No. 1
2025

ARHIMEDE
MATHEMATICAL JOURNAL

Articles
Problems
Mathlessons

ARHIMEDE ASSOCIATION

CONTENTS

Solutions

Articles

Arhimede Mathematical Journal aims to publish interesting and attractive papers with elegant mathematical exposition. Articles should include examples, applications and illustrations, whenever possible. Manuscripts submitted should not be currently submitted to or accepted for publication in another journal.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

jose.luis.diaz@upc.edu

The half–form $\sqrt{dx}$

Simone Camosso

Abstract

In this brief paper we try to find an "interpretation" for the formalism $\sqrt{dx}$.

1 Introduction

The problem to make sense of $\int \sqrt{dx}$ is treated in this article. In what follows we explore a solution called "the corrected" integral, formally we denote it by ${}_{\gamma}\int_{[a,b]}\sqrt{dx}$. Some inspiration for this study can be drawn from Ramanujan [4] and his work on convergent series. The problem is approached from the point of view of the calculus, with some relations to the fractional calculus in Ross [6]. A well definition of the symbol $\sqrt{dx}$ can be useful in geometric quantization (Woodhouse [8]) and different areas of mathematics. In the contest of geometric quantization the symbol $\sqrt{dx}$ denotes a section of the quantum line bundle defined on a compact, complex, symplectic manifold M.

2 Integration of $\sqrt{dx}$ with a Riemannian integral

In this section we start to examine the situation evaluating the following integral:

$$\int_{[a,b]} \sqrt{dx}. \tag{1}$$

Let $\mathcal{P} = \{x_0 = a, x_1, \dots, x_n = b\}$ be a partition of the interval $[a, b]$ in n subintervals of amplitude $\frac{b-a}{n}$. Thus

$$\int_{[a,b]} \sqrt{dx} = \lim_{n\to+\infty} \sum_{i=1}^{n} \sqrt{\Delta x_i}, \tag{2}$$

where $\Delta x_i = x_i - x_{i-1}$ for $i = 1, \dots, n$. This is the Riemann integral where, instead to consider as "base of rectangles" the quantities Δx_i, we consider $\sqrt{\Delta x_i}$. The result is:

$$\begin{aligned}\int_{[a,b]} \sqrt{dx} &= \lim_{n\to+\infty} \sum_{i=1}^{n} \sqrt{\frac{b-a}{n}} = \\ &= \lim_{n\to+\infty} \sqrt{(b-a)n} = +\infty.\end{aligned} \tag{3}$$

The integral diverges at $+\infty$, the result is not satisfactory at all.

3 On a Ramanujan sum

In order to find a way to make this infinity "disappear", let us consider this result due to Ramanujan[4]:

$$\begin{aligned}&\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{n} = \\ &= \frac{2}{3} n\sqrt{n} + \frac{1}{2}\sqrt{n} - \frac{\zeta\left(\frac{3}{2}\right)}{4\pi} + \frac{1}{24\sqrt{n}} + o\left(\frac{1}{n^{\frac{5}{2}}}\right).\end{aligned} \tag{4}$$

This is an asymptotic expansion of the sum of the square roots of the first n natural numbers. The main term in the expansion, when n goes to infinity, is $\frac{2}{3} n^{\frac{3}{2}}$. Another similar series, always due to Ramanujan is:

$$\begin{aligned}&\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} = \\ &= 2\sqrt{n} + \frac{1}{2\sqrt{n}} + \zeta\left(\frac{1}{2}\right) + o\left(\frac{1}{n^{\frac{3}{2}}}\right).\end{aligned} \tag{5}$$

The idea can be to modify the sum of rectangles using the factor $\gamma(n) = \frac{1}{\sqrt{n}}$. In this case:

$$\begin{aligned} \int_{[a,b]} \sqrt{dx} &= \lim_{n\to+\infty} \gamma(n) \cdot \sum_{i=1}^{n} \sqrt{\frac{b-a}{n}} = \\ &= \sqrt{(b-a)} \lim_{n\to+\infty} \frac{\sqrt{n}}{\sqrt{n}} = \sqrt{b-a}. \end{aligned} \tag{6}$$

This is not the Riemannian integral used before but it is a variation with a normalized sum. The problem was in fact the divergence of the series of square roots. This sort of "correction" bring to a finite result that corresponds to the square root of the initial interval.

We denote this corrected integral with the notation:

$$ {}_{\gamma}\int_{[a,b]} \sqrt{dx} = \lim_{n\to+\infty} {}_{\gamma}\sum_{i=1}^{n} \sqrt{\Delta x_i}, \tag{7}$$

where ${}_{\gamma}\sum$ is the sum corrected by the factor $\gamma(n)$ and ${}_{\gamma}\int$ is the "corrected" integral.

4 The geometric quantization program and the definition of $\sqrt{dx}$

An attempt to define the half–form $\sqrt{dx}$ has been done in the program of geometric quantization. The geometric quantization is a process that associate to a symplectic manifold an Hilbert space representing the space of quantum states (for references about the geometric quantization program see Woodhouse [8] or works of Kostant [3] and Souriau [7]).

The idea is to see the half–form $\sqrt{dx}$ as a section of the square root of the canonical line bundle associated to the polarization adopted during the process of geometric quantization. The concept of $\frac{1}{2}$–form is strictly correlated to the concept of $\frac{1}{2}$–density. In particular we have that $\frac{1}{2}$–densities can be identified with $\frac{1}{2}$–forms. Let us consider the case of $\mathbb{R}$. The space $\mathbb{R}$ is a vectorial space where we can choose an orientation. If $F(\mathbb{R})$ represents the set of frames of $\mathbb{R}$ we can consider the action of $GL(1,\mathbb{R})$ on $F(\mathbb{R})$ that is simply the scalar multiplication. We define the set of $\frac{1}{2}$-densities as:

$$|\mathbb{R}|^{\frac{1}{2}} = \Big\{\nu : F(\mathbb{R}) \to \mathbb{C} : \nu(a \cdot v) = \nu(v)|\det a|^{\frac{1}{2}}, \\ \forall v \in F(\mathbb{R}), a \in GL(1,\mathbb{R})\Big\}. \tag{8}$$

This set is a line bundle denoted by $|\mathbb{R}|^{\frac{1}{2}} \to \mathbb{R}$ where the 1–dimensional fiber at $x \in \mathbb{R}$ is $|\mathbb{R}_x|^{\frac{1}{2}} \to \mathbb{R}_x$. Rigorous definitions of $\frac{1}{2}$–forms and $\frac{1}{2}$–densities with its properties can be found in Guillemin and Sternberg [1], Hall [2] and Rawnsley [5].

In geometric quantization the $\frac{1}{2}$–forms define, intrinsically, an half–form Hilbert space with an inner product and a norm. Let us consider the case of $\mathbb{R}$. Then the symplectic manifold is $M = T^{\vee}\mathbb{R} = \mathbb{R}^2$. Let P be the vertical polarization of M with the orientation of $\mathbb{R}$ so that oriented 1–forms are positive multiple of dx. Let $\sqrt{K_P}$ to be the trivial bundle with trivializing section $\sqrt{dx}$ such that $\sqrt{dx} \otimes \sqrt{dx} = dx$. Then the half–form $\sigma = f(x)\sqrt{dx}$, for some real function $f(x)$, has the norm:

$$\|\sigma\|^2 = \int_{\mathbb{R}} |f(x)|^2 dx. \tag{9}$$

5 The corrected integral as application from the $\frac{1}{2}$–forms to $\mathbb{R}$

Let us consider the following $\frac{1}{2}$–form $\sigma = \sqrt{dx}$. In order to be precise, we must view this form as the form $1 \otimes \sqrt{dx}$ of the quantum line bundle $L \otimes \sqrt{K_P}$. In this case we have that:

$$\|\sigma\|^2 = \int_{[a,b]} dx = b - a, \tag{10}$$

where we considered as a base space the interval $[a, b]$. Now the question is if exists a square root of the following equation:

$$\sigma \cdot \sigma = b - a, \tag{11}$$

where the product here is the squared norm in the Hilbert space of half–forms. In order to answer the question let us consider the corrected integral:

$$_{\gamma}\int_{[a,b]} \sqrt{dx} = \sqrt{b-a}. \tag{12}$$

We can see that:

$$_{\gamma}\int_{[a,b]} \sqrt{dx} \cdot {}_{\gamma}\int_{[a,b]} \sqrt{dx} = (b-a). \tag{13}$$

So it is possible to apply the corrected integral in order to find the corrected result. In other terms we can see the corrected integral as a map $_{\gamma}\int : |\mathbb{R}|^{\frac{1}{2}} \to \mathbb{R}$, omitting the quantum line bundle associated.

6 Interesting integrals in $\sqrt{dx}$

Let us consider the integral:

$$\int_{[a,b]} x \sqrt{dx}. \tag{14}$$

Without the correction the integral diverges. We can try with:

$$_{\gamma}\int_{[a,b]} x \sqrt{dx}. \tag{15}$$

Let us consider the following partition of $[a, b]$ with $x_i = a + (i -$

$1) \cdot h$ and $x_{i+1}, = a + i \cdot h$, where $h = \frac{b-a}{n}$. Thus:

$$\begin{aligned} {}_{\gamma}\int_{[a,b]} x\sqrt{dx} &= \lim_{n\to+\infty} \gamma(n) \sum_{i=1}^{n} (a + i \cdot h)\sqrt{h} = \\ &= \lim_{n\to+\infty} a \cdot \sqrt{b-a} + \frac{\sqrt{(b-a)^3}}{n^2} \sum_{i=1}^{n} i = \\ &= a \cdot \sqrt{b-a} + \frac{\sqrt{(b-a)^3}}{2}, \end{aligned} \tag{16}$$

where we used the fact that $\sum_{i=1}^{n} i = \frac{n\cdot(n+1)}{2}$.

A similar result is the following:

$$\begin{aligned} {}_{\gamma}\int_{[a,b]} x^2\sqrt{dx} &= a^2\sqrt{b-a} + \\ &+ (b-a)\sqrt{b-a} + \frac{1}{3}(b-a)^2\sqrt{b-a}. \end{aligned} \tag{17}$$

The calculations are similar to the previous case where the formula used now is $\sum_{i=1}^{n} i^2 = \frac{n\cdot(n+1)\cdot(2n+1)}{6}$.

7 The corrected integral as application from the $\frac{1}{2}$–forms to $\mathbb{R}$

In this section we see as our definition of corrected fractional integral is in perfect agreement with the definition from the fractional calculus [6]. We start recalling the formula for the $\frac{1}{2}$–integral between a and b, this is given by the formula:

$$D^{-\frac{1}{2}}_{[a,b]} f(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_a^b (b-t)^{-\frac{1}{2}} f(t)dt \tag{18}$$

Now we observe that if we compute the $\frac{1}{2}$–integral for the constant function $f(t) = 1$ we find that:

$$D^{-\frac{1}{2}}_{[a,b]} 1 = \frac{2}{\Gamma\left(\frac{1}{2}\right)} \sqrt{b-a}. \tag{19}$$

We observe that we have considered the fractional integral over the interval $[a, b]$. Usually the integral (18) is considered on the intervall $[0, x]$ and, for general calculations of (18), we need to use the beta integral:

$$\int_0^x (x-y)^d y^b dy = \frac{\Gamma(d+1)\Gamma(b+1)}{\Gamma(b+d+2)} x^{b+d+1}. \tag{20}$$

We can compare the result from the theory of fractional calculus $\frac{2}{\Gamma(\frac{1}{2})}\sqrt{b-a}$ with our definition using the corrected integral (12) that gives $\sqrt{b-a}$ (only a constant factor of difference!). In fact:

$$_\gamma\int_{[a,b]} 1\,\sqrt{dx} = \frac{\Gamma(\frac{1}{2})}{2} D_{[a,b]}^{-\frac{1}{2}} 1. \tag{21}$$

8 Observations and conclusion

The main observation is that we have realized the following relation:

$$_\gamma\int_{[a,b]} 1\,\sqrt{dx} = \sqrt{\int_{[a,b]} 1\,dx}. \tag{22}$$

The relation seems not true in general $_\gamma\int_{[a,b]} f(x)\,\sqrt{dx} \neq \sqrt{\int_{[a,b]} f(x)\,dx}$. The fact is clear observing that for $f(x) = x$ the previous relation is false. We can try to define an integral function $F(x) = {_\gamma\int_{[0,x]}} 1\,\sqrt{ds}$, in this case:

$$F(x) = {_\gamma\int_{[0,x]}} 1\,\sqrt{ds} = \sqrt{x}. \tag{23}$$

Other questions remain open. For instance, does a similar definition exist for $\sqrt[n]{dx}$ when $n = 3, 4, \ldots$? Additionally, an important discussion topic concerns the geometrical interpretation of this correction.

References

[1] Guillemin, V. and Sternberg, S. “Geometric Asymptotics”. *AMS Surveys* 14 (1990), pp. 251–258.

[2] Hall, B. C. *Quantum theory for mathematicians*. 267. GTM Springer-Verlag New York, 2013.

[3] Kostant, B. *Quantization and unitary representations*. 170. Lecture Notes in Mathematics. Springer, Berlin, Heidelberg.

[4] Ramanujan, S. “On the sum of the square roots of the first n natural numbers”. *J. Indian Math. Soc.* VII (1915), pp. 173–175.

[5] Rawnsley, J. H. *Some properties of half-forms*. 676. Lecture Notes in Mathematics. Springer, Berlin, Heidelberg, 1978.

[6] Ross, B. “Fractional Calculus”. *Mathematics Magazine* 50 (1977), pp. 115–122.

[7] Souriau, J. M. *Structure des systèmes dynamiques*. Dunod, Paris, 1970.

[8] Woodhouse, N. M. J. *Geometric quantization*. Clarendon Press Oxford, 2e, 1991.

Simone Camosso
Mathematics and physics department
Saluzzo (CN), “I.I.S. G.Soleri-A.Bertoni”, Italy.
r.camosso@alice.it

Convexity and discrete inequalities

José Luis Díaz-Barrero

Abstract

In this short note, using the convexity of real functions some discrete inequalities are obtained.

1 Introduction

As is well-known, a function $f : \mathbb{R} \to \mathbb{R}$ is convex if the secant line between any two points on its graph lies above or on the graph itself. Specifically, for any $t \in [0,1]$, the value of the secant line at $x = at + (1-t)b$ is given by $tf(a) + (1-t)f(b)$, while the value of the function at $x = at + (1-t)b$ is $f(at + (1-t)b)$. Thus, f is convex if and only if

$$f(at + (1-t)b) \le tf(a) + (1-t)f(b) \quad \text{for all } a, b \in \mathbb{R}, t \in [0,1].$$

When f is concave the inequality reverses. Equality case occurs when $a = b$ or f is linear, as can be easily checked. Putting $q_1 = t, q_2 = 1-t$, $x_1 = a$ and $x_2 = b$ the last inequality can be written as

$$f(q_1x_1 + q_2x_2) \le q_1f(x_1) + q_2f(x_2) \quad \text{for all } x_1, x_2 \in \mathbb{R}, q_1 + q_2 = 1.$$

Jensen's inequality [3] generalizes the previous statement, which corresponds to the special case of two points in Jensen's inequality. For ease of reference, the statement and its proof are provided below.

Theorem 1. *For a convex function $f : \mathbb{R} \to \mathbb{R}$, numbers $x_1, x_2, \ldots, x_n$ in its domain, and positive weights q_i such that $q_1 + q_2 + \ldots + q_n = S$, Jensen's inequality can be stated as*

$$f\left(\frac{1}{S}\sum_{i=1}^{n} q_i x_i\right) \leq \frac{1}{S}\sum_{i=1}^{n} q_i f(x_i),$$

with the inequality reversed if f is concave.

Proof. To prove this finite form of Jensen's inequality, we argue by induction, assuming without loss of generality (WLOG) that $S = 1$. By the convexity hypothesis, the statement holds for $n = 2$. Now, suppose the statement is true for some n. That is, we assume

$$f\left(\sum_{i=1}^{n} q_i x_i\right) \leq \sum_{i=1}^{n} q_i f(x_i)$$

for any $q_1, q_2, \ldots, q_n$ such that $q_1 + q_2 + \ldots + q_n = 1$. Now, we want to prove it for $n+1$. Since at least one of the q_i is strictly smaller than 1, say q_{n+1} then by convexity inequality, we have

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} q_i x_i\right) &= f\left((1 - q_{n+1})\sum_{i=1}^{n} \frac{q_i}{1 - q_{n+1}} x_i + q_{n+1} x_{n+1}\right) \\ &\leq (1 - q_{n+1}) f\left(\sum_{i=1}^{n} \frac{q_i}{1 - q_{n+1}} x_i\right) + q_{n+1}\, f(x_{n+1}). \end{aligned}$$

Since $q_1 + q_2 + \ldots + q_{n+1} = 1$, then $\sum_{i=1}^{n} \frac{q_i}{1 - q_{n+1}} = 1$, and applying the inductive hypothesis gives

$$f\left(\sum_{i=1}^{n} \frac{q_i}{1 - q_{n+1}} x_i\right) \leq \sum_{i=1}^{n} \frac{q_i}{1 - q_{n+1}} f(x_i)$$

therefore

$$\begin{aligned} f\left(\sum_{i=1}^{n+1} q_i x_i\right) &\leq (1 - q_{n+1})\sum_{i=1}^{n} \frac{q_i}{1 - q_{n+1}} f(x_i) + q_{n+1}\, f(x_{n+1}) \\ &= \sum_{i=1}^{n+1} q_i f(x_i). \end{aligned}$$

We deduce the inequality is true for $n+1$, and by PMI it follows that the result is also true for all integer n greater or equal than 2.

In this paper, the key idea is to use the convexity of a function $f:\mathbb{R}\to\mathbb{R}$ and apply Jensen's inequality to derive useful discrete inequalities, similar to those published in [2].

2 Discrete inequalities

In this section, some discrete inequalities are presented. We begin with the following generalization of Nesbitt's inequality [4].

Theorem 2. *Let $x_1, x_2, \ldots, x_n$ be $n \geq 2$ positive real numbers. Then, it holds:*

$$\frac{x_1}{x_2+x_3+\ldots+x_n}+\frac{x_2}{x_1+x_3+\ldots+x_n}+\ldots+\frac{x_n}{x_1+x_2+\ldots+x_{n-1}}$$

$$\geq \frac{n}{n-1}.$$

Proof. Let $s = x_1 + x_2 + \ldots + x_n$ and $\mathbb{D} = \{x \in \mathbb{R} \mid x > s\}$. Consider the function $f:\mathbb{D}\to\mathbb{R}$ defined by

$$f(x) = \frac{x}{s-x}$$

which is convex on $\mathbb{D}$ because

$$f'(x) = \frac{s}{(s-x)^2} > 0$$

and

$$f''(x) = \frac{2s}{(s-x)^3} > 0.$$

Applying Jensen's inequality with $q_i = 1/n, 1 \leq i \leq n$, yields

$$\frac{1}{n}\sum_{i=1}^{n} f(x_i) \geq f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) \Leftrightarrow \frac{1}{n}\sum_{i=1}^{n}\frac{x_i}{s-x_i} \geq f(s/n) = \frac{1}{n-1}$$

from which the statement follows. Equality holds when $x_1 = x_2 = \ldots = x_n$.

Putting $n = 3$, the preceding result becomes Nesbitt's inequality.

Next, we present the following interesting inequality.

Theorem 3. *Let a_k, $(1 \leq k \leq n)$ be real numbers such that $0 < a_k \leq 1$ and let α be a positive real number. Then*

$$\frac{1}{n}\sum_{k=1}^{n}\frac{a_k - \alpha}{1 + a_k} \geq \frac{G - \alpha}{1 + G},$$

where G is the geometric mean of $a_1, a_2, \ldots, a_n$.

Proof. Let us denote by $x_k = \ln a_k$ for $1 \leq k \leq n$. We have $x_k \leq 0$. Let α be a positive constant and consider the function $f : [-\infty, 0] \to \mathbb{R}$ defined by $f(x) = \dfrac{e^x - \alpha}{1 + e^x}$. Since $f'(x) = \dfrac{(\alpha + 1)e^x}{(1 + e^x)^2}$ and $f''(x) = \dfrac{(\alpha + 1)e^x(1 - e^x)}{(1 + e^x)^3} \geq 0$, then f is convex, and applying Jensen's inequality, yields

$$\frac{1}{n}\sum_{k=1}^{n}\frac{e^{x_k} - \alpha}{1 + e^{x_k}} \geq \frac{e^A - \alpha}{1 + e^A},$$

where A represents the arithmetic mean of the x_k. In terms of a_k the preceding inequality becomes

$$\frac{1}{n}\sum_{k=1}^{n}\frac{a_k - \alpha}{1 + a_k} \geq \frac{G - \alpha}{1 + G},$$

where G represents the geometric mean of the a_k's. This completes the proof.

An immediate consequence of the preceding result are the following corollaries.

Corollary 1. *Let* a_k, $(1 \leq k \leq n)$ *be real numbers such that* $0 < a_k \leq 1$. *Then,*

$$\sum_{k=1}^{n} \frac{na_k - G}{1 + a_k} \geq \binom{n}{2} \frac{2G}{1 + G},$$

where G *is the geometric mean of* $a_1, a_2, \ldots, a_n$.

Proof. Putting $\alpha = G/n$ in Theorem 3, and rearranging terms, the statement follows. Equality holds when $a_1 = a_2 = \ldots = a_n$.

Corollary 2. *Let* a_k, $(1 \leq k \leq n)$ *be real numbers such that* $0 < a_k \leq 1$. *Then,*

$$\frac{1}{n} \sum_{k=1}^{n} \frac{na_k - m}{1 + a_k} \geq \frac{nG - m}{1 + G},$$

where m *is the minimum of* $a_1, a_2, \ldots, a_n$.

Proof. Putting $\alpha = m/n$ in Theorem 3, and rearranging terms, the statement follows. Equality holds when $a_1 = a_2 = \ldots = a_n$.

Convexity will be used to obtain the following constrain inequality.

Theorem 4. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a convex function, and let* a, b, c *be positive reals such that* $abc = 1$. *Then, it holds:*

$$\frac{f(a)}{1 + a + ab} + \frac{f(b)}{1 + b + bc} + \frac{f(c)}{1 + c + ca} \geq f(1).$$

Proof. Let a, b, c be positive reals such that $abc = 1$, then the following identities hold:

$$\sum_{cyc} \frac{1}{1 + a + ab} = 1 \quad \text{and} \quad \sum_{cyc} \frac{a}{1 + a + ab} = 1,$$

as can be easily checked. Applying Jensen's inequality, we get

$$\sum_{cyc} \frac{f(a)}{1 + a + ab} \geq f\left(\sum_{cyc} \frac{a}{1 + a + ab}\right) = f(1),$$

and therefore,

$$\sum_{cyc} \frac{f(a)}{1+a+ab} \geq f(1).$$

Applying the previous result to the function $\Gamma(x)$ instead of $f(x)$, which is convex for all $x > 0$ and satisfies $\Gamma(1) = 1$ (a well-known property) the following inequality was proposed in [1].

Let a, b, c be three positive reals numbers such that $abc = 1$. Prove that

$$\frac{\Gamma(a)}{1+a+ab} + \frac{\Gamma(b)}{1+b+bc} + \frac{\Gamma(c)}{1+c+ca} \geq 1,$$

where Γ is the gamma function.

Finally, we close this section with another constrain inequality.

Theorem 5. *Let $x_1, x_2, \ldots, x_n$ be $n \geq 2$ real numbers such that $0 \leq x_i < 1$ and $x_1 + x_2 + \ldots + x_n = 1$. Then, it holds:*

$$\prod_{i=1}^{n} \frac{1+x_i}{1-x_i} \geq \left(\frac{n+1}{n-1}\right)^n.$$

Proof. Consider the real function $f : (-1,1) \to \mathbb{R}$ defined by $f(x) = \ln\left(\frac{1+x}{1-x}\right)$. Since $f'(x) = \frac{2}{1-x^2} > 0$ and $f''(x) = \frac{4x}{(1-x^2)^2} \geq 0$, then f is convex. Applying Jensen's inequality and taking into account the constrain $x_1 + x_2 + \ldots + x_n = 1$, yields

$$\frac{1}{n} \sum_{i=1}^{n} f(x_i) \geq f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right),$$

or equivalently,

$$\ln\left(\frac{1+x_1}{1-x_1}\right) + \ln\left(\frac{1+x_2}{1-x_2}\right) + \ldots + \ln\left(\frac{1+x_n}{1-x_n}\right)$$

$$\geq n\ \ln\left(\frac{n+x_1+x_2+\ldots+x_n}{n-x_1+x_2+\ldots+x_n}\right) = \ln\left(\frac{n+1}{n-1}\right)^n.$$

On account of the injectivity of the logarithm function, from the preceding the statement follows. Equality holds when $x_1 = x_2 = \ldots = x_n = 1/n$.

3 Applications

Hereafter, we give some applications of the previous results.

Problem 1. If a, b, c are positive reals such that $a + b + c = 1$, then find the minimum value of the expression

$$\left(a+\frac{1}{a}\right)^{100} + \left(b+\frac{1}{b}\right)^{100} + \left(c+\frac{1}{c}\right)^{100}.$$

Solution. First, we observe that $0 < a, b, c < 1$ and we consider the real function $f : (0, 1) \to \mathbb{R}$ defined by $f(x) = \left(x+\frac{1}{x}\right)^{100}$. Since $f'(x) = 100\left(x+\frac{1}{x}\right)^{99} \cdot \left(1-\frac{1}{x^2}\right)$ and

$$f''(x) = 100\cdot 99\left(x+\frac{1}{x}\right)^{98} \cdot \left(1-\frac{1}{x^2}\right)^{2} + 100\left(x+\frac{1}{x}\right)^{99} \cdot \left(\frac{2}{x^3}\right) > 0,$$

then f is convex on $(0, 1)$ and by Jensen's inequality, we have

$$\left(a+\frac{1}{a}\right)^{100} + \left(b+\frac{1}{b}\right)^{100} + \left(c+\frac{1}{c}\right)^{100} = f(a) + f(b) + f(c)$$

$$\geq 3\, f\left(\frac{a+b+c}{3}\right) = 3\, f\left(\frac{1}{3}\right) = 10\left(\frac{10}{3}\right)^{99}.$$

The equality holds when $a = b = c = 1/3$, and the minimum value of the given expression is

$$10\left(\frac{10}{3}\right)^{99}.$$

Problem 2. (Mediterranean Mathematical Competition 2025) If a, b, c are positive reals no larger than one, then prove that it holds

$$\frac{2a - \sqrt[3]{abc}}{1+a} + \frac{2b - \sqrt[3]{abc}}{1+b} + \frac{2c - \sqrt[3]{abc}}{1+c} \geq \frac{3\sqrt[3]{abc}}{1+\sqrt[3]{abc}}.$$

Solution. Putting $\alpha = G/2$ in Theorem 3, we obtain

$$\frac{1}{3}\left(\frac{2a - \sqrt[3]{abc}}{2(1+a)} + \frac{2b - \sqrt[3]{abc}}{2(1+b)} + \frac{2c - \sqrt[3]{abc}}{2(1+c)}\right) \geq \frac{\sqrt[3]{abc}}{2(1+\sqrt[3]{abc})}$$

from which, after simplifying and rearranging terms, the statement follows. Equality holds when $a = b = c$.

Problem 3. If a, b, c are positive reals no larger than one, then prove that it holds

$$\frac{2a - m}{1+a} + \frac{2b - m}{1+b} + \frac{2c - m}{1+c} \geq 3\left(\frac{2\sqrt[3]{abc} - m}{1+\sqrt[3]{abc}}\right),$$

where $m = \min\{a, b, c\}$.

Solution. Putting $\alpha = m/2$ in Theorem 3, we obtain

$$\frac{1}{3}\left(\frac{2a - m}{2(1+a)} + \frac{2b - m}{2(1+b)} + \frac{2c - m}{2(1+c)}\right) \geq \frac{2\sqrt[3]{abc} - m}{2(1+\sqrt[3]{abc})}$$

from which, after simplifying and rearranging terms, the statement follows. Equality holds when $a = b = c$, as can be easily checked.

References

[1] Diaz-Barrero, J. L. “Advanced Problem A-76”. *Arhimede math. j.* 2 (2019), p. 132.

[2] Diaz-Barrero, J. L. and Popescu, P. G. “Some Elementary Inequalities for Convex Functions”. *J. Ineq. Pure and Appl. Math.* 2 (2006), Article 20.

[3] Jensen, J. L. W. V. “Sur les fonctions convexes et les inégalités entre les valeurs moyennes”. *Acta Math.* (1906), pp. 175–193.

[4] Nesbitt, A. M. “Problem 15114”. *The Educational Times* (1902), p. 233.

José Luis Díaz-Barrero
Barcelona Math Circle
Barcelona, Spain
jose.luis.diaz@upc.edu

Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.

2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

October 30, 2025

Elementary Problems

E–137. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let a, b, c be three positive integer numbers such that $7ab = 2c^6$. Is the number $a^7 + b^7 + c^7$ prime or composite?

E–138. *Proposed by Michel Bataille, Rouen, France.* Let x, a_1, a_2, a_3, a_4 be complex numbers such that $a_1+a_2+a_3+a_4 = 0$. Evaluate

$$\sum_{1\le i<j<k\le 4}(x+a_i+a_j+a_k)^3 - \sum_{1\le i<j\le 4}(x+a_i+a_j)^3 + \sum_{1\le i\le 4}(x+a_i)^3.$$

E–139. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Consider the square $ABCD$, with a point M on the side DC. If the value of the sum $S = \tan(MAB) + \tan(MBA) + \tan(AMB)$ is minimal, calculate the value of the ratio MD/AB.

E–140. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* In the lottery game of Springfield, five numbers are drawn out of 50 every week. Homer fills out a single lottery ticket with the same five numbers in each of the 52 weeks of the year. Flanders uses a different scheme. He plays only once a year with 52 tickets simultaneously: he fills them out in pairwise different ways. Is it true that both of them have the same chance of having a ticket with five correct numbers?

E–141. *Proposed by Goran Conar, Varaždin, Croatia.* Let $a_1, a_2, \ldots, a_n$ be positive real numbers satisfying $a_1 + a_2 + \cdots + a_n = n$. Prove that

$$\frac{n}{2} + \frac{1}{2}\sum_{i=1}^{n} a_i^4 \ge \sum_{i=1}^{n}\sqrt{\frac{a_i^8 + a_i^4 + 1}{a_i^2 + a_i + 1}} \ge \sum_{i=1}^{n} a_i\sqrt{a_i}.$$

E–142. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Find all real roots of the equation $9x^4 - 24x^3 - 23x^2 + 58x + 26 = 0$, if it is known that it has four distinct real roots, two of which add up to 2.

Easy–Medium Problems

EM–137. *Proposed by Michel Bataille, Rouen, France.* Let n be a nonnegative integer. Evaluate

$$\sum_{k=0}^{n}(k+1)\binom{n+2}{k+2}.$$

EM–138. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.* We denote by A, B the endpoints of the diameter of a semicircle Γ of radius a and construct the rectangle $ABCD$, with $\overline{BC}=a\sqrt{2}$, which contains it. If E is a point on Γ, distinct from A and B, and the lines DE, CE intersect the line AB at points F, G respectively, prove that

$$AG^2+BF^2=4a^2.$$

EM–139. *Proposed by Goran Conar, Varaždin, Croatia.* Let a,b,c be positive real numbers such that $a+b+c=4$. Prove that

$$\sqrt[4]{a^a b^b c^c}\geq\frac{4}{3}.$$

EM–140. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $z_1,z_2,\ldots,z_n$ be complex numbers. Prove that

$$\frac{1}{2}\left(\sum_{k=1}^{n}|z_k|+\left|\sum_{k=1}^{n}z_k\right|\right)^2\geq\sum_{k=1}^{n}|z_k|^2+\left|\sum_{k=1}^{n}z_k\right|^2.$$

EM–141. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a scalene triangle with incenter I and centroid G. Let G_a be the orthogonal projection of G on BC. Knowing that the points A,I,G_a are collinear, find the ratio AI/IG_a.

EM–142. *Proposed by José Luis Díaz-Barrero, Barcelona, Spaion.* What is the maximum number of regions into which a circle can be divided by segments connecting n points on its boundary?

Medium–Hard Problems

MH–137. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Is it possible to delete one number from the set $\{1, 2, \ldots, 10000\}$ so that the remaining 9999 numbers can be arranged in an order $a_1, a_2, \ldots, a_{9999}$, such that the differences $|a_1 - a_2|, |a_2 - a_3|, \ldots, |a_{9998} - a_{9999}|, |a_{9999} - a_1|$ are all distinct?

MH–138. *Proposed by Michel Bataille, Rouen, France.* Let I be the incenter of triangle ABC and let the line AI intersect BC at D. Let E be the circumcenter of ΔABD and let the line EC intersect the circumcircle of ΔBIC again at F. Prove that ED is tangent to the circumcircle of ΔCDF.

MH–139. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Lisa and Bart play the following game. They first choose a positive integer N, and then they take turns writing numbers on a blackboard. Lisa starts by writing 1. Thereafter, when one of them has written the number n, the next player writes down either $n + 1$ or $2n$, provided the number is not greater than N. The player who writes N on the blackboard wins.

(a) Determine which player has a winning strategy if $N = 2025$.
(b) Find the number of positive integers $N \leq 2025$ for which Bart has a winning strategy.

MH–140. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a non-right triangle with $AB \neq AC$ and let Γ be its circumcircle with center O. Let I be an arbitrary point in the interior of triangle ABC $(I \neq O)$. Let D be the intersect point of BC and AO and E be the second intersect point of AI with Γ. Let F be the point symmetric to point A about the OI. Knowing that $DEFI$ is a cyclic quadrilateral, prove that $DI = FI$.

MH–141. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Óscar Rivero Salgado, Santiago de Compostela, Spain.* For any positive number n, let $\tau(n)$ denote the number of positive divisors of n. We say that a positive number is **nice** if $2\,\tau(10n) < 5\,\tau(n)$. Determine how many positive numbers smaller or equal than 2025 are **nice**.

MH–142. *Proposed by José Luis Díaz-Barrero Barcelona, Spain.* Let n, k be integers with $n \geq 2$ and $1 \leq k \leq n$. Show that

$$(-1)^{n-k}\binom{n}{k}\sum_{\substack{j=1\\ j\neq k}}^{n}\frac{k}{k-j}+\sum_{\substack{j=1\\ j\neq k}}^{n}(-1)^{n-j}\binom{n}{j}\frac{j}{k-j}=0.$$

Advanced Problems

A–137. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Compute the following limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{h=1}^{n} \sum_{k=1}^{n} \frac{h+k}{h^2+k^2}.$$

A–138. *Proposed by Michel Bataille, Rouen, France.* Let n be a positive integer. Evaluate

$$\int_0^{\pi} (\cos x)^{2n} (\cos 2x + \cdots + \cos(2nx))\, dx.$$

A–139. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain.* Determine the smallest positive integer λ such that if A and B are any 2×2 integer matrices satisfying that all of the matrices

$$A,\ A+B,\ A+2B,\ \ldots,\ A+\lambda B$$

are invertible and their inverses also have integer entries, then for any integer t, the matrix $A + tB$ is also invertible and its inverse has integer entries.

A–140. *Proposed by Vasile Mircea Popa, "Lucian Blaga" University of Sibiu, Romania.* Calculate the following integral:

$$\int_0^{\infty} \frac{\ln x}{(x+1)(x^2+1)(x^4+1)} dx.$$

A–141. *Proposed by Joseph Santmyer, Las Cruces, New Mexico, USA.* The n^{th} harmonic number H_n is defined as $H_n = \sum_{k=1}^{n} \frac{1}{k}$ and the generalized harmonic number of order m is defined as

$H_{n,m} = \sum_{k=1}^{n} \frac{1}{k^m}$. Show that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{H_n^2}{2^n} &= \zeta(2) + \ln^2(2) && (1) \\ \sum_{n=1}^{\infty} \frac{H_{n,2}}{2^n} &= \zeta(2) - \ln^2(2) && (2) \\ \sum_{n=1}^{\infty} \frac{1}{2^n} \sum_{k=1}^{n-1} \frac{H_k}{k(k+1)} &= \frac{\zeta(2)}{2} - \ln^2(2) && (3) \end{aligned}$$

where ζ is the Riemann zeta function.

A–142. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a non-right triangle with $AB \neq AC$ and let G be its centroid, I be its incenter and Γ be its circumcircle with center O. Let D be the foot of the perpendicular drawn from C to IO. Let G_a, I_b be the feet of the perpendiculars drawn from G to BC and from I to AC respectively. Let M_b be the midpoint of AC and E be the reflection of I_b in the point M_b. Knowing that I_bIDG_a is a cyclic quadrilateral, prove that $ABIE$ is a cyclic quadrilateral too.

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to
jose.luis.diaz@upc.edu

Some relations involving the elements of a triangle

Jordi Ferré Garcia and José Luis Díaz-Barrero

1 Introduction

In this math lesson, various relationships between the elements of a triangle such as those appearing in [1] are explored. These relations, among others, will be introduced, discussed, and worked in the annual training sessions organized by the Barcelona Math Circle (BMC) for very young students who have a general interest in mathematics and, more specifically, in mathematical contests.

2 The relations

Hereafter, we present some well-known relations between the elements of a triangle. During this lecture, we will denote by a, b, and c the lengths of sides BC, AC, and AB, respectively; by h the length of the altitude from vertex A; by r and R the inradius and circumradius; and by s the semiperimeter of triangle ABC.

We begin by giving formulas to compute the area of $\triangle ABC$.

Theorem 1 (Area of a triangle). *In any triangle ABC, it holds:*

$$[ABC] = \frac{abc}{4R} = sr = \sqrt{s(s-a)(s-b)(s-c)}$$

The last expression is known as Heron's formula.

Proof. We have

$$[ABC] = \frac{bh}{2} = \frac{bc\sin A}{2} = \frac{abc}{4R}$$

because $\dfrac{a}{\sin A} = 2R$ on account of Sine Law. To prove Heron'a formula, we have on account of Cosine Law, that

$$\begin{aligned}[ABC] &= \frac{1}{2}\,ac\sin B = \frac{1}{2}\,ac\sqrt{1-\cos^2 B} = \frac{1}{2}\,ac\sqrt{1-\left(\frac{a^2+c^2-b^2}{2ac}\right)^2} \\ &= \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)} \\ &= \sqrt{s(s-a)(s-b)(s-c)}.\end{aligned}$$

From the preceding, it immediately follows that

$$R = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \quad \text{and} \quad r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

Next, we give an interesting and most useful relation.

Theorem 2. *In any triangle ABC, it holds $ab + bc + ca = s^2 + r^2 + 4rR$.*

Proof. From $r = \sqrt{\dfrac{(s-a)(s-b)(s-c)}{s}}$ we get

$$sr^2 = (s-a)(s-b)(s-c) = s^3 - s^2(a+b+c) + s(ab+bc+ca) - abc.$$

on account that $a + b + c = 2s$ and $abc = 4Rrs$, the preceding expression becomes

$$sr^2 = -s^3 + s(ab+bc+ca) - 4Rrs \Leftrightarrow r^2 = -s^2 + ab + bc + ca - 4Rr$$

from which the statement follows.

An immediate consequence of the las result is the following

Corollary 1. *In any triangle ABC with the usual notations, it holds that $a^2+b^2+c^2=2(s^2-r^2-4Rr)$.*

Proof. Putting $a+b+c=2s$ and $ab+bc+ca=s^2+r^2+4Rr$ in

$$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca),$$

yields $a^2+b^2+c^2=4s^2-2(s^2+r^2+4Rr)=2(s^2-r^2-4Rr)$.

Let ABC be a triangle with side lengths a,b,c. An useful substitution is the following: $a=x+y, b=y+z$, and $c=z+x$ where x,y,z are positive real numbers. From, the preceding immediately follows $x=\frac{1}{2}(c+a-b)$, $y=\frac{1}{2}(a+b-c)$, and $z=\frac{1}{2}(b+c-a)$. Using this transformation, we get

$$R=\frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}=\frac{(x+y)(y+z)(z+x)}{4\sqrt{xyz(x+y+z)}}$$

$$r=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}=\frac{\sqrt{xyz(x+y+z)}}{x+y+z}$$

From the preceding expressions, and taking into account the well-known Cesaro's inequality: $(x+y)(y+z)(z+x)\geq 8xyz$, $(x>0, y>0, z>0)$, immediately follows

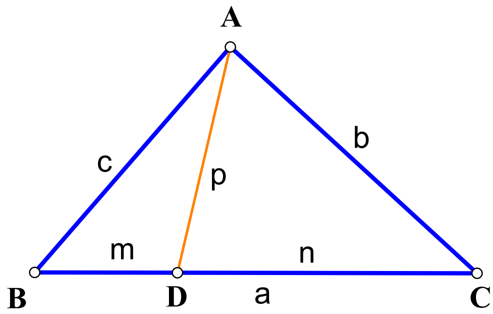
$$\frac{R}{r}\geq 2 \quad \text{(Euler's inequality).}$$

The following result allow us to compute the length of the cevian in triangle ABC.

Theorem 3 (Stewart). *Let D be a point in the side BC of triangle ABC. Then, it holds*

$$b^2m+c^2n=a(p^2+mn),$$

where $m=BD, n=DC$, and $p=AD$, respectively.



Scheme for proving Stewart's theorem.

Proof. Since $\angle ADB + \angle ADC = \pi$, then $\cos \widehat{ADB} + \cos \widehat{ADC} = 0$. Taking into account Cosine's Law, we have

$$\frac{m^2 + p^2 - c^2}{2mp} + \frac{n^2 + p^2 - b^2}{2np} = 0,$$

or equivalently, $n(m^2 + p^2 - c^2) + m(n^2 + p^2 - b^2) = 0$ from which follows $b^2 m + c^2 n = (m + n)(p^2 + mn) = a(p^2 + mn)$, as claimed.

From the preceding the length of the cevian p is given by

$$p = \sqrt{\frac{b^2 m + c^2 n - amn}{a}}.$$

Corollary 2 (Length of the median). *In any triangle ABC the length of the median is given by the expression*

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \quad \textit{(cyclic)}$$

Proof. Applying Stewart's theorem with $m = n = a/2$ and $p = m_a$, we have $\dfrac{b^2 a}{2} + \dfrac{c^2 a}{2} = a\left(m_a^2 + \dfrac{a^2}{4}\right)$ from which we get $4m_a^2 = 2b^2 + 2c^2 - a^2$, and the statement follows.

The preceding expressions are also known as *Apollonius formulae*.

Adding up the expressions $4m_a^2 = 2b^2 + 2c^2 - a^2$, $4m_b^2 = 2c^2 + 2a^2 - b^2$ and $4m_c^2 = 2a^2 + 2b^2 - c^2$, we get the following relation

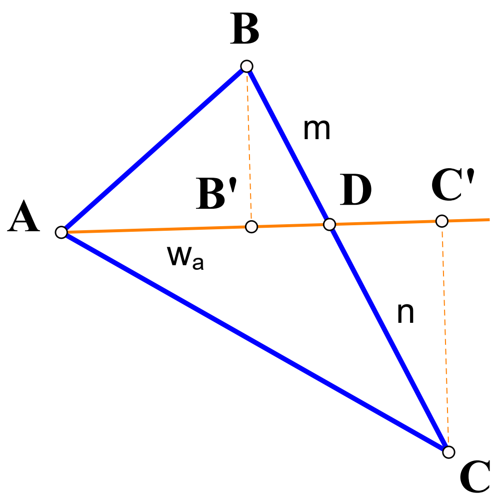
$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2).$$

Now, we will derive a formula to compute the length of the angle bisector. It follows from

Theorem 4. *Let D be a point in the side BC of triangle ABC. Then,*

$$\frac{BD}{DC} = \frac{AB \sin \angle DAB}{AC \sin \angle DAC}$$

Proof. Notice that triangles $BB'D$ and $CC'D$ are similar because both are right triangles $\angle BDB' = \angle CDC'$. So, $\angle B'BD = \angle C'CD$. Then, $BB' = AB \sin \angle DAB$ and $CC' = AC \sin \angle CAD$.



Scheme for bisector.

Therefore,

$$\frac{BD}{DC} = \frac{BB'}{CC'} = \frac{AB \sin \angle DAB}{AC \sin \angle DAC}$$

and the proof is complete.

Corollary 3 (Length of angle bisector). *In any triangle ABC the length of the angle bisector interior and exterior are given by*

$$\begin{aligned} w_a &= \frac{1}{b+c}\sqrt{(a+b+c)(b+c-a)bc} \\ &= \frac{2}{b+c}\sqrt{s(s-a)bc} \end{aligned}$$

$$\begin{aligned} w_{1a} &= \frac{1}{b-c}\sqrt{(a+b-c)(c+a-b)bc} \\ &= \frac{2}{b-c}\sqrt{(s-b)(s-c)bc} \quad (b>c). \end{aligned}$$

Proof. Applying the preceding result when AD is the angle bisector w_a, we have $\frac{BD}{DC} = \frac{AB}{AC}$ or $\frac{m}{n} = \frac{c}{b}$ that jointly with $m+n=a$ gives $m = \frac{ca}{b+c}$ and $n = \frac{ba}{b+c}$. Applying Stewart's theorem, yields

$$b^2\frac{ca}{b+c} + c^2\frac{ab}{b+c} = a\left(w_a^2 + \frac{bca^2}{(b+c)^2}\right)$$

from which we get

$$w_a^2 = bc\left[1 - \left(\frac{a}{b+c}\right)^2\right]$$

and

$$\begin{aligned} w_a &= \sqrt{bc\left[1-\left(\frac{a}{b+c}\right)^2\right]} = \frac{1}{b+c}\sqrt{bc\left((b+c)^2-a^2\right)} \\ &= \frac{2}{b+c}\sqrt{s(s-a)bc}. \end{aligned}$$

Likewise, the length of the external bisector w_{1a} is given by

$$w_{1a} = \frac{2}{b-c}\sqrt{(s-b)(s-c)bc}.$$

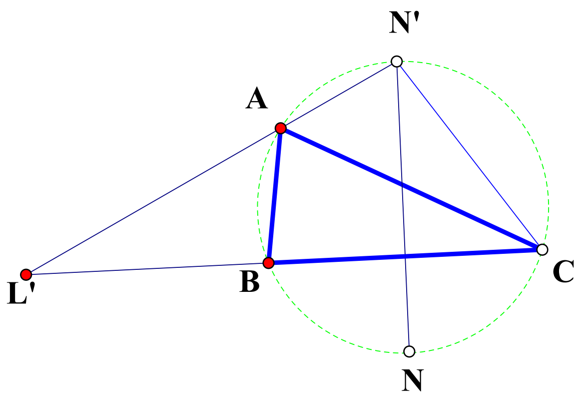
Indeed, suppose that AL' the exterior bisector on angle A meets again the circumcircle of $\triangle ABC$ at N' opposite diametrically to point N. Then $\triangle ACN'$ and $\triangle ABL'$ are similar because $\angle N'AC = \angle BAL'$ on account that AL' is a bisector. Furthermore, $\angle AN'C = \angle ABL'$ because they are supplementary of $\angle ABC$. Thus, we have

$$\frac{AB}{AL'} = \frac{AN'}{AC}$$

or

$$AB \cdot AC = AL' \cdot AN' = AL'(L'N' - AL') = AL' \cdot L'N' - AL'^2.$$

On account of the power point respect to a circle, we have



Scheme for exterior bisector.

$$AL' \cdot L'N' = L'B \cdot L'C.$$

Thus, $AB \cdot AC = L'B \cdot L'C - AL'^2$ and $AL'^2 = L'B \cdot L'C - AB \cdot AC$. Implied by Theorem 4 we obtain that $\dfrac{L'B}{L'C} = \dfrac{AB}{AC}$, which, after rearranging terms, turns into

$$\frac{L'B}{AB} = \frac{L'C}{AC} = \frac{L'C - L'B}{AC - AB} = \frac{BC}{AC - AB}$$

from which $AL' = \dfrac{ac}{b-c}$ and $L'C = \dfrac{ab}{b-c}$ follows. Finally,

$$AL'^2 = \frac{a^2bc}{(b-c)^2} - bc = \frac{bc(a+b-c)(a-b+c)}{(b-c)^2} = \frac{4bc(s-b)(s-c)}{(b-c)^2}$$

from which we get $AL' = \dfrac{2}{b-c}\sqrt{(s-b)(s-c)bc} = w_{1a}$.

Finally, we conclude this note by presenting a result that is a combination of the preceding ones.

Theorem 5. *Let ABC be a triangle. Then, the diameter of its circumcircle is give by*

$$2R = \frac{w_a^2}{h_a}\sqrt{\frac{m_a^2 - h_a^2}{w_a^2 - h_a^2}}.$$

Proof. In Figure 1, we observe that $m_a^2 - h_a^2 = AM^2 - AD^2 = DM^2$

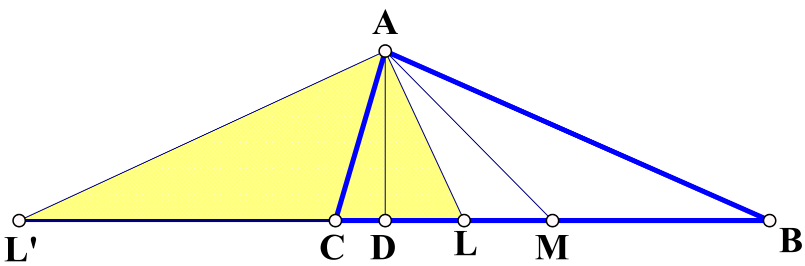


Figure 1.

and $w_a^2 - h_a^2 = AL^2 - AD^2 = DL^2$. We have to prove that

$$2R = \frac{AL^2}{AD} \cdot \frac{DM}{DL}.$$

Let AL' be the exterior bisector of angle A. Since $\triangle ALL' \sim \triangle ADL$ then

$$\frac{AL}{DL} = \frac{LL'}{AL} \Leftrightarrow \frac{AL^2}{DL} = LL'.$$

To compute LL' we consider the right triangle ALL' and we have

$$LL'^2 = w_a^2 + w_{1a}^2 = \frac{4}{(b+c)^2}\, s(s-a)bc + \frac{4}{(b-c)^2}\,(s-b)(s-c)bc$$

$$= \frac{4a^2b^2c^2}{(b^2-c^2)^2} \quad \text{and} \quad LL' = \frac{2abc}{b^2-c^2}.$$

On the other hand, applying Cosine Law, we have

$$AC^2 = AM^2 + MC^2 - 2AM \cdot MC \cos(AM, MC)$$
$$= AM^2 + MC^2 + 2MC \cdot DM$$

and

$$AB^2 = AM^2 + MB^2 - 2AM \cdot MB \cos(AM, MB)$$
$$= AM^2 + MB^2 - 2MB \cdot DM.$$

Subtracting the second expression from the first one, yields

$$b^2 - c^2 = 2a \cdot DM \Rightarrow DM = \frac{b^2-c^2}{2a}.$$

Finally, we have

$$\frac{AL^2}{AD} \cdot \frac{DM}{DL} = \frac{2abc}{b^2-c^2} \cdot \frac{b^2-c^2}{2ah_a} = \frac{abc}{ah_a} = \frac{4R[ABC]}{2[ABC]} = 2R.$$

References

[1] F.G-M. *Exercises de Géométrie.* Sixiéme édition. Jacques Gabay, Sceaux, France, 1991, pp. xxiv+1302. ISBN: 2-87647-083-7.

Jordi Ferré Garcia
BarcelonaTech CFIS
Barcelona, Spain
jordi.ferre@estudiantat.upc.edu

José Luis Díaz-Barrero
Barcelona Math Circle
Barcelona, Spain
jose.luis.diaz@upc.edu

A Unique Number Pattern

N. Thiruniraiselvi and M. A. Gopalan

Abstract

Let a, b, c be three non-zero single digit integers respectively. We show that there exists an integer N such that the three digit integer $abc = N + 2$ and its reverse integer $cab = 2N$.

Keywords: Pair of equations, Integer solutions, Linear Diophantine equation.

1 Introduction

Number is the essence of calculation. Numbers have varieties of range and richness [1-3]. Number patterns are groups of numbers that follow rules. Recognizing number patterns is a vital problem solving skill. In this short note, a typical number pattern is illustrated.

2 Methodology

Given three non-zero single digit integers a, b, c, we have to find positive integer N satisfying the following pair of equations

$$\begin{aligned} abc &= N + 2, \\ cab &= 2N. \end{aligned}$$

To do that we start by eliminating N between the above equations, and we obtain the following relation between the digits a, b, c:

$$2abc - cab = 4. \tag{1}$$

Now, using the decimal number system, (1) may be written as

$$2\,(100a + 10b + c) - (100c + 10b + a) = 4,$$

or

$$199a + 10b - 98c = 4 \tag{2}$$

which is a linear Diophantine equation in three variables. The integer solution to the above equation (2) exists because the greatest common divisor $(199, 10, 98)$ divides 4 on the RHS of (2), as it is well-known. The procedure to solve it is illustrated below:

From (2), we write

$$b = 9c - 19a + \frac{4 + 8c - 9a}{10}. \tag{3}$$

Putting $d = \dfrac{4 + 8c - 9a}{10}$ expression (2) becomes $b = 9c - 19a + d$. Substituting the above equation in (2), yields $199a - 10\,(9c - 19a + d) - 98c = 4$ or $9a - 8c + 10d = 4$. Writing the last expression in the form $8c = 8(a + d) + (a + 2d - 4)$, we get

$$c = a + d + \frac{a + 2d - 4}{8}. \tag{4}$$

Now, putting $e = \dfrac{a + 2d - 4}{8}$ the above becomes $c = a + d + e$. From (4) we get $a = 8e - 2d + 4$. Combining the last two expressions, yields $c = 9e - d + 4$.

Substituting the above expressions in $b = 9c - 19a + d$, we obtain

$$b = 9c - 19a + d = 9\,(9e - d + 4) - 19\,(8e - 2d + 4) + d = 30d - 71e - 40.$$

Choose d, e in the preceding expressions such that the values of a, b, c are single digit integers respectively. By inspection, $d = 4, e = 1$ leads to $a = 4, b = 9, c = 9$ and we get

$$N = 497.$$

Hence, coming back to the first equations, we observe that

$$\begin{aligned}499 &= 497 + 2\\ 994 &= 2 \cdot 497.\end{aligned}$$

Following the above reasoning, one may observe the following patterns

$$\begin{aligned}4999 &= 4997 + 2, \quad 9994 = 2 \cdot 4997\\ 49999 &= 49997 + 2, \quad 99994 = 2 \cdot 49997\end{aligned}$$

and so on.

3 Conclusion

Understanding number patterns is essential for students of all ages to appreciate the beauty and presence of mathematics in everyday life. The study of number patterns remains a source of fascination for both amateur and professional mathematicians, as they can be explored both algebraically and geometrically.

In conclusion, readers and researchers in this field may explore additional number patterns, further uncovering their significance and applications.

References

[1] Dickson, L. E. *History of the Theory of Numbers*. Vol. 256. Chelsea Publishing Company, 1952.

[2] Mordell, L. J. *Diophantine Equations*. Academic Press, 1969.

[3] Sierpiński, W. *Pythagorean Triangles*. Vol. 9. New York: Dover Publications, Inc., 2003.

Dr. Thiruniraiselvi
Assistant Professor, Department of Mathematics
School of Engineering and Technology
Dhanalakshmi Srinivasan University
Samayapuram,Trichy- 621 112, Tamil Nadu, India.
drntsmaths@gmail.com

Dr. M. A. Gopalan
Professor, Department of Mathematics
Shrimati Indira Gandhi College
Trichy-620 002, Tamil Nadu, India.
mayilgopalan@gmail.com

Contests

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to find interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

Problems and solutions from the 12th edition of BarcelonaTech Mathcontest

O. Rivero Salgado and J. L. Díaz-Barrero

1 Problems and solutions

Hereafter, we present the four problems that appeared in the paper given to the contestants of the BarcelonaTech Mathcontest 2025, as well as their official solutions.

Problem 1. Let n be a positive integer. Show that the equation

$$\sqrt[3]{x} + \sqrt[3]{y} = \sqrt[3]{n}$$

has solution in positive integers x, y if and only if the number n is divisible by the cube of an integer greater than 1.

Solution. Suppose that the positive integers x, y, n satisfy the the given equation. Then the positive numbers $\alpha = \sqrt[3]{\dfrac{x}{n}}$ and $\beta = \sqrt[3]{\dfrac{y}{n}}$ satisfy the equation $\alpha + \beta = 1$. Their cubes are rational numbers. From the identity

$$\begin{aligned} 3(\alpha+\beta)(\alpha^3-\beta^3) &= (\alpha-\beta)\,3(\alpha+\beta)\,(\alpha^2+\alpha\beta+\beta^2) \\ &= (\alpha-\beta)(2(\alpha+\beta)^3+(\alpha^3+\beta^3)) \end{aligned}$$

we conclude that $\alpha - \beta$ is a rational number, and consequently the numbers α, β themselves are rational, as can be easily checked.

Writing the number α in the form of an irreducible fraction $\alpha = k/m$ we have the equality $xm^3 = nk^3$, from which it follows that n is divisible by m^3. Since $x < n$, then $k < m$, so the number n is divisible by the cube of the number m, greater than 1.

Conversely, assuming that the number n has a divisor of the form m^3 $(m > 1)$, we assume

$$x = \frac{n}{m^3}, \quad y = \frac{n(m-1)^3}{m^3}$$

from which we get the required equation $\sqrt[3]{x} + \sqrt[3]{y} = \sqrt[3]{n}$.

Problem 2. In the chess tournament played in a round-robin system, only the first-year and second-year students participated. Despite the fact that there were three times as many second-year students as first-year students, they together scored only 3 points more than the first-year students. How many students participated in the tournament?

Solution. Let n be the number of first-year students who participated in the tournament. The number of second-year students was then $3n$, and the total number of students was $4n$. They played a total of $\frac{1}{2}4n(4n-1)$ matches between them. Since one point is awarded for a win, half a point for a draw, and no points for a loss, the total number of points distributed is equal to the number of matches played. If the first-year students scored p points and the second-year students scored d points, then $p + d = 2n(4n-1)$. The second-year students scored only three points more than the first-year students, so $p = d - 3$. By substituting the second equation into the first, we get

$$d = n(4n-1) + \frac{3}{2}.$$

The number of points the second-year students scored is at least equal to the number of points they earned in the matches between themselves. Thus

$$n(4n-1) + \frac{3}{2} \geq \frac{3n(3n-1)}{2}.$$

By simplifying, we get the inequality $n + 2 \geq n^2$. In the set of natural numbers, this inequality is satisfied only by the numbers $n = 1$ and $n = 2$, because for $n \geq 3$, $n + 2 \leq 2n < n^2$.
The tournament could thus have been participated in by either 3 second-year students and 1 first-year student, or 6 second-year students and 2 first-year students. In both cases, we need to confirm that the tournament could have proceeded in a way that satisfies the conditions of the problem. Indeed,

- In the case of four participants, the first-year student scored 1.5 points in matches against the second-year students. The second-year students then scored the remaining 4.5 points.
- In the case of 8 participants, in all the matches between a first-year and a second-year student, except for one that ended in a draw, the first-year student won. The first-year students thus scored 12.5 points and the second-year students scored 15.5 points.

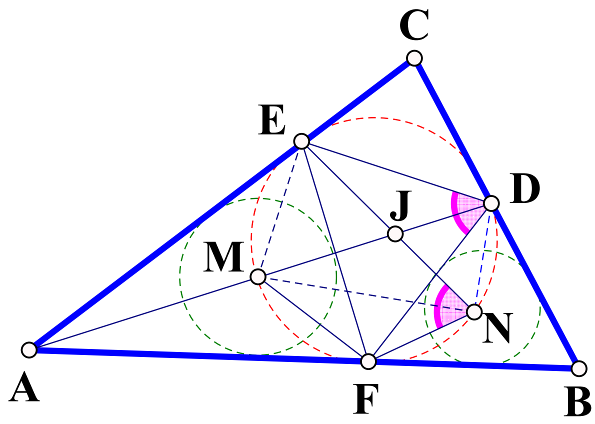
Thus, the number of participants is 4: (first, second)$= (1, 3)$ or 8: (first, second)$= (2, 6)$.

Problem 3. The incircle of triangle ABC is tangent to the sides BC, CA, and AB at points D, E, and F, respectively. Points M, N, and J are the centers of the incircles of the triangles AEF, BDF, and DEF, respectively. Prove that the points F and J are symmetric with respect to the line MN.

Solution. We will first prove that the points M and N are the centers of the shorter arcs FE and FD of the incircle of triangle ABC, respectively. Indeed, let M' be the center of the shorter arc FE of this circle. The line AC is tangent to the circle at point E, so

$$\angle AEM' = \angle EFM'. \qquad (1)$$

Since point M' is the center of the arc EF, the triangle $EM'F$ is isosceles. Therefore, $\angle EFM' = \angle FEM'$, which, together with equality (1), shows that point M' lies on the angle bisector of $\angle AEF$. Similarly, we prove that this point lies on the angle bisector



Scheme for solving problem 3.

of $\angle AFE$, so it coincides with the center M of the incircle of triangle AEF. We proceed in the same manner for point N.

The incircle of triangle ABC is simultaneously the circumcircle of triangle DEF. Since M is the center of the shorter arc EF of this circle, the line DM contains the angle bisector of $\angle EDF$. Therefore, the point J lies on this line, and J is the center of the incircle of this triangle. Similarly, we conclude that point J lies on the line EN.

To justify that points J and F are symmetric with respect to the line MN, it is sufficient to show that triangles MFN and MJN are congruent, as this would imply that they are symmetric with respect to the line MN. However, we have

$$\angle JMN = \angle DMN = \angle FMN,$$

where the second equality follows from the fact that N is the center of the arc FD. Similarly, we obtain $\angle JNM = \angle FNM$. The triangles MFN and MJN therefore have equal corresponding angles and a common side MN, so they are congruent (by the Angle-Side-Angle criterion). This concludes the solution.

Problem 4. Find the largest possible value of the expression

$$x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n + x_nx_1$$

for a given $n \geq 3$, where $x_1, x_2, \ldots, x_n$ is an arbitrary arrangement of the integer numbers $1, 2, \ldots, n$.

Solution. Let $S_n(x_1, x_2, \ldots, x_n)$ denote the above expression and let M_n be its maximal value. Since $S_3(x_1, x_2, x_3) = 1{\cdot}2+2{\cdot}3+3{\cdot}1 = 11$ when $\{x_1, x_2, x_3\} = \{1, 2, 3\}$, we have $M_3 = 11$. Since the given expression is independent of cyclic permutations, it suffices for $n > 3$ to consider only those arrangements $x_1, x_2, \ldots, x_n$ of $1, 2, \ldots, n$ for which $x_1 = n$. We then obtain

$$S_n(n, x_2, x_3, \ldots, x_n) = S_{n-1}(x_2, x_3, \ldots, x_n) - x_2x_n + nx_2 + nx_n$$

$$= S_{n-1}(x_2, x_3, \ldots, x_n) + n^2 - (n - x_2)(n - x_n) \leq M_{n-1} + n^2 - 1 \cdot 2,$$

with equality when $S_{n-1}(x_2, x_3, \ldots, x_n) = M_{n-1}$ and, at the same time, $\{x_2, x_n\} = \{n-2, n-1\}$. If we make the induction hypothesis T_n that there exist an arrangement $y_1, y_2, \ldots, y_{n-1}$ of $1, 2, \ldots, n-1$ for which $y_1 = n-1, y_{n-1} = n-2$, and $S_{n-1}(y_1, y_2, \ldots, y_{n-1}) = M_{n-1}$ (T_n holds for $n = 4$), we obtain the recurrence relation $M_n = M_{n-1} + n^2 - 2$, where $n, y_{n-1}, y_{n-2}, \ldots, y_2, y_1$ is the arrangement of $1, 2, \ldots, n$ belonging to the assumption T_{n+1}. Now we evaluate

$$\begin{aligned} M_n &= M_3 + (4^2 - 2) + (5^2 - 2) + \ldots + (n^2 - 2) \\ &= 11 + (4^2 + 5^2 + \ldots + n^2) - 2(n - 3) \\ &= \frac{1}{6}(2n^3 + 3n^2 - 11n + 18). \end{aligned}$$

Óscar Rivero Salgado
Universidad de Santiago de Compostela
Spain
riverosalgado@gmail.com

José Luis Díaz-Barrero
Barcelona Math Circle (BMC)
Barcelona, Spain
jose.luis.diaz@upc.edu

Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

jose.luis.diaz@upc.edu

Elementary Problems

E–131. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let α, β, γ be real numbers. If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$, then prove that

$$\frac{\sin 5\alpha + \sin 5\beta + \sin 5\gamma}{\cos 5\alpha + \cos 5\beta + \cos 5\gamma} = \tan(\alpha + \beta + \gamma).$$

Solution 1 by the proposer. Let $a = \cos\alpha + i\,\sin\alpha$, $b = \cos\beta + i\,\sin\beta$ and $c = \cos\gamma + i\,\sin\gamma$. Since $a + b + c = 0$, then the identity $a^5 + b^5 + c^5 - 5abc = (a+b+c)(a^4 + b^4 + c^4 - ab^3 - a^3b - b^3c - bc^3 - c^3a - ca^3 + 5abc)$ becomes $a^5 + b^5 + c^5 = 5abc$. Putting the values of the complex numbers a, b, c in the last expression, yields

$$e^{i5\alpha} + e^{i5\beta} + e^{i5\gamma} = 5e^{i\alpha}e^{i\beta}e^{i\gamma} = 5e^{i(\alpha+\beta+\gamma)}.$$

Identifying, real and imaginary parts, we get $\cos 5\alpha + \cos 5\beta + \cos 5\gamma = 5\cos(\alpha + \beta + \gamma)$ and $\sin 5\alpha + \sin 5\beta + \sin 5\gamma = 5\sin(\alpha + \beta + \gamma)$ from which the statement immediately follows.

Solution 2 by Michel Bataille, Rouen, France. Let $z_1 = e^{i\alpha}$, $z_2 = e^{i\beta}$, $z_3 = e^{i\gamma}$ and $p = z_1z_2z_3$. From the hypothesis, we have $z_1 + z_2 + z_3 = 0$. It follows that $\overline{z_1} + \overline{z_2} + \overline{z_3} = 0$, that is, $\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = 0$ (since $|z_1| = |z_2| = |z_3| = 1$). Therefore $\frac{z_1z_2 + z_2z_3 + z_3z_1}{p} = 0$ so that $z_1z_2 + z_2z_3 + z_3z_1 = 0$.
Note that z_1, z_2, z_3 are distinct (if for example $z_1 = z_2$, then $2z_1 = -z_3$, a contradiction when we take moduli).
We deduce that $(z - z_1)(z - z_2)(z - z_3) = z^3 - p$ so that z_1, z_2, z_3 are the three cubic roots of the complex p. As a result, we have

$$\{z_1, z_2, z_3\} = \{z_1, \omega z_1, \omega^2 z_1\}$$

where $\omega = \exp(2\pi i/3)$.
It follows that

$$z_1^5 + z_2^5 + z_3^5 = z_1^5 + \omega^5 z_1^5 + \omega^{10} z_1^5 = (1 + \omega + \omega^2) z_1^5 = 0$$

(since $\omega^5 = \omega^2, \omega^{10} = \omega$), that is,

$$e^{5i\alpha} + e^{5i\beta} + e^{5i\gamma} = 0.$$

As a result, $\cos 5\alpha + \cos 5\beta + \cos 5\gamma = \sin 5\alpha + \sin 5\beta + \sin 5\gamma = 0$ (and the required relation is meaningless).

Solution 3 by Albert Stadler, Herrliberg, Switzerland. The equation is equivalent to

$$(\sin(5\alpha) + \sin(5\beta) + \sin(5\gamma))\cos(\alpha + \beta + \gamma)$$

$$= (\cos(5\alpha) + \cos(5\beta) + \cos(5\gamma))\sin(\alpha + \beta + \gamma)$$

which in turn is equivalent to

$$\sin(4\alpha - \beta - \gamma) + \sin(4\beta - \gamma - \alpha) + \sin(4\gamma - \alpha - \beta) = 0,$$

since $\sin(5\alpha)\cos(\alpha + \beta + \gamma) - \cos(5\alpha)\sin(\alpha + \beta + \gamma))$

$$= \sin(5\alpha - (\alpha + \beta + \gamma)),$$

and similarly for the cyclic variants.

Let $a = e^{i\alpha}, b = e^{i\beta}, c = e^{i\gamma}$. Then,

$\cos\alpha + \cos\beta + \cos\gamma = 0$ is equivalent to $a + \frac{1}{a} + b + \frac{1}{b} + c + \frac{1}{c} = 0$.

$\sin\alpha + \sin\beta + \sin\gamma = 0$ is equivalent to $a - \frac{1}{a} + b - \frac{1}{b} + c - \frac{1}{c} = 0$.

Hence $a + b + c = 0$ and $1/a + 1/b + 1/c = 0$ or equivalently $ab + bc + ca = 0$. Thus, $\sin(4\alpha - \beta - \gamma) + \sin(4\beta - \gamma - \alpha) + \sin(4\gamma - \alpha - \beta) = 0$ is equivalent to

$$\frac{a^4}{bc} - \frac{bc}{a^4} + \frac{b^4}{ca} - \frac{ca}{b^4} + \frac{c^4}{ab} - \frac{ab}{c^4} = 0$$

which in turn is equivalent to

$$-a^5b^5 + a^8b^3c^3 + a^3b^8c^3 - a^5c^5 - b^5c^5 + a^3b^3c^8 = 0.$$

This is a symmetric function in a, b, c and can therefore be expressed according the symmetric polynomials theorem as a polynomial in $u = a + b + c, v = ab + bc + ca, w = abc$. We find

$$-a^5b^5 + a^8b^3c^3 + a^3b^8c^3 - a^5c^5 - b^5c^5 + a^3b^3c^8$$
$$= -v^5 + w^3u^5 + 5uv^3w - 5u^2vw^2 - 5v^2w^2 + 5uw^3$$
$$-5u^3vw^3 + 5uv^2w^3 + 5u^2w^4 - 5vw^4.$$

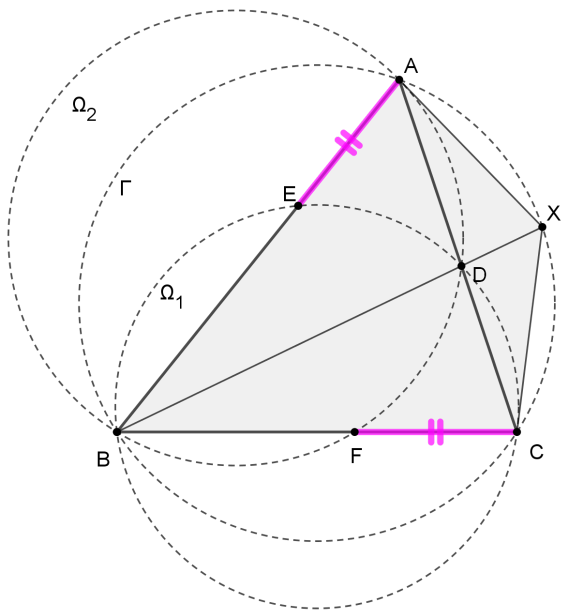
However $u = v = 0$. Hence

$$-a^5b^5 + a^8b^3c^3 + a^3b^8c^3 - a^5c^5 - b^5c^5 + a^3b^3c^8 = 0,$$

and we are done.

E–132. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* **(Correction).** Let ABC be a triangle with circumcircle Γ. Choose an arbitrary point D on side AC, and let X be the intersection of line BD with Γ, different from B. If the circumcircle Ω_1 of the triangle BDC cut AB in E, the circumcircle Ω_2 of the triangle ABD cut BC in F and $AE = CE$, show that

$$\frac{AB + BC}{AC} = \frac{BX}{XA}.$$



Solution by the proposer. B, C, D, E concyclic points $\Rightarrow$ from the power of a point A with respect to the circle Ω_1:

$$AD \cdot AC = AE \cdot AB \quad (1)$$

A, B, F, D concyclic points $\Rightarrow$ from the power of a point C with respect to the circle Ω_1:

$$CD \cdot AC = CF \cdot BC \quad (2)$$

Divide (1) by (2) $\Rightarrow \dfrac{AD \cdot AC}{CD \cdot AC} = \dfrac{AE \cdot AB}{CF \cdot BC}$, with $AE = CF \Rightarrow$ $\dfrac{AD}{CD} = \dfrac{AB}{BC}$ and from the hypothesis $\Rightarrow$

$$AE = CF \Rightarrow BD \text{ bisector of } \angle ABC \Rightarrow AX = XC$$

Apply Ptolemy's theorem in the cyclic quadrilateral $ABCX$

$$AB \cdot XC + BC \cdot AX = AC \cdot BX \text{ where } XA = XC$$

$$\Rightarrow XA(AB + BC) = AC \cdot BX \Rightarrow \frac{AB + BC}{AC} = \frac{BX}{XA}.$$

Also solved by *José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain and José Luis Díaz-Barrero, Barcelona, Spain.*

E–133. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain and José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain.* Let a, b, c be positive reals such that $a+b+c = abc$. Prove that

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \geq \frac{a+b+c}{4}.$$

Solution 1 by Sarah B. Seales, Arizona State University, USA; Miquel Amengual Covas, Cala Figuera, Mallorca, Spain; Brian Bradie, Christopher Newport University, Newport News, VA; Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA; Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania; Ioan Viorel Codreanu, Satulung, Maramures, Romania, and Daniel Văcaru, National Economic College „Maria Teiuleanu", Pitești, Romania (same solution). The inequality is equivalent to

$$\frac{a^2}{a(1+bc)} + \frac{b^2}{b(1+ca)} + \frac{c^2}{c(1+ab)} \geq \frac{a+b+c}{4}.$$

From Bergström's inequality,

$$\frac{a^2}{a(1+bc)} + \frac{b^2}{b(1+ca)} + \frac{c^2}{c(1+ab)} \geq \frac{(a+b+c)^2}{a+b+c+3abc}$$

and using the condition, the right hand side becomes

$$\frac{(a+b+c)^2}{4(a+b+c)} = \frac{a+b+c}{4}.$$

Equality occurs only if $a=b=c=\sqrt{3}$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland. We have

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} - \frac{a+b+c}{4}$$

$$= \frac{a}{\frac{abc}{a+b+c}+bc} + \frac{b}{\frac{abc}{a+b+c}+ca} + \frac{c}{\frac{abc}{a+b+c}+ab} - \frac{a+b+c}{\frac{4abc}{a+b+c}}$$

$$= \frac{(a+b+c)^2\left(2a^3+2b^3+2c^3+a^2b+ab^2+a^2c+c^2a+b^2c+bc^2-12abc\right)}{4abc(2a+b+c)(a+2b+c)(a+b+2c)} \geq 0,$$

by the AM-GM inequality.

Solution 3 by the proposers. Multiplying each term of the fractions in the LHS by its numerator, we get

$$\frac{a^2}{a+abc}+\frac{b^2}{b+bca}+\frac{c^2}{c+cab} \geq \frac{a+b+c}{4}.$$

Putting $s = a+b+c$ in the preceding expression, we obtain on account of the constraint,

$$\frac{a^2}{a+s}+\frac{b^2}{b+s}+\frac{c^2}{c+s} \geq \frac{a+b+c}{4}.$$

The LHS of the previous expression suggest to consider the function

$$f(x) = \frac{x^2}{s+x}$$

which is convex in the set of positive numbers. Indeed,

$$f'(x) = \frac{x(2s+x)}{(s+x)^2} \quad \text{and} \quad f''(x) = \frac{2s^2}{(s+x)^3} \geq 0.$$

Then, applying Jensen's inequality, we have

$$f(a)+f(b)+f(c) \geq 3\,f\left(\frac{a+b+c}{3}\right)$$

or

$$\frac{a^2}{a+s}+\frac{b^2}{b+s}+\frac{c^2}{c+s} \geq 3\,f\left(\frac{s}{3}\right)$$

from which the statement follows. Equality holds when $a = b = c = \sqrt{3}$.

Solution 4 by Michel Bataille, Rouen, France. Let L denote the left-hand side of the inequality and let $s = a+b+c$. From the hypothesis, we deduce that

$$L = \frac{a^2}{a+s}+\frac{b^2}{b+s}+\frac{c^2}{c+s} = a-s+\frac{s^2}{a+s}+b-s+\frac{s^2}{b+s}+c-s+\frac{s^2}{c+s},$$

that is,

$$L = s^2\left(\frac{1}{a+s} + \frac{1}{b+s} + \frac{1}{c+s}\right) - 2s.$$

Now, from the convexity of $x \mapsto \frac{1}{x}$ on $(0, \infty)$, we have

$$\frac{1}{a+s} + \frac{1}{b+s} + \frac{1}{c+s} \geq 3 \cdot \frac{1}{\frac{s+3s}{3}} = \frac{9}{4s},$$

hence

$$L \geq s^2 \cdot \frac{9}{4s} - 2s = \frac{s}{4},$$

as required.

Solution 5 by Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam. First, we rewrite the given inequality in the form:

$$12(a(b+c) + b(c+a) + c(a+b)) \leq (a+b+c)^2(a+b)(b+c)(c+a)$$

or

$$24(ab+bc+ca) \leq (a+b+c)^2(a+b)(b+c)(c+a)$$

By the AM-GM inequality, we obtain:

$$a+b+c \geq 3\sqrt[3]{abc} \geq 3$$

We recall the well-known result from the paper *A Nice Identity* ([see here](https://www.math.hkust.edu.hk/excalibur/v14-n1.pdf)):

$$(a+b)(b+c)(c+a) \geq \frac{8}{9}(a+b+c)(ab+bc+ca)$$

Therefore

$$(a+b+c)^2(a+b)(b+c)(c+a) \geq \frac{8}{9}(a+b+c)^3(ab+bc+ca)$$

$$\geq 24(ab+bc+ca).$$

Equality holds if and only if $a = b = c = 1$.

E–134. *Proposed by Michel Bataille, Rouen, France.* Let n be a nonnegative integer. Prove that

$$\sum_{k=0}^{n}\binom{2n}{2k}288^k289^{n-k}=\sum_{k=0}^{4n}\binom{8n}{2k}2^k.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. By the binomial theorem,

$$(x+y)^{2n}=\sum_{k=0}^{2n}\binom{2n}{k}x^ky^{2n-k}$$

and

$$(x-y)^{2n}=\sum_{k=0}^{2n}\binom{2n}{k}x^k(-y)^{2n-k},$$

so

$$\sum_{k=0}^{n}\binom{2n}{2k}x^{2k}y^{2(n-k)}=\frac{1}{2}((x+y)^{2n}+(x-y)^{2n})$$

and

$$\begin{aligned}\sum_{k=0}^{n}\binom{2n}{2k}288^k289^{n-k}&=\sum_{k=0}^{n}\binom{2n}{2k}(12\sqrt{2})^{2k}17^{2(n-k)}\\&=\frac{1}{2}\Big((12\sqrt{2}+17)^{2n}+(12\sqrt{2}-17)^{2n}\Big)\\&=\frac{1}{2}\Big((17+12\sqrt{2})^{2n}+(17-12\sqrt{2})^{2n}\Big).\end{aligned}$$

On the other hand, again by the binomial theorem,

$$(1+x)^{8n}=\sum_{k=0}^{8n}\binom{8n}{k}x^k$$

and

$$(1-x)^{8n}=\sum_{k=0}^{8n}\binom{8n}{k}(-x)^k,$$

so

$$\sum_{k=0}^{4n}\binom{8n}{2k}x^{2k}=\frac{1}{2}((1+x)^{8n}+(1-x)^{8n})$$

and

$$\begin{aligned}\sum_{k=0}^{4n}\binom{8n}{2k}2^k &= \sum_{k=0}^{4n}\binom{8n}{2k}(\sqrt{2})^{2k}\\ &= \frac{1}{2}((1+\sqrt{2})^{8n}+(1-\sqrt{2})^{8n})\\ &= \frac{1}{2}((3+2\sqrt{2})^{4n}+(3-2\sqrt{2})^{4n})\\ &= \frac{1}{2}((17+12\sqrt{2})^{2n}+(17-12\sqrt{2})^{2n}).\end{aligned}$$

Thus,

$$\sum_{k=0}^{n}\binom{2n}{2k}288^k289^{n-k} = \sum_{k=0}^{4n}\binom{8n}{2k}2^k.$$

Solution 2 by Michel Bataille, Rouen, France. The key remark is the following: from the binomial theorem, we have

$$(a+b)^{2m}+(a-b)^{2m} = 2\sum_{k=0}^{m}\binom{2m}{2k}a^{2m-2k}b^{2k}.$$

From this identity, we deduce that

$$S_n = \sum_{k=0}^{4n}\binom{8n}{2k}2^k = \frac{1}{2}\big((1+\sqrt{2})^{8n}+(1-\sqrt{2})^{8n}\big).$$

Since $(1\pm\sqrt{2})^4 = 17\pm 12\sqrt{2}$, we see that

$$S_n = \frac{1}{2}((17+12\sqrt{2})^{2n}+(17-12\sqrt{2})^{2n}) = \sum_{k=0}^{n}\binom{2n}{2k}17^{2n-2k}(12\sqrt{2})^{2k}$$

and therefore

$$S_n = \sum_{k=0}^{n}\binom{2n}{2k}289^{n-k}\,288^k.$$

Solution 3 by Albert Stadler, Herrliberg, Switzerland. We have, by the binomial theorem,

$$\sum_{k=0}^{n}\binom{2n}{2k}288^k289^{n-k} = \sum_{k=0}^{2n}\binom{2n}{k}\frac{1+(-1)^k}{2}\big(\sqrt{288}\big)^k17^{2n-k}$$

$$= \frac{1}{2}\left(12\sqrt{2}+17\right)^{2n} + \frac{1}{2}\left(-12\sqrt{2}+17\right)^{2n}$$

$$= \frac{1}{2}\left(\sqrt{2}+1\right)^{8n} + \frac{1}{2}\left(-\sqrt{2}+1\right)^{8n}$$

$$= \sum_{k=0}^{8n} \binom{8n}{k} \frac{1+(-1)^k}{2}\left(\sqrt{2}\right)^k = \sum_{k=0}^{4n} \binom{8n}{2k} 2^k.$$

Also solved by *José Luis Díaz-Barrero, Barcelona, Spain.*

E–135. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let a, b, n be integers such that $0 < a \leq b < n$. Prove that there exist a prime p that divide both $\binom{n}{a}$ and $\binom{n}{b}$.

Solution 1 by Michel Bataille, Rouen, France. We suppose that $0 < a < b < n$ (the result is obvious if $a = b$), so that $n \geq 3$.
We remark that

$$\binom{n}{b}\binom{b}{a} = \binom{n}{a}\binom{n-a}{b-a}$$

(both sides equal $\frac{n!}{a!(n-b)!(b-a)!}$).
Also, we have $\binom{n}{a} > \binom{b}{a}$: indeed, this is equivalent to $\frac{n!}{b!} > \frac{(n-a)!}{(b-a)!}$, that is, to $n(n-1)\cdots(b+1) > (n-a)(n-a-1)\cdots(b+1-a)$, which holds because $n > n-a,\ n-1 > n-a-1, \ldots, b+1 > b+1-a$. Now, from a theorem of Gauss, if the integers $\binom{n}{a}$ and $\binom{n}{b}$ were coprime, then $\binom{n}{a}$ would divide $\binom{b}{a}$, a contradiction since $\binom{n}{a} > \binom{b}{a}$. Thus, $\binom{n}{a}$ and $\binom{n}{b}$ are not coprime, hence, being greater than 1, they have a common prime divisor.

Solution 2 by the proposers. We argue by contradiction. Suppose that $\binom{n}{a}$ and $\binom{n}{b}$ are relatively prime. We need the following identity:

$$\binom{n}{b}\binom{b}{a} = \binom{n}{a}\binom{n-a}{b-a}$$

valid for $0 \leq a \leq b \leq n$. Indeed, suppose we have a class with n students. Then we may choose $\binom{n}{b}$ committees of b students and for each b students we have $\binom{n}{a}$ subcommittees with a girls. Thus, the total number of students may be counted (committed) in two ways:

$$\binom{n}{b}\binom{b}{a} = \binom{n}{a}\binom{n-a}{b-a}.$$

From the preceding, we get that

$$\binom{n}{a} \Big| \binom{n}{b}\binom{b}{a}.$$

But since $\binom{n}{a}$ and $\binom{n}{b}$ are coprime, it follows that $\binom{n}{a}$ divides $\binom{b}{a}$ which is impossible because it is clear that $\binom{b}{a} < \binom{n}{a}$. Thus,

$$GCD\left(\binom{n}{a}, \binom{n}{b}\right) > 1,$$

and we are done.

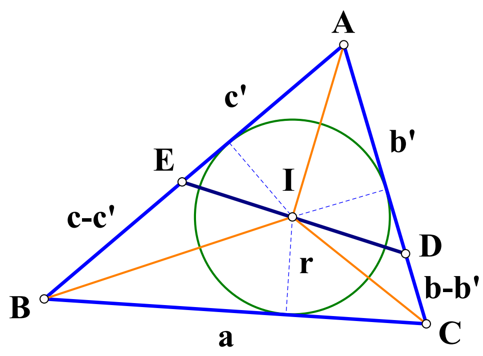
E–136. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let ℓ be a line that divides triangle ABC into two parts. Prove that ℓ divides the area and the perimeter of ABC in the same proportion if and only if ℓ passes through the incenter of triangle ABC.

Solution 1 by the proposers. $\Rightarrow$) We first assume that ℓ divides the area and the perimeter of $\triangle ABC$ in the same ratio. Using the standard notation the ratio of areas is

$$\frac{[EBCD]}{[AED]} = \frac{[ABC] - [AED]}{[AED]} = \frac{[ABC]}{[AED]} - 1$$

while the ratio of perimeters is

$$\frac{EB + BC + CD}{DA + AE} = \frac{(c - c') + a + (b - b')}{b' + c'}$$



Scheme for solving problem E-136.

The right-hand sides are equal when

$$\frac{[ABC]}{[AED]} = 1 + \frac{(c-c') + a + (b-b')}{b'+c'} = \frac{a+b+c}{b'+c'}$$

from which it follows

$$[AED] = \frac{b'+c'}{a+b+c}[ABC] = \frac{b'+c'}{a+b+c} \cdot (a+b+c)\frac{r}{2} = (b'+c')\frac{r}{2}$$

We want to prove that I lies on DE. To this end we let the bisector of $\angle BAC$ meet DE at F. The perpendicular distances from F to AC and AE have the same value, say d. Therefore,

$$[AED] = [AEF] + [AFD] = \frac{c'd}{2} + \frac{b'd}{2} = (b'+c')\frac{d}{2}$$

Equating both expressions for $[AED]$ we get $d = r$. Because I is the unique point on the angle bisector AI at a distance of r from AB and AC, it follows that F coincides with I, whence I lies on DE as claimed.

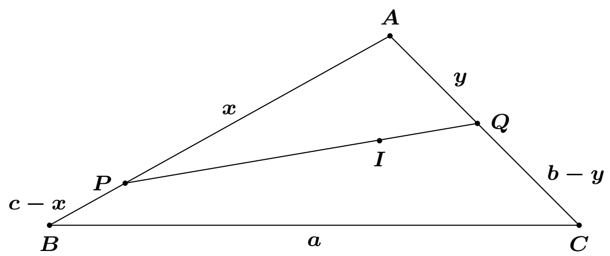
$\Leftarrow$) For the converse we are given that I lies on DE. We start the chain of equalities with the ratio of areas,

$$\begin{aligned}
\frac{[EBCD]}{[AED]} &= \frac{[ABC]-[AED]}{[AED]} \\
&= \frac{\frac{r}{2}\cdot(a+b+c)-([AEI]+[AID])}{[AEI]+[AID]} \\
&= \frac{\frac{r}{2}\cdot(a+b+c)-\left(b'\cdot\frac{r}{2}+c'\cdot\frac{r}{2}\right)}{b'\cdot\frac{r}{2}+c'\cdot\frac{r}{2}} \\
&= \frac{(c-c')+a+(b-b')}{b'+c'} = \frac{EB+BC+CD}{DA+AE},
\end{aligned}$$

and end with the ratio of the two pieces of the perimeter. That is, the two ratios are equal as desired.

Solution 2 by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain. We may assume without loss of generality that ℓ intersects the sides AB and AC at P and Q, respectively.

We set $a = BC$, $b = CA$, $c = AB$, $x = AP$, and $y = AQ$. Let I denotes the incenter of $\triangle ABC$.



From [1], we have that PQ passes through I if and only if

$$bc\left(\frac{1}{x}+\frac{1}{y}\right) = a+b+c$$

or, equivalently,

$$\frac{bc}{xy} = \frac{a+b+c}{x+y}. \tag{1}$$

Let us agree to denote the area of any figure by the name of the figure enclosed in brackets. Since the area of a triangle is equal to one-half the product of the length of two of the sides and the sinus of their included angle, equation (1) may therefore be written as

$$\frac{[ABC]}{[APQ]} = \frac{a+b+c}{x+y}$$

Subtracting 1 from each side gives

$$\frac{[PBCQ]}{[APQ]} = \frac{a+b+c-x-y}{x+y} = \frac{(c-x)+a+(b-y)}{x+y}$$
$$= \frac{PB+BC+CQ}{AP+AQ},$$

and the conclusion follows.

Reference.

[1] The Olympiad Corner No. 250, Crux Mathematicorum, Vol. 31, No. 8, December 2005, pp. 534-36.

Solution 3 by Michel Bataille, Rouen, France. Without loss of generality we suppose that ℓ intersects the sides AC and AB in D and E, respectively. We set $BC = a, CA = b, AB = c, AD = d, AE = e$ and use barycentric coordinates relatively to (A, B, C). Let $I = (a : b : c)$ be the incenter of ABC.
Since $D = ((b-d) : 0 : d)$ and $E = (c-e : e : 0)$, the equation of $\ell = DE$ is $dex - d(c-e)y - e(b-d)z = 0$, hence I is on the line ℓ if and only if $dea - d(c-e)b - e(b-d)c = 0$, that is,

$$\frac{a+b+c}{bc} = \frac{d+e}{de}. \tag{1}$$

Let $[\cdot]$ denote area. We have $2[ABC] = bc\sin A$ and $2[ADE] = de\sin A$, hence $\dfrac{[ADE]}{[ABC]} = \dfrac{de}{bc}$ The line ℓ divides the perimeter of ABC in the ratio $\dfrac{d+e}{a+b+c}$. Thus, ℓ divides the area and the perimeter in the same proportion if and only $\dfrac{de}{bc} = \dfrac{d+e}{a+b+c}$, that is, if and only if (1) holds, or if and only if ℓ passes through I.

Easy–Medium Problems

EM–131. *Proposed by Michel Bataille, Rouen, France.* Let ABC be a triangle neither equilateral nor right-angled and let O be its circumcentre. Let A', B', C' be the respective reflections of A, B, C about O and let U, V, and W be the circumcentres of $\Delta OB'C', \Delta OC'A'$, and $\Delta OA'B'$, respectively. Prove that the lines AU, BV, CW are concurrent.

Solution by the proposer. We embed the problem in the complex plane and without loos of generality we suppose that the circumcircle of ΔABC is the unit circle Γ (centre O, radius 1). We denote by m (lower-case letter) the affix of the point M (upper-case letter). With these notations, we have $|a| = |b| = |c| = 1$ and $a' = -a, b' = -b, c' = -c$. From the definition of U, we have $|u|^2 = |u - b'|^2 = |u - c'|^2$, that is, $|u|^2 = |u + b|^2 = |u + c|^2$. Since $\overline{b} = \frac{1}{b}$ and $\overline{c} = \frac{1}{c}$, we obtain

$$u\overline{u} = (u + b)\left(\overline{u} + \frac{1}{b}\right) = (u + c)\left(\overline{u} + \frac{1}{c}\right)$$

and readily deduce that $u = \frac{-bc}{b+c}$ (and $\overline{u} = \frac{-1}{b+c}$).
Now, let the line AU intersect Γ at D ($D \neq A$). The equation of the line $AD = AU$ is $z + ad\overline{z} = a + d$ with $u + ad\overline{u} = a + d$. We deduce that $d = -\dfrac{ab + bc + ca}{a + b + c}$ (note that $a + b + c \neq 0$ since ΔABC is not equilateral). This affix d does not change when a, b, c are permuted, hence D is also on BV and CW and we conclude that AU, BV, CW are concurrent (and their common point is on circle Γ).

Also solved by *José Luis Díaz-Barrero, Barcelona, Spain.*

EM–132. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let a, b be positive integers such that $(a, b) = d$. Prove that

$$\frac{\varphi(d)}{d} \leq \frac{1}{2}\left(\frac{\varphi^2(a) + \varphi^2(b)}{\varphi(ab)}\right),$$

where $\varphi(n)$ is the Euler's totient function.

Solution 1 by Michel Bataille, Rouen, France. If $d = 1$, then $\varphi(ab) = \varphi(a)\varphi(b)$ and $\varphi(d) = 1$, hence the inequality writes as $(\varphi(a) - \varphi(b))^2 \geq 0$, which clearly holds.
Suppose now that $d > 1$ and let $\{p_1, p_2, \ldots, p_k\}$ denote the set of all the prime divisors of d. The integers a, b, d then write as

$$a = p_1^{r_1} \cdots p_k^{r_k} \cdot a_1, \quad b = p_1^{s_1} \cdots p_k^{s_k} \cdot b_1, \quad d = p_1^{u_1} \cdots p_k^{u_k}$$

where a_1, b_1 and $r_i, s_i, u_i = \min(r_i, s_i)$ $(i = 1, \ldots, k))$ are positive integers, none of $p_1, \ldots, p_k$ dividing a_1 or b_1. Note also that a_1 and b_1 are coprime (a common prime factor would divide d, hence be among $p_1, \ldots, p_k$) and that the least common multiple of a and b is $m = p_1^{v_1} \cdots p_k^{v_k} \cdot a_1 b_1$ where $v_i = \max(r_i, s_i)$ $(i = 1, \ldots, k)$.
Using well-known results about φ and setting $h = \prod\limits_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$, we obtain

$$\varphi(a) = h\varphi(a_1) \prod_{i=1}^{k} p_i^{r_i}, \quad \varphi(b) = h\varphi(b_1) \prod_{i=1}^{k} p_i^{s_i}, \quad \varphi(d) = h \prod_{i=1}^{k} p_i^{u_i},$$

$$\varphi(m) = h\varphi(a_1)\varphi(b_1) \prod_{i=1}^{k} p_i^{v_i}, \quad \varphi(ab) = h\varphi(a_1)\varphi(b_1) \prod_{i=1}^{k} p_i^{r_i+s_i}.$$

Using $r_i + s_i = u_i + v_i$ $(i = 1, \ldots, k)$, we readily deduce that

$$\varphi(a)\varphi(b) = \varphi(m)\varphi(d), \qquad \varphi(ab) = d\varphi(m).$$

Since $\varphi^2(a) + \varphi^2(b) \geq 2\varphi(a)\varphi(b)$, it follows that

$$\frac{1}{2}\left(\frac{\varphi^2(a) + \varphi^2(b)}{\varphi(ab)}\right) \geq \frac{\varphi(a)\varphi(b)}{\varphi(ab)} = \frac{\varphi(d)\varphi(m)}{d\varphi(m)} = \frac{\varphi(d)}{d},$$

as desired.

Solution 2 by Ricardo, Westchester Area Math Circle, Purchase, New York, USA. Each prime divisor of ab is either a prime divisor of a or a prime divisor of b—or possibly a prime divisor of both a and b, Thus

$$\frac{\varphi(ab)}{ab} = \prod_{p|ab}\left(1 - \frac{1}{p}\right) = \frac{\prod_{p|a}\left(1 - \frac{1}{p}\right)\prod_{p|b}\left(1 - \frac{1}{p}\right)}{\prod_{p|(a,b)}\left(1 - \frac{1}{p}\right)} = \frac{\frac{\varphi(a)}{a}\frac{\varphi(b)}{b}}{\frac{\varphi(d)}{d}}.$$

It follows that
$$\frac{\varphi(d)}{d} = \frac{\varphi(a)\varphi(b)}{\varphi(ab)},$$
and so, applying the Power Mean inequality ($GM \le QM$), we see that
$$\frac{\varphi(d)}{d} = \frac{\varphi(a)\varphi(b)}{\varphi(ab)} \le \frac{\frac{\varphi^2(a)+\varphi^2(b)}{2}}{\varphi(ab)} = \frac{1}{2}\left(\frac{\varphi^2(a)+\varphi^2(b)}{\varphi(ab)}\right).$$
Equality holds if and only if $\varphi(a) = \varphi(b)$.

Solution 3 by the proposers. Since $(a,b) = d$, then there exist two positive integers m, n such that $a = dm$, $b = dn$, where $(m,n) = 1$. Let $d = \prod_k p_k^{\alpha_k}$. Then
$$ab = mn \prod_k p_k^{2\alpha_k}$$
and
$$\varphi(ab) = \varphi(mn)\prod_k \varphi(p_k^{2\alpha_k}) = \varphi(mn)\prod_k \left(p^{2\alpha_k - 1}\right)(p_k - 1)$$
On the other hand,
$$\begin{aligned}\varphi(a)\varphi(b) &= \varphi(m)\varphi\left(\prod_k p_k^{\alpha_k}\right)\varphi(n)\varphi\left(\prod_k p_k^{\alpha_k}\right)\\ &= \varphi(m)\varphi(n)\left(\prod_k p_k^{\alpha_k - 1}(p_k - 1)\right)^2\\ &= \varphi(m)\varphi(n)\prod_k p_k^{2\alpha_k - 1}\frac{(p_k-1)^2}{p_k}\\ &= \varphi(ab)\prod_k\left(1 - \frac{1}{p_k}\right) = \frac{\varphi(ab)\varphi(d)}{d}\end{aligned}$$
Taking into account GM-QM inequality, we have
$$\sqrt{\frac{\varphi(ab)\varphi(d)}{d}} = \sqrt{\varphi(a)\varphi(b)} \le \sqrt{\frac{\varphi^2(a)+\varphi^2(b)}{2}}$$

and

$$\frac{\varphi(ab)}{\varphi^2(a)+\varphi^2(b)} \le \frac{d}{2\varphi(d)}$$

from which after inverting terms the statement follows. Equality holds when $d=(a,b)=1$ and $\varphi(a)=\varphi(b)$.

Also solved by *Albert Stadler, Herrliberg, Switzerland.*

EM–133. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Find all real solutions of the equation:

$$\sqrt{x}+\sqrt[3]{x}+\sqrt[4]{x}+\sqrt[5]{x}+\ldots+\sqrt[2024]{x}=\sqrt[2025]{x+2024^{2025}-1}$$

Solution by the proposer. Denoting by $f(x)=\sqrt[3]{x}+\sqrt[4]{x}+\sqrt[5]{x}+\ldots+\sqrt[2024]{x}$ the equation claimed may be written as

$$\sqrt{x}+f(x)=\sqrt[2025]{x+2024^{2025}-1}.$$

Raising to 2025 both terms of the above expression, yields

$$x^{2025/2}+\sum_{k=1}^{2025}\binom{2025}{k}\sqrt{x}^{2025-k}f^k(x)=x+2024^{2025}-1$$

It is obvious that $x\ge 0$. So, we consider two cases:

1. If $x<1$ then $x^{2025/2}<x$. Let $g:(0,1)\to\mathbb{R}$ be the function defined by

$$g(x)=\sum_{k=1}^{2025}\binom{2025}{k}\sqrt{x}^{2025-k}f^k(x).$$

We observe that $f(1)=2023$ and $g(x)<g(1)$, as can be easily checked. That is,

$$g(x)<g(1)=\sum_{k=1}^{2025}\binom{2025}{k}f^k(1)=\sum_{k=1}^{2025}\binom{2025}{k}2023^k=2024^{2025}-1$$

on account of the above. Therefore, for $x<1$ holds

$$x^{2025/2}+\sum_{k=1}^{2025}\binom{2025}{k}\sqrt{x}^{2025-k}f^k(x)<x+2024^{2025}-1$$

2. If $x > 1$ then $x^{2025/2} > x$ and $g(x) > g(1)$. Then, holds

$$x^{2025/2} + \sum_{k=1}^{2025} \binom{2025}{k} \sqrt{x}^{2025-k} f^k(x) > x + 2024^{2025} - 1$$

From the preceding we conclude that

$$x^{2025/2} + \sum_{k=1}^{2025} \binom{2025}{k} \sqrt{x}^{2025-k} f^k(x) = x + 2024^{2025} - 1$$

holds only when $x = 1$ which is the unique solution of the equation given in the statement.

Also solved by *José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain.*

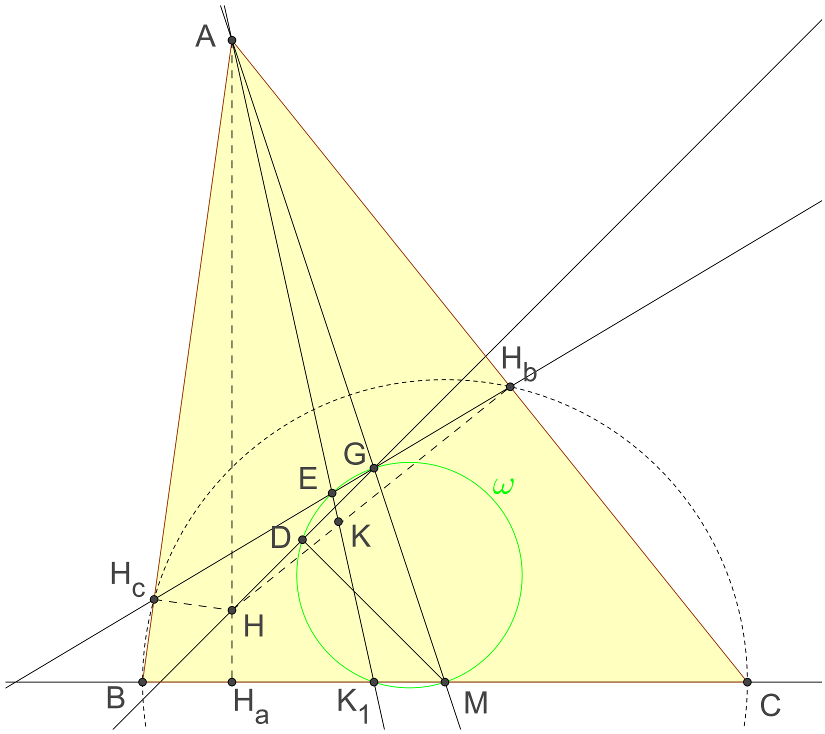
EM–134. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a non-right triangle with $AB \neq AC$ and let H be its orthocenter, G be its centroid, K be its symmedian point. Let H_b, H_c be the feet of the altitudes drawn from B, C respectively. Let M, E be the midpoints of BC, H_bH_c respectively and let D be the foot of the perpendicular from M to the line GH. Let $K_1 = AK \cap BC$. Knowing that G lies on H_bH_c, prove that the points M, D, E, K_1 are concyclic.

Solution 1 by the proposer. Let ω be the circle with diameter GM. Since $DM \perp GH$, hence $\angle MDG = 90^\circ$ and $D \in \omega$.

Since $\angle BH_bC = \angle BH_cC = 90^\circ$, the points H_b, H_c lie on the circle with diameter BC, centered at M, so $MH_b = MH_c$. ME is a median in $\triangle MH_bH_c$ and so $ME \perp H_bH_c$ and since $G \in H_bH_c$, hence $\angle MEG = 90^\circ, E \in \omega$.

The centroid G and the symmedian point K are isogonal conjugates, so $\angle BAK = \angle CAG$.

It is well-known that $\angle ABC = \angle AH_bH_c$ and $\triangle ABC \sim \triangle AH_bH_C$. Let $\mathcal{S}$ be the transformation consisting of a homothety centered at A, followed by a reflection with respect to the internal angle bisector of angle $\angle BAC$, sending H_bH_c to BC. Then, $\mathcal{S} : E \to M$



Scheme for Solution of EM-134.

as midpoints and $\mathcal{S} : AE \to AM$, $\mathcal{S} : \triangle AEH_c \to \triangle AMC$ or $\angle EAH_c = \angle MAC$ so $E \in AK$.

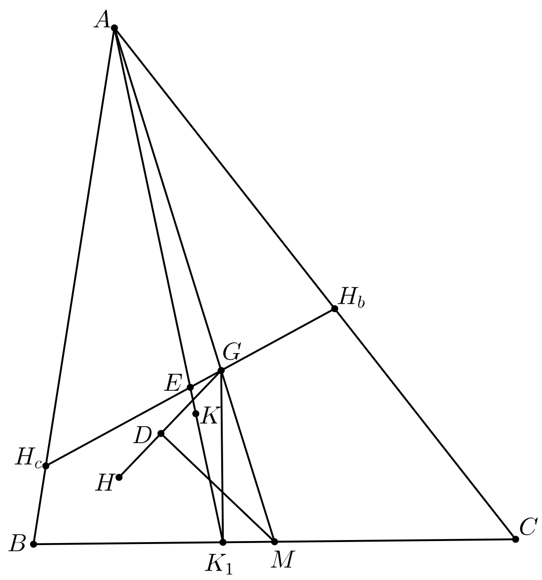
Now $\mathcal{S} : AG \to AK_1$, $\mathcal{S} : G \to K_1$. Hence $\mathcal{S} : \triangle AEG \to \triangle AMK_1$ or $\triangle AEG \sim \triangle AMK_1$. So $\angle AGE = \angle AK_1M \equiv \angle EK_1M$ and $\angle EK_1M + \angle EGM = \angle AK_1M + (180^\circ - \angle AGE) = 180^\circ$. Hence the points G, E, K_1, M lie on circle ω.

It follows that the points D, E, G, K_1, M are concyclic.

Solution 2 by Michel Bataille, Rouen, France. Let $a = BC, b = CA, c = AB$ as usual. In barycentric coordinates relatively to (A, B, C), we have

$$M = (0 : 1 : 1), G = (1 : 1 : 1),\ K = (a^2 : b^2 : c^2),\ K_1 = (0 : b^2 : c^2),$$

and $H = (S_BS_C : S_CS_A : S_AS_B)$ where $S_A = \frac{b^2+c^2-a^2}{2}$ $S_B = \frac{c^2+a^2-b^2}{2}$, $S_C = \frac{a^2+b^2-c^2}{2}$. For later use, note that S_A is the dot



Scheme for solving problem EM-134.

product $\overrightarrow{AB}\cdot\overrightarrow{AC}$.
We deduce that $H_b = (S_C : 0 : S_A), H_c = (S_B : S_A : 0)$ so that the equation of the line H_bH_c is $xS_A - yS_B - zS_C = 0$. Expressing that G is on this line gives $S_A = S_B + S_C = a^2$, hence $3a^2 = b^2 + c^2$. Taking this into account, we obtain

$$\begin{aligned} 2b^2c^2E &= (b^2c^2)(H_b + H_c) = c^2(b^2H_b) + b^2(c^2H_c) \\ &= (c^2S_C + b^2S_B)A + (b^2S_A)B + (c^2S_A)C \\ &= mA + a^2b^2B + a^2c^2C \end{aligned}$$

where we have set $m = b^2S_B + c^2S_C$.
Now, from $3a^2K_1 = b^2B + c^2C$ and $3a^2G = a^2A + a^2B + a^2C$, we obtain $3a^2\overrightarrow{K_1G} = (a^2 - b^2)\overrightarrow{AB} + (a^2 - c^2)\overrightarrow{AC}$ so that

$$3a^2\overrightarrow{K_1G}\cdot\overrightarrow{CB} = ((a^2 - b^2)\overrightarrow{AB} + (a^2 - c^2)\overrightarrow{AC})\cdot(\overrightarrow{AB} - \overrightarrow{AC}).$$

Since $\overrightarrow{AB}^2 = c^2, \overrightarrow{AC}^2 = b^2$ and $\overrightarrow{AB}\cdot\overrightarrow{AC} = S_A = a^2$, a simple calculation gives $\overrightarrow{K_1G}\cdot\overrightarrow{CB} = 0$. Therefore $GK_1 \perp K_1M$ and K_1 is on the circle Γ with diameter GM. Since $MD \perp DG$, D is also on this circle. Thus, it just remains to show that E is on Γ, that

is, $GE \perp EM$.
Similarly, from $6b^2c^2E = 3mA + 3a^2b^2B + 3a^2c^2C$, $6b^2c^2M = 3b^2c^2B + 3b^2c^2C$ and $6b^2c^2G = 2b^2c^2A + 2b^2c^2B + 2b^2c^2C$, we get $6b^2c^2\overrightarrow{EM} = 3b^2(c^2 - a^2)\overrightarrow{AB} + 3c^2(b^2 - a^2)\overrightarrow{AC}$ and $6b^2c^2\overrightarrow{GE} = (b^2 - c^2)(b^2\overrightarrow{AB} - c^2\overrightarrow{AC})$. Again, by a simple calculation, we obtain $\overrightarrow{GE} \cdot \overrightarrow{EM} = 0$, that is, $GE \perp EM$.

EM–135. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Compute the value of

$$\cos\left(\frac{2\pi}{17}\right)\cos\left(\frac{4\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\dots\cos\left(\frac{16\pi}{17}\right).$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. With

$$\cos\left(\frac{10\pi}{17}\right) = -\cos\left(\frac{7\pi}{17}\right), \quad \cos\left(\frac{12\pi}{17}\right) = -\cos\left(\frac{5\pi}{17}\right),$$

$$\cos\left(\frac{14\pi}{17}\right) = -\cos\left(\frac{3\pi}{17}\right), \quad \text{and} \quad \cos\left(\frac{16\pi}{17}\right) = -\cos\left(\frac{\pi}{17}\right),$$

the desired product is equal to

$$\prod_{k=1}^{8}\cos\left(\frac{k\pi}{17}\right).$$

Now, let n be a positive integer, and let $\omega_n = e^{i(2\pi/n)}$. Then

$$1 + z + z^2 + \dots + z^{n-1} = \frac{z^n - 1}{z - 1} = \prod_{k=1}^{n-1}(z - \omega_n^k).$$

Substituting $z = -1$ and taking the modulus of both sides yields

$$\frac{1 - (-1)^n}{2} = \prod_{k=1}^{n-1}|1 + \omega_n^k|.$$

But,

$$1 + \omega_n^k = 1 + e^{i(2\pi k/n)} = \left(e^{-i(k\pi/n)} + e^{i(k\pi/n)}\right)e^{i(k\pi/n)}$$

$$= 2\cos\left(\frac{k\pi}{n}\right)e^{i(k\pi/n)},$$

so

$$|1+\omega_n^k| = 2\left|\cos\left(\frac{k\pi}{n}\right)\right|$$

and

$$\prod_{k=1}^{n-1}\left|\cos\left(\frac{k\pi}{n}\right)\right| = \frac{1-(-1)^n}{2^n}.$$

If n is an odd integer, $n = 2m+1$, then

$$\prod_{k=1}^{2m}\left|\cos\left(\frac{k\pi}{2m+1}\right)\right| = \frac{1}{2^{2m}}.$$

Because

$$\cos\left(\frac{k\pi}{2m+1}\right) > 0 \quad \text{for } 1 \le k \le m,$$

and

$$\cos\left(\frac{(2m+1-k)\pi}{2m+1}\right) = -\cos\left(\frac{k\pi}{2m+1}\right),$$

it follows that

$$\prod_{k=1}^{m}\cos\left(\frac{k\pi}{2m+1}\right) = \frac{1}{2^m}.$$

Finally,

$$\begin{aligned}\cos\left(\frac{2\pi}{17}\right)\cos\left(\frac{4\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\cdots\cos\left(\frac{16\pi}{17}\right) &= \prod_{k=1}^{8}\cos\left(\frac{k\pi}{2(8)+1}\right)\\ &= \frac{1}{2^8} = \frac{1}{256}.\end{aligned}$$

Solution 2 by Michel Bataille, Rouen, France. Let P denote the required product. We show that $P = \frac{1}{256}$.

Since $\cos x = -\cos y$ if $x + y = \pi$, we have

$$\begin{aligned}&\cos\left(\frac{10\pi}{17}\right)\cos\left(\frac{12\pi}{17}\right)\cos\left(\frac{14\pi}{17}\right)\cos\left(\frac{16\pi}{17}\right)\\ &= \cos\left(\frac{7\pi}{17}\right)\cos\left(\frac{5\pi}{17}\right)\cos\left(\frac{3\pi}{17}\right)\cos\left(\frac{\pi}{17}\right),\end{aligned}$$

hence $P = Q \cdot R$ where

$$Q = \cos\left(\frac{\pi}{17}\right)\cos\left(\frac{2\pi}{17}\right)\cos\left(\frac{4\pi}{17}\right)\cos\left(\frac{8\pi}{17}\right)$$

and

$$R = \cos\left(\frac{3\pi}{17}\right)\cos\left(\frac{5\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\cos\left(\frac{7\pi}{17}\right).$$

Now, since $\sin a \cos a = \frac{1}{2}\sin(2a)$ and $\sin(\pi - x) = \sin x$, we have

$$\begin{aligned}\left(\sin\left(\frac{\pi}{17}\right)\right)Q &= \frac{1}{2}\sin\left(\frac{2\pi}{17}\right)\cos\left(\frac{2\pi}{17}\right)\cos\left(\frac{4\pi}{17}\right)\cos\left(\frac{8\pi}{17}\right)\\ &= \frac{1}{4}\sin\left(\frac{4\pi}{17}\right)\cos\left(\frac{4\pi}{17}\right)\cos\left(\frac{8\pi}{17}\right)\\ &= \frac{1}{8}\sin\left(\frac{8\pi}{17}\right)\cos\left(\frac{8\pi}{17}\right)\\ &= \frac{1}{16}\sin\left(\frac{16\pi}{17}\right) = \frac{1}{16}\sin\left(\frac{\pi}{17}\right)\end{aligned}$$

so that $Q = \frac{1}{16}$ (since $\sin\left(\frac{\pi}{17}\right) \neq 0$).
In a similar way, we obtain

$$\begin{aligned}\left(\sin\left(\frac{3\pi}{17}\right)\right)R &= \frac{1}{2}\sin\left(\frac{6\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\cos\left(\frac{5\pi}{17}\right)\cos\left(\frac{7\pi}{17}\right)\\ &= \frac{1}{4}\sin\left(\frac{12\pi}{17}\right)\cos\left(\frac{5\pi}{17}\right)\cos\left(\frac{7\pi}{17}\right)\\ &= \frac{1}{4}\sin\left(\frac{5\pi}{17}\right)\cos\left(\frac{5\pi}{17}\right)\cos\left(\frac{7\pi}{17}\right)\\ &= \frac{1}{8}\sin\left(\frac{10\pi}{17}\right)\cos\left(\frac{7\pi}{17}\right)\\ &= \frac{1}{16}\sin\left(\frac{14\pi}{17}\right) = \frac{1}{16}\sin\left(\frac{3\pi}{17}\right)\end{aligned}$$

and $R = \dfrac{1}{16}$ follows. We conclude that $P = \dfrac{1}{16} \cdot \dfrac{1}{16} = \dfrac{1}{256}$.

Solution 3 by Albert Stadler, Herrliberg, Switzerland. We start with the identity

$$\prod_{k=1}^{n}\left(1 - xe^{\frac{2\pi ik}{n}}\right) = 1 - x^n,$$

which holds true, since both sides are polynomials with the same set of zeros and the same constant coefficient. So they must be identical. If n is odd then

$$\prod_{k=1}^{n}\left(1-x^2e^{\frac{4\pi ik}{n}}\right)=\prod_{k=1}^{n}\left(1-xe^{\frac{2\pi ik}{n}}\right)\prod_{k=1}^{n}\left(1+xe^{\frac{2\pi ik}{n}}\right)$$

$$=(1-x^n)(1+x^n)=\left(1-x^{2n}\right).$$

We set $x=i$ and find

$$\prod_{k=1}^{n-1}\left(1+e^{\frac{4\pi ik}{n}}\right)=1.$$

We note that $\cos\left(\frac{2\pi k}{17}\right)>0,\ k=1,2,3,4,$ and $\cos\left(\frac{2\pi k}{17}\right)<0,\ k=5,6,7,8,$ and $\cos\left(\frac{2\pi k}{17}\right)=\cos\left(\frac{2\pi(17-k)}{17}\right)$.

Hence

$\cos\left(\frac{2\pi}{17}\right)\cos\left(\frac{4\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\cos\left(\frac{8\pi}{17}\right)\cos\left(\frac{10\pi}{17}\right)\cos\left(\frac{12\pi}{17}\right)\cos\left(\frac{14\pi}{17}\right)\cos\left(\frac{16\pi}{17}\right)>0$ and

$$\cos\left(\frac{2\pi}{17}\right)\cos\left(\frac{4\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\cos\left(\frac{8\pi}{17}\right)\cos\left(\frac{10\pi}{17}\right)\cos\left(\frac{12\pi}{17}\right)\cos\left(\frac{14\pi}{17}\right)\cos\left(\frac{16\pi}{17}\right)$$

$$=\sqrt{\prod_{k=1}^{16}\cos\left(\frac{2\pi k}{17}\right)}=\sqrt{\prod_{k=1}^{16}\left(\frac{e^{\frac{2\pi ik}{17}}+e^{-\frac{2\pi ik}{17}}}{2}\right)}$$

$$=\frac{1}{256}\sqrt{e^{-\sum_{k=1}^{16}\frac{2\pi ik}{17}}\prod_{k=1}^{16}\left(1+e^{\frac{4\pi ik}{17}}\right)}=\frac{1}{256}.$$

Also solved by *Arkady Alt, San Jose, California, USA; Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA; Sarah B. Seales Arizona State University, USA, and the proposers.*

EM–136. *Proposed by Michel Bataille, Rouen, France.* Let $a,b,$ and c be positive real numbers such that $abc\geq 1$. Prove that

$$\frac{a}{(a+b)(a+c)}+\frac{b}{(b+c)(b+a)}+\frac{c}{(c+a)(c+b)}\leq\frac{(a+b+c)^2}{12}.$$

Solution 1 by Michel Bataille, Rouen, France. It suffices to show that for all $a, b, c > 0$,

$$abc\left(\frac{a}{(a+b)(a+c)} + \frac{b}{(b+c)(b+a)} + \frac{c}{(c+a)(c+b)}\right) \le \frac{(a+b+c)^2}{12}.$$

By homogeneity, we may suppose that $a+b+c=1$. Then the above inequality becomes

$$12abc(a(1-a)+b(1-b)+c(1-c)) \le (1-a)(1-b)(1-c)$$

or

$$24abc(ab+bc+ca) \le ab+bc+ca-abc.$$

(since $a^2+b^2+c^2 = (a+b+c)^2-2(ab+bc+ca) = 1-2(ab+bc+ca)$).

Recalling that $abc \le \frac{(a+b+c)^3}{27} = \frac{1}{27}$, it is sufficient to prove that

$$8(ab+bc+ca) \le 9(ab+bc+ca-abc),$$

that is,

$$9 \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

We are done since by the Cauchy-Schwartz inequality, we have

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)(a+b+c) \ge (1+1+1)^2 = 9.$$

Solution 2 by Ioan Viorel Codreanu, Satulung, Maramures, Romania. We have

$$\sum \frac{a}{(a+b)(a+c)} \le \sum \frac{a}{2\sqrt{ab}\cdot 2\sqrt{ac}} = \frac{1}{4}\sum \frac{1}{\sqrt{b}\cdot\sqrt{c}} \le \frac{1}{4}\sum \frac{1}{a}$$

$$= \frac{1}{4}\cdot\frac{\sum ab}{\prod a} \le \frac{\sum ab}{4} \le \frac{(\sum a)^2}{12}.$$

Solution 3 by Henry Ricardo, Westchester Area Math Circle,

Purchase, New York, USA. We have

$$\begin{aligned}\sum_{cyclic} \frac{a}{(a+b)(a+c)} &\overset{\text{AGM}}{\leq} \sum_{cyclic} \frac{a}{2\sqrt{ab}\cdot 2\sqrt{ac}} \\ &= \frac{1}{4}\sum_{cyclic} \frac{\sqrt{bc}}{bc} \\ &= \frac{1}{4abc}\sum_{cyclic} a\sqrt{bc} \\ &\overset{\text{AGM}}{\leq} \frac{1}{4}\sum_{cyclic} a\left(\frac{b+c}{2}\right) \\ &= \frac{1}{4}(ab+bc+ca) \\ &\overset{\text{M}}{\leq} \frac{(a+b+c)^2}{12}.\end{aligned}$$

Equality holds if and only if $a = b = c = 1$.

Solution 4 by Sarah B. Seales, Arizona State University, USA.
We have

$$\frac{a}{(a+b)(a+c)}+\frac{b}{(b+c)(b+a)}+\frac{c}{(c+a)(c+b)} = \frac{2(ab+bc+ca)}{(a+b)(b+c)(c+b)}$$

so it's enough to show

$$\frac{2(ab+bc+ca)}{(a+b)(b+c)(c+b)} \leq \frac{(a+b+c)^2}{12}.$$

Since $(a+b)(b+c)(c+a) \geq 8abc \geq 8$ from AM-GM and the given condition, we need to show $\frac{1}{4}(ab+bc+ca) \leq \frac{(a+b+c)^2}{12}$, which simplifies to

$$3(ab+bc+ca) \leq (a+b+c)^2.$$

This is equivalent to

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0,$$

which is true by the trivial inequality.

Solution 5 by Albert Stadler, Herrliberg, Switzerland. We have

$$\frac{(a+b+c)^2}{12}-\frac{a}{(a+b)(a+c)}-\frac{b}{(b+c)(b+a)}-\frac{c}{(c+a)(c+b)}$$

$$\geq \frac{(a+b+c)^2}{12abc}-\frac{a}{(a+b)(a+c)}-\frac{b}{(b+c)(b+a)}-\frac{c}{(c+a)(c+b)}$$

$$=\frac{\sum\limits_{symm} a^4b+3\sum\limits_{symm} a^3b^2+3\sum\limits_{symm} a^3bc-7\sum\limits_{symm} a^2b^2c}{12abc(a+b)(b+c)(c+a)}\geq 0,$$

since by Muirhead's inequality

$$\sum_{symm} a^4b \geq \sum_{symm} a^2b^2c,\quad \sum_{symm} a^3b^2 \geq \sum_{symm} a^2b^2c,\quad \sum_{symm} a^3bc \geq \sum_{symm} a^2b^2c.$$

Also solved by *Arkady Alt, San Jose, California, USA, and Cao Minh Quang, Nguyen Binh Khiem High School, Vinh Long, Vietnam.*

Medium–Hard Problems

MH–131. *Proposed by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain.* On the sides AB, BC, CA of a triangle ABC points C', A', B' are marked respectively. It turns out that

$$\frac{AC'}{AB} = \frac{BA'}{BC} = \frac{CB'}{CA} = \frac{1}{3}.$$

Prove that:

1. The sides of $\triangle A'B'C'$ are parallel to the medians of $\triangle ABC$ and $\frac{2}{3}$ as the length of the correspondent median.
2. Each of the sides of $\triangle A'B'C'$ is trisected by two medians of $\triangle ABC$.
3. Each of the medians of $\triangle A'B'C'$ is parallel to a side of $\triangle ABC$.

Solution 1 by the proposer. Let M, N be the midpoints of sides BC, CA, respectively.

1. Since $A'M = BM - A'M = \frac{1}{2}BC - \frac{1}{3}BC = \frac{1}{6}BC$, we have (FIGURE 1).

   $$\frac{BA'}{A'M} = \frac{BA'}{BC}\cdot\frac{BC}{A'M} = \frac{1}{3}\times 6 = 2 = (\text{since } C' \text{ trisects } AB) = \frac{BC'}{C'A}.$$

   By the Thales's theorem, it follows that $A'C'$ is parallel to AM.
   Consequently, triangles $BA'C'$ and BMA are similar. From the proportional sides, then,

   $$\frac{C'A'}{AM} = \frac{BA'}{BM} = \frac{BA'/BC}{BM/BC} = \frac{1/3}{1/2} = \frac{2}{3}$$

   and
   $$C'A' = \frac{2}{3}AM.$$

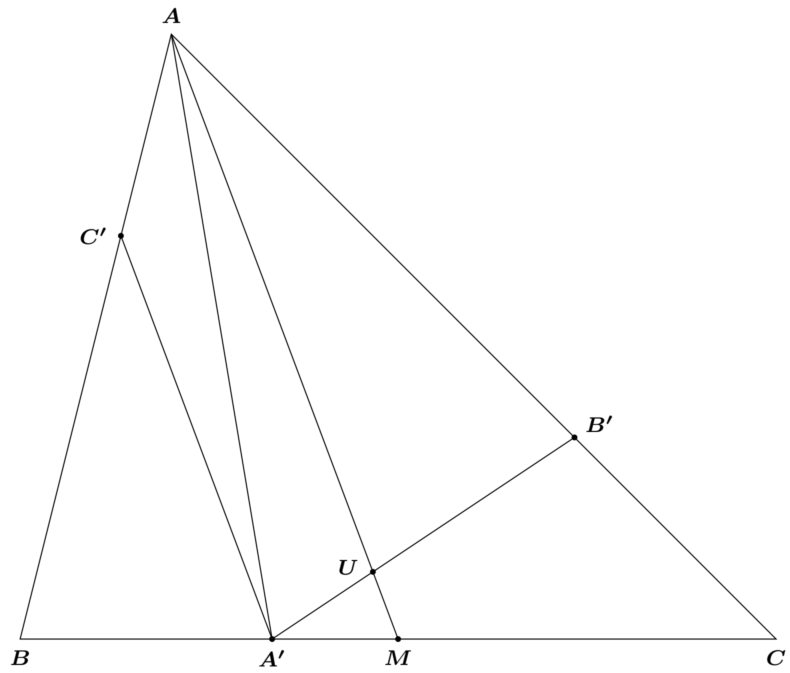


FIGURE 1

2. Let $\{U\} = AM \cap A'B'$. By the Menelaus's theorem, applied to $\triangle A'CB'$ and transversal AUM,

$$\frac{A'M}{MC} \cdot \frac{CA}{AB'} \cdot \frac{B'U}{UA'} = 1,$$

where $\dfrac{A'M}{MC} = \dfrac{A'M/BC}{MC/BC} = \dfrac{1/6}{1/2} = \dfrac{1}{3}$ and $\dfrac{CA}{AB'} = \frac{3}{2}$. Therefore, $\dfrac{B'U}{UA'} = 2$. That is,

$$B'U = 2 \cdot UA'. \tag{1}$$

Let $\{V\} = CN \cap A'B'$ and let $\{X\} = CN \cap AA'$. (FIGURE 2).

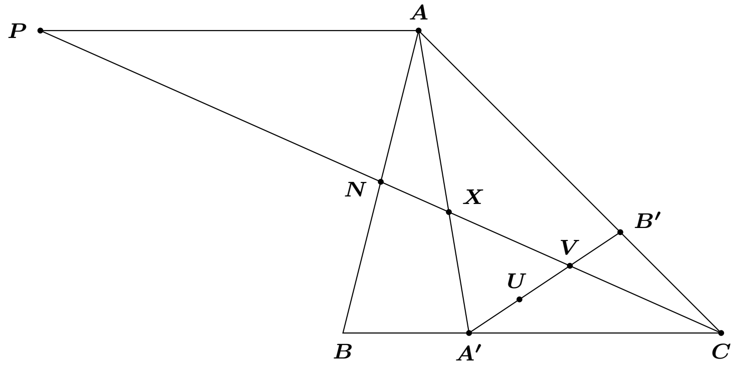


FIGURE 2

If P denotes the point of median CN such that AP is parallel to BC, triangles APX and $A'CX$ are similar (angle-angle-angle).
In turn, triangles APN and BCN are congruent (they are similar with $AN = NB$).
Thus, $AP = BC$. This yields

$$\frac{AX}{XA'} = \frac{AP}{A'C} = \frac{AP}{BC} \cdot \frac{BC}{A'C} = 1 \times \frac{3}{2} = \frac{3}{2}.$$

By the Menelaus's theorem, applied to $\triangle AA'B'$ and transversal XVC,

$$\frac{AX}{XA'} \cdot \frac{A'V}{VB'} \cdot \frac{B'C}{CA} = 1,$$

and therefore,

$$\frac{3}{2} \cdot \frac{A'V}{VB'} \cdot \frac{1}{3} = 1,$$

yielding

$$A'V = 2 \cdot VB'. \qquad (2)$$

By (1) and (2),

$$A'U = UV = VB'.$$

3. Let B'' ($\neq B'$) trisect the side CA of $\triangle ABC$ and let $\{Q\} = A'B'' \cap B'C'$ (FIGURE 3).

Since

$$\frac{AB''}{BA'} = \left(\frac{\frac{1}{3}CA}{\frac{1}{3}BC}\right) = \frac{B''B'}{A'A''},$$

we have $A'B'' \parallel AB$.
That is,

$$A'Q \parallel AB.$$

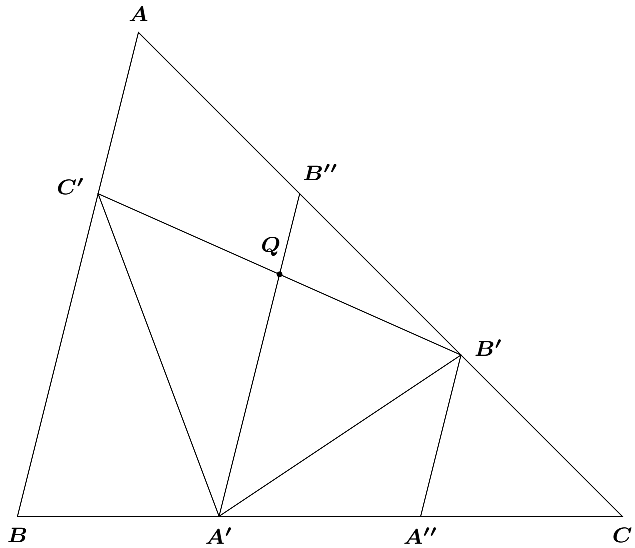


FIGURE 3

Since $AB'' = B''B'$, we also have $C'Q = QB'$, making $A'Q$ the median to $B'C'$ in $\triangle A'B'C'$, which we have just seen is parallel to side AB of $\triangle ABC$.
Analogously, the median from B' in $\triangle A'B'C'$ is parallel to BC and the median from C' is parallel to CA.

Solution 2 by Michel Bataille, Rouen, France. We have $3\overrightarrow{AC'} = \overrightarrow{AB}$, $3\overrightarrow{BA'} = \overrightarrow{BC}$, $3\overrightarrow{CB'} = \overrightarrow{CA}$, that is, $3A' = 2B + C$, $3B' = 2C + A$, $3C' = 2A + B$, or, in barycentric coordinates relatively to (A, B, C),

$$A' = (0 : 2 : 1),\ B' = (1 : 0 : 2),\ C' = (2 : 1 : 0).$$

1. Let $D = (0 : 1 : 1)$, $E = (1 : 0 : 1)$, $F = (1 : 1 : 0)$ denote the

midpoints of BC, CA, AB, respectively. Then, we obtain

$$3\overrightarrow{C'A'} = 3A' - 3C' = -2A + B + C = \overrightarrow{AB} + \overrightarrow{AC} = 2\overrightarrow{AD}$$

hence $\overrightarrow{C'A'} = \frac{2}{3}\overrightarrow{AD}$, proving that $C'A'$ is parallel to the median AD and that $C'A' = \frac{2}{3}AD$.
Similarly, $A'B' \parallel BE$ with $A'B' = \frac{2}{3}BE$ and $B'C' \parallel CF$ with $B'C' = \frac{2}{3}CF$.

2. The equations of the medians AD and BE are $y = z$ and $x = z$, respectively, and the equation of the line $B'C'$ is $2x - 4y - z = 0$. It follows that $B'C'$ intersect AD at $L = (5 : 2 : 2)$ and BE at $M = (4 : 1 : 4)$. Now,

$$9\overrightarrow{C'L} = 5A+2B+2C-6A-3B = -A-B+2C = 3B'-3C' = 3\overrightarrow{C'B'},$$

hence $\overrightarrow{C'L} = \frac{1}{3}\overrightarrow{C'B'}$. A similar calculation shows that $\overrightarrow{B'M} = \frac{1}{3}\overrightarrow{B'C'}$ and therefore the segment $B'C'$ is trisected by the medians AD and BE. Cyclically, we see that the sides $C'A'$ and $A'B'$ are also trisected by two medians of $\Delta A'B'C'$.

3. Let U, V, W be the midpoints of $B'C', C'A', A'B'$, respectively. We have $6U = 3B' + 3C' = 3A + B + 2C$ and $6A' = 4B + 2C$, hence

$$6\overrightarrow{A'U} = 6U - 6A' = 3A - 3B = 3\overrightarrow{BA},$$

that is, $\overrightarrow{A'U} = \frac{1}{2}\overrightarrow{BA}$ and therefore the median $A'U$ is parallel to AB. Similarly (or cyclically), we see that $B'V$ is parallel to BC and $C'W$ is parallel to CA.

Also solved by *Adil Allahveranov, Baku, Azerbaijan.*

MH–132. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*
Let S_n denote the set of all permutations of $\{1, 2, \ldots, n\}$. Show that the number

$$N = \frac{3}{n!} \sum_{\sigma \in S_n} \sum_{k=1}^{n} |k - \sigma(k)|$$

is an integer number and determine its value.

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. For each $i, j \in \{1, 2, \ldots, n\}$, there are $(n-1)!$ permutations in S_n such that $\sigma(i) = j$. Moreover, for each $j \in \{1, 2, \ldots, n-1\}$, there are $2(n-j)$ ordered pairs $(k, \sigma(k))$ such that $|k - \sigma(k)| = j$. Thus,

$$\begin{aligned}
\sum_{\sigma \in S_n} \sum_{k=1}^{n} &= 2(n-1)! \sum_{j=1}^{n-1} j(n-j) \\
&= 2(n-1)! \left(n \sum_{j=1}^{n-1} j - \sum_{j=1}^{n-1} j^2 \right) \\
&= n!(n-1)\left(n - \frac{2n-1}{3} \right) = \frac{n!}{3}(n^2-1),
\end{aligned}$$

and

$$N = \frac{3}{n!} \sum_{\sigma \in S_n} \sum_{k=1}^{n} |k - \sigma(k)| = n^2 - 1,$$

which is an integer number.

Solution 2 by Michel Bataille, Rouen, France. We claim $N = n^2 - 1$. In fact, let $S(n) = \sum_{\sigma \in S_n} \sum_{k=1}^{n} |k - \sigma(k)| = \sum_{k=1}^{n} \sum_{\sigma \in S_n} |k - \sigma(k)|$. The claim will follow if we prove that

$$S(n) = \frac{n!(n^2-1)}{3}.$$

Let k and j be integers of $\{1, 2, \ldots, n\}$. There exist $(n-1)!$ elements of S_n such that $\sigma(k) = j$ (as many as bijections from $\{1, 2, \ldots, n\} - \{k\}$ onto $\{1, 2, \ldots, n\} - \{j\}$), hence

$$\begin{aligned}
\sum_{\sigma \in S_n} |k - \sigma(k)| &= (n-1)! \sum_{j=1}^{n} |k - j| = (n-1)! \left(\sum_{j=1}^{k-1} j + \sum_{j=1}^{n-k} j \right) \\
&= (n-1)! \left(\frac{(k-1)k}{2} + \frac{(n-k)(n-k+1)}{2} \right).
\end{aligned}$$

As a result, we have

$$\begin{aligned} S(n) &= (n-1)!\left(\sum_{k=1}^{n}\frac{(k-1)k}{2}+\sum_{k=1}^{n}\frac{(n-k)(n-k+1)}{2}\right) \\ &= (n-1)!\left(\sum_{k=1}^{n}\frac{(k-1)k}{2}+\sum_{\ell=1}^{n}\frac{(\ell-1)\ell}{2}\right) \\ &= (n-1)!\sum_{k=1}^{n}(k^2-k) = (n-1)!\left(\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right) \\ &= (n-1)!\cdot\frac{n(n+1)(n-1)}{3} = \frac{n!(n^2-1)}{3} \end{aligned}$$

and we are done.

Solution 3 by José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain. Reversing the sum gives

$$N = \frac{3}{n!}\sum_{\sigma\in S_n}\sum_{k=1}^{n}|k-\sigma(k)| = \frac{3}{n!}\sum_{k=1}^{n}\sum_{\sigma\in S_n}|k-\sigma(k)|$$

Now, consider k fixed. For each $j\in\{1,2,\ldots,n\}$ observe that $\sigma(k)=j$ for exactly $(n-1)!$ permutations of S_n. That is,

$$N = \frac{3}{n!}\sum_{k=1}^{n}\sum_{\sigma\in S_n}|k-\sigma(k)| = \frac{3}{n!}\sum_{k=1}^{n}\sum_{j=1}^{n}(n-1)!|k-j| = \frac{3}{n}\sum_{k=1}^{n}\sum_{j=1}^{n}|k-j|$$

Notice that the $j=k$ terms are 0 and

$$\begin{aligned} N &= \frac{3}{n}\sum_{1\le j\le k\le n}|k-j| + \frac{3}{n}\sum_{1\le k\le j\le n}|k-j| \\ &= \frac{6}{n}\sum_{1\le j\le k\le n}(k-j) = \frac{6}{n}\sum_{k=1}^{n}\sum_{j=1}^{k}(k-j) \end{aligned}$$

Then,

$$\begin{aligned} N &= \frac{6}{n}\sum_{k=1}^{n}\left(k^2-\frac{k(k+1)}{2}\right) = \frac{6}{n}\sum_{k=1}^{n}(k^2-k) \\ &= \frac{3}{n}\left(\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}\right) = n^2-1. \end{aligned}$$

Also solved by *the proposer.*

MH–133. *Proposed by Titu Zvonaru, Comăneşti, Romania.* Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Find the greatest k such that the following inequality holds:

$$\frac{a^2+b^2}{a+b+2} + \frac{b^2+c^2}{b+c+2} + \frac{c^2+a^2}{c+a+2} \geq \frac{k}{4}(a+b+c) - \frac{(3k-6)}{4}.$$

Solution by the proposer. Since $ab + bc + ca = 3$, we have $a + b + c \geq 3$. It is easy to see that if the inequality is true for k_0, then it is true for any $k < k_0$.

Clearing the denominators, the inequality is equivalent to

$$(8-k)\sum_{\text{cyc}} a^3b + (8-k)\sum_{\text{cyc}} ab^3 + 2(8-k)\sum_{\text{cyc}} a^3 - 2(k-4)\sum_{\text{cyc}} a^2b^2$$
$$-4(k-2)\sum_{\text{cyc}} a^2bc - (5k-18)\sum_{\text{cyc}} a^2b - (5k-18)\sum_{\text{cyc}} ab^2$$
$$-2(k-10)\sum_{\text{cyc}} a^2 - 12(k+1)abc + 2(k-18)\sum_{\text{cyc}} ab$$
$$+16(k-3)\sum_{\text{cyc}} a + 24(k-2) \geq 0 \qquad (1)$$

We denote $p = a+b+c$, $q = ab+bc+ca = 3$, and $r = abc$. Using the formulas

$$\sum_{\text{cyc}} a^3b + \sum_{\text{cyc}} ab^3 = p^2q - 2q^2 - pr,$$
$$\sum_{\text{cyc}} a^3 = p^3 - 3pq + 3r,$$
$$\sum_{\text{cyc}} a^2b^2 = q^2 - 2pr,$$
$$\sum_{\text{cyc}} a^2b + \sum_{\text{cyc}} ab^2 = pq - 3r,$$
$$\sum_{\text{cyc}} a^2 = p^2 - 2q,$$

the inequality (1) becomes

$$2(8-k)p^3 + (44-5k)p^2 + (k-16)pr$$

$$+(19k-138)p-(3k+18)r+6(7k-58)\geq 0 \tag{2}$$

Looking at (2), we will try $k=8$. We have to prove that

$$2p^2+7p\geq 4pr+21r+6. \tag{3}$$

Since $pq\geq 9r$ implies $p\geq 3r$, it remains to prove that

$$2p^2\geq 4pr+6. \tag{4}$$

Using $p\geq 3$, we obtain

$$2p^2=\frac{4}{3}p^2+\frac{2}{3}p^2\geq\frac{4}{3}p(3r)+2/3\cdot 9=4pr+6.$$

Suppose that $k\geq 8$ and let $c=0$. The sum p can be very large, hence we can suppose that $2p^3+5p^2-19p-42>0$. The inequality (2) is equivalent to

$$2(8-k)p^3+(44-5k)p^2+(19k-138)p+6(7k-58)\geq 0,$$

and

$$k\leq\frac{16p^3+44p^2-139p-340}{2p^3+5p^2-19p-42}.$$

For p tending to infinity, it results that $k\leq 8$. Hence, the greatest k is $k=8$.

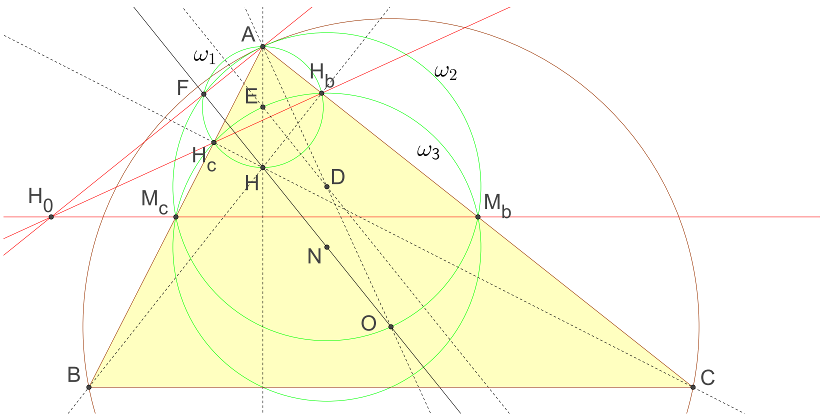
Also solved by *Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania.*

MH–134. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a non-right triangle with $AB\neq AC$. Let H be its orthocenter, O be its circumcenter, M_a, M_b, M_c be the midpoints of the sides BC, CA, AB respectively, and H_a, H_b, H_c be the feet of the altitudes drawn from A, B, C respectively. Let D be the intersect point of AO and BC, N be the midpoint of OH and N_0 be the reflection of N in BC. Let M_0 be the midpoint of AM_a and H_0 be the intersect point of H_bH_c and M_bM_c. Knowing that $AB\cdot AC=4\cdot AH_a\cdot AD$, prove that the points A, N_0, M_0, H_0 are concyclic.

Solution by the proposer. We first establish a lemma.

Lemma 1. *For any non-right triangle ABC with $AB \neq AC$, $OH \perp AH_0$.*

Proof.

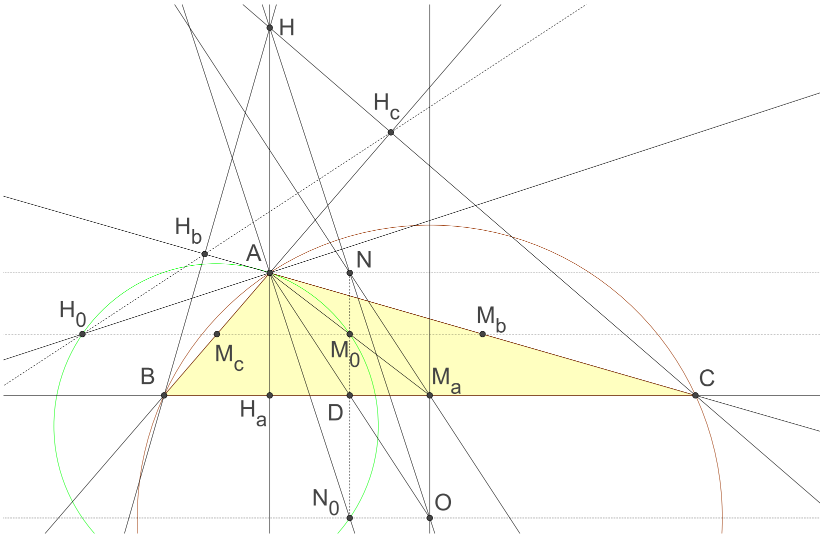


Scheme for Solution of MH-134.

Let D, E, N be the midpoints of AO, AH, OH respectively. Denote with ω_1 the circumcircle with diameter AH centered at E, with ω_2 the circumcircle with diameter AO centered at D and with ω_3 the nine point circle centered at N. Since $\angle AH_bH = \angle AH_cH = 90°$, we have $H_b, H_c \in \omega_1$. Since $\angle AM_bO = \angle AM_cO = 90°$, we have $M_b, M_c \in \omega_2$. But $H_b, H_c, M_b, M_c \in \omega_3$. Denote with F the second intersect point of ω_1, ω_2.

Now, $\omega_1 \cap \omega_2 = \{A, F\}, \omega_2 \cap \omega_3 = \{M_b, M_c\}, \omega_1 \cap \omega_3 = \{H_b, H_c\}$, hence AF, M_bM_c, H_bH_c are their pairwise radical axes and they concur at the radical center of these three circles and this is $H_0 = M_bM_c \cap H_bH_c$. Therefore H_0, A, F are collinear. Since the radical axis of two circles is perpendicular to the line connecting the centers of the circles, hence $AF \perp DE$. But DE is a midsegment in the triangle AHO, so $DE \parallel HO$ and therefore $AH_0 \equiv AF \perp OH$. □

Let $BC = a, CA = b, AB = c$ are the side lengths of the triangle ABC.



Scheme for Solution of MH-134.

Denote with $h_a = AH_a$ a altitude and with $R = AO$ the radius of the circumcircle of the triangle ABC. Let S be the area of the $\triangle ABC$.

$$S = \frac{1}{2}ah_a = \frac{abc}{4R}$$

Hence

$$AB \cdot AC = 2h_aR = 2 \cdot AH_a \cdot AO = 4 \cdot AH_a \cdot AD$$

so D is the midpoint of AO and O and A are in different half-planes about BC, so $\angle BAC > 90^\circ$ and A is between H and H_a. Since O lies on the perpendicular bisector of BC, hence $OM_a \perp BC$ and $OM_a \parallel AH_a$, so $\triangle AH_aD \cong \triangle OM_aD$, $H_aD = DM_a$, $OM_a = AH_a = h_a$. M_0D is a midsegment in the triangle AM_aO and since $OM_a \perp BC$, hence $M_0D \perp BC$. On the other hand ND is a midsegment in the triangle HAO and $ND \perp BC$, so the points N, M_0, D, N_0 are collinear and from $M_bM_c \parallel BC$ and since H_0, M_0, M_b, M_c are collinear, hence $\angle H_0M_0N_0 = 90^\circ$.

Since D bisects NN_0 and AO, $ANON_0$ is a parallelogram and $AN_0 \parallel NO \equiv OH$. From the Lemma 1 we get $OH \perp AH_0$, or $AN_0 \perp AH_0$, $\angle H_0AN_0 = \angle H_0M_0N_0 = 90^\circ$ and therefore A, N_0, M_0, H_0 are concyclic.

MH–135. *Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploieşti, Romania Romania.* Let $a \geq b \geq c \geq 1 \geq d \geq e \geq 0$ such that $ab + bc + cd + de + ea = 5$. Prove that

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3} + \frac{1}{d+3} + \frac{1}{e+3} \geq \frac{5}{4}.$$

Solution by the proposer. For fixed a, d and e, from the equality constraint we may assume that b is a function of c. By differentiating the constraint, we get

$$(a+c)b' + b + d = 0, \qquad -b' = \frac{b+d}{a+c} \leq 1.$$

Denoting the left side of the desired inequality by $f(c)$, we have

$$f'(c) = \frac{-b'}{(b+3)^2} - \frac{1}{(c+3)^2} \leq \frac{1}{(b+3)^2} - \frac{1}{(c+3)^2} \leq 0.$$

Thus, $f(c)$ is decreasing and has the minimum value when c is maximum, that is when $c = b$. So, we only need to show that

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{d+3} + \frac{1}{e+3} \geq \frac{5}{4}$$

for

$$ab + b^2 + bd + de + ea = 5, \qquad a \geq b \geq 1 \geq d \geq e.$$

For fixed a and e, from the equality constraint, we may assume that b is a decreasing function of d. By differentiating the constraint, we get

$$(a+2b+d)b' + b + e = 0, \qquad -b' = \frac{b+e}{a+2b+d} \leq \frac{1}{2}.$$

Denoting the left side of the desired inequality by $g(d)$, we have

$$g'(d) = \frac{-2b'}{(b+3)^2} - \frac{1}{(d+3)^2} \leq \frac{1}{(b+3)^2} - \frac{1}{(d+3)^2} \leq 0.$$

Thus, $g(d)$ is decreasing and has the minimum value when d is maximum (b is minimum), that is when $d = 1$ (because $d \leq 1$) or $b = 1$ (because $b \geq 1$). So, it suffices to consider these cases.

Case 1: $d = 1$. We need to show that

$$\frac{1}{a+3} + \frac{2}{b+3} + \frac{1}{e+3} \geq 1$$

for

$$ab + b^2 + b + e + ea = 5, \qquad a \geq b \geq 1 \geq e.$$

Since

$$e = \frac{5 - b - b^2 - ab}{1+a}, \qquad e + 3 = \frac{8 - b - b^2 + (3-b)a}{1+a},$$

we need to show that

$$\frac{1}{a+3} + \frac{2}{b+3} + \frac{1+a}{8 - b - b^2 + (3-b)a} \geq 1,$$

which is equivalent to

$$\frac{1}{a+3} + \frac{1+a}{8 - b - b^2 + (3-b)a} \geq \frac{b+1}{b+3},$$

$$(b-1)[ba^2 + (b^2 + 5b - 4)a + 2b^2 + 4b - 9] \geq 0.$$

Since $a \geq b \geq 1$, we have

$$ba^2 + (b^2 + 5b - 4)a + 2b^2 + 4b - 9 \geq a^2 + 2a - 3 = (a-1)(a+3) \geq 0.$$

Case 2: $b = 1$. We need to show that

$$\frac{1}{a+3} + \frac{1}{d+3} + \frac{1}{e+3} \geq \frac{3}{4}$$

for

$$(a+d)(e+1) = 4, \qquad a \geq 1 \geq d \geq e.$$

Write the desired inequality as

$$\frac{a+d+6}{(a+3)(d+3)} + \frac{1}{e+3} \geq \frac{3}{4}.$$

From $(a-1)(d-1) \leq 0$, we get $ad \leq a + d - 1$, hence

$$(a+3)(d+3) = (a-1)(d-1) + 4(a+d) + 8 \leq 4(a+d) + 8$$

and

$$\frac{a+d+6}{(a+3)(d+3)} \geq \frac{a+d+6}{4(a+d)+8} = \frac{4/(e+1)+6}{16/(e+1)+8} = \frac{3e+5}{4(e+3)}.$$

So, it suffices to show that

$$\frac{3e+5}{4(e+3)} + \frac{1}{e+3} \geq \frac{3}{4},$$

which is an identity.

The equality occurs for $b = c = d = 1$ and $a + e + ae = 3$, $a \geq 1 \geq e$.

MH–136. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* If $a > b > 1$ are integers such that $(a+b)|(ab+1)$ and $(a-b)|(ab-1)$, then prove that $a < b\sqrt{3}$.

Solution 1 by Michel Bataille, Rouen, France. The hypothesis ensures that $ab + 1 = k(a + b)$ and $ab - 1 = \ell(a - b)$ for some positive integers k, ℓ. We deduce that

$$a(b-k) = kb - 1, \qquad a(\ell - b) = \ell b - 1 \tag{1}$$

and therefore a and b are co-prime integers. Also, we see that $\ell > b > k$.

By subtraction, (1) yields $b(\ell - k) = a(k + \ell - 2b)$, and since a, b is co-prime, a divides $\ell - k$. Thus, $\ell - k = au$ for some positive integer u and consequently $bu = k + \ell - 2b$.

Now, since $(k + \ell)b - a(\ell - k) = 2$ (from (1)), we obtain

$$a^2 = b^2 + 2\frac{b^2-1}{u}.$$

Furthermore, we have

$$b^2 - \frac{b^2-1}{u} = \frac{b^2(u-1)+1}{u} > 0,$$

hence $\frac{b^2-1}{u} < b^2$ and therefore $a^2 < b^2 + 2b^2 = 3b^2$. Taking square roots, we deduce $a < b\sqrt{3}$ (since $a, b > 0$).

Solution 2 by the proposers. Since $(a+b)\,|\,(ab+1)$ then $(a+b)\,|\,[b(a+b)-(ab+1)] = b^2-1$, and from $(a-b)\,|\,(ab-1)$, we get that $(a-b)\,|\,[-b(a-b)+(ab-1)] = b^2-1$. Hence,

$$[a-b, a+b]\,|\,(b^2-1) \quad \text{and} \quad [a-b, a+b] \le b^2-1$$

Let $d = (a,b)$ then $d\,|\,a\,|\,ab$ and $d\,|\,(a+b)\,|\,(ab+1)$ from which follows that $d\,|\,(ab+1-ab) = 1$. Then $d = 1$ and a and b are coprime. Let $e = (a-b, a+b)$, then $e\,|\,[(a+b)+(a-b)] = 2a$ and $e\,|\,[(a+b)-(a-b)] = 2b$ from which follows that $e\,|\,(2a, 2b) = 2(a,b) = 2$. Therefore, $e \le 2$. From the identity $[x,y]\,(x,y) = xy$, we get

$$[a-b, a+b] = \frac{(a-b)(a+b)}{(a-b, a+b)} \ge \frac{a^2-b^2}{2}$$

that jointly with $[a-b, a+b] \le b^2-1$, we obtain

$$\frac{a^2-b^2}{2} \le b^2-1 \Leftrightarrow a^2 \le 3b^2-2 \le 3b^2$$

and $a < b\sqrt{3}$ follows. Since $b > a > 1$, then the equality does not hold.

Advanced Problems

A–131. *Proposed by Mihaela Berindeanu, Bucharest, Romania.*
For $x > 0$, calculate

$$\lim_{x\to\infty} \int\limits_{[x]}^{x+1} \frac{(\sin t)^{2025}}{t^2+4}\,dt.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland. Clearly, 1≤x+1-[x]<2 and |sin t|≤1. Hence

$$\left|\int_{[x]}^{x+1} \frac{(\sin t\ )^{2025}}{t^2+4}dt\right| \le \frac{2}{[x]^2+4}$$

and $\lim_{x\to\infty} \int_{[x]}^{x+1} \frac{(\sin t\ )^{2025}}{t^2+4}dt\ = 0.$

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA. The limit is 0. We have

$$\begin{aligned}
0 &\le \left|\int_{\lfloor x\rfloor}^{x+1} \frac{(\sin t)^{2025}}{t^2+4}\,dt\right| \\
&\le \int_{\lfloor x\rfloor}^{x+1} \frac{|(\sin t)^{2025}|}{t^2+4}\,dt \\
&\le \int_{\lfloor x\rfloor}^{x+1} \frac{1}{t^2+4}\,dt \\
&\overset{\text{AGM}}{\le} \frac{1}{4}\cdot\int_{\lfloor x\rfloor}^{x+1} \frac{dt}{t} \\
&= \frac{1}{4}\cdot\ln\left(\frac{x+1}{\lfloor x\rfloor}\right). \qquad (1)
\end{aligned}$$

With $\{x\}$ denoting the fractional part of x, we have

$$\frac{x+1}{\lfloor x\rfloor} = \frac{\lfloor x\rfloor+\{x\}+1}{\lfloor x\rfloor} = 1+\frac{\{x\}}{\lfloor x\rfloor}+\frac{1}{\lfloor x\rfloor} \to 1 \quad \text{as } x\to\infty,$$

and we see that the expression in (1) goes to 0, giving us the claimed limit by the squeeze principle.

Solution 3 by Moti Levy, Rehovot, Israel.

$$I=\int_{[x]}^{x+1}\frac{(\sin(t))^{2025}}{t^2+4}dt$$

$$|I|\leq\int_{[x]}^{x+1}\left|\frac{(\sin(t))^{2025}}{t^2+4}\right|dt\leq\int_{[x]}^{x+1}\frac{1}{t^2+4}dt$$

$$\begin{aligned}\int_{[x]}^{x+1}\frac{1}{t^2+4}dt&\overset{t=ux}{=}\int_{\frac{[x]}{x}}^{1+\frac{1}{x}}\frac{1}{u^2x^2+4}xdu\\&=\frac{1}{x}\int_{\frac{[x]}{x}}^{1+\frac{1}{x}}\frac{1}{u^2+\frac{4}{x^2}}du\leq\frac{1}{x}\int_{\frac{[x]}{x}}^{1+\frac{1}{x}}\frac{1}{u^2}du\\&=\frac{1}{[x]}-\frac{1}{x+1}=\frac{x+1-[x]}{[x](x+1)}=\frac{1+\{x\}}{[x](x+1)}\leq\frac{2}{[x](x+1)}\end{aligned}$$

$$\lim_{x\to\infty}|I|\leq\lim_{x\to\infty}\frac{2}{[x](x+1)}=0$$

$$\lim_{x\to\infty}|I|=0\qquad\Longrightarrow\qquad\lim_{x\to\infty}I\ =0$$

Solution 4 by Michel Bataille, Rouen, France. (We assume that $[x]$ denotes the integral part of x.)

For $t\geq 0$, let $f(t)=\dfrac{(\sin t)^{2025}}{t^2+4}$. The function f is continuous, hence integrable on every interval $[0,X]$ $(X>0)$. In addition, we have

$$|f(t)|\leq\frac{1}{t^2+4}$$

for all $t\geq 0$ (since $|\sin t|\leq 1$). Since $\int_0^\infty\dfrac{dt}{t^2+4}$ exists, the integral $\displaystyle\int_0^\infty f(t)\,dt$ is convergent. Let I be its value. We have

$$\int_{[x]}^{x+1}\frac{(\sin t)^{2025}}{t^2+4}\,dt=\int_0^{x+1}f(t)\,dt-\int_0^{[x]}f(t)\,dt$$

and

$$\lim_{x\to\infty}\int_0^{x+1}f(t)\,dt=I,\qquad\lim_{x\to\infty}\int_0^{[x]}f(t)\,dt=I$$

(since $\lim_{x\to\infty}(x+1) = \lim_{x\to\infty}[x] = \infty$).
We conclude that

$$\lim_{x\to\infty}\int_{[x]}^{x+1}\frac{(\sin t)^{2025}}{t^2+4}\,dt = 0.$$

Solution 5 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. Let $x > 0$. For $t \in [\lfloor x\rfloor, x+1]$, $\frac{1}{t^2+4}$ does not change sign, so by the Weighted Mean Value Theorem for Integrals, there exists $\xi \in [\lfloor x\rfloor, x+1]$ such that

$$\int_{\lfloor x\rfloor}^{x+1}\frac{(\sin t)^{2025}}{t^2+4}\,dt = (\sin\xi)^{2025}\int_{\lfloor x\rfloor}^{x+1}\frac{1}{t^2+4}\,dt.$$

Now,

$$\frac{1}{(x+1)^2+4} < \frac{1}{t^2+4} < \frac{1}{\lfloor x\rfloor^2+4}$$

for $t \in [\lfloor x\rfloor, x+1]$, and

$$1 \le x+1-\lfloor x\rfloor < 2,$$

so

$$\frac{1}{(x+1)^2+4} < \int_{\lfloor x\rfloor}^{x+1}\frac{1}{t^2+4}\,dt < \frac{2}{\lfloor x\rfloor^2+4}.$$

Because

$$\lim_{x\to\infty}\frac{1}{(x+1)^2+4} = \lim_{x\to\infty}\frac{2}{\lfloor x\rfloor^2+4} = 0,$$

it follows from the squeeze theorem that

$$\lim_{x\to\infty}\int_{\lfloor x\rfloor}^{x+1}\frac{1}{t^2+4}\,dt = 0.$$

Additionally, because $(\sin\xi)^{2025}$ is bounded as $x\to\infty$, it follows that

$$\lim_{x\to\infty}(\sin\xi)^{2025}\int_{\lfloor x\rfloor}^{x+1}\frac{1}{t^2+4}\,dt = 0.$$

Thus,

$$\lim_{x\to\infty}\int_{\lfloor x\rfloor}^{x+1}\frac{(\sin t)^{2025}}{t^2+4}\,dt = 0.$$

Also solved by *the proposer.*

A–132. *Proposed by Vasile Mircea Popa, Affiliate Professor, "Lucian Blaga" University of Sibiu, Romania.* Calculate the following integral:

$$\int_0^{\infty} \frac{\arctan(x)}{\sqrt[3]{x^6+1}}\,dx.$$

Solution 1 by Joseph Santmyer, Las Cruces, New Mexico, USA. Let $u = 1/x$. Then $x = 1/u$. If $x = 0$ then $u = \infty$ and if $x = \infty$ then $u = 0$. Also, $\frac{du}{dx} = -x^{-2}$, that is, $-\frac{du}{u^2} = dx$. Hence

$$\begin{aligned}
I &= \int_{\infty}^{0} \frac{\tan^{-1}\left(\frac{1}{u}\right)}{\sqrt[3]{1+\left(\frac{1}{u}\right)^6}}\left[-\frac{du}{u^2}\right] \\
&= \int_0^{\infty} \frac{u^2\cot^{-1}(u)}{\sqrt[3]{1+u^6}}\left[\frac{du}{u^2}\right] \\
&= \int_0^{\infty} \frac{\cot^{-1}(u)}{\sqrt[3]{1+u^6}}du \\
&= \int_0^{\infty} \frac{\frac{\pi}{2}-\tan^{-1}(u)}{\sqrt[3]{1+u^6}}du \\
&= \frac{\pi}{2}\int_0^{\infty} \frac{du}{\sqrt[3]{1+u^6}} - \int_0^{\infty} \frac{\tan^{-1}(u)}{\sqrt[3]{1+u^6}}du \\
&= \frac{\pi}{2}\int_0^{\infty} \frac{du}{\sqrt[3]{1+u^6}} - I \\
2I &= \frac{\pi}{2}\int_0^{\infty} \frac{du}{\sqrt[3]{1+u^6}} \\
I &= \frac{\pi}{4}\int_0^{\infty} \frac{du}{\sqrt[3]{1+u^6}}.
\end{aligned}$$

Let $u = t^{\frac{1}{6}}$ then $u^6 = t$. If $u = 0$ then $t = 0$ and if $u = \infty$ then $t = \infty$. Also, $\frac{dt}{du} = 6u^5 = 6t^{\frac{5}{6}}$, that is, $\frac{dt}{6t^{\frac{5}{6}}} = du$. Hence

$$I = \frac{\pi}{4}\int_0^{\infty} \frac{du}{\sqrt[3]{1+u^6}} = \frac{\pi}{4}\int_0^{\infty} \frac{\frac{dt}{6t^{\frac{5}{6}}}}{\sqrt[3]{1+t}} = \frac{\pi}{24}\int_0^{\infty} \frac{t^{-\frac{5}{6}}}{\sqrt[3]{1+t}}dt$$

$$= \frac{\pi}{24}\int_0^{\infty} \frac{t^{-\frac{5}{6}}}{(1+t)^{\frac{2}{6}}}dt = \frac{\pi}{24}\int_0^{\infty} \frac{t^{-\frac{5}{6}}}{(1+t)^{\frac{2}{6}}}dt = \frac{\pi}{24}\int_0^{\infty} \frac{t^{\frac{1}{6}-1}}{(1+t)^{\frac{1}{6}+\frac{1}{6}}}dt.$$

By 17.2 and 17.5 on p 103 in (1) we get

$$I = \frac{\pi}{24}B\left(\frac{1}{6},\frac{1}{6}\right) = \frac{\pi}{24}\left[\frac{\Gamma^2\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{3}\right)}\right]$$

where $B(r,s)$ is the beta function and $\Gamma(r)$ is the gamma function.

REFERENCE.

[1] Spiegel, M. R., “Mathematical Handbook", Schaum's Outline Series, McGraw-Hill Book Company, 1968.

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. With the change of variables $x \to 1/x$ and the identity $\arctan(x) + \arctan(\frac{1}{x}) = \frac{\pi}{2}$,

$$\int_1^\infty \frac{\arctan(x)}{\sqrt[3]{x^6+1}}\,dx = \int_1^0 \frac{\arctan(\frac{1}{x})}{\sqrt[3]{x^{-6}+1}}\left(-\frac{dx}{x^2}\right) = \int_0^1 \frac{\frac{\pi}{2}-\arctan(x)}{\sqrt[3]{x^6+1}}\,dx.$$

Thus,

$$\int_0^\infty \frac{\arctan(x)}{\sqrt[3]{x^6+1}}\,dx = \frac{\pi}{2}\int_0^1 \frac{1}{\sqrt[3]{x^6+1}}.$$

Now, in the integral on the right side, make the substitution $x = u^{1/6}$ to obtain

$$\int_0^1 \frac{1}{\sqrt[3]{x^6+1}}\,dx = \frac{1}{6}\int_0^1 \frac{u^{-5/6}}{(u+1)^{1/3}}\,du.$$

With the substitution $w = 1/u$, it follows that

$$\int_0^1 \frac{u^{-5/6}}{(u+1)^{1/3}}\,du = \int_1^\infty \frac{w^{-5/6}}{(w+1)^{1/3}}\,dw$$

and

$$\int_0^1 \frac{1}{\sqrt[3]{x^6+1}}\,dx = \frac{1}{12}\int_0^\infty \frac{u^{-5/6}}{(u+1)^{1/3}}\,du.$$

Next, let $u = z/(1-z)$ to obtain

$$\int_0^1 \frac{1}{\sqrt[3]{x^6+1}}\,dx = \frac{1}{12}\int_0^1 z^{-5/6}(1-z)^{-5/6}\,dz = \frac{1}{12}B\left(\frac{1}{6},\frac{1}{6}\right) = \frac{1}{12}\frac{\Gamma\left(\frac{1}{6}\right)^2}{\Gamma\left(\frac{1}{3}\right)},$$

where $B(\cdot,\cdot)$ is the beta function and $\Gamma(\cdot)$ is the gamma function. Finally,

$$\int_0^\infty \frac{\arctan(x)}{\sqrt[3]{x^6+1}}\,dx = \frac{\pi}{24}\frac{\Gamma\left(\frac{1}{6}\right)^2}{\Gamma\left(\frac{1}{3}\right)}.$$

Solution 3 by Michel Bataille, Rouen, France. The change of variables $x = 1/u$ gives

$$\begin{aligned}\int_1^\infty \frac{\arctan(x)}{\sqrt[3]{x^6+1}}\,dx &= \int_1^0 \frac{\arctan(1/u)}{\sqrt[3]{(1/u)^6+1}}\cdot\frac{-du}{u^2}\\ &= \int_0^1 \frac{\pi/2-\arctan(u)}{\sqrt[3]{u^6+1}}\,du\\ &= \frac{\pi}{2}\int_0^1 \frac{du}{\sqrt[3]{u^6+1}} - \int_0^1 \frac{\arctan(u)}{\sqrt[3]{u^6+1}}\,du.\end{aligned}$$

It follows that the required integral I is equal to

$$\frac{\pi}{2}\int_0^1 \frac{du}{\sqrt[3]{u^6+1}}.$$

Now, the change of variables $u = 1/x$ leads to

$$\int_0^1 \frac{du}{\sqrt[3]{u^6+1}} = \int_1^\infty \frac{dx}{\sqrt[3]{x^6+1}}$$

from which we deduce that $\displaystyle\int_0^1 \frac{du}{\sqrt[3]{u^6+1}} = \frac{J}{2}$ where

$$J = \int_0^\infty \frac{dx}{\sqrt[3]{x^6+1}} = \frac{1}{6}\int_0^\infty \frac{t^{-5/6}}{(1+t)^{1/3}}\,dt = \frac{1}{6}B(1/6,1/6)$$

(since $\displaystyle\int_0^\infty \frac{t^{x-1}}{(1+t)^{x+y}}\,dt = B(x,y)$ where $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, a well-known result). We deduce that

$$J = \frac{(\Gamma(1/6))^2}{6\Gamma(1/3)}$$

and conclude

$$I = \frac{\pi}{24}\cdot\frac{(\Gamma(1/6))^2}{\Gamma(1/3)}.$$

Solution 4 by Moti Levy, Rehovot, Israel.

$$I = \int_0^\infty \frac{\arctan(x)}{\sqrt[3]{x^6+1}} dx.$$

By change of integration variable $t = 1/x$

$$I = \int_0^\infty \frac{\arctan\left(\frac{1}{t}\right)}{\sqrt[3]{t^6+1}}\, dt$$

$$2I = \int_0^\infty \frac{\arctan\left(\frac{1}{x}\right) + \arctan(x)}{\sqrt[3]{x^6+1}}\, dx = \int_0^\infty \frac{\arctan\left(\frac{1}{x}\right) + \arctan(x)}{\sqrt[3]{x^6+1}}\, dx$$

$$= \int_0^\infty \frac{\frac{\pi}{2}}{\sqrt[3]{x^6+1}}\, dx.$$

$$\arctan\left(\frac{1}{x}\right) + \arctan(x) = \frac{\pi}{2},$$

$$2I = \int_0^\infty \frac{\frac{\pi}{2}}{\sqrt[3]{x^6+1}}\, dx.$$

$$I = \frac{\pi}{4} \int_0^\infty \frac{1}{\sqrt[3]{x^6+1}}\, dx$$

By change of integration variable $t = x^6$,

$$I = \frac{\pi}{24} \int_0^\infty \frac{t^{-\frac{5}{6}}}{(t+1)^{\frac{1}{3}}}\, dt.$$

One of the integral representations of the Beta function is

$$B(z_1, z_2) = \int_0^\infty \frac{t^{z_1-1}}{(t+1)^{z_1+z_2}}\, dt.$$

$$I = \frac{\pi}{24} B\left(\frac{1}{6}, \frac{1}{6}\right) = \frac{\pi}{24} \frac{\Gamma^2\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{3}\right)} \cong 1.51395$$

Solution 5 by Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA. Let I denote the given integral. The substitution $x = 1/u,\ dx = -u^{-2}du$ and the identity $\arctan(1/u)+$ $\arctan u - \pi/2$ give us

$$I = \int_0^\infty \frac{(\pi/2 - \arctan u)}{\sqrt[3]{u^6+1}}\,du = \frac{\pi}{2}\int_0^\infty \frac{du}{\sqrt[3]{u^6+1}} - \int_0^\infty \frac{\arctan u}{\sqrt[3]{u^6+1}}\,du,$$

so that

$$2I = \frac{\pi}{2}\int_0^\infty \frac{du}{\sqrt[3]{u^6+1}}, \quad \text{or} \quad I = \frac{\pi}{4}\int_0^\infty \frac{du}{\sqrt[3]{u^6+1}}.$$

The successive substitutions $t = u^6$ and $x = t/(1+t)$ yield the value of the last integral:

$$\int_0^\infty \frac{du}{\sqrt[3]{u^6+1}} = \frac{1}{6}\int_0^\infty \frac{dt}{t^{5/6}(t+1)^{1/3}} = \frac{1}{6}\int_0^1 x^{1/6-1}(1-x)^{1/6-1}\,dx$$
$$= \frac{1}{6}B\left(\frac{1}{6},\frac{1}{6}\right),$$

where $B(x,y)$ denotes Euler's beta function. Now we can write

$$I = \frac{\pi}{4}\cdot\frac{1}{6}\cdot B\left(\frac{1}{6},\frac{1}{6}\right) = \frac{\pi}{24}\cdot B\left(\frac{1}{6},\frac{1}{6}\right) = \frac{\pi}{24}\cdot\frac{\Gamma^2\left(\frac{1}{6}\right)}{\Gamma\left(\frac{1}{3}\right)} \approx 1.51395,$$

where we have used a well known relation between the beta and gamma functions.

Solution 6 by Albert Stadler, Herrliberg, Switzerland. The variable transform x→1/x yields

$$\int_0^\infty \frac{\arctan(x)}{\sqrt[3]{x^6+1}}dx = \int_0^\infty \frac{\arctan\left(\frac{1}{x}\right)}{\sqrt[3]{x^6+1}}dx.$$

Hence

$$\int_0^\infty \frac{\arctan(x)}{\sqrt[3]{x^6+1}}dx = \frac{1}{2}\int_0^\infty \frac{\arctan(x)\ +\arctan\left(\frac{1}{x}\right)}{\sqrt[3]{x^6+1}}dx = \frac{\pi}{4}\int_0^\infty \frac{1}{\sqrt[3]{x^6+1}}dx.$$

We express the last integral in terms of the Euler beta function by means of the substitution $y = 1/(x^6+1)$ implying $x = \dfrac{(1-y)^{1/6}}{y^{1/6}},\ dx = -\dfrac{1}{6(1-y)^{5/6}y^{7/6}}dy$ and obtain

$$\frac{\pi}{4}\int_0^\infty \frac{1}{\sqrt[3]{x^6+1}}dx = \frac{\pi}{24}\int_0^1 y^{\frac{1}{3}-\frac{7}{6}}\frac{1}{(1-y)^{5/6}}dy = \frac{\pi}{24}\frac{{}^2\left(\frac{1}{6}\right)}{\left(\frac{1}{3}\right)}.$$

We use Legendre's duplication formula

$$\left(\frac{x}{2}\right)\left(\frac{x+1}{2}\right) = \frac{\sqrt{\pi}}{2^{x-1}}(x)$$

as well as Euler's reflection formula

$$(x)(1-x) = \frac{\pi}{\sin(\pi x)}$$

with $x = 1/3$ and get

$$\left(\frac{1}{6}\right)\left(\frac{2}{3}\right) = 2^{\frac{2}{3}}\sqrt{\pi}\left(\frac{1}{3}\right),$$

$$\left(\frac{1}{6}\right)\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = 2^{\frac{2}{3}}\sqrt{\pi}^2\left(\frac{1}{3}\right),$$

$$\left(\frac{1}{6}\right) = \frac{\sqrt{3}}{2^{\frac{1}{3}}\sqrt{\pi}}^{\;2}\left(\frac{1}{3}\right).$$

Finally

$$\int_0^\infty \frac{\arctan(x)}{\sqrt[3]{x^6+1}}dx = \frac{\pi}{24}\frac{{}^2\left(\frac{1}{6}\right)}{\left(\frac{1}{3}\right)} = \frac{\sqrt[3]{2}\;{}^3\left(\frac{1}{3}\right)}{16}.$$

Note: $\Gamma(1/3)$ was shown to be transcendental by G. V. Chudnovsky. As $\sqrt[3]{2}/16$ is algebraic, the integral

$$\int_0^\infty \frac{\arctan(x)}{\sqrt[3]{x^6+1}}dx$$

is transcendental.

Also solved by *Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania and the proposer.*

A–133. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a triangle with side lengths $BC = a, CA = b, AB = c$, centroid G and circumcircle Γ with circumcenter O. Let D be the reflection of G in the line BC and E be the second intersect point of AG and Γ. Knowing that $AG = DE$, find the maximum possible value of

$$S = \frac{(a+b+c)(a^3+b^3+c^3)}{a^4+b^4+c^4}$$

and determine where the maximum holds.

Solution by the proposer. Let $\angle BAC = \alpha$. Use barycentric coordinates relative to the vertices A, B, C of the triangle; then the centroid $G = (1:1:1)$. The equation of the circumcircle of $\triangle ABC$ is

$$\Gamma: \; a^2yz + b^2zx + c^2xy = 0$$

The equation of the line AG is

$$\begin{vmatrix} x & 1 & 1 \\ y & 0 & 1 \\ z & 0 & 1 \end{vmatrix} = 0 \text{ so } z - y = 0$$

Since E is the intersection point of AG and Γ, we get

$$\begin{aligned} E =& \frac{a^2}{a^2-2b^2-2c^2}A + \frac{-b^2-c^2}{a^2-2b^2-2c^2}B + \frac{-b^2-c^2}{a^2-2b^2-2c^2}C \\ &= (a^2 : -b^2-c^2 : -b^2-c^2) \end{aligned}$$

Let G_a be the orthogonal projection of G on BC. The equation of the line BC is $x = 0$ and the point $G_a = (0 : 1-x : x)$ for some $x \in \mathbb{R}$. The line $GG_a \perp BC$ so $\overrightarrow{GG_a} \cdot \overrightarrow{BC} = 0$.

$$\begin{aligned} 0 &= \left(\overrightarrow{AG_a} - \overrightarrow{AG}\right) \cdot \left(\overrightarrow{AC} - \overrightarrow{AB}\right) = \\ &= \left((1-x)\overrightarrow{AB} + x\overrightarrow{AC} - \frac{1}{3}\overrightarrow{AB} - \frac{1}{3}\overrightarrow{AC}\right) \cdot \left(\overrightarrow{AC} - \overrightarrow{AB}\right) \\ &= \frac{3x-2}{3}c^2 + \frac{3x-1}{3}b^2 + \frac{1}{2}(1-2x)(b^2+c^2-a^2) \\ &= \frac{-3a^2+b^2-c^2}{6} + a^2x; \quad x = \frac{3a^2-b^2+c^2}{6a^2} \\ G_a &= \frac{3a^2+b^2-c^2}{6a^2}B + \frac{3a^2-b^2+c^2}{6a^2}C \\ &= (0 : 3a^2+b^2-c^2 : 3a^2-b^2+c^2) \end{aligned}$$

where we have

$$\overrightarrow{AB} \cdot \overrightarrow{AB} = c^2; \quad \overrightarrow{AC} \cdot \overrightarrow{AC} = b^2;$$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = bc\cos\alpha = bc\frac{b^2+c^2-a^2}{2bc} = \frac{b^2+c^2-a^2}{2} \tag{1}$$

Since G_a is the midpoint of GD, hence $\overrightarrow{G_aD} = \overrightarrow{GG_a}$.

$$\begin{aligned} D = 2G_a - G &= \frac{3a^2+b^2-c^2}{3a^2}B + \frac{3a^2-b^2+c^2}{3a^2}C - \frac{1}{3}(A+B+C) \\ &= -\frac{1}{3}A + \frac{2a^2+b^2-c^2}{3a^2}B + \frac{2a^2-b^2+c^2}{3a^2}C \\ &= (-a^2 : 2a^2+b^2-c^2 : 2a^2-b^2+c^2) \end{aligned}$$

It is well known that squared length of the A-median is

$$m_a{}^2 = \frac{2b^2+2c^2-a^2}{4}$$

Since $AG = \frac{2}{3}m_a$,

$$AG^2 = \frac{1}{9}(2b^2+2c^2-a^2)$$

$$\begin{aligned} DE^2 &= \overrightarrow{DE}\cdot\overrightarrow{DE} = \left(\overrightarrow{AE}-\overrightarrow{AD}\right)\cdot\left(\overrightarrow{AE}-\overrightarrow{AD}\right) \\ &= \left(\frac{-b^2-c^2}{a^2-2b^2-2c^2}\overrightarrow{AB} + \frac{-b^2-c^2}{a^2-2b^2-2c^2}\overrightarrow{AC}\right. \\ &\quad \left. - \frac{2a^2+b^2-c^2}{3a^2}\overrightarrow{AB} - \frac{2a^2-b^2+c^2}{3a^2}\overrightarrow{AC}\right)^2 \\ &= \left(\frac{-2(a^4-b^4-a^2c^2+c^4)}{3a^2(a^2-2b^2-2c^2)}\overrightarrow{AB} + \frac{-2(a^4-a^2b^2+b^4-c^4)}{3a^2(a^2-2b^2-2c^2)}\overrightarrow{AC}\right)^2 \\ &\overset{(1)}{=} \frac{4(a^4-a^2b^2+b^4-a^2c^2-b^2c^2+c^4)}{9(2b^2+2c^2-a^2)} \end{aligned}$$

$$\begin{aligned} AG^2 - DE^2 &= \frac{1}{9}(2b^2+2c^2-a^2) - \frac{4(a^4-a^2b^2+b^4-a^2c^2-b^2c^2+c^4)}{9(2b^2+2c^2-a^2)} \\ &= \frac{4b^2c^2-a^4}{3(2b^2+2c^2-a^2)} = \frac{(2bc-a^2)(2bc+a^2)}{3(2b^2+2c^2-a^2)} \end{aligned}$$

So $AG = DE$ if and only if

$$a^2 = 2bc \tag{2}$$

Now let $b + c = ua$ for some $u > 0$, then from AM-GM:

$$\frac{ua}{2} = \frac{b+c}{2} \geq \sqrt{bc} = \sqrt{\frac{a^2}{2}}, \text{ so } u \geq \sqrt{2}$$

with equality if and only if $b = c$ and $u = \sqrt{2}$.

$$\begin{aligned} b^2 + c^2 &= (b+c)^2 - 2bc = a^2(u^2 - 1) \\ b^3 + c^3 &= (b+c)^3 - 3bc(b+c) = a^3\left(u^3 - \frac{3}{2}u\right) \\ b^4 + c^4 &= (b^2+c^2)^2 - 2b^2c^2 \\ &= a^4\left((u^2-1)^2 - \frac{1}{2}\right) = a^4\left(u^4 - 2u^2 + \frac{1}{2}\right) \end{aligned}$$

$$\begin{aligned} S &= \frac{(a+b+c)(a^3+b^3+c^3)}{a^4+b^4+c^4} \\ &= \frac{a^4+b^4+c^4+a^3(b+c)+a(b^3+c^3)+bc(b^2+c^2)}{a^4+b^4+c^4} \\ &= 1 + \frac{u + (u^3 - \frac{3}{2}u) + \frac{1}{2}(u^2-1)}{1 + u^4 - 2u^2 + \frac{1}{2}} \\ &= 1 + \frac{2u^3 + u^2 - u - 1}{2u^4 - 4u^2 + 3} \end{aligned}$$

We will prove that $S \leq \frac{4}{3} + \sqrt{2}$. Let

$$\begin{aligned} Q &= \frac{4}{3} + \sqrt{2} - S = \frac{1}{3} + \sqrt{2} - \frac{2u^3 + u^2 - u - 1}{2u^4 - 4u^2 + 3} \\ &= \frac{2(1+3\sqrt{2})u^4 - 6u^3 - (7+12\sqrt{2})u^2 + 3u + 3(2+3\sqrt{2})}{3(2u^4 - 4u^2 + 3)} \\ &= \frac{(u-\sqrt{2})\big((2+6\sqrt{2})u^3 + 2(3+\sqrt{2})u^2 - 3(1+2\sqrt{2})u - 3(3+\sqrt{2})\big)}{3(2u^4 - 4u^2 + 3)} \\ &= \frac{u - \sqrt{2}}{6u^2(\sqrt{2}+u)(u-\sqrt{2}) + 9}\big(15 + 2\sqrt{2} \\ &\quad + (u-\sqrt{2})((2+6\sqrt{2})u^2 + 2(9+2\sqrt{2})u + 5 + 12\sqrt{2})\big) \end{aligned}$$

Since

$$15+2\sqrt{2}+(u-\sqrt{2})((2+6\sqrt{2})u^2+2(9+2\sqrt{2})u+5+12\sqrt{2})>0;$$

$$6u^2(\sqrt{2}+u)(u-\sqrt{2})+9>0$$

for $u \geq \sqrt{2}$, hence $Q \geq 0$ with equality if and only if $u=\sqrt{2}$ and $b=c$.

It follows that

$$S=\frac{4}{3}+\sqrt{2}-Q \leq \frac{4}{3}+\sqrt{2}$$

The maximum value

$$S_{\max}=\frac{4}{3}+\sqrt{2}$$

occurs if $b=c$, and since $a^2=2bc$, it follows that $a=b\sqrt{2}$.

Finally the maximum value holds for isosceles triangle ABC with $(a;b;c)=(t\sqrt{2};t;t), t>0$.

A–134. *Proposed by Michel Bataille, Rouen, France.* Let F_m be the m-th Fibonacci number ($F_0=0, F_1=1$ and $F_{m+1}=F_m+F_{m-1}$ for $m \geq 1$). Let

$$A_n=\sum_{k=1}^{n} \frac{k}{F_{n+1-k}F_{n+3-k}} \quad (n \geq 1) \quad \text{and} \quad S=\sum_{k=1}^{\infty} \frac{1}{F_kF_{k+1}}.$$

Prove that

$$\lim_{n\to\infty}(A_n-n)=1-S.$$

Solution 1 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. Multiply the numerator and denominator of the summand inside A_n by F_{n+2-k}, writing

$$F_{n+2-k}=F_{n+3-k}-F_{n+1-k}$$

in the numerator. Then

$$\begin{aligned}
A_n &= \sum_{k=1}^{n} \frac{k(F_{n+3-k} - F_{n+1-k})}{F_{n+1-k}F_{n+2-k}F_{n+3-k}} \\
&= \sum_{k=1}^{n} k\left(\frac{1}{F_{n+1-k}F_{n+2-k}} - \frac{1}{F_{n+2-k}F_{n+3-k}}\right) \\
&= \sum_{k=1}^{n} \frac{k}{F_{n+1-k}F_{n+2-k}} - \sum_{k=0}^{n-1} \frac{k+1}{F_{n+1-k}F_{n+2-k}} \\
&= \frac{n}{F_1F_2} - \sum_{k=1}^{n-1} \frac{1}{F_{n+1-k}F_{n+2-k}} - \frac{1}{F_{n+1}F_{n+2}} \\
&= n - \sum_{k=2}^{n} \frac{1}{F_kF_{k+1}} - \frac{1}{F_{n+1}F_{n+2}} \\
&= n + 1 - \sum_{k=1}^{n+1} \frac{1}{F_kF_{k+1}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\lim_{n\to\infty}(A_n - n) &= \lim_{n\to\infty}\left(1 - \sum_{k=1}^{n+1} \frac{1}{F_kF_{k+1}}\right) \\
&= 1 - \sum_{k=1}^{\infty} \frac{1}{F_kF_{k+1}} = 1 - S.
\end{aligned}$$

For completeness, we can show that the sum in S converges by the ratio test:

$$\lim_{k\to\infty} \frac{\frac{1}{F_{k+1}F_{k+2}}}{\frac{1}{F_kF_{k+1}}} = \lim_{k\to\infty} \frac{F_k}{F_{k+2}} = \frac{1}{\phi^2} < 1,$$

where

$$\phi = \frac{1+\sqrt{5}}{2}.$$

Solution 2 by Moti Levy, Rehovot, Israel. By changing the direction of the summation index, we get

$$\begin{aligned}
A_n &= \sum_{k=1}^{n} \frac{k}{F_{n+1-k}F_{n+3-k}} = \sum_{k=1}^{n} \frac{n+1-k}{F_kF_{k+2}} \\
&= (n+1)\sum_{k=1}^{n} \frac{1}{F_kF_{k+2}} - \sum_{k=1}^{n} \frac{k}{F_kF_{k+2}} \\
&= \sum_{k=1}^{n} \frac{1}{F_kF_{k+2}} + \sum_{k=1}^{n} \frac{n-k}{F_kF_{k+2}} = \sum_{k=1}^{n} \frac{1}{F_kF_{k+2}} + \sum_{k=1}^{n-1} \frac{n-k}{F_kF_{k+2}}. \qquad (1)
\end{aligned}$$

We use the following identity,

$$\sum_{k=1}^{n-1}(n-k)b_k = \sum_{k=1}^{n-1}\sum_{m=1}^{k} b_m,$$

to obtain

$$\sum_{k=1}^{n-1} \frac{n-k}{F_kF_{k+2}} = \sum_{k=1}^{n-1}\sum_{m=1}^{k} \frac{1}{F_mF_{m+2}}. \qquad (2)$$

We substitute (2) in (1) to get this expression for A_n,

$$A_n = \sum_{k=1}^{n} \frac{1}{F_kF_{k+2}} + \sum_{k=1}^{n-1}\sum_{m=1}^{k} \frac{1}{F_mF_{m+2}}. \qquad (3)$$

Now, the following identity is known (also can easily be proved by mathematical induction)

$$\sum_{k=1}^{n} \frac{1}{F_kF_{k+2}} = 1 - \frac{1}{F_{n+1}F_{n+2}}. \qquad (4)$$

We substitute (4) twice in (3) to get another expression for A_n,

$$\begin{aligned}
A_n &= 1 - \frac{1}{F_{n+1}F_{n+2}} + \sum_{k=1}^{n-1}\left(1 - \frac{1}{F_{k+1}F_{k+2}}\right) \\
&= 1 - \frac{1}{F_{n+1}F_{n+2}} + n - 1 - \sum_{k=1}^{n-1} \frac{1}{F_{k+1}F_{k+2}} \\
&= n - \sum_{k=1}^{n} \frac{1}{F_{k+1}F_{k+2}} = n + 1 - \sum_{k=1}^{n+1} \frac{1}{F_kF_{k+1}}. \qquad (5)
\end{aligned}$$

Equation (5) implies

$$A_n - n = 1 - \sum_{k=1}^{n+1} \frac{1}{F_k F_{k+1}}.$$

By taking the limit $n \to \infty$, we obtain the required result.

Solution 3 by the proposer. The change of index $j = n + 1 - k$ gives

$$A_n = \sum_{j=1}^{n} \frac{n+1-j}{F_j F_{j+2}} = (n+1) \sum_{j=1}^{n} \frac{1}{F_j F_{j+2}} - \sum_{j=1}^{n} \frac{j}{F_j F_{j+2}}. \qquad (1)$$

For later use note that

$$\frac{1}{F_j F_{j+2}} = \frac{F_{j+2} - F_j}{F_j F_{j+1} F_{j+2}} = \frac{1}{F_j F_{j+1}} - \frac{1}{F_{j+1} F_{j+2}}$$

so that for $m \geq 1$,

$$\sum_{j=m}^{\infty} \frac{1}{F_j F_{j+2}} = \frac{1}{F_m F_{m+1}}. \qquad (2)$$

In particular $\sum_{j=1}^{\infty} \frac{1}{F_j F_{j+2}} = 1$.

Let $U_n = A_n - n$. We evaluate $U_{n+1} - U_n = A_{n+1} - A_n - 1$ with the help of (1) and (2):

$$U_{n+1} - U_n = (n+2) \sum_{j=1}^{n+1} \frac{1}{F_j F_{j+2}} - \frac{n+1}{F_{n+1} F_{n+3}} - (n+1) \sum_{j=1}^{n} \frac{1}{F_j F_{j+2}} - 1$$
$$= \sum_{j=1}^{n+1} \frac{1}{F_j F_{j+2}} - 1 = - \sum_{j=n+2}^{\infty} \frac{1}{F_j F_{j+2}} = - \frac{1}{F_{n+2} F_{n+3}}$$

We deduce that for $n \geq 2$, we have

$$U_n = U_1 + \sum_{j=1}^{n-1} (U_{j+1} - U_j) = A_1 - 1 - \sum_{j=1}^{n-1} \frac{1}{F_{j+2} F_{j+3}}$$

and therefore

$$\lim_{n \to \infty} (A_n - n) = \lim_{n \to \infty} U_n = -\frac{1}{2} - \left(S - \frac{1}{F_1 F_2} - \frac{1}{F_2 F_3} \right) = 1 - S.$$

Also solved by *José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain, and José Luis Díaz-Barrero, Barcelona, Spain.*

A–135. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ be sequences of real numbers satisfying

$$x_n^2+y_n^2+x_{n-1}^2+y_{n-1}^2 = (y_n x_{n-1} - x_n y_{n-1}) + \sqrt{3}(x_n x_{n-1} + y_n y_{n-1}).$$

Show that they are periodic and determine their periods.

Solution 1 by Michel Bataille, Rouen, France. Using the hypothesis, we readily see that

$$\left(x_n + \frac{1}{2}\,y_{n-1} - \frac{\sqrt{3}}{2}\,x_{n-1}\right)^2 + \left(y_n - \frac{1}{2}\,x_{n-1} - \frac{\sqrt{3}}{2}\,y_{n-1}\right)^2 = 0.$$

It follows that the sequences $\{x_n\}$ and $\{y_n\}$ satisfy the recursions

$$x_n = \frac{\sqrt{3}}{2}\,x_{n-1} - \frac{1}{2}\,y_{n-1}, \qquad y_n = \frac{1}{2}\,x_{n-1} + \frac{\sqrt{3}}{2}\,y_{n-1},$$

that is, $\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}$ where $A = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$.
The characteristic polynomial of A is

$$\lambda^2 - \sqrt{3}\lambda + 1 = (\lambda - e^{i\pi/6})(\lambda - e^{-i\pi/6}),$$

hence $A = PDP^{-1}$ for some invertible matrix P where

$$D = \begin{pmatrix} e^{i\pi/6} & 0 \\ 0 & e^{-i\pi/6} \end{pmatrix}.$$

It follows that for nonnegative integers n, k we have

$$\begin{pmatrix} x_{n+k} \\ y_{n+k} \end{pmatrix} = A^k \begin{pmatrix} x_n \\ y_n \end{pmatrix} = PD^kP^{-1} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

Since $D^k = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ if and only if $k = 12m$ for some integer m, we deduce that $\{x_n\}$ and $\{y_n\}$ are periodic with group of periods $12\mathbb{Z}$.

Solution 2 by Brian Bradie, Christopher Newport University, Newport News, VA, USA. Let $x_n = r_n \cos\theta_n$ and $y_n = r_n \sin\theta_n$. Substituting these expressions into

$$x_n^2 + y_n^2 + x_{n-1}^2 + y_{n-1}^2 = (y_n x_{n-1} - x_n y_{n-1}) + \sqrt{3}(x_n x_{n-1} + y_n y_{n-1})$$

yields

$$\begin{aligned} r_n^2 + r_{n-1}^2 &= r_n r_{n-1}(\sin\theta_n \cos\theta_{n-1} - \cos\theta_n \sin\theta_{n-1}) \\ &+ \sqrt{3} r_n r_{n-1}(\cos\theta_n \cos\theta_{n-1} + \sin\theta_n \sin\theta_{n-1}) \\ &= 2r_n r_{n-1}\left(\frac{1}{2}\sin(\theta_n - \theta_{n-1}) + \frac{\sqrt{3}}{2}\cos(\theta_n - \theta_{n-1})\right) \\ &= 2r_n r_{n-1}\cos\left(\theta_n - \theta_{n-1} - \frac{\pi}{6}\right). \end{aligned}$$

Now, by the AM-GM inequality, $r_n^2 + r_{n-1}^2 \geq 2r_n r_{n-1}$ with equality holding if and only if $r_n = r_{n-1}$. On the other hand,

$$2r_n r_{n-1}\cos\left(\theta_n - \theta_{n-1} - \frac{\pi}{6}\right) \leq 2r_n r_{n-1}$$

with equality holding if and only if

$$\theta_n - \theta_{n-1} - \frac{\pi}{6} = 2k\pi$$

for some integer k. Thus, r_n must equal r_{n-1} and

$$\theta_n = 2k\pi + \theta_{n-1} + \frac{\pi}{6}$$

for some integer k. It follows that $r_n = R$ for all n for some real number R and

$$\theta_{n+11} = (24k+1)\pi + \theta_{n-1},$$

which implies $x_{n+11} = x_{n-1}$ and $y_{n+11} = y_{n-1}$ for all n. The sequences $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are therefore periodic, and each has period 12.

Solution 3 by Moti Levy, Rehovot, Israel. Let us define a sequence $(z_n)_{n\geq 0}$ of complex numbers by$z_n = x_n + iy_n$. Then

$$\begin{aligned} |z_n|^2 &= x_n^2 + y_n^2, \\ \mathrm{Re}(z_n \overline{z_{n-1}}) &= x_n x_{n-1} + y_n y_{n-1}, \\ \mathrm{Im}(z_n \overline{z_{n-1}}) &= x_n y_{n-1} - y_n x_{n-1}. \end{aligned}$$

Thus, the equation $x_n^2 + y_n^2 + x_{n-1}^2 + y_{n-1}^2 = (y_n x_{n-1} - x_n y_{n-1}) + \sqrt{3}(x_n x_{n-1} + y_n y_{n-1})$ becomes

$$|z_n|^2 + |z_{n-1}|^2 = -\operatorname{Im}(z_n \overline{z_{n-1}}) + \sqrt{3}\operatorname{Re}(z_n \overline{z_{n-1}}).$$

$$\begin{aligned} |z_n|^2 + |z_{n-1}|^2 &= \operatorname{Re}\Big(\sqrt{3} z_n \overline{z_{n-1}} - i z_n \overline{z_{n-1}}\Big) \\ &= \operatorname{Re}\Big(\big(\sqrt{3} - i\big) z_n \overline{z_{n-1}}\Big). \end{aligned}$$

Now, let us define a sequence $(T_n)_{n\geq 0}$ of complex numbers by

$$T_n = \frac{z_n}{z_{n-1}}, \quad \text{or} \quad z_n = T_n z_{n-1}.$$

We have

$$|z_n|^2 = z_n \overline{z}_n = T_n z_{n-1} \overline{T_n}\,\overline{z_{n-1}} = |T_n|^2 |z_{n-1}|^2.$$

$$|z_n|^2 + |z_{n-1}|^2 = \big(|T_n|^2 + 1\big)|z_{n-1}|^2.$$

$$\big(|T_n|^2 + 1\big)|z_{n-1}|^2 = \operatorname{Re}\Big(\big(\sqrt{3} - i\big) T_n z_{n-1} \overline{z_{n-1}}\Big),$$

$$\big(|T_n|^2 + 1\big)|z_{n-1}|^2 = \operatorname{Re}\Big(\big(\sqrt{3} - i\big) T_n |z_{n-1}|^2\Big).$$

Dividing by $|z_{n-1}|^2$, the original equation is equivalent to

$$|T_n|^2 + 1 = \operatorname{Re}\Big(\big(\sqrt{3} - i\big) T_n\Big).$$

(assuming that $z_{n-1} \neq 0$, otherwise the sequences must be constant zeros).
It follows that T_n is constant (does not depend on n), so we may write,

$$T_n = \alpha e^{i\theta}, \quad \alpha \geq 0.$$

$$\alpha^2 + 1 = 2\alpha \operatorname{Re}\big(e^{-i\frac{\pi}{6}} e^{i\theta}\big)$$

$$\alpha^2 - 2\alpha \cos\Big(\theta - \frac{\pi}{6}\Big) + 1 = 0.$$

$\cos\big(\theta - \frac{\pi}{6}\big)$ must be positive, hence

$$0 = \alpha^2 - 2\alpha \cos\Big(\theta - \frac{\pi}{6}\Big) + 1 \geq (\alpha - 1)^2.$$

It follows that

$$\alpha = 1.$$
$$2 - 2\cos\Big(\theta - \frac{\pi}{6}\Big) = 0$$
$$\theta = -\frac{\pi}{6} + 2\pi k, \quad k \in \mathbb{Z}.$$
$$T_n = e^{-i\frac{\pi}{6}},$$

and

$$z_n = e^{-i\frac{\pi}{6}} z_{n-1}.$$

We conclude that each z_n is a rotation of z_{n-1} by $\frac{\pi}{6}$, hence the sequence $\{x_n, y_n\}$ is periodic with period 12.

Solution 4 by the proposer. Multiplying all the terms by 4, yields

$$4\,(x_n^2+x_{n-1}^2+y_n^2+y_{n-1}^2) = 4\,(y_n x_{n-1} - x_n y_{n-1}) + 4\sqrt{3}\,(x_n x_{n-1} + y_n y_{n-1}).$$

Rearranging terms, we get

$$4\,(x_n^2+y_n^2)+4\,(x_{n-1}^2+y_{n-1}^2)-4\,(x_n y_{n-1}-y_n x_{n-1})-4\sqrt{3}\,(x_n x_{n-1}+y_n y_{n-1}) = 0.$$

Completing squares, we obtain

$$\Big(2x_n - \sqrt{3}\,x_{n-1} - y_{n-1}\Big)^2 + \Big(2y_n + x_{n-1} - \sqrt{3}\,y_{n-1}\Big)^2 = 0$$

from which it follows

$$\begin{aligned} 2x_n - \sqrt{3}\,x_{n-1} - y_{n-1} &= 0, \\ 2y_n + x_{n-1} - \sqrt{3}\,y_{n-1} &= 0, \end{aligned}$$

or

$$\begin{aligned} x_n &= \frac{\sqrt{3}}{2}\,x_{n-1} + \frac{1}{2}y_{n-1} = \cos\frac{\pi}{6}\,x_{n-1} + \sin\frac{\pi}{6}\,y_{n-1}, \\ y_n &= -\frac{1}{2}\,x_{n-1} + \frac{\sqrt{3}}{2}\,y_{n-1} = -\sin\frac{\pi}{6}\,x_{n-1} + \cos\frac{\pi}{6}\,y_{n-1}. \end{aligned}$$

In matrix from, we have

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix} \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Using induction, it is easy to see that

$$\begin{pmatrix} \cos\frac{\pi}{6} & \sin\frac{\pi}{6} \\ -\sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}^n = \begin{pmatrix} \cos\frac{n\pi}{6} & \sin\frac{n\pi}{6} \\ -\sin\frac{n\pi}{6} & \cos\frac{n\pi}{6} \end{pmatrix},$$

and then we have

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \cos\frac{n\pi}{6} & \sin\frac{n\pi}{6} \\ -\sin\frac{n\pi}{6} & \cos\frac{n\pi}{6} \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

or

$$\begin{aligned} x_n &= x_0 \cos\frac{n\pi}{6} + y_0 \sin\frac{n\pi}{6}, \\ y_n &= -x_0 \sin\frac{n\pi}{6} + y_0 \cos\frac{n\pi}{6}. \end{aligned}$$

Then,

$$\begin{aligned} x_{n+12} &= x_0 \cos\frac{(n+12)\pi}{6} + y_0 \sin\frac{(n+12)\pi}{6} = x_n, \\ y_{n+12} &= -x_0 \sin\frac{(n+12)\pi}{6} + y_0 \cos\frac{(n+12)\pi}{6} = y_n. \end{aligned}$$

So, $x_{n+12} = x_n$, $y_{n+12} = y_n$ and both sequences are periodic with period $T = 12$.

A–136. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain and Óscar Rivero Salgado, Santiago de Compostela, Spain.* Find all n-tuples $(a_1, a_2, \dots, a_n)$ of positive integers such that

$$(a_1! - 1)(a_2! - 1)\dots(a_n! - 1) = 4^2 + k^2,$$

where k is an integer number.

Solution by the proposers. First, we claim that *if p is a prime of the form $4n + 3$ and a and b are integers such that $p \mid a^2 + b^2$, then p divides a and p divides b.* Indeed, assume that p does not divide a then there exists a positive integer c such that $ca \equiv 1 \pmod p$. Since $p \mid a^2+b^2$ then $p \mid c^2(a^2+b^2) = (ca)^2+(cb)^2$ and we have that $1 + (bc)^2 \equiv 0 \pmod p$ from which $(bc)^2 \equiv -1 \pmod p$ follows. Since $p \nmid bc$ then it holds

$$(-1)^{\frac{p-1}{2}} \equiv (bc)^{p-1} \equiv 1 \pmod p$$

where the last congruence is true on account of Fermat's Little Theorem (FLT). On the other hand, since $p \equiv 3 \pmod 4$ then

$$(-1)^{\frac{p-1}{2}} = -1$$

(contradiction) and the claim is proven.

Now, we suppose that

$$(a_1! - 1)(a_2! - 1) \ldots (a_n! - 1) - 16 = k^2.$$

We will see that $a_i \in \{2, 3\}$ for all i, $(1 \le i \le n)$. Obviously, $a_i \neq 1$ for $1 \le i \le n$, so assume that $a_i > 3$ for some i. Then $a_i! - 1 \equiv 3 \pmod 4$, thus there is a prime $p \equiv 3 \pmod 4$ such that $p \mid a_i! - 1$. Then, p divides $4^2 + k^2$, a contradiction by the claim. So, actually $a_i \in \{2, 3\}$.

Let m be the number of a_i in the sequence $(a_1, a_2, \ldots, a_n)$ and the remaining $n - m$ equal to 2. In this case, the equation

$$(a_1! - 1)(a_2! - 1) \ldots (a_n! - 1) - 16 = k^2$$

becomes $5^m - 16 = k^2$. Since k is odd, then considering the last equation $\pmod 8$ we get that m is even. But then

$$\left(5^{m/2} - k\right)\left(5^{m/2} + k\right) = 16.$$

Since the divisors of 16 are $1, 2, 4, 8, 16$, then by inspection, immediately it follows that $\left(5^{m/2} - k\right) = 2$ and $\left(5^{m/2} + k\right) = 8$ from which we get $m = 2$. Thus the sequence $(a_1, a_2, \ldots, a_n)$ contains two terms equal 3 and $n - 2$ equal 2. Clearly, all such sequences are solutions of the problem.

Also solved by *Moti Levy, Rehovot, Israel.*

Previous issue solutions

Ioan Viorel Codreanu, Satulung, Maramures, Romania sent solutions to proposals E-125,E-126,E-127, EM-128 and MH-128.

Arhimede Mathematical Journal

Volume 12, No. 1 Spring 2025

Aim and Scope

The goal of *Arhimede Mathematical Journal* is to provide a means of publication of useful materials to train students for Mathematical Contests at all levels. Potential contributions include any work involving fresh ideas and techniques, problems and lessons helpful to train contestants, all written in a clear and elegant mathematical style. All areas of mathematics, including algebra, combinatorics, geometry, number theory and real and complex analysis, are considered appropriate for the journal.

Information for Authors

A detailed statement of author guidelines is available at the journal's website:

http://www.amj-math.com

Edited in Barcelona by the Arhimede Association

ISSN 2462-537X