a = h_b = h_c$, that is, when $\triangle ABC$ is equilateral.

Note that $\frac{h_a}{h_b+h_c}+\frac{h_b}{h_c+h_a}+\frac{h_c}{h_a+h_b} \geq \frac{3}{2}$ is Nesbitt's inequality applied to the heights of the triangle.

References

[1] Ferré Gracia, J. and Diaz-Barrero, J. L. “Some relations involving the elements of a triangle”. *Arhimede Math. J.* 12.1 (2025), pp. 30–38.

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Contests

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to find interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Problems and solutions from the 2nd Catalonian Girls Mathematical Olympiad

O. Rivero Salgado and J. L. Díaz-Barrero

1 Problems and solutions

Hereafter, we present the four problems that appeared in the paper given to the contestants of the Catalonian Girls Mathematical Olympiad 2025, as well as their official solutions.

Problem 1. Bart Simpson is grounded and has written all the numbers from 1 to 10000 on the board. Lisa erases all the multiples of 3, then those of 5, and continues the same process with the multiples of all the remaining primes in the list largest than 5. What are the numbers that Lisa has not erased?

Solution. By the Sieve of Eratosthenes, Lisa will erase every integer with an odd factor. Therefore, the numbers that remain are the powers of 2, say 2^n. Since $2^{13} = 8192$ and $2^{14} = 16384$, we must have $0 \leq n \leq 13$, for a total of 14 values. That is, the numbers not erased are:

$$1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, 4096, 8192$$

Problem 2. Let a, b, c be positive real numbers such that $a + b + c = 3$. Show that

$$\sqrt[3]{a(b+2c)} + \sqrt[3]{b(c+2a)} + \sqrt[3]{c(a+2b)} \leq 3\sqrt[3]{3}.$$

Solution. On account of AM-GM inequality and the constrain, we have

$$\sqrt[3]{a(b+2c)} + \sqrt[3]{b(c+2a)} + \sqrt[3]{c(a+2b)}$$

$$= \frac{1}{\sqrt[3]{9}} \cdot \sqrt[3]{3a(b+2c)\cdot 3} + \frac{1}{\sqrt[3]{9}} \cdot \sqrt[3]{3b(c+2a)\cdot 3} + \frac{1}{\sqrt[3]{9}} \cdot \sqrt[3]{3c(a+2b)\cdot 3}$$

$$\leq \frac{1}{\sqrt[3]{9}} \cdot \frac{3a+(b+2c)+3}{3} + \frac{1}{\sqrt[3]{9}} \cdot \frac{3b+(c+2a)+3}{3} + \frac{1}{\sqrt[3]{9}} \cdot \frac{3c+(a+2b)+3}{3}$$

$$= \frac{1}{\sqrt[3]{9}} \cdot \frac{4a+2b+3}{3} + \frac{1}{\sqrt[3]{9}} \cdot \frac{4b+2c+3}{3} + \frac{1}{\sqrt[3]{9}} \cdot \frac{4c+2a+3}{3}$$

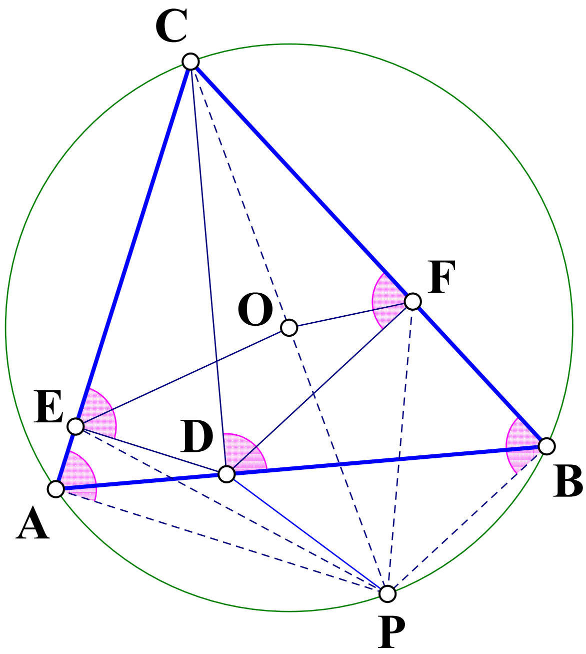
$$= \frac{1}{\sqrt[3]{9}} \cdot \frac{6(a+b+c)+9}{3} = 3\sqrt[3]{3}.$$

Equality holds when $a = b = c = 1$ and the proof is complete.

Problem 3. The acute triangle ABC is inscribed in a circle with center O. Point D is the orthogonal projection of point C on line AB, and points E and F are the orthogonal projections of point D on lines AC and BC, respectively. Show that the area of the triangle ABC is double of the area of quadrilateral $EOFC$.

Solution. Let P be a point symmetrical to point C with respect to point O. Since triangle ABC is acute, so the points A, P, B and C lie on a circle about the center of O in that order. The areas of triangles COE and POE are equal, since these triangles have a common altitude dropped from the vertex E, and the segments CO and OP are of equal length. Similarly, the areas of triangles COF and POF are equal.

Hence, we get $[EOFC] = \frac{1}{2} \cdot [EPFC]$. To complete the solution it is necessary to prove that $[EPFC] = [ABC]$. Indeed, the segment CP is the diameter of the circle described by the triangle ABC, so the lines AP and AC are perpendicular. The line DE



Scheme for solving problem 3.

is perpendicular to the line AC, so it is parallel to the line AP. From here it follows that the areas of triangles DEA and DEP are equal. Similarly, the areas of triangles DFB and DFP are equal. Thus,

$$\begin{aligned}[EPFC] &= [EDFC] + [DEP] + [DFP[EDF] \\ &= [EDFC] + [DEA] + [DFB] \\ &= [ABC].\end{aligned}$$

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Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Elementary Problems

E–137. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let a, b, c be three positive integer numbers such that $7ab = 2c^6$. Is the number $a^7 + b^7 + c^7$ prime or composite?

Solution 1 by the proposer. Using the well-known identity

$$x^7 + y^7 + z^7 - 7xyz$$

$$= (x+y+z)(x^6+y^6+z^6-x^5y-x^5z-y^5x-y^5z-z^5x-z^5y+7x^2y^2z^2)$$

with $x = a, y = b$ and $z = -c$, we have

$$a^7 + b^7 + c^7 = a^7 + b^7 - c^7 + 7abc$$

$$= (a+b-c)(a^6+b^6+c^6-a^5b+a^5c-b^5a-c^5a+b^5c-c^5b+7a^2b^2c^2).$$

Thus, the given expression is divisible by $a + b - c$ and $a^7 + b^7 + c^7$ is composite.

Solution 2 by Cao Minh Quang, Nguyen Binh Khiem High School for the Gifted, Vinh Long, Vietnam. Write $v_7(\cdot)$ for the 7-adic valuation. From $7ab = 2c^6$ we immediately get $7 \mid c$ (since the right-hand side is divisible by 7), hence $c \equiv 0 \pmod 7$. The right-hand side is even, so at least one of a, b is even.

We consider two cases:

- **Case 1:** c **is odd** Then $2c^7$ has valuation $v_7(2c^7) = 1$, so exactly one of a, b is even and the other is odd. Therefore

$$N = a^7 + b^7 + c^7 = \text{(even)} + \text{(odd)} + \text{(odd)}$$

is even. Since $N > 2$, it follows that N is composite.

- **Case 2:** c **is even** Write $c = 2^\ell c_0$ with $\ell \geq 1$ and c_0 odd. Then

$$v_7(2c^7) = 1 + \ell + 7 \geq 7 \quad \Rightarrow \quad v_7(ab) = v_7(c) \geq 7.$$

Hence at least one of a, b is even; if both are even then $2^7 \mid a^7$ and $2^7 \mid b^7$, so $2 \mid N$ and $N > 2$, whence N is composite. Moreover, from $v_7(7ab) = v_7(2c^6)$ we obtain

$$1 + v_7(a) + v_7(b) = 6\, v_7(c) \quad \Rightarrow \quad v_7(a) + v_7(b) \geq 5,$$

so at least one of a, b is divisible by 7. If in fact both a and b are divisible by 7, then $7 \mid a^7$, $7 \mid b^7$ and, since $7 \mid c$, also $7 \mid c^7$. Hence $7 \mid N$ and $N > 7$, so N is composite.

In all subcases above we conclude that N is not prime. That is,

$$N = a^7 + b^7 + c^7$$

is composite.

Also solved by *José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain.*

E–138. *Proposed by Michel Bataille, Rouen, France.* Let x, a_1, a_2, a_3, a_4 be complex numbers such that $a_1+a_2+a_3+a_4 = 0$. Evaluate

$$\sum_{1\le i<j<k\le 4}(x+a_i+a_j+a_k)^3 - \sum_{1\le i<j\le 4}(x+a_i+a_j)^3 + \sum_{1\le i\le 4}(x+a_i)^3.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland. Clearly,

$$(x-a)^3 + (x+a)^3 = 6a^2x + 2x^3,$$

$$\sum_{1\le i<j<k\le 4}(x+a_i+a_j+a_k)^3 = \sum_{1\le i\le 4}(x-a_i)^3,$$

$$\sum_{1\le i<j\le 4}(x+a_i+a_j)^3$$

$$= \frac{1}{2}\sum_{1\le i<j\le 4}(x+a_i+a_j)^3 + \frac{1}{2}\sum_{1\le i<j\le 4}(x+a_i+a_j-a_1-a_2-a_3-a_4)^3$$

$$= \frac{1}{2}\sum_{1\le i<j\le 4}(x+a_i+a_j)^3 + \frac{1}{2}\sum_{1\le i<j\le 4}(x-a_i-a_j)^3.$$

Hence

$$\sum_{1\le i<j<k\le 4}(x+a_i+a_j+a_k)^3 - \sum_{1\le i<j\le 4}(x+a_i+a_j)^3 + \sum_{1\le i\le 4}(x+a_i)^3$$

$$= \sum_{1\le i\le 4}(x+a_i)^3 + \sum_{1\le i\le 4}(x-a_i)^3 - \frac{1}{2}\sum_{1\le i<j\le 4}(x+a_i+a_j)^3$$

$$-\frac{1}{2}\sum_{1\le i<j\le 4}(x-a_i-a_j)^3$$

$$= 8x^3 + 6x\sum_{1\le i\le 4}a_i^2 - 6x^3 - 3x\sum_{1\le i<j\le 4}(a_i+a_j)^2$$

$$= 2x^3 + 6x\sum_{1\le i\le 4}a_i^2 - 3x\left(3\sum_{1\le i\le 4}a_i^2 + 2\sum_{1\le i<j\le 4}a_ia_j\right)$$

$$= 2x^3 - 3x\left(\sum_{1\le i\le 4}a_i^2 + 2\sum_{1\le i<j\le 4}a_ia_j\right)$$

$$= 2x^3 - 3x(a_1+a_2+a_3+a_4)^2 = 2x^3.$$

Solution 2 by Daniel Văcaru, „Maria Teiuleanu" National Economic College, Pitești, Romania. One has

$$\sum_{1\le i<j<k\le 4}(x+a_i+a_j+a_k)^3=\sum_{1\le i\le 4}(x-a_i)^3. \tag{1}$$

It follows that

$$\sum_{1\le i<j<k\le 4}(x+a_i+a_j+a_k)^3+\sum_{1\le i\le 4}(x+a_i)^3\overset{(1)}{=}\sum_{1\le i\le 4}(x-a_i)^3+\sum_{1\le i\le 4}(x+a_i)^3. \tag{2}$$

But for all $x\in\mathbb{C}$, we have

$$(x-a)^3+(x+a)^3=2x^3+6a^2x. \tag{3}$$

For all $x\in\mathbb{C}$ it follows that

$$\begin{aligned}\sum_{1\le i<j<k\le 4}(x+a_i+a_j+a_k)^3+\sum_{1\le i\le 4}(x+a_i)^3 &= (2x^3+6a_1x)+(2x^3+6a_2x)\\ &+(2x^3+6a_3x)+(2x^3+6a_4x)=8x^3,\end{aligned} \tag{4}$$

because $\sum_{i=1}^4 a_i=0$.

On the other hand, one has

$$\begin{aligned}\sum_{1\le i<j\le 4}(x+a_i+a_j)^3 &= [x^3+3x^2(a_1+a_2)+3x(a_1+a_2)^2+(a_1+a_2)^3]\\ &+[x^3+3x^2(a_1+a_3)+3x(a_1+a_3)^2+(a_1+a_3)^3]\\ &+[x^3+3x^2(a_1+a_4)+3x(a_1+a_4)^2+(a_1+a_4)^3]\\ &+[x^3+3x^2(a_2+a_3)+3x(a_2+a_3)^2+(a_2+a_3)^3]\\ &+[x^3+3x^2(a_2+a_4)+3x(a_2+a_4)^2+(a_2+a_4)^3]\\ &+[x^3+3x^2(a_3+a_4)+3x(a_3+a_4)^2+(a_3+a_4)^3].\end{aligned} \tag{5}$$

which is equal to

$$6x^3+9x^2(a_1+a_2+a_3+a_4)+9x\left[\sum_{i=1}^4 a_i^2+2\sum_{1\le i<j\le 4}a_ia_j\right]+\sum_{1\le i<j\le 4}(a_i+a_j)^3. \tag{6}$$

But
$$\sum_{i=1}^{4} a_i^2 + 2 \sum_{1\le i<j\le 4} a_i a_j = (a_1 + a_2 + a_3 + a_4)^2 = 0, \tag{7}$$
and
$$(a_i + a_j)^3 + (a_{4-i} + a_{4-j})^3 = 0. \tag{8}$$
That follows from the fact that
$$\begin{aligned}(a_i + a_j)^3 + (a_{4-i} + a_{4-j})^3 &= [(a_i + a_j) + (a_{4-i} + a_{4-j})] \\ \times\ [(a_i + a_j)^2 - (a_i + a_j)(a_{4-i} + a_{4-j}) &+ (a_{4-i} + a_{4-j})^2],\end{aligned} \tag{9}$$
which is 0. For all $x \in \mathbb{C}$ it follows that
$$\sum_{1\le i<j\le 4} (x + a_i + a_j)^3 = 6x^3, \tag{10}$$

Using relationships (4) and (10), for all $x \in \mathbb{C}$ yields
$$\sum_{1\le i<j<k\le 4} (x+a_i+a_j+a_k)^3 - \sum_{1\le i<j\le 4} (x+a_i+a_j)^3 + \sum_{1\le i\le 4} (x+a_i)^3 = 2x^3. \tag{11}$$

Solution 3 by the proposer. Let $P(x)$ be the given polynomial. We show that $P(x) = 2x^3$.

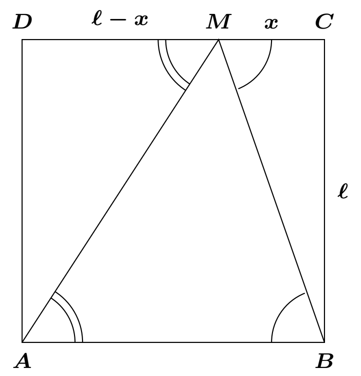
Since $a_i + a_j + a_k = -a_\ell$ with $\ell \ne i, j, k$, we have
$$\sum_{1\le i<j<k\le 4} (x + a_i + a_j + a_k)^3 = \sum_{1\le \ell\le 4} (x - a_\ell)^3.$$
Noting the identity $(x + y)^3 + (x - y)^3 = 2x^3 + 6xy^2$, we see that
$$\sum_{1\le i<j<k\le 4} (x+a_i+a_j+a_k)^3 + \sum_{1\le i\le 4} (x+a_i)^3 = 8x^3+6x(a_1^2+a_2^2+a_3^2+a_4^2).$$
From $a_i + a_j = -(a_k + a_\ell)$ if $\{i, j, k, \ell\} = \{1, 2, 3, 4\}$, we similarly obtain
$$\sum_{1\le i<j\le 4} (x + a_i + a_j)^3 = 6x^3 + 6x \sum_{1\le i<j\le 3} (a_i + a_j)^2.$$
Since $a_4^2 = (a_1 + a_2 + a_3)^2$, it follows that
$$\begin{aligned}P(x) &= 2x^3 + 6x\left[a_1^2 + a_2^2 + a_3^2 + (a_1 + a_2 + a_3)^2 - \sum_{1\le i<j\le 3} (a_i + a_j)^2\right] \\ &= 2x^3.\end{aligned}$$

Also solved by *José Luis Díaz-Barrero, Barcelona, Spain.*

E–139. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Consider the square $ABCD$, with a point M on the side DC. If the value of the sum $S = \tan(MAB) + \tan(MBA) + \tan(AMB)$ is minimal, calculate the value of the ratio MD/AB.

Solution 1 by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain. Let ℓ denote the length of the side of the given square. We put $CM = x$. Then $MD = \ell - x$ and in $\triangle ABM$ we have

$$\begin{aligned} \tan(\angle MAB) &= \tan(\angle AMD) &&= \tfrac{\ell}{\ell-x} \\ \tan(\angle MBA) &= \tan(\angle BMC) &&= \tfrac{\ell}{x} \\ \tan(\angle AMB) &= -\tan(\angle MAB + \angle MBA) &&= -\tfrac{\frac{\ell}{\ell-x}+\frac{\ell}{x}}{1-\frac{\ell}{\ell-x}\cdot\frac{\ell}{x}} = \tfrac{\ell^2}{\ell^2-x(\ell-x)} \end{aligned}$$



Therefore

$$S = \left(\frac{\ell}{\ell-x} + \frac{\ell}{x} + \frac{\ell^2}{\ell^2 - x(\ell-x)}\right) = \frac{\ell^3}{x(\ell-x)[\ell^2 - x(\ell-x)]},$$

and hence minimize S is equivalent to maximize $x(\ell-x)[\ell^2 - x(\ell-x)]$.

Let $f : (0,\ell) \to (0,\infty)$ be given by $f(x) = x(\ell-x)[\ell^2 - x(\ell-x)]$. From the derivative $f'(x) = (\ell - 2x)\left[(x-\ell)^2 + x^2\right]$ we see that $f'(x) \geq 0$ for $x \in \left(0, \frac{\ell}{2}\right)$ while $f'(x) \leq 0$ for $x \in \left(\frac{\ell}{2}, \ell\right)$.

Therefore, $f\left(\frac{\ell}{2}\right)$ is the absolute maximum value of f on $(0,\ell)$, Thus S takes its minimum value $S=\frac{16}{3}$ for $x=\frac{\ell}{2}$, making the value of the desired ratio
$$\frac{MD}{AB}=\frac{\ell-\frac{\ell}{2}}{\ell}=\frac{1}{2}.$$

Solution 2 by Michel Bataille, Rouen, France. Without loss of generality, we suppose that the sidelength of $ABCD$ is 1. Let $x=MD/AB$ and let N be the point of AB such that $MN\perp AB$ (note that $x\in(0,1)$ and $MN=1$). We have

$$\tan(MAB)=\frac{MN}{AN}=\frac{1}{MD}=\frac{1}{x},\qquad \tan(MBA)=\frac{MN}{CM}=\frac{1}{1-x},$$

hence

$$\tan(AMB)=-\tan(MAB+MBA)=-\frac{\frac{1}{x}+\frac{1}{1-x}}{1-\frac{1}{x(1-x)}}=\frac{1}{1-x(1-x)}.$$

Setting $y=x(1-x)$, it follows that $S=S(y)=\frac{1}{y}+\frac{1}{1-y}$.
Since $0<\sqrt{x(1-x)}\leq\frac{x+1-x}{2}=\frac{1}{2}$ with equality if and only if $x=\frac{1}{2}$, we have $0<y\leq\frac{1}{4}$ with $y=\frac{1}{4}$ if and only if $x=\frac{1}{2}$. Since $S'(y)=\frac{2y-1}{y^2(1-y)^2}$, the function S is decreasing on $(0,\frac{1}{4}]$ and therefore the minimal value of S is attained when $y=\frac{1}{4}$, that is, when the value of $x=MD/AB$ is $\frac{1}{2}$.

Solution 3 by the proposer. Let $ME\perp AB$, $\measuredangle MAB=\alpha$, $\measuredangle MBA=\beta$, $\measuredangle AMB=\gamma$. Using the equality

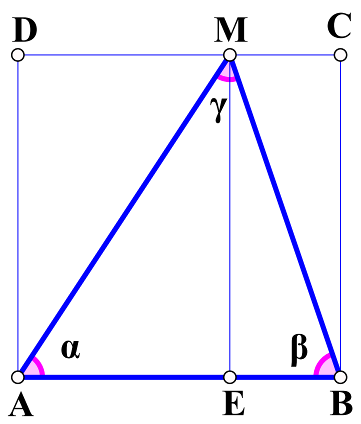
$$\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\cdot\tan\beta\cdot\tan\gamma \tag{1}$$

$\cot\alpha+\cot\beta=\dfrac{AE}{ME}+\dfrac{EB}{ME}=\dfrac{AB}{ME}$ with $AB=ME\Rightarrow\cot\alpha+\cot\beta=\dfrac{AB}{AB}=1$, so $\dfrac{1}{\tan\alpha}+\dfrac{1}{\tan\beta}=1\Rightarrow$

$$\tan\alpha+\tan\beta=\tan\alpha\tan\beta$$

From AM–GM inequality

$$(\tan\alpha+\tan\beta)^2\geq 4\tan\alpha\tan\beta$$



If $\tan\alpha+\tan\beta=x\Rightarrow$

$$\underbrace{(\tan\alpha+\tan\beta)^2}_{x^2}\geq 4\underbrace{\tan\alpha\tan\beta}_{x}\Rightarrow x^2\geq 4x\Rightarrow x\geq 4$$

Replace in (1)

$$\tan\gamma+x=x\tan\gamma\Rightarrow\tan\gamma(x-1)=x\Rightarrow\tan\gamma=\frac{x}{x-1}$$

$$S=\tan\alpha+\tan\beta+\tan\gamma=x+\frac{x}{x-1}=\frac{x^2}{x-1}$$

Prove that $\dfrac{x^2}{x-1}\geq\dfrac{16}{3}\Leftrightarrow$

$3x^2-16x+16\geq 0\Leftrightarrow(x-4)(3x-4)\geq 0$ True, because $x\geq 4$

The minimum value of the sum $S=\tan(MAB)+\tan(MBA)+\tan(AMB)$ is $\dfrac{16}{3}$ and is reached for $x=4$

$$\begin{cases}\tan\alpha+\tan\beta=4\\ \tan\alpha\cdot\tan\beta=4\end{cases}\Rightarrow\tan\alpha=\tan\beta=2\Rightarrow\frac{ME}{AE}=2$$

$$\Rightarrow\frac{MD}{AB}=\frac{AE}{AB}=\frac{1}{2}$$

E–140. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* In the lottery game of Springfield, five numbers are drawn out of 50 every week. Homer fills out a single lottery ticket with the same five numbers in each of the 52 weeks of the year. Flanders uses a different scheme. He plays only once a year with 52 tickets simultaneously: he fills them out in pairwise different ways. Is it true that both of them have the same chance of having a ticket with five correct numbers?

Solution 1 by Pau Matarrodona Moragas, BarcelonaTech, Terrassa, Spain. In each lottery draw, there are

$$\binom{50}{5}$$

possible distinct tickets, exactly one of which is winning. Thus, the probability that a fixed ticket wins is $p = 1/\binom{50}{5}$. If Homer plays the same ticket n times (with $n \geq 2$ an integer), the probability that his ticket loses a single draw is $1 - p$, and therefore the probability that it loses in all n draws is $(1 - p)^n$

Since every lottery game is independent, the probability of Homer of winning at least one lottery game (P_H) can be computed as 1 minus the chance of losing every single game, $P_H = 1 - (1 - p)^n$. Hence, Homer's probability of winning at least once is

$$P_H = 1 - (1 - p)^n$$

Flanders, by contrast, purchases n distinct tickets in a single draw. Since all possible winning tickets are equally likely, the probability that he holds the winning ticket (P_F) is

$$\frac{n}{\binom{50}{5}} = np$$

We now prove, by induction on $n \geq 2$, that

$$P_H < P_F.$$

Base case ($n = 2$): Expanding P_H gives

$$P_H = 1 - (1-p)^2 = 2p - p^2.$$

Since $P_H = 2p - p^2 < 2p = P_F$, the inequality holds.

Inductive step: Assume that $P_H < P_F$ holds for $n = k$

$$1 - (1-p)^k < kp \Rightarrow -(1-p)^k < kp - 1$$

For $n = k+1$, we compute Homer's probability

$$\begin{aligned} P_H &= 1 - (1-p)^{k+1} \\ &= 1 - (1-p)^k(1-p) \end{aligned}$$

By the inductive hypothesis, we have $1 - (1-p)^k < kp$, which implies

$$-(1-p)^k < kp - 1.$$

Substituting this into the expression for P_H, we obtain

$$P_H = 1 - (1-p)^k(1-p) < 1 - (kp-1)(1-p) = kp - kp^2 + p.$$

On the other hand, Flanders has probability

$$P_F = (k+1)p = kp + p.$$

Comparing both bounds yields

$$\begin{gathered} P_H < P_F, \\ kp - kp^2 + p < kp + p, \\ -kp^2 < 0, \end{gathered}$$

which is always true.

Thus, by induction, the inequality $P_H < P_F$ holds for all $n \geq 2$. In particular, for the study case $n = 52$, Flanders has a strictly higher probability of holding the winning ticket than Homer.

Solution 2 by Pau Matarrodona Moragas, BarcelonaTech, Terrassa, Spain. In each lottery draw, there are

$$\binom{50}{5}$$

possible distinct tickets, exactly one of which is winning. Thus, the probability that a fixed ticket wins is $p = 1/\binom{50}{5}$. Homer plays the same ticket in 52 independent draws. The probability that his ticket loses a single draw is $1 - p$, and therefore the probability that it loses in all 52 draws is $(1 - p)^{52}$

Since every lottery game is independent, the probability of Homer of winning at least one lottery game (P_H) can be computed as 1 minus the chance of losing every single game, $P_H = 1 - (1 - p)^{52}$. Hence, Homer's probability of winning at least once is

$$P_H = 1 - (1 - p)^{52}$$

Flanders, by contrast, purchases 52 distinct tickets in a single draw. Since all possible winning tickets are equally likely, the probability that he holds the winning ticket is

$$\frac{52}{\binom{50}{5}} = 52p$$

To compare P_H and P_F, we expand P_H using the binomial theorem:

$$P_H = 1 - (1 - p)^{52} = 1 - \sum_{k=0}^{52} \binom{52}{k} (-1)^{52-k} p^k =$$
$$1 - \left[1 - 52p + \binom{52}{2} p^2 - \binom{52}{3} p^3 + \binom{52}{4} p^4 + \ldots + p^{52}\right] =$$
$$52p - \left[\binom{52}{2} p^2 - \binom{52}{3} p^3 + \binom{52}{4} p^4 + \ldots\right]$$

Define

$$\delta = \binom{52}{2}p^2 - \binom{52}{3}p^3 + \binom{52}{4}p^4 + .. > 0$$

This series is strictly positive because $0 < p < 1$ and the alternating expansion of $(1-p)^{52}$ ensures $(1-p)^{52} > 1 - 52p$, which is equivalent to $\delta > 0$. Therefore, $P_H = 52 - \delta = P_F - \delta$ and since $\delta > 0$, it follows that $P_H < P_F$.

In conclusion, Flanders has a higher probability of holding a winning ticket than Homer.

Solution 3 by the proposer. The answer is NOT and $p_F > p_H$. Indeed, since there are a total of $\binom{50}{5}$ ways to fill out a lottery ticket, the probability that Flander's ticket will be a jackpot ticket is:

$$p_F := 52 \times \frac{1}{\binom{50}{5}} = 52p$$

where $p = 1/\binom{50}{5}$. Homers' ticket has a probability of $p = 1/\binom{50}{5}$ of being a jackpot ticket every week, so the probability that he does not have a jackpot ticket in any given week is:

$$\left(1 - \frac{1}{\binom{50}{5}}\right)^{52}.$$

Therefore, the probability that Homer will have a jackpot ticket is:

$$p_H := 1 - \left(1 - \frac{1}{\binom{50}{5}}\right)^{52} = 1 - (1-p)^{52}.$$

Now, we claim that for any $k \geq 2$, it holds:

$$(1-p)^k > 1 - kp.$$

To prove the claim we argue by induction on k. If $k = 2$, then indeed:

$$(1-p)^2 = 1 - 2p + p^2 > 1 - 2p, \quad \text{since}\, p > 0.$$

The induction step from k to $k+1$ can be verified as follows:

$$\begin{aligned}(1-p)^{k+1} &= (1-p)^k(1-p) > (1-kp)(1-p)\\ &= 1-(k+1)p+kp^2 > 1-(k+1)p,\end{aligned}$$

and the claim is proven.

Then, according to the claim

$$(1-p)^{52} > 1-52p,$$

and thus

$$p_F = 52p > 1-(1-p)^{52} = p_H.$$

Also solved by *Alberto Espuny Díaz, Universitat de Barcelona, Spain.*

E–141. *Proposed by Goran Conar, Varaždin, Croatia.* Let $a_1, a_2, \ldots, a_n$ be positive real numbers satisfying $a_1+a_2+\cdots+a_n = n$. Prove that

$$\frac{n}{2}+\frac{1}{2}\sum_{i=1}^{n} a_i^4 \geq \sum_{i=1}^{n}\sqrt{\frac{a_i^8+a_i^4+1}{a_i^2+a_i+1}} \geq \sum_{i=1}^{n} a_i\sqrt{a_i}.$$

Solution 1 by Michel Bataille, Rouen, France. Let x be any real number. From $x^4+x^2+1 = (x^2+x+1)(x^2-x+1)$, we obtain

$$x^8+x^4+1 = (x^2+x+1)(x^2-x+1)(x^4-x^2+1)$$

and therefore $R(x) = \frac{x^8+x^4+1}{x^2+x+1}$ is equal to $(x^2-x+1)(x^4-x^2+1)$. Thus, for positive x we have

$$R(x) = ((x-1)^2+x)((x^2-1)^2+x^2) \geq x^3$$

and using $\sqrt{XY} \leq \frac{X+Y}{2}$,

$$\sqrt{R(x)} \leq \frac{x^4-x+2}{2}.$$

We deduce that

$$\sum_{i=1}^{n}\sqrt{a_i^3} \le \sum_{i=1}^{n}\sqrt{R(a_i)} \le \frac{1}{2}\sum_{i=1}^{n}(a_i^4 - a_i + 2) = \frac{1}{2}\sum_{i=1}^{n}a_i^4 - \frac{1}{2}\sum_{i=1}^{n}a_i + n.$$

Since $\sqrt{a_i^3} = a_i\sqrt{a_i}$ and $\sum\limits_{i=1}^{n} a_i = n$, the required inequality immediately follows.

Solution 2 by Albert Stadler, Herrliberg, Switzerland. Using GM-AM inequality and a little algebra, for $x \ge 0$ we have

$$\frac{x^8 + x^4 + 1}{x^2 + x + 1} = \left(1 - x + x^2\right)\left(1 - x^2 + x^4\right),$$

$$\left(1 - x + x^2\right)\left(1 - x^2 + x^4\right) - x^3 = (1 - x)^2\left(1 + x + x^2 + x^3 + x^4\right) \ge 0,$$

$$\sqrt{\frac{x^8 + x^4 + 1}{x^2 + x + 1}} = \sqrt{(1 - x + x^2)(1 - x^2 + x^4)}$$

$$\le \frac{(1 - x + x^2) + (1 - x^2 + x^4)}{2} = 1 - \frac{x}{2} + \frac{x^4}{2}.$$

Hence

$$\sum_{i=1}^{n} a_i\sqrt{a_i} \le \sum_{i=1}^{n}\sqrt{\frac{a_i^8 + a_i^4 + 1}{a_i^2 + a_i + 1}} \le \sum_{i=1}^{n}\left(1 - \frac{a_i}{2} + \frac{a_i^4}{2}\right) = \frac{n}{2} + \frac{1}{2}\sum_{i=1}^{n}a_i^4.$$

Solution 3 by the proposer. For every $i \in \{1, 2, \ldots, n\}$ it is satisfied

$$\frac{a_i^8 + a_i^4 + 1}{a_i^2 + a_i + 1} = (a_i^2 - a_i + 1)(a_i^4 - a_i^2 + 1) \ge a_i \cdot a_i^2 = a_i^3$$

because $(a_i - 1)^2 \ge 0$ and $(a_i^2 - 1)^2 \ge 0$. Thus

$$\sqrt{\frac{a_i^8 + a_i^4 + 1}{a_i^2 + a_i + 1}} \ge a_i\sqrt{a_i}$$

for every $i \in \{1, 2, \ldots, n\}$. Adding this n inequalities we get the right hand side inequality. To prove left hand side we use arithmetic mean - geometric mean inequality:

$$1 + \frac{a_i^4 - a_i}{2} = \frac{(a_i^2 - a_i + 1) + (a_i^4 - a_i^2 + 1)}{2}$$

$$\overset{AM-GM}{\overbrace{\geq}} \sqrt{(a_i^2-a_i+1)(a_i^4-a_i^2+1)} = \sqrt{\frac{a_i^8+a_i^4+1}{a_i^2+a_i+1}}.$$

Adding this n inequalities and putting $\sum_{i=1}^n a_i = n$ we get

$$\frac{n}{2}+\frac{1}{2}\sum_{i=1}^n a_i^4 = n+\sum_{i=1}^n \frac{a_i^4-a_i}{2} \geq \sum_{i=1}^n \sqrt{\frac{a_i^8+a_i^4+1}{a_i^2+a_i+1}}$$

which prove the inequality claimed. Equality occurs if and only if $a_i^2-a_i+1 = a_i^4-a_i^2+1$ and $(a_i^2-a_i+1)(a_i^4-a_i^2+1) = a_i^3$ which implies $a_i = 1$, for every $i \in \{1, 2, \ldots, n\}$.

Solution 4 by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam. For any positive real number x, we note that

$$x^8+x^4+1 = \left(x^4+1\right)^2-x^4 = \left(x^2-x+1\right)\left(x^2+x+1\right)\left(x^4-x^2+1\right).$$

Therefore,

$$\sqrt{\frac{x^8+x^4+1}{x^2+x+1}} = \sqrt{(x^2-x+1)(x^4-x^2+1)}$$

$$= \sqrt{((x-1)^2+x)((x^2-1)^2+x^2)} \geq \sqrt{x^3} = x\sqrt{x}.$$

From this, we obtain

$$\sum_{i=1}^n \sqrt{\frac{a_i^8+a_i^4+1}{a_i^2+a_i+1}} \geq \sum_{i=1}^n a_i\sqrt{a_i}.$$

On the other hand, by using the AM-GM inequality, we have

$$\sqrt{\frac{x^8+x^4+1}{x^2+x+1}} = \sqrt{(x^2-x+1)(x^4-x^2+1)}$$

$$\leq \frac{1}{2}\left(x^2-x+1+x^4-x^2+1\right) = \frac{x^4-x+2}{2}.$$

Hence,

$$\sum_{i=1}^n \sqrt{\frac{a_i^8+a_i^4+1}{a_i^2+a_i+1}} \leq \sum_{i=1}^n \frac{a_i^4-a_i+2}{2}$$

$$= \frac{1}{2}\sum_{i=1}^{n} a_i^4 - \frac{1}{2}\sum_{i=1}^{n} a_i + \frac{2n}{2} = \frac{1}{2}\sum_{i=1}^{n} a_i^4 + \frac{n}{2}.$$

We are done. Equality holds if and only if $a_1 = a_2 = \cdots = a_n = 1$.

Also solved by *Daniel Văcaru, „Maria Teiuleanu" National Economic College, Pitești, Romania and José Luis Díaz-Barrero, Barcelona, Spain.*

E–142. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Find all real roots of the equation $9x^4 - 24x^3 - 23x^2 + 58x + 26 = 0$, if it is known that it has four distinct real roots, two of which add up to 2.

Solution 1 by Albert Stadler, Herrliberg, Switzerland. By Vietà's theorem, $9x^4 - 24x^3 - 23x^2 + 58x + 26$ is divisible by $x^2 - 2x + c$ for some c. We find

$$9x^4 - 24x^3 - 23x^2 + 58x + 26 = \left(9x^2 - 6x - 9c - 35\right)\left(x^2 - 2x + c\right)$$

$$+9c^2 + 35c + 26 - (12c + 12)x.$$

The remainder vanishes for c=-1 and we have

$$9x^4 - 24x^3 - 23x^2 + 58x + 26 = (x^2 - 2x - 1)(9x^2 - 6x - 26)$$

Therefore the roots of equation $9x^4 - 24x^3 - 23x^2 + 58x + 26 = 0$ are

$$\left\{1 - \sqrt{2}, 1 + \sqrt{2}, \frac{1}{3}\left(1 - 3\sqrt{3}\right), \frac{1}{3}\left(1 + 3\sqrt{3}\right)\right\}.$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Without loss of generality, we may consider the real roots of the polynomial

$$x^4 - \frac{8}{3}x^3 - \frac{23}{9}x^2 + \frac{58}{9}x + \frac{26}{9}.$$

Let r_1, r_2, r_3, and r_4 denote the four distinct real roots of this polynomial. By Viete's formulas,

$$r_1 + r_2 + r_3 + r_4 = \frac{8}{3}.$$

Suppose the two roots which add up to 2 are r_1 and r_2; then

$$r_3 + r_4 = \frac{2}{3}.$$

It follows that

$$(x-r_1)(x-r_2) = x^2-2x+a \quad \text{and} \quad (x-r_3)(x-r_4) = x^2-\frac{2}{3}x+b$$

for some constants a and b. Thus

$$x^4 - \frac{8}{3}x^3 - \frac{23}{9}x^2 + \frac{58}{9}x + \frac{26}{9} = (x^2 - 2x + a)\left(x^2 - \frac{2}{3}x + b\right)$$

$$= x^4 - \frac{8}{3}x^3 + \left(a + b + \frac{4}{3}\right)x^2 - \left(2b + \frac{2}{3}a\right)x + ab.$$

Equating coefficients of x^2, x, and the constant term yields the system of equations

$$a + b + \frac{4}{3} = -\frac{23}{9}, \quad 2b + \frac{2}{3}a = -\frac{58}{9}, \quad ab = \frac{26}{9},$$

whose solution is

$$a = -1 \quad \text{and} \quad b = -\frac{26}{9}.$$

The polynomial $x^2 - 2x - 1$ has roots $1 \pm \sqrt{2}$, and the polynomial

$$x^2 - \frac{2}{3}x - \frac{26}{9} \quad \text{has roots} \quad \frac{1}{3} \pm \sqrt{3}.$$

Hence, the real roots of the equation $9x^4-24x^3-23x^2+58x+26 = 0$ are

$$1 \pm \sqrt{2} \quad \text{and} \quad \frac{1}{3} \pm \sqrt{3}.$$

Solution 3 by Michel Bataille, Rouen, France. Let x_1, x_2, x_3, x_4 be the roots of the equation. From Viete's relations, we have

$$x_1+x_2+x_3+x_4 = \frac{24}{9}, \quad x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4 = -\frac{23}{9}$$

$$x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4 + x_1x_2x_3 = -\frac{58}{9}, \quad x_1x_2x_3x_4 = \frac{26}{9}.$$

Without loss of generality, we may assume that $x_1 + x_2 = 2$. Then we obtain $x_3 + x_4 = \frac{2}{3}$ so that $2x_3x_4 + \frac{2}{3}x_1x_2 = -\frac{58}{9}$. Also, we have $x_1x_2 + 2(x_3 + x_4) + x_3x_4 = -\frac{23}{9}$, hence $x_1x_2 + x_3x_4 = -\frac{35}{9}$. Solving for x_1x_2 and x_3x_4 yields $x_1x_2 = -1$ and $x_3x_4 = -\frac{26}{9}$. It follows that x_1, x_2 (resp. x_3, x_4) are the solutions of the quadratic $X^2 - 2X - 1 = 0$ (resp. $X^2 - \frac{2X}{3} - \frac{26}{9} = 0$), hence

$$x_1, x_2 = 1 \pm \sqrt{2}, \qquad x_3, x_4 = \frac{1}{3} \pm \sqrt{3}.$$

We conclude that the desired roots are

$$1 + \sqrt{2}, \quad 1 - \sqrt{2}, \quad \frac{1}{3} + \sqrt{3}, \quad \frac{1}{3} - \sqrt{3}.$$

Solution 4 by Miquel Amengual Covas, Cala Figuera, Mallorcs, Spain. Let x_1, x_2, x_3, x_4 be the required roots. Their sum is $\frac{8}{3}$ since the coefficient of x^3 is -24 and the leading coefficient is 9. We may assume that $x_1 + x_2 = 2$. Then

$$x_3 + x_4 = \frac{8}{3} - (x_1 + x_2) = \frac{2}{3},$$

and

$$(x_1 + x_2)(x_3 + x_4) = 2 \cdot \frac{2}{3} = \frac{4}{3}.$$

In turn, we have $\sum\limits_{1 \le i < j \le 4} x_ix_j = -\dfrac{23}{9}$, since the coefficient of x^2 is -23. That is

$$x_1x_2 + (x_1 + x_2)(x_3 + x_4) + x_3x_4 = -\frac{23}{9}$$

and

$$x_1x_2 + x_3x_4 = -\frac{23}{9} - (x_1 + x_2)(x_3 + x_4) = -\frac{23}{9} - \frac{4}{3} = -\frac{35}{9}. \quad (1)$$

Moreover, we have $x_1x_2x_3x_4 = \frac{26}{9}$, since the constant term is 26. Thus

$$(x_1x_2) \cdot (x_3x_4) = \frac{26}{9} \quad (2)$$

We solve the system (1) and (2) by considering x_1x_2 and x_3x_4 as roots of a quadratic equation with coefficients determined by the sum (1) and the product (2) of the roots:

$$w^2 + \frac{35}{9}w + \frac{26}{9} = 0$$

The roots of this equation are $w = -1$ and $w = -\frac{26}{9}$. Hence, either

$$\begin{array}{lll} \text{(i)} & x_1x_2 = -1, & x_1 + x_2 = 2 \\ & \text{or} & \\ \text{(ii)} & x_1x_2 = -\frac{26}{9}, & x_1 + x_1 = 2 \end{array}$$

In the first case, $\{x_1, x_2\} = \left\{1 + \sqrt{2}, 1 - \sqrt{2}\right\}$ and these values satisfy our equation. In the second case, $\{x_1, x_2\} = \left\{1 + \frac{\sqrt{35}}{3}, 1 - \frac{\sqrt{35}}{3}\right\}$ and these values does not satisfy our equation.

It follows that x_3, x_4 are the roots of the polynomial

$$\begin{aligned} \frac{9x^4 - 24x^3 - 23x^2 + 58x + 26}{(x - x_1)(x - x_2)} &= \frac{9x^4 - 24x^3 - 23x^2 + 58x + 26}{\left(x - \left(1 + \sqrt{2}\right)\right)\left(x - \left(1 - \sqrt{2}\right)\right)} \\ &= 9x^2 - 6x - 26, \end{aligned}$$

namely $\frac{1}{3} \pm \sqrt{3}$.

We conclude that the roots of the given polynomial are $1 \pm \sqrt{2}$, $\frac{1}{3} \pm \sqrt{3}$.

Solution 5 by Daniel Văcaru, „Maria Teiuleanu" National Economic College, Pitești, Romania and the proposer (same solution). Denote the roots by x_1, x_2, x_3, x_4 in such a way that $x_1 + x_2 = 2$. Then

$$9x^4 - 24x^3 - 23x^2 + 58x + 26 = 9(x - x_1)(x - x_2)(x - x_3)(x - x_4).$$

Comparing the coefficients at the corresponding powers of x, we obtain the well-known Vièta's relations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 8/3, \\ x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 &= -23/9, \\ x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 &= -58/9, \\ x_1x_2x_3x_4 &= 26/9. \end{aligned}$$

Since $x_1+x_2=2$, it follows from the first equation that $x_3+x_4=2/3$. We rewrite the second and third equations in the form

$$\begin{aligned}(x_1+x_2)(x_3+x_4)+x_1x_2+x_3x_4 &= -23/9,\\ (x_1+x_2)x_3x_4+(x_3+x_4)x_1x_2 &= -58/9.\end{aligned}$$

Substituting $x_1+x_2=2$ and $x_3+x_4=2/3$, yields

$$\begin{aligned}x_1x_2+x_3x_4 &= -35/9,\\ x_1x_2+3x_3x_4 &= -87/9.\end{aligned}$$

Form the preceding, we get $x_1x_2=-1$ and $x_3x_4=-26/9$. Note that $x_1x_2x_3x_4=26/9$. Now we consider two cases:

- Case 1: $x_1+x_2=2$ and $x_1x_2=-1$ from which we obtain the quadratic equation $x^2-2x-1=0$ with roots: $1\pm\sqrt{2}$.
- Case 2: $x_3+x_4=2/3$ and $x_3x_4=-26/9$ from which we obtain the quadratic equation $9x^2-6x-26=0$ with roots: $1/3\pm\sqrt{3}$.

Thus, the four roots of the given equation are:

$$1-\sqrt{2},\quad 1+\sqrt{2},\quad \frac{1}{3}-\sqrt{3},\quad \frac{1}{3}-\sqrt{3}.$$

Easy–Medium Problems

EM–137. *Proposed by Michel Bataille, Rouen, France.* Let n be a nonnegative integer. Evaluate

$$\sum_{k=0}^{n}(k+1)\binom{n+2}{k+2}.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY, USA. We see that

$$\binom{n+2}{k+2}=\frac{(n+2)(n+1)}{(k+2)(k+1)}\cdot\binom{n}{k},$$

giving us

$$\sum_{k=0}^{n}(k+1)\binom{n+2}{k+2}=(n+2)(n+1)\sum_{k=0}^{n}\binom{n}{k}\cdot\frac{1}{k+2}. \qquad (1)$$

Now we have

$$\begin{aligned}
(x+1)^n &= \sum_{k=0}^{n}\binom{n}{k}x^k,\\
x(x+1)^n &= \sum_{k=0}^{n}\binom{n}{k}x^{k+1},\\
\int_0^1 x(x+1)^n\,dx &= \sum_{k=0}^{n}\binom{n}{k}\int_0^1 x^{k+1}\,dx, \text{ putting } x=u-1 \text{ yields}\\
\int_1^2 (u-1)u^n\,du &= \sum_{k=0}^{n}\binom{n}{k}\cdot\frac{1}{k+2},
\end{aligned}$$

or

$$\frac{n2^{n+1}+1}{(n+1)(n+2)}=\sum_{k=0}^{n}\binom{n}{k}\frac{1}{k+2}.$$

Comparing this last equality with (1), we find

$$\sum_{k=0}^{n}(k+1)\binom{n+2}{k+2}=n2^{n+1}+1.$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Note

$$\binom{n+2}{k+2} = \frac{(n+2)!}{(k+2)!(n-k)!} = \frac{(n+2)(n+1)}{(k+2)(k+1)}\binom{n}{k},$$

so

$$(k+1)\binom{n+2}{k+2} = \frac{(n+2)(n+1)}{k+2}\binom{n}{k}$$

and

$$\sum_{k=0}^{n}(k+1)\binom{n+2}{k+2} = (n+2)(n+1)\sum_{k=0}^{n}\frac{1}{k+2}\binom{n}{k}.$$

By the binomial theorem,

$$\sum_{k=0}^{n}\binom{n}{k}x^{k+1} = (1+x)^n x,$$

from which it follows that

$$\sum_{k=0}^{n}\frac{1}{k+2}\binom{n}{k} = \int_0^1 (1+x)^n x\,dx.$$

With the substitution $u = 1+x$,

$$\begin{aligned}
\int_0^1 (1+x)^n x\,dx &= \int_1^2 u^n(u-1)\,du = \left.\left(\frac{u^{n+2}}{n+2} - \frac{u^{n+1}}{n+1}\right)\right|_1^2 \\
&= \frac{2^{n+2}}{n+2} - \frac{2^{n+1}}{n+2} - \frac{1}{n+2} + \frac{1}{n+1}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sum_{k=0}^{n}(k+1)\binom{n+2}{k+2} &= 2^{n+2}(n+1) - 2^{n+1}(n+2) - (n+1) + (n+2) \\
&= 2^{n+1}n + 1.
\end{aligned}$$

Solution 3 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. By the binomial theorem

$$(1+x)^{n+2} = 1+(n+2)x+\sum_{k=2}^{n+2}\binom{n+2}{k}x^k = 1+(n+2)x+\sum_{k=0}^{n}\binom{n+2}{k+2}x^{k+2}.$$

Rearranging terms yields

$$\frac{(1+x)^{n+2}-1-(n+2)x}{x}=\sum_{k=0}^{n}\binom{n+2}{k+2}x^{k+1}.$$

Upon taking the derivative on both sides of this last equation, we obtain

$$\frac{x(n+2)(1+x)^{n+1}-(1+x)^{n+2}}{x^2}+\frac{1}{x^2}=\sum_{k=0}^{n}(k+1)\binom{n+2}{k+2}x^{k}.$$

Finally, substitution of $x=1$ yields

$$\sum_{k=0}^{n}(k+1)\binom{n+2}{k+2}=(n+2)2^{n+1}-2^{n+2}+1=2^{n+1}n+1.$$

Solution 4 by Ioan Viorel Codreanu, Satulung, Maramures, Romania and Albert Stadler, Herrliberg, Switzerland (same solution). Using the well-known identities:

$$\sum_{k=1}^{n}k\binom{n}{k}=n\cdot 2^{n-1}\quad\text{and}\quad\sum_{k=0}^{n}\binom{n}{k}=2^{n},$$

we get:

$$\begin{aligned}\sum_{k=0}^{n}(k+1)\binom{n+2}{k+2}&=\sum_{k=0}^{n}(k+2)\binom{n+2}{k+2}-\sum_{k=0}^{n}\binom{n+2}{k+2}\\&=\sum_{k=2}^{n+2}k\binom{n+2}{k}-\sum_{k=2}^{n+2}\binom{n+2}{k}\end{aligned}$$

$$=\left(\sum_{k=1}^{n+2}k\binom{n+2}{k}-\binom{n+2}{1}\right)-\left(\sum_{k=0}^{n+2}\binom{n+2}{k}-\binom{n+2}{1}-\binom{n+2}{0}\right)$$

$$=(n+2)\cdot 2^{n+1}-2^{n+2}+1=n\cdot 2^{n+1}+1.$$

Solution 5 by the proposer. Let S_n denote the sum to be evaluated. Since

$$\binom{n+2}{k+2}=\sum_{j=k+1}^{n+1}\binom{j}{k+1}=\sum_{i=k}^{n}\binom{i+1}{k+1},$$

(using the well-known $\sum\limits_{j=p}^{q}\binom{j}{p}=\binom{q+1}{p+1}$ for integers p,q such that $0\le p\le q$), we obtain

$$S_n=\sum_{k=0}^{n}\sum_{i=k}^{n}(k+1)\binom{i+1}{k+1}$$

and, changing the order of summation, we deduce that

$$\begin{aligned}S_n&=\sum_{i=0}^{n}\sum_{k=0}^{i}(k+1)\binom{i+1}{k+1}\\&=\sum_{i=0}^{n}\sum_{k=0}^{i}(i+1)\binom{i}{k}\\&=\sum_{i=0}^{n}(i+1)\sum_{k=0}^{i}\binom{i}{k}=\sum_{i=0}^{n}(i+1)2^i.\end{aligned}$$

Now, for $x\neq 1$, we have $\sum\limits_{i=0}^{n}x^{i+1}=\frac{x^{n+2}-x}{x-1}$, hence by differentiation,

$$\sum_{i=0}^{n}(i+1)x^i=\frac{(n+1)x^{n+2}-(n+2)x^{n+1}+1}{(x-1)^2}.$$

Thus,

$$S_n=(n+1)2^{n+2}-(n+2)2^{n+1}+1=n2^{n+1}+1.$$

EM–138. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.* We denote by A, B the endpoints of the diameter of a semicircle Γ of radius a and construct the rectangle $ABCD$, with $\overline{BC}=a\sqrt{2}$, which contains it. If E is a point on Γ, distinct from A and B, and the lines DE, CE intersect the line AB at points F, G respectively, prove that

$$AG^2+BF^2=4a^2.$$

Solution 1 by Michel Bataille, Rouen, France. Let H be the foot of the altitude from E in ΔAEB and let $h=EH$. Since $\angle AEB=90^\circ$, we know that the dot product $\overrightarrow{HA}\cdot\overrightarrow{HB}$ is equal to $-h^2$.

Since $EH \parallel CB$, the homothety with center G transforming H into B transforms E into C. Therefore $\overrightarrow{GH} = \frac{h}{a\sqrt{2}}\overrightarrow{GB}$, that is, $\overrightarrow{GA} + \overrightarrow{AH} = \frac{h}{a\sqrt{2}}(\overrightarrow{GA} + \overrightarrow{AB})$, hence

$$\left(1 - \frac{h}{a\sqrt{2}}\right)\overrightarrow{GA} = \overrightarrow{HA} + \frac{h}{a\sqrt{2}}\overrightarrow{AB}. \tag{1}$$

In a similar way, we obtain

$$\left(1 - \frac{h}{a\sqrt{2}}\right)\overrightarrow{FB} = \overrightarrow{HB} + \frac{h}{a\sqrt{2}}\overrightarrow{BA}. \tag{2}$$

Using $AB = 2a$, (1), (2) and setting $k = 1 - \frac{h}{a\sqrt{2}}$, we calculate

$$\begin{aligned} k^2(AG^2 + FB^2) &= \left(\overrightarrow{HA} + \frac{h}{a\sqrt{2}}\overrightarrow{AB}\right)^2 + \left(\overrightarrow{HB} + \frac{h}{a\sqrt{2}}\overrightarrow{BA}\right)^2 \\ &= HA^2 + HB^2 + 4h^2 + \frac{2h}{a\sqrt{2}}\overrightarrow{AB} \cdot (\overrightarrow{HA} - \overrightarrow{HB}) \\ &= 4h^2 + (\overrightarrow{HA} - \overrightarrow{HB})^2 + 2\overrightarrow{HA} \cdot \overrightarrow{HB} - 4ah\sqrt{2} \\ &= 4h^2 + 4a^2 - 2h^2 - 4ah\sqrt{2} = 4a^2k^2. \end{aligned}$$

Thus, $AG^2 + BF^2 = 4a^2$ (note that $h \le a < a\sqrt{2}$, hence $k = 1 - \frac{h}{a\sqrt{2}} \neq 0$).

Solution 2 by the proposer. Let H be the foot of the perpendicular from E to AB.

The pairs of similar triangles HEG, BCG and EHF, DAF yield

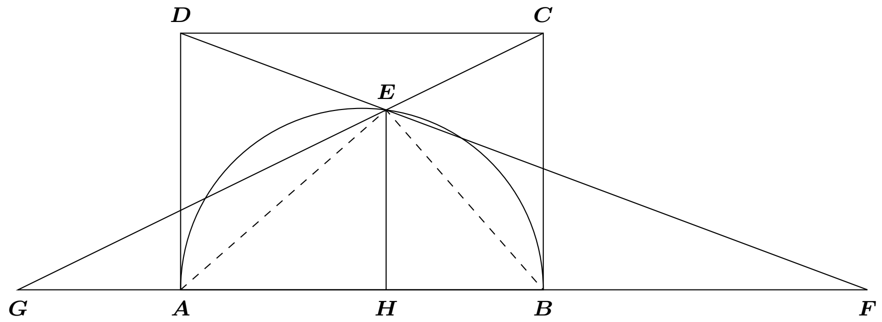
$$\frac{CB}{EH} = \frac{BG}{HG}, \qquad \frac{DA}{EH} = \frac{AF}{HF},$$

or

$$\frac{CB}{EH} = \frac{BA + AG}{HA + AG}, \qquad \frac{DA}{EH} = \frac{AB + BF}{HB + BF}.$$

Hence

$$AG = \frac{BA \cdot EH - CB \cdot HA}{BC - EH}, \qquad BF = \frac{AB \cdot EH - DA \cdot HB}{DA - EH}. \tag{1}$$



We put $\overline{AH} = x$. Then $\overline{HB} = 2a - x$.

Triangle AEB is right-angled at E. By a standard mean proportion, then,

$$EH = \sqrt{AH \cdot HB} = \sqrt{x(2a-x)}.$$

In terms of x, (1) becomes

$$\begin{aligned} AG &= \frac{2a\sqrt{x(2a-x)} - a\sqrt{2} \cdot x}{a\sqrt{2} - \sqrt{x(2a-x)}}, \\ BF &= \frac{2a\sqrt{x(2a-x)} - a\sqrt{2} \cdot (2a-x)}{a\sqrt{2} - \sqrt{x(2a-x)}}, \end{aligned}$$

which may be rewritten as

$$\begin{aligned} AG &= \frac{\sqrt{2x(2a-x)} - x}{a\sqrt{2} - \sqrt{x(2a-x)}} \cdot a\sqrt{2}, \\ BF &= \frac{\left(\sqrt{2x(2a-x)} + x\right) - 2a}{a\sqrt{2} - \sqrt{x(2a-x)}} \cdot a\sqrt{2}. \end{aligned}$$

Hence,

$$AG^2 + BF^2 = \frac{2a^2}{\left(a\sqrt{2} - \sqrt{x(2a-x)}\right)^2} \times$$

$$\left[\left(\sqrt{2x(2a-x)} - x\right)^2 + \left(\left(\sqrt{2x(2a-x)} + x\right) - 2a\right)^2\right]$$

$$\begin{aligned}
&= \frac{4a^2}{\left(a\sqrt{2}-\sqrt{x(2a-x)}\right)^2}\left(2a^2+2ax-2a\sqrt{2x(2a-x)}-x^2\right)\\
&= \frac{4a^2}{\left(a\sqrt{2}-\sqrt{x(2a-x)}\right)^2}\left(2a^2+x(2a-x)-2a\sqrt{2x(2a-x)}\right)\\
&= \frac{4a^2}{\left(a\sqrt{2}-\sqrt{x(2a-x)}\right)^2}\left(a\sqrt{2}-\sqrt{x(2a-x)}\right)^2\\
&= 4a^2,
\end{aligned}$$

as desired.

EM–139. *Proposed by Goran Conar, Varaždin, Croatia.* Let a, b, c be positive real numbers such that $a+b+c=4$. Prove that

$$\sqrt[4]{a^a b^b c^c} \geq \frac{4}{3}.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY, USA. The weighted Geometric-Harmonic Mean inequality states that

$$a^{w_1}b^{w_2}c^{w_3} \geq \frac{w_1+w_2+w_3}{\frac{w_1}{a}+\frac{w_2}{b}+\frac{w_3}{c}},$$

where $a, b, c > 0$ and $w_1+w_2+w_3=1$. In our problem, let $w_1=a/4, w_2=b/4, w_3=c/4$ to get

$$\sqrt[4]{a^a b^b c^c} \;\geq\; \frac{1}{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}} \;=\; \frac{4}{3}.$$

Solution 2 by Henry Ricardo, Westchester Area Math Circle, NY, USA. Taking logarithms, the problem has the equivalent form

$$\frac{1}{4}\sum_{cyc} a\ln a \geq \ln\left(\frac{4}{3}\right).$$

Since the function $f(x)=x\ln x$ is convex—$f''(x)=1/x>0$ on $(0,\infty)$—we can apply Jensen's inequality to get

$$\sum_{cyc} f(a) = \sum_{cyc} a\ln a \geq 3f\left(\frac{a+b+c}{3}\right)$$

$$= 3\,f\left(\frac{4}{3}\right) = 3\cdot\frac{4}{3}\ln\left(\frac{4}{3}\right) = 4\ln\left(\frac{4}{3}\right).$$

Finally, dividing each side by 4 yields the desired result.

Solution 3 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Let $f(x) = x\ln x$. Then

$$f'(x) = 1 + \ln x \quad \text{and} \quad f''(x) = \frac{1}{x}.$$

Because $f''(x) > 0$ for $x > 0$, it follows from Jensen's inequality that

$$\begin{aligned} a\ln a + b\ln b + c\ln c &= f(a) + f(b) + f(c) \\ &\geq 3f\left(\frac{a+b+c}{3}\right) = 3f\left(\frac{4}{3}\right) = 4\ln\frac{4}{3}. \end{aligned}$$

Exponentiating both sides of this inequality yields

$$a^a b^b c^c \geq \left(\frac{4}{3}\right)^4, \quad \text{or} \quad \sqrt[4]{a^a b^b c^c} \geq \frac{4}{3}.$$

Equality holds if and only if $a = b = c = \frac{4}{3}$.

Solution 4 by Michel Bataille, Rouen, France and Ioan Viorel Codreanu, Satulung, Maramures, Romania (same solution). The inequality is equivalent to

$$\ln(\sqrt[4]{a^a b^b c^c}) \geq \ln(4/3),$$

that is, to

$$a\ln a + b\ln b + c\ln c \geq 4\ln(4/3). \tag{1}$$

Now, the function $x \mapsto x\ln x$ is convex on $(0,\infty)$ (its second derivative is the positive function $x \mapsto \frac{1}{x}$), hence from Jensen's inequality we have

$$a\ln a + b\ln b + c\ln c \geq 3\,\frac{a+b+c}{3}\ln\frac{a+b+c}{3} = 3\cdot\frac{4}{3}\ln(4/3)$$

and (1) holds.

Solution 5 by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam. We rewrite the above inequality in the form

$$a^a b^b c^c \geq \left(\frac{4}{3}\right)^4.$$

By the weighted AM–GM inequality, we obtain

$$\frac{1}{a^a b^b c^c} = \left(\frac{1}{a}\right)^a \left(\frac{1}{b}\right)^b \left(\frac{1}{c}\right)^c$$

$$\leq \left(\frac{a \cdot \frac{1}{a} + b \cdot \frac{1}{b} + c \cdot \frac{1}{c}}{a+b+c}\right)^{a+b+c} = \left(\frac{3}{a+b+c}\right)^{a+b+c}.$$

Thus,

$$a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c} = \left(\frac{4}{3}\right)^4.$$

Equality holds if and only if

$$a = b = c = \tfrac{4}{3}.$$

Solution 6 by the proposer. We will use the following claim: *For* $a_1, a_2, \ldots, a_n > 0$, *the following inequality holds*

$$a_1^{a_1} a_2^{a_2} \cdot \ldots \cdot a_n^{a_n} \geq \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^{a_1 + a_2 + \ldots + a_n}.$$

Indeed, function $f(x) = x \ln x$ is convex on $\langle 0, +\infty\rangle$ because f'' is strictly positive there: $f'(x) = \ln x + 1$, $f''(x) = \frac{1}{x}$. f' is decreasing on $\langle 0, 1/e\rangle$ (because $f' < 0$ on that set) and increasing on $\langle 1/e, +\infty\rangle$ (where is $f' > 0$), and second derivation is $f''(x) > 0$, $\forall x \in \langle 0, +\infty\rangle$. Thus, using Jensen's inequality we have

$$\frac{1}{n}(a_1 \ln a_1 + a_2 \ln a_2 + \ldots + a_n \ln a_n)$$

$$\geq \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right) \ln\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right) \Leftrightarrow$$

$$\Leftrightarrow \quad \frac{1}{n} \ln\left(a_1^{a_1} a_2^{a_2} \cdot \ldots \cdot a_n^{a_n}\right) \geq \ln\left(\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^{\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)}\right)$$

$$\Leftrightarrow \quad a_1^{a_1} a_2^{a_2} \cdot \ldots \cdot a_n^{a_n} \geq \left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^{a_1+a_2+\ldots+a_n}.$$

Now, applying the claim on a, b, c and using $a + b + c = 4$ we get desired inequality ($\frac{256}{81}$ is exactly $\left(\frac{4}{3}\right)^4$).

Also solved by *Albert Stadler, Herrliberg, Switzerland and José Luis Díaz-Barrero, Barcelona, Spain.*

EM–140. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $z_1, z_2, \ldots, z_n$ be complex numbers. Prove that

$$\frac{1}{2}\left(\sum_{k=1}^{n} |z_k| + \left|\sum_{k=1}^{n} z_k\right|\right)^2 \geq \sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k\right|^2.$$

Solution 1 by Michel Bataille, Rouen, France. The required inequality is equivalent to

$$\left(\sum_{k=1}^{n} |z_k|\right)^2 + 2\left|\sum_{k=1}^{n} z_k\right| \cdot \sum_{k=1}^{n} |z_k| \geq 2\sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k\right|^2.$$

Since $\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|$, the left-hand side is greater than or equal to $\left(\sum_{k=1}^{n} |z_k|\right)^2 + 2\left|\sum_{k=1}^{n} z_k\right|^2$ and consequently, it is sufficient to prove that

$$\left(\sum_{k=1}^{n} |z_k|\right)^2 + \left|\sum_{k=1}^{n} z_k\right|^2 \geq 2\sum_{k=1}^{n} |z_k|^2. \tag{1}$$

Now, the left-hand side of (1) successively writes as

$$\sum_{k=1}^{n} |z_k|^2 + 2\sum_{1\leq i<j\leq n} |z_i||z_j| + \left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \overline{z_k}\right)$$

$$2\sum_{1\leq i<j\leq n} |z_i z_j| + 2\sum_{k=1}^{n} |z_k|^2 + 2\sum_{1\leq i<j\leq n} \Re(z_i\overline{z_j})$$

(since $z_i\overline{z_j} + \overline{z_i}z_j = 2\Re(z_i\overline{z_j})$). It follows that (1) reduces to

$$\sum_{1\leq i<j\leq n} \left(|z_i z_j| + \Re(z_i\overline{z_j})\right) \geq 0.$$

We are done since the latter directly follows from

$$-\Re(z_i\overline{z_j}) \leq |\Re(z_i\overline{z_j})| \leq |z_i\overline{z_j}| = |z_iz_j|.$$

Solution 2 by the proposer. For any $1 \leq k \leq n$, we claim that

$$|z_k| \leq \sum_{\substack{1\leq j\leq n\\ j\neq k}} |z_j| + \left|\sum_{j=1}^{n} z_j\right| \quad (1)$$

Indeed,

$$|z_k| = \left|\sum_{j=1}^{n} z_j - \sum_{\substack{1\leq j\leq n\\ j\neq k}} z_j\right| \leq \sum_{\substack{1\leq j\leq n\\ j\neq k}} |z_j| + \left|\sum_{j=1}^{n} z_j\right|,$$

as claimed. For all $1 \leq k \leq n$, let $S_k = \sum\limits_{\substack{1\leq j\leq n\\ j\neq k}} |z_j|$, then

$$|z_k| \leq S_k + \left|\sum_{j=1}^{n} z_j\right| \quad (2)$$

Moreover, $\left|\sum\limits_{j=1}^{n} z_j\right| \leq \sum\limits_{j=1}^{n} |z_j|$. Multiplying the inequalities (1) by $|z_k|$ and the inequalities (2) by $|z_1 + z_2 + \ldots + z_n|$, we obtain after addition,

$$\sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k\right|^2 \leq 2\left|\sum_{k=1}^{n} z_k\right| \sum_{k=1}^{n} |z_k| + \sum_{k=1}^{n} |z_k| S_k.$$

Adding the expression $\sum\limits_{k=1}^{n} |z_k|^2 + \left|\sum\limits_{k=1}^{n} z_k\right|^2$ to both sides of the inequality yields

$$2\left(\sum_{k=1}^{n} |z_k|^2 + \left|\sum_{k=1}^{n} z_k\right|^2\right) \leq \left(\sum_{k=1}^{n} |z_k| + \left|\sum_{k=1}^{n} z_k\right|\right)^2$$

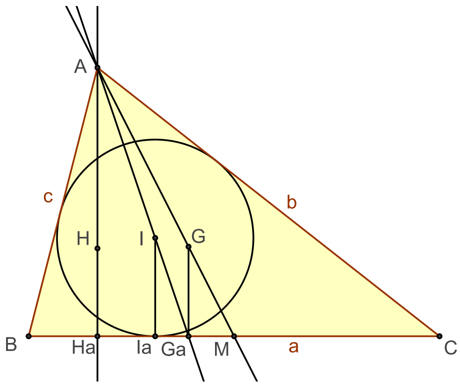
from which the statement follows. Equality holds when $n = 1$.

EM–141. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a scalene triangle with incenter I and centroid G. Let G_a be the orthogonal projection of G on BC. Knowing that the points A, I, G_a are collinear, find the ratio AI/IG_a.

Solution 1 by the proposer. The answer is

$$\frac{AI}{IG_a} = \sqrt{3}.$$

Let $BC = a, CA = b, AB = c$ are the side lengths of the triangle ABC. Let H be the orthocenter of the triangle ABC, H_a be the orthogonal projection of H on BC, I_a be the orthogonal projection of I on BC and M be the midpoint of BC.



Scheme for solving problem EM-141.

Without loss of generality we may assume that $b > c$, hence M is between C and the points H_a, I_a, G_a.

$AH_a \perp BC, GG_a \perp BC$ and A, G, M are collinear, hence $\triangle AH_aM \sim \triangle GG_aM$ and

$$\frac{G_aM}{H_aM} = \frac{GM}{AM} = \frac{1}{3}.$$

$$G_a = AI \cap BC; \ \frac{BG_a}{G_aC} = \frac{c}{b}; \ BG_a = \frac{ac}{b+c}.$$

$$H_aM = H_aC - MC = b\cos C - \frac{a}{2} = \frac{a^2+b^2-c^2}{2a} - \frac{a}{2} = \frac{b^2-c^2}{2a}$$
$$G_aM = BM - BG_a = \frac{a}{2} - \frac{ac}{b+c} = \frac{a(b+c)-2ac}{2(b+c)} = \frac{a(b-c)}{2(b+c)}$$

$$H_aM = 3G_aM \Leftrightarrow \ \frac{b^2-c^2}{2a} = \frac{3a(b-c)}{2(b+c)}$$
$$\Leftrightarrow \ (b+c)^2 = 3a^2 \Leftrightarrow \ (b+c+a\sqrt{3})(b+c-a\sqrt{3}) = 0,$$

but $b+c+a\sqrt{3} > 0$ and from here, the condition that the points A, I, G_a are collinear is

$$a\sqrt{3} = b+c. \tag{1}$$

Now, the triangles $\triangle AH_aG_a \sim \triangle II_aG_a$ and

$$\frac{AI}{IG_a} = \frac{H_aI_a}{I_aG_a} = \frac{H_aG_a}{I_aG_a} - 1$$
$$H_aG_a = H_aM - G_aM = 2G_aM = \frac{a(b-c)}{b+c}$$

$$I_aG_a = BM - BI_a - G_aM = \frac{a}{2} - \frac{a-b+c}{2} - \frac{a(b-c)}{2(b+c)}$$
$$= \frac{(b-c)(b+c-a)}{2(b+c)}$$
$$\frac{AI}{IG_a} = \frac{H_aG_a}{I_aG_a} - 1 = \frac{\frac{a(b-c)}{b+c}}{\frac{(b-c)(b+c-a)}{2(b+c)}} - 1$$
$$= \frac{2a}{b+c-a} - 1 \stackrel{(1)}{=} \frac{2a}{a\sqrt{3}-a} - 1 = \sqrt{3}$$

Solution 2 by Michel Bataille, Rouen, France. Let $a = BC, b = CA, c = AB$ and let $H = tB + (1-t)C$ denote the foot of the altitude from A in ΔABC. Then,

$$0 = \overrightarrow{AH} \cdot \overrightarrow{BC} = (t\overrightarrow{AB} + (1-t)\overrightarrow{AC}) \cdot (\overrightarrow{AC} - \overrightarrow{AB}).$$

Using $2\overrightarrow{AB}\cdot\overrightarrow{AC} = b^2+c^2-a^2$, we obtain $t = \frac{a^2+b^2-c^2}{2a^2}$ so that

$$2a^2H = (a^2+b^2-c^2)B + (a^2-b^2+c^2)C.$$

Let M be the midpoint of BC. Since $GG_a \parallel AH$ and $\overrightarrow{MG} = \frac{1}{3}\overrightarrow{MA}$, we have $\overrightarrow{MG_a} = \frac{1}{3}\overrightarrow{MH}$, from which we get $3G_a = 2M+H$, that is,

$$6a^2G_a = (3a^2+b^2-c^2)B + (3a^2-b^2+c^2)C.$$

Now, if A, I, G_a are collinear, then G_a coincides with the point of intersection of the internal bisector of $\angle A$ and the side BC so that

$$\frac{3a^2+b^2-c^2}{b} = \frac{3a^2-b^2+c^2}{c}.$$

This rewrites as $(b-c)(3a^2-(b+c)^2) = 0$ and, ΔABC being scalene, implies that $a\sqrt{3} = b+c$.
Recalling that $(a+b+c)I = aA+bB+cC$, we see that $(a+b+c)I = aA+(b+c)G_a$, and therefore

$$\frac{AI}{IG_a} = \frac{b+c}{a} = \sqrt{3}.$$

Solution 3 by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain. Suppose $\triangle ABC$ with sides a, b, c in the usual order. Let s and h represent, respectively, the semiperimeter and the length of the altitude from A.

Since the condition in the statement of the problem implies that AG_a bisects $\angle CAB$, AG_a divides BC in the ratio of the sides about A, i.e., $\frac{BG_a}{G_aC} = \frac{c}{b}$; and then

$$BG_a = \frac{ac}{b+c}$$

On the one hand, the area of $\triangle ABC$ is given by $\frac{1}{2}ah$ and also $\sqrt{s(s-a)(s-b)(s-c)}$; hence $ah = 2\sqrt{s(s-a)(s-b)(s-c)}$. In turn, since G trisects the median from A, equation $GG_a = \frac{1}{3}h$ must hold. Thus

$$GG_a = \frac{2}{3a}\sqrt{s(s-a)(s-b)(s-c)}.$$

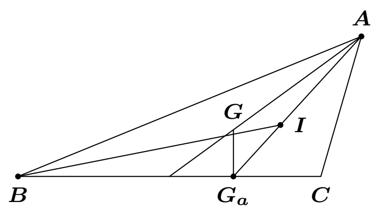
On the other hand, we have

$$BG = \left(\frac{2}{3} \text{ of the length of the median from } B\right)$$

$$= \frac{1}{3}\sqrt{2c^2 + 2a^2 - b^2}.$$

Now Pythagorean theorem, applied to $\triangle GBG_a$, asserts that

$$\overline{BG}^2 - \overline{GG_a}^2 = \overline{BG_a}^2$$



and therefore

$$\frac{2c^2 + 2a^2 - b^2}{9} - \frac{4s(s-a)(s-b)(s-c)}{9a^2} = \left(\frac{ac}{b+c}\right)^2.$$

This may be written in the form

$$\frac{2c^2 + 2a^2 - b^2}{9} - \frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{36a^2}$$

$$= \left(\frac{ac}{b+c}\right)^2,$$

or, equivalently,

$$\frac{9a^4 + b^4 + c^4 - 6a^2b^2 - 2b^2c^2 + 6a^2c^2}{36a^2} = \frac{a^2c^2}{(b+c)^2},$$

that is,

$$\left(3a^2 - b^2 + c^2\right)^2 (b+c)^2 = 36a^4c^2,$$

which we rewrite in the form

$$\left[\left(3a^2-b^2+c^2\right)(b+c)+6a^2c\right]\left[\left(3a^2-b^2+c^2\right)(b+c)-6a^2c\right]=0$$

and may assume without loss of generality that $b<c$. Otherwise, interchange B and C.

When $(3a^2-b^2+c^2)(b+c)+6a^2c=0$, $a^2=\frac{(b-c)(b+c)^2}{3(b+3c)}$. This implies that $b-c>0$, which contradicts our assumption.

When $(3a^2-b^2+c^2)(b+c)-6a^2c=0$ i.e., $(b-c)\left(3a^2-(b+c)^2\right)=0$, we get $a^2=\frac{(b+c)^2}{3}$ (being $(b-c\neq 0)$ so $\dfrac{b+c}{a}=\sqrt{3}$. Thus,

$$\frac{AI}{IGa}=\sqrt{3},$$

since $\frac{AI}{IGa}=\frac{b+c}{a}$ holds by the internal angle bisector theorem applied to $\triangle ABG_a$ at B.

EM–142. *Proposed by José Luis Díaz-Barrero, Barcelona, Spaion.* What is the maximum number of regions into which a circle can be divided by segments connecting n points on its boundary?

Solution 1 by the proposer. Let R_n be the maximal number of regions for n points. Clearly, this occur when any three chords are not concurrent. Let us consider the configuration with $n+1$ points, say $A_1,\ldots,A_{n+1}$ in that order around the circle. For each $i=1,\ldots,n$ the chord $A_{n+1}A_i$ intersects each of the chords which have endpoints on the opposite sides of it, and none of the others. Since there are $i-1$ points in one side and $n-i$ on the opposite side, there are $(i-1)(n-i)$ intersections. The number of regions which have to be added with this chord is the number of intersected chord plus 1. Thus,

$$R_{n+1}=R_n+\sum_{i=1}^{n}[(i-1)(n-i)+1]=R_n+\frac{n^3-3n^2+8n}{6}.$$

Using that $R_1=1$ it follows that

$$R_{n+1}=1+\sum_{i=1}^{n}\frac{i^3-3i^2+8i}{6}=1+\frac{n(n+1)(n^2-3n+14)}{24},$$

from which we easily obtain

$$R_n = R_{n+1} - \frac{n^3 - 3n^2 + 8n}{6} = 1 + \frac{n(n-1)(n^2 - 5n + 18)}{24}.$$

Solution 2 by Alberto Espuny Díaz, Universitat de Barcelona, Barcelona, Spain. Given a configuration $\mathcal{P} = \{p_1, \ldots, p_n\}$ of n points on a circle (where we assume that they appear clockwise in this order), let $\mathcal{Q} = \mathcal{Q}(\mathcal{P})$ denote the set of all points which lie in the intersection of two line segments between points of $\mathcal{P}$ (not including the points in $\mathcal{P}$ themselves). We call a configuration $\mathcal{P}$ *maximising* if no three line segments with endpoints in $\mathcal{P}$ intersect in the same point; such configurations would clearly maximise $|\mathcal{Q}|$. Note that every configuration on at most five points is maximising, as any obstruction for being maximising (i.e., any three line segments intersecting in a single point) must have endpoints in six points on the circle. We first claim that, for every integer n, there exist maximising configurations of n points on a circle.

Indeed, assume this is not the case and let n be the smallest integer such that there does not exist a maximising configuration. Let $\mathcal{P}' = \{p_1, \ldots, p_{n-1}\}$ be a maximising set of $n-1$ points on the circle, and let $\mathcal{P} := \mathcal{P}' \cup \{p_n\}$, where p_n is placed arbitrarily between p_{n-1} and p_1 on the circle. There must be three line segments between points of $\mathcal{P}$ which intersect in a single point. Clearly, since $\mathcal{P}'$ is maximising, for every such set of three line segments, one of the three segments must have p_n as one of its endpoints. Let $I \subseteq \{1, \ldots, n-1\}$ denote the set of all indices i such that $\overline{p_i p_n}$ intersects two other line segments in the same point. Now, for each $i \in I$ and each pair of indices $j \in \{1, \ldots, i-1\}$ and $k \in \{i+1, \ldots, n-1\}$, let $\alpha_{i,j,k}$ denote the minimum distance that p_n would have to be moved counterclockwise along the circle so that $\overline{p_i p_n}$ would now intersect $\overline{p_j p_k}$ at some point from $\mathcal{P}' \cup \mathcal{Q}(\mathcal{P}')$. (Note that $\alpha_{i,j,k} = 0$ if $\overline{p_i p_n}$ already intersects $\overline{p_j p_k}$ at some point in $\mathcal{Q}(\mathcal{P}')$, and that we only consider indices j and k as above since otherwise $\overline{p_i p_n}$ and $\overline{p_j p_k}$ do not intersect.) Let

$$\alpha := \min\{\alpha_{i,j,k} : i \in I, j \in \{1, \ldots, i-1\},$$
$$k \in \{i+1, \ldots, n-1\}, \alpha_{j,k} > 0\},$$

and let $\mathcal{P}^*$ be obtained from $\mathcal{P}$ by displacing p_n counterclockwise by a distance of $\alpha/2$. We remark that $\alpha_{i,1,n-1} = \operatorname{dist}(p_{n-1}, p_n)$ for any $i \in I$, and hence the relative order of the points does not change when creating $\mathcal{P}^*$. Clearly, this operation ensures that any triple intersection of $\overline{p_i p_n}$ with two other line segments has now disappeared, for all $i \in I$. On the other hand, by the choice of the displacement, no new triple intersection has been created. This leads to a maximising configuration $\mathcal{P}^*$ on n points, which is a contradiction. Hence, such configurations must exist for all n.

Next, we consider the size of $\mathcal{Q}$ for any maximising configuration $\mathcal{P}$ on n points. In order to count the number of crossings, we are going to fix each possible line segment and count the number of other segments which intersect it; in this way, we count each crossing exactly twice. Let $\mathcal{P} = \{p_1, \ldots, p_n\}$ and recall that we assume the points appear clockwise in this order. Each pair of indices i and j with $1 \le i < j \le n$ corresponds uniquely to one of the possible line segments. Moreover, given such i and j, the points p_i and p_j split the circle into two arcs and every other line segment intersecting $\overline{p_i p_j}$ must have one endpoint in each of the arcs; as such, the number of segments which intersect this segment is $(j-i-1)(n-(j-i)-1)$. Note, lastly, that the number of choices for indices i and j such that $j - i = k$ is $n - k$. All in all, we conclude that

$$\begin{aligned}
|\mathcal{Q}| &= \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (j-i-1)(n-(j-i)-1) \\
&= \frac{1}{2} \sum_{k=1}^{n-1} (n-k)(k-1)(n-k-1) \\
&= \frac{1}{2} \left(\sum_{k=1}^{n-1} k^3 - 2n \sum_{k=1}^{n-1} k^2 + (n^2+n-1) \sum_{k=1}^{n-1} k - n(n-1)^2 \right).
\end{aligned}$$

Using the standard formulas for the sum of the first $n-1$ integers, squares and cubes, we conclude that

$$|\mathcal{Q}| = \frac{1}{2}\Bigg(\frac{n^2(n-1)^2}{4} - 2n\frac{(n-1)n(2n-1)}{6} + (n^2+n-1)\frac{n(n-1)}{2} - n(n-1)^2\Bigg) = \frac{n(n-1)(n^2-5n+6)}{24}.$$

We are now ready to solve the problem. Consider an arbitrary configuration $\mathcal{P}$ of n points on the circle, and an arbitrary labelling of the $\binom{n}{2}$ different line segments formed by $\mathcal{P}$. We are going to add the segments one by one, following this order, and study how the number of regions evolves. Clearly, before adding any segment we have exactly one region. Each time we add a new segment, each region it intersects is divided into two, so the number of regions increases by as much as the number of regions intersected by the new segment. And the number of such regions is precisely one more than the number of points of $\mathcal{Q}$ which involve the newly added segment together with segments that had been added previously. As the segments are added one by one, every point in $\mathcal{Q}$ will be relevant exactly once and every segment also counts once, so we conclude that the total number of regions, denoted by $P(\mathcal{P})$, satisfies

$$P(\mathcal{P}) = 1 + |\mathcal{Q}| + \binom{n}{2}.$$

By choosing a maximal configuration $\mathcal{P}$, we conclude that the desired maximum number of regions, P_n, satisfies

$$P_n = 1 + \frac{n(n-1)(n^2-5n+6)}{24} + \binom{n}{2} = 1 + \frac{n(n-1)(n^2-5n+18)}{24}.$$

Also solved by *Albert Stadler, Herrliberg, Switzerland.*

Medium–Hard Problems

MH–137. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Is it possible to delete one number from the set $\{1, 2, \ldots, 10000\}$ so that the remaining 9999 numbers can be arranged in an order $a_1, a_2, \ldots, a_{9999}$, such that the differences $|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_{9998} - a_{9999}|, |a_{9999} - a_1|$ are all distinct?

Solution by the proposers. We will show that for every $n = 4k$, among the numbers $1, 2, \ldots, 4k$, after omitting the number $k+1$, there exists an arrangement of the remaining $n-1$ numbers, $a_1, a_2, \ldots, a_{n-1}$, such that the absolute differences $|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_{n-2} - a_{n-1}|, |a_{n-1} - a_1|$ are all distinct. Indeed, consider the following arrangement of the numbers:

$$1, 4k, 2, 4k{-}1, \ldots, k, 3k{+}1, k{+}2, 3k, k{+}3, 3k{-}1, \ldots, 2k, 2k{+}2, 2k{+}1,$$

or equivalently, let $a_{2i-1} = i$ for $1 \leq i \leq k$, $a_{2i-1} = i+1$ for $k+1 \leq i \leq 2k$, and finally, $a_{2i} = 4k+1-i$ for $1 \leq i \leq 2k-1$. Then, for $1 \leq i \leq 2k-1$, $|a_i - a_{i+1}| = 4k - i$, meaning that we get the differences $4k-1, 4k-2, \ldots, 2k+1$ in order. For $2k \leq i \leq 4k-2$, $|a_i - a_{i+1}| = 4k - i - 1$, giving the differences $2k-1, 2k-2, \ldots, 1$ in order. Finally, $|a_{4k-1} - a_1| = 2k$, which indeed gives $4k-1$ distinct differences.

In our particular case, we have $n = 4k = 4 \cdot 2500$ and deleting the number $k+1 = 2501$ the following arrangement of the remaining 9999 numbers, say

$$\{a_1, a_2, a_3, a_4, \ldots, a_{9997}, a_{9998}, a_{9999}\}$$

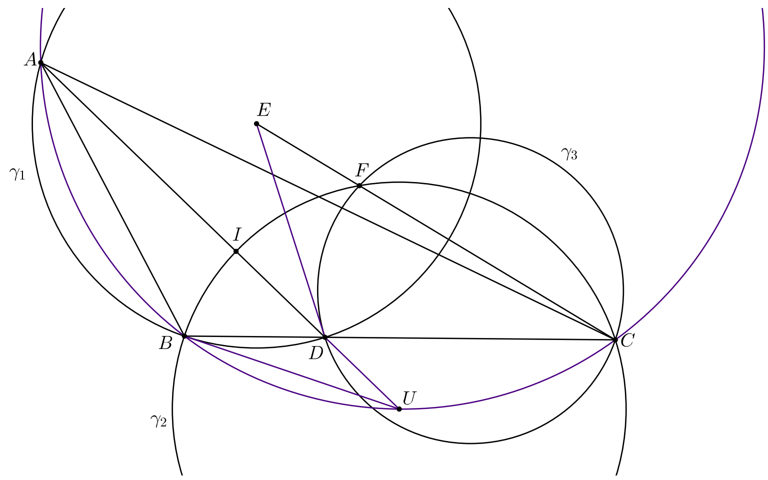
$$= \{1, 10000, 2, 9999, \ldots, 2500, 7501, 2501,$$

$$2502, 7500, 2503, 7499, \ldots, 5000, 5002, 5001\}$$

satisfies the conditions of the statement.

MH–138. *Proposed by Michel Bataille, Rouen, France.* Let I be the incenter of triangle ABC and let the line AI intersect BC at D. Let E be the circumcenter of ΔABD and let the line EC intersect the circumcircle of ΔBIC again at F. Prove that ED is tangent to the circumcircle of ΔCDF.

Solution by the proposer. Let γ_1, γ_2, and γ_3 be the circumcircles of the triangles ABD, BIC, and CDF, respectively (see figure). We first show that the circles γ_1, γ_2 are orthogonal.



Let AI intersect again the circumcircle of ABC at U. It is well-known that U is the center of γ_2.
We have $\angle(BU, BC) = \angle(AU, AC) = \angle(AB, AU)$ (since A, B, C, U are concyclic and AU bisects $\angle BAC$), hence

$$\angle(BU, BD) = \angle(AB, AD)$$

and therefore BU is tangent to γ_1 at B and γ_1, γ_2 are orthogonal.
Now, let $\mathcal{I}$ be the inversion with center C such that $\mathcal{I}(B) = D$. Then $\mathcal{I}(\gamma_1) = \gamma_1$, hence $\mathcal{I}(\gamma_2)$ is a line through D and orthogonal to γ_1, that is, the diameter DE of γ_1. It follows that $\mathcal{I}(F) = E$ so that γ_3 inverts into the line BE. But this line is a diameter of γ_1, hence γ_3 is orthogonal to γ_1 and therefore ED is tangent to γ_3.

MH–139. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Lisa and Bart play the following game. They first choose a positive integer N, and then they take turns writing numbers on a blackboard. Lisa starts by writing 1. Thereafter, when one of them has written the number n, the next player writes down either $n+1$ or $2n$, provided the number is not greater than N. The player who writes N on the blackboard wins.

(a) Determine which player has a winning strategy if $N = 2025$.
(b) Find the number of positive integers $N \leq 2025$ for which Bart has a winning strategy.

Solution by the proposer. **(a)** Lisa has a winning strategy for odd N, and so wins when $N = 2025$. Observe that, whenever a player writes down an odd number, the next is forced to write down an even number. By adding 1 to that number, the first player can write down another odd number. Since Lisa starts the game by writing down an odd number, she can force Bart to write down even numbers only. Since N is odd, Lisa will win the game, and in particular, she wins for $N = 2025$.

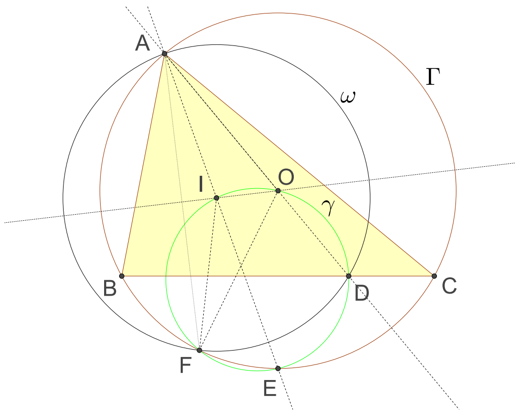
(b) For even N we consider two cases, according to the value of $N \pmod 4$.

- Let $N = 4k$. If any player is forced to write down a number $m \in \{k+1, k+2, \dots, 2k\}$, the other player wins by writing down $2m \in \{2k+2, 2k+4, \dots, 4k\}$, for the players will then have to write down the remaining numbers one after the other. Since there is an even number of numbers remaining, the latter player wins. This implies that the player who can write down k (that is, has a winning strategy for $N = k$), wins the game for $N = 4k$.
- Similarly, let $N = 4k+2$. If any player is forced to write down a number $m \in \{k+1, k+2, \dots, 2k+1\}$, the other player wins the game by writing down $2m \in \{2k+2, 2k+4, \dots, 4k+2\}$, as in the previous case. Analogously, this implies that the player who has a winning strategy for $N = k$ wins the game for $N = 4k+2$.

Since Lisa wins the game for $N = 1, 3$, while Bart wins the game for $N = 2$, Bart wins the game for $N = 8, 10$ as well, and thus for $N = 32, 34, 40, 42$ too. Then Bart wins the game for a further 8 values of N between 128 and 170, and then a further 16 values between 512 and 682, and for no other values with $N \leq 2025$. Hence Burt has a winning strategy for precisely 31 values of N with $N \leq 2025$.

MH–140. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a non-right triangle with $AB \neq AC$ and let Γ be its circumcircle with center O. Let I be an arbitrary point in the interior of triangle ABC $(I \neq O)$. Let D be the intersection point of BC and AO and E be the second intersection point of AI with Γ. Let F be the point symmetric to point A about the OI. Knowing that $DEFI$ is a cyclic quadrilateral, prove that $DI = FI$.

Solution 1 by the proposer. Let ω be the circle with center I and radius AI and let γ be the circle (IOE).



Scheme for solving problem MH-140.

Since F is the symmetric point of A with respect to IO, hence $AO = OF$, $AI = IF$, $\angle IAO = \angle IFO$ and so $F \in \Gamma$, $F \in \omega$. From $\triangle AOE$ we have $AO = OE$ or $\angle IAO \equiv \angle EAO = \angle AEO \equiv \angle IEO$ since I is between A and E. Therefore $\angle IFO = \angle IEO$ so $F \in \gamma$.

But from the condition that the points D, E, F, I are concyclic, it follows that $D \in \gamma$.

1. $\angle BAC < 90°$, O is between A and D.

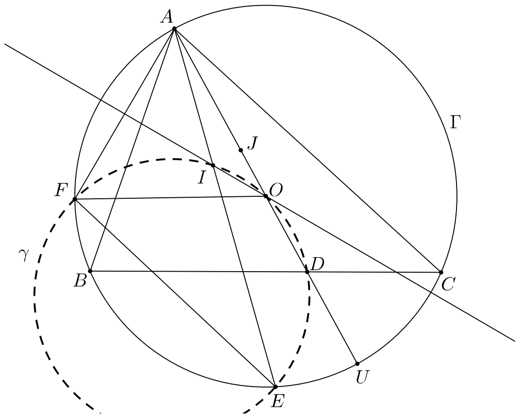
$\angle IDO \overset{\gamma}{=} \angle IEO = \angle IAO$, $\angle IDA = \angle IAD$ and from $\triangle AID$ hence $AI = ID$.

2. $\angle BAC > 90°$, D is between A and O.

$\angle IDO \overset{\gamma}{=} 180° - \angle IEO = 180° - \angle IAO$, $\angle IDA = 180° - \angle IDO = \angle IAO \equiv \angle IAD$ and from $\triangle AID$ hence $AI = ID$.

Therefore $D \in \omega$ and $ID = IF$.

Solution 2 by Michel Bataille, Rouen, France. Let AD intersect Γ at U ($U \neq A$) and let J be the midpoint of AD. We consider the inversion $\mathbf{i}$ with center A exchanging O and D (note that $D \neq O$ since ABC is not right). Since $AJ \cdot AU = \frac{1}{2}AD \cdot 2AO = AD \cdot AO$, we have $\mathbf{i}(U) = J$ and therefore $\mathbf{i}(\Gamma)$ is the perpendicular bisector m of AD. Note that F is on Γ (since A is on Γ and OI is a diameter of Γ).



Scheme for solving problem MH-140.

Now, suppose that D, E, F, I are on a circle γ. From

$$\angle(EF, EI) = \angle(EF, EA) = \frac{1}{2}\angle(\overrightarrow{OF}, \overrightarrow{OA}) = \angle(OF, OI)$$

we deduce that O is a point of γ as well. Since $\mathbf{i}(O) = D$, the circle γ satisfies $\mathbf{i}(\gamma) = \gamma$. It follows that $\mathbf{i}(E) = I$ and since E is on Γ, I is on m. Thus, $IA = ID$ and $DI = FI$ (since $IA = IF$).

MH–141. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Óscar Rivero Salgado, Santiago de Compostela, Spain.* For any positive number n, let $\tau(n)$ denote the number of positive divisors of n. We say that a positive number is **nice** if $2\,\tau(10n) < 5\,\tau(n)$. Determine how many positive numbers smaller or equal than 2025 are **nice**.

Solution 1 by Michel Bataille, Rouen, France. We show that the answer is 305. To this aim, we distinguish four mutually exclusive cases for n (where $1 \le n \le 2025$). We set $\delta_n = 5\tau(n) - 2\tau(10n)$ and recall that τ is a multiplicative function such that $\tau(p^k) = k+1$ if p is a prime. In what follows m always denotes a positive integer which is coprime to 2 and 5.
• If $n = m$, then $\delta_n = 5\tau(m) - 2\tau(2 \times 5 \times m) = 5\tau(m) - 2 \times 2 \times 2\tau(m) < 0$, hence n is not nice.
• If n is of the form $2^r \cdot 5^s \cdot m$ (with $r, s \ge 1$), then $\delta_n > 0$ as

$$\delta_n = (5(r+1)(s+1) - 2(r+2)(s+2))\tau(m) = (3(rs-1)+r+s)\tau(m).$$

Thus, the 202 multiples of 10 between 1 and 2025 are all nice.
• If n is of the form $5^u \cdot m$ ($u \ge 1$), then $\delta_n = 5(u+1)\tau(m) - 2(u+2) \cdot 2\tau(m) = (u-3)\tau(m)$, hence n is nice if and only if $u \ge 4$. There are two suitable numbers, namely 5^4 and $5^4 \times 3$.
• If n is of the form $2^v \cdot m$ ($v \ge 1$), then similarly, $\delta_n = (v-3)\tau(m)$, hence n is nice if and only if $v \ge 4$. Since $2^{11} > 2025$, there are seven possibilities for v: $4, 5, \ldots, 10$.

We observe that between 1 and a positive integer k (inclusive), there are $\left\lfloor \frac{k}{2} \right\rfloor$ multiples of 2, $\left\lfloor \frac{k}{5} \right\rfloor$ multiples of 5, hence $\left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k}{5} \right\rfloor - \left\lfloor \frac{k}{10} \right\rfloor$ multiples of 2 or 5. Thus, $a_k = k - \left\lfloor \frac{k}{2} \right\rfloor - \left\lfloor \frac{k}{5} \right\rfloor + \left\lfloor \frac{k}{10} \right\rfloor$ are coprime to 2 and 5.
This said, we examine the seven possibilities for v in turn:
$(v = 4, m \le 126, a_{126} = 50)$, $(v = 5, m \le 63, a_{63} = 26)$, $(v = 6, m \le 31, a_{31} = 13)$, $(v = 7, m \le 15, a_{15} = 6)$, $(v = 8, m \le 7, a_7 = 3)$, $(v = 9, m \le 3, a_3 = 2)$, $(v = 10, m \le 1, a_1 = 1)$.
This provides $50 + 26 + 13 + 6 + 3 + 2 + 1 = 101$ nice integers.
Together, we see that between 1 and 2025, there are $202+2+101 = 305$ nice numbers.

Solution 2 by Cao Minh Quang, Nguyen Binh Khiem high school for the gifted, Vinh Long, Vietnam. We write $n = 2^a 5^b m$ with

$a, b \geq 0$ and $\gcd(m, 10) = 1$. Then $\tau(n) = (a+1)(b+1)\tau(m)$, and $\tau(10n) = \tau(2^{a+1}5^{b+1}m) = (a+2)(b+2)\tau(m)$. Hence the condition $2\tau(10n) < 5\tau(n)$ is equivalent to

$$2(a+2)(b+2) < 5(a+1)(b+1) \iff 3ab + a + b > 3. \qquad (*)$$

Therefore n is *not* nice iff $3ab + a + b \leq 3$. That is,

$$(a, b) \in \{(0,0), (0,1), (0,2), (0,3), (1,0), (2,0), (3,0)\}.$$

Consequently:

- If $a, b \geq 1$ (i.e. $10 \mid n$), then n is always nice.
- If exactly one of a, b is nonzero and that exponent is ≥ 4, then n is nice.
- All remaining cases (no factor 10, and any nonzero exponent ≤ 3) are not nice.

Thus all nice numbers ≤ 2025 fall into the following disjoint families:

(I) Multiples of 10**.** Count: $\left\lfloor \dfrac{2025}{10} \right\rfloor = 202$.

(II) Multiples of 16 **not divisible by** 5 (i.e. $a \geq 4$, $b = 0$). There are $\left\lfloor \dfrac{2025}{16} \right\rfloor = 126$ multiples of 16 up to 2025, among which $\left\lfloor \dfrac{2025}{80} \right\rfloor = 25$ are also multiples of 5 (i.e. multiples of $\operatorname{lcm}(16, 5) = 80$). Hence the count here is $126 - 25 = 101$.

(III) Odd multiples of 625 (i.e. $a = 0$, $b \geq 4$ and $2 \nmid n$). Up to 2025 these are 625 and 1875, so 2 numbers.

Adding up, the total number of nice integers is

$$202 + 101 + 2 = 305.$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland. We have the inequality:

$$2\tau(10n) < 5\tau(n)$$

which is equivalent to

$$2(a+2)(b+2) < 5(a+1)(b+1)$$

and further equivalent to

$$3ab + a + b > 3.$$

So: If $a = 0$, then $b > 3$. If $b = 0$, then $a > 3$. If $a > 0$ and $b > 0$, then the inequality holds true. Thus, a number is *nice* if and only if n is divisible by 10, 16, or 625. By the inclusion-exclusion principle, the number of nice numbers ≤ 2025 equals

$$\sum_{10|n \,\vee\, 16|n \,\vee\, 625|n} 1$$

$$= \sum_{10|n} 1 + \sum_{16|n} 1 + \sum_{625|n} 1 - \sum_{10|n \,\wedge\, 16|n} 1 - \sum_{16|n \,\wedge\, 625|n} 1$$

$$- \sum_{10|n \,\wedge\, 625|n} 1 + \sum_{10|n \,\wedge\, 16|n \,\wedge\, 625|n} 1$$

$$= 202 + 126 + 3 - 25 - 0 - 1 + 0 = 305.$$

Solution 4 by the proposers. Let

$$n = 2^a \cdot 5^b \cdot \prod_{i=1}^{r} p_i^{k_i}$$

be the prime decomposition of a positive number n, with $a, b \geq 0$, $p_i \neq 2, 5$ and $k_i \geq 1$. Then,

$$10n = 2^{a+1} \cdot 5^{b+1} \cdot \prod_{i=1}^{r} p_i^{k_i}$$

and consequently

$$\frac{\tau(10n)}{\tau(n)} = \frac{(a+2)(b+2)\prod_{i=1}^{r}(k_i+1)}{(a+1)(b+1)\prod_{i=1}^{r}(k_i+1)} = \Big(1 + \frac{1}{a+1}\Big)\Big(1 + \frac{1}{b+1}\Big).$$

If both $a, b \geq 1$, the quotient is smaller or equal than $9/4$. On the other hand, if one of them is 0 (say a) then it must hold that

$$1 + \frac{1}{b+1} < \frac{5}{4},$$

or what is the same $b \geq 4$. Hence, the condition holds if and only if at least one of the following conditions holds:

- The number is a multiple of 10.
- The number is a multiple of 16.
- The number is a multiple of 625.

Fix now a positive number M. By the inclusion-exclusion principle, the amount of number that are smaller or equal than M and satisfy some of the conditions is

$$\left\lfloor \frac{N}{10} \right\rfloor + \left\lfloor \frac{N}{16} \right\rfloor + \left\lfloor \frac{N}{625} \right\rfloor - \left\lfloor \frac{N}{80} \right\rfloor - \left\lfloor \frac{N}{1250} \right\rfloor - \left\lfloor \frac{N}{10000} \right\rfloor + \left\lfloor \frac{N}{10000} \right\rfloor$$
$$= \left\lfloor \frac{N}{10} \right\rfloor + \left\lfloor \frac{N}{16} \right\rfloor + \left\lfloor \frac{N}{625} \right\rfloor - \left\lfloor \frac{N}{80} \right\rfloor - \left\lfloor \frac{N}{1250} \right\rfloor.$$

Hence, substituting N by 2025 we get

$$202 + 126 + 3 - 25 - 1 = 305.$$

MH–142. *Proposed by José Luis Díaz-Barrero Barcelona, Spain.* Let n, k be integers with $n \geq 2$ and $1 \leq k \leq n$. Show that

$$(-1)^{n-k}\binom{n}{k}\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{k}{k-j} + \sum_{\substack{j=1\\ j\neq k}}^{n}(-1)^{n-j}\binom{n}{j}\frac{j}{k-j} = 0.$$

Solution 1 by Michel Bataille, Rouen, France. Let $P(x)$ be the polynomial $(x-1)(x-2)\cdots(x-n)$ and for $k = 1, 2, \ldots, n$ let P_k be defined by $P(x) = (x-k)P_k(x)$. The decomposition of $\frac{n!}{P(x)}$ into partial fractions is

$$\frac{n!}{P(x)} = \sum_{j=1}^{n}\frac{a_j}{x-j}$$

with

$$a_j = \frac{n!}{P_j(j)} = \frac{n!}{(j-1)!(-1)^{n-j}(n-j)!} = (-1)^{n-j}\binom{n}{j}j.$$

Also, $\sum\limits_{j=1}^{n} \frac{1}{x-j} = \frac{P'(x)}{P(x)}$, hence

$$a_k \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{x-j} + \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{a_j}{x-j} = a_k \frac{P'(x)}{P(x)} - \frac{a_k}{x-k} + \frac{n!}{P(x)} - \frac{a_k}{x-k} = \frac{Q(x)}{P(x)}$$

where $Q(x) = a_k P'(x) - 2a_k P_k(x) + n!$.
Now, we have $P'(x) = P_k(x) + (x-k)P_k'(x)$, hence $P'(k) = P_k(k) = \frac{n!}{a_k}$ and therefore $Q(k) = n! - 2n! + n! = 0$. In addition,

$$Q'(x) = a_k(P''(x) - 2P_k'(x)) = a_k(2P_k'(x) + (x-k)P_k''(x) - 2P_k'(x))$$

so that $Q'(k) = 0$.
It follows that the polynomial $Q(x)$ is divisible by $(x-k)^2$, say, $Q(x) = (x-k)^2 R(x)$ and we obtain $\frac{Q(x)}{P(x)} = (x-k)\frac{R(x)}{P_k(x)}$. Finally,

$$a_k \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{k-j} + \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{a_j}{k-j} = \left[(x-k)\frac{R(x)}{P_k(x)}\right]_{x=k} = 0,$$

as required.

Solution 2 by the proposer. We will show equivalently that

$$(-1)^{n-k}\binom{n}{k} \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{k}{k-j} = -\sum_{\substack{j=1\\ j\neq k}}^{n} (-1)^{n-j}\binom{n}{j}\frac{j}{k-j}.$$

Indeed, consider the rational function

$$F(X) = \frac{n!}{(X-1)(X-2)\cdots(X-n)}.$$

Function F has a partial fraction decomposition of the form

$$F(X) = \sum_{j=1}^{n} \frac{\lambda_j}{X-j},$$

where

$$\lambda_j = \lim_{t\to j}(t-j)F(t) = \frac{n!}{(j-1)(j-2)\cdots 1(-1)(-2)\cdots(j-n)}$$
$$= (-1)^{n-j}\binom{n}{j}j.$$

Thus,

$$\frac{n!}{(X-1)(X-2)\cdots(X-n)} = \sum_{j=1}^{n}\frac{(-1)^{n-j}\binom{n}{j}j}{X-j}. \tag{1}$$

Now, for $1 \le k \le n$, let

$$F_k(X) = (X-k)F(X) = \frac{n!}{\prod_{\substack{\ell=1\\ \ell\neq k}}^{n}(X-\ell)}.$$

From (1), we conclude that

$$\frac{F_k(t)-F_k(k)}{t-k} = \sum_{\substack{j=1\\ j\neq k}}^{n}\frac{(-1)^{n-j}\binom{n}{j}j}{t-j}.$$

So, letting t tend to k, we see that

$$\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{(-1)^{n-j}\binom{n}{j}j}{k-j} = F_k'(k). \tag{2}$$

On the other hand, taking the logarithmic derivative of F_k, we get

$$F_k'(k) = -F_k(k)\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{k-j} = -(-1)^{n-k}\binom{n}{k}k\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{1}{k-j}. \tag{3}$$

and the desired conclusion follows from (2) and (3).

Advanced Problems

A–137. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*
Compute the following limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{h=1}^{n} \sum_{k=1}^{n} \frac{h+k}{h^2+k^2}.$$

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Write

$$\frac{1}{n} \sum_{h=1}^{n} \sum_{k=1}^{n} \frac{h+k}{h^2+k^2} = \frac{1}{n^2} \sum_{h=1}^{n} \sum_{k=1}^{n} \frac{h/n+k/n}{(h/n)^2+(k/n)^2},$$

and recognize this as a Riemann sum. Thus,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{h=1}^{n} \sum_{k=1}^{n} \frac{h+k}{h^2+k^2} = \int_0^1 \int_0^1 \frac{x+y}{x^2+y^2}\, dy\, dx.$$

Now,

$$\begin{aligned}
\int_0^1 \frac{x+y}{x^2+y^2}\, dy &= \left(\tan^{-1}\frac{y}{x} + \frac{1}{2}\ln(x^2+y^2)\right)\Bigg|_0^1 \\
&= \tan^{-1}\frac{1}{x} + \frac{1}{2}\ln(x^2+1) - \ln x \\
&= \frac{\pi}{2} - \tan^{-1} x + \frac{1}{2}\ln(x^2+1) - \ln x,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \frac{\pi}{2}\,dx &= \frac{\pi}{2},\\
\int_0^1 \tan^{-1}x\,dx &= x\tan^{-1}x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx\\
&= \frac{\pi}{4} - \frac{1}{2}\ln(1+x^2)\Big|_0^1 = \frac{\pi}{4} - \frac{1}{2}\ln 2,\\
\frac{1}{2}\int_0^1 \ln(x^2+1)\,dx &= \frac{1}{2}x\ln(x^2+1)\Big|_0^1 - \int_0^1 \frac{x^2}{x^2+1}\,dx\\
&= \frac{1}{2}\ln 2 - \left(x - \tan^{-1}x\right)\Big|_0^1 = \frac{1}{2}\ln 2 - 1 + \frac{\pi}{4},\\
\int_0^1 \ln x\,dx &= x\ln x\Big|_0^1 - \int_0^1 dx = -1.
\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{h=1}^{n}\sum_{k=1}^{n}\frac{h+k}{h^2+k^2} = \frac{\pi}{2}-\frac{\pi}{4}+\frac{1}{2}\ln 2+\frac{1}{2}\ln 2-1+\frac{\pi}{4}+1 = \frac{\pi}{2}+\ln 2.$$

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA and Moti Levy, Rehovot, Israel (same solution). We have

$$\begin{aligned}
\sum_{h=1}^{n}\sum_{k=1}^{n}\frac{h+k}{h^2+k^2} &= \sum_{h=1}^{n}\sum_{k=1}^{n}\frac{n(\frac{h}{n}+\frac{k}{n})}{n^2(\frac{h^2}{n^2}+\frac{k^2}{n^2})}\\
&= \frac{1}{n}\sum_{h=1}^{n}\sum_{k=1}^{n}\frac{\frac{h}{n}+\frac{k}{n}}{\frac{h^2}{n^2}+\frac{k^2}{n^2}},
\end{aligned}$$

so that we recognize a Riemann sum for a double integral:

$$\begin{aligned}
\frac{1}{n}\sum_{h=1}^{n}\sum_{k=1}^{n}\frac{h+k}{h^2+k^2} &= \frac{1}{n^2}\sum_{h=1}^{n}\sum_{k=1}^{n}\frac{\frac{h}{n}+\frac{k}{n}}{\frac{h^2}{n^2}+\frac{k^2}{n^2}} \to \int_0^1\int_0^1 \frac{x+y}{x^2+y^2}\,dx\,dy\\
&= \frac{\pi}{2} + \ln 2.
\end{aligned}$$

Solution 3 by Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA and the proposer. Applying Stolz's criterium to the sequences of the general term

$$a_n = \sum_{h=1}^{n} \sum_{k=1}^{n} \frac{h+k}{h^2+k^2}$$

and $b_n = n$, we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{h=1}^{n} \sum_{k=1}^{n} \frac{h+k}{h^2+k^2} = \lim_{n\to\infty} \left(\sum_{h=1}^{n+1} \sum_{h=1}^{n+1} \frac{h+k}{h^2+k^2} - \sum_{h=1}^{n} \sum_{k=1}^{n} \frac{h+k}{h^2+k^2} \right).$$

Since

$$\sum_{h=1}^{n+1} \sum_{h=1}^{n+1} \frac{h+k}{h^2+k^2} - \sum_{h=1}^{n} \sum_{k=1}^{n} \frac{h+k}{h^2+k^2}$$

$$= 2\left(\frac{(n+1)+1}{(n+1)^2+1^2} + \frac{(n+1)+2}{(n+1)^2+2^2} + \cdots + \frac{(n+1)+n}{(n+1)^2+n^2} \right) + \frac{1}{n+1}$$

$$= \frac{2}{n+1} \sum_{i=1}^{n} \left(\frac{1+\dfrac{i}{n+1}}{1+\left(\dfrac{i}{n+1}\right)^2} \right) + \frac{1}{n+1},$$

then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{h=1}^{n} \sum_{k=1}^{n} \frac{h+k}{h^2+k^2}$$

$$= \lim_{n\to\infty} \left\{ \frac{2}{n+1} \sum_{i=1}^{n} \left(\frac{1+\dfrac{i}{n+1}}{1+\left(\dfrac{i}{n+1}\right)^2} \right) + \frac{1}{n+1} \right\}$$

$$= 2\int_0^1 \frac{1+x}{1+x^2}\, dx = 2\left[\arctan(x) + \frac{1}{2}\ln(1+x^2) \right]_0^1 = \frac{\pi}{2} + \ln 2.$$

Solution 4 by Michel Bataille, Rouen, France. Let $S_0 = 0$ and

$S_n = \sum_{h=1}^{n}\sum_{k=1}^{n} \frac{h+k}{h^2+k^2}$ for $n \geq 1$. For all $n \geq 0$, we have

$$\begin{aligned} S_{n+1} &= \sum_{h=1}^{n}\sum_{k=1}^{n+1} \frac{h+k}{h^2+k^2} + \sum_{k=1}^{n+1} \frac{(n+1)+k}{(n+1)^2+k^2} \\ &= S_n + \sum_{h=1}^{n} \frac{h+(n+1)}{h^2+(n+1)^2} + \sum_{k=1}^{n+1} \frac{(n+1)+k}{(n+1)^2+k^2} \\ &= S_n + 2\sum_{h=1}^{n+1} \frac{h+(n+1)}{h^2+(n+1)^2} - \frac{1}{n+1}. \end{aligned}$$

The sum $\sum_{h=1}^{n} \frac{h+n}{h^2+n^2} = \frac{1}{n}\sum_{h=1}^{n} \frac{1+\frac{h}{n}}{1+\frac{h^2}{n^2}}$ is a Riemann sum for the function $x \mapsto \frac{1+x}{1+x^2}$ on $[0,1]$, hence

$$\lim_{n\to\infty} \sum_{h=1}^{n} \frac{h+n}{h^2+n^2} = \int_0^1 \frac{1+x}{1+x^2}\,dx = \left[\arctan x + \frac{\ln(1+x^2)}{2}\right]_0^1 = \frac{\pi}{4} + \frac{\ln 2}{2}.$$

It follows that $\lim\limits_{n\to\infty}(S_n - S_{n-1}) = \frac{\pi}{2} + \ln 2$, that is,

$$S_n - S_{n-1} \sim \frac{\pi}{2} + \ln 2$$

as $n \to \infty$. The Stolz-Cesaro Theorem then gives

$$S_n = \sum_{k=1}^{n}(S_k - S_{k-1}) \sim n\Big(\frac{\pi}{2} + \ln 2\Big)$$

as $n \to \infty$. In consequence

$$\lim_{n\to\infty} \frac{1}{n} S_n = \frac{\pi}{2} + \ln 2.$$

Solution 5 by Albert Stadler, Herrliberg, Switzerland. We have

$$\sum_{h=1}^{n}\sum_{k=1}^{n} \frac{h+k}{h^2+k^2} = 2\sum_{h=1}^{n}\sum_{k=1}^{h} \frac{h+k}{h^2+k^2} - \sum_{h=1}^{n} \frac{2h}{2h^2}$$

$$= 2\sum_{h=1}^{n}\sum_{k=1}^{h} \frac{h+k}{h^2+k^2} + O(\log n\ ).$$

The inner sum of the double sum is a Riemann sum. So

$$\sum_{k=1}^{h} \frac{h+k}{h^2+k^2} = \frac{1}{h}\sum_{k=1}^{h} \frac{1+\frac{k}{h}}{1+\left(\frac{k}{h}\right)^2} = \int_0^1 \frac{1+x}{1+x^2}dx + o(1),$$

as h tends to infinity. We have

$$\int_0^1 \frac{1+x}{1+x^2}dx = \left[\arctan x \ + \frac{1}{2}\log\left(1+x^2\right)\ \right]_{x=0}^{x=1} = \frac{\pi}{4} + \frac{1}{2}\log 2\ .$$

Hence

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{h=1}^{n}\sum_{k=1}^{n}\frac{h+k}{h^2+k^2} \ &= \lim_{n\to\infty}\frac{2}{n}\sum_{h=1}^{n}\left(\frac{\pi}{4}+\frac{1}{2}\log 2\ + o(1)\right)\\ &= \frac{\pi}{2} + \log 2\ .\end{aligned}$$

Also solved by *José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain.*

A–138. *Proposed by Michel Bataille, Rouen, France.* Let n be a positive integer. Evaluate

$$\int_0^{\pi} (\cos x)^{2n}(\cos 2x + \cdots + \cos(2nx))\, dx.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland. We have

$$\int_0^{\pi}\ (\cos x\ )^{2n}(\cos 2x\ \ + \cdots + \cos(2nx)\ )dx$$

$$= \frac{1}{2^{2n+1}}\int_0^{\pi}\left(e^{ix}+e^{-ix}\right)^{2n}\left(e^{2ix}+e^{-2ix}+\cdots+e^{2inx}+e^{-2inx}\right)dx$$

$$= \frac{1}{2^{2n+1}}\int_0^{\pi}\sum_{k=0}^{2n}\binom{2n}{k}e^{i(k-2n+k)x}\left(-1+\sum_{j=-n}^{n}e^{2ijx}\right)dx$$

$$= -\frac{\pi}{2^{2n+1}}\binom{2n}{n} + \frac{\pi}{2^{2n+1}}\sum_{\substack{0\le k\le 2n,\, -n\le j\le n\\ j=n-k}}\binom{2n}{k}$$

$$= -\frac{\pi}{2^{2n+1}}\binom{2n}{n} + \frac{\pi}{2^{2n+1}}\sum_{k=0}^{2n}\binom{2n}{k}$$
$$= -\frac{\pi}{2^{2n+1}}\binom{2n}{n} + \frac{\pi}{2} = \frac{\pi}{2}\left(1 - \frac{1}{2^{2n}}\binom{2n}{n}\right).$$

Solution 2 by Joe Santmyer, Las Cruces, NM, USA. It will be shown that $I_n = \frac{\pi}{2}\left[1 - \frac{1}{4^n}\binom{2n}{n}\right]$.

By the formula

$$\sum_{k=1}^{n}\cos(kx) = \frac{\sin\left[\left(n+\frac{1}{2}\right)x\right]}{2\sin\left(\frac{x}{2}\right)} - \frac{1}{2}$$

on p 460 in [**fulks69**] we get

$$\begin{aligned}\sum_{k=1}^{n}\cos(2kx) &= \frac{\sin\left[\left(n+\frac{1}{2}\right)2x\right]}{2\sin\left(\frac{2x}{2}\right)} - \frac{1}{2}\\ &= \frac{\sin[(2n+1)x]}{2\sin(x)} - \frac{1}{2}.\end{aligned}$$

Hence

$$I_n = \int_{x=0}^{\pi}\cos^{2n}(x)\left[\frac{\sin[(2n+1)x]}{2\sin(x)} - \frac{1}{2}\right]dx.$$

By (3.16) on p 9 in Volume 6 of [2] we have

$$\frac{\sin[(2n+1)x]}{2\sin(x)} = \frac{1}{2}\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}2^{2n-2k}\cos^{2n-2k}(x).$$

Hence

$$\begin{aligned}I_n &= \int_{x=0}^{\pi}\cos^{2n}(x)\left[\frac{1}{2}\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}2^{2n-2k}\cos^{2n-2k}(x) - \frac{1}{2}\right]dx\\ &= \frac{1}{2}\left[\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}2^{2n-2k}\int_{x=0}^{\pi}\cos^{2(2n-k)}(x)dx - \int_{x=0}^{\pi}\cos^{2n}(x)dx\right]\\ &= \sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}4^{n-k}\int_{x=0}^{\frac{\pi}{2}}\cos^{2(2n-k)}(x)dx - \int_{x=0}^{\frac{\pi}{2}}\cos^{2n}(x)dx.\end{aligned}$$

By 15.30 on p 96 in [3] we get

$$\begin{aligned}
I_n &= \sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}4^{n-k}\left[\frac{\pi}{2}\left(\frac{1}{4^{2n-k}}\right)\binom{4n-2k}{2n-k}\right]-\frac{\pi}{2}\left(\frac{1}{4^n}\right)\binom{2n}{n}\\
&= \sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}\left[\frac{\pi}{2}\left(\frac{1}{4^{n}}\right)\binom{4n-2k}{2n-k}\right]-\frac{\pi}{2}\left(\frac{1}{4^n}\right)\binom{2n}{n}\\
&= \frac{\pi}{2}\left(\frac{1}{4^n}\right)\left[\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}\binom{4n-2k}{2n-k}-\binom{2n}{n}\right].
\end{aligned}$$

It is easy to see that

$$\binom{2n-k}{k}\binom{4n-2k}{2n-k} = \binom{2n}{k}\binom{4n-2k}{2n}.$$

Hence

$$\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}\binom{4n-2k}{2n-k} = \sum_{k=0}^{n}(-1)^k\binom{2n}{k}\binom{4n-2k}{2n}.$$

By (10.6) on p 40 in Volume 5 of [2] we have

$$\sum_{k=0}^{\lfloor\frac{m}{2}\rfloor}(-1)^k\binom{m}{k}\binom{2m-2k}{m} = 2^m.$$

Let $m=2n$ to get

$$\begin{aligned}
\sum_{k=0}^{\lfloor\frac{2n}{2}\rfloor}(-1)^k\binom{2n}{k}\binom{4n-2k}{2n} &= 2^{2n}\\
\sum_{k=0}^{n}(-1)^k\binom{2n}{k}\binom{4n-2k}{2n} &= 4^n.
\end{aligned}$$

Consequently

$$\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}\binom{4n-2k}{2n-k} = 4^n.$$

Hence

$$\begin{aligned}
I_n &= \frac{\pi}{2}\left(\frac{1}{4^n}\right)\left[4^n-\binom{2n}{n}\right]\\
&= \frac{\pi}{2}\left[1-\frac{1}{4^n}\binom{2n}{n}\right].
\end{aligned}$$

References

[1] Fulks, W., "Advanced Calculus", John Wiley and Sons, Inc., New York, 1969.

[2] Gould, H., W., "Tables of Combinatorial Identities", 9193344/http://www.math.wvu.edu/ gould/, 2010.

[3] Spiegel, M. R., "Mathematical Handbook", Schaum's Outline Series, McGraw-Hill Book Company, 1968.

Solution 3 by the proposer. Let I_n be the integral. We show that $I_n = \frac{\pi}{2}\left(1 - \frac{\binom{2n}{n}}{2^{2n}}\right)$.
We use $2\cos a\cos b = \cos(a+b) + \cos(a-b)$ to calculate I_n as follows

$$\begin{aligned} I_n &= \int_0^{\pi} (\cos x)^{2n-1}(\cos x\cos 2x + \cdots + \cos x\cos 2nx)\,dx \\ &= \frac{1}{2}\int_0^{\pi} (\cos x)^{2n-1}(\cos x + \cos 3x + \cdots + \cos((2n+1)x))\,dx \\ &= \frac{K_n}{2} + \int_0^{\pi} (\cos x)^{2n-1}(\cos 3x + \cdots + \cos((2n-1)x))\,dx \end{aligned}$$

where we set $K_n = \int_0^{\pi}(\cos x)^{2n}\,dx$ and used the following result: if p, q are integers such that $q > p \geq 0$, then $\int_0^{\pi}(\cos x)^p(\cos(qx))\,dx = 0$. [This is easily proved by induction on p using $(\cos x)^{p+1}(\cos(qx)) = \frac{1}{2}(\cos x)^p(\cos((q-1)x) + \cos((q+1)x))$.

We introduce

$$J_n = \int_0^{\pi} (\cos x)^{2n-1}(\cos x + \cos 3x + \cdots + \cos((2n-1)x))\,dx$$

so that

$$I_n = \frac{K_n}{2} + J_n - K_n = J_n - \frac{K_n}{2}. \tag{1}$$

Similarly, for any positive integer n, we have

$$\begin{aligned} J_{n+1} &= \int_0^\pi (\cos x)^{2n}(\cos^2 x + \cos x \cos 3x + \cdots + \cos x \cos((2n+1)x))\,dx \\ &= \frac{1}{2}\int_0^\pi (\cos x)^{2n}(1 + \cos 2x + \cos 2x + \cdots + \cos((2n+2)x))\,dx \\ &= \frac{K_n}{2} + I_n = J_n \quad \text{(using (1))} \end{aligned}$$

so that $J_n = J_1 = \int_0^\pi (\cos x)^2\,dx = \frac{\pi}{2}$ (2).
In addition, we have

$$K_n = \int_{-\pi/2}^{\pi/2} (\cos x)^{2n}\,dx = 2\int_0^{\pi/2} (\cos x)^{2n}\,dx = \pi \cdot \frac{\binom{2n}{n}}{2^{2n}} \qquad (3)$$

(the last equality from the well-known Wallis integrals).
The announced result follows from $(1), (2), (3)$.

Solution 4 by Moti Levy, Rehovot, Israel. We prove the identity

$$(\cos x)^{2n} = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{k=1}^{n}\binom{2n}{n-k}\cos(2kx). \qquad (1)$$

Write $\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$. Then

$$(\cos x)^{2n} = \frac{1}{2^{2n}}(e^{ix} + e^{-ix})^{2n} = \frac{1}{2^{2n}}\sum_{j=0}^{2n}\binom{2n}{j}e^{i(2n-2j)x}.$$

Let $k = n - j$. As j runs from 0 to $2n$, k runs from n to $-n$, and $e^{i(2n-2j)x} = e^{i\,2kx}$. Hence

$$\begin{aligned} (\cos x)^{2n} &= \frac{1}{2^{2n}}\sum_{k=-n}^{n}\binom{2n}{n-k}e^{i\,2kx} \\ &= \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n}}\sum_{k=-n}^{-1}\binom{2n}{n-k}e^{i\,2kx} + \frac{1}{2^{2n}}\sum_{k=1}^{n}\binom{2n}{n-k}e^{i\,2kx}. \end{aligned}$$

Since $\binom{2n}{n-k} = \frac{(2n)!}{(n+k)!(n-k)!} = \binom{2n}{n+k}$,

$$\begin{aligned} (\cos x)^{2n} &= \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n}}\sum_{k=1}^{n}\binom{2n}{n-k}\left(e^{i\,2kx} + e^{-i\,2kx}\right) \\ &= \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{k=1}^{n}\binom{2n}{n-k}\cos(2kx). \end{aligned}$$

Now we state the orthogonality property of $\{\cos(2nx)\}$:

For integers $k, m \geq 1$,

$$\int_0^{\pi} \cos(2kx)\cos(2mx)\,dx = \begin{cases} 0, & k \neq m, \\ \dfrac{\pi}{2}, & k = m\ (\geq 1). \end{cases} \tag{2}$$

Using (1),

$$\begin{aligned} I_n &= \sum_{m=1}^{n} \int_0^{\pi} (\cos x)^{2n} \cos(2mx)\,dx \\ &= \sum_{m=1}^{n} \int_0^{\pi} \left[\frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}} \sum_{k=1}^{n} \binom{2n}{n-k} \cos(2kx)\right] \cos(2mx)\,dx \end{aligned}$$

The constant term integrates to 0 against $\cos(2mx)$ on $[0, \pi]$. By (2), only the terms with $k = m$ are not zero:

$$I_n = \frac{1}{2^{2n-1}} \sum_{m=1}^{n} \binom{2n}{n-m} \int_0^{\pi} \cos^2(2mx)\,dx = \frac{1}{2^{2n-1}} \sum_{m=1}^{n} \binom{2n}{n-m} \cdot \frac{\pi}{2}.$$

Thus

$$I_n = \frac{\pi}{2^{2n}} \sum_{m=1}^{n} \binom{2n}{n-m} = \frac{\pi}{2^{2n}} \sum_{r=0}^{n-1} \binom{2n}{r} = \frac{\pi}{2} - \frac{\pi}{2^{\,2n+1}} \binom{2n}{n},$$

since $\sum_{r=0}^{n-1} \binom{2n}{r} = \frac{1}{2}\left(2^{2n} - \binom{2n}{n}\right)$. 2

A–139. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain.* Determine the smallest positive integer λ such that if A and B are any 2×2 integer matrices satisfying that all of the matrices

$$A,\ A + B,\ A + 2B,\ \ldots,\ A + \lambda B$$

are invertible and their inverses also have integer entries, then for any integer t, the matrix $A + tB$ is also invertible and its inverse has integer entries.

Solution 1 by by Moti Levy, Rehovot, Israel. For an integer 2×2 matrix M, the inverse M^{-1} has integer entries if and only if $\det M = \pm 1$.

A unimodular matrix M is a square integer matrix having determinant $+1$ or -1.

Thus we can rephrase the problem statement as follows:

Determine the smallest positive integer λ with the following property: whenever A, B are 2×2 unimodular matrices, then for every integer t, the matrix $A + tB$ is also unimodular.

Now we define the function $f(t)$ by

$$f(t) = \det(A + tB).$$

For 2×2 matrices f is a polynomial of degree at most 2:

$$f(t) = \det(B)\, t^2 + w\, t + \det(A).$$

Assume that $A, A + B, \ldots, A + 4B$ are unimodular. then their determinant is ± 1, that is,

$$f(0),\ f(1),\ f(2),\ f(3),\ f(4) \in \{\pm 1\}.$$

Consider $g(t) = f(t)^2 - 1$. Since $\deg f \leq 2$, we have $\deg g \leq 4$. But $g(k) = 0$ for $k = 0, 1, 2, 3, 4$, i.e. g has 5 distinct roots. Therefore $g \equiv 0$ as a polynomial, whence $f(t)^2 \equiv 1$. Over the integral domain $\mathbb{Z}[t]$ this implies $f \equiv 1$ or $f \equiv -1$. In particular, $f(t) = \pm 1$ for every integer t, i.e. every $A + tB$ is unimodular. Hence each $(A + tB)^{-1}$ has integer entries. This proves that the property holds for $\lambda = 4$.

Now we give an example, where the property does not hole for $\lambda = 3$. Let $A = I_2$ and

$$B = \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}.$$

Then

$$\det(A + tB) = \det(I_2 + tB) = t^2 - 3t + 1.$$

Hence

$$\det(A + tB) \in \{1, -1, -1, 1\} \quad \text{for } t = 0, 1, 2, 3,$$

so $A, A + B, A + 2B, A + 3B$ are unimodular. However $\det(A + 4B) = 4^2 - 3 \cdot 4 + 1 = 5$, so $A + 4B$ is not unimodular and its inverse is not an integer matrix.

We conclude that the smallest λ with the property is 4.

Solution 2 by Michel Bataille, Rouen, France. We show that the desired integer λ_{min} is 4. Indeed, let $GL_2(\mathbb{Z})$ denote the set of all invertible matrices with integer entries whose inverse has integer entries. It is well-known that a matrix M with integer entries is in $GL_2(\mathbb{Z})$ if and only if $\det(M) = \pm 1$ (*quick proof*: if M is in $GL_2(\mathbb{Z})$, the integers $\det(M), \det(M^{-1})$ satisfy $\det(M) \cdot \det(M^{-1}) = 1$, hence $\det(M) = \pm 1$; conversely, if the latter holds, M is invertible and for any given column vector Y with integer entries, the solution X to the equation $MX = Y$ has integer entries as seen by the Cramer formulas, hence M^{-1} has integer entries).

Now, let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}$. A simple calculation gives

$$I + B = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}, \quad I + 2B = \begin{pmatrix} 1 & -2 \\ 2 & -5 \end{pmatrix}, \quad I + 3B = \begin{pmatrix} 1 & -3 \\ 3 & -8 \end{pmatrix}$$

and $I + 4B = \begin{pmatrix} 1 & -4 \\ 4 & -11 \end{pmatrix}$, so that $I, I + B, I + 2B, I + 3B$ are in $GL_2(\mathbb{Z})$, but $I + 4B$ is not. Thus, a suitable λ must satisfy $\lambda \geq 4$.

Conversely, consider $A \in GL_2(\mathbb{Z})$ and B with integer entries. If $A,\ A + B,\ A + 2B,\ A + 3B,\ A + 4B$ are in $GL_2(\mathbb{Z})$, then the polynomial $p(x) = \det(A + xB)$ takes the value 1 or -1 for $x = 0, 1, 2, 3, 4$, hence takes either the value 1 or the value -1 for at least three of the five values of x, say the former. It follows that $p(x) - 1$ has degree at most 2 and at least three roots, hence $p(x) - 1$ must be the zero polynomial. This means that $p(x) = 1$ for all x, and therefore $A + xB$ is in $GL_2(\mathbb{Z})$ for all integer x.
In consequence $\lambda_{\text{min}} = 4$.

Solution 3 by Albert Stadler, Herrliberg, Switzerland. Invertible integer $n \times n$ matrices with integer inverses are unimodular, i.e. $\det M = \pm 1$, since if $M \in M_n(\mathbb{Z})$ and M^{-1} has integer entries, then

$$M^{-1}M = I_n$$

and

$$\det\left(M^{-1}\right)\det(M) = 1.$$

So $\det M = \pm 1$, as $\det(M^{-1})$ and $\det(M)$ are integers.

So all given matrices $A + kB$ satisfy

$$\det(A + kB) = \pm 1.$$

Let

$$f(t) = \det(A + tB)$$

be the determinant polynomial. Since the entries of A, B are integers, $f(t)$ is a quadratic polynomial with integer coefficients:

$$f(t) = at^2 + bt + c, \quad a, b, c \in \mathbb{Z}.$$

The given condition is

$$f(0), f(1), \ldots, f(\lambda) \in \{\pm 1\}.$$

We need the smallest λ such that this forces $f(t) \in \{\pm 1\}$ for **all** integers t.

We claim that $\lambda > 3$. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}.$$

Then

$$f(t) = \det(A + tB) = \det\begin{pmatrix} 1 & -t \\ t & 1 - 3t \end{pmatrix} = t^2 - 3t + 1$$

with $f(0) = 1, f(1) = -1, f(2) = -1, f(3) = 1$, but $f(4) = 5$. So $A, A + B, A + 2B, A + 3B$ are unimodular, but $A + 4B$ is not.

We next claim that $\lambda = 4$ forces constancy. Let $f(x)$ be a quadratic polynomial and suppose $f(0), f(1), f(2), f(3), f(4) \in \{-1, 1\}$. Consider the polynomial

$$P(x) = f(x)^2 - 1.$$

Because f is quadratic, $f(x)^2$ is a polynomial of degree at most 4, so $P(x)$ has degree at most 4.

By hypothesis $f(t) = \pm 1$ for $t = 0, 1, 2, 3, 4$. Hence for each such t we have $P(t) = f(t)^2 - 1 = 0$. Thus $P(x)$ has five distinct zeros $0, 1, 2, 3, 4$. A nonzero polynomial of degree ≤ 4 can have at most 4 distinct roots. Since P has 5 distinct roots it must be the zero polynomial:

$$P(x) \equiv 0 \text{ for all } x,$$

so $f(x)^2 - 1 \equiv 0$, i.e. $f(x)^2 \equiv 1$. Hence $f(x)$ takes only the values $+1$ or -1 for every x. But a nonconstant polynomial cannot take only two values on infinitely many inputs; therefore f must be constant. From $f^2 \equiv 1$ we get $f \equiv 1$ or $f \equiv -1$.

Conclusion: the smallest positive integer λ satisfying the imposed condition is $\lambda = 4$.

Solution 4 by the proposer. The answer is $\lambda = 4$. We begin by showing that if A, $A + B$, $A + 2B$, $A + 3B$, and $A + 4B$ are invertible and their inverses have integer entries, then $A + tB$ is always invertible and its inverse has integer entries.

Observe that if a matrix C has integer entries and its inverse also has integer entries, then $\det C \cdot \det C^{-1} = 1$, so $\det C \in \{1, -1\}$. Setting $C(t) = A + tB$, we can express its determinant as a quadratic polynomial:

$$\det C(t) = \det(A + tB) = \alpha t^2 + \beta t + \gamma,$$

where the coefficients depend on the matrices A and B.

Since $\det C(t)$ takes values in $\{1, -1\}$ for $t = 0, 1, 2, 3, 4$, we note that a non-constant quadratic polynomial can have at most two distinct roots. This forces $\det C(t)$ to be constant and equal to either 1 or -1 for all integers t. By Cramer's rule, since $A+tB$ has integer entries, the inverse of $A+tB$ will always have integer entries.

To show that $\lambda = 3$ does not work, consider the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}.$$

For these choices, we compute:

$$\det(A) = 1, \quad \det(A+B) = -1, \quad \det(A+2B) = -1, \quad \det(A+3B) = 1.$$

Thus, all four matrices are invertible and their inverses have integer coefficients. However, for $t = 4$,

$$A + 4B = \begin{bmatrix} 4 & -11 \\ -1 & 4 \end{bmatrix},$$

which is still invertible, but its inverse is

$$(A+4B)^{-1} = \begin{bmatrix} 4/5 & 11/5 \\ 1/5 & 4/5 \end{bmatrix},$$

which has non-integer entries, contradicting the given conditions. Thus, $\lambda = 3$ is insufficient, and the smallest valid value is $\lambda = 4$.

Also solved by *Daniel Văcaru, National Economic College „Maria Teiuleanu", Pitești, Romania.*

A–140. *Proposed by Vasile Mircea Popa, "Lucian Blaga" University of Sibiu, Romania.* Calculate the following integral:

$$\int_0^\infty \frac{\ln x}{(x+1)(x^2+1)(x^4+1)} dx.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA. A partial fraction decomposition gives us

$$\int_0^\infty \frac{\ln x}{(x+1)(x^2+1)(x^4+1)}\, dx$$

$$= \int_0^\infty \ln x \left(\left(\frac{1}{4(x+1)} + \frac{1-x}{4(x^2+1)} \right) + \frac{1-x}{2(x^4+1)} \right) dx$$

$$= \frac{1}{2} \int_0^\infty \frac{\ln x}{(x+1)(x^2+1)}\, dx + \frac{1}{2} \int_0^\infty \frac{(1-x)\ln x}{x^4+1}\, dx$$

$$= \frac{1}{2} \left(\int_0^\infty \frac{(x-1)\ln x}{x^4-1}\, dx + \int_0^\infty \frac{(1-x)\ln x}{x^4+1}\, dx \right) = \frac{1}{2}(A + B).$$

Formula 4.235.1 in *Table of Integrals, Series, and Products* (*Sixth Edition*) by Gradshteyn and Ryzhik states that

$$\int_0^\infty \frac{\ln(x)(1-x)x^{n-2}}{1-x^{2n}}\, dx = -\frac{\pi^2}{4n^2} \tan^2 \frac{\pi}{2n}.$$

With $n = 2$, we have

$$A = \int_0^\infty \frac{(x-1)\ln x}{x^4-1}\, dx = -\frac{\pi^2}{16} \cdot \tan^2 \frac{\pi}{4} = -\frac{\pi^2}{16}.$$

Formula 4.254.5 in the same compendium gives

$$\int_0^\infty \frac{x^{p-1}\ln x}{1+x^q}\, dx = -\frac{\pi^2}{q^2} \cdot \frac{\cos \frac{p\pi}{q}}{\sin^2 \frac{p\pi}{q}} \quad (0 < p < q).$$

Letting $p = 1, q = 4$ and then $p = 2, q = 4$, we get

$$B = \int_0^\infty \frac{(1-x)\ln x}{x^4+1}\, dx = \int_0^\infty \frac{\ln x}{x^4+1}\, dx - \int_0^\infty \frac{x\ln x}{x^4+1}\, dx$$

$$= -\frac{\pi^2}{16} \cdot \frac{\cos \frac{\pi}{4}}{\sin^2 \frac{\pi}{4}} - \left(\frac{\pi^2}{16} \cdot \frac{\cos \frac{\pi}{2}}{\sin^2 \frac{\pi}{2}} \right) = -\frac{\sqrt{2}\,\pi^2}{16}.$$

Finally, the given integral equals

$$\frac{1}{2}(A+B) = \frac{1}{2} \left(-\frac{\pi^2}{16} - \frac{\sqrt{2}\,\pi^2}{16} \right) = -\frac{(\sqrt{2}+1)\pi^2}{32}.$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. With the change of variables $x \to \frac{1}{x}$,

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{(1+x)(1+x^2)(1+x^4)}\,dx &= \int_1^0 \frac{\ln\left(\frac{1}{x}\right)}{\left(\frac{1}{x}+1\right)\left(\frac{1}{x^2}+1\right)\left(\frac{1}{x^4}+1\right)} \cdot -\frac{dx}{x^2}\\
&= -\int_0^1 \frac{\ln x}{\left(\frac{1}{x}+1\right)(1+x^2)\left(\frac{1}{x^4}+1\right)}\,dx\\
&= -\int_0^1 \frac{x^5 \ln x}{(1+x)(1+x^2)(1+x^4)}\,dx;
\end{aligned}$$

thus,

$$\begin{aligned}
\int_0^\infty \frac{\ln x}{(1+x)(1+x^2)(1+x^4)}\,dx &= \int_0^1 \frac{(1-x^5)\ln x}{(1+x)(1+x^2)(1+x^4)}\,dx\\
&= \int_0^1 \frac{(1-x-x^5+x^6)\ln x}{1-x^8}\,dx.
\end{aligned}$$

Now, introduce the geometric series

$$\frac{1}{1-x^8} = \sum_{n=0}^\infty x^{8n}$$

to obtain

$$\begin{aligned}
&\int_0^\infty \frac{\ln x}{(1+x)(1+x^2)(1+x^4)}\,dx\\
&= \int_0^1 \sum_{n=0}^\infty \left(x^{8n} - x^{8n+1} - x^{8n+5} + x^{8n+6}\right)\ln x\,dx\\
&= \sum_{n=0}^\infty \int_0^1 \left(x^{8n} - x^{8n+1} - x^{8n+5} + x^{8n+6}\right)\ln x\,dx\\
&= -\sum_{n=0}^\infty \left(\frac{1}{(8n+1)^2} - \frac{1}{(8n+2)^2} - \frac{1}{(8n+6)^2} + \frac{1}{(8n+7)^2}\right)\\
&= -\frac{1}{64}\sum_{n=0}^\infty \left(\frac{1}{(n+\frac{1}{8})^2} - \frac{1}{(n+\frac{1}{4})^2} - \frac{1}{(n+\frac{3}{4})^2} + \frac{1}{(n+\frac{7}{8})^2}\right)\\
&= -\left(\psi_1\left(\frac{1}{8}\right) + \psi_1\left(\frac{7}{8}\right) - \psi_1\left(\frac{1}{4}\right) - \psi_1\left(\frac{3}{4}\right)\right),
\end{aligned}$$

where $\psi_1(x)$ is the trigamma function. By the reflection formula for the trigamma function,

$$\psi_1(1-z)+\psi_1(z)=-\pi\frac{d}{dz}\cot\pi z=\pi^2\csc^2\pi z,$$

it follows that

$$\psi_1\left(\frac{1}{4}\right)+\psi_1\left(\frac{3}{4}\right)=\pi^2\csc^2\frac{\pi}{4}=2\pi^2$$

and

$$\psi_1\left(\frac{1}{8}\right)+\psi_1\left(\frac{7}{8}\right)=\pi^2\csc^2\frac{\pi}{8}.$$

But

$$\sin^2\frac{\pi}{8}=\frac{1-\frac{\sqrt{2}}{2}}{2}=\frac{2-\sqrt{2}}{4},$$

so

$$\csc^2\frac{\pi}{8}=\frac{4}{2-\sqrt{2}}=4+2\sqrt{2}.$$

Finally,

$$\int_0^\infty\frac{\ln x}{(1+x)(1+x^2)(1+x^4)}\,dx=-\frac{\pi^2}{64}(4+2\sqrt{2}-2)=-\frac{(1+\sqrt{2})\pi^2}{32}.$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

A.

$$\int_0^\infty\frac{\ln x}{(x+1)(x^2+1)(x^4+1)}dx$$

$$=\int_0^1\frac{\ln x}{(x+1)(x^2+1)(x^4+1)}dx+\int_1^\infty\frac{\ln x}{(x+1)(x^2+1)(x^4+1)}dx$$

$$=\int_0^1\frac{(1-x^5)\ln x}{(x+1)(x^2+1)(x^4+1)}dx$$

$$=\frac{1}{2}\int_0^1\left(\frac{1}{1+x}-\frac{x}{1+x^2}+\frac{1-x^2}{1+x^4}\right)\ln x\ dx.$$

We have

$$\int_0^\infty\frac{\ln x}{1+x}dx=\sum_{k=0}^\infty(-1)^k\int_0^\infty x^k\ln x\ dx$$

$$= -\sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^2} = -\left(1 - \frac{1}{2}\right)\zeta(2) = -\frac{\pi^2}{12},$$

$$\int_0^1 \frac{x\ln x}{1+x^2}dx = \sum_{k=0}^{\infty} (-1)^k \int_0^{\infty} x^{2k+1}\ln x\ dx = -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+2)^2} = -\frac{\pi^2}{48},$$

$$\int_0^1 \frac{1-x^2}{1+x^4}\ln x\ dx = \sum_{k=0}^{\infty} (-1)^k \int_0^{\infty} \left(x^{4k} - x^{4k+2}\right)\ln x\ dx$$

$$= -\sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+1)^2} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(4k+3)^2}$$

$$= -\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(4k+1)^2} = -\frac{\pi^2}{8\sqrt{2}},$$

since differentiation of the partial fraction decomposition of the cotangent yields

$$\frac{\pi^2}{\sin^2(\pi x)} = \sum_{k=-\infty}^{\infty} \frac{1}{(x+n)^2},$$

and so

$$-\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{(4k+1)^2} = -\sum_{k=-\infty}^{\infty} \frac{1}{(8k+1)^2} + \sum_{k=-\infty}^{\infty} \frac{1}{(8k+5)^2}$$

$$= -\frac{\pi^2}{64\sin^2(\pi x)}\bigg|_{x=\frac{1}{8}} + \frac{\pi^2}{64\sin^2(\pi x)}\bigg|_{x=\frac{5}{8}}$$

$$= -\frac{\pi^2}{16\left(2-\sqrt{2}\right)} + \frac{\pi^2}{16\left(2+\sqrt{2}\right)} = -\frac{\pi^2}{8\sqrt{2}}\ .$$

Finally

$$\int_0^{\infty} \frac{\ln x}{(x+1)(x^2+1)(x^4+1)}dx = -\frac{\pi^2}{32}\left(1+\sqrt{2}\right).$$

B.

We consider the function

$$f(s) = \int_0^{\infty} \frac{x^s}{(x+1)(x^2+1)(x^4+1)}dx, \quad -1 < s < 6.$$

Chapter 5.3 of [1] deals with the evaluation of definite integrals of various types. Specifically section 4 of that chapter is devoted to the evalution of integrals of the form

$$\int_0^\infty x^\alpha R(x)dx,$$

where the exponent α is real and may be supposed to lie in the interval $0<\alpha<1$, and R(z) is a rational function that has a zero of at least order two at ∞, at most a simple pole at the origin and no poles on the positive real axis. Let $A=\left\{e^{i\pi}, e^{\frac{i\pi}{2}}, e^{\frac{3i\pi}{2}}, e^{\frac{i\pi}{4}}, e^{\frac{3i\pi}{4}}, e^{\frac{5i\pi}{4}}, e^{\frac{7i\pi}{4}}\right\}$, then we find that

$$f(s)=\frac{2\pi i}{1-e^{2\pi is}}\sum_{\omega\in A}\operatorname{res}\left(\frac{x^s}{(x+1)(x^2+1)(x^4+1)};\, z=\omega\right) \tag{1}$$

$$=\frac{2\pi i}{1-e^{2\pi is}}\left(\frac{e^{\pi is}}{4}+\frac{e^{\pi is/2}}{(i+1)(2i)(2)}+\frac{e^{3\pi is/2}}{(-i+1)(-2i)(2)}\right. \tag{2}$$

$$-\frac{e^{\pi is/4}e^{i\pi/4}}{(e^{i\pi/4}+1)(i+1)(4)}-\frac{e^{3\pi is/4}e^{3i\pi/4}}{(e^{3i\pi/4}+1)(-i+1)(4)} \tag{3}$$

$$\left.-\frac{e^{5\pi is/4}e^{5i\pi/4}}{(e^{5i\pi/4}+1)(i+1)(4)}-\frac{e^{7\pi is/4}e^{7i\pi/4}}{(e^{7i\pi/4}+1)(-i+1)(4)}\right) \tag{4}$$

$$=\frac{\pi}{16}\sqrt{2-\sqrt{2}}\frac{1}{\sin\left(\frac{\pi}{4}+\frac{\pi s}{8}\right)\sin\left(\frac{\pi}{8}+\frac{\pi s}{8}\right)}.$$

Then

$$\int_0^\infty \frac{\ln x}{(x+1)(x^2+1)(x^4+1)}dx=f'(0)$$

$$=\frac{d}{ds}\left(\frac{\pi}{16}\sqrt{2-\sqrt{2}}\frac{1}{\sin\left(\frac{\pi}{4}+\frac{\pi s}{8}\right)\sin\left(\frac{\pi}{8}+\frac{\pi s}{8}\right)}\right)\Bigg|_{s=0}=-\frac{\pi^2}{32}\left(1+\sqrt{2}\right),$$

as is seen after some tedious calculations.

References

[1] Lars V. Ahlfors, Complex Analysis, McGraw-Hill Book Company, second edition, 1966.

Solution 4 by the proposer. Let us denote:

$$I = \int_0^\infty \frac{\ln x}{(x+1)(x^2+1)(x^4+1)}dx;$$

$$A = \int_0^1 \frac{\ln x}{(x+1)(x^2+1)(x^4+1)}dx = \int_0^1 \frac{(1-x)\ln x}{1-x^8}dx$$

$$B = \int_1^\infty \frac{\ln x}{(x+1)(x^2+1)(x^4+1)}dx = \int_1^\infty \frac{(1-x)\ln x}{1-x^8}dx$$

We consider the integral A. We have:

$$A = \int_0^1 \frac{\ln x}{1-x^8}dx - \int_0^1 \frac{x\ln x}{1-x^8}dx$$

We will calculate:

$$A_1 = \int_0^1 \frac{\ln x}{1-x^8}dx.$$

Making the change of variable $x^8 = y;\ x = y^{\frac{1}{8}}$ We obtain:

$$A_1 = \frac{1}{64}\int_0^1 \frac{y^{-\frac{7}{8}}\ln y}{1-y}dy$$

We will use the following known relationship:

$$\int_0^1 \frac{z^a \ln z}{1-z}dz = -\psi_1(a+1), \quad a \in \mathbb{R} \setminus \{-1,-2\ldots\}$$

where $\psi_1(x)$ is the trigamma function. We obtain:

$$A_1 = -\frac{1}{64}\psi_1\left(\frac{1}{8}\right)$$

We will calculate

$$A_2 = \int_0^1 \frac{x\ln x}{1-x^8}dx$$

Proceeding similarly, we obtain:

$$A_2 = -\frac{1}{64}\psi_1\left(\frac{2}{8}\right)$$

We obtain the value of the integral A:

$$A = A_1 - A_2 = \frac{1}{64}\left[-\Psi_1\left(\frac{1}{8}\right) + \psi_1\left(\frac{2}{8}\right)\right]$$

We consider the integral B. We make the variable change: $x = \frac{1}{t}$. We obtain:

$$B = \int_0^1 \frac{(t-1)t^5 \ln t}{1-t^8} dt$$

By proceeding similarly to the integral A, we obtain:

$$B = \frac{1}{64}\left[\psi_1\left(\frac{6}{8}\right) - \psi_1\left(\frac{7}{8}\right)\right]$$

Result:

$$I = A + B$$

$$I = \frac{1}{64}\left[-\psi_1\left(\frac{1}{8}\right) + \psi_1\left(\frac{2}{8}\right) + \psi_1\left(\frac{6}{8}\right) - \psi_1\left(\frac{7}{8}\right)\right]$$

We use the reflection formula:

$$\psi_1(x) + \psi_1(1-x) = \frac{\pi^2}{\sin^2(\pi x)}$$

We obtain:

$$\psi_1\left(\frac{1}{8}\right) + \psi_1\left(\frac{7}{8}\right) = \frac{\pi^2}{\sin^2 \frac{\pi}{8}}$$

We have:

$$\sin\frac{\pi}{8} = \frac{1}{2}\sqrt{2-\sqrt{2}}$$

Result:

$$\psi_1\left(\frac{1}{8}\right) + \psi_1\left(\frac{7}{8}\right) = 2\left(2+\sqrt{2}\right)\pi^2$$

The following special values of the trigamma functions are known:

$$\Psi_1\left(\frac{1}{4}\right) = \pi^2 + 8G; \psi_1\left(\frac{3}{4}\right) = \pi^2 - 8G$$

where G is Catalan's constant Result:

$$I = -\frac{1}{32}\pi^2\left(1+\sqrt{2}\right)$$

Thus, the problem is solved.

Solution 5 by Michel Bataille, Rouen, France. We show that the given integral I is equal to $-\frac{\pi^2(1+\sqrt{2})}{32}$. Indeed, we have

$$I = \int_0^\infty \frac{(x-1)\ln x}{x^8-1}\,dx = \int_0^1 \frac{(1-x)\ln x}{1-x^8}\,dx + \int_1^\infty \frac{(x-1)\ln x}{x^8-1}\,dx$$

or, after the change of variables $x = \frac{1}{u}$ in the rightmost integral,

$$I = \int_0^1 \frac{(1-x)\ln x}{1-x^8}\,dx - \int_0^1 \frac{x^5(1-x)\ln x}{1-x^8}\,dx.$$

Now, the substitution $x = t^{1/8}$ gives

$$\begin{aligned}\int_0^1 \frac{(1-x)\ln x}{1-x^8}\,dx &= \frac{1}{64}\left(\int_0^1 \frac{t^{-7/8}\ln t}{1-t}\,dt - \int_0^1 \frac{t^{-3/4}\ln t}{1-t}\,dt\right)\\ &= \frac{1}{64}(\psi_1(1/4) - \psi_1(1/8))\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \frac{x^5(1-x)\ln x}{1-x^8}\,dx &= \frac{1}{64}\left(\int_0^1 \frac{t^{-1/4}\ln t}{1-t}\,dt - \int_0^1 \frac{t^{-1/8}\ln t}{1-t}\,dt\right)\\ &= \frac{1}{64}(\psi_1(7/8) - \psi_1(3/4))\end{aligned}$$

where ψ_1 denotes the trigamma function defined by

$$\psi_1(z) = -\int_0^1 \frac{t^{z-1}\ln t}{1-t}\,dt.$$

From the known relation $\psi_1(z) + \psi_1(1-z) = \frac{\pi^2}{\sin^2(\pi z)}$, we are led to

$$\begin{aligned} I &= \frac{1}{64}((\psi_1(1/4) + \psi_1(3/4)) - (\psi_1(1/8) + \psi_1(7/8))) \\ &= \frac{1}{64}\left(\frac{\pi^2}{\sin^2(\pi/4)} - \frac{\pi^2}{\sin^2(\pi/8)}\right) \\ &= \frac{\pi^2}{64}\big(2 - 2(2+\sqrt{2})\big) = -\frac{\pi^2(1+\sqrt{2})}{32}. \end{aligned}$$

Solution 6 by Moti Levy, Rehovot, Israel. We have

$$\frac{1}{(1+x)(1+x^2)(1+x^4)} = \frac{1-x}{1-x^8},$$

so that,

$$I = \int_0^\infty \frac{(1-x)\ln(x)}{1-x^8}\,dx.$$

$$I = \int_1^\infty \frac{(1-x)\ln(x)}{1-x^8}\,dx + \int_0^1 \frac{(1-x)\ln(x)}{1-x^8}\,dx.$$

$$\int_1^\infty \frac{(1-x)\ln(x)}{1-x^8}\,dx \overset{x\to\frac{1}{u}}{=} \int_0^1 \frac{u^5(u-1)\ln(u)}{1-u^8}du$$

It follows that

$$I = \int_0^1 \frac{(1-x-x^5+x^6)\ln x}{1-x^8}\,dx.$$

Now we expand $\frac{1}{1-x^8} = \sum_{n=0}^\infty x^{8n}$ and integrate term by term using

$$\int_0^1 x^\alpha \ln x\,dx = -1/(\alpha+1)^2.$$

$$I = \sum_{n=0}^\infty\left(-\frac{1}{(8n+1)^2} + \frac{1}{(8n+2)^2} + \frac{1}{(8n+6)^2} - \frac{1}{(8n+7)^2}\right).$$

The trigamma function is defined as $\psi_1(z) = \sum_{n=0}^{\infty} \frac{1}{(n+z)^2}$. By this definition,

$$I = \frac{1}{64}\left(-\psi_1(\tfrac{1}{8}) + \psi_1(\tfrac{1}{4}) + \psi_1(\tfrac{3}{4}) - \psi_1(\tfrac{7}{8})\right).$$

By the reflection formula $\psi_1(1-z) + \psi_1(z) = \frac{\pi^2}{\sin^2(\pi z)}$ we obtain,

$$\begin{aligned} I &= \frac{1}{64}\left(\psi_1(\tfrac{1}{4}) + \psi_1(\tfrac{3}{4}) - \psi_1(\tfrac{1}{8}) - \psi_1(\tfrac{7}{8}))\right) \\ &= \frac{\pi^2}{64}\left(\frac{1}{\sin^2(\frac{\pi}{4})} - \frac{1}{\sin^2(\frac{\pi}{8})}\right) = \frac{\pi^2}{64}\left(2 - \frac{1}{\sin^2(\frac{\pi}{8})}\right) \end{aligned}$$

Finally, $\sin^2(\pi/8) = \frac{2-\sqrt{2}}{4}$,

$$I = -\frac{\pi^2}{32}(1+\sqrt{2}) \cong -0.7446$$

Solution 7 by Joe Santmyer, Las Cruces, New Mexico, USA. *Mathematica* can find the exact value of I and produces $I = -\frac{1}{32}(1+\sqrt{2})\pi^2$. The derivation below justifies this answer.

Now

$$\begin{aligned} I &= \int_0^1 \frac{\ln(x)}{(x+1)(x^2+1)(x^4+1)}dx + \int_1^{\infty} \frac{\ln(x)}{(x+1)(x^2+1)(x^4+1)}dx \\ &= I_1 + I_2 \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_0^1 \frac{\ln(x)}{(x+1)(x^2+1)(x^4+1)}dx \\ I_2 &= \int_1^{\infty} \frac{\ln(x)}{(x+1)(x^2+1)(x^4+1)}dx. \end{aligned}$$

Consider I_2. Let $u = \frac{1}{x}$ then $x = \frac{1}{u}$. If $x = 1$ then $u = 1$ and if $x = \infty$ then $u = 0$. Also, $\frac{du}{dx} = -x^{-2}$, that is, $-x^2 du = dx$, that is, $-\frac{du}{u^2} = dx$. This subsitution yields

$$I_2 = -\int_0^1 \frac{u^5 \ln(u)}{(1+u)(1+u^2)(1+u^4)}du.$$

Hence

$$\begin{aligned}
I &= \int_0^1 \frac{\ln(x)}{(x+1)(x^2+1)(x^4+1)}dx - \int_0^1 \frac{x^5\ln(x)}{(1+x)(1+x^2)(1+x^4)}dx \\
&= \int_0^1 \frac{(1-x^5)\ln(x)}{(x+1)(x^2+1)(x^4+1)}dx.
\end{aligned}$$

Partial fractions gives

$$\begin{aligned}
I &= \int_0^1 \ln(x)\left(\frac{1}{2(x+1)} + \frac{1}{2(x^4+1)} - \frac{x^2}{2(x^4+1)} - \frac{x}{2(x^2+1)}\right)dx \\
I &= \frac{1}{2}\left[\int_0^1 \ln(x)\left(\frac{1}{x+1} + \frac{1}{x^4+1} - \frac{x^2}{x^4+1} - \frac{x}{x^2+1}\right)dx\right] \\
2I &= \int_0^1 \ln(x)\left(\frac{1}{x+1} + \frac{1}{x^4+1} - \frac{x^2}{x^4+1} - \frac{x}{x^2+1}\right)dx \\
&= \int_0^1 \frac{\ln(x)}{x+1}dx + \int_0^1 \frac{(1-x^2)\ln(x)}{x^4+1}dx - \int_0^1 \frac{x\ln(x)}{x^2+1}dx \\
&= J_1 + J_2 - J_3
\end{aligned}$$

where

$$\begin{aligned}
J_1 &= \int_0^1 \frac{\ln(x)}{x+1}dx \\
J_2 &= \int_0^1 \frac{(1-x^2)\ln(x)}{x^4+1}dx \\
J_3 &= \int_0^1 \frac{x\ln(x)}{x^2+1}dx.
\end{aligned}$$

By 15.91 on p 99 in [2] we have $J_1 = -\frac{\pi^2}{12}$.

Mathematica can find the exact value of J_2 and produces $J_2 = -\frac{\pi^2\sqrt{2}}{16}$. We will now justify this result. Let $u = \frac{1}{x}$ then $x = \frac{1}{u}$. If $x = 0$ then $u = \infty$ and if $x = 1$ then $u = 1$. Also, $\frac{du}{dx} = -x^{-2}$, that is, $-x^2du = dx$, that is, $-\frac{du}{u^2} = dx$. This substitution gives

$$J_2 = \int_1^\infty \frac{(1-u^2)\ln(u)}{1+u^4}du.$$

Hence

$$\begin{aligned} 2J_2 &= \int_0^1 \frac{(1-x^2)\ln(x)}{1+x^4}dx + \int_1^\infty \frac{(1-x^2)\ln(x)}{1+x^4}dx \\ &= \int_0^\infty \frac{(1-x^2)\ln(x)}{1+x^4}dx. \end{aligned}$$

To evaluate J_2 residue theory from complex analysis is applied. Now

$$\begin{aligned} 2J_2 &= \frac{1}{2}\int_{-\infty+i0}^{\infty+i0} \frac{(1-x^2)\log(x)}{1+x^4}dx \\ 4J_2 &= \int_{-\infty+i0}^{\infty+i0} \frac{(1-x^2)\log(x)}{1+x^4}dx \end{aligned}$$

where log is the principal branch of the complex logarithm. By the residue theorem, see for example integral type 1 in table 4.2 on p 253 in [1], we get

$$\begin{aligned} 4J_2 &= \int_{-\infty}^{\infty} f(x)dx \\ &= 2\pi i \sum_{\mathrm{Im}(z)>0} \mathrm{res}(f,z) \end{aligned}$$

where $f(z) = \frac{(1-z^2)\log(z)}{1+z^4}$, $\mathrm{res}(f,z)$ is the residue of f at z and the sum is taken over all poles z of f in the upper half complex plane. Function f has 4 simple poles at $z_1 = e^{\frac{\pi i}{4}}$, $z_2 = e^{\frac{3\pi i}{4}}$, $z_3 = e^{\frac{5\pi i}{4}}$ and $z_4 = e^{\frac{7\pi i}{4}}$. Only z_1 and z_2 are in the upper half plane. Hence

$$4J_2 = 2\pi i[\mathrm{res}(f,z_1) + \mathrm{res}(f,z_2)].$$

A standard result such as theorem 2 on p 206 in [1] is used to calculate each residue. Now $f(z) = \frac{(1-z^2)\log(z)}{1+z^4} = \frac{g(z)}{h(z)}$ where $g(z) = (1-z^2)\log(z)$, $h(z) = 1+z^4$ and $h'(z) = 4z^3$. By theorem

2 on page 206 in [1] we get

$$\begin{aligned}
\operatorname{res}(f, z_1) &= \frac{g(z_1)}{h'(z_1)} \\
&= \frac{(1-z_1^2)\log(z_1)}{4z_1^3} \\
&= \frac{\left(1-e^{\frac{\pi i}{2}}\right)\log(e^{\frac{\pi i}{4}})}{4e^{\frac{3\pi i}{4}}} \\
&= \frac{-i\pi\sqrt{2}}{16}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\operatorname{res}(f, z_2) &= \frac{g(z_2)}{h'(z_2)} = \frac{(1-z_2^2)\log(z_2)}{4z_2^3} \\
&= \frac{\left(1-e^{\frac{3\pi i}{2}}\right)\log(e^{\frac{3\pi i}{4}})}{4e^{\frac{9\pi i}{4}}} = \frac{3\sqrt{2}\pi i}{16}.
\end{aligned}$$

Hence

$$\begin{aligned}
4J_2 &= 2\pi i\left[\frac{-i\pi\sqrt{2}}{16} + \frac{3\sqrt{2}\pi i}{16}\right] \\
4J_2 &= 2\pi i\left[\frac{2\sqrt{2}\pi i}{16}\right] \\
4J_2 &= -\frac{\pi^2\sqrt{2}}{4} \\
J_2 &= -\frac{\pi^2\sqrt{2}}{16}.
\end{aligned}$$

Consider J_3. Let $v = \frac{1}{2}\ln(x^2+1)$ then $v' = \dfrac{1}{2}\left[\dfrac{2x}{x^2+1}\right] = \dfrac{x}{x^2+1}$.

Let $u = \ln(x)$ then $u' = \frac{1}{x}$. Integration by parts gives

$$\begin{aligned}
J_4 &= \frac{1}{2}\Big[\ln(x^2+1)\ln(x)\Big]_{x=0}^{1} - \frac{1}{2}\int_0^1 \frac{\ln(x^2+1)}{x}dx \\
&= \frac{1}{2}\Big[\lim_{x\to 1}\ln(x^2+1)\ln(x) - \lim_{x\to 0}\ln(x^2+1)\ln(x)\Big] - \frac{1}{2}\int_0^1 \frac{\ln(x^2+1)}{x}dx \\
&= \frac{1}{2}[0-0] - \frac{1}{2}\int_0^1 \frac{\ln(x^2+1)}{x}dx = -\frac{1}{2}\int_0^1 \frac{\ln(x^2+1)}{x}dx \\
&= -\frac{1}{2}\int_0^1 \frac{1}{x}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}x^{2k}}{k}dx = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\int_{x=0}^{1} x^{2k-1}dx \\
&= \frac{1}{4}\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2} = \frac{1}{4}\mathrm{Li}_2(-1) = \frac{1}{4}\left[-\frac{\pi^2}{12}\right] \\
&= -\frac{\pi^2}{48}
\end{aligned}$$

where Li_2 is the dilogarithm function. Hence

$$\begin{aligned}
2I &= J_1 + J_2 - J_3 \\
2I &= -\frac{\pi^2}{12} - \frac{\pi^2\sqrt{2}}{16} + \frac{\pi^2}{48} \\
2I &= -\frac{\pi^2}{16} - \frac{\pi^2\sqrt{2}}{16} \\
2I &= -\frac{\pi^2(1+\sqrt{2})}{16} \\
I &= -\frac{\pi^2(1+\sqrt{2})}{32}.
\end{aligned}$$

References

[1] Marsden, J. E., “Basic Complex Analysis”, Freeman and Co., 1973.

[2] Spiegel, M. R., “Mathematical Handbook", Schaum’s Outline Series, McGraw-Hill Book Company, 1968.

A–141. *Proposed by Joseph Santmyer, Las Cruces, New Mexico, USA.* The n^{th} harmonic number H_n is defined as $H_n = \sum\limits_{k=1}^{n} \frac{1}{k}$ and the generalized harmonic number of order m is defined as $H_{n,m} = \sum\limits_{k=1}^{n} \frac{1}{k^m}$. Show that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^2}{2^n} &= \zeta(2) + \ln^2(2) && (1)\\
\sum_{n=1}^{\infty} \frac{H_{n,2}}{2^n} &= \zeta(2) - \ln^2(2) && (2)\\
\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} &= \frac{\zeta(2)}{2} - \ln^2(2) && (3)
\end{aligned}$$

where ζ is the Riemann zeta function.

Solution 1 by the proposer. By either (4.6) on p 284 in [3] or the website Harmonic Number we know that

$$\sum_{n=1}^{\infty} z^n H_{n,2} = \frac{\mathrm{Li}_2(z)}{1-z}$$

where Li_2 is the dilogarithm function. Let $z = \frac{1}{2}$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_{n,2}}{2^n} &= \frac{\mathrm{Li}_2\left(\frac{1}{2}\right)}{1-\frac{1}{2}}\\
&= 2\mathrm{Li}_2\left(\frac{1}{2}\right).
\end{aligned}$$

By the website Dilogarithm Function we get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_{n,2}}{2^n} &= 2\left(\frac{\pi^2}{12} - \frac{1}{2}\ln^2(2)\right)\\
&= \frac{\pi^2}{6} - \ln^2(2)\\
&= \zeta(2) - \ln^2(2).
\end{aligned}$$

This establishes (2).

Again, by the website Harmonic Number we have

$$\begin{aligned}
H_{n,2} &= \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} + \frac{H_n}{n} \\
\sum_{n=1}^{\infty} \frac{H_{n,2}}{2^n} &= \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} + \sum_{n=1}^{\infty} \frac{H_n}{n2^n} \\
\zeta(2) - \ln^2(2) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} + \frac{\pi^2}{12} \\
&= \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} + \frac{\zeta(2)}{2} \\
\frac{\zeta(2)}{2} - \ln^2(2) &= \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)}.
\end{aligned}$$

This establishes (3).

Consider (1). Here are two ways to justify (1).

First, apply (4.7) on p 284 in [3], namely

$$\sum_{n=1}^{\infty} x^n H_n^2 = \frac{\ln^2(1-x) + \mathrm{Li}_2(x)}{1-x}$$

for $|x| < 1$. Let $x = \frac{1}{2}$ to get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n^2}{2^n} &= \frac{\ln^2\left(1-\frac{1}{2}\right) + \mathrm{Li}_2\left(1-\frac{1}{2}\right)}{1-\frac{1}{2}} \\
&= 2\ln^2(2) + 2\mathrm{Li}_2\left(\frac{1}{2}\right) \\
&= 2\left[\ln^2(2) + \frac{\pi^2}{12} - \frac{1}{2}\ln^2(2)\right] \\
&= 2\left[\frac{1}{2}\ln^2(2) + \frac{\zeta(2)}{2}\right] \\
&= \ln^2(2) + \zeta(2).
\end{aligned}$$

This etsablishes (1).

Second, consider

$$S_n = \sum_{k=1}^{n} \binom{n}{k} \frac{(-1)^{k+1}}{k^2}.$$

Using the Pascal triangle formula $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ it is relatively easy to show that $S_n = S_{n-1} + \frac{H_n}{n}$. The details are left to the reader. Now $S_1 = 1$ and $S_n = 0$ for $n \leq 0$. Let $S = \sum\limits_{n=1}^{\infty} \frac{S_n}{2^n}$. Then

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} \frac{S_n}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n}\left[S_{n-1} + \frac{H_n}{n}\right] = \sum_{n=1}^{\infty} \frac{S_{n-1}}{2^n} + \sum_{n=1}^{\infty} \frac{H_n}{n2^n} \\
&= \frac{1}{2}\sum_{n=1}^{\infty} \frac{S_{n-1}}{2^{n-1}} + \frac{\zeta(2)}{2} = \frac{1}{2}\sum_{n=0}^{\infty} \frac{S_n}{2^n} + \frac{\zeta(2)}{2} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{S_n}{2^n} + \frac{\zeta(2)}{2} \\
&= \frac{1}{2}S + \frac{\zeta(2)}{2} \\
\frac{1}{2}S &= \frac{\zeta(2)}{2} \implies S = \zeta(2).
\end{aligned}$$

By the recurrence formula $S_n = S_{n-1} + \frac{H_n}{n}$ we have $S_n = \sum\limits_{k=1}^{n} \frac{H_k}{k}$. By an example on p 850 in [2] or formula (1.30) in [1] we have

$$S_n = \sum_{k=1]}^{n} \frac{H_k}{k} = \frac{H_n^2 + H_{n,2}}{2}.$$

Consequently

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} \frac{1}{2^n}\left[\frac{H_n^2 + H_{n,2}}{2}\right] \\
\zeta(2) &= \sum_{n=1}^{\infty} \frac{H_n^2}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{H_{n,2}}{2^{n+1}} \\
2\zeta(2) &= \sum_{n=1}^{\infty} \frac{H_n^2}{2^n} + \sum_{n=1}^{\infty} \frac{H_{n,2}}{2^n} \\
2\zeta(2) &= \sum_{n=1}^{\infty} \frac{H_n^2}{2^n} + \zeta(2) - \ln^2(2) \\
\zeta(2) + \ln^2(2) &= \sum_{n=1}^{\infty} \frac{H_n^2}{2^n}.
\end{aligned}$$

This establishes (1).

References

[1] Choi, J., Srivastava, H., M., "Some summation formulas involving harmonic numbers and generalized harmonic numbers", *Mathematical and Computer Modelling*, Vol. 54, Issues 9-10, Nov 2011, pp 2220-2234.

[2] Spieß, J., Some identities involving harmonic numbers, *Math. of Comp.*, Vol. 55, No. 192, Oct. 1990, 839-803.

[3] Vălean, C., I., "(Almost) Impossible Integrals, Sums, and Series", Problem Books in Mathematics, Springer, 2019.

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA.

(i) $\sum_{n=1}^{\infty} \frac{H_{n,2}}{2^n}$:

The Cauchy product of

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{and} \quad \mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2},$$

where $\mathrm{Li}_2(x)$ denotes the dilogarithm function, yields

$$\sum_{n=0}^{\infty} H_{n,2} x^n = \frac{\mathrm{Li}_2(x)}{1-x}.$$

Then

$$\sum_{n=0}^{\infty} \frac{H_{n,2}}{2^n} = 2\mathrm{Li}_2\left(\frac{1}{2}\right).$$

Substituting $x = \frac{1}{2}$ into the reflection formula for the dilogarithm function,

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \zeta(2) - \ln x \ln(1-x),$$

yields

$$2\mathrm{Li}_2\left(\frac{1}{2}\right) = \zeta(2) - \ln^2(2);$$

thus

$$\sum_{n=1}^{\infty} \frac{H_{n,2}}{2^n} = \zeta(2) - \ln^2(2).$$

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)}$:

We start with

$$\begin{aligned}
\sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} &= \sum_{k=1}^{n-1}\left(\frac{H_k}{k} - \frac{H_k}{k+1}\right) = \sum_{k=1}^{n-1}\left(\frac{H_k}{k} - \frac{H_{k+1}}{k+1} + \frac{1}{(k+1)^2}\right) \\
&= 1 - \frac{H_n}{n} + \sum_{k=2}^{n} \frac{1}{k^2} = H_{n,2} - \frac{H_n}{n},
\end{aligned}$$

so

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} = \sum_{n=1}^{\infty} \frac{H_{n,2}}{2^n} - \sum_{n=1}^{\infty} \frac{H_n}{2^n n} = \zeta(2) - \ln^2(2) - \sum_{n=1}^{\infty} \frac{H_n}{2^n n}.$$

Now, from the generating function for the harmonic numbers,

$$\sum_{n=1}^{\infty} H_n x^n = -\frac{\ln(1-x)}{1-x},$$

it follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{H_n}{n} x^n &= -\int \frac{\ln(1-x)}{x(1-x)}\, dx = -\int \frac{\ln(1-x)}{x}\, dx - \int \frac{\ln(1-x)}{1-x}\, dx \\
&= \mathrm{Li}_2(x) + \frac{1}{2}\ln^2(1-x),
\end{aligned}$$

where the value of the constant of integration is determined to be zero by substituting $x = 0$. Thus,

$$\sum_{n=1}^{\infty} \frac{H_n}{2^n n} = \mathrm{Li}_2\left(\frac{1}{2}\right) + \frac{1}{2}\ln^2(2) = \frac{1}{2}\zeta(2),$$

and

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} = \frac{\zeta(2)}{2} - \ln^2(2).$$

(iii) $\sum_{n=1}^{\infty} \frac{H_n^2}{2^n}$:

Note

$$H_n^2 = \sum_{k=1}^{n} \frac{1}{k^2} + 2\sum_{i=2}^{n}\sum_{j=1}^{i-1} \frac{1}{ij} = H_{n,2} + 2\sum_{i=2}^{n} \frac{H_{i-1}}{i} = H_{n,2} + 2\sum_{i=1}^{n-1} \frac{H_i}{i+1}.$$

Upon integrating the generating function for the harmonic numbers from 0 to x, we obtain

$$\sum_{n=1}^{\infty} \frac{H_n}{n+1} x^{n+1} = \frac{1}{2}\ln^2(1-x),$$

from which it then follows that

$$\frac{\ln^2(1-x)}{1-x} = 2\sum_{n=1}^{\infty} \left(\sum_{i=1}^{n-1} \frac{H_i}{i+1}\right) x^n.$$

Thus,

$$\sum_{n=1}^{\infty} H_n^2 x^n = \frac{\mathrm{Li}_2(x) + \ln^2(1-x)}{1-x},$$

and

$$\sum_{n=1}^{\infty} \frac{H_n^2}{2^n} = 2\mathrm{Li}_2\left(\frac{1}{2}\right) + 2\ln^2(2) = \zeta(2) + \ln^2(2).$$

Solution 3 by Michel Bataille, Rouen, France. We will use the two following known results:

$$\sum_{n=1}^{\infty} \frac{H_n x^{n+1}}{n+1} = \frac{(\ln(1-x))^2}{2} \quad (x \in [0,1)); \qquad \sum_{n=1}^{\infty} \frac{1}{2^n n^2} = \frac{\pi^2}{12} - \frac{(\ln 2)^2}{2}.$$

(For completeness, a quick proof is given at the end.)

Let S_1 and S_2 be the sums to be calculated in (1) and (2), respectively. From the following identities (easily proved by induction on $n \geq 1$)

$$H_n^2 - H_{n,2} = \sum_{k=1}^{n-1} \frac{2H_k}{k+1}, \qquad H_n^2 + H_{n,2} = \sum_{k=1}^{n} \frac{2H_k}{k}$$

we deduce that

$$S_1 - S_2 = \sum_{k=1}^{\infty} \frac{H_k}{k+1} \sum_{n=k+1}^{\infty} \frac{1}{2^{n-1}} = 4 \sum_{k=1}^{\infty} \frac{H_k}{k+1} \cdot \frac{1}{2^{k+1}} = 4 \cdot \frac{(\ln 2)^2}{2} = 2(\ln 2)^2$$

and that

$$S_1 + S_2 = 2 \sum_{k=1}^{\infty} \frac{H_k}{k} \sum_{n=k}^{\infty} \frac{1}{2^n} = 4 \sum_{k=1}^{\infty} \frac{H_k}{k 2^k} = 2\zeta(2)$$

since

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{H_k}{k 2^k} &= \sum_{k=1}^{\infty} \frac{\frac{1}{k} + H_{k-1}}{k 2^k} \quad (\text{where } H_0 = 0) \\ &= \sum_{k=0}^{\infty} \frac{H_k}{(k+1) 2^{k+1}} + \sum_{k=1}^{\infty} \frac{1}{k^2 2^k} \\ &= \frac{(\ln 2)^2}{2} + \frac{\pi^2}{12} - \frac{(\ln 2)^2}{2} = \frac{\zeta(2)}{2}. \end{aligned}$$

The required results (1) and (2) follow immediately.
As for (3), we use the following identity (again easily checked by induction):

$$\sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} = H_{n,2} - \frac{H_n}{n}.$$

which leads to

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} = S_2 - \sum_{n=1}^{\infty} \frac{H_n}{n 2^n} = S_2 - \frac{\zeta(2)}{2} = \frac{\zeta(2)}{2} - \ln^2(2).$$

Proofs of the results mentioned at the beginning.

1/Let $f(x) = \sum_{n=1}^{\infty} \frac{H_n}{n+1} x^{n+1}$. Then $f'(x) = \sum_{n=1}^{\infty} H_n x^n$, hence

$$(1-x) f'(x) = x + \sum_{n=2}^{\infty} (H_n - H_{n-1}) x^n = x + \sum_{n=2}^{\infty} \frac{x^n}{n} = -\ln(1-x)$$

from which we first deduce $f'(x) = \left(\frac{-1}{1-x}\right) \ln(1-x)$ and then, since $f(0) = 0,\ f(x) = \frac{1}{2}[\ln(1-x)]^2$.

2/ From $\frac{-\ln(1-x)}{x}=\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}$, we deduce

$$\int_0^1 \frac{-\ln(1-x)}{x}\,dx=\zeta(2)=\frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty}\frac{1}{2^n n^2}=\int_0^{1/2}\frac{-\ln(1-x)}{x}\,dx.$$

Using integration by parts,

$$\int_0^{1/2}\frac{\ln(1-x)}{x}\,dx=(\ln 2)^2+\int_0^{1/2}\frac{\ln x}{1-x}\,dx=(\ln 2)^2+\int_{1/2}^1\frac{\ln(1-u)}{u}\,du,$$

hence

$$2\int_0^{1/2}\frac{\ln(1-x)}{x}\,dx=(\ln 2)^2+\int_0^1\frac{\ln(1-x)}{x}\,dx=(\ln 2)^2-\frac{\pi^2}{6}$$

and the result follows.

Solution 4 by Moti Levy, Rehovot, Israel. We will use the machinery of Ordinary Generating Function (OGF) to get somewhat more general results. A good reference is "*Concrete Mathematics*" by Graham, Knut, Patashnik.

Properties of OGF that we use are: Let $P(z):=\sum_{n=1}^{\infty}p_n z^n$ be the OGF of the sequence $(p_n)_{n\geq 1}$, then

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}p_k\right)z^n=\frac{P(z)}{1-z}, \qquad \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n-1}p_k\right)z^n=\frac{zP(z)}{1-z}, \tag{4}$$

$$\sum_{n=1}^{\infty}\frac{p_n}{n}z^n=\int_0^z\frac{P(t)}{t}dt, \tag{5}$$

$$\sum_{n=1}^{\infty}\frac{p_n}{n+1}z^n=\frac{1}{z}\int_0^z P(t)dt. \tag{6}$$

OGF of $(H_n)_{n\geq 1},\left(\frac{H_n}{n}\right)_{n\geq 1},\left(\frac{H_n}{n+1}\right)_{n\geq 1}$

The OGF of the sequence $\left(\frac{1}{n}\right)_{n\geq 1}$ is $-\ln(1-z)$. Then by (4), the OGF of $(H_n)_{n\geq 1}$ is:

$$F(z):=\sum_{n=1}^{\infty}H_n z^n=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{1}{k}\right)z^n=-\frac{\ln(1-z)}{1-z}. \tag{7}$$

By (5)

$$\sum_{n=1}^{\infty}\frac{H_n}{n}z^n=\int_0^z\frac{F(t)}{t}dt=\int_0^z-\frac{\ln(1-t)}{(1-t)t}dt=\mathrm{Li}_2(z)+\frac{1}{2}\ln^2(1-z). \tag{8}$$

By (6)

$$\sum_{n=1}^{\infty}\frac{H_n}{n+1}z^n=-\frac{1}{z}\int_0^z\frac{\ln(1-t)}{(1-t)}dt=\frac{\ln^2(1-z)}{2z}. \tag{9}$$

OGF of $\left(H_n^2\right)_{n\geq 1}$

Note that $H_{n-1}^2=\left(H_n-\frac{1}{n}\right)^2=H_n^2+\frac{1}{n^2}-2\frac{H_n}{n}$.

$$\sum_{n=1}^{\infty}H_{n-1}^2z^n=G(z)+\sum_{n=1}^{\infty}\frac{z^n}{n^2}-2\sum_{n=1}^{\infty}\frac{H_n}{n}z^n. \tag{10}$$

By definition $\sum_{n=1}^{\infty}\frac{z^n}{n^2}=\mathrm{Li}_2(z)$, and (10), we get:

$$\sum_{n=1}^{\infty}H_{n-1}^2z^n=G(z)+\mathrm{Li}_2(z)-2\left(\mathrm{Li}_2(z)+\frac{1}{2}\ln^2(1-z)\right).$$

Since $\sum_{n=1}^{\infty}H_{n-1}^2z^n=zG(z)$,

$$zG(z)=G(z)+\mathrm{Li}_2(z)-2\left(\mathrm{Li}_2(z)+\frac{1}{2}\ln^2(1-z)\right).$$

Solving for $G(z)$, we obtain

$$G(z):=\sum_{n=1}^{\infty}H_n^2z^n=\frac{1}{1-z}\left(\mathrm{Li}_2(z)+\ln^2(1-z)\right). \tag{11}$$

$$\mathrm{Li}_2\left(\frac{1}{2}\right)=\frac{\zeta(2)-\ln^2(2)}{2}$$

It follows that

$$\sum_{n=1}^{\infty}H_n^2\left(\frac{1}{2}\right)^n=\frac{1}{1-\frac{1}{2}}\left(\frac{\zeta(2)-\ln^2(2)}{2}+\ln^2\left(1-\frac{1}{2}\right)\right)=\zeta(2)+\ln^2(2).$$

OGF of $\left(H_{n,2}\right)_{n\geq 1}$

$$\sum_{n=1}^{\infty} H_{n,2} z^n = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k^2} \right) z^n = \frac{1}{1-z} \mathrm{Li}_2(z). \tag{12}$$

It follows that

$$\sum_{n=1}^{\infty} H_{n,2} \left(\frac{1}{2} \right)^n = \frac{1}{1-\frac{1}{2}} \mathrm{Li}_2\left(\frac{1}{2} \right) = \zeta(2) - \ln^2(2).$$

OGF of $\left(\sum_{k=1}^{n-1} \frac{H_k}{k(k+1)}\right)_{n\geq 1}$

Using partial sum of $\frac{1}{k(k+1)}$,

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} \right) z^n = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k}{k} \right) z^n - \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k}{k+1} \right) z^n.$$

By property (4) and (8),

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k}{k} \right) z^n = \frac{z}{1-z} \left(\mathrm{Li}_2(z) + \frac{1}{2} \ln^2(1-z) \right).$$

By property (4) and (9),

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k}{k+1} \right) z^n = \frac{z}{1-z} \frac{\ln^2(1-z)}{2z} = \frac{1}{2(1-z)} \ln^2(1-z).$$

$$\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} \right) z^n = \frac{z}{1-z} \mathrm{Li}_2(z) - \frac{1}{2} \ln^2(1-z).$$

It follows that

$$\begin{aligned}\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} \right) \left(\frac{1}{2} \right)^n &= \frac{\frac{1}{2}}{1-\frac{1}{2}} \frac{\zeta(2) - \ln^2(2)}{2} - \frac{1}{2} \ln^2\left(1 - \frac{1}{2} \right) \\ &= \frac{\zeta(2)}{2} - \ln^2(2).\end{aligned}$$

Solution 5 by Albert Stadler, Herrliberg, Switzerland. We recall a few fact about harmonic numbers and related function values.

By [1],

$$\sum_{n\geq 1} H_n^2 x^n = \frac{\ln^2(1-x) \ + Li_2(x)}{1-x}.$$

By [2],

$$Li_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{\ln^2(2)}{2}.$$

By [3],

$$\sum_{n\geq 1} H_{n,2} x^n = \frac{Li_2(x)}{1-x}.$$

By [4],

$$\sum_{n=1}^{\infty} \frac{H_n}{n2^n} = \frac{\pi^2}{12}.$$

So

$$\sum_{n=1}^{\infty} \frac{H_n^2}{2^n} = \frac{\ln^2\left(1-\frac{1}{2}\right) \ + Li_2\left(\frac{1}{2}\right)}{1-\frac{1}{2}} = 2\ln^2(2) \ + 2\left(\frac{\pi^2}{12} - \frac{\ln^2(2)}{2}\right)$$

$$= \zeta(2) + \ln^2(2) \ ,$$

$$\sum_{n=1}^{\infty} \frac{H_{n,2}}{2^n} = \frac{Li_2\left(\frac{1}{2}\right)}{1-\frac{1}{2}} = 2\left(\frac{\pi^2}{12} - \frac{\ln^2(2)}{2}\right) = \zeta(2) - \ln^2(2) \ .$$

We note that

$$\sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} = \sum_{k=1}^{n-1} \left(\frac{H_k}{k} - \frac{H_k}{k+1}\right) = \sum_{k=1}^{n-1} \left(\frac{H_k}{k} - \frac{H_{k+1}}{k+1} + \frac{1}{(k+1)^2}\right)$$

$$= 1 - \frac{H_n}{n} + H_{n,2} - 1 = H_{n,2} - \frac{H_n}{n}$$

and conclude that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} = \sum_{n=1}^{\infty} \frac{H_{n,2}}{2^n} - \sum_{n=1}^{\infty} \frac{H_n}{n2^n} = \zeta(2) - \ln^2(2) \ - \frac{\pi^2}{12}$$

$$= \frac{\zeta(2)}{2} - \ln^2(2) \ .$$

References

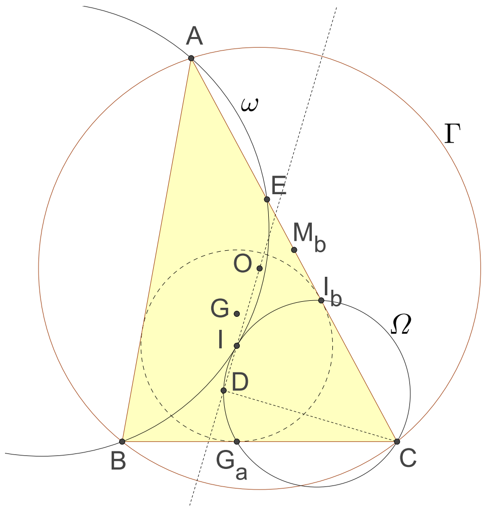
[1] https://math.stackexchange.com/questions/2531689/generating-function-of-square-harmonic-numbers

[2] https://en.wikipedia.org/wiki/Dilogarithm

[3] https://functions.wolfram.com/GammaBetaErf/HarmonicNumber2/11/

[4] https://en.wikipedia.org/wiki/Harmonic_number

A–142. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a non-right triangle with $AB \neq AC$ and let G be its centroid, I be its incenter and Γ be its circumcircle with center O. Let D be the foot of the perpendicular drawn from C to IO. Let G_a, I_b be the feet of the perpendiculars drawn from G to BC and from I to AC respectively. Let M_b be the midpoint of AC and E be the reflection of I_b in the point M_b. Knowing that I_bIDG_a is a cyclic quadrilateral, prove that $ABIE$ is a cyclic quadrilateral too.



Scheme for solving problem A-142.

Solution 1 by the proposer. Let Ω denote the circle with diameter CI. Since $\angle II_bC = \angle IDC = 90°$, the points $I_b, D \in \Omega$. But I_bIDG_a is a cyclic quadrilateral, hence $G_a \in \Omega$ and $IG_a \perp BC$. But $GG_a \perp BC$,so I, G, G_a are collinear.

Let $BC = a, CA = b, AB = c$ are the side lengths of the triangle ABC. Use barycentric coordinates relative to the vertices A, B, C; then $G = (1 : 1 : 1), I = (a : b : c)$

The lines $GI \perp BC$ if and only if $\overrightarrow{GI} \cdot \overrightarrow{BC} = 0$.

$$\begin{aligned}
\overrightarrow{AG} &= \frac{1}{3}\overrightarrow{AB} + \frac{1}{3}\overrightarrow{AC} \\
\overrightarrow{AI} &= \frac{b}{a+b+c}\overrightarrow{AB} + \frac{c}{a+b+c}\overrightarrow{AC} \\
\overrightarrow{GI} &= \overrightarrow{AI} - \overrightarrow{AG} = \frac{-a+2b-c}{3(a+b+c)}\overrightarrow{AB} + \frac{-a-b+2c}{3(a+b+c)}\overrightarrow{AC} \\
\overrightarrow{BC} &= -\overrightarrow{AB} + \overrightarrow{AC}
\end{aligned}$$

We have

$$\overrightarrow{AB} \cdot \overrightarrow{AB} = c^2; \quad \overrightarrow{AC} \cdot \overrightarrow{AC} = b^2$$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = bc\cos A = bc\frac{b^2+c^2-a^2}{2bc} = \frac{b^2+c^2-a^2}{2}$$

So

$$\begin{aligned}
0 = \overrightarrow{GI} \cdot \overrightarrow{BC} &= -\frac{-a+2b-c}{3(a+b+c)}c^2 + \frac{-a-b+2c}{3(a+b+c)}b^2 + \\
&+ \left(\frac{-a+2b-c}{3(a+b+c)} - \frac{-a-b+2c}{3(a+b+c)}\right)\frac{b^2+c^2-a^2}{2} = \\
&= \frac{1}{6}(3a-b-c)(c-b)
\end{aligned}$$

Since $b \neq c$, it follows that the condition I_bIDG_a be a cyclic quadrilateral is equivalent to

$$3a - b - c = 0 \tag{1}$$

Every circle can be represented by an equation of the form

$$a^2yz + b^2zx + c^2xy - (x+y+z)(px+qy+rz) = 0$$

The line $px + qy + rz = 0$ is the radical axis of this circle and the circumcircle of $\triangle ABC$.

Let ω denote the circle (ABI). The line AB is the radical axis of the circles Γ and ω. The equation of the line AB is $z=0$, so $p=0, q=0$ and

$$\omega : a^2yz + b^2zx + c^2xy - rz(x+y+z) = 0 \text{ for some scalar } r$$

Since $I=(a:b:c)\in\omega$, $x=a; y=b; z=c$ is a root of the equation above, or $c(a+b+c)(ab-r)=0$, $r=ab$

$$\omega : a^2yz + b^2zx + c^2xy - ab(x+y+z)z = 0$$

The point E is the reflection of I_b in the point M_b. So $AE = I_bC = \frac{a+b-c}{2}$, $\overrightarrow{AE} = \frac{a+b-c}{2b}\overrightarrow{AC}$, $E=(-a+b+c:0:a+b-c)$. Substituting the coordinates of E into the equation of ω, we get $b^2(a+b-c)(-3a+b+c)=0$. This is true under the condition (1).

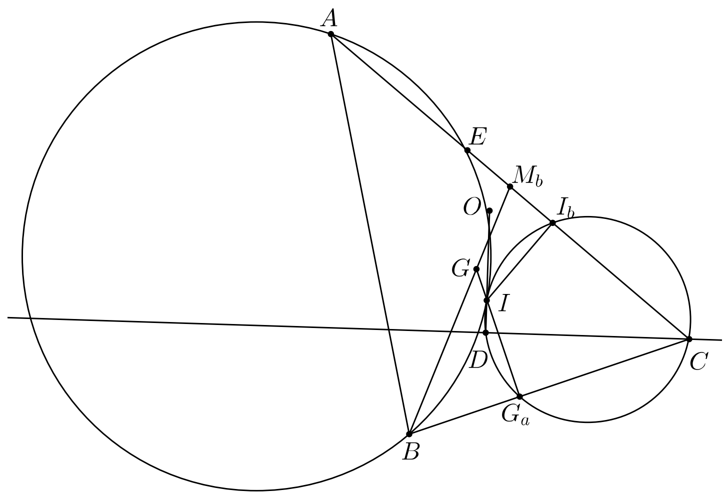
It follows that $E\in\omega$ if I_bIDG_a is a cyclic quadrilateral.

Solution 2 by Michel Bataille, Rouen, France. Let $BC=a, CA=b, AB=c$ and $s=\frac{a+b+c}{2}$, as usual. We prove that the conditions I_bIDG_a is a cyclic quadrilateral and $ABIE$ is a cyclic quadrilateral are both equivalent to $3a=b+c$.
We use barycentric coordinates relatively to (A,B,C). We know that

$$G=(1:1:1),\ M_b=(1:0:1),\ I=(a:b:c),\ I_b=(s-c:0:s-a),$$

and $H=(S_BS_C : S_CS_A : S_AS_B)$ where H is the orthocenter of ΔABC and $S_A=\frac{b^2+c^2-a^2}{2}, S_B=\frac{c^2+a^2-b^2}{2}, S_C=\frac{a^2+b^2-c^2}{2}$ (Conway's notations).
It follows that the foot of the altitude from A is $H_a=(0:S_C:S_B)$. If $M_a=(0:1:1)$ is the midpoint of BC, we have $\overrightarrow{M_aG}=\frac{1}{3}\overrightarrow{M_aA}$, hence $\overrightarrow{M_aG_a}=\frac{1}{3}\overrightarrow{M_aH_a}$ and therefore $6a^2G_a = 4a^2M_a + 2a^2H_a = 2a^2B + 2a^2C + (a^2+b^2-c^2)B + (c^2+a^2-b^2)C$ so that

$$G_a = (0 : 3a^2+b^2-c^2 : 3a^2-b^2+c^2).$$



Scheme for solving problem A-142.

This said, we observe that the circle with diameter CI passes through D and I_b, hence I_bIDG_a is a cyclic quadrilateral if and only if $IG_a \perp CG_a$, that is, if and only if I, G, G_a are collinear (since the lines CG_a coincides with BC). Since the equation of the line IG is $(b-c)x + (c-a)y + (a-b)z = 0$, the latter is equivalent to

$$(c-a)(3a^2 + b^2 - c^2) + (a-b)(3a^2 - b^2 + c^2) = 0.$$

A short calculation shows that this condition writes as

$$(c-b)(a+b+c)(3a-b-c) = 0,$$

hence I_bIDG_a is a cyclic quadrilateral if and only if $3a = b + c$.
On the other hand, from $\overrightarrow{M_bE} = \overrightarrow{I_bM_b}$ we obtain

$$bE = 2bM_b - bI_b = bA + bC - (s-c)A - (s-a)C = (s-a)A + (s-c)C,$$

hence $E = (s-a : 0 : s-c)$.
Since the equation of the circumcirle of ΔABI is

$$a^2yz + b^2zx + c^2xy - abz(x+y+z) = 0$$

(as it is readily checked), $ABIE$ is a cyclic quadrilateral if and only if $b^2(s-a)(s-c) - ab(s-c)(b)$, that is, if and only if $s - a = a$, or $b + c = 3a$. This completes the proof.

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