No. 2
2024

ARHIMEDE
MATHEMATICAL JOURNAL

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Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

jose.luis.diaz@upc.edu

Goldbach's Conjecture mod n

Joe Santmyer

Abstract

An entertaining, and sometimes interesting exercise, is to take a mathematical statement or formula that applies to the set of integers and reformulate it into a mod n analog. The statements in mathematics that I find most intriguing are conjectures, those claims that may have a wealth of supporting evidence but have resisted mathematical proof. In this paper a mod n version of a well known conjecture is formulated and discussed.

1 Variants of the Goldbach Conjecture, GC

Consider the famous *Goldbach Conjecture*, GC.

Goldbach Conjecture, GC: Every even integer greater than 2 is the sum of two, not necessarily distinct, prime numbers.

Now change prime to relatively prime. Relatively prime to what? An integer n needs to be introduced so that the integers that sum to the even integer are each relatively prime to n. Consequently, the mod n version is stated as follows.

GC mod n: Let n be a positive integer. Every even integer is the sum of two, not necessarily distinct, integers relatively prime to n.

GC is a statement about the set Z of integers. GC mod n is a statement about the set $Z_n = \{0, 1, 2, \ldots, n-1\}$ of integers under addition mod n. The set of integers in Z_n relatively prime to n is the set of units of the ring Z_n under addition and multiplication mod n. The set of units is commonly denoted as Z_n^*. With this notation introduced, GC mod n can be reformulated as follows.

GC mod n: Let n be a positive integer. If $s \in Z_n$ and s is even then there exist integers $x, y \in Z_n^*$ that satisfy the congruence $s \equiv (x + y) \bmod n$.

In fact, the following stronger statement is true.

Theorem 1. *Let n be a positive integer and $s \in Z_n$. There exist integers $x, y \in Z_n^*$ that satisfy the congruence $s \equiv (x + y) \bmod n$ except when s is odd and n is even.*

The previous result is stated as a theorem because it can be proved. In fact, a formula can be derived that *counts* the number of solutions. The formula was motivated by several results mentioned in [5]. Variations and reformulations of GC are not new. Others can be found in [1] and [3].

The primary goal of this paper is to prove theorem 1 and derive a formula for counting the number of solutions in Z_n to the congruence $s \equiv (x + y) \bmod n$ with x and y relatively prime to n. To get started the following lemma is proved.

Lemma 1. *Let n be a positive even integer and $s \in Z_n$ with s odd. No integers $x, y \in Z_n^*$ satisfy the congruence $s \equiv (x + y) \bmod n$.*

Proof. Suppose $x, y \in Z_n^*$. Since $x, y \in Z_n^*$, x and y are relatively prime to n. Since n is even both x and y must be odd. Consequently, $x + y$ is even. Since s is odd, x and y cannot satisfy the congruence $s \equiv (x + y) \bmod n$. Therefore, no $x, y \in Z_n^*$ satisfy the congruence. This completes the proof of the lemma. □

2 A Recurrence Formula

The goal now is to prove theorem 1 and derive a formula which *counts* the number of solutions. A solution to $s \equiv (x + y) \bmod n$

is either a two element set $\{x, y\}$ or the set $\{x\}$ if $y = x$. The following definition will be useful in proving theorem 1 and counting solutions.

Definition 1. *Let n be a natural number and $s \in Z_n$. Define the set $R_{n,s}$ as*

$$R_{n,s} = \{x \in Z_n^* : \text{ there exists } y \in Z_n^* \text{ with } s \equiv (x+y) \bmod n\}.$$

Observation. *Note that $R_{n,0} = Z_n^*$ since for each $x \in Z_n^*$, if $y = n - x \in Z_n^*$ then $0 \equiv (x+y) \bmod n$.*

Let $|R_{n,s}|$ denote the number of elements in $R_{n,s}$. By lemma 1, $|R_{n,s}| = 0$ when n is even and s is odd.

With the aid of *Mathematica* consider some examples.

Example 1. *If $n = 24$ then $Z_{24}^* = \{1, 5, 7, 11, 13, 17, 19, 23\}$. For $s \in Z_{24}$ and s odd we have $R_{24,s} = \emptyset$. Otherwise*

$$\begin{aligned} R_{24,0} &= R_{24,6} = R_{24,12} = R_{24,18} = Z_{24}^* \\ R_{24,2} &= R_{24,8} = R_{24,14} = R_{24,20} = \{1, 7, 13, 19\} \\ R_{24,4} &= R_{24,10} = R_{24,16} = R_{24,22} = \{5, 11, 17, 23\}. \end{aligned}$$

Example 2. *If $n = 45$ then*

$$Z_{45}^* = \{1, 2, 4, 7, 8, 11, 13, 14, 16, 17, 19, 22, 23,$$

$$26, 28, 29, 31, 32, 34, 37, 38, 41, 43, 44\}.$$

If $s = 13$, $s = 27$ and $s = 38$ we have

$$R_{45,13} = \{2, 11, 14, 17, 26, 29, 32, 41, 44\}$$
$$R_{45,27} = \{1, 4, 8, 11, 13, 14, 16, 19, 23, 26, 28, 29, 31, 34, 38, 41, 43, 44\}$$
$$R_{45,38} = \{1, 4, 7, 16, 19, 22, 31, 34, 37\}.$$

The following result provides a recurrence that can be used to prove a counting formula for $|R_{n,s}|$ by mathematical induction.

Lemma 2. *Let $n \geq 2$ be an integer with prime factorization*

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{k-1}^{\alpha_{k-1}} p_k^{\alpha_k}$$

where $\alpha_i > 0$ for $1 \leq i \leq k$ and $p_1 < p_2 < \cdots < p_k$ is an increasing sequence of primes (the p_i are not necessarily the first k primes). Let $m = n/p_k^{\alpha_k}$. Let $s \in Z_n$ and $s = mq + s_0$ where $s_0 \in Z_m$ (by the Euclidean algorithm given s and m such a representation of s is always possible). Then

$$|R_{n,s}| = \begin{cases} p_k^{\alpha_k - 1}(p_k - 1)|R_{m,s_0}| & \text{if } p_k \mid s, \\ p_k^{\alpha_k - 1}(p_k - 2)|R_{m,s_0}| & \text{otherwise.} \end{cases}$$

Proof. The overall idea of the proof is to show that for each $x \in R_{m,s_0}$ there are $p_k^{\alpha_k - 1}(p_k - 1)$ integers y relatively prime to p_k satisfying the congruence $s \equiv (x+y) \bmod n$ when $p_k \mid s$, otherwise there are $p_k^{\alpha_k - 1}(p_k - 2)$ such integers.

Let $x \in R_{m,s_0}$. Consider the integer $u = ma + x$ where $0 \leq a < p_k$. Since m and p_k are relatively prime, the equation $ma \equiv -x \bmod p_k$ has a unique solution mod p_k. Let a_0 be such a solution and $u_0 = ma_0 + x$. Then $p_k \mid u_0$ and for the remaining $p_k - 1$ values $u = ma + x$ where $a \in [0, p_k)$ and $a \neq a_0$, we have $\gcd(u, p_k) = 1$.

Similarly, $a_i = a_0 + ip_k$ is the unique solution mod p_k to $ma \equiv -x \bmod p_k$ where $a_i \in [ip_k, (i+1)p_k)$ for $i = 1, 2, \ldots, p^{\alpha_k - 1} - 1$. Let $u_i = ma_i + x$. Then $p_k \mid u_i$ and for the remaining $p_k - 1$ values $u = ma + x$ where $a \in [ip_k, (i+1)p_k)$ and $a \neq a_i$, we have $\gcd(u, p_k) = 1$.

Altogether there are $p_k - 1 + (p_k^{\alpha_k - 1} - 1)(p_k - 1) = p_k^{\alpha_k - 1}(p_k - 1)$ values $u = ma + x$ where $a \in [0, p^{\alpha_k})$ such that $\gcd(u, p_k) = 1$.

First, assume $p_k \mid s$. Let u be one of the values relatively prime to p_k. Then $s - u$ is relatively prime to p_k. If not, then $p_k \mid s - u$. Since $p_k \mid s$ then $p_k \mid u$. But this contradicts the fact that u and p_k are relatively prime. So $s - u$ and p_k are relatively prime.

We claim that u and $s - u$ are both relatively prime to n when $p_k \mid s$. Let d be a common divisor of u and n. Since $d \mid u$ and

u and p_k are relatively prime then d and p_k are relatively prime. Since $d \mid n$, $n = mp_k^{\alpha_k}$ and m and p_k are relatively prime then $d \mid m$. Since $d \mid u$, $d \mid m$ and $u = ma + x$ then $d \mid x$. Since $d \mid m$, $d \mid x$ and $x \in R_{m,s_0}$ then $d = 1$. Since d was a common divisor of u and n then u and n are relatively prime. Next, let d be a common divisor of $s - u$ and n. Since $d \mid s - u$ and $s - u$ and p_k are relatively prime then d and p_k are relatively prime. Since $d \mid n$, $n = mp_k^{\alpha_k}$ and m and p_k are relatively prime then $d \mid m$. So $d \mid m$ and $d \mid s - u$. So $d \mid mq + s_0 - (ma + x)$. That is, $d \mid m(q-a) + s_0 - x$. Since $d \mid m$ then $d \mid s_0 - x$. Since $x \in R_{m,s_0}$ and $x + (s_0 - x) = s_0$ then $s_0 - x \in R_{m,s_0}$. Since $d \mid m$, $d \mid s_0 - x$ and $s_0 - x \in R_{m,s_0}$ then $d = 1$. Since d was a common divisor of $s - u$ and n then $s - u$ and n are relatively prime. This shows that $u \in R_{n,s}$. Therefore if $p_k \mid s$, for each $x \in R_{m,s_0}$ if $u = ma + x$ and u and p_k are relatively prime then $u \in R_{n,s}$. Since there are $p_k^{\alpha_k} - p_k^{\alpha_k - 1}$ values u relatively prime to p_k then

$$|R_{n,s}| = (p_k^{\alpha_k} - p_k^{\alpha_k-1})|R_{m,s_0}| = p_k^{\alpha_k-1}(p_k - 1)|R_{m,s_0}|.$$

Second, assume that p_k does not divide s. Consider the p_k values $u = ma + x$ where $a = 0, 1, 2, \ldots, p_k - 1$. The equation $ma \equiv (s-x) \bmod p_k$ has a unique solution $a_0 \bmod p_k$. So $p_k \mid ma_0 + x - s$, that is, $p_k \mid u_0 - s$ where $u_0 = ma_0 + x$. And so, $p_k \mid s - u_0$. Now p_k does not divide u_0. For, if $p_k \mid u_0$, then, since $p_k \mid s - u_0$, this would imply $p_k \mid s$, contradicting the assumption that p_k does not divide s. Since p_k does not divide u_0, p_k and u_0 are relatively prime.

Now $u_0 \notin A_{n,s}$. Otherwise, there exists $v_0 \in Z_n^*$ such that $u_0 + v_0 \equiv s \bmod n$. So $n \mid u_0 + v_0 - s$. So $p_k \mid u_0 + v_0 - s$ and so $p_k \mid (u_0 - s) + v_0$. Since $p_k \mid u_0 - s$ then $p_k \mid v_0$. But this contradicts the fact that $v_0 \in Z_n^*$. Thus $u_0 \notin A_{n,s}$. So, among the $p_k - 1$ values $u = ma + x$ relatively prime to p_k with $a \in [0, p_k)$ exactly one of them $u_0 \notin A_{n,s}$. The remaining $p_k - 2$ values u must be in $A_{n,s}$ because both u and $s - u$ are relatively prime to p_k.

Similarly, there is exactly one $u_i = ma_i + x$ where

a. $a_i \in [ip_k, (i+1)p_k)$,

b. $\gcd(u_i, p_k) = 1$,
c. $p_k \mid s - u_i$,
d. $u_i \notin A_{n,s}$.

So, among the $p_k - 1$ values u relatively prime to p_k, $p_k - 2$ are in $A_{n,s}$. The remaining $p_k - 2$ values u must be in $A_{n,s}$ because both u and $s - u$ are relatively prime to p_k.

Therefore, if p_k does not divide s, altogether there are

$$p_k - 2 + (p_k^{\alpha_k - 1} - 1)(p_k - 2) = p_k^{\alpha_k - 1}(p_k - 2)$$

values u that are in $A_{n,s}$. Since this is true for each $x \in R_{m,s_0}$ then

$$|R_{n,s}| = p_k^{\alpha_k - 1}(p_k - 2)|R_{m,s_0}|.$$

This completes the proof of the lemma. □

3 A Counting Formula

The next result gives a *counting* formula for $|R_{n,s}|$ based on the prime factorization of n.

Theorem 2. *Let $n \geq 2$ be an integer with prime factorization*

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{k-1}^{\alpha_{k-1}} p_k^{\alpha_k}$$

where $\alpha_i > 0$ for $1 \leq i \leq k$ and $p_1 < p_2 < \cdots < p_k$ is an increasing sequence of primes (the p_i are not necessarily the first k primes). Let $P_n = \{p_i : 1 \leq i \leq k\}$ and $D_s = \{p_i \in P_n : p_i \mid s\}$. Then

$$|R_{n,s}| = \left(\prod_{i=1}^{k} p_i^{\alpha_i - 1}\right)\left(\prod_{p_i \in D_s} (p_i - 1)\right)\left(\prod_{p_i \in P_n \setminus D_s} (p_i - 2)\right). \quad (1)$$

In (1) we assume that if $D_s = \emptyset$ or $P_n \setminus D_s = \emptyset$ then

$$\prod_{p_i \in D_s} (p_i - 1) = 1 \qquad \text{and} \qquad \prod_{p_i \in P_n \setminus D_s} (p_i - 2) = 1.$$

Note that if n is even and s is odd then $p_1 = 2 \in P_n \setminus D_s$. This means that the third factor in parentheses in (1) contains $(p_1 - 2) = (2 - 2) = 0$ and so $|R_{n,s}| = 0$ which agrees with lemma 1. Now consider the proof of the theorem.

Proof. It is easy to establish the formula for $n = 2$ and $n = 3$. By the induction hypothesis, assume the formula holds for $m < n$. Let $s \in Z_n$ and $m = n/p_k^{\alpha_k}$. By the Euclidean algorithm given s and m there exist integers q and s_0 with $0 \le s_0 < m$ such that $s = mq + s_0$. It is easy to see that $P_n = P_m \cup \{p_k\}$, that

$$D_s = \begin{cases} D_{s_0} \cup \{p_k\} & \text{if } p_k \mid s, \\ D_{s_0} & \text{otherwise} \end{cases}$$

and that

$$P_n \setminus D_s = \begin{cases} P_m \setminus D_{s_0} & \text{if } p_k \mid s, \\ (P_m \cup \{p_k\}) \setminus D_{s_0} & \text{otherwise.} \end{cases}$$

By lemma 2

$$|R_{n,s}| = \begin{cases} p_k^{\alpha_k - 1}(p_k - 1)|R_{m,s_0}| & \text{if } p_k \mid s, \\ p_k^{\alpha_k - 1}(p_k - 2)|R_{m,s_0}| & \text{otherwise.} \end{cases}$$

By the induction hypothesis

$$|R_{m,s_0}| = \left(\prod_{i=1}^{k-1} p_i^{\alpha_i - 1}\right)\left(\prod_{p_i \in D_{s_0}} (p_i - 1)\right)\left(\prod_{p_i \in P_m \setminus D_{s_0}} (p_i - 2)\right).$$

First, suppose $p_k \mid s$. Then

$$\begin{aligned} |R_{n,s}| &= p_k^{\alpha_k - 1}(p_k - 1)\left(\prod_{i=1}^{k-1} p_i^{\alpha_i - 1}\right)\left(\prod_{p_i \in D_{s_0}} (p_i - 1)\right)\left(\prod_{p_i \in P_m \setminus D_{s_0}} (p_i - 2)\right) \\ &= \left(\prod_{i=1}^{k} p_i^{\alpha_i - 1}\right)\left(\prod_{p_i \in D_{s_0} \cup \{p_k\}} (p_i - 1)\right)\left(\prod_{p_i \in P_m \setminus D_{s_0}} (p_i - 2)\right) \\ &= \left(\prod_{i=1}^{k} p_i^{\alpha_i - 1}\right)\left(\prod_{p_i \in D_s} (p_i - 1)\right)\left(\prod_{p_i \in P_n \setminus D_s} (p_i - 2)\right). \end{aligned}$$

Second, suppose p_k does not divide s. Then

$$\begin{aligned}|R_{n,s}| &= p_k^{\alpha_k-1}(p_k-2)\left(\prod_{i=1}^{k-1} p_i^{\alpha_i-1}\right)\left(\prod_{p_i\in D_{s_0}}(p_i-1)\right)\left(\prod_{p_i\in P_m\setminus D_{s_0}}(p_i-2)\right)\\ &= \left(\prod_{i=1}^{k} p_i^{\alpha_i-1}\right)\left(\prod_{p_i\in D_{s_0}}(p_i-1)\right)\left(\prod_{p_i\in (P_m\cup\{p_k\})\setminus D_{s_0}}(p_i-2)\right)\\ &= \left(\prod_{i=1}^{k} p_i^{\alpha_i-1}\right)\left(\prod_{p_i\in D_{s}}(p_i-1)\right)\left(\prod_{p_i\in P_n\setminus D_{s}}(p_i-2)\right).\end{aligned}$$

In both cases the formula holds for n. Therefore, by mathematical induction the formula holds for all positive integers $n \geq 2$. This completes the proof of the theorem. □

Corollary 1. *If n is odd and $s \in Z_n$ then $R_{n,s} \neq \emptyset$.*

Proof. The only factor in (1) that can be zero is $p_i - 2$. And so, $p_i = 2$. Since 2 is the smallest prime then $i = 1$. That is, $p_1 = 2$. Since $p_1|n$ then $2|n$. But n is odd. Hence, no factor in (1) can be zero. That is, $|R_{n,s}| \neq 0$, that is, $R_{n,s} \neq \emptyset$. This completes the proof of the corollary. □

Corollary 2. *If p is a prime then $R_{p,s} = Z_p^*$ for all $s \in Z_p$.*

Proof. By (1) we have $|R_{p,s}| = p^{1-1}(p-1)(1) = p-1$ for all $s \in Z_p$. Since $R_{p,s} \subseteq Z_p^*$ and $|Z_p^*| = p-1$ then $R_{p,s} = Z_p^*$. This completes the proof of the corollary. □

Consider some examples aided by *Mathematica*.

Example 3. *If $n = 24$ and $s = 6$ or $s = 10$ we have*

$$\begin{aligned}R_{24,6} &= \{1, 5, 7, 11, 13, 17, 19, 23\}\\ R_{24,10} &= \{5, 11, 17, 23\}.\end{aligned}$$

Now $n = 24 = 2^3 3$, $P_n = P_{24} = \{2, 3\}$. If $s = 6$ then $D_s = D_6 = \{2, 3\}$ and $P_n \setminus D_s = P_{24} \setminus D_6 = \emptyset$. Formula (1) gives

$$|R_{24,6}| = (2^{3-1}3^{1-1})((2-1)(3-1))(1) = 8.$$

If $s = 10$ then $D_s = D_{10} = \{2\}$ and $P_n \setminus D_s = P_{24} \setminus D_6 = \{3\}$. Formula (1) gives

$$|R_{24,10}| = (2^{3-1}3^{1-1})(2-1)(3-2) = 4.$$

Example 4. *If $n = 45$ and $s = 27$ or $s = 38$ we have*

$$\begin{aligned} R_{45,27} &= \{1,4,8,11,13,14,16,19,23,26,28,29,31,34,38,41,43,44\} \\ R_{45,38} &= \{1,4,7,16,19,22,31,34,37\}. \end{aligned}$$

Now $n = 45 = 3^2 5$, $P_n = P_{45} = \{3,5\}$. If $s = 27$ then $D_s = D_{27} = \{3\}$ and $P_n \setminus D_s = P_{45} \setminus D_{27} = \{5\}$. Formula (1) gives

$$|R_{45,27}| = (3^{2-1}5^{1-1})(3-1)(5-2) = 18.$$

If $s = 38$ then $D_s = D_{38} = \emptyset$ and $P_n \setminus D_s = P_{45} \setminus D_{38} = \{3,5\}$. Formula (1) gives

$$|R_{45,27}| = (3^{2-1}5^{1-1})(1)(3-2)(5-2) = 9.$$

Example 5. *Let $n = 25200$ and $s = 196$ or $s = 6250$. The sets $R_{n,s}$ are too large to explicitly write out but* Mathematica *was used to generate these sets and count the size of the resulting sets to produce $|R_{25200,196}| = 2160$ and $|R_{25200,6250}| = 2400$. Now $n = 25200 = 2^4 3^2 5^2 7$.*

If $s = 196 = 2^2 7^2$ then $D_s = \{2,7\}$ and $P_n \setminus D_s = \{3,5\}$. Formula (1) gives

$$|R_{25200,196}| = (2^{4-1}3^{2-1}5^{2-1}7)((2-1)(7-1))((3-2)(5-2)) = 2160.$$

If $s = 6250 = 2^1 5^5$ then $D_s = \{2,5\}$ and $P_n \setminus D_s = \{3,7\}$. Formula (1) gives

$$|R_{25200,6250}| = (2^{4-1}3^{2-1}5^{2-1}7)((2-1)(5-1))((3-2)(7-2)) = 2400.$$

4 Counting Equations

Let $E_{n,s}$ represent the number of congruence equations $s \equiv (x + y) \bmod n$ with $s \in Z_n$ and $x, y \in Z_n^*$. Since $x + y = y + x$ the congruence equation $s \equiv (y + x) \bmod n$ is considered the same as $s \equiv (x + y) \bmod n$ when calculating $E_{n,s}$. Note that $E_{n,s}$ is also the number of solutions (either two element sets $\{x, y\}$ or one element sets $\{x\}$ when $y = x$) to $s \equiv (x + y) \bmod n$.

Lemma 3. *Let $n \geq 2$ be an integer, $s \in Z_n$ and e be the number of one element solution sets,* i.e., *the number of equations of the form $s \equiv 2x \bmod n$. Then either $e = 0$, $e = 1$ or $e = 2$ and*

$$E_{n,s} = \frac{|R_{n,s}| + e}{2}. \tag{2}$$

Proof. The formula for calculating $E_{n,s}$ follows from the way $R_{n,s}$ is defined since each two element solution set $\{x, y\}$ corresponds to two elements in $R_{n,s}$ and each one element set $\{x\}$ corresponds to one element in $R_{n,s}$. The task is to show that e can only have the values 0, 1 or 2. If there is no congruence equation of the form $s \equiv 2x \bmod n$ then $e = 0$. So, suppose such an equation is possible. Then s must be even.

Now either n is odd or even. Suppose n is odd. Since $s \equiv 2x \bmod n$, we also have $2x \equiv s \bmod n$. Since n is odd, x is the unique solution mod n. Therefore, in all other equations $s \equiv (x + y) \bmod n$, $y \neq x$. Consequently, there is exactly one solution to $s \equiv 2x \bmod n$ and $e = 1$. Next, suppose n is even. Since $s \equiv 2x \bmod n$ then $\frac{s}{2} \equiv x \bmod \frac{n}{2}$. Consequently, $x \equiv \frac{s}{2} \bmod \frac{n}{2}$. Since x and $\frac{n}{2}$ are relatively prime, x is the unique solution mod $\frac{n}{2}$. And so, there are exactly two solutions mod n to $s \equiv 2x \bmod n$, namely, $x_1 = x$ and $x_2 = x + \frac{n}{2}$. Moreover, $x_2 \in Z_n^*$ since $x \in Z_n^*$. In this case $e = 2$. Thus, e can only have the values 0, 1 or 2. This completes the proof of the lemma. □

Consider some examples illustrating formula (2).

Example 6. *Let $n = 15$ and $s = 2$. Then $|R_{15,2}| = 3$. The distinct equations are $2 \equiv (1+1) \bmod 15$ and $2 \equiv (4+13) \bmod 15$. Hence, $E_{15,2} = 2$. Now $e = 1$ and by (2) we get $E_{15,2} = \frac{3+1}{2} = 2$.*

Example 7. *Let $n = 15$ and $s = 9$. Then $|R_{15,9}| = 6$. The distinct equations are $9 \equiv (1 + 8) \bmod 15$, $9 \equiv (2 + 7) \bmod 15$ and $9 \equiv$*

$(11+13) \bmod 15$. *Hence,* $E_{15,9} = 3$. *Now* $e = 0$ *and by (2) we get* $E_{15,9} = \frac{6+0}{2} = 3$.

Example 8. *Let* $n = 20$ *and* $s = 6$. *Then* $|R_{20,6}| = 6$. *The distinct equations are* $6 \equiv (3+3) \bmod 20$, $6 \equiv (13+13) \bmod 20$, $6 \equiv (7+19) \bmod 20$ *and* $6 \equiv (9+17) \bmod 20$. *Hence,* $E_{20,6} = 4$. *Now* $e = 2$ *and by (2) we get* $E_{20,6} = \frac{6+2}{2} = 4$.

Example 9. *Let* $n = 40$ *and* $s = 14$. *Then* $|R_{40,14}| = 12$. *The distinct equations are* $14 \equiv (7+7) \bmod 40$, $14 \equiv (27+27) \bmod 40$, $14 \equiv (1+13) \bmod 40$, $14 \equiv (3+11) \bmod 40$, $14 \equiv (17+37) \bmod 40$, $14 \equiv (21+33) \bmod 40$ *and* $14 \equiv (23+31) \bmod 40$. *Hence,* $E_{40,14} = 7$. *Now* $e = 2$ *and by (2) we get* $E_{20,6} = \frac{12+2}{2} = 7$.

Theorem 1 and lemma 1 now follow from theoerem 2 and lemma 3. To see this, suppose n is even and s is odd. Then the prime 2 is a factor of n but not a factor of s. In equation (1), $p_1 = 2$. Since p_1 does not divide s, $p_1 - 2 = 2 - 2 = 0$ is a factor in the product on the right side of equation (1). Consequently, $|R_{n,s}| = 0$. Clearly, the value of e in equation (2) is also zero. Therefore, when n is even and s is odd there are no solutions to the congruence $s \equiv (x+y) \bmod n$. In addition to justifying theorem 1 and lemma 1, equations (1) and (2) provide a way of *counting* the number of solutions.

5 An Equivalence Relation

Examples indicate that $|R_{n,s'}| = |R_{n,s}|$ for many $s' \neq s$. Define relation R on Z_n as sRs' if $|R_{n,s'}| = |R_{n,s}|$. It is easy to see that R is an equivalence relation. However, this relation sheds little light on exactly how s and s' are related to each other. A more revealing relation can be defined in terms of the factors of s and s'.

Consider the set

$$A_{n,s} = \begin{cases} \{1\} & \text{if } n \text{ is even and } s \text{ is odd} \\ \{x : x \text{ is a common factor of } s \text{ and } n\} & \text{otherwise.} \end{cases}$$

Define relation E on Z_n as sEs' if $A_{n,s} = A_{n,s'}$. Again, it is easy to see that E is an equivalence relation on Z_n. Moreover, the following corollary holds.

Corollary 3. *Let $s, s' \in Z_n$ and sEs'. Then $|R_{n,s'}| = |R_{n,s}|$.*

Proof. This is a consequence of theorem 2. Let n have prime factorization $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{k-1}^{\alpha_{k-1}} p_k^{\alpha_k}$. As in theorem 2, let $P_n = \{p_i : 1 \leq i \leq k\}$, $D_s = \{p_i \in P_n : p_i \mid s\}$ and $D_{s'} = \{p_i \in P_n : p_i \mid s'\}$. Since sEs' then $A_{n,s} = A_{n,s'}$. This clearly implies that $D_s = D_{s'}$. From equation (1) it follows that $|R_{n,s}| = |R_{n,s'}|$. This completes the proof of the corollary. □

Example 10. *Consider $n = 30$. Now $R_{30,0} = Z_{30}^*$ and $R_{30,s} = \emptyset$ if s is odd. Otherwise,* Mathematica *can be used to produce*

$$\begin{aligned}
R_{30,2} &= \{1, 13, 19\} \\
R_{30,4} &= \{11, 17, 23\} \\
R_{30,8} &= \{1, 7, 19\} \\
R_{30,14} &= \{1, 7, 13\} \\
R_{30,16} &= \{17, 23, 29\} \\
R_{30,22} &= \{11, 23, 29\} \\
R_{30,26} &= \{7, 13, 19\} \\
R_{30,28} &= \{11, 17, 29\} \\
R_{30,10} &= \{11, 17, 23, 29\} \\
R_{30,20} &= \{1, 7, 13, 19\} \\
R_{30,6} &= \{7, 13, 17, 19, 23, 29\} \\
R_{30,12} &= \{1, 11, 13, 19, 23, 29\} \\
R_{30,18} &= \{1, 7, 11, 17, 19, 29\} \\
R_{30,24} &= \{1, 7, 11, 13, 17, 23\}.
\end{aligned}$$

If s is odd then $\{1\} = A_{30,s}$ otherwise

$$\begin{aligned}
\{1, 2, 3, 5\} &= A_{30,0} \\
\{1, 2\} &= A_{30,2} = A_{30,4} = A_{30,8} = A_{30,14} \\
&= A_{30,16} = A_{30,22} = A_{30,26} = A_{30,28} \\
\{1, 2, 5\} &= A_{30,10} = A_{30,20} \\
\{1, 2, 3\} &= A_{30,6} = A_{30,12} = A_{30,18} = A_{30,24}.
\end{aligned}$$

By corollary 3, $|R_{30,s}|$ all have the same value if s is odd and $R_{30,0}$ is the only set with size $|R_{30,0}|$. Otherwise

$$\begin{aligned}
|R_{30,2}| &= |R_{30,4}| = |R_{30,8}| = |R_{30,14}| = |R_{30,16}| \\
&= |R_{30,22}| = |R_{30,26}| = |R_{30,28}| \\
|R_{30,10}| &= |R_{30,20}| \\
|R_{30,6}| &= |R_{30,12}| = |R_{30,18}| = |R_{30,24}|.
\end{aligned}$$

What corollary 3 calculates agrees with the size of the sets generated with Mathematica.

Example 11. *Consider $n = 15$. Now $R_{15,0} = Z^*_{15}$ and* Mathematica *can be used to produce*

$$\begin{aligned}
R_{15,1} &= \{2,8,14\} \\
R_{15,2} &= \{1,4,13\} \\
R_{15,4} &= \{2,8,11\} \\
R_{15,7} &= \{8,11,14\} \\
R_{15,8} &= \{1,4,7\} \\
R_{15,11} &= \{4,7,13\} \\
R_{15,13} &= \{2,11,14\} \\
R_{15,14} &= \{1,7,13\} \\
R_{15,5} &= \{1,4,7,13\} \\
R_{15,10} &= \{2,8,11,14\} \\
R_{15,3} &= \{1,2,4,7,11,14\} \\
R_{15,6} &= \{2,4,7,8,13,14\} \\
R_{15,9} &= \{1,2,7,8,11,13\} \\
R_{15,12} &= \{1,4,8,11,13,14\}.
\end{aligned}$$

Now

$$\begin{aligned}
\{1,3,5\} &= A_{15,0} \\
\{1\} &= A_{15,1} = A_{15,2} = A_{15,4} = A_{15,7} \\
&= A_{15,8} = A_{15,11} = A_{15,13} = A_{15,14} \\
\{1,5\} &= A_{15,5} = A_{15,10} \\
\{1,3\} &= A_{15,3} = A_{15,6} = A_{15,9} = A_{15,12}.
\end{aligned}$$

By corollary 3, $R_{15,0}$ is the only set with size $|R_{15,0}|$. Otherwise

$$\begin{aligned} |R_{15,1}| &= |R_{15,2}| = |R_{15,4}| = |R_{15,7}| = |R_{15,8}| \\ &= |R_{15,11}| = |R_{15,13}| = |R_{15,14}| \\ |R_{15,5}| &= |R_{15,10}| \\ |R_{15,3}| &= |R_{15,6}| = |R_{15,9}| = |R_{15,12}|. \end{aligned}$$

What corollary 3 calculates agrees with the size of the sets generated with Mathematica.

Example 12. *Consider $n = 12$. Now $R_{12,0} = Z^*_{12}$, $R_{12,s} = \emptyset$ if s is odd and* Mathematica *can be used to produce*

$$\begin{aligned} R_{12,2} &= R_{12,8} = \{1, 7\} \\ R_{12,4} &= R_{12,10} = \{5, 11\} \\ R_{12,6} &= Z^*_{12}. \end{aligned}$$

Now

$$\begin{aligned} \{1, 2, 3\} &= A_{12,0} = A_{12,6} \\ \{1\} &= A_{12,1} = A_{12,3} = A_{12,5} = A_{12,7} = A_{12,9} \\ \{1, 2\} &= A_{12,2} = A_{12,4} = A_{12,8} = A_{12,10}. \end{aligned}$$

By corollary 3, $R_{12,s}$ for s odd have the same size. Otherwise

$$\begin{aligned} |R_{12,0}| &= |R_{12,6}| \\ |R_{12,2}| &= |R_{12,4}| = |R_{12,8}| = |R_{12,10}|. \end{aligned}$$

What corollary 3 calculates agrees with the size of the sets generated with Mathematica.

6 Goldbach's Comet

A plot of the *number* of prime pairs that sum to an even integer is given in [2]. The graph is called *Goldbach's comet* for obvious reasons.

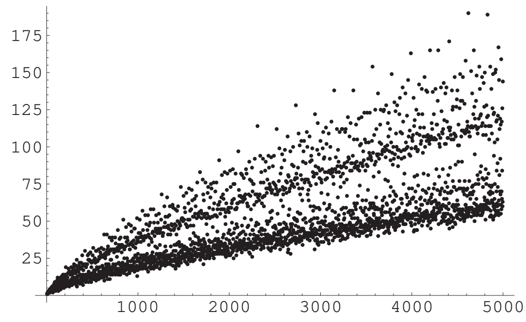

Figure 1: Goldbach comet for $n \leq 5000$

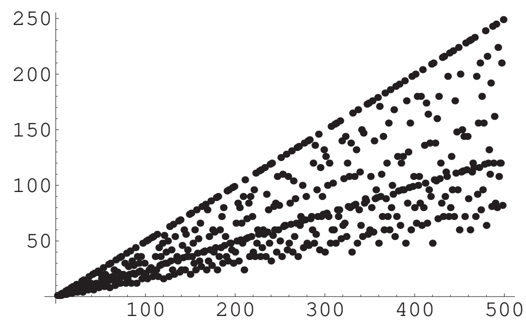

Figure 2: Goldbach mod n comet for $s = 0$ and $n \leq 500$

Fig. 1 shows Goldbach's comet for $n \leq 5000$. There is no analogous 2 dimensional plot for GC mod n since there are 3 parameters, n, s and a count of the pairs relatively prime to n whose sum is s. However, if s is fixed the points (n, c) where c is a count of the pairs relatively prime to n that sum to s mod n can be plotted. A good choice for s is $s = 0$ since it belongs to all Z_n. Fig. 2 shows such a plot for $n \leq 500$. It bares some resemblance to Goldbach's comet but the “bands" are more linear in nature.

7 Conclusion

As the above discussion indicates, some interesting results can be obtained by taking a statement, formula, *etc.*, proven or unproven, that holds for all integers and reformulating it into a mod

n analog. This might be a source for coming up with a topic for an undergraduate research project. In conclusion, when examining a mathematical statement see if it can be reformulated into a mod n version and explore the implications.

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Joe Santmyer
Las Cruces, NM 88011,
santmyerjoe@yahoo.com

A Portrayal of Integer Solutions to Non-homogeneous Ternary Cubic Diophantine Equation $6(x^2+y^2)-11xy=2z^3$

J. Shanthi, N. Thiruniraiselvi and M. A. Gopalan

Abstract

This paper is concerned with the problem of determining varieties of non-zero distinct integer solutions to the non homogeneous ternary cubic Diophantine equation $6(x^2+y^2)-11xy=2z^3$. Different sets of integer solutions to the above equation are obtained by reducing it to the equation, which is solvable, through employing suitable transformations and applying the method of factorization.

1 Introduction

One of the interesting areas of Number Theory is the subject of Diophantine equations which has fascinated and motivated both Amateurs and Mathematicians alike. It is well-known that Diophantine equation is a polynomial equation in two or more unknowns requiring only integer solutions. It is quite obvious that Diophantine equations are rich in variety playing a significant role in the development of Mathematics .The theory of Diophantine equations is popular in recent years providing a fertile ground for both Professionals and Amateurs. In addition to known results, this abounds with unsolved problems. Although many of its results can be stated in simple and elegant terms, their proofs are sometimes long and complicated. There is no well unified body of knowledge concerning general methods. A Diophantine problem is considered as solved if a method is available to decide whether the problem is solvable or not and in case of its solvability, to exhibit all integers satisfying the requirements set forth in the problem. The successful completion of exhibiting all integers satisfying the requirements set forth in the problem add to further progress of Number Theory as they offer good applications in the field of Graph theory, Modular theory, Coding and Cryptography, Engineering, Music and so on. Integers have repeatedly played a crucial role in the evolution of the Natural Sciences. The theory of integers provides answers to real world problems.

It is well-known that Diophantine equations, homogeneous or non-homogeneous, have aroused the interest of many mathematicians. It is worth to observe that Cubic Diophantine equations fall in to the theory of Elliptic curves which are used in Cryptography. In particular, one may refer [2] - [10] for cubic equations with three and four unknowns.

The main thrust of this paper is to exhibit different sets of integer solutions to an interesting ternary non-homogeneous cubic equation given by $6(x^2+y^2)-11xy = 2z^3$ by using elementary algebraic methods. The outstanding results in this study of Diophantine equation will be useful for all readers.

2 Method of analysis

The non-homogeneous ternary cubic Diophantine equation to be solved is

$$6(x^2 + y^2) - 11xy = 2z^3 \tag{1}$$

Two different approaches of obtaining distinct integer solutions to (1) are presented below.

1. Introduction of the linear transformations

$$x = (u + v), y = (u - v), u \neq v \neq 0 \tag{2}$$

in (1) leads to

$$u^2 + 23v^2 = 2z^3 \tag{3}$$

The process of obtaining different sets of integer solutions to (1) is illustrated below:

(a) It is observed that (3) is satisfied by

$$u = 6\alpha^{3s}, v = 2\alpha^{3s} \tag{4}$$

and

$$z = 4\alpha^{2s} \tag{5}$$

Using (4) in (2) , we get

$$x = 8\alpha^{3s}, y = 4\alpha^{3s} \tag{6}$$

Thus, (5) and (6) represent the integer solutions to (1).

Note 1. Choosing suitably the values of u, v in (3), other solutions to (1) are obtained. For example, the choice

$$u = 8\alpha^{3s}, v = 4\alpha^{3s}$$

gives

$$x = 12\alpha^{3s}, y = 4\alpha^{3s}, z = 6\alpha^{2s}$$

(b) Taking

$$u = sv, s \geq 1 \tag{7}$$

in (3) leads to

$$(s^2 + 23)v^2 = 2z^3$$

which is satisfied by

$$v = 4(s^2 + 23)t^{3\alpha} \tag{8}$$

and

$$z = 2(s^2 + 23)t^{2\alpha} \tag{9}$$

In view of (7) , note that

$$u = 4s(s^2 + 23)t^{3\alpha} \tag{10}$$

Using (8) and (10) in (2), we get

$$x = 4\,(s^2 + 23)(s + 1)t^{3s},\ y = 4\,(s^2 + 23)^2(s - 1)t^{3s} \tag{11}$$

Thus, (9) and (11) represent the integer solutions to (1).

(c) Taking

$$v = su, s \geq 1 \tag{12}$$

in (3) leads to

$$(23s^2 + 1)u^2 = 2z^3$$

which is satisfied by

$$u = 4(23s^2 + 1)t^{3\alpha} \tag{13}$$

and

$$z = 2(23s^2 + 1)t^{2\alpha} \tag{14}$$

In view of (12), note that

$$v = 4s(23s^2 + 1)t^{3\alpha} \tag{15}$$

Using (13) and (15) in (2) , we get

$$x = 4\,(s + 1)(23s^2 + 1)t^{3\alpha},\ y = 4\,(1 - s)(23s^2 + 1)t^{3\alpha} \tag{16}$$

Thus, (14) and (16) represent the integer solutions to (1).

(d) Taking

$$v = sz \tag{17}$$

in (3), it is written as

$$u^2 = z^2(2z - 23s^2) \tag{18}$$

which is satisfied by

$$z = (2k^2 - 2k + 12)s^2 \tag{19}$$

and

$$u = (2k - 1)(2k^2 - 2k + 12)s^3 \tag{20}$$

In view of (17) note that

$$v = (2k^2 - 2k + 12)s^3 \tag{21}$$

Using (20) and (21) in (2) , we have

$$x = 2k(2k^2-2k+12)s^3,\ y = (2k-2)(2k^2-2k+12)s^3 \tag{22}$$

Thus, (19) and (22) represent the integer solutions to (1).

(e) Taking

$$u = sz \tag{23}$$

in (3) , it is written as

$$23v^2 = z^2(2z - s^2) \tag{24}$$

which is satisfied by

$$z = (46k^2 - 46k + 12)s^2 \tag{25}$$

and

$$v = (2k - 1)(46k^2 - 46k + 12)s^3 \tag{26}$$

In view of (23) note that

$$u = (46k^2 - 46k + 12)s^3 \tag{27}$$

Using (26) and (27) in (2), we have

$$x = 2k(46k^2-46k+12)s^3, y = (2-2k)(46k^2-46k+12)s^3 \tag{28}$$

Thus, (25) and (28) represent the integer solutions to (1).

(f) Assume

$$z = a^2 + 23b^2 \tag{29}$$

Express the integer 2 on the R.H.S. of (3) as the product of complex Conjugates as follows

$$2 = \frac{(3 + i\sqrt{23})(3 - i\sqrt{23})}{16} \tag{30}$$

Substituting (29) and (30) in (3) and employing the method of factorization,
consider

$$u + i\sqrt{23}v = \frac{3 + i\sqrt{23}}{4}(a + i\sqrt{23}b)^3 \tag{31}$$

Equating the real and imaginary parts in (31) , the values of u, v are found.
In view of (2) ,the corresponding integer values of x, y, z satisfying (1) are given by

$$\begin{aligned} x &= 8(a^3 - 69ab^2) - 40(3a^2b - 23b^3), \\ y &= 4(a^3 - 69ab^2) - 52(3a^2b - 23b^3), \\ z &= 4(a^2 + 23b^2) \end{aligned} \tag{32}$$

Note 2. Apart from (30), one may consider the integer 2 on the R.H.S. of (3) as

$$2 = \frac{(7 + i\sqrt{23})(7 - i\sqrt{23})}{36}, 2 = \frac{(25 + i\sqrt{23})(25 - i\sqrt{23})}{324}$$

Giving different sets of integer solutions to (1).

(g) Write (3) as

$$u^2 + 23v^= 2z^3 \cdot 1 \tag{33}$$

Express the integer 1 on the R.H.S. of (33) as the product of complex conjugates as below :

$$1 = \frac{(11 + i\sqrt{23})(11 - i\sqrt{23})}{144} \tag{34}$$

Substituting (29), (30) and (34) in (33) and employing the method of factorization, consider

$$u + i\sqrt{23}v = \frac{(3 + i\sqrt{23})(11 + i\sqrt{23})}{12}(a + i\sqrt{23b})^3$$

from which , on equating the real and imaginary parts , we get the values of u, v. In view of (2), the corresponding integer values of x, y, z satisfying (1) are given by

$$x = 6 \cdot 12^2[(a^3 - 69ab^2) - 13(3a^2b - 23b^3)],$$

$$y = 12^2[-(a^3 - 69ab^2) - 83(3a^2b - 23b^3)],$$
$$z = 12^2(a^23 + 23b^2)$$

Note 3. Apart from (34), one may consider the integer 1 on the R.H.S. of (3) as

$$1 = \frac{(7 + i3\sqrt{23})(7 - i3\sqrt{23})}{256}$$

$$1 = \frac{(23r^2 - s^2 + i\sqrt{23}rs)((23r^2 - s^2 - i\sqrt{23}rs))}{(23r^2 + s^2)^2}$$

giving different sets of integer solutions to (1).

Note 4. By considering different combinations for the integers 2, 1 on the R.H.S. of (33) from Note 2 and Note 3 correspondingly, other sets of integer solutions to (1) are obtained.

2. Treating (1) as a quadratic in and solving for, one obtains

$$x = \frac{11y \pm \sqrt{48z^3 - 23y^2}}{12} \tag{35}$$

It is possible to choose y, z so that the square-root on the R.H.S. of (35) is removed and the corresponding value of x is an integer. For brevity, a few examples are exhibited in Table 1 below:

Table 1: Examples

x	y	z
$36s^3$	$27s^3$	$9s^2$
$20s^3$	$8s^3$	$8s^2$
$-18s^3$	$27s^3$	$18s^2$
$288s^3$	$216s^3$	$36s^2$
$44s^3$	$24s^3$	$12s^2$

However, to eliminate the square-root on the R.H.S. of (35) , one may also employ the following process:
Assume

$$\alpha^2 + 23y^2 = 48z^3 \tag{36}$$

Write the integer 48 on the R.H.S. of (36) as the product of complex conjugates as below :

$$48 = (5 + i\sqrt{23})(5 - i\sqrt{23}) \tag{37}$$

Substituting (29) and (37) in (36) and employing the method of factorization,
define

$$\alpha + i\sqrt{23}y = (5 + i\sqrt{23})(a + i\sqrt{23}b)^3$$

Following the procedure as in Illustration 6 , the corresponding two sets of integer solutions to (1) are found to be:

Set 1:

$$x = 36\left(a^3 - 69ab^2\right) + 72\left(3a^2b - 23b^3\right),$$
$$y = 27\left[\left(a^3 - 69ab^2\right) + 5\left(3a^2b - 23b^3\right)\right],$$
$$z = 9\left(a^2 + 23b^2\right)$$

Set 2:

$$x = 4\left(a^3 - 69ab^2\right) + 52\left(3a^2b - 23b^3\right),$$
$$y = 8\left[\left(a^3 - 69ab^2\right) + 5\left(3a^2b - 23b^3\right)\right],$$
$$z = 4\left(a^2 + 23b^2\right)$$

Note 5. In addition to (37), one may write 48 as below:

$$48 = \frac{(13 + \mathrm{i}\sqrt{23})(13 - \mathrm{i}\sqrt{23})}{4},$$
$$48 = \frac{(8 + \mathrm{i}4\sqrt{23})(8 - \mathrm{i}4\sqrt{23})}{9}$$

For these choices, different sets of integer solutions to (1) are obtained.

3 Conclusion

In this paper, we have made an attempt to find infinitely many non-zero distinct integer solutions to the non- homogeneous cubic equation with three unknowns given by $6(x^2+y^2)-11xy=2z^3$. To conclude, one may search for other choices of solutions to the considered cubic equation with three unknowns and higher degree Diophantine equations with multiple variables.

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J. Shanthi
Assistant Professor, Department of Mathematics
Shrimati Indira Gandhi College
(Affiliated to Bharathidasan University)
Trichy- 620 002, Tamil Nadu, India.
shanthivishvaa@gmaill.com

N. Thiruniraiselvi
Assistant Professor, Department of Mathematics
School of Engineering and Technology
Dhanalakshmi Srinivasan University
Samayapuram, Trichy-621 112, Tamil Nadu, India.
drntsmaths@gmail.com

M. A. Gopalan
Professor, Department of Mathematics
Shrimati Indira Gandhi College
(Affiliated to Bharathidasan University)
Trichy- 620 002, Tamil Nadu, India.
mayilgopalan@gmail.com

Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.
2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

April 30, 2025

Elementary Problems

E–131. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let α, β, γ be real numbers. If $\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$, then prove that

$$\frac{\sin 5\alpha + \sin 5\beta + \sin 5\gamma}{\cos 5\alpha + \cos 5\beta + \cos 5\gamma} = \tan(\alpha + \beta + \gamma).$$

E–132. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let ABC be a triangle with D an arbitrary point chosen on AC and the circumcircle Γ. If the circumcircle Ω_1 of the triangle BDC cut AB in E, the circumcircle Ω_2 of the triangle ABD cut BC in F and $AE = CE$, show that

$$\frac{AB + BC}{AC} = \frac{BX}{XA}.$$

E–133. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain and José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain.* Let a, b, c be positive reals such that $a + b + c = abc$. Prove that

$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \geq \frac{a+b+c}{4}.$$

E–134. *Proposed by Michel Bataille, Rouen, France.* Let n be a nonnegative integer. Prove that

$$\sum_{k=0}^{n} \binom{2n}{2k} 288^k 289^{n-k} = \sum_{k=0}^{4n} \binom{8n}{2k} 2^k.$$

E–135. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let a, b, n be integers such that $0 < a \leq b < n$. Prove that there exist a prime p that divides both $\binom{n}{a}$ and $\binom{n}{b}$.

E–136. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let ℓ be a line that divides triangle ABC into two parts. Prove that ℓ divides the area and the perimeter of ABC in the same proportion if and only if ℓ passes through the incenter of triangle ABC.

Easy–Medium Problems

EM–131. *Proposed by Michel Bataille, Rouen, France.* Let ABC be a triangle neither equilateral nor right-angled and let O be its circumcentre. Let A', B', C' be the respective reflections of A, B, C about O and let U, V, and W be the circumcentres of $\Delta OB'C', \Delta OC'A'$, and $\Delta OA'B'$, respectively. Prove that the lines AU, BV, CW are concurrent.

EM–132. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let a, b be positive integers such that $(a, b) = d$. Prove that

$$\frac{\varphi(d)}{d} \leq \frac{1}{2}\left(\frac{\varphi^2(a) + \varphi^2(b)}{\varphi(ab)}\right),$$

where $\varphi(n)$ is the Euler's totient function.

EM–133. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Find all real solutions of the equation

$$\sqrt{x} + \sqrt[3]{x} + \sqrt[4]{x} + \sqrt[5]{x} + \ldots + \sqrt[2024]{x} = \sqrt[2025]{x + 2024^{2025} - 1}.$$

EM–134. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a non-right triangle with $AB \neq AC$ and let H be its orthocenter, G be its centroid, K be its symmedian point. Let H_b, H_c be the feet of the altitudes drawn from B, C respectively. Let M, E be the midpoints of BC, H_bH_c respectively and let D be the foot of the perpendicular from M to the line GH. Let $K_1 = AK \cap BC$. Knowing that G lies on H_bH_c, prove that the points M, D, E, K_1 are concyclic.

EM–135. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Compute the value of

$$\cos\left(\frac{2\pi}{17}\right)\cos\left(\frac{4\pi}{17}\right)\cos\left(\frac{6\pi}{17}\right)\ldots\cos\left(\frac{16\pi}{17}\right).$$

EM–136. *Proposed by Michel Bataille, Rouen, France.* Let a, b, and c be positive real numbers such that $abc \geq 1$. Prove that

$$\frac{a}{(a+b)(a+c)} + \frac{b}{(b+c)(b+a)} + \frac{c}{(c+a)(c+b)} \leq \frac{(a+b+c)^2}{12}.$$

Medium–Hard Problems

MH–131. *Proposed by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain.* On the sides AB, BC, CA of a triangle ABC points C', A', B' are marked respectively. It turns out that

$$\frac{AC'}{AB} = \frac{BA'}{BC} = \frac{CB'}{CA} = \frac{1}{3}.$$

Prove that:

1. The sides of $\triangle A'B'C'$ are parallel to the medians of $\triangle ABC$ and $\frac{2}{3}$ as the length of the correspondent median.
2. Each of the sides of $\triangle A'B'C'$ is trisected by two medians of $\triangle ABC$.
3. Each of the medians of $\triangle A'B'C'$ is parallel to a side of $\triangle ABC$.

MH–132. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let S_n denote the set of all permutations of $\{1, 2, \ldots, n\}$. Show that the number

$$N = \frac{3}{n!} \sum_{\sigma \in S_n} \sum_{k=1}^{n} |k - \sigma(k)|$$

is an integer number and determine its value.

MH–133. *Proposed by Titu Zvonaru, Comănești, Romania.* Let a, b, c be positive real numbers such that $ab + bc + ca = 3$. Find the greatest k such that the following inequality holds:

$$\frac{a^2 + b^2}{a + b + 2} + \frac{b^2 + c^2}{b + c + 2} + \frac{c^2 + a^2}{c + a + 2} \geq \frac{k}{4}(a + b + c) - \frac{(3k - 6)}{4}.$$

MH–134. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a non-right triangle with $AB \neq AC$. Let H be its orthocenter, O be its circumcenter, M_a, M_b, M_c be the midpoints of the sides BC, CA, AB respectively, and H_a, H_b, H_c be the feet of the altitudes drawn from A, B, C respectively. Let D be the intersect point of AO and BC, N be the midpoint of OH and N_0 be the reflection of N in BC. Let M_0 be the midpoint of AM_a and H_0 be the intersect point of H_bH_c and M_bM_c. Knowing that $AB \cdot AC = 4 \cdot AH_a \cdot AD$, prove that the points A, N_0, M_0, H_0 are concyclic.

MH–135. *Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiești, Romania Romania.* Let $a \geq b \geq c \geq 1 \geq d \geq e \geq 0$ such that $ab + bc + cd + de + ea = 5$. Prove that

$$\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3} + \frac{1}{d+3} + \frac{1}{e+3} \geq \frac{5}{4}.$$

MH–136. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* If $a > b > 1$ are integers such that $(a+b)|(ab+1)$ and $(a-b)|(ab-1)$, then prove that $a < b\sqrt{3}$.

Advanced Problems

A–131. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* For $x > 0$, calculate

$$\lim_{x\to\infty} \int\limits_{[x]}^{x+1} \frac{(\sin t)^{2025}}{t^2+4}\, dt.$$

A–132. *Proposed by Vasile Mircea Popa, Affiliate Professor, "Lucian Blaga" University of Sibiu, Romania.* Calculate the following integral:

$$\int_0^\infty \frac{\arctan(x)}{\sqrt[3]{x^6+1}}\, dx.$$

A–133. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a triangle with side lengths $BC = a, CA = b, AB = c$, centroid G and circumcircle Γ with circumcenter O. Let D be the reflection of G in the line BC and E be the second intersect point of AG and Γ. Knowing that $AG = DE$, find the maximum possible value of

$$S = \frac{(a+b+c)(a^3+b^3+c^3)}{a^4+b^4+c^4}$$

and determine where the maximum holds.

A–134. *Proposed by Michel Bataille, Rouen, France.* Let F_m be the m-th Fibonacci number ($F_0 = 0, F_1 = 1$ and $F_{m+1} = F_m + F_{m-1}$ for $m \geq 1$). Let

$$A_n = \sum_{k=1}^{n} \frac{k}{F_{n+1-k}F_{n+3-k}} \quad (n \geq 1) \quad \text{and} \quad S = \sum_{k=1}^{\infty} \frac{1}{F_k F_{k+1}}.$$

Prove that

$$\lim_{n\to\infty} (A_n - n) = 1 - S.$$

A–135. *Proposed by José Lui Díaz-Barrero, Barcelona, Spain.* Let $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ be sequences of real numbers satisfying

$$x_n^2 + y_n^2 + x_{n-1}^2 + y_{n-1}^2 = (y_n x_{n-1} - x_n y_{n-1}) + \sqrt{3}(x_n x_{n-1} + y_n y_{n-1}).$$

Show that they are periodic and determine their periods.

A–136. *Proposed by José Lui Díaz-Barrero, Barcelona, Spain and Óscar Rivero Salgado, Santiago de Compostela, Spain.* Find all n-tuples $(a_1, a_2, \ldots, a_n)$ of positive integers such that

$$(a_1! - 1)(a_2! - 1)\ldots(a_n! - 1) = 4^2 + k^2,$$

where k is an integer number.

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to
`jose.luis.diaz@upc.edu`

The W function and some of its applications

Simone Camosso

Abstract

In this short paper we examine the definition of the Lambert function and some applications in different fields as: algebra, chemistry and quantum statistic. The "potentiality" of this mathematical tool seems have no limit in recent years.

1 Introduction

The special function $W(x)$ was introduced by J. H. Lambert (1758 in [9], [8] according to [7]) in order to solve the equation:

$$x^m - x = q, \tag{1}$$

where $m = 1, 2, 3, \ldots$ and q is fixed. The solution of (1) is expressed as a power series respect q. The function $W(x)$ was called the "Lambert function" and the equation (1) the Lambert equation. In a second moment, L. Euler [4] symmetrized the Lambert equation in the following form:

$$x^\alpha - x^\beta = (\alpha - \beta)vx^{\alpha+\beta}, \tag{2}$$

with $x \mapsto x^{-\beta}$, $m = \alpha\beta$ and $q = (\alpha - \beta)v$.

The Lambert function W is related to an important mathematical operation called the tetration operation. The tetration is an operation defined for $x > 0$ as

$$^{\alpha}x = x^{x^{x^{\cdot^{\cdot^{\cdot^{x}}}}}}, \tag{3}$$

where $\alpha = 0, 1, 2, 3, \ldots$ is called the level of the "power tower" (with ${}^{0}x = 1$). The reader interested in the tetration operation can read the accurate article of [1]. The case of infinite tetration ${}^{+\infty}x$ is treated in [5], [6] and [3].

The tetration operation has different properties: it is not an associative operation, always evaluated for top to bottom, and admits an inverse function that is W. Il particular the Lambert function W satisfy the interesting equation:

$$W(x)e^{W(x)} = x. \tag{4}$$

For this connection to the tetration operation, the Lambert function can be used to solve various exponential and logarithmic equations of self-exponential nature. Details on the relation between $W(x)$ and the tetration operation are in [2] and [4]. Let us examine the graph of the Lambert function. First, for $x \in \mathbb{R}$ the function W has two possible values for $x \in \left[-\frac{1}{e}, 0\right)$. The branch where $W(x) > -1$ will be called $W_0(x)$ or simply $W(x)$ and, the branch where $W(x) < -1$, will be denoted with $W_{-1}(x)$. In the figures are represented the two graph of respective branches using GeoGebra (respectively Figure 1 and Figure 2).

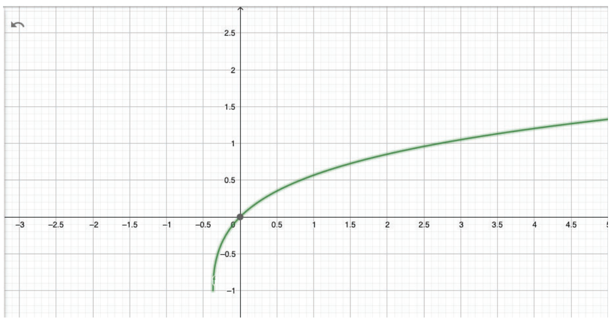

Figure 1: W_0

Second, the function W is analytic in $z = 0$ and admits the following power expansion:

$$W(z) = \sum_{j=1}^{+\infty} \frac{(-j)^{j-1}}{j!} z^j = z - z^2 + \frac{3}{2} z^3 + \ldots \tag{5}$$

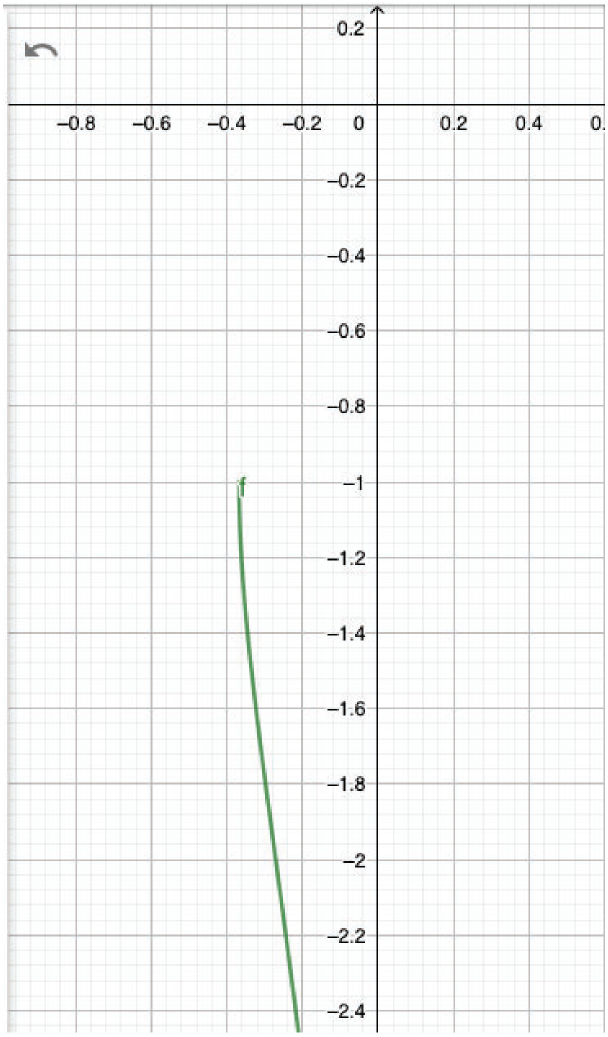

Figure 2: W_{-1}

Third, the inverse of the Lambert function can be expanded as:

$$W^{-1}(z) = ze^z = \sum_{j=0}^{+\infty} \frac{z^{j+1}}{j!} = z + z^2 + \frac{z^3}{2!} + \frac{z^4}{3!} \dots \qquad (6)$$

2 The results

In what follows the main results of this paper are presented.

2.1 Pythagorean Theorem

In this section we consider an alternative version of the pythagorean theorem.

Proposition 1. *Let (a, b, c) be a pythagorean triple, then*

$$(x, y, z) = \left(e^{W(2\ln(a))}, e^{W(2\ln(b))}, e^{W(2\ln(c))}\right) \tag{7}$$

is a point of the surface Σ generated by the equation:

$$x^x + y^y = z^z. \tag{8}$$

Let us consider the following equation:

$$x^x = a^2.$$

It is equivalent to $x \ln x = \ln a^2$ and, by the substitution $x = e^{\ln x}$ we have that:

$$e^{\ln x} \ln x = 2 \ln a.$$

Thus we find that $x = e^{W(2\ln a)}$. The same can be done for $y^y = b^2$ and $z^z = c^2$, proving the proposition.

2.2 Tetrated quadratic equations

Let us consider an equation of the following form:

$$ax^{2x} + bx^x + c = 0, \tag{9}$$

where $a, b, c \in \mathbb{R}$. We can try a substitution setting $t = x^x$ recovering an equation of second order of the form:

$$at^2 + bt + c = 0. \tag{10}$$

Now, if we try to solve the equation we have three cases: the equation is impossible in $\mathbb{R}$. Second, we have the solution $t = -\frac{b}{2a}$. Third we have two solutions $t_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$. If the quantity t or $t_{1,2}$ are strictly positive we can try to solve an elementary equation of the form:

$$x^x = -\frac{b}{2a}, \tag{11}$$

for the second case or,

$$x^x = t_{1,2} \tag{12}$$

for the third case finding all the solution (or solutions).

Example 1. *Let us consider the equation:*

$$x^{2x} - 3x^x + 2 = 0. \tag{13}$$

By the substitution we have to solve $t^2 - 3t + 2 = 0$. *This is equivalent to* $(t-1)(t-2) = 0$, *then* $t_1 = 1$ *and* $t_2 = 2$. *From* $x^x = 1$ *we have that* $x = 1$ *and, from* $x^x = 2$ *we have that* $x = e^{W(\ln 2)}$.

2.3 Chemistry: equilibrium asymptotic expansion of concentrations

Let us consider a simple chemical reaction at equilibrium:

$$A \leftrightarrow B. \tag{14}$$

The quantity A is the reactant and B is the product of the chemical reaction (14). In this case the stoichiometric coefficients α and β are simply 1. If the reaction is in equilibrium, the following first order differential equation holds:

$$-\frac{d[A]}{dt} = k[A], \tag{15}$$

where $[A]$ is the concentration of the reactant A and k the equilibrium constant. The solution is given by the exponential:

$$[A](t) = [A_0]e^{-kt}, \tag{16}$$

where $[A_0]$ is the value of initial concentration of A. Now, this other equation must be true:

$$[A](t) = [A_0]e^{-\frac{[B]}{[A]}t}. \tag{17}$$

We can try to solve the equation (17) by the Lambert function W, we have that:

$$1 = \frac{[A_0]}{[A]}e^{-\frac{[B]}{[A]}t} \Leftrightarrow \frac{[B]}{[A_0]}t = \frac{[B]}{[A]}te^{-\frac{[B]}{[A]}t},$$

so by the definition of W:

$$-\frac{[B]}{[A]}t = W\left(-\frac{[B]}{[A_0]}t\right).$$

If we expand in power series W, we find that:

$$[A](t) = \frac{[B]t}{\frac{[B]}{[A_0]}t + \frac{[B]^2}{[A_0]^2}t^2 + \frac{3}{2}\frac{[B]^3}{[A_0]^3}t^3 + \ldots}, \tag{18}$$

that is:

$$[A](t) = \frac{1}{\frac{1}{[A_0]} + \frac{[B]}{[A_0]^2}t + \frac{3}{2}\frac{[B]^2}{[A_0]^3}t^2 + \ldots}. \tag{19}$$

We observe that if $t = 0$ then $[A] = [A_0]$ and, second now $[A](t)$ is now a function of two variables $[B]$ and t. In the following figure (Figure 3) there is a graph of the function $[A](t, [B])$.

We observe that if $t \to 0$ then $[A] \to [A_0]$ (that we fixed to 1 in the graph elaborated by GeoGebra).

2.4 Physics: Quantum Statistic

In this section let us consider some arbitrary physical quantity M that can be expressed as a "generalized" moments of the Energy E:

$$M = A\int_0^{+\infty} h(E)dE, \tag{20}$$

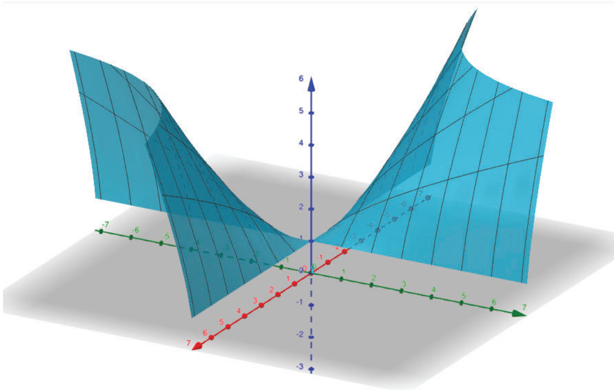

Figure 3: $[A](t,[B])$

where

$$h(E) = \frac{E^{f(r)}}{e^{\beta E+\alpha}+\gamma},$$

$f(r)$ is a real valued continuous positive function, $A \in \mathbb{R}$ is fixed, $\beta = \frac{1}{kT}$ with T the temperature and k is the Boltzmann constant and at the end, α, r and γ are parameters.

The problem, as treated in [10], is to find extrema in E. Assume E to be a continuous variable and we fix the parameters α, r, T and γ. Now the problem is to find the extrema in E on the interval $(0, +\infty)$. Let us consider the partial derivative of h:

$$\partial_E h(E) = \frac{f(r)E^{f(r)-1}\cdot\left(e^{\beta E+\alpha}+\gamma\right) - \beta E^{f(r)}e^{\beta E+\alpha}}{\left[e^{\beta E+\alpha}+\gamma\right]^2},$$

We want to find the solution of this trascendental equation:

$$f(r)E^{f(r)-1}\cdot\left(e^{\beta E+\alpha}+\gamma\right) - \beta E^{f(r)}e^{\beta E+\alpha} = 0.$$

By algebraic manipulations, we have to solve:

$$\beta E e^{\beta E+\alpha} - f(r)e^{\beta E+\alpha} = f(r)\gamma.$$

We can multiply both sides by $e^{-f(r)-\alpha}$ so we have that:

$$[\beta E - f(r)]e^{\beta E - f(r)} = f(r)\gamma e^{-f(r)-\alpha}. \qquad (21)$$

The equation (21) can be solved using the W function. Thus we can find the following critical value for the energy E:

$$\overline{E} = kT\big[f(r) + W_j\big(f(r)\gamma e^{-f(r)-\alpha}\big)\big], \qquad (22)$$

where $j \in \{0, -1\}$ (Details can be found in [10]).

2.5 On trascendental equations

Through algebraic manipulations it is possible to find impressive results on equations that classically doesn't have a closed form solution. For example let us consider the following theorem.

Theorem 1. *Let $a > 1$ be a natural number, the equation:*

$$a^x = x^a, \qquad (23)$$

has the following formal solution:

$$x = e^{-W\left(-\frac{\ln a}{a}\right)}. \qquad (24)$$

Proof. We can start the proof taking the $\ln$ to both sides of (23), so we have that:

$$x \ln a = a \ln x.$$

The last equation is equivalent to the following expression:

$$\frac{\ln a}{a} = \frac{\ln x}{x}.$$

Now, let us consider the substitution $x = e^{-t}$, we have that:

$$\frac{\ln a}{a} = -t \cdot e^{t},$$

so we find that:

$$t = W\left(-\frac{\ln a}{a}\right),$$

or

$$x = e^{-W\left(-\frac{\ln a}{a}\right)}.$$

Example 2. *Another interesting example not considered in the previous theorem is the equation:*

$$2^x = \frac{1}{x}.$$

It is a simple exercise to find the solution: $x = \frac{W(\ln 2)}{\ln 2}$. *The reader can try it!*

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Simone Camosso
Mathematics and physics department
Saluzzo (CN), "I.I.S. G.Soleri-A.Bertoni", Italy.
r.camosso@alice.it

Contests

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to find interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

jose.luis.diaz@upc.edu

Problems and solutions from the 2nd edition of the Barcelona Spring Matholympiad

O. Rivero Salgado and J. L. Díaz-Barrero

1 Problems and solutions

Hereafter, we present the four problems that appeared in the paper given to the contestants of the Barcelona Spring Matholympiad, as well as their official solutions.

Problem 1. Consider a permutation $(a_1, a_2, \ldots, a_{2021})$ of the numbers $(1, 2, \ldots, 2021)$. Find the minimum and the maximum value that can take the expression

$$a_1^2 + \ldots + a_8^2 + a_9 a_{10} \ldots a_{2021}.$$

Solution. Consider the sequence for which that value is maximum. Without loss of generality we may assume that $a_1 > a_2 > \ldots > a_8$ and that $a_9 < \ldots < a_{2021}$. Let us suppose that $a_1 > a_9$ and reach a contradiction by swapping the values of a_1 and a_9. In particular, we will prove that

$$a_9^2 + \ldots + a_8^2 + a_1 a_{10} \cdots a_{2021} > a_1^2 + \ldots a_8^2 + a_9 a_{10} \cdots a_{2021}.$$

This is equivalent to

$$(a_1 - a_9) a_{10} \cdots a_{2021} > (a_1 + a_9)(a_1 - a_9).$$

But once we divide by $a_1 - a_9$, this follows by observing that

$$a_{10} \cdots a_{2021} > 2011! > 4042 > a_1 + a_9.$$

Then, the maximum value is

$$1^2 + 2^2 + \ldots + 8^2 + 9 \cdot 10 \cdots 2021.$$

The same argument shows that the minimum is attained when the numbers $(a_1, \ldots, a_8)$ correspond to $(2014, \ldots, 2021)$. Then, the minimum values is

$$2014^2 + \ldots + 2021^2 + 1 \cdot 2 \cdots 2013.$$

Problem 2. Let ABC be an acute-angled triangle and let H be its orthocenter If h_a, h_b and h_c are the lengths of the corresponding altitudes, then prove that

$$\frac{AH + BH + CH}{2} \leq \max\{h_a, h_b, h_c\}.$$

Solution 1. Let $\angle C$ be the smallest angle, so that $CA \geq AB$ and $CB \geq AB$. In this case h_c the altitude through C is the longest one. Let the altitude through C meet AB in D and let H be the orthocentre of $\triangle ABC$. Let CD extended meet the circumcircle of $\triangle ABC$ in point K.

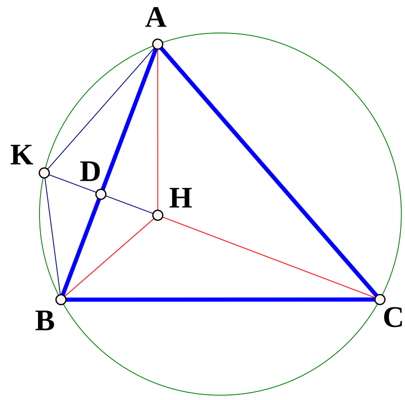
We have $h_c = CD$ so that the inequality to be proven is

$$AH + BH + CH \leq 2CD.$$

On account that $CD = CH + HD$, the above reduces to $AH + BH \leq CD + HD$. Since K is the symmetric of H respect to D then we have $HD = DK$ and also that right triangles DBK and DBH are similar. So, $BH = BK$. Likewise, we have $AH = AK$.

Thus we need to prove that $AK + BK \leq CK$. Applying Ptolemy's theorem to the cyclic quadrilateral $BCAK$, we get

$$AB \cdot CK = AC \cdot BK + BC \cdot KA.$$



Scheme for solving problem 2.

On account that $CA \geq AB$ and $CB \geq AB$, we have

$$AB \cdot CK \geq AB \cdot BK + AB \cdot AK$$

from which $AK + BK \leq CK$ follows.

Solution 2. We first note that $AH = 2R\cos\alpha$, and similarly for the other lengths BH and CH. Using that $h_a = c\sin\beta = b\sin\gamma$, the statement is equivalent to

$$R(\cos\alpha+\cos\beta+\cos\gamma) \leq 2R\max\{\sin\beta\sin\gamma, \sin\gamma\sin\alpha, \sin\alpha\sin\beta\}.$$

Without loss of generality, we may assume that $a \geq b \geq c$, and therefore $\sin\alpha \geq \sin\beta \geq \sin\gamma$. By virtue of the cosine rule, the inequality we want to prove may now be rewritten as

$$\frac{a^2b + a^2c + b^2c + b^2a + c^2a + c^2b - a^3 - b^3 - c^3}{2abc}$$

$$\leq 2\sin\alpha\sin\beta = \frac{8S^2}{abc^2},$$

where the last equality follows from the relations $a = 2R\sin\alpha$, $b = 2R\sin\beta$ and $abc = 4SR$. Here, as usual, S is the area of the triangle. Using now Heron's formula, we may write $16S^2$ as a function of the sides a, b and c, and the inequality is then equivalent to

$$a^2bc + a^2c^2 + b^2c^2 + b^2ca + c^3a + c^3b - a^3c - b^3c - c^4$$

$$\leq 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4.$$

But rearranging the terms, this is the same as

$$a^3(c-a)+b^3(c-b)+bc^2(b-c)+ac^2(a-c)+ab^2(a-c)+ba^2(b-c)$$

$$= (a-c)a(b^2+c^2-a^2) + (b-c)b(b^2+c^2-a^2) \geq 0,$$

which clearly holds since $a, b \geq c$ (by assumption), and $b^2+c^2 \geq a^2$ since the triangle is acute-angled.

Problem 3. Let a, b and c be positive integers for which $a \mid b^2, b \mid c^2$, $c \mid a^2$ hold. Determine whether or not the following statements are true or false, justifying your answer:

(a) All numbers a, b, c that satisfy the above conditions also verify that abc divides $(a+b+c)^6$?

(b) All numbers a, b, c that satisfy the above conditions verify that abc divides $(a+b+c)^7$?

Solution. (a) The answer is NOT. Indeed, triple $a = 4, b = 2, c = 16$ is a counterexample. We can see that $4 \mid 2^2, 2 \mid 16^2$, and $16 \mid 4^2$, but $abc = 2^7$ does not divide $(a+b+c)^6 = 22^6$.

(b) Expanding $(a+b+c)^7$, we get the following sum:

$$(a+b+c)^7 = \sum_{i+j+k=7} \binom{7}{i,j,k} a^i b^j c^k = \sum_{i+j+k=7} \frac{7!}{i!j!k!} a^i b^j c^k,$$

where $0 \leq i, j, k \leq 7$ and each $\binom{7}{i,j,k}$ is some positive integer. We will show that each term in the above sum is divisible by abc. We distinguish the following cases:

1. If $i, j, k \geq 1$, then $abc \mid a^i b^j c^k$ obviously holds.

2. If two of i, j, k are 0, then we can assume $i = j = 0$ and so $k = 7$. From $b \mid c^2$ we get $b^2 \mid c^4$ and so $a \mid b^2 \mid c^4$. Therefore $abc \mid c^4 \cdot c^2 \cdot c = c^7 = a^i b^j c^k$. The other two cases follow by an analogue reasoning.
3. If exactly one of i, j, k is 0, then we can assume it is i. In this case $j + k = 7$. If $j \geq 3$, then $ab \mid b^j$ and $c \mid c^k$, therefore $abc \mid b^j c^k = a^i b^j c^k$. If $j \leq 2$, then $k \geq 5$. In this case $ac \mid c^4 \cdot c \mid c^k$ and $b \mid b^j$, so again $abc \mid b^j c^k = a^i b^j c^k$.

Problem 4. Let x, y be two relatively prime positive integers and $p \geq 3$ be a prime number. Compute

$$\gcd\left(x + y, \frac{x^p + y^p}{x + y}\right).$$

Solution. We attempt to simplify the problem to the case when $y = 1$. Our goal is to compute

$$\gcd\left(x + 1, \frac{x^p + 1}{x + 1}\right).$$

Factoring gives

$$\frac{x^p + 1}{x + 1} = x^{p-1} - x^{p-2} + x^{p-3} - x^{p-4} + \ldots - x + 1.$$

In order to calculate

$$\gcd\left(x + 1, \frac{x^p + 1}{x + 1}\right),$$

we attempt to reduce the above expression mod $x + 1$. Using the fact that p is an odd prime, we know that $p - 1$ is even, therefore

$$\begin{aligned}
\frac{x^p + 1}{x + 1} &= x^{p-1} - x^{p-2} + \ldots + x^{2a} - x^{2a-1} + \ldots - x + 1 \\
&\equiv (-1)^{p-1} - (-1)^{p-2} + \ldots + (-1)^{2a} - (-1)^{2a-1} + \ldots - x + 1 \\
&\equiv \underbrace{1 + 1 + \ldots + 1 + 1}_{p} \equiv p \pmod{x + 1}
\end{aligned}$$

Now, by the Euclidan Algorithm, we have

$$\gcd\left(x+1, \frac{x^p+1}{x+1}\right) = \gcd(x+1, p).$$

Since p is a prime, the above expression can only be equal to 1 or p, depending on x. We have now solved the problem for $y = 1$. We wish to generalize the method to any y.

Using a similar factorization as above, we have

$$\frac{x^p+y^p}{x+y} = x^{p-1} - x^{p-2}y + x^{p-3}y^2 - x^{p-4}y^3 + \ldots - xy^{p-2} + y^{p-1}.$$

In order to invoke the Euclidean Algorithm, we wish to evaluate this expression mod $x+y$. Using the fact that $x \equiv -y \pmod{x+y}$ and that $p-1$ is even, we can simplify as follows:

$$\begin{aligned} x^{p-1} - x^{p-2}y + \ldots - xy^{p-2} + y^{p-1} &\equiv (-y)^{p-1} - (-y)^{p-2} + \ldots + y^{p-1} \\ &\equiv (-1)^{p-1}\left(\underbrace{y^{p-1} + y^{p-1} + \ldots + y^{p-1}}_{p}\right) \\ &\equiv py^{p-1} \pmod{x+y} \end{aligned}$$

Therefore, by the Euclidean Algorithm, we arrive at

$$\gcd\left(x+y, \frac{x^p+y^p}{x+y}\right) = \gcd(x+y, py^{p-1}).$$

Now, in the problem statement, it was given that x and y are relatively prime. Hence, similarly, $\gcd(x+y, y) = 1$, and we can simplify the above expression further:

$$\gcd\left(x+y, \frac{x^p+y^p}{x+y}\right) = \gcd(x+y, py^{p-1}) = \gcd(x+y, p) = 1 \text{ or } p,$$

depending on whether p divides $x+y$ or not.

Óscar Rivero Salgado
University of Santiago de Compostela
Spain
riverosalgado@gmail.com

José Luis Díaz-Barrero
Civil and Environmental Engineering
BarcelonaTech
Barcelona, Spain
jose.luis.diaz@upc.edu

Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

Elementary Problems

E–125. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* If α is an irrational number then prove that α^{17} and α^{19} cannot be both rational numbers.

Solution 1 by Titu Zvonaru, Comăneşti, and Daniel Văcaru, "Maria Teiuleanu" National Economic College, Pitești, both in Romania (same solution). Suppose that α^{17} and α^{19} are rational numbers. Then $\alpha^{17}/\alpha^{19} = \alpha^2$ is a rational number, and $\alpha^{17}/(\alpha^2)^8 = \alpha$ is also a rational number. Contradiction!

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA and Titu Zvonaru, Comăneşti, Romania (same solution). We generalize this result, assuming that α is irrational, p, q are any relatively prime numbers, and α^p, α^q are rational. Since p and q are relatively prime, there exist integers b and c such that $pb + qc = 1$. Then

$$\alpha = \alpha^{pb+qc} = (\alpha^p)^b \cdot (\alpha^q)^c$$

is a rational number because α^{pb}, α^{qc}, and their product must be rational. This contradiction establishes the result.

Solution 3 by the proposer. First, we observe that if α is rational, say $\alpha = p/q$ with $(p,q) = 1$, then for all integer $m \geq 1$, the power

$$\alpha^m = \left(\frac{p}{q}\right)^m = \frac{p^m}{q^m}, \quad (p,q) = 1$$

is also a rational number. To solve the problem we argue by contradiction. Indeed, suppose that α^{17} and α^{19} are both rational. Then, on account that the powers of rational numbers are also rational, we have that

$$\begin{array}{cccccccccc} \alpha^{17} & \alpha^{34} & \alpha^{51} & \alpha^{68} & \alpha^{85} & \alpha^{92} & \alpha^{119} & \alpha^{136} & \alpha^{153} & \cdots \\ \alpha^{19} & \alpha^{38} & \alpha^{57} & \alpha^{76} & \alpha^{95} & \alpha^{114} & \alpha^{133} & \alpha^{152} & \alpha^{171} & \alpha^{190} \cdots \end{array}$$

are also rational. Therefore, taking into account that the quotient of two rational numbers is also rational, we get that $\alpha^{153}/\alpha^{152} = \alpha$ is rational. Contradiction. Thus, if α is an irrational number, then at least one of α^{17} and α^{19} is a rational number.

Also solved by *Michel Bataille, Rouen, France; Rovsen Pirguliyev, Sumgait, Azerbaijan, and Victoria Farina (student), SUNY Brockport, USA.*

E–126. *Proposed by Michel Bataille, Rouen, France.* Let n be a positive integer and a a non-negative real number. Prove that

$$(1+a)^{n+1} \geq 1 + (n+1)a\sqrt{(1+a)^n}.$$

Solution 1 by Titu Zvonaru, Comăneşti, Romania and the proposer (same solution). Using AM-GM for the inequality, we have

$$\begin{aligned}(1+a)^{n+1} - 1 &= ((1+a)-1)\big(1 + (1+a) + (1+a)^2 + \cdots + (1+a)^n\big) \\ &\geq a(n+1)\big(1 \cdot (1+a) \cdot (1+a)^2 \cdots (1+a)^n\big)^{1/(n+1)} \\ &= a(n+1)\big((1+a)^{n(n+1)/2}\big)^{1/(n+1)} \\ &= a(n+1)(1+a)^{n/2},\end{aligned}$$

that is, $(1+a)^{n+1} \geq 1+(n+1)a\sqrt{(1+a)^n}$.

Notes:

- Equality holds if and only if $a=0$.
- The inequality improves Bernoulli's inequality $(1+a)^{n+1} \geq 1+a(n+1)$.

Solution 2 by Victoria Farina (student), SUNY Brockport, USA.
We will prove the inequality by induction.

For $n=1$ we get

$$(1+a)^2 \geq 1+2a\sqrt{(1+a)} \iff 1+2a+a^2 \geq 1+2a\sqrt{(1+a)}$$

$$\iff 2a+a^2 \geq 2a\sqrt{1+a} \iff 2+a \geq 2\sqrt{1+a}$$

$$\iff 4+4a+a^2 \geq 2+4a \iff a^2 \geq 0$$

which is true.

We assume that

$$(1+a)^{n+1} \geq 1+(n+1)a\sqrt{(1+a)^n}$$

and we want to prove that

$$(1+a)^{n+2} \geq 1+(n+2)a\sqrt{(1+a)^{n+1}}$$

We start from

$$(1+a)^{n+1} \geq 1+(n+1)a\sqrt{(1+a)^n}$$

and we multiply by $(1+a)$. We get

$$(1+a)^{n+2} \geq (1+a)(1+(n+1)a\sqrt{(1+a)^n})$$

In order to finish the induction it suffices to prove that

$$(1+a)(1+(n+1)a\sqrt{(1+a)^n}) \geq 1+(n+2)a\sqrt{(1+a)^{n+1}}$$

$$\iff 1+(n+1)a\sqrt{(1+a)^n}) + a + (n+1)a^2\sqrt{(1+a)^n})$$

$$\geq 1+(n+2)a\sqrt{(1+a)^{n+1}}$$

$$\iff (n+1)a\sqrt{(1+a)^n}) + a + (n+1)a^2\sqrt{(1+a)^n})$$

$$\geq (n+2)a\sqrt{(1+a)^{n+1}}$$

$$\iff (n+1)\sqrt{(1+a)^n}) + 1 + (n+1)a\sqrt{(1+a)^n})$$

$$\geq (n+2)\sqrt{(1+a)^{n+1}}$$

$$\iff 1+(n+1)(1+a)\sqrt{(1+a)^n})$$

$$\geq (n+2)\sqrt{(1+a)^{n+1}}$$

We will make the substitution $t = \sqrt{1+a} > 1$. Thus

$$\iff 1+(n+1)t^{n+2} - (n+2)t^{n+1} \geq 0$$

$$\iff (n+1)t^{n+1}(t-1) + 1 - t^{n+1} \geq 0$$

$$\iff (n+1)t^{n+1}(t-1) - (t-1)(t^n + t^{n-1} + \cdots + t + 1) \geq 0$$

$$\iff (t-1)((n+1)t^{n+1} - t^n - t^{n-1} - \cdots - t - 1) \geq 0$$

$$\iff (t-1)(t^{n+1} - t^n + t^{n+1} - t^{n-1} + \cdots + t^{n+1} - t + t^{n+1} - 1) \geq 0$$

$$\iff (t-1)(t^n(t-1) + t^{n+1}(t^2-1) + \cdots + t(t^n-1) + (t^{n+1}-1)) \geq 0$$

which is true because $t > 1$.

Also solved by *José Luis Díaz-Barrero, Barcelona, Spain.*

E–127. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Points D, E, F lie on the sides BC, CA and AB of triangle ABC respectively, such that $BD = 3DC, CE = 3A$ and $AF = 3FB$. Point P is the intersection point of BE and CF, Q is the intersection point of CF and AD and R is the intersection point of AD and BE. Determine $[PQR]/[ABC]$.

Solution 1 by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain. The answer is $\frac{4}{13}$.

More generally, we suppose

$$BD : DC = CE : EA = AF : FB = p : q$$

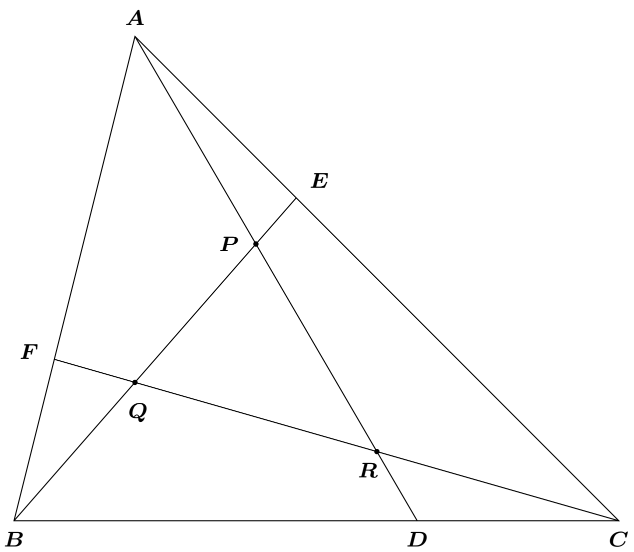
and apply the Menelaus's theorem to the two triads of points CRF, BQE on the sides of the two triangles ABD, ADC, obtaining

$$\frac{AF}{FB}\cdot\frac{BC}{CD}\cdot\frac{DR}{RA}=1, \qquad \frac{AP}{PD}\cdot\frac{DB}{BC}\cdot\frac{CE}{EA}=1,$$

where $\frac{AF}{FB}=\frac{p}{q}=\frac{CE}{EA}$, $\frac{BC}{CD}=\frac{p+q}{q}$, $\frac{DB}{BC}=\frac{p}{p+q}$.

Therefore,

$$\frac{DR}{RA}=\frac{q^2}{p(p+q)}, \qquad \frac{PD}{AP}=\frac{p^2}{q(p+q)}. \tag{1}$$



Adding 1 to each side of both equalities in (1), gives

$$\frac{AD}{RA}=\frac{p^2+pq+q^2}{p(p+q)}, \qquad \frac{AD}{AP}=\frac{p^2+pq+q^2}{q(p+q)},$$

and therefore

$$RA=\frac{p(p+q)}{p^2+pq+q^2}AD, \qquad AP=\frac{q(p+q)}{p^2+pq+q^2}AD.$$

Hence

$$PR=AR-AP=\frac{p(p+q)-q(p+q)}{p^2+pq+q^2}AD=\frac{p^2-q^2}{p^2+pq+q^2}AD.$$

Analogously,

$$RQ = \frac{p^2 - q^2}{p^2 + pq + q^2} CF, \qquad QP = \frac{p^2 - q^2}{p^2 + pq + q^2} BE.$$

Consequently, exists a triangle XYZ similar to $\triangle PQR$ and whose sides are equal and parallel to AD, BE, CF.

Since the ratio of the areas of two similar triangles is equal to the square of the ratio of any pair of the corresponding sides of the similar triangles, we have

$$[PQR] = \frac{(p^2 - q^2)^2}{(p^2 + pq + q^2)^2} [XYZ]. \tag{2}$$

Let α, β, γ denote the lengths of cevians AD, BE, CF, respectively.

By the Stewart's theorem,

$$\begin{aligned}
(p+q)^2 \alpha^2 &= (p+q)pb^2 + (p+q)qc^2 - pqa^2 \\
(p+q)^2 \beta^2 &= (p+q)pc^2 + (p+q)qa^2 - pqb^2 \\
(p+q)^2 \gamma^2 &= (p+q)pa^2 + (p+q)qb^2 - pqc^2
\end{aligned}$$

We add these three equations. Next square these and add again. We obtain

$$\alpha^2 + \beta^2 + \gamma^2 = \frac{p^2 + pq + q^2}{(p+q)^2} \left(a^2 + b^2 + c^2\right),$$

$$\alpha^4 + \beta^4 + \gamma^4 = \frac{(p^2 + pq + q^2)^2}{(p+q)^4} \left(a^4 + b^4 + c^4\right).$$

from which we deduce

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = \frac{(p^2 + pq + q^2)^2}{(p+q)^4} \left(a^2b^2 + b^2c^2 + c^2a^2\right).$$

By the Heron's formula,

$$\begin{aligned}16[XYZ]^2 &= 2(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)-(\alpha^4+\beta^4+\gamma^4)\\ &= \tfrac{\left(p^2+pq+q^2\right)^2}{(p+q)^4}(2(a^2b^2+b^2c^2+c^2a^2)-(a^4+b^4+c^4))\\ &= \tfrac{\left(p^2+pq+q^2\right)^2}{(p+q)^4}\cdot 16[ABC]^2\end{aligned}$$

and

$$[XYZ]=\frac{p^2+pq+q^2}{(p+q)^2}[ABC]. \qquad (3)$$

By eliminating $[XYZ]$ between (2) and (3), we obtain

$$\frac{[PQR]}{[ABC]}=\frac{(p-q)^2}{p^2+pq+q^2}.$$

Substituting $p=3$ and $q=1$, we obtain $\frac{[PQR]}{[ABC]}=\frac{4}{13}$, as claimed.

Solution 2 by Michel Bataille, Rouen, France. In barycentric coordinates relatively to (A,B,C), we have

$$D=(0:1:3),\quad E=(3:0:1),\quad F=(1:3:0),$$

hence the respective equations of the lines AD, BE, CF are

$$3y-z=0,\quad x-3z=0,\quad 3x-y=0.$$

We deduce that $P=(3:9:1)$, $Q=(1:3:9)$, $R=(9:1:3)$ and therefore $[PQR]/[ABC]=|\delta|$ where

$$\delta=\frac{1}{13^3}\begin{vmatrix}3&1&9\\9&3&1\\1&9&3\end{vmatrix}=\frac{1}{13^3}\begin{vmatrix}0&1&0\\0&3&-26\\-26&9&-78\end{vmatrix}=\frac{26^2}{13^3}=\frac{4}{13}.$$

Thus, $\frac{[PQR]}{[ABC]}=\dfrac{4}{13}$.

Solution 3 by Titu Zvonaru, Comăneşti, Romania. More generally, let $x = \frac{BD}{DC}, y = \frac{CE}{EA}, z = \frac{AF}{FB}$. Then,

$$BD = \frac{ax}{x+1}, \quad DC = \frac{a}{x+1}, \quad CE = \frac{by}{y+1},$$

$$EA = \frac{b}{y+1}, \quad AF = \frac{cz}{z+1}, \quad FB = \frac{c}{z+1}.$$

We have

$$\frac{[BFC]}{[ABC]} = \frac{BF}{BA} = \frac{1}{z+1} \quad \Rightarrow \quad [BFC] = \frac{[ABC]}{z+1}.$$

By Menelaus' theorem in triangle AFC and transversal $B-P-E$, we obtain

$$\frac{BF}{BA} \cdot \frac{EA}{EC} \cdot \frac{PC}{PF} = 1 \quad \Leftrightarrow \quad \frac{1}{z+1} \cdot \frac{1}{y} \cdot \frac{PC}{PF} = 1 \quad \Leftrightarrow \quad \frac{PC}{PF} = y(z+1)$$

$$\Rightarrow \quad \frac{PC}{CF} = \frac{y(z+1)}{yz+y+1}.$$

It results that

$$\frac{[BCP]}{[BCF]} = \frac{PC}{CF} = \frac{y(z+1)}{yz+y+1}$$

$$\Rightarrow \quad [BCP] = \frac{y(z+1)}{yz+y+1}[BCF], \quad [BCF] = \frac{y}{yz+y+1}[ABC].$$

We obtain

$$[PQR] = [ABC] - [BCP] - [ABR] - [CAQ]$$

$$= [ABC]\left(1 - \frac{y}{yz+y+1} - \frac{z}{zx+z+1} - \frac{x}{xy+x+1}\right)$$

$$= [ABC] \cdot \frac{(xyz-1)^2}{(xy+x+1)(yz+y+1)(zx+z+1)}.$$

For $x = y = z = 3$, we get

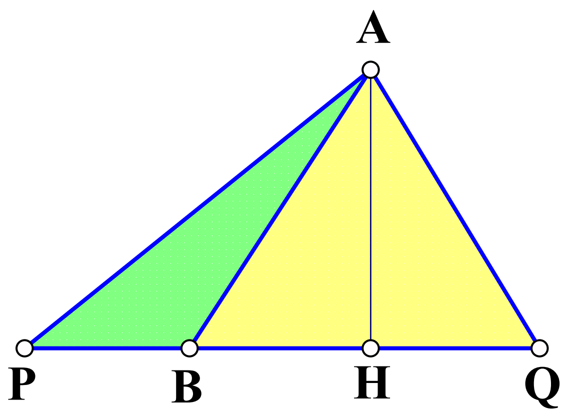
$$\frac{[PQR]}{[ABC]} = \frac{26^2}{13^3} = \frac{4}{13}.$$

Solution 4 by the proposer. We claim that if B is a point on line PQ distinct from P and Q, and A is a point not lying on this line. Then, it holds that

$$\frac{[PAB]}{[QAB]} = \frac{PB}{QB}.$$

Indeed, let AH be the altitude drop from A on side PQ. Then, we have

$$\frac{[PAB]}{[QAB]} = \frac{\frac{1}{2}PB \cdot AH}{\frac{1}{2}QB \cdot AH} = \frac{PB}{QB}.$$

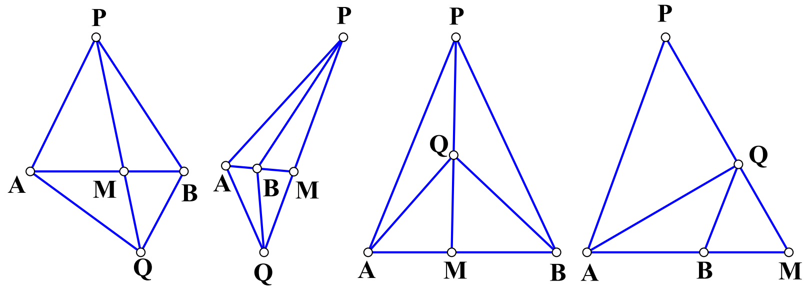


We also claim that if PAB and QAB are triangles such that the lines AB and PQ meet at M. Then, it holds that

$$\frac{[PAB]}{[QAB]} = \frac{PM}{QM}.$$

Indeed, we may assume that M is distinct from A, B, P, and Q as otherwise it reduces to the previous special case. By the first claim, we have

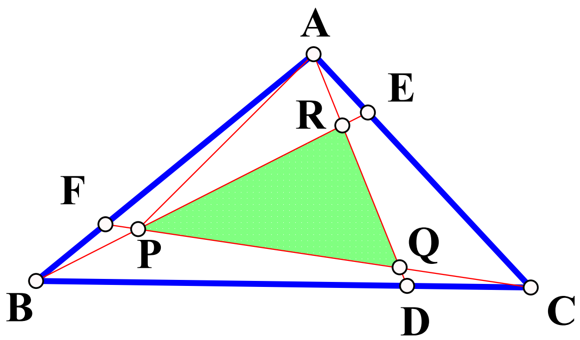
$$\frac{[PAB]}{[QAB]} = \frac{[PAB]}{[PMB]} \cdot \frac{[PMB]}{[QMB]} \cdot \frac{[QMB]}{[QAB]} = \frac{AB}{MB} \cdot \frac{PM}{QM} \cdot \frac{MB}{AB} = \frac{PM}{QM}.$$



On account of the last claim, we have

$$\frac{[PBC]}{[PAC]} = \frac{BF}{AF} = \frac{1}{3} \quad \text{and} \quad \frac{[PBC]}{[PBA]} = \frac{CE}{AE} = 3.$$

Hence,



Scheme for solving problem E127.

$$[ABC] = [PBC] + [PCA] + [PAB] = \left(1 + 3 + \frac{1}{3}\right)[PBC],$$

and $[PBC] = \frac{3}{13}[ABC]$. Likewise, $[QCA] = [RAB] = \frac{3}{13}[ABC]$. Now, from $[ABC] = [QCA] + [RAB] + [PBC] + [PQR]$ it follows $[PQR] = \frac{4}{13}[ABC]$, and $\frac{[PQR]}{[ABC]} = \frac{4}{13}$.

E–128. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* In how many different ways may you change an euro? That is, in how many different ways can you pay 100 cents using six different kinds of coins, $1, 2, 5, 10, 20$ and 50 cents, respectively.

Solution 1 by Michel Bataille, Rouen, France. We show that the required number N of all 6-uples of nonnegative integers $(x_1, \ldots, x_6)$ satisfying

$$x_1 + 2x_2 + 5x_3 + 10x_4 + 20x_5 + 50x_6 = 100 \qquad (1)$$

is 4562.

We remark that (1) writes as $(x_1+2x_2+5x_3)+10(x_4+2x_5+5x_6) = 100$, so that $0 \leq x_4 + 2x_5 + 5x_6 \leq 10$, and we first consider the equation $x + 2y + 5z = n$. If $N(n)$ is the number of its solutions (x, y, z) in nonnegative integers, then

$$N = \sum_{k=0}^{10} N(k)N(100 - 10k).$$

Now, there are $\left\lfloor \frac{n}{2} \right\rfloor + 1$ solutions to $x + 2y = n$ (the solutions are the pairs $(n - 2a, a)$ for $0 \leq a \leq \frac{n}{2}$ if n is even, $0 \leq a \leq \frac{n-1}{2}$ if n is odd), hence

$$N(n) = \sum_{k=0}^{\lfloor n/5 \rfloor} \left(\left\lfloor \frac{n-5k}{2} \right\rfloor + 1 \right).$$

Therefore, we have $N(n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$ if $0 \leq n \leq 4$, $N(n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n-5}{2} \right\rfloor + 2$ if $5 \leq n \leq 9$, that is, $N(0) = N(1) = 1$, $N(2) = N(3) = 2$, $N(4) = 3$, $N(5) = 4$, $N(6) = 5$, $N(7) = 6$, $N(8) = 7$, $N(9) = 8$.
Since

$$N(10m) = \sum_{k=0}^{2m} \left(\left\lfloor \frac{10m - 5k}{2} \right\rfloor + 1 \right) = \sum_{j=0}^{2m} \left(\left\lfloor \frac{5j}{2} \right\rfloor + 1 \right)$$

we have

$$\begin{aligned} N(10(m+1)) &= N(10m) + \left\lfloor \frac{5(2m+1)}{2} \right\rfloor + \left\lfloor \frac{5(2m+2)}{2} \right\rfloor + 2 \\ &= N(10m) + 5m + 2 + 5m + 5 + 2 \\ &= N(10m) + 10m + 9. \end{aligned}$$

Since $N(10) = 6+3+1 = 10$, we then obtain $N(20) = 29$, $N(30) = 58$, $N(40) = 97$, $N(50) = 146$, $N(60) = 205$, $N(70) = 274$, $N(80) = 353$, $N(90) = 442$, $N(100) = 541$ and deduce that

$$N = 541 + 442 + 2 \cdot 353 + 2 \cdot 274 + 3 \cdot 205 + 4 \cdot 146 + 5 \cdot 97 + 6 \cdot 58 \\ + 7 \cdot 29 + 8 \cdot 10 + 10 = 4562.$$

Solution 2 by the proposers. Using generating functions, we have to find the coefficient corresponding to x^{100} in

$$\begin{aligned} A(x) &= (1 + x + x^2 + x^3 + x^4 \dots) \\ &\times (1 + x^2 + x^4 + x^6 + x^8 \dots) \\ &\times (1 + x^5 + x^{10} + x^{15} + x^{20} \dots) \\ &\times (1 + x^{10} + x^{20} + x^{30} + x^{40} \dots) \\ &\times (1 + x^{20} + x^{40} + x^{60} + x^{80} \dots) \\ &\times (1 + x^{50} + x^{100} + x^{150} + x^{200} \dots) \\ &= \frac{1}{(1-x)(1-x^2)(1-x^5)(1-x^{10})(1-x^{50})} \\ &= \sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

Since

$$\frac{1}{(1-x)(1-x^2)(1-x^5)(1-x^{10})(1-x^{50})}$$

$= 1+x+2x^2+2x^3+3x^4+4x^5+\dots+4219x^{98}+4366x^{99}+4562x^{100}+\dots$

Then, $[x^{100}] = 4562$, and we are done.

E–129. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let a, b, c be positive real numbers and let

$$x = \left(\frac{bc}{a}\right)^{lg\frac{b}{c}},\ y = \left(\frac{ca}{b}\right)^{lg\frac{c}{a}},\ z = \left(\frac{ab}{c}\right)^{lg\frac{a}{b}}.$$

If $x + y + z$ and $x^3 + y^3 + z^3$ are rational, then show that $x^{2024} + y^{2024} + z^{2024}$ is also rational.

Solution by the proposer. We argue as follows:

- Calculate the product xyz

$$\lg xyz = \lg x + \lg y + \lg z = \lg \frac{b}{c} \cdot \lg \frac{bc}{a} + \lg \frac{c}{a} \cdot \lg \frac{ca}{b} + \lg \frac{a}{b} \cdot \lg \frac{ab}{c}$$

$$= (\lg b - \lg c)(\lg b + \lg c - \lg a)$$

$$+(\lg c - \lg a)(\lg c + \lg a - \lg b) + (\lg a - \lg b)(\lg a + \lg b - \lg c) = 0$$

$$\lg xyz = 0 \Rightarrow \boxed{xyz = 1}$$

- Use the identity

$$x^3 + y^3 + z^3 - 3\underbrace{xyz}_{1} = (x+y+z)\left[x^2 + y^2 + z^2 - (xy + xz + yz)\right]$$

$$\text{where} \begin{cases} x^3 + y^3 + z^3 - 3 \in \mathbb{Q} \\ x + y + z \in \mathbb{Q}^* \end{cases} \Rightarrow$$

$$x^2 + y^2 + z^2 - (xy + xz + yz) \in \mathbb{Q} \Rightarrow (x+y+z)^2 - 3(xy + xz + yz) \in \mathbb{Q}$$

$$\Rightarrow \boxed{xy + xz + yz \in \mathbb{Q}}$$

Note: $\begin{cases} x + y + z = p \in \mathbb{Q}^* \\ xy + xz + yz = q \in \mathbb{Q} \\ xyz = 1 \end{cases} \Rightarrow x, y, z$ are the solutions of the equation $x^3 - px^2 + qx - 1 = 0$

If $S_k = x^k + y^k + z^k \Rightarrow S_{k+3} - pS_{p+2} + qS_{k+1} - S_k = 0$

Inductively, $S_n \in \mathbb{Q}$, $\forall n \in \mathbb{N}$, so:

$$x^{2024} + y^{2024} + z^{2024} \in \mathbb{Q}$$

Also solved by *José Luis Díaz-Barrero, Barcelona, Spain.*

E–130. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Find the largest positive integer k for which $\dfrac{1000!}{10^k}$ is an integer number and determine the maximum power of 2 that divides it.

Solution 1 by Michel Bataille, Rouen, France. Clearly, the maximum value of k is the exact number of zeros at the end of the decimal expression of $1000!$. Now, from a well-known result, the exponent of 5 in the standard factorization of $1000!$ is

$$\sum_{j=1}^{\infty}\left\lfloor\frac{1000}{5^j}\right\rfloor = 200+40+8+1=249$$

and the exponent of 2 is

$$\sum_{j=1}^{\infty}\left\lfloor\frac{1000}{2^j}\right\rfloor = 500+250+125+62+31+15+7+3+1=994.$$

We deduce that $1000! = 2^{994}\times 5^{249}\times \ell$ where ℓ is coprime with 10. We can conclude: $\max k = 249$ and the the maximum power of 2 that divides $\frac{1000!}{10^{249}}$ is $2^{994-249}=2^{745}$.

Solution 2 by Titu Zvonaru, Comănești, Romania. We have $2^9 < 1000 < 2^{10}$ and $5^4 < 1000 < 5^5$. We denote by $[m]$ the largest integer less than or equal to m. Since

$$\left\lfloor\frac{1000}{2}\right\rfloor+\left\lfloor\frac{1000}{2^2}\right\rfloor+\left\lfloor\frac{1000}{2^3}\right\rfloor+\left\lfloor\frac{1000}{2^4}\right\rfloor$$
$$+\left\lfloor\frac{1000}{2^5}\right\rfloor+\left\lfloor\frac{1000}{2^6}\right\rfloor+\left\lfloor\frac{1000}{2^7}\right\rfloor+\left\lfloor\frac{1000}{2^8}\right\rfloor+\left\lfloor\frac{1000}{2^9}\right\rfloor$$
$$=500+250+125+62+31+15+7+3+1=994,$$

and

$$\left\lfloor\frac{1000}{5}\right\rfloor+\left\lfloor\frac{1000}{5^2}\right\rfloor+\left\lfloor\frac{1000}{5^3}\right\rfloor+\left\lfloor\frac{1000}{5^4}\right\rfloor=200+40+8+1=249.$$

It results that

$$n=\frac{1000!}{10^k}=\frac{2^{994}\cdot 5^{249}\cdot m}{2^k\cdot 5^k},$$

where m is not divisible by 2 or 5. It follows that the largest positive integer k for which n is an integer is $k=249$, and the maximum power of 2 that divides n is $994-249=745$.

Solution 3 by the proposers. The decimal representation of a number n will end in a 0 if $10 \mid n$. Furthermore, the number of 0's that trail the decimal representation of n is equal to the highest power of 10 that divides n. But 10 can be factored as $2 \cdot 5$, so the highest power of 10 that divides n is the minimum of the highest power that 2 can divide n and the highest power of 5 that can divide n.

By Legendre's formulae, we see that $2^{\alpha} \mid 1000!$ where

$$\begin{aligned}\alpha &= \sum_{i=1}^{\infty}\left\lfloor\frac{1000}{2^i}\right\rfloor = \sum_{i=1}^{9}\left\lfloor\frac{1000}{2^i}\right\rfloor \\ &= \left\lfloor\frac{1000}{2^1}\right\rfloor + \left\lfloor\frac{1000}{2^2}\right\rfloor + \left\lfloor\frac{1000}{2^3}\right\rfloor + \left\lfloor\frac{1000}{2^4}\right\rfloor \\ &+ \left\lfloor\frac{1000}{2^5}\right\rfloor + \left\lfloor\frac{1000}{2^6}\right\rfloor + \left\lfloor\frac{1000}{2^7}\right\rfloor + \left\lfloor\frac{1000}{2^8}\right\rfloor + \left\lfloor\frac{1000}{2^9}\right\rfloor \\ &= 500 + 250 + 125 + 62 + 31 + 15 + 7 + 3 + 1 \\ &= 994\end{aligned}$$

Likewise, we see that $5^{\beta} \mid 1000!$ where

$$\begin{aligned}\beta &= \sum_{i=1}^{\infty}\left\lfloor\frac{1000}{5^i}\right\rfloor = \sum_{i=1}^{4}\left\lfloor\frac{1000}{5^i}\right\rfloor \\ &= \left\lfloor\frac{1000}{5^1}\right\rfloor + \left\lfloor\frac{1000}{5^2}\right\rfloor + \left\lfloor\frac{1000}{5^3}\right\rfloor + \left\lfloor\frac{1000}{5^4}\right\rfloor \\ &= 200 + 40 + 8 + 1 \\ &= 249\end{aligned}$$

We see that $\min\{994, 249\} = 249$, so $10^{249} \mid 1000!$, and the maximum value of k satisfying the statement is 249.

Furthermore, the number $\dfrac{1000!}{10^{249}} = 2^{745} p_2^{\alpha_2} p_3^{\alpha_3} \cdots$ is obviously composite and 2^{745} is the maximum power 2 that divides it.

Also solved by *Victoria Farina (studen)t, SUNY Brockport, USA.*

Easy–Medium Problems

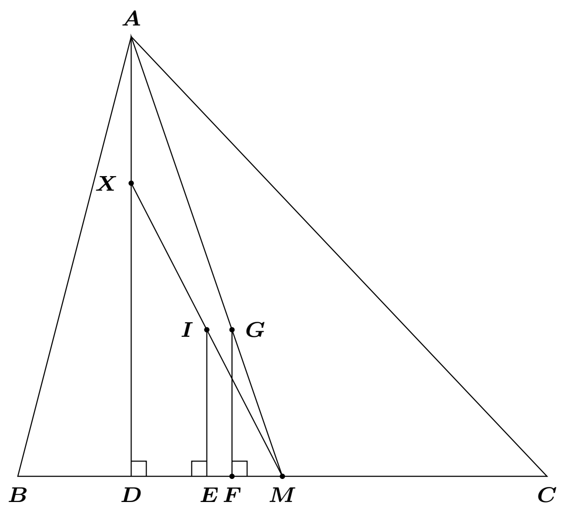
EM–125. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let ABC be a triangle with $AB < AC$. Let M be the midpoint of the side BC, I be its incenter, G be its centroid and D be the foot of the altitude drawn from A. If $MI \cap AD = \{X\}$ and $3\,AX = AD$, then show that $IG \parallel BC$.

Solution 1 by Miquel Amengual Covas, Cala Figuera, Mallorca, Spain. Suppose $\triangle ABC$ with inradius r, semiperimeter s and sides a, b, c opposite A, B, C respectively.

We are told that $b > c$.

Let E be the foot of the perpendicular from I to BC. Then IE is the inradius to the point of contact with BC and hence $BE = s-b$.

Let M be the midpoint of side BC and let h denote the length of the altitude AD.



We have

$$EM = BM - BE = \frac{a}{2} - (s-b) = \frac{b-c}{2}.$$

From the similarity of right-angled triangles XDM and IEM it follows that
$$\frac{IE}{EM} = \frac{XD}{DM}.$$
Therefore, since X trisects AD, we have
$$\frac{r}{\frac{b-c}{2}} = \frac{\frac{2h}{3}}{DM}$$
and
$$DM = \frac{(b-c)h}{3r}.$$
Since AM is the median to side BC, we have $AM = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$.

On the one hand, the area of $\triangle ABC$ is $\frac{1}{2}ah$ and also, according to Heron's formula,
$$\frac{1}{4}\sqrt{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}.$$
Hence $ah = \frac{1}{2}\sqrt{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}$ and
$$h = \frac{\sqrt{2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4}}{2a}.$$
By the Pythagorean theorem, applied to $\triangle ADM$,
$$DM^2 = AM^2 - AD^2,$$
which, on substitution, yields
$$\begin{aligned} \left(\frac{(b-c)h}{3r}\right)^2 &= \frac{2b^2+2c^2-a^2}{4} - \frac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{4a^2} \\ &= \frac{\left(b^2-c^2\right)^2}{4a^2}. \end{aligned}$$
This equality is now equivalent to
$$(b-c)^2\left(\frac{h^2}{9r^2} - \frac{(b+c)^2}{4a^2}\right) = 0,$$

which we rewrite as

$$(b-c)^2\left(\frac{h}{3r}+\frac{b+c}{2a}\right)\left(\frac{h}{3r}-\frac{b+c}{2a}\right)=0$$

Since $b \neq c$, it follows that $\frac{h}{3r}-\frac{b+c}{2a}=0$, and therefore

$$\frac{ah}{r}=\frac{3(b+c)}{2}. \tag{1}$$

On the other hand, the area of $\triangle ABC$ may be expressed also as rs; hence $\frac{1}{2}ah = rs$. We substitute $2rs$ for ah into (1), simplify, and obtain

$$b+c=2a. \tag{2}$$

By eliminating a between (1) and (2), we obtain

$$r=\frac{h}{3}.$$

That is,

$$IE=\frac{AD}{3}.$$

Let F the foot of the perpendicular from G to BC. Since $\triangle ADM \sim \triangle GFM$, the segment GF has the same ratio to AD as GM has to AM. Since G trisects AM, we have $GF = AD/3$.

Thus, $IE = GF$.

Since G and I lie on the same half-plane with respect to the line BC, we conclude that $IG \| BC$, as desired.

Solution 2 by Michel Bataille, Rouen, France. In barycentric coordinates relatively to (A, B, C), we have $G = (1 : 1 : 1)$ and $I = (a : b : c)$ where $BC = a, CA = b, AB = c$. The point at infinity of the line BC ($x = 0$) is $(0 : 1 : -1)$, hence the desired conclusion $IG \parallel BC$ is equivalent to $\begin{vmatrix} a & 1 & 0 \\ b & 1 & 1 \\ c & 1 & -1 \end{vmatrix} = 0$, that is, to $2a = b + c$ (this is a well-known result).
Thus, the problem boils down to proving that $b + c = 2a$.

Let $S_A = \frac{b^2+c^2-a^2}{2}, S_B = \frac{c^2+a^2-b^2}{2}, S_C = \frac{a^2+b^2-c^2}{2}$ (Conway's notation). Then $D = (0 : S_C : S_B)$ and the equation of the line AD is $yS_B - zS_C = 0$. The equation of MI is $(b-c)x - ay + az = 0$ (since $M = (0 : 1 : 1)$) and we deduce that $X = (a(b+c) : S_C : S_B)$. It follows that

$$a(a+b+c)X = a(b+c)A + S_C B + S_B C = a(b+c)A + a^2 D.$$

The latter writes as $(a+b+c)\overrightarrow{AX} = a\overrightarrow{AD}$. Since $3AX = AD$, we must have $3a = a+b+c$, hence $2a = b+c$, as desired.

Solution 3 by Titu Zvonaru, Comăneşti, Romania. Let AA' be the bisector of the angle $\angle BAC$. Using usual notations and formulas, we obtain

$$BD = c\cos B = \frac{a^2+c^2-b^2}{2a},$$

$$DM = BM - BD = \frac{a}{2} - \frac{a^2+c^2-b^2}{2a} = \frac{b^2-c^2}{2a},$$

$$BA' = \frac{ac}{b+c}, \quad A'M = BM - BA' = \frac{a}{2} - \frac{ac}{b+c} = \frac{a(b-c)}{2(b+c)}.$$

By Van Aubel's theorem, we have

$$\frac{AI}{IA'} = \frac{b+c}{a}.$$

Since $3AX = AD$ implies $\frac{XD}{XA} = 2$, applying Menelaus' theorem for the triangle ADA' and the transversal $M - I - X$, it follows that

$$\frac{MA'}{MD} \cdot \frac{XD}{XA} \cdot \frac{IA}{IA'} = 1 \iff \frac{a(b-c)}{2(b+c)} \cdot \frac{2a}{b^2-c^2} \cdot 2 \cdot \frac{b+c}{a} = 1$$

$$\iff b+c = 2a \iff \frac{AI}{IA'} = 2 = \frac{AG}{GM},$$

hence $IG \parallel BC$.

Also solved by *the proposer.*

EM–126. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $n \geq 1$ be an integer. Find the first decimal figure of the real number $\sqrt{n^2+15n+55}$.

Solution 1 by Michel Bataille, Rouen, France. Answer: 4.

Let $a_n = n^2+15n+55$. We want to show that

$$\lfloor 10\sqrt{a_n}\rfloor - 10\lfloor\sqrt{a_n}\rfloor = 4.$$

Since

$$(n+7)^2 = n^2+14n+49 < a_n < n^2+16n+64 = (n+8)^2$$

we have $\lfloor\sqrt{a_n}\rfloor = n+7$.
Let $\theta_n = \sqrt{a_n} - \lfloor\sqrt{a_n}\rfloor$. Since $\lfloor k+x\rfloor = k+\lfloor x\rfloor$ if $k\in\mathbb{Z}, x\in\mathbb{R}$, all boils down to showing that $\lfloor 10\theta_n\rfloor = 4$.
Now,

$$10\theta_n = 10(\sqrt{a_n}-(7+n)) = 10\cdot\frac{a_n-(7+n)^2}{7+n+\sqrt{a_n}} = \frac{10n+60}{7+n+\sqrt{a_n}}$$

and therefore

$$\frac{10n+60}{2n+15} = \frac{10n+60}{7+n+8+n} < 10\theta_n < \frac{10n+60}{7+n+7+n} = \frac{10n+60}{2n+14}.$$

Since

$$\frac{10n+60}{2n+15} = 4+\frac{2n}{2n+15} > 4 \quad\text{and}\quad \frac{10n+60}{2n+14} = 5-\frac{10}{2n+14} < 5$$

we see that $4 < 10\theta_n < 5$ and $\lfloor 10\theta_n\rfloor = 4$ follows.

Solution 2 by Rovsen Pirguliyev, Sumgait, Azerbaijan and Victoria Farina (student), SUNY Brockport, USA. For all integer $n \geq 1$, we claim that

$$n+7.4 < \sqrt{n^2+15n+55} < n+7.5.$$

Indeed, $(n+7.4)^2 < n^2+15n+55 < (n+7.5)^2 \iff n^2+14.8n+54.76 < n^2+15n+55 < n^2+15n+56.25$ which is true. This shows that the first decimal figure of the given number is 4.

Solution 3 by the proposer. Note that $(n+7)^2 < n^2+15n+55 < (n+8)^2$ and therefore,

$$n + 7 < \sqrt{n^2 + 15n + 55} < n + 8$$

from which it follows that $\lfloor\sqrt{n^2 + 11n + 30}\rfloor = n + 7$. Let α be the first decimal figure of $\sqrt{n^2 + 15n + 55} - (n + 7)$. Then $\sqrt{n^2 + 11n + 30} - (n + 5) = 0.\alpha\beta\gamma\dots$ and

$$10\Big(\sqrt{n^2 + 11n + 30} - (n + 7)\Big) = \alpha.\beta\gamma\dots$$

This shows that α is the integer part of the number $x = 10\big(\sqrt{n^2 + 15n + 55} - (n + 7)\big)$, and for all $n \geq 1$, it holds:

$$\alpha \leq 10\Big(\sqrt{n^2 + 15n + 55} - (n + 7)\Big) < \alpha + 1,$$

or equivalently

$$n + 7 + \frac{\alpha}{10} \leq \sqrt{n^2 + 15n + 55} < n + 7 + \frac{\alpha + 1}{10}.$$

Squaring, we get

$$n^2 + 2\Big(7 + \frac{\alpha}{10}\Big)n + \Big(7 + \frac{\alpha}{10}\Big)^2$$

$$\leq n^2 + 15n + 55 < n^2 + 2\left(7 + \frac{\alpha + 1}{10}\right)n + \left(7 + \frac{\alpha + 1}{10}\right)^2 \quad (1)$$

A necessary condition for the preceding to hold is that the next relations between the coefficients of n in the three terms of the relation (1) hold:

$$2\Big(7 + \frac{\alpha}{10}\Big) \leq 15 \leq 2\left(7 + \frac{\alpha + 1}{10}\right),$$

otherwise the preceding relation (1) could not be fulfilled for any given n.

The above relations lead to the double inequality $\alpha \leq 5 \leq \alpha + 1$, satisfied for $\alpha \in \{4, 5\}$, but only $\alpha = 4$ can be accepted because this value of α is sufficient as well for (1) to hold, since $55 < \left(7+\frac{1}{2}\right)^2$. The value $\alpha = 5$ fails to hold (1), since $55 < \left(7+\frac{1}{2}\right)^2$ contradicting the LHS inequality of (1). Finally, we conclude that the first decimal figure of $\sqrt{n^2+11n+30}$ is $\alpha = 4$ for all $n \geq 1$.

Also solved by *Titu Zvonaru, Comăneşti, Romania, and Albert Stadler, Herrliberg, Switzerland.*

EM–127. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a scalene triangle with incenter I and centroid G. Let G_a be the orthogonal projection of G on BC. Let M be the midpoint of BC and A_M be the reflection of A in M. Denote with P the second intersection point of the line AI and the circumcircle of the triangle ABC. Knowing that the points A_M, P, G_a are collinear, find the ratio AI/IP.

Solution 1 by Michel Bataille, Rouen, France. Let $BC = a, CA = b, AB = c$ and let D be the foot of the altitude from A. In barycentric coordinates relatively to (A, B, C), we have $G = (1 : 1 : 1)$, $I = (a : b : c)$ and $D = (0 : S_C : S_b)$ where $S_B = \frac{c^2+a^2-b^2}{2}$, $S_C = \frac{a^2+b^2-c^2}{2}$.
The line AI $(cy - bz = 0)$ intersects the circumcircle of ABC $(a^2yz + b^2zx + c^2xy = 0)$ at $A = (1 : 0 : 0)$ and

$$P = (a^2 : -b(b+c) : -c(b+c)).$$

From $\overrightarrow{MG_a} = \frac{1}{3}\overrightarrow{MD}$ and $2M = B + C$, we obtain $3a^2G_a = a^2D + 2a^2M = (S_C + a^2)B + (S_B + a^2)C$. Since $A_M = -A + B + C$, the hypothesis about A_M, P, G_a writes as

$$\begin{vmatrix} -1 & 0 & a^2 \\ 1 & S_C + a^2 & -b(b+c) \\ 1 & S_B + a^2 & -c(b+c) \end{vmatrix} = 0,$$

that is, $(b^2 - c^2)[(b+c)^2 - 5a^2] = 0$ after a simple calculation. Since ABC is scalene, we have $b^2 \neq c^2$ and therefore $b + c = a\sqrt{5}$.
We deduce that $P = (a : -b\sqrt{5} : -c\sqrt{5})$, hence

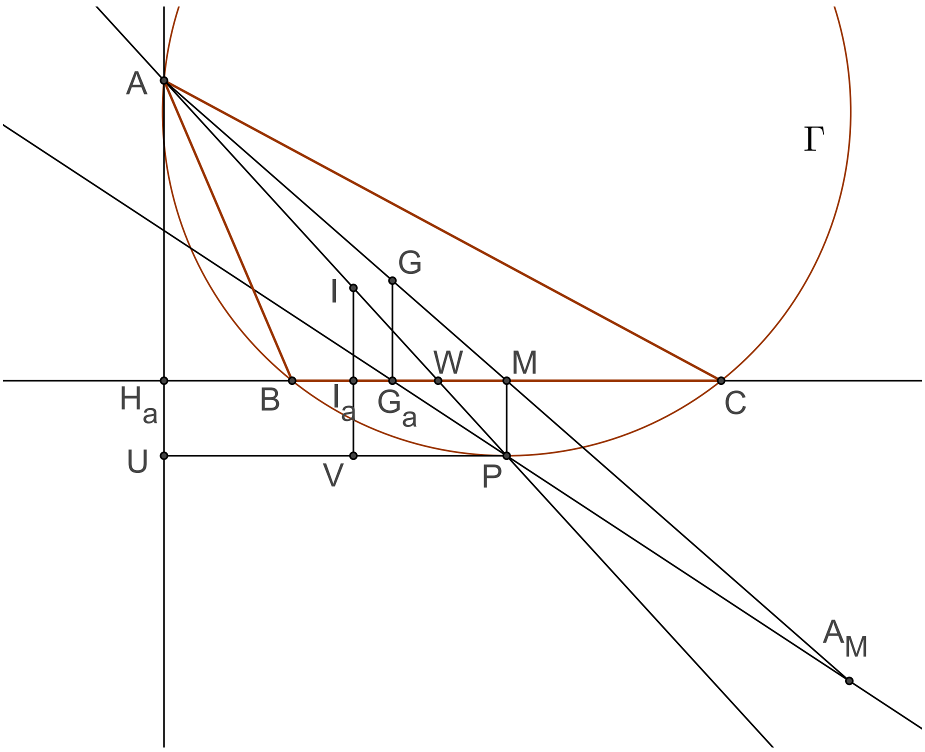
$$-4aP = aA - b\sqrt{5}B - c\sqrt{5}C = a(1+\sqrt{5})(A - \sqrt{5}I)$$

so that $(\sqrt{5}-1)\overrightarrow{IP} = \overrightarrow{AI}$. We conclude: $\frac{AI}{IP} = \sqrt{5}-1$.

Solution 2 by the proposer. The answer is

$$\frac{AI}{IP} = \sqrt{5}-1.$$

Let $BC = a, CA = b, AB = c$ are the side lengths of the triangle ABC. Let H_a, I_a are the orthogonal projections of A, I respectively on the line BC.



Scheme for solving problem EM-127

Since AI is the angle bisector of $\angle BAC$, the point P is the midpoint of the arc BC and P lies on the perpendicular bisector of BC, so $PM \perp BC$, $PM \parallel GG_a$. Hence $\triangle A_M GG_a \sim \triangle A_M MP$ and

$$\frac{PM}{GG_a} = \frac{A_M M}{A_M G} = \frac{AM}{AM + MG} = \frac{3}{4}$$

From $\triangle MGG_a \sim \triangle MAH_a$ hence

$$\frac{GG_a}{AH_a} = \frac{G_aM}{H_aM} = \frac{GM}{AM} = \frac{1}{3}; \quad GG_a = \frac{1}{3}h_a; \quad PM = \frac{1}{4}h_a,$$

where $h_a = AH_a$ is the A-altitude of $\triangle ABC$.

Let U, V are the orthogonal projections of P on the lines AH_a, II_a respectively. Hence $\triangle PAU \sim \triangle PIV$ and

$$\frac{AI}{IP} = \frac{AP - IP}{IP} = \frac{AP}{IP} - 1 = \frac{AU}{IV} - 1 = \frac{AH_a + H_aU}{II_a + I_aV} - 1$$

$$= \frac{AH_a + PM}{II_a + PM} - 1$$

since $H_aU = I_aV = PM$. Denote with $r = II_a$ the radius of the incircle of the triangle ABC. Let S be the area of the $\triangle ABC$.

$$S = \frac{1}{2}ah_a = \frac{1}{2}r(a+b+c); \quad \frac{h_a}{r} = \frac{a+b+c}{a}$$

$$\frac{AI}{IP} = \frac{h_a + \frac{1}{4}h_a}{r + \frac{1}{4}h_a} - 1 = \frac{5h_a}{4r + h_a} - 1 = \frac{5\frac{h_a}{r}}{4 + \frac{h_a}{r}} - 1 = \frac{5\frac{a+b+c}{a}}{4 + \frac{a+b+c}{a}} - 1$$

$$= \frac{4(b+c)}{5a+b+c}$$

Denote with $W = AI \cap BC$. Without loss of generality we may assume that $b > c$, hence M is between C and the points H_a, I_a, G_a, W.

$$\frac{BW}{WC} = \frac{c}{b}; \quad BW = \frac{ac}{b+c}$$

$$H_aM = H_aC - MC = b\cos C - \frac{a}{2} = \frac{a^2+b^2-c^2}{2a} - \frac{a}{2} = \frac{b^2-c^2}{2a}$$

$$WM = BM - BW = \frac{a}{2} - \frac{ac}{b+c} = \frac{a(b+c) - 2ac}{2(b+c)} = \frac{a(b-c)}{2(b+c)}$$

The triangle $\triangle WPM \sim \triangle WAH_a$, so

$$\frac{H_aW}{WM} = \frac{AH_a}{PM} = 4; \quad \frac{H_aW}{WM} = \frac{H_aM - WM}{WM} = \frac{H_aM}{WM} - 1$$

$$4 = \frac{\frac{b^2-c^2}{2a}}{\frac{a(b-c)}{2(b+c)}} - 1 \;\Leftrightarrow\; 5 = \frac{2(b+c)(b^2-c^2)}{2a^2(b-c)}$$

$$\Leftrightarrow\; 0 = \frac{(b+c)^2}{a^2} - 5 \;\Leftrightarrow\; 0 = (b+c)^2 - 5a^2$$

$$\Leftrightarrow\; 0 = \left(b + c + \sqrt{5}a\right)\left(b + c - \sqrt{5}a\right)$$

but $b+c+a\sqrt{5} > 0$ and so, the condition that the points A_M, P, G_a are collinear is

$$a\sqrt{5} = b + c.$$

Now

$$\frac{AI}{IP} = \frac{4(b+c)}{5a+b+c} = \frac{4a\sqrt{5}}{5a+a\sqrt{5}} = \frac{4}{\sqrt{5}+1} = \sqrt{5} - 1$$

EM–128. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let u, v, z, w be complex numbers. Prove that

$$2\text{Re}\left(uz + vw\right) \leq 2\left(|u|^2 + |v|^2\right) + \frac{1}{2}\left(|z|^2 + |w|^2\right).$$

Solution 1 by José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain. Let a_1, a_2, b_1, b_2 and α be complex numbers. Then, from

$$\begin{aligned}\sum_{k=1}^{2} |a_k - \alpha\overline{b}_k|^2 &= \sum_{k=1}^{2} (a_k - \alpha\overline{b}_k)(\overline{a}_k - \overline{\alpha} b_k) \\ &= \sum_{k=1}^{2} |a_k|^2 + |\alpha|^2 \sum_{k=1}^{2} |b_k|^2 - 2\text{Re}\left(\overline{\alpha} \sum_{k=1}^{2} a_k b_k\right) \\ &\geq 0,\end{aligned}$$

we get

$$\text{Re}\left(\overline{\alpha} \sum_{k=1}^{2} a_k b_k\right) \leq \frac{1}{2}\left(\sum_{k=1}^{2} |a_k|^2 + |\alpha|^2 \sum_{k=1}^{2} |b_k|^2\right).$$

Now setting $a_1 = u, a_2 = v, b_1 = z, b_2 = w$ and $\alpha = 1/2$ into the preceding inequality, we have

$$\frac{1}{2}\mathrm{Re}\Big(uz + vw\Big) \le \frac{1}{2}\Big[\Big(|u|^2 + |v|^2\Big) + \frac{1}{4}\Big(|z|^2 + |w|^2\Big)\Big].$$

Multiplying by 4 both term of the preceding inequality the statement immediately follows, and we are done.

Solution 2 by Michel Bataille, Rouen, France. For any complex number Z, we have $2\mathrm{Re}(Z) = Z + \overline{Z}$ and $|Z|^2 = Z\overline{Z}$. We deduce that

$$\begin{aligned}
&2(|u|^2 + |v|^2) + \frac{1}{2}(|z|^2 + |w|^2) - 2\mathrm{Re}(uz + vw)\\
&= (2u\overline{u} + \frac{1}{2}z\overline{z} - uz - \overline{uz}) + (2v\overline{v} + \frac{1}{2}w\overline{w} - vw - \overline{vw})\\
&= \frac{1}{2}((2u - \overline{z})(2\overline{u} - z) + (2v - \overline{w})(2\overline{v} - w))\\
&= \frac{1}{2}(|2u - \overline{z}|^2 + |2v - \overline{w}|^2),
\end{aligned}$$

hence $2(|u|^2 + |v|^2) + \frac{1}{2}(|z|^2 + |w|^2) - 2\mathrm{Re}(uz + vw) \ge 0$ and the result follows.

Solution 3 by Albert Stadler, Herrliberg, Switzerland. Let $u = a + ib, v = c + id, w = e + if, z = g + ih$ with a, b, c, d, e, f, g, h real. Then

$$2\big(|u|^2 + |v|^2\big) + \frac{1}{2}\big(|z|^2 + |w|^2\big) - 2\mathrm{Re}(uz + vw)$$

$$= 2\big(a^2 + b^2 + c^2 + d^2\big) + \frac{1}{2}\big(e^2 + f^2 + g^2 + h^2\big) - 2(ag - bh + ce - df)$$

$$= 2\left(\left(a - \frac{g}{2}\right)^2 + \left(b + \frac{h}{2}\right)^2 + \left(c - \frac{e}{2}\right)^2 + \left(d + \frac{f}{2}\right)^2\right) \ge 0.$$

Solution 4 by Titu Zvonaru, Comănești, Romania. Let $u = u_1 + iu_2$, $z = z_1 + iz_2$, with u_1, u_2, z_1, z_2 real numbers. We have

$$2\Re(uz) \le 2|u|^2 + \frac{1}{2}|z|^2 \iff 4(u_1 z_1 - u_2 z_2) \le 4(u_1^2 + u_2^2) + z_1^2 + z_2^2$$

$$\iff \quad (2u_1 - z_1)^2 + (2u_2 + z_2)^2 \geq 0.$$

It follows that

$$2\Re(uz) \leq 2|u|^2 + \frac{1}{2}|z|^2,$$

and similarly,

$$2\Re(vw) \leq 2|v|^2 + \frac{1}{2}|w|^2.$$

Adding these inequalities, we obtain the desired inequality.

Equality holds if and only if $z_1 = 2u_1$, $z_2 = -2u_2$, $w_1 = 2v_1$, and $w_2 = -2v_2$.

Solution 5 by Victoria Farina (studen)t, SUNY Brockport, USA.
By AM - GM inequality

$$2(|u|^2 + |v|^2) + \frac{1}{2}(|z|^2 + |w|^2) \geq 2\sqrt{2(|u|^2 + |v|^2) \cdot \frac{1}{2}(|z|^2 + |w|^2)}$$

$$= 2\sqrt{(|u|^2 + |v|^2) \cdot (|z|^2 + |w|^2)}.$$

By Cauchy - Buniakovsky - Schwarz inequality

$$\sqrt{(|u|^2 + |v|^2) \cdot (|z|^2 + |w|^2)} \geq |uz + vw|$$

$$\geq |Re(uz + vw)| \geq Re(uz + vw).$$

Therefore

$$2(|u|^2 + |v|^2) + \frac{1}{2}(|z|^2 + |w|^2) \geq 2Re(uz + vw).$$

Also solved by *the proposer.*

EM–129. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let $\alpha \in \{1,2,3,4,5,6,7,8,9\}$. Prove that every positive real number is the sum of nine numbers consisting only of 0 and α in their digits and decimal part.

Solution by the proposers. First we claim that any positive number a may be represented as a sum of nine positive numbers, whose decimal representation consists only of the digits 0 and 1. Indeed, let

$$a = \overline{a_k a_{k-1} \cdots a_0 . a_{-1} a_{-2} \cdots},$$

where a_i $(i = k, k-1, \cdots, 0, -1, -2, \cdots)$ are the consecutive decimal digits of a. Consider the nine numbers

$$a^{(j)} = \overline{a_k^{(j)} a_{k-1}^{(j)} \cdots a_0^{(j)} . a_{-1}^{(j)} a_{-2}^{(j)} \cdots}, \quad 1 \leq j \leq 9,$$

where

$$a_i^{(j)} = \begin{cases} 1, & \text{if } j \leq a_i, \\ 0, & \text{if } j > a_i. \end{cases}$$

Each digit of $a^{(j)}$ $(j = 1, 2, \ldots, 9)$ is either 0 or 1 and

$$a^{(1)} + a^{(2)} + \ldots + a^{(9)} = a,$$

therefore the result claimed above is true.

For example, let $a = 198726.67240351$ and consider the numbers

$$\begin{aligned}
a^{(1)} &= 111111.11110111, \\
a^{(2)} &= 011111.11110110, \\
a^{(3)} &= 011101.11010110, \\
a^{(4)} &= 011101.11010010, \\
a^{(5)} &= 011101.11000010, \\
a^{(6)} &= 011101.11000000, \\
a^{(7)} &= 011100.01000000, \\
a^{(8)} &= 011000.00000000, \\
a^{(9)} &= 010000.00000000,
\end{aligned}$$

then a is sum of $a^{(j)}$'s.

Now let b be any positive number. Then $a = b/\alpha$ may be represented as a sum of nine numbers whose digits are 0's and 1's, according to the claim. Let

$$a = a^{(1)} + a^{(2)} + \ldots + a^{(9)}$$

be such a representation. Then each of the numbers $\alpha\, a^{(j)}$ $(j = 1, 2, \ldots, 9)$ has a decimal representation consisting only of the digits 0 and α. Therefore the representation

$$b = \left(\alpha a^{(1)}\right) + \left(\alpha a^{(2)}\right) + \ldots + \left(\alpha a^{(9)}\right)$$

has the desired properties.

EM–130. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Lisa and Bart play the following game. They first choose a positive integer N, and then they take turns writing numbers on a blackboard. Lisa starts by writing 1. Thereafter, when one of them has written the number n, the next player writes down either $n+1$ or $2n$, provided the number is not greater than N. The player who writes N on the blackboard wins.

(a) Determine which player has a winning strategy if $N = 2025$.
(b) Find the number of positive integers $N \leq 2025$ for which Bart has a winning strategy.

Solution by the proposer. **(a)** Lisa has a winning strategy for odd N, and so wins when $N = 2025$. Observe that, whenever a player writes down an odd number, the next is forced to write down an even number. By adding 1 to that number, the first player can write down another odd number. Since Lisa starts the game by writing down an odd number, she can force Bart to write down even numbers only. Since N is odd, Lisa will win the game, and in particular, she wins for $N = 2025$.

(b) For even N we consider two cases, according to the value of $N \pmod 4$.

- Let $N = 4k$. If any player is forced to write down a number $m \in \{k+1, k+2, \ldots, 2k\}$, the other player wins by writing down $2m \in \{2k+2, 2k+4, \ldots, 4k\}$, for the players will then have to write down the remaining numbers one after the other. Since there is an even number of numbers remaining, the latter player wins. This implies that the player who can write down k (that is, has a winning strategy for $N = k$), wins the game for $N = 4k$.
- Similarly, let $N = 4k + 2$. If any player is forced to write down a number $m \in \{k+1, k+2, \ldots, 2k+1\}$, the other player wins the game by writing down $2m \in \{2k+2, 2k+4, \ldots, 4k+2\}$, as in the previous case. Analogously, this implies that the player who has a winning strategy for $N = k$ wins the game for $N = 4k + 2$.

Since Lisa wins the game for $N = 1, 3$, while Bart wins the game for $N = 2$, Bart wins the game for $N = 8, 10$ as well, and thus for $N = 32, 34, 40, 42$ too. Then Bart wins the game for a further 8 values of N between 128 and 170, and then a further 16 values between 512 and 682, and for no other values with $N \leq 2025$. Hence Burt has a winning strategy for precisely 31 values of N with $N \leq 2025$.

Medium–Hard Problems

MH–125. *Proposed by Michel Bataille, Rouen, France.* For a positive integer x, let $v(x)$ denote the greatest of the integers $r \geq 0$ such that 2^r divides x and let m, n be positive integers. Prove that $v(2023^m - 1) = v(2025^n - 1)$ if and only if $v(m) = v(n) \neq 0$.

Solution by the proposer. Suppose that we have proved that $v(2025^n - 1) = v(n) + 3$ for all positive integer n and that $v(2023^m - 1) = 1$ if $v(m) = 0$ while $v(2023^m - 1) = v(m) + 3$ if $v(m) \geq 1$. Then, if $v(m) = v(n) \geq 1$, we clearly have $v(2023^m - 1) = v(2025^n - 1) (= v(m) + 3 = v(n) + 3)$. Conversely, if $v(2023^m - 1) = v(2025^n - 1)$, then $v(2023^m - 1)$ must be greater than 1, hence $v(m) \neq 0$ and $v(2023^m - 1) = v(m) + 3$. Thus, $v(n) + 3 = v(m) + 3$ and $v(n) = v(m) \neq 0$ follows.

We now consider the above formulas about $v(2025^n - 1)$ and $v(2023^m - 1)$.

Let $a = 2024 = 8 \times 253$, so that $v(a) = 3$. Using the binomial theorem, we immediately obtain $v(2025^n + 1) = v((a+1)^n + 1) = 1$ (for any $n \geq 1$), $v(2023^m + 1) = v((a-1)^m + 1) = 1$ for any even m, $v(2023^m - 1) = v(a-1)^m - 1) = 1$ for any odd m. Also, if m, n are odd, then $\frac{(a-1)^m+1}{a}$ and $\frac{(a+1)^n-1}{a}$ are odd integers, hence $v(2023^m + 1) = v((a-1)^m + 1) = v(a) = 3$ and $v(2025^n - 1) = v((a+1)^n - 1) = v(a) = 3$.

To complete the calculation, we examine the case when m, n are even. We use the following remark: if x and r are positive integers and s is odd, then

$$x^{s \cdot 2^r} - 1 = (x^s - 1)(x^s + 1) \cdot \prod_{k=1}^{r-1} (x^{s \cdot 2^k} + 1).$$

Taking $x = 2023$ and $m = s \cdot 2^r$, we deduce that $v(2023^m - 1) = 1 + 3 + \sum_{k=1}^{r-1} 1 = r + 3 = v(m) + 3$ and taking $x = 2025, n = s \cdot 2^r$, $v(2025^n - 1) = 3 + 1 + \sum_{k=1}^{r-1} 1 = 3 + r = 3 + v(n)$. This completes the proof of the formulas assumed at the beginning.

MH–126. *Proposed by Jordi Ferré García, CFIS, BarcelonaTech, Barcelona, Spain.* **(Correction)** Let ABC be a triangle with $AB < AC$, and H be its orthocenter. Let E and F be the intersection of lines BH and CH with AC and AB respectively, and H' be a point on line EF such that $HH' \perp EF$. If we let M be the midpoint of side BC, and T be the intersection of line AM with the circumcircle of ABC, show that lines MH' and TH intersect on the circumcircle of triangle MEF.

Solution 1 by the proposer. *During the following proof, for given points A, B, C, we'll denote (ABC) as the circumcircle of the triangle with vertices A, B and C.*

We start by letting $S = MH' \cap (MEF)$, so now the problem becomes equivalent to showing that points S, H and T lie on the same line.

To start, define $D = AH \cap BC$ and $G = EF \cap BC$. Now notice that the quadrilateral $GH'HD$ is clearly cyclic, as

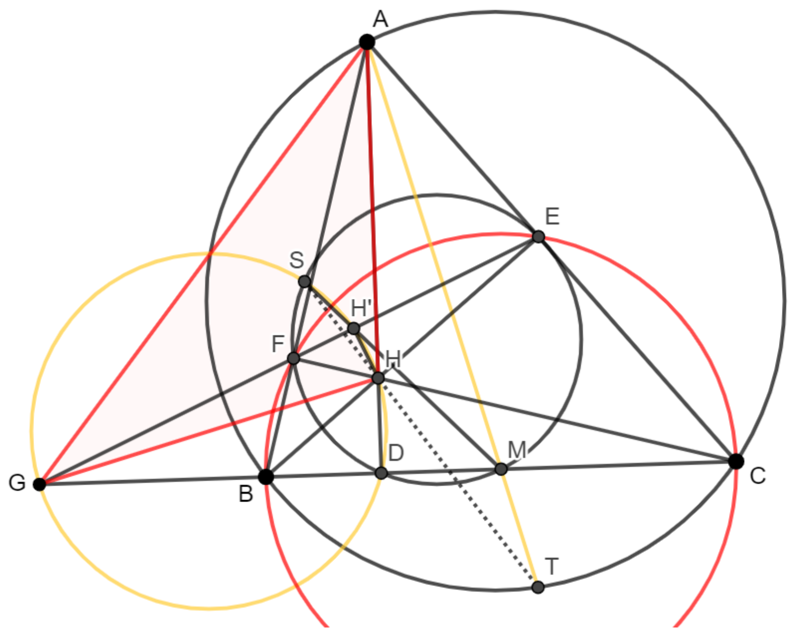
$$\angle GH'H = \angle HDG = 90^\circ.$$

We claim that S also lies in this circumference. To show this, consider the inversion around the circumference centered at M with radius $MB = MF = ME = MC$, which we'll denote by $\varphi : \mathbb{R}^2 \setminus \{M\} \to \mathbb{R}^2$. Notice that, by definition, $\varphi(F) = F$ and $\varphi(E) = E$, so we conclude that $\varphi(EF) = (MFE)$. This implies that $\varphi(D) = G$ and $\varphi(H') = S$, which means that $MH' \cdot MS = MD \cdot MG$, which is equivalent to the desired cyclic, so our claim is already proven.

Now, we consider $\psi : \mathbb{R}^2 \setminus \{H\} \to \mathbb{R}^2$ as the inversion centered at H with radius $\sqrt{HA \cdot HD}$, followed by a reflection across H. We will show that this map sends S to T, which will clearly end the problem. We will denote $\psi(S) \coloneqq S'$ for now.

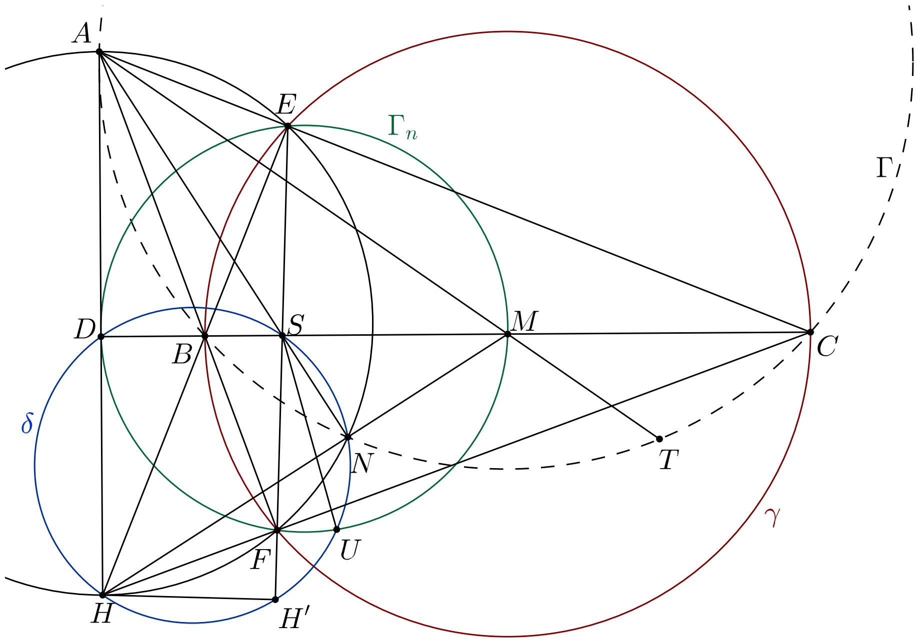
In order to do this first notice that $S = (DEF) \cap (GDHH')$, so $S' = \psi((DEF)) \cap \psi((GDHH'))$. We first see that (DEF) is sent to (ABC), as $\psi(D) = E$, $\psi(E) = B$ and $\psi(F) = C$ follows from the fact that $HD \cdot HA = HE \cdot HB = HF \cdot HC$. So S lying over (DEF) implies that $S' \in (ABC)$.

Now it suffices to show that $(GDHH')$ is sent to line AM. See how $(GDHH')$ passes through H so its inverse is a line, and $\psi(D) = A$ implies that this line goes through A. Now, as GH is a diameter of the circumference, we get that $\psi(GH'HD)$ is the line through A perpendicular to GH. So it suffices to show that $AM \perp GH$. But this is just a straight-forward implication of Brokard's Theorem applied to quadrilateral $BFEC$, as this tells us that triangle GHA is autopolar with respect to this circumference, and as M is its center, we obtain that $GH \perp AM$, which ends the problem.



Solution 2 by Michel Bataille, Rouen, France. Let Γ be the circumcircle of ABC, Γ_n its Euler circle (the circumcircle of MEF) and γ be the circle with diameter BC (and center M). Let D be the foot of the altitude from A (see figure). The points B, C, E, F are on γ and CE, BF intersect at A, EB, CF intersect at H and EF, BC intersect at, say, S so that ΔASH is self-polar with respect to γ. It follows that the line AS intersects HM at, say, N such that $AN \perp HM$. As a result, D, S, N, H' lie on the circle δ with diameter HS.

Let δ and Γ_n intersect at D and U ($U \neq D$). We answer the



problem by showing that U is on the lines HT and MH'.

First, we consider the inversion $\mathcal{I}$ with center H such that $\mathcal{I}(E) = B$. Then $\mathcal{I}(\gamma) = \gamma$ and $\mathcal{I}(F) = C$. The circle with diameter AH, which passes through E and F inverts into the line BC, hence $\mathcal{I}(A) = D$. Since $\mathcal{I}(\gamma) = \gamma$, the center M of γ inverts into the foot of the polar of H with respect to γ, that is, $\mathcal{I}(M) = N$. We deduce that $\mathcal{I}(\Gamma) = \Gamma_n$ and $\mathcal{I}(AM) = \delta$. Thus, $\mathcal{I}(T) = U$ and H, U, T are collinear.

Second, we consider the inversion $\mathcal{J}$ in the circle γ. We have $\mathcal{J}(E) = E, \mathcal{J}(F) = F$ and $\mathcal{J}(S) = D$ (since AH is the polar of S and $MS \perp AH$). It follows that $\mathcal{J}(\delta) = \delta$. Since $\mathcal{J}(EF) = \Gamma_n$, we see that $\mathcal{J}(H')$ is on δ and Γ_n, hence $\mathcal{J}(H') = U$ and H', U, M are collinear. The proof is complete.

MH–127. *Proposed by Ruben Mason Carpenter, Yale University, New Haven, USA.* Let p be a prime, and let $a_1, \ldots, a_p$ be positive integers, none of them divisible by p. Prove that, for every integer

n, there is a nonempty subset $S \subset \{1, 2, \ldots, p\}$ such that

$$n - \sum_{s \in S} a_s$$

is divisible by p.

Solution by the proposer. For any set S, let $\Sigma(S)$ denote the set of (nonempty) possible sums of elements in S, modulo p:

$$\Sigma(S) = \left\{ \sum_{a \in A} x_a \pmod p \mid \emptyset \neq A \subseteq \{1, 2, \ldots, p\} \right\}.$$

The key idea is to keep track of the effect of adding each element, one by one. This is made precise by the following claim.

Claim. *For any $1 \leq k \leq p$, we have $|\Sigma(\{a_1, \ldots, a_k\})| \geq k$.*

Proof. The proof is combinatorial: induct on k, with the base case $k = 1$ trivial. Observe that the set $\Sigma(\{a_1, \ldots, a_k\})$ contains all the elements of $S_{k-1} := \Sigma(\{a_1, \ldots, a_{k-1}\})$ (by not including a_k in the set A), and contains a copy of each element in S_k, shifted by a_k (by including a_k in the set A).

By induction hypothesis $|S_{k-1}| \geq k - 1$. If it is at least k, then we are done, so assume that $|S_{k-1}| = |S_k| = k - 1$. By the above observation, this means that if $s \in S_{k-1}$, then $s + a_k \in S_{k-1}$ as well. But a_k is nonzero modulo p, so this means that

$$s, s + a_k, s + 2a_k, \ldots, s + (p-1)a_k \in S_{k-1}$$

cycles through all possible residues modulo p, so
$S_{k-1} = \{0, 1, \ldots, p-1\}$. This is a contradiction, so the inductive step is complete. $\square$

The problem now follows by setting $k = p$ in the above.

MH–128. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let a, b, c be positive real numbers such that the sum of their inverses equals the inverse of their product. Find the maximum value of

$$\frac{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}}{a^4+b^4+c^4}.$$

Solution 1 by Rovsen Pirguliyev, Sumgait, Azerbaijan. Clearing denominators in the given condition $a^{-1}+b^{-1}+c^{-1}=abc^{-1}$ we get $ab+bc+ca=1$. On account of AM-GM inequality, we have

$$\frac{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}}{a^4+b^4+c^4} \le \frac{a\cdot\dfrac{b+c}{2}+b\cdot\dfrac{c+a}{2}+c\cdot\dfrac{a+bc}{2}}{a^4+b^4+c^4}$$

$$=\frac{ab+bc+ca}{a^4+b^4+c^4}=\frac{1}{a^4+b^4+c^4}.$$

On the other hand,

$$a^2+b^4+c^4 \ge 3\left(\frac{a^2+b^2+c^2}{3}\right)^2 \ge 3\left(\frac{ab+bc+ca}{3}\right)^2 = 3\left(\frac{1}{3}\right)^2=\frac{1}{3}.$$

In the above we have used the inequality $a^2+b^2+c^2 \ge ab+bc+ca$. Combining the previous results, we obtain

$$\frac{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}}{a^4+b^4+c^4} \le \frac{1}{a^4+b^4+c^4} \le 3.$$

The maximum is 3 which is attained when $a=b=c=1/\sqrt{3}$.

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. We use the following well-known results: (1) The Cauchy-Schwarz inequality; (2) $[(a^4+b^4+c^4)/3]^{1/4} \ge [(a^2+b^2+c^2)/3]^{1/2} \ge (a+b+c)/3$; (3) Maclaurin's inequality: $(a+b+c)/3 \ge [(ab+bc+ca)/3]^{1/2}$.

First we note that $1/a+1/b+1/c=1/abc$, or $(ab+bc+ca)/abc=$

$1/abc$, implies $ab+bc+ca=1$. Now we see that

$$\sum_{cyclic}\frac{a\sqrt{bc}}{a^4+b^4+c^4} \overset{(1)}{\leq} \frac{\sqrt{a^2+b^2+c^2}\cdot\sqrt{ab+bc+ca}}{a^4+b^4+c^4} = \frac{\sqrt{a^2+b^2+c^2}}{a^4+b^4+c^4}$$

$$\overset{(2)}{\leq} 3\cdot\frac{\sqrt{a^2+b^2+c^2}}{a^2+b^2+c^2} = \frac{3}{\sqrt{a^2+b^2+c^2}}$$

$$\overset{(2),(3)}{\leq} \frac{3}{1} = 3.$$

Equality holds if and only if $a=b=c=1/\sqrt{3}$.

Solution 3 by Michel Bataille, Rouen, France. The condition on a,b,c, that is, $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{abc}$, is equivalent to $ab+bc+ca=1$.

Let $X=\frac{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}}{a^4+b^4+c^4}$. We show that the maximum value of X under this constraint is 3.

If $a=b=c=\frac{1}{\sqrt{3}}$, then a,b,c satisfy $ab+bc+ca=1$ and $X=3$. To complete the proof, it remains to prove that $X\leq 3$ whenever $ab+bc+ca=1$.

Suppose that a,b,c satisfy the constraint and let $s=a+b+c$, $p=abc$. From the general inequality $x^2+y^2+z^2\geq xy+yz+zx$, we successively obtain

$$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}=\sqrt{ab}\sqrt{ca}+\sqrt{ca}\sqrt{ab}+\sqrt{ab}\sqrt{bc}\leq ab+bc+ca=1$$

$$a^2b^2+b^2c^2+c^2a^2\geq ab^2c+bc^2a+cab^2=sp$$

and

$$a^4+b^4+c^4\geq a^2b^2+b^2c^2+c^2a^2=(ab+bc+ca)^2-2abc(a+b+c)=1-2sp.$$

Note that $1-2sp\geq sp$ so that $3sp\leq 1$ and therefore

$$a^4+b^4+c^4\geq 1-2sp\geq 1-\frac{2}{3}=\frac{1}{3}.$$

We are done since we deduce that

$$X\leq\frac{1}{1-2sp}\leq\frac{1}{1/3}=3.$$

Solution 4 by Albert Stadler, Herrliberg, Switzerland. The constraint on a, b, c is equivalent to $ab + bc + ca = 1$. By the Cauchy-Schwarz inequality,

$$a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab} \le \sqrt{a^2 + b^2 + c^2}\,\sqrt{ab + bc + ca}$$

and

$$ab + bc + ca \ \le a^2 + b^2 + c^2.$$

Hence

$$\frac{a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}}{a^4 + b^4 + c^4} = \frac{\left(a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}\right)(ab + bc + ca\ )}{a^4 + b^4 + c^4}$$

$$\le \frac{\sqrt{a^2 + b^2 + c^2}(ab + bc + ca\ )^{\frac{3}{2}}}{a^4 + b^4 + c^4} \le \frac{\left(a^2 + b^2 + c^2\right)^2}{a^4 + b^4 + c^4} \le 3$$

as the last inequality is equivalent to

$$2\left(a^4 + b^4 + c^4\right) \ge 2\left(a^2b^2 + b^2c^2 + c^2a^2\right)$$

which holds true by the Cauchy-Schwarz inequality. We have equality if and only if $a = b = c$. So the maximum value equals 3, and is assumed when $a = b = c = \sqrt{3}/3$.

Solution 5 by the proposers. Let $f(a, b, c) = \dfrac{a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}}{a^4 + b^4 + c^4}$.
Since $f(\sqrt{3}/3, \sqrt{3}/3, \sqrt{3}/3) = 3$ and $f(1/2, 1/3, 1) = \dfrac{216}{1393}(\sqrt{2} + \sqrt{3} + \sqrt{6}) = 0.867$, we conjecture that if $a^{-1} + b^{-1} + c^{-1} = (abc)^{-1}$, then

$$\frac{a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}}{a^4 + b^4 + c^4} \le 3.$$

Indeed, taking into account the well-known inequality $a^2 + b^2 + c^2 \ge ab + bc + ca$, we get $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) \ge 3(ab + bc + ca)$. On the other hand, using AM-GM inequality two times, we have

$$\begin{aligned}(a+b+c)^2 &= (a^2+b^2+c^2+ab+bc+ca)+\frac{1}{2}(2ab+2bc+2ca)\\ &= (a^2+bc)+(b^2+ca)+(c^2+ab)\\ &+ \frac{1}{2}[(ab+ca)+(bc+ab)+(ca+bc)]\\ &\geq 2a\sqrt{bc}+2b\sqrt{ca}+2c\sqrt{ab}\\ &+ \frac{1}{2}\Big(2a\sqrt{bc}+2b\sqrt{ca}+2c\sqrt{ab}\Big)\\ &= 3\Big(a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\Big).\end{aligned}$$

Multiplying up the preceding inequalities and taking into account the constrain, that can be written as $ab+bc+ca=1$, yields

$$(a+b+c)^4 \geq 9(ab+bc+ca)\Big(a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\Big) = 9\left(a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\right)$$

or equivalently,

$$9\Big(\frac{a+b+c}{3}\Big)^4 \geq \left(a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\right).$$

Since $\dfrac{a+b+c}{3} \leq \sqrt[4]{\dfrac{a^4+b^4+c^4}{3}}$, on account of mean inequalities, then

$$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab} \leq 9\Big(\frac{a+b+c}{3}\Big)^4 \leq 3\left(a^4+b^4+c^4\right)$$

from which the claim follows. Equality holds when $a=b=c=\sqrt{3}/3$, and we are done.

Also solved by *Titu Zvonaru, Comăneşti, Romania*

MH–129. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let $1=d_1<d_2<\cdots<d_k=n$ be all divisors of a positive integer n. Find all n, such that $k\geq 6$ and

$$\frac{{d_6}^2+2024}{5d_4}=n.$$

Solution 1 by the proposer. Since $n = d_6 d_{k-5}$,

$$d_6{}^2 + 2024 = 5d_4 n = 5d_4 d_6 d_{k-5}$$

and $d_6 \mid 2024 = 2^3 \cdot 11 \cdot 23$.

From $5 \mid d_6{}^2 + 2024$ hence $0 \equiv d_6{}^2 + 2024 \equiv d_6{}^2 - 1 \pmod 5$, or $d_6 \equiv \pm 1 \pmod 5$.

Since $d_4 \geq d_2 + 2 \geq 4$ and $d_6 \geq d_4 + 2 \geq 6$, we have the following cases:

(1) $d_6 = 11$

$$n = \frac{d_6{}^2 + 2024}{5d_4} = \frac{2145}{5d_4} = \frac{429}{d_4}$$

$d_4 \mid 429 = 3 \cdot 11 \cdot 13$, and $d_4 < d_6 = 11$, hence $d_4 = 3 < 4$, a contradiction.

(2) $d_6 = 44$

$$n = \frac{d_6{}^2 + 2024}{5d_4} = \frac{3960}{5d_4} = \frac{792}{d_4}$$

$d_4 \mid 792 = 2^3 \cdot 3^2 \cdot 11$, and so $d_2 = 2$, $d_3 = 3$, $n \geq 44$, hence $d_4 \leq 18$ and $d_4 \in \{4, 6, 8, 9, 12, 18\}$. Since d_4 is either a prime or a product of two primes, d_4 is impossible to be one of $8 = 2^3$, $12 = 2^2 \cdot 3$, $18 = 2 \cdot 3^2$.

d_4	n	
4	198	$4 \nmid n$
6	132	$d_6 = 11 \neq 44$
9	88	$9 \nmid n$

and no solution in this case.

(3) $d_6 = 46$, hence $d_2 = 2$, $d_5 \leq 23$,

$$n = \frac{d_6{}^2 + 2024}{5d_4} = \frac{4140}{5d_4} = \frac{828}{d_4}$$

$d_4 \mid 828 = 2^2 \cdot 3^2 \cdot 23$, and so $d_2 = 2$, $d_3 = 3$. $n \geq 46$, hence $d_4 \leq 18$ and the possible values for d_4 are $6, 9, 12$. Since d_4 is either a prime or a product of two primes, d_4 is impossible to be $12 = 2^2 \cdot 3$.

If $d_4 = 6$, then $n = 138$ and this is a solution.

If $d_4 = 9$, then $n = 92$ but $9 \nmid n$, a contradiction.

(4) $d_6 = 184 = 2^3 \cdot 23$, hence $d_2 = 2, d_3 \leq 4, d_4 \leq 8, d_5 \leq 23, d_6 \leq 46$, but this is impossible since $d_6 = 184$.

(5) $d_6 = 506 = 2 \cdot 11 \cdot 23$, hence $d_2 = 2, d_3 \leq 11, d_4 \leq 22, d_5 \leq 23, d_6 \leq 46$, but this is impossible since $d_6 = 506$.

(6) $d_6 = 2024 = 2^3 \cdot 11 \cdot 23$, hence $d_2 = 2, d_3 \leq 4, d_4 \leq 8, d_5 \leq 11, d_6 \leq 22$, but this is impossible since $d_6 = 2024$.

In summary, the only solution is $n = 138$.

Solution 2 by Albert Stadler, Herrliberg, Switzerland. We note that

$$d_6\left(5\frac{n}{d_6}d_4 - d_6\right) = 2024.$$

So d_6 is a divisor of 2024 and therefore $d_6 \in \{1, 2, 4, 8, 11, 22, 23, 44, 46, 88, 92, 184, 253, 506, 1012, 2024\}$. Clearly, $d_4 \geq 4$, $d_6 \geq 6$.

$d_6^2 + 2024$ is divisible by 5 which implies that either $d_6 \equiv (1 \bmod 5)$ or $d_6 \equiv (4 \bmod 5)$. This reduces the set of feasible values of d_6 to $\{11, 44, 46, 184, 506, 2024\}$. We are left with the following cases:

1. d_6=11: $n\,d_4 = 429 = 3 \cdot 11 \cdot 13$
2. d_6=44: $n\,d_4 = 792 = 2^3 \cdot 3^2 \cdot 11$
3. d_6=46: $n\,d_4 = 828 = 2^2 \cdot 3^2 \cdot 23$
4. d_6=184: $n\,d_4 = 7176 = 2^3 \cdot 3 \cdot 13 \cdot 23$
5. d_6=506: $n\,d_4 = 51612 = 2^2 \cdot 3 \cdot 11 \cdot 17 \cdot 23$
6. d_6=2024: $n\,d_4 = 819720 = 2^3 \cdot 3^4 \cdot 5 \cdot 11 \cdot 23$

We have $n\,d_4 = (d_4)^2 \frac{n}{d_4}$. It follows that $n\,d_4$ is not square free and so case 1 is not feasible. n is divisible by 2 and 3, since $d_4 n$ is and d_4 is a divisor of n. So d_1=1, d_2=2, d_3=3. We have d_4=4 or d_4=5 or d_4=6. d_4=4 is absurd, since $d_4 n$ is not divisible by 16. In particular, n is not divisible by 4. d_4=5 is absurd, since $d_4 n$ is not divisible by 25. So d_4=6, d_4n is divisible by 36, and n=6m and m is odd. So cases 2, 4, 5, 6 are not feasible. We are left with case 3, and the only feasible n is $n = 138$. For $n = 138$ we have d_1=1, d_2=2, d_3=3, d_4=6, d_5=23, d_6=46, d_7=69, d_8=138 and $\frac{d_6^2+2024}{5d_4} = n$.

Solution 3 by Titu Zvonaru, Comăneşti, Romania. We have $d_4 \geq 4$, $d_6 \geq 6$. Since n is divisible by d_6, by the relation

$$d_6^2 + 2024 = 5d_4n \tag{1}$$

we deduce that d_6 is a divisor of $2024 = 2^3 \cdot 11 \cdot 23$. Since any divisor of d_6 is a divisor of n, it follows that d_6 has at most 5 divisors. This gives the possibilities for d_6:

$$d_6 = 8, 11, 23, 2 \cdot 11, 2 \cdot 23, 11 \cdot 23.$$

The left side of equation (1) is divisible by 5; hence, the possible values for d_6 are 11 and $2 \cdot 23$.

Let $n = d_6a$. For $d_6 = 11$, we obtain

$$121 + 2024 = 5d_4n \quad \Leftrightarrow \quad 429 = d_4d_6a \quad \Leftrightarrow \quad 39 = d_4a,$$

which gives $d_4 = 13$ or $d_4 = 39$, a contradiction with $d_6 = 11$.

For $d_6 = 2 \cdot 23$, we obtain

$$2116 + 2024 = 5d_4n \quad \Leftrightarrow \quad 828 = d_4d_6a \quad \Leftrightarrow \quad 18 = d_4a,$$

which gives the pairs $(d_4, a) = (6, 3), (9, 2), (18, 1)$. If $d_4 = 9$, $a = 2$ or $d_4 = 18$, $a = 1$, it follows that n is not divisible by 3, which is a contradiction.

Thus, we conclude that $n = 2 \cdot 3 \cdot 23 = 138$, with divisors

$$d_1 = 1,\ d_2 = 2,\ d_3 = 3,\ d_4 = 6,\ d_5 = 23,\ d_6 = 46,\ d_7 = 69,\ d_8 = 138.$$

Also solved by *José Luis Díaz Barrero, Barcelona, Spain.*

MH–130. *Proposed by Ander Lamaison Vidarte, Brno, Czech Republic.* An increasing sequence of positive integers $a_1 < a_2 < \cdots < a_n$ is **requenense** if for every $2 \le k \le n-1$ we have that $a_{k-1}a_{k+1}$ divides a_k^4.

- Prove that there exists a requenense sequence of length 10^6+1 with $a_{10^6} = 6^{2024}$.
- Prove that there does not exist a requenense sequence of length $10^8 + 1$ with $a_{10^8} = 6^{2024}$.

Solution by the proposer. We start by noticing that, in both cases, every term a_k in the sequence must be of the form 2^b3^c. Indeed, this is true for $k = 10^6$ or $k = 10^8$. On the other hand, by the requenense condition, if it holds for some k which is neither the first or the last then $a_{k-1}a_{k+1}$ divides a_k^4, implying that both a_{k-1} and a_{k+1} are of the form 2^b3^c.

For the first part, we look at the numbers of the form 2^b3^c with $1025 \le b, c \le 2024$. There are 10^6 numbers of this form, and the largest is 6^{2024}. If we take any three of them, say $2^{b_1}3^{c_1}$, $2^{b_2}3^{c_2}$ and $2^{b_3}3^{c_3}$, then $b_1 + b_2 \le 2 \cdot 2024 < 4 \cdot 1025 \le 4b_3$, and similarly $c_1 + c_2 \le 4c_3$. Thus if we take $a_1 < a_2 < \cdots < a_{10^6}$ to be these 10^6 numbers in increasing order, the sequence will be requenense. In order to reach length $10^6 + 1$ we just need to choose $a_{10^6+1} > 6^{2024}$ such that

$$\frac{6^{2024}}{2}\, a_{10^6+1} = a_{10^6-1}a_{10^6+1}$$

divides $6^{4\cdot 2024}$. We can choose for example $a_{10^6+1} = 2 \cdot 6^{2024}$.

For the second part, we just need to show that there are fewer than 10^8 numbers of the form 2^b3^c less than or equal to 6^{2024}. If $2^b3^c \le 6^{2024}$, then $2^b \le 6^{2024} < 8^{2024} = 2^{3\cdot 2024}$ and $3^c \le 6^{2024} < 9^{2024} = 3^{2\cdot 2024}$, so $b < 3 \cdot 2024$ and $c < 2 \cdot 2024$. The number of possible pairs (b, c) is at most $6 \cdot 2024^2 < 10 \cdot 3000^2 = 9 \cdot 10^7 < 10^8$.

Also solved by *José Luis Díaz Barrero, Barcelona, Spain.*

Advanced Problems

A–125. *Proposed by José Luis Díaz Barrero, Barcelona, Spain.*
Compute

$$\lim_{n\to\infty}\frac{1}{n^n}\prod_{k=1}^{n}\left(\frac{n\sqrt{n}+(n+1)\sqrt{k}}{\sqrt{n}+\sqrt{k}}\right).$$

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University Newport News, VA, USA. Let

$$\begin{aligned} P_n &= \frac{1}{n^n}\prod_{k=1}^{n}\left(\frac{n\sqrt{n}+(n+1)\sqrt{k}}{\sqrt{n}+\sqrt{k}}\right)=\prod_{k=1}^{n}\left(\frac{\sqrt{n}+(1+\frac{1}{n})\sqrt{k}}{\sqrt{n}+\sqrt{k}}\right)\\ &= \prod_{k=1}^{n}\left(1+\frac{\sqrt{k}}{n(\sqrt{n}+\sqrt{k})}\right)=\prod_{k=1}^{n}\left(1+\frac{\sqrt{\frac{k}{n}}}{n\left(1+\sqrt{\frac{k}{n}}\right)}\right). \end{aligned}$$

Then

$$\begin{aligned} \ln P_n &= \sum_{k=1}^{n}\ln\left(1+\frac{\sqrt{\frac{k}{n}}}{n\left(1+\sqrt{\frac{k}{n}}\right)}\right)\\ &= \sum_{k=1}^{n}\frac{\sqrt{\frac{k}{n}}}{n\left(1+\sqrt{\frac{k}{n}}\right)}+\sum_{j=2}^{\infty}(-1)^{j-1}\frac{1}{j}\sum_{k=1}^{n}\left(\frac{\sqrt{\frac{k}{n}}}{n\left(1+\sqrt{\frac{k}{n}}\right)}\right)^{j}. \end{aligned}$$

As $n\to\infty$,

$$\begin{aligned} \sum_{k=1}^{n}\frac{\sqrt{\frac{k}{n}}}{n\left(1+\sqrt{\frac{k}{n}}\right)} &\to \int_0^1\frac{\sqrt{x}}{1+\sqrt{x}}\,dx=2\int_0^1\frac{u^2}{1+u}\,du\\ &= 2\int_0^1\left(u-1+\frac{1}{1+u}\right)du=2\left(\frac{1}{2}u^2-u+\ln(1+u)\right)\Bigg|_0^1\\ &= -1+\ln 4. \end{aligned}$$

Now, for $j\geq 2$,

$$\sum_{k=1}^{n}\left(\frac{\sqrt{\frac{k}{n}}}{n\left(1+\sqrt{\frac{k}{n}}\right)}\right)^{j}\to\frac{1}{n^{j-1}}\int_0^1\left(\frac{\sqrt{x}}{1+\sqrt{x}}\right)^{j}dx\to 0$$

as $n \to \infty$. Thus,

$$\lim_{n\to\infty} \ln P_n = -1 + \ln 4,$$

and

$$\lim_{n\to\infty} P_n = \exp(-1 + \ln 4) = \frac{4}{e}.$$

Solution 2 by Michel Bataille, Rouen, France. We claim that the limit is $\frac{4}{e}$.

We have $\frac{n\sqrt{n}+(n+1)\sqrt{k}}{\sqrt{n}+\sqrt{k}} = n\left(1 + \frac{1}{n} \cdot \frac{\sqrt{k/n}}{1+\sqrt{k/n}}\right)$, hence the problem reduces to finding the limit of

$$P_n = \prod_{k=1}^{n} \left(1 + \frac{1}{n} \cdot \frac{\sqrt{k/n}}{1 + \sqrt{k/n}}\right).$$

We know that $x - \frac{x^2}{2} \leq \ln(1+x) \leq x$ for $x > 0$ and we deduce

$$\frac{1}{n}\sum_{k=1}^{n} \frac{\sqrt{k/n}}{1+\sqrt{k/n}} - \frac{1}{2n^2}\sum_{k=1}^{n}\left(\frac{\sqrt{k/n}}{1+\sqrt{k/n}}\right)^2 \leq \ln(P_n) \leq \frac{1}{n}\sum_{k=1}^{n} \frac{\sqrt{k/n}}{1+\sqrt{k/n}}.$$

Now, we have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{\sqrt{k/n}}{1+\sqrt{k/n}} &= \int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx \\ &= 2\int_0^1 \frac{u^2}{1+u}\,du = 2\int_0^1 \left(u - 1 + \frac{1}{1+u}\right) du \\ &= 2\left(-\frac{1}{2} + \ln 2\right) = \ln(4/e).\end{aligned}$$

Also, since $\frac{k/n}{(1+\sqrt{k/n})^2} \leq 1$, we have $\sum_{k=1}^{n}\left(\frac{\sqrt{k/n}}{1+\sqrt{k/n}}\right)^2 \leq n$ and therefore

$$0 \leq \frac{1}{2n^2}\sum_{k=1}^{n}\left(\frac{\sqrt{k/n}}{1+\sqrt{k/n}}\right)^2 \leq \frac{1}{2n}.$$

By the Squeeze Principle, we first deduce that

$$\lim_{n\to\infty} \frac{1}{2n^2} \sum_{k=1}^{n} \left(\frac{\sqrt{k/n}}{1+\sqrt{k/n}} \right)^2 = 0$$

and then that

$$\lim_{n\to\infty} \ln(P_n) = \ln(4/e).$$

The claim follows.

Solution 3 by Albert Stadler, Herrliberg, Switzerland. We have

$$\frac{1}{n^n} \prod_{k=1}^{n} \left(\frac{n\sqrt{n}+(n+1)\sqrt{k}}{\sqrt{n}+\sqrt{k}} \right) = \prod_{k=1}^{n} \left(\frac{1+\left(1+\frac{1}{n}\right)\sqrt{\frac{k}{n}}}{1+\sqrt{\frac{k}{n}}} \right)$$

$$= \prod_{k=1}^{n} \left(1 + \frac{\sqrt{\frac{k}{n}}}{n\left(1+\sqrt{\frac{k}{n}}\right)} \right)$$

$$= \exp\left(\sum_{k=1}^{n} \ln\left(1 + \frac{\sqrt{\frac{k}{n}}}{n\left(1+\sqrt{\frac{k}{n}}\right)} \right) \right) = \exp\left(\sum_{k=1}^{n} \frac{\sqrt{\frac{k}{n}}}{n\left(1+\sqrt{\frac{k}{n}}\right)} + \sum_{k=1}^{n} O\left(\frac{1}{n^2}\right) \right).$$

We note that $\sum_{k=1}^{n} \frac{\sqrt{\frac{k}{n}}}{n\left(1+\sqrt{\frac{k}{n}}\right)}$ is a Riemann sum that tends to

$$\int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}dx = \int_0^1 \frac{2y^2}{1+y}dy = 2\int_0^1 \left(y-1+\frac{1}{1+y}\right)dy = -1+2\ln 2 .$$

So

$$\lim_{n\to\infty} \frac{1}{n^n} \prod_{k=1}^{n} \left(\frac{n\sqrt{n}+(n+1)\sqrt{k}}{\sqrt{n}+\sqrt{k}} \right) = \frac{4}{e}.$$

Solution 4 by the proposer. We begin with a lemma.

Lemma 1. *Let $f : [0,1] \to (0,+\infty)$ be an integrable function. Then,*

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left[1 + \frac{1}{n} f\left(\frac{k}{n}\right) \right] = e^{\int_0^1 f(x)\,dx}.$$

Proof. Since f is integrable, then f is bounded. That is, there exists $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in [0,1]$. Now, calling $L = \lim_{n\to\infty} \prod_{k=1}^{n} \left[1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right]$, we have $\ln L = \sum_{k=1}^{n} \ln\left[1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right]$. Putting $x = \frac{1}{n} f\left(\frac{k}{n}\right)$, $1 \leq k \leq n$ into the well-known inequality

$$x - \frac{x^2}{2} \leq \ln(1+x) \leq x, \forall x \in [0,1]$$

and adding the resulting inequalities, yields

$$\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) - \sum_{k=1}^{n} \frac{1}{2n^2} f^2\left(\frac{k}{n}\right) \leq \sum_{k=1}^{n} \ln\left[1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right] \leq \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right).$$

Taking limits when $n \to \infty$, we get

$$\int_0^1 f(x)\,dx - \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2n^2} f^2\left(\frac{k}{n}\right) \leq \lim_{n\to\infty} \sum_{k=1}^{n} \ln\left[1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right] \leq \int_0^1 f(x)\,dx.$$

Since $0 \leq \sum_{k=1}^{n} \frac{1}{2n^2} f^2\left(\frac{k}{n}\right) \leq \frac{M^2}{2n}$, then when $n \to \infty$ is

$$0 \leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2n^2} f^2\left(\frac{k}{n}\right) \leq \lim_{n\to\infty} \frac{M^2}{2n} = 0$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left[1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right] = \int_0^1 f(x)\,dx$$

from which the statement follows and the proof is complete. □

Applying the Lemma to the function $f(x) = \frac{\sqrt{x}}{1+\sqrt{x}}$, we have

$$\lim_{n\to\infty} \frac{1}{n^n} \prod_{k=1}^{n} \left(\frac{n\sqrt{n} + (n+1)\sqrt{k}}{\sqrt{n}+\sqrt{k}}\right) = \frac{1}{n^n} \lim_{n\to\infty} \prod_{k=1}^{n} \left(n + \frac{\sqrt{k}}{\sqrt{n}+\sqrt{k}}\right)$$

$$= \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{1}{n} \frac{\sqrt{k}}{\sqrt{n}+\sqrt{k}}\right) = \lim_{n\to\infty} \prod_{k=1}^{n} \left[1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right]$$

$$= \exp\left(\int_0^1 \frac{\sqrt{x}}{1+\sqrt{x}}\,dx\right) = \exp\left(\left[x - 2\sqrt{x} + 2\ln(1+\sqrt{x})\right]_0^1\right)$$

$$= e^{2\ln 2 - 1} = \ln\frac{4}{e} \approx 1.4715.$$

Solution 5 by Moti Levy, Rehovot, Israel. Let

$$L := \frac{1}{n^n}\prod_{k=1}^{n}\left(\frac{n\sqrt{n}+(n+1)\sqrt{k}}{\sqrt{n}+\sqrt{k}}\right) = \prod_{k=1}^{n}\left(\frac{1+\sqrt{\frac{k}{n}}+\frac{1}{n}\sqrt{\frac{k}{n}}}{1+\sqrt{\frac{k}{n}}}\right), \quad (1)$$

$$\ln(L) = \sum_{k=1}^{n}\left(\ln\left(1+\sqrt{\frac{k}{n}}+\frac{1}{n}\sqrt{\frac{k}{n}}\right) - \ln\left(1+\sqrt{\frac{k}{n}}\right)\right). \quad (2)$$

By Taylor's theorem,

$$\ln(1+y+h) = \ln(1+y) + h\frac{1}{1+y} - \frac{h^2}{2}\frac{1}{(1+\eta)^2}, \quad 1+y \le \eta \le 1+y+h. \quad (3)$$

Setting $y = \sqrt{\frac{k}{n}}$ and $h = \frac{1}{n}\sqrt{\frac{k}{n}}$ in (3),

$$\ln\left(1+\sqrt{\frac{k}{n}}+\frac{1}{n}\sqrt{\frac{k}{n}}\right) = \ln\left(1+\sqrt{\frac{k}{n}}\right) + \frac{1}{n}\sqrt{\frac{k}{n}}\frac{1}{1+\sqrt{\frac{k}{n}}} - \left(\frac{1}{n}\sqrt{\frac{k}{n}}\right)^2\frac{1}{(1+\eta)^2},$$

$$1+\sqrt{\frac{k}{n}} \le \eta \le 1+\sqrt{\frac{k}{n}}+\frac{1}{n}\sqrt{\frac{k}{n}}. \quad (4)$$

$$\ln\left(1+\sqrt{\frac{k}{n}}+\frac{1}{n}\sqrt{\frac{k}{n}}\right) - \ln\left(1+\sqrt{\frac{k}{n}}\right) = \frac{1}{n}\frac{\sqrt{\frac{k}{n}}-1}{\frac{k}{n}-1}\sqrt{\frac{k}{n}} + kO\left(\frac{1}{n^3}\right).$$

$$\begin{aligned}\ln(L) &= \sum_{k=1}^{n}\left(\ln\left(1+\sqrt{\frac{k}{n}}+\frac{1}{n}\sqrt{\frac{k}{n}}\right) - \ln\left(1+\sqrt{\frac{k}{n}}\right)\right)\\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{\sqrt{\frac{k}{n}}-1}{\frac{k}{n}-1}\sqrt{\frac{k}{n}} + \sum_{k=1}^{n}kO\left(\frac{1}{n^3}\right)\\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{\sqrt{\frac{k}{n}}-1}{\frac{k}{n}-1}\sqrt{\frac{k}{n}} + O\left(\frac{1}{n}\right).\end{aligned}$$

$$\lim_{n\to\infty}\ln(L)=\lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{\left(\sqrt{\frac{k}{n}}-1\right)\sqrt{\frac{k}{n}}}{\frac{k}{n}-1}+O\left(\frac{1}{n}\right)\right)=\int_0^1\frac{(\sqrt{x}-1)\sqrt{x}}{x-1}dx$$
$$=-1+2\ln(2).$$
$$\lim_{n\to\infty}L=e^{-1+2\ln(2)}=\frac{4}{e}\cong 1.4715.$$

Also solved by *Shivam Sharma, Delhi University, New Delhi, India.*

A–126. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.* Consider in $\mathbb{R}^3$ a sphere and one of its equatorial planes. Find the geometric locus of the vertices of the cones circumscribed to the sphere and whose trace on the equatorial plane is a parabola.

Solution 1 by Michel Bataille, Rouen, France. Let $\mathcal{S}$ and $\mathcal{P}$ be the given sphere and equatorial plane, respectively. Without loss of generality, we suppose that their equations are $x^2+y^2+z^2=R^2$ (with $R>0$) and $z=0$, respectively. We consider a cone $\mathcal{C}$ with vertex $\Omega(a,b,c)$ circumscribed to $\mathcal{S}$. Note that Ω is exterior to $\mathcal{S}$ so we must have $a^2+b^2+c^2>R^2$. The equation of $\mathcal{C}$ is

$$[a(x-a)+b(y-b)+c(z-c)]^2=m[(x-a)^2+(y-b)^2+(z-c)^2].$$

where $m=a^2+b^2+c^2-R^2$.
(This equation is obtained by writing that $M(x,y,z)$ with $M\neq\Omega$ is on $\mathcal{C}$ if and only if the line of points

$$(a+t(x-a),b+t(y-b),c+t(z-c))\quad(t\in\mathbb{R})$$

is tangent to $\mathcal{S}$, that is, if and only if the quadratic equation

$$t^2[(x-a)^2+(y-b)^2+(z-c)^2]+2t[a(x-a)+b(y-b)+c(z-c)]+m=0$$

has a double solution for t.)
Now, a point Ω, exterior to $\mathcal{S}$, is a point of the required locus if and only if $\Omega\notin\mathcal{P}$ and $\mathcal{P}$ is parallel to a generatrix of $\mathcal{C}$. Since the generatrix through $M(x,y,z)\in\mathcal{C}$ is directed by the vector $(x-a,y-b,z-c)$ this is achieved if and only if $z=c\neq 0$. Thus,

Ω is suitable if and only if $c \neq 0$ and for some x, y such that $(x-a, y-b) \neq (0,0)$ we have

$$[a(x-a)+b(y-b)]^2 = m[(x-a)^2+(y-b)^2]$$

or

$$(b^2+c^2-R^2)(x-a)^2-2ab(x-a)(y-b)+(a^2+c^2-R^2)(y-b)^2 = 0.$$

But if $\alpha, \beta, \gamma \in \mathbb{R}$, then $\alpha X^2 + 2\beta XY + \gamma Y^2 = 0$ for some real pair $(X, Y) \neq (0,0)$ if and only if $\beta^2 \geq \alpha\gamma$. Thus, the conditions for Ω to belong to the locus are $a^2+b^2+c^2 > R^2, c \neq 0$ and $a^2b^2 \geq (b^2+c^2-R^2)(a^2+c^2-R^2)$. The latter rewriting as $(c^2-R^2)(a^2+b^2+c^2-R^2) \leq 0$, we finally get the conditions $a^2+b^2+c^2 > R^2, c \neq 0, -R \leq c \leq R$ and we conclude that the required locus is the set of points exterior to $\mathcal{S}$, lying on one of the planes strictly parallel to $\mathcal{P}$, between (and including) those tangent to $\mathcal{S}$.

Solution 2 by the proposer. We consider a rectangular coordinate system with the unit of measurement the same along the three axes.

Let r denote the radius of the given sphere, place its centre at $(0,0,0)$ and the simpler equation

$$x^2+y^2+z^2 = r^2 \tag{1}$$

now represents the sphere with radius r.

Le us choose the xy-plane as the considered equatorial plane.

Let $P(x_\circ, y_\circ, z_\circ)$ be a point of the required locus. The circumscribed cone with vertex at P is the locus of the straight lines

$$\ell : \begin{cases} x &= x_\circ + \lambda m \\ y &= y_\circ + \lambda n \\ z &= z_\circ + \lambda p \end{cases} \tag{2}$$

which cut the sphere in coincident points.

Without lost of generality, we suppose $\overrightarrow{v} = (m, n, p)$ to be a unit vector, i.e.,

$$m^2 + n^2 + p^2 = 1. \tag{3}$$

Solving (1) and (2) simultaneously gives

$$(x_\circ + \lambda m)^2 + (y_\circ + \lambda n)^2 + (z_\circ + \lambda p)^2 = r^2.$$

Expanding and collecting terms,

$$\left(m^2 + n^2 + p^2\right)\lambda^2 + 2(mx_\circ + ny_\circ + pz_\circ)\lambda + x_\circ^2 + y_\circ^2 + z_\circ^2 - r^2 = 0$$

Substituting from (3) for $m^2 + n^2 + p^2$, we obtain

$$\lambda^2 + 2(mx_\circ + ny_\circ + pz_\circ)\lambda + x_\circ^2 + y_\circ^2 + z_\circ^2 - r^2 = 0. \tag{4}$$

Since the lines ℓ above are tangents to the sphere, the roots of (3) must be equal, that is, the discriminant must be equal to zero. Hence,

$$(mx_\circ + ny_\circ + pz_\circ)^2 - x_\circ^2 - y_\circ^2 - z_\circ^2 + r^2 = 0. \tag{5}$$

In (3) we substitute $\frac{x-x_\circ}{\lambda}$, $\frac{y-y_\circ}{\lambda}$, $\frac{z-z_\circ}{\lambda}$ for m, n, p from (2), obtaining

$$\lambda^2 = (x - x_\circ)^2 + (y - y_\circ)^2 + (z - z_\circ)^2.$$

Likewise, we obtain

$$\begin{aligned} & (mx_\circ + ny_\circ + pz_\circ)^2 \\ = \ & \left(\tfrac{x-x_\circ}{\lambda}x_\circ + \tfrac{y-y_\circ}{\lambda}y_\circ + \tfrac{z-z_\circ}{\lambda}z_\circ\right)^2 \\ = \ & \tfrac{[(x-x_\circ)x_\circ+(y-y_\circ)y_\circ+(z-z_\circ)z_\circ]^2}{\lambda^2}. \end{aligned}$$

The elimination of λ^2 yields the equation

$$(mx_\circ + ny_\circ + pz_\circ)^2 = \frac{[(x - x_\circ)x_\circ + (y - y_\circ)y_\circ + (z - z_\circ)z_\circ]^2}{(x - x_\circ)^2 + (y - y_\circ)^2 + (z - z_\circ)^2},$$

so that (5) now becomes

$$\frac{\left[(x-x_\circ)x_\circ+(y-y_\circ)y_\circ+(z-z_\circ)z_\circ\right]^2}{(x-x_\circ)^2+(y-y_\circ)^2+(z-z_\circ)^2}-x_\circ^2-y_\circ^2-z_\circ^2+r^2=0,$$

which is the equation of the circumscribed cone, whose intersection with the equatorial plane $z=0$ is

$$\frac{\left[(x-x_\circ)x_\circ+(y-y_\circ)y_\circ-z_\circ^2\right]^2}{(x-x_\circ)^2+(y-y_\circ)^2+z_\circ^2}-x_\circ^2-y_\circ^2-z_\circ^2+r^2=0.$$

We rewrite the last equation as the conic

$$\begin{aligned}\left(r^2-y_\circ^2-z_\circ^2\right)x^2 \;&-\; 2x_\circ y_\circ xy+\left(r^2-x_\circ^2-z_\circ^2\right)y^2\\ &-\; 2\left(2x_\circ^3+2x_\circ y_\circ^2+2x_\circ z_\circ^2-x_\circ r^2\right)x\\ &-\; 2\left(2y_\circ^3+2y_\circ x_\circ^2+2y_\circ z_\circ^2-y_\circ r^2\right)y\\ &+\; \left(z_\circ^4-r^2\left(x_\circ^2+y_\circ^2+z_\circ^2\right)\right)=0.\end{aligned}\tag{6}$$

Hence in order for (6) represents a parabola, we require (†) that

$$\begin{vmatrix} r^2-y_\circ^2-z_\circ^2 & -x_\circ y_\circ\\ -x_\circ y_\circ & r^2-x_\circ^2-z_\circ^2\end{vmatrix}=\left(z_\circ^2-r^2\right)\left(x_\circ^2+y_\circ^2+z_\circ^2-r^2\right)=0$$

This is satisfied when

$x_\circ^2+y_\circ^2+z_\circ^2-r^2=0$ i.e., when P lies on the sphere and then the cone is degenerate; or $z_\circ^2-r^2=0$.

The locus of P is obtained by allowing the coordinates $x_\circ$, $y_\circ$, $z_\circ$ to become variable; this is achieved by omiting the suffix $\circ$.

We then find that the required locus is given by

$$z^2-r^2=0,\quad \text{except the points } (0,0,\pm r).$$

(†) When

$$\begin{vmatrix} A & B \\ B & C \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix} \neq 0,$$

the general quadratic equation $Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0$ represents a parabola.

A–127. *Proposed by Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA.* Let the function $f(x)$ have a continuous second derivative on $[-a, a]$. Prove that if $f(0) = 0$, there exists $\xi \in (-a, a)$ such that $f''(\xi) = (1/a^2)[f(a) + f(-a)]$.

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University Newport News, VA, USA. Let f be a function with a continuous second derivative on $[-a, a]]$ with $f(0) = 0$. By Taylor's theorem, there exists $\xi_+ \in (0, a)$ and $\xi_- \in (-a, 0)$ such that

$$f(a) = f'(0)a + \frac{f''(\xi_+)}{2}a^2$$

and

$$f(-a) = -f'(0)a + \frac{f''(\xi_-)}{2}a^2.$$

Adding these two equations yields

$$f(a) + f(-a) = \frac{a^2}{2}(f''(\xi_+) + f''(\xi_-))$$

or

$$\frac{f(a) + f(-a)}{a^2} = \frac{1}{2}(f''(\xi_+) + f''(\xi_-)).$$

Next, because the second derivative is continuous on $[-a, a]$, by the Extreme Value Theorem, there exist real numbers m and M such that

$$m \leq f''(x) \leq M$$

for all $x \in [-a, a]$. In particular

$$m \le f''(\xi_+) \le M \quad \text{and} \quad m \le f''(\xi_-) \le M.$$

Add these two equations and divide by two to obtain

$$m \le \frac{1}{2}(f''(\xi_+) + f''(\xi_-)) \le M.$$

Now, by the Intermediate Value Theorem, there exists $\xi \in (\xi_-, \xi_+) \subset (-a, a)$ such that

$$f''(\xi) = \frac{1}{2}(f''(\xi_+) + f''(\xi_-)).$$

Thus, there exists $\xi \in (-a, a)$ such that

$$f''(\xi) = \frac{f(a) + f(-a)}{a^2}.$$

Solution 2 by Michel Bataille, Rouen, France. By Taylor's Theorem and $f(0) = 0$, we have

$$f(a) = af'(0) + \frac{a^2}{2}f''(\alpha) \quad \text{and} \quad f(-a) = -af'(0) + \frac{a^2}{2}f''(\beta)$$

for some α, β with $\alpha \in (0, a), \beta \in (-a, 0)$.

We deduce that

$$\frac{f(a) + f(-a)}{a^2} = \frac{1}{2}(f''(\alpha) + f''(\beta)).$$

The function f'' is continuous on $[\beta, \alpha]$ and $\frac{1}{2}(f''(\alpha) + f''(\beta))$ is a real number between $f''(\alpha)$ and $f''(\beta)$, hence

$$\frac{1}{2}(f''(\alpha) + f''(\beta)) = f''(\xi)$$

for some $\xi \in [\beta, \alpha]$ (by the Intermediate Value Theorem). Since $[\beta, \alpha] \subset (-a, a)$, the conclusion follows.

Solution 3 by Moti Levy, Rehovot, Israel. By Taylor's theorem,

$$f(a) = f(0) + af^{'}(0) + \frac{a^2}{2} f^{''}(\xi_1), \quad 0 \leq \xi_1 \leq a, \tag{1}$$

$$f(-a) = f(0) - af^{'}(0) + \frac{a^2}{2} f^{''}(\xi_2), \quad -a \leq \xi_2 \leq 0. \tag{2}$$

Adding both sides of (1) and (2), we get,

$$f(a) + f(-a) = a^2 \frac{f^{''}(\xi_1) + f^{''}(\xi_2)}{2}. \tag{3}$$

Since the function $f(x)$ have a continuous second derivative on $[-a, a]$, and

$$f^{''}(\xi_1) \leq \frac{f^{''}(\xi_1) + f^{''}(\xi_2)}{2} \leq f^{''}(\xi_2),$$

or

$$f^{''}(\xi_2) \leq \frac{f^{''}(\xi_1) + f^{''}(\xi_2)}{2} \leq f^{''}(\xi_1),$$

then by the *intermediate value theorem* there is a point ξ, such that

$$\xi \in (-a, a), \quad \text{and} \quad f^{''}(\xi) = \frac{f^{''}(\xi_1) + f^{''}(\xi_2)}{2}.$$

Also solved by *Albert Stadler, Herrliberg, Switzerland, and the proposer.*

A–128. *Proposed by Vasile Mircea Popa, Lucian Blaga University of Sibiu, Romania.* Prove that it holds:

$$\int_0^\infty \frac{|\cos(x)|}{1+x^2} dx = \frac{e^2+1}{e} \arctan\left(\frac{1}{e}\right).$$

Solution 1 by Michel Bataille, Rouen, France. From a classical expansion in Fourier series (see **[1]**, p. 26, for example), we have

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\cos(2nx)}{4n^2-1} \qquad (x \in \mathbb{R}).$$

We deduce that

$$\int_0^\infty \frac{|\cos x|}{1+x^2}\,dx = \frac{2}{\pi}\int_0^\infty \frac{dx}{1+x^2} + \frac{4}{\pi}\int_0^\infty \left(\sum_{n=1}^\infty (-1)^{n+1}\frac{\cos(2nx)}{4n^2-1}\right)\frac{dx}{1+x^2}.$$

Since

$$\sum_{n=1}^\infty \int_0^\infty \left|(-1)^{n+1}\frac{\cos(2nx)}{(4n^2-1)(1+x^2)}\right| dx \le \frac{\pi}{2}\sum_{n=1}^\infty \frac{1}{4n^2-1} < \infty$$

we can interchange series and integral and obtain

$$\begin{aligned}\int_0^\infty \frac{|\cos x|}{1+x^2}\,dx &= 1 + \frac{4}{\pi}\sum_{n=1}^\infty \left(\frac{(-1)^{n+1}}{4n^2-1}\int_0^\infty \frac{\cos(2nx)}{1+x^2}\,x\right)\\ &= 1 + 2\sum_{n=1}^\infty \frac{(-1)^{n+1}e^{-2n}}{4n^2-1}.\end{aligned}$$

(We have used the well-known $\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$ and a widespread exercise that gives $\int_0^\infty \frac{\cos(ax)}{1+x^2}\,dx = \frac{\pi}{2}e^{-a}$ for $a > 0$ (see **[1]**, p. 193 or **[2]**, for example).)

Now, from $\frac{2}{4n^2-1} = \frac{1}{2n-1} - \frac{1}{2n+1}$ and $\arctan x = \sum_{n=1}^\infty \frac{(-1)^{n-1}x^{2n-1}}{2n-1}$ if $|x| < 1$, we deduce

$$\begin{aligned}\int_0^\infty \frac{|\cos x|}{1+x^2}\,dx &= 1 + \frac{1}{e}\sum_{n=1}^\infty \frac{(-1)^{n-1}(e^{-1})^{2n-1}}{2n-1} - (-e)\sum_{n=1}^\infty \frac{(-1)^{n+2}(e^{-1})^{2n+1}}{2n+1}\\ &= 1 + \frac{1}{e}\arctan(1/e) - (-e)(\arctan(1/e) - (1/e))\\ &= \left(\frac{1}{e} + e\right)\arctan\left(\frac{1}{e}\right),\end{aligned}$$

as desired.

[1] G. P. Tolstov, *Fourier Series*, Dover, 1962

[2] M. R. Spiegel, *Complex Variables*, McGraw Hill, 2009, Ex. 7-16, p. 219

Solution 2 by Albert Stadler, Herrliberg, Switzerland. We expand $|\cos(x)|$ into a Fourier series and obtain

$$|\cos(x)| = \frac{a_0}{2} + \sum_{n=1}^\infty a_n\cos(2nx)$$

with

$$a_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} |\cos(x)|\cos(2nx)\,dx = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos(x)\cos(2nx)\,dx$$
$$= \frac{-4(-1)^n}{\pi(4n^2-1)}.$$

The residue calculus gives

$$\int_0^{\infty} \frac{\cos(2nx)}{1+x^2}\,dx = \frac{1}{2}\int_0^{\infty} \frac{e^{2inx}+e^{-2inx}}{1+x^2}\,dx$$
$$= 2\pi i\left(\operatorname{res}\left(\frac{e^{2inx}}{2(1+x^2)}; x=i\right) - \operatorname{res}\left(\frac{e^{-2inx}}{2(1+x^2)}; x=-i\right)\right)$$
$$= \frac{\pi}{2}e^{-2n}, \quad n \geq 0.$$

Hence,

$$\int_0^{\infty} \frac{|\cos(x)|}{1+x^2}\,dx = 1 - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2-1}e^{-2n}$$
$$= 1 - \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)e^{-2n}$$
$$= -\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{2n+1}e^{-2(n+1)} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}e^{-2n}$$
$$= \left(\frac{1}{e^2}+1\right)e\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}e^{-2n-1}$$
$$= \frac{e^2+1}{e}\arctan\left(\frac{1}{e}\right).$$

Solution 3 by the proposer. Let us denote:

$$I = \int_0^{\infty} \frac{|\cos(x)|}{1+x^2}dx. \tag{1}$$

The function $f(x) = |\cos(x)|$ is periodic with period π and satisfies Dirichlet's conditions. Also, the function are even. We expand the function in the Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(2nx)$$

where:

$$a_0 = \frac{1}{\pi}\int_0^{\pi} |\cos(x)| dx;\ \ a_n = \frac{2}{\pi}\int_0^{\pi} |\cos(x)| \cos(2nx) dx.$$

Calculating these integrals, we obtain:

$$a_0 = \frac{2}{\pi}; a_n = -\frac{4}{\pi}\cdot\frac{(-1)^n}{4n^2-1}.$$

We have:

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\cos(2nx)dx.$$

Substituting $f(x)$ in the expression (1) of I, result:

$$I = \frac{2}{\pi}\int_0^{\infty}\frac{1}{1+x^2}dx - \frac{4}{\pi}\int_0^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^n\cos(2nx)}{(4n^2-1)(1+x^2)}dx.$$

So:

$$I = 1 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}\int_0^{\infty}\frac{\cos(2nx)}{1+x^2}dx.$$

We now use the following relationship:

$$\int_0^{\infty}\frac{\cos(mx)}{1+x^2}dx = \frac{\pi}{2}e^{-m},\ \text{where } m > 0.$$

This relation is Laplace's integral and is well known. It is easily proved for example using the properties of the Laplace transform. We obtained the value of the integral I:

$$I = 1 - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{(4n^2-1)e^{2n}}. \tag{2}$$

We will calculate the sum of the series:

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{(4n^2 - 1)e^{2n}}$$

We have:

$$\frac{(-1)^n}{(4n^2 - 1)e^{2n}} = \frac{1}{2}\left[\frac{(-1)^n}{(2n-1)e^{2n}} - \frac{(-1)^n}{(2n+1)e^{2n}}\right]$$

So:

$$S = \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n-1)e^{2n}} - \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)e^{2n}}\right] = \frac{1}{2}(A - B).$$

We calculate B:

$$B = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)e^{2n}}.$$

We have:

$$B = \sum_{n=0}^{\infty} \int_0^1 (-1)^n \frac{x^{2n}}{e^{2n}} dx = \int_0^1 \sum_{n=0}^{\infty} \left(-\frac{x^2}{e^2}\right)^n = \int_0^1 \frac{1}{1 + \frac{x^2}{e^2}} dx$$

(we have $0 < x < 1$ and $-1 < \dfrac{x^2}{e^2} < 0$).

We get immediately:

$$B = e \arctan\left(\frac{1}{e}\right).$$

Likewise it is calculated A and we obtain:

$$A = -1 - \frac{1}{e} \arctan\left(\frac{1}{e}\right).$$

Results:

$$S=\frac{1}{2}(A-B)=\frac{-e-\arctan\left(\frac{1}{e}\right)-e^2\arctan\left(\frac{1}{e}\right)}{2e}$$

We have:

$$S=\sum_{n=0}^{\infty}\frac{(-1)^n}{(4n^2-1)e^{2n}}=\sum_{n=1}^{\infty}\frac{(-1)^n}{(4n^2-1)e^{2n}}-1$$

Result:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(4n^2-1)e^{2n}}=S+1=\frac{e-\arctan\left(\frac{1}{e}\right)-e^2\arctan\left(\frac{1}{e}\right)}{2e}$$

Replacing in the relationship (2) we obtain:

$$\int_0^{\infty}\frac{\cos(x)}{1+x^2}dx=\frac{e^2+1}{e}\arctan\left(\frac{1}{e}\right).$$

Thus, the problem is solved.

Solution 4 by Moti Levy, Rehovot, Israel. Solution A: Using Fourier series:

The Fourier series of $|\cos(x)|$ in the interval $[-\pi,\pi]$ is

$$|\cos(x)|=\frac{2}{\pi}+\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{1-4m^2}\cos(2mx)$$

By interchanging the order of summation and integration,

$$I=\int_0^{\infty}\frac{|\cos(x)|}{1+x^2}dx=\frac{2}{\pi}\int_0^{\infty}\frac{1}{1+x^2}dx+\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m}{1-4m^2}\int_0^{\infty}\frac{\cos(2mx)}{1+x^2}dx$$

$$\int_0^{\infty}\frac{\cos(2mx)}{1+x^2}dx=\frac{\pi}{2}e^{-2m}.$$

$$I = \int_0^\infty \frac{|\cos(x)|}{1+x^2}dx = 1 + 2\sum_{m=1}^\infty \frac{(-1)^m}{1-4m^2}e^{-2m}$$

The Taylor series for $\arctan(z)$ is

$$\arctan(z) = \sum_{m=0}^\infty \frac{(-1)^m}{2m+1}z^{2m+1}, \quad |z| \leq 1.$$

It follows that

$$\sum_{m=0}^\infty \frac{(-1)^m}{2m+1}zx^m = \frac{\arctan(\sqrt{z})}{\sqrt{z}},$$
$$\sum_{m=0}^\infty \frac{(-1)^m}{2m-1}z^m = -1 - \sqrt{z}\arctan(\sqrt{z}).$$

The partial fractions of $\frac{(-1)^m}{2m-1}$ is

$$\frac{(-1)^m}{2m-1} = \frac{1}{2}\frac{(-1)^m}{2m+1} - \frac{1}{2}\frac{(-1)^m}{2m-1}.$$

It follows that

$$\sum_{m=1}^\infty \frac{(-1)^m}{1-4m^2}z^m = -\frac{1}{2} + \frac{(1+z)\arctan(\sqrt{z})}{\sqrt{z}}.$$

Setting $z = e^{-2}$, we get,

$$\int_0^\infty \frac{|\cos(x)|}{1+x^2}dx = \frac{(1+e^{-2})\arctan(e^{-1})}{e^{-1}} = \frac{e^2+1}{e}\arctan\left(\frac{1}{e}\right).$$

Solution B: By partitioning the integration interval:

Let

$$\begin{aligned}
I &:= \int_0^\infty \frac{|\cos(x)|}{1+x^2}dx = \sum_{k=0}^{\infty} \int_{k\pi}^{k\pi+\frac{\pi}{2}} \frac{\cos(x)}{1+x^2}dx - \int_{k\pi+\frac{\pi}{2}}^{(k+1)\pi} \frac{\cos(x)}{1+x^2}dx \\
&= \frac{1}{2}\left(\sum_{k=-\infty}^{\infty}\left(\int_{2k\pi+\frac{3\pi}{2}}^{2k\pi+\frac{5\pi}{2}} \frac{\cos(x)}{1+x^2}dx - \int_{2k\pi+\frac{\pi}{2}}^{2k\pi+\frac{3\pi}{2}} \frac{\cos(x)}{1+x^2}dx\right)\right) \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty}\left(\int_0^\pi \frac{\cos\left(y+2k\pi+\frac{3\pi}{2}\right)}{1+\left(y+2k\pi+\frac{3\pi}{2}\right)^2}dy - \int_0^\pi \frac{\cos\left(y+2k\pi+\frac{\pi}{2}\right)}{1+\left(y+2k\pi+\frac{\pi}{2}\right)^2}dy\right) \\
&= \frac{1}{2}\sum_{k=-\infty}^{\infty}\left(\int_0^\pi \frac{\sin(y)}{1+\left(y+2k\pi+\frac{3\pi}{2}\right)^2}dy + \int_0^\pi \frac{\sin(y)}{1+\left(y+2k\pi+\frac{\pi}{2}\right)^2}dy\right).
\end{aligned}$$

By interchanging the order of summation and integration,

$$\begin{aligned}
I &= \frac{1}{2}\int_0^\pi \sin(y)\sum_{k=-\infty}^{\infty}\left(\frac{1}{1+\left(y+2k\pi+\frac{3\pi}{2}\right)^2} + \frac{1}{1+\left(y+2k\pi+\frac{\pi}{2}\right)^2}\right)dy. \\
&= \frac{1}{2}\int_0^\pi \sin(y)\sum_{k=-\infty}^{\infty}\left(\frac{1}{1+\left(y+2k\pi+\frac{3\pi}{2}\right)^2} + \frac{1}{1+\left(y+2k\pi+\frac{\pi}{2}\right)^2}\right)dy
\end{aligned}$$

$$\sum_{k=-\infty}^{\infty}\frac{1}{1+\left(y+2k\pi+\frac{3\pi}{2}\right)^2} = \frac{e^2-1}{2+2e^2-4e\sin(y)}$$
$$\sum_{k=-\infty}^{\infty}\frac{1}{1+\left(y+2k\pi+\frac{\pi}{2}\right)^2} = \frac{e^2-1}{2+2e^2+4e\sin(y)}$$

$$\begin{aligned}
&\frac{e^2-1}{2+2e^2-4e\sin(y)} + \frac{e^2-1}{2+2e^2+4e\sin(y)} \\
&= \frac{e^4-1}{e^4+1+2e^2\cos(2y)}
\end{aligned}$$

$$\begin{aligned}
I &= \frac{1}{2}\int_0^{\pi} \frac{(e^4-1)\sin y}{e^4+1+2e^2\cos(2y)}dy \\
&= \frac{1}{2}\int_0^{\pi} \frac{(e^4-1)\sin y}{e^4+1+2e^2(2\cos^2(y)-1)}dy \\
&= \frac{1}{2}\int_0^{\pi} \frac{(e^4-1)\sin y}{\left(e^2-1\right)^2+4e^2\cos^2(y)}dy \\
&= \frac{1}{2}\frac{(e^4-1)}{\left(e^2-1\right)^2}\int_0^{\pi} \frac{\sin y}{1+\left(\frac{2e}{e^2-1}\right)^2\cos^2(y)}dy \\
&= \frac{1}{2}\frac{(e^4-1)}{\left(e^2-1\right)^2}\frac{2\arctan\left(\frac{2e}{e^2-1}\right)}{\frac{2e}{e^2-1}} \\
&= \frac{1}{2}\frac{e^2+1}{e}\arctan\left(\frac{\frac{1}{e}+\frac{1}{e}}{1-\frac{1}{e}\frac{1}{e}}\right) = \frac{e^2+1}{e}\arctan\left(\frac{1}{e}\right) \cong 1.0879.
\end{aligned}$$

A–129. *Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploieşti, Romania Romania.* Prove that $\frac{7}{6}$ is the least positive value of the constant k such that

$$x_1^k + x_2^k + x_3^k + x_4^k + x_5^k \geq 5$$

for any nonnegative real numbers x_i with at most one $x_i < 1$ and $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 5$.

Solution 1 by the proposer. Assuming $x_1 = x_2 := x$, $x_3 = x_5 = 1$ and $x_4 = \dfrac{5-2x-x^2}{2}$, the constraints are satisfied for $x \in [1, \sqrt{6}-1]$, while the inequality becomes $f(x) \geq 0$, where $f(x) = 2x^k + \left(\dfrac{5-2x-x^2}{2}\right)^k - 3$. From

$$\frac{1}{k}f'(x) = 2x^{k-1} - (x+1)\left(\frac{5-2x-x^2}{2}\right)^{k-1},$$

$$\frac{1}{k}f''(x) = 2(k-1)x^{k-2} - \left(\frac{5-2x-x^2}{2}\right)^{k-1}$$

$$+(k-1)(x+1)^2\left(\frac{5-2x-x^2}{2}\right)^{k-2},$$

we find $f(1) = f'(1) = 0$ and $f''(1) = k(6k-7)$. From the necessary condition $f''(1) \geq 0$, we get $k \geq 7/6$. To show that $7/6$ is the least positive value of k, we need to prove the required inequality for $k = 7/6$. By Lemma below, it suffices to show that $E(a,b,c,d,e) \geq 0$ for $a \geq b \geq c \geq d \geq 1 \geq e \geq 0$ such that $ab + ac + bd + ce + de = 5$, where

$$E(a,b,c,d,e) = a^k + b^k + c^k + d^k + e^k - 5.$$

For fixed b, c and e, we may assume that a and E are functions of d. By differentiating the equality constraint, we get

$$(b+c)a' + b + e = 0, \quad a' = \frac{-(b+e)}{b+c} \geq \frac{-(b+e)}{b+d} = \frac{d-e}{b+d} - 1$$
$$\geq \frac{d-e}{a+d} - 1 = \frac{-(a+e)}{a+d}.$$

Denoting $E(a,b,c,d,e)$ by $f(d)$, we have

$$\frac{6f'(d)}{7} = d^{1/6} + a^{1/6}a' \geq d^{1/6} - \frac{a^{1/6}(a+e)}{a+d}.$$

We claim that $f'(d) \geq 0$. To prove this, it suffices to show that $\frac{a+d}{a+e} \geq \left(\frac{a}{d}\right)^{1/6}$. By Bernoulli's inequality,

$$\left(\frac{a}{d}\right)^{1/6} = \left(1 + \frac{a-d}{d}\right)^{1/6} \leq 1 + \frac{a-d}{6d} = \frac{a+5d}{6d}.$$

So, it is enough to show that $\frac{a+d}{a+e} \geq \frac{a+5d}{6d}$. From $5 = ab + ac + bd + ce + de \geq ad + ad + d^2 + de + de$, we get $e \leq \frac{5-2ad-d^2}{2d}$ and $a + e \leq \frac{5-d^2}{2d}$, therefore

$$\frac{a+d}{a+e} - \frac{a+5d}{6d} \geq \frac{2d(a+d)}{5-d^2} - \frac{a+5d}{6d} = \frac{a(13d^2-5) + d(17d^2-25)}{6d(5-d^2)}$$

$$\geq \frac{d(13d^2-5)+d(17d^2-25)}{6d(5-d^2)} = \frac{5(d^2-1)}{5-d^2} \geq 0.$$

Since $f'(d) \geq 0$, $f(d)$ is increasing and has the minimum value when d is minimum, hence when $d = 1$. So, we need to show that

$$a^{7/6} + b^{7/6} + c^{7/6} + e^{7/6} \geq 4$$

for $a \geq b \geq c \geq 1 \geq e \geq 0$ such that $ab + ac + b + ce + e = 5$.

For fixed a and e, we may assume that b is a decreasing function of c. By differentiating the equality constraint, we get $(a+1)b' + a + e = 0$. Denoting the left side of the desired inequality by $g(c)$, we have

$$\frac{6g'(c)}{7} = c^{1/6} + b^{1/6}b' = c^{1/6} - \frac{b^{1/6}(a+e)}{a+1} \geq 1 - \frac{a^{1/6}(a+e)}{a+1}.$$

We claim that $g'(d) \geq 0$. To prove this, it suffices to show that $\frac{a+1}{a+e} \geq a^{1/6}$. By Bernoulli's inequality,

$$a^{1/6} = [1+(a-1)]^{1/6} \leq 1 + \frac{a-1}{6} = \frac{a+5}{6}.$$

So, it suffices to show that $\frac{a+1}{a+e} \geq \frac{a+5}{6}$. From $5 = ab + ac + b + ce + e \geq a + a + 1 + e + e$, we get $a + e \leq 2$, therefore

$$\frac{a+1}{a+e} - \frac{a+5}{6} \geq \frac{a+1}{2} - \frac{a+5}{6} = \frac{a-1}{3} \geq 0.$$

Since $g'(c) \geq 0$, $g(c)$ is increasing and has the minimum value when c is minimum (b is maximum), that is when $c = 1$ or $b = a$. Consider now these cases.

Case 1: $c = 1$. We need to show that $a^{7/6} + b^{7/6} + e^{7/6} \geq 3$ for $a \geq b \geq 1 \geq e \geq 0$ such that $ab + a + b + 2e = 5$. Let $x = \frac{a+b}{2} \geq 1$. Since, by Jensen's inequality and Bernoulli's inequality,

$$a^{7/6} + b^{7/6} \geq 2x^{7/6} = 2[1+(x-1)]^{7/6} \geq 2\left[1 + \frac{7(x-1)}{6}\right] = \frac{7x-1}{3},$$

we have

$$a^{7/6}+b^{7/6}+e^{7/6}-3\geq\frac{7x-1}{3}-3=\frac{7x-10}{3}\geq 0$$

for $x\geq 10/7$. For $x\in[1,10/7]$, since $e=\dfrac{5-2x-ab}{2}\geq\dfrac{5-2x-x^2}{2}>0$, we have

$$a^{7/6}+b^{7/6}+e^{7/6}-3\geq 2x^{7/6}+\left(\frac{5-2x-x^2}{2}\right)^{7/6}-3:=G(x).$$

If $G'(x)\geq 0$, then $G(x)$ is increasing, therefore $G(x)\geq G(1)=0$. Since

$$\begin{aligned}G'(x)&=\frac{7}{3}x^{1/6}-\frac{7}{6}(x+1)\left(\frac{5-2x-x^2}{2}\right)^{1/6}\\&=\frac{7}{6}x^{1/6}(x+1)\left[\frac{2}{x+1}-\left(\frac{5-2x-x^2}{2x}\right)^{1/6}\right],\end{aligned}$$

we need to show that $H(x)\geq 0$, where $H(x)=\left(\dfrac{2}{x+1}\right)^6-\dfrac{5-2x-x^2}{2x}$. Indeed, since $\left(\dfrac{2}{x+1}\right)^6\geq 2\left(\dfrac{2}{x+1}\right)^3-1$, we have

$$\begin{aligned}H(x)&\geq\frac{16}{(x+1)^3}-1-\frac{5-2x-x^2}{2x}=\frac{16}{(x+1)^3}-\frac{5-x^2}{2x}\\&=\frac{x^5+3x^4-2x^3-14x^2+17x-5}{2x(x+1)^3}\\&=\frac{(x-1)^2(x^3+5x^2+7x-5)}{2x(x+1)^3}\geq 0.\end{aligned}$$

Case 2: $b=a$. We need to show that $2a^{7/6}+c^{7/6}+e^{7/6}\geq 4$ for $a\geq c\geq 1\geq e\geq 0$ such that $a^2+ac+a+ce+e=5$. For fixed e, we may assume that a is a function of c. By differentiating the equality constraint, we get

$$(2a+c+1)a'+a+e=0,\qquad a'=\frac{-(a+e)}{2a+c+1}\geq\frac{-(a+e)}{2(a+1)}.$$

Denoting the left side of the desired inequality by $h(c)$, we have

$$\frac{6h'(c)}{7} = c^{1/6} + 2a^{1/6}a' = c^{1/6} - \frac{a^{1/6}(a+e)}{a+1} \geq 1 - \frac{a^{1/6}(a+e)}{a+1} \geq 0.$$

The last inequality was proved before. Since $h'(c) \geq 0$, $h(c)$ is increasing and has the minimum value when c is minimum, hence when $c = 1$. So, we need to show that $2a^{7/6} + e^{7/6} \geq 3$ for $a \geq 1 \geq e \geq 0$ such that $a^2 + 2a + 2e = 5$. If $a \geq 10/7$, then

$$2a^{7/6} + e^{7/6} - 3 \geq 2a^{7/6} - 3 > 0,$$

and if $a \in [1, 10/7]$, then

$$2a^{7/6} + e^{7/6} - 3 = 2a^{7/6} + \left(\frac{5 - 2a - a^2}{2}\right)^{7/6} - 3 \geq 0.$$

The last inequality was proved at Case 1.

The proof is completed. The equality occurs for $x_1 = x_2 = x_3 = x_4 = x_5 = 1$.

Lemma. *Let x_1, x_2, x_3, x_4, x_5 be nonnegative real numbers such that at most one of them is less than 1 and $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 5$, and let $E(x_1, x_2, x_3, x_4, x_5)$ be a symmetric and increasing function with respect to each variable. If $E(a, b, c, d, e) \geq 0$ for any $a \geq b \geq c \geq d \geq 1 \geq e \geq 0$ such that $ab + ac + bd + ce + de = 5$, then $E(x_1, x_2, x_3, x_4, x_5) \geq 0$.*

Proof. Let $T = (T_1, T_2, T_3, T_4, T_5)$ and $t = (t_1, t_2, t_3, t_4, t_5)$ be two decreasing sequences of positive real numbers. By Karamata majorization inequality applied to the convex function $f(x) = e^x$, if $T_1 \cdots T_j \geq t_1 \cdots t_j$ for $j = 1, 2, 3, 4, 5$, then

$$T_1 + T_2 + T_3 + T_4 + T_5 \geq t_1 + t_2 + t_3 + t_4 + t_5.$$

If (a, b, c, d, e) is a permutation of $(x_1, x_2, x_3, x_4, x_5)$ such that $a \geq b \geq c \geq d \geq 1 \geq e \geq 0$, then

$$E(a, b, c, d, e) = E(x_1, x_2, x_3, x_4, x_5).$$

Let $T = (ab, ac, bd, ce, de)$ be a decreasing sequence, and t a decreasing permutation of the sequence $(x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$. Since $T_1 \cdots T_j \geq t_1 \cdots t_j$ for $j = 1, 2, 3, 4, 5$, by Karamata's inequality we have

$$ab + ac + bd + ce + de \geq x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 5.$$

In the case $ab+ac+bd+ce+de > 5$, by decreasing the numbers a, b, c, d, e to have $ab + ac + bd + ce + de = 5$ and to keep the constraint $a \geq b \geq c \geq d \geq 1 \geq e \geq 0$, the function $E(a, b, c, d, e)$ decreases, therefore

$$E(a, b, c, d, e) \leq E(x_1, x_2, x_3, x_4, x_5).$$

On the other hand, by hypothesis, $E(a, b, c, d, e) \geq 0$. So, we have

$$E(x_1, x_2, x_3, x_4, x_5) \geq E(a, b, c, d, e) \geq 0.$$

Solution 2 by Moti Levy, Rehovot, Israel. Let

$$x_1 = 1 - e_1,\ x_2 = 1 + e_2,\ x_3 = 1 + e_3,\ x_4 = 1 + e_4,\ x_5 = 1 + e_5.$$

Then $1 > e_1 \geq 0$, and $e_2, e_3, e_4, e_5 \geq 0$.

In terms of e_i the cost function S_k becomes

$$S_k := \sum_{i=1}^{5} x_i^k = (1 - e_1)^k + (1 + e_2)^k + (1 + e_3)^k + (1 + e_4)^k + (1 + e_5)^k,$$

and the constraint becomes

$$\begin{gathered} 2e_2 - 2e_1 + 2e_3 + 2e_4 + 2e_5 - e_1e_2 + e_2e_3 - e_1e_5 + e_3e_4 + e_4e_5 = 0, \\ 1 > e_1 \geq 0, \text{and } e_2, e_3, e_4, e_5 \geq 0. \end{gathered} \tag{1}$$

Let us proceed using method of Lagrange multipliers:

The Lagrange function is

$$L(e_1, e_2, e_3, e_4, e_5, \lambda)$$

$$= S_k + \lambda(2e_2 - 2e_1 + 2e_3 + 2e_4 + 2e_5 - e_1e_2 + e_2e_3 - e_1e_5 + e_3e_4 + e_4e_5).$$

We compute the partial derivatives and set them to zero

$$\frac{\partial L}{\partial e_1} = -k(1-e_1)^{k-1} - \lambda(2+e_2+e_5) = 0,$$
$$\frac{\partial L}{\partial e_2} = k(1+e_2)^{k-1} + \lambda(2-e_1+e_3) = 0,$$
$$\frac{\partial L}{\partial e_3} = k(1+e_3)^{k-1} + \lambda(2+e_2+e_4) = 0,$$
$$\frac{\partial L}{\partial e_4} = k(1+e_4)^{k-1} + \lambda(2+e_3+e_5) = 0,$$
$$\frac{\partial L}{\partial e_5} = k(1+e_5)^{k-1} + \lambda(2+e_4-e_1) = 0, \tag{2}$$
$$\frac{\partial L}{\partial \lambda} = 2e_2 - 2e_1 + 2e_3 + 2e_4 + 2e_5 - e_1e_2 + e_2e_3 - e_1e_5 + e_3e_4 + e_4e_5.$$

which imply

$$\lambda = \frac{k(1-e_1)^{k-1}}{2+e_2+e_5},$$
$$\lambda = \frac{k(1+e_2)^{k-1}}{2-e_1+e_3},$$
$$\lambda = \frac{k(1+e_3)^{k-1}}{2+e_2+e_4},$$
$$\lambda = \frac{k(1+e_4)^{k-1}}{2+e_3+e_5},$$
$$\lambda = \frac{k(1+e_5)^{k-1}}{2+e_4-e_1}. \tag{3}$$

Now we use the symmetries in the constraint function and the cost function to simplify the system of equations (3).
Observing the original constraint equation $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 5$,we can see that x_2 and x_5 play symmetric roles, as do x_3 and x_4.
This symmetry in the constraint suggests that a solution with $e_3 = e_4$ and $e_2 = e_5$ might indeed exist.

Our partial derivative equations (2) under these assumptions reduces to 3 equations:

$$\lambda = \frac{k(1-e_1)^{k-1}}{2(1+e_2)},$$
$$\lambda = \frac{k(1+e_2)^{k-1}}{2-e_1+e_3},$$
$$\lambda = \frac{k(1+e_3)^{k-1}}{2+e_2+e_3}. \tag{4}$$

The reduced system of equations (4) is consistent with the original system (3).

The simplified problem is find minimum of

$$\begin{aligned} S_k(e_1) &= (1-e_1)^k + 2 + 2(1+e_3)^k \\ &= (1-e_1)^k + 2 + 2\left(\sqrt{2}\sqrt{e_1+2}-1\right)^k. \end{aligned}$$

subject to the constraint

$$e_3 = \sqrt{2}\sqrt{e_1+2} - 2, \qquad 1 > e_1 \geq 0.$$

We conclude that it is indeed possible to deduce that a solution exists with $e_3 = e_4$ and $e_2 = e_5$.
This deduction is based on:

1. The symmetry in the original problem.

2. The consistency of this assumption with the Lagrange multiplier equations.

3. The existence of feasible solutions in the simplified system.

Moreover, this symmetry suggests that if there is a unique minimum, it must have this form ($e_3 = e_4$ and $e_2 = e_5$).
If we find a solution with this symmetry, we can be confident that it's the global minimum.

Now we want to show that $k \geq \frac{7}{6}$ is necessary and sufficient condition for $S_k \geq 5$ under the constraint (1).

a) *Necessary condition*:

$$\frac{\partial S_k}{\partial e_1} = \frac{\partial\Big((1-e_1)^k + 2 + 2\big(\sqrt{2}\sqrt{e_1+2}-1\big)^k\Big)}{\partial e_1}$$
$$= k\frac{\sqrt{2}\big(\sqrt{2}\sqrt{e_1+2}-1\big)^{k-1} - (1-e_1)^{k-1}\sqrt{e_1+2}}{\sqrt{e_1+2}}.$$

Let

$$f(e_1) = \sqrt{2}\Big(\sqrt{2}\sqrt{e_1+2}-1\Big)^{k-1} - (1-e_1)^{k-1}\sqrt{e_1+2} \geq 0, 0 \leq e_1 < 1. \tag{5}$$

The sign of $\dfrac{\partial S_k}{\partial e_1}$ is the same of the sign of $f(e_1)$. Thus, showing that $f(e_1) \geq 0$ for $0 \leq e < 1$ is equivalent to showing that

$$\frac{1}{2}\ln(2) + (k-1)\ln\Big(\sqrt{2}\sqrt{e_1+2}-1\Big)$$
$$\geq (k-1)\ln(1-e_1) + \frac{1}{2}\ln(e_1+2). \tag{6}$$

Let

$$g_k(e_1) = \frac{1}{2}\ln(2) + (k-1)\ln\Big(\sqrt{2}\sqrt{e_1+2}-1\Big)$$
$$-(k-1)\ln(1-e_1) - \frac{1}{2}\ln(e_1+2), \tag{7}$$

then (6) is equivalent to

$$g_k(e_1) \geq 0. \tag{8}$$

We have $g_k(0) = 0$. For $g_k(e_1)$ to be non-negative for $e_1 > 0$, a necessary condition is $\frac{\partial g}{\partial e_1}\Big|_{e_1=0} \geq 0$.

$$\frac{\partial g_k}{\partial e_1} = \frac{k-1}{\sqrt{2}\big(\sqrt{2}\sqrt{e_1+2}-1\big)\sqrt{e_1+2}} + \frac{k-1}{1-e_1} - \frac{1}{2(e_1+2)} \tag{9}$$

$$\frac{\partial g_k}{\partial e_1}\Big|_{e_1=0} = \frac{3}{2}k - \frac{7}{4} \geq 0. \tag{10}$$

It follows that necessary condition is $k \geq \frac{7}{6}$.

b) *Sufficient condition*

By definition of the minimum we have

$$\min\{S_k(e_i)\} \leq \min\{S_k(e_3 = e_4 \; and \; e_2 = e_5 = 0)\}.$$

Since $S_k(e_3 = e_4 \; and \; e_2 = e_5 = 0) \leq S_k(e_i)$, then

$$\min\{S_k(e_i)\} \geq \min\{S_k(e_3 = e_4 \; and \; e_2 = e_5 = 0)\}.$$

We conclude that

$$\min\{S_k(e_i)\} = \min\{S_k(e_3 = e_4 \; and \; e_2 = e_5 = 0)\},$$

which implies that

$$S_k(e_i) \geq \min\{S_k(e_3 = e_4 \; and \; e_2 = e_5 = 0)\}$$

We have $S_{\frac{7}{6}}(e_3 = e_4 \text{ and } e_2 = e_5 = e_1 = 0) = 5$, hence if we show that the derivative $\frac{\partial g_{\frac{7}{6}}}{\partial e_1} \geq 0$ for $0 \leq e < 1$ then

$$\min\Big\{S_{\frac{7}{6}}(e_3 = e_4 \; and \; e_2 = e_5 = 0)\Big\} = 5$$

and we are done.

$$\frac{\partial g_{\frac{7}{6}}}{\partial e_1} = \frac{1}{6\sqrt{2}\big(\sqrt{2}\sqrt{e_1+2}-1\big)\sqrt{e_1+2}} + \frac{1}{6(1-e_1)} - \frac{1}{2(e_1+2)} \geq 0$$

which is equivalent to

$$\frac{\sqrt{e_1+2}}{3\sqrt{2}\big(\sqrt{2}\sqrt{e_1+2}-1\big)} + \frac{e_1+2}{3(1-e_1)} \geq 1, \text{ for } \; 0 \leq e < 1$$

Setting $t := \sqrt{e_1+2}$,

$$\frac{t}{3\sqrt{2}\big(\sqrt{2}t-1\big)} + \frac{t^2}{3(3-t^2)} \geq 1, \text{ for } \; \sqrt{2} \leq t < \sqrt{3}$$

or

$$\frac{7\big(t-\sqrt{2}\big)\big(t^2+\frac{3}{7}\sqrt{2}t-\frac{9}{7}\big)}{3\sqrt{2}\big(\sqrt{2}t-1\big)(3-t^2)} \geq 0 \text{ for } \; \sqrt{2} \leq t < \sqrt{3}$$

One can easily check that that $t^2+\frac{3}{7}\sqrt{2}t-\frac{9}{7} \geq 0$ for $\sqrt{2} \leq t < \sqrt{3}$.

A–130. *Proposed by Michel Bataille, Rouen, France.* For each positive integer n, let

$$I_n = \int_0^1 \frac{x(x^{2n}-1)\ln(x+1)}{x^2-1}\,dx.$$

Prove that there exist real numbers a, b such that

$$\lim_{n\to\infty}(I_n - (a + b\ln n)) = 0.$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland. We prove in a first step that

$$\int_0^1 \frac{x(x^{2n}-1)\ln(x+1)}{x^2-1}dx = \sum_{k=1}^{n}\frac{H_{2k}-H_k}{2k}$$

from which will follow that a and b exist. (H_k denotes as usual the k*th* harmonic number.)

In a second step we prove that $a = -\frac{\pi^2}{24} + \frac{1}{2}\gamma\ln 2 + \frac{\ln^2 2}{2}$, $b = \frac{\ln 2}{2}$.

We have

$$\int_0^1 x^{k-1}\ln(1-x)\,dx = \frac{x^k-1}{k}\ln(1-x)\,\Bigg|_{x=0}^{x=1} - \frac{1}{k}\int_0^1 \frac{x^k-1}{x-1}dx = -\frac{H_k}{k}.$$

So

$$\int_0^1 \frac{x(x^{2n}-1)\ln(x+1)}{x^2-1}dx = \int_0^1 \frac{x(x^{2n}-1)\ln\left(\frac{1-x^2}{1-x}\right)}{x^2-1}dx =$$

$$= \int_0^1 \frac{x(x^{2n}-1)\ln(1-x^2)}{x^2-1}dx - \int_0^1 \frac{x(x^{2n}-1)\ln(1-x)}{x^2-1}dx =$$

$$= \frac{1}{2}\int_0^1 \frac{(x^n-1)\ln(1-x)}{x-1}dx - \int_0^1 \frac{x(x^{2n}-1)\ln(1-x)}{x^2-1}dx =$$

$$= \sum_{k=1}^{n}\frac{1}{2}\int_0^1 x^{k-1}\ln(1-x)\,dx - \sum_{k=1}^{n}\int_0^1 x^{2k-1}\ln(1-x)\,dx = \sum_{k=1}^{n}\frac{H_{2k}-H_k}{2k}.$$

In the third last step we have performed the substitution y=x^2 and then renamed y as x.

It is known (see for instance https://en.wikipedia.org/wiki/Harmonic_number) that

$$H_n = \ln n \ + \gamma + O\left(\frac{1}{n}\right)$$

as n tends to infinity, where γ is the Euler-Mascheroni constant.
Put $a_k := H_{2k} - H_k - \ln 2$. Then $a_k = O\left(\frac{1}{k}\right)$ and

$$\sum_{k=1}^{n} \frac{H_{2k} - H_k}{2k} = \frac{\ln 2}{2} \sum_{k=1}^{n} \frac{1}{k} + \sum_{k=1}^{n} \frac{a_k}{2k} = \frac{\ln 2}{2} H_n + \sum_{k=1}^{\infty} \frac{a_k}{2k} - \sum_{k=n+1}^{\infty} \frac{a_k}{2k}$$

$$= \frac{\ln 2}{2} \ln n \ + \frac{\ln 2}{2}\gamma + \sum_{k=1}^{\infty} \frac{a_k}{2k} - \sum_{k=n+1}^{\infty} \frac{a_k}{2k} + O\left(\frac{1}{n}\right)$$

$$= \frac{\ln 2}{2} \ln n \ + \frac{\ln 2}{2}\gamma + \sum_{k=1}^{\infty} \frac{a_k}{2k} + O\left(\frac{1}{n}\right).$$

In particular, if $a := \frac{\ln 2}{2}\gamma + \sum_{k=1}^{\infty} \frac{H_{2k} - H_k - \ln 2}{2k},\ b := \frac{\ln 2}{2}$, then

$$\lim_{n\to\infty} (I_n - (a + b \ln n)) = \lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{H_{2k} - H_k}{2k} - (a + b \ln n\ ) \right) \ = 0.$$

So there are real numbers a, b such that $\lim_{n\to\infty} (I_n - (a + b \ln n)) = 0.$

We finally evaluate a (although this is not required).

We will establish the following equations

1. $\sum_{k=1}^{n} \frac{H_k}{k} = \frac{1}{2} H_n^2 + \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k^2}$
2. $\sum_{k=1}^{\infty} (-1)^{k-1} \frac{H_k}{k} = \frac{\pi^2}{12} - \frac{1}{2} \ln^2 2$

Proof of (i)

We proceed by induction. The claim holds true for n=1. Then

$$\sum_{k=1}^{n+1} \frac{H_k}{k} = \frac{H_{n+1}}{n+1} + \sum_{k=1}^{n} \frac{H_k}{k} = \frac{H_{n+1}}{n+1} + \frac{1}{2} H_n^2 + \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k^2}$$

$$= \frac{H_{n+1}}{n+1} + \frac{1}{2} \left(H_{n+1} - \frac{1}{n+1} \right)^2 + \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k^2}$$

$$= \frac{1}{2} H_{n+1}^2 + \frac{1}{2} \sum_{k=1}^{n+1} \frac{1}{k^2}.$$

Proof of (ii)

A generating function of the harmonic numbers is (see for instance https://en.wikipedia.org/wiki/Harmonic_number)

$$\sum_{k=1}^{\infty} H_k z^k = \frac{-\ln(1-z)}{1-z}.$$

Hence

$$\sum_{k=1}^{\infty} (-1)^{k-1} H_k z^{k-1} = \frac{\ln(1+z)}{z(1+z)}.$$

We integrate in the limits 0 and 1 and get

$$\sum_{k=1}^{\infty} (-1)^{k-1} \frac{H_k}{k} = \int_0^1 \frac{\ln(1+z)}{z(1+z)} dz = \int_0^1 \left(\frac{1}{z} - \frac{1}{z+1}\right) \ln(1+z)\, dz$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \int_0^1 z^{k-1} dz - \frac{1}{2}\ln^2(1+z) \Big|_{z=0}^{z=1}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} - \frac{1}{2}\ln^2 2 = \left(1 - \frac{2}{4}\right) \sum_{k=1}^{\infty} \frac{1}{k^2} - \frac{1}{2}\ln^2 2 = \frac{\pi^2}{12} - \frac{1}{2}\ln^2 2 .$$

It follows from (i) and (ii) that

$$\sum_{k=1}^{n} \frac{H_{2k} - H_k}{2k} = \frac{1}{2}\left(\sum_{k=1}^{2n} \frac{H_k}{k} - \sum_{k=1}^{2n} (-1)^{k-1} \frac{H_k}{k} - \sum_{k=1}^{n} \frac{H_k}{k}\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}H_{2n}^2 + \frac{1}{2}\sum_{k=1}^{2n} \frac{1}{k^2} - \sum_{k=1}^{\infty} (-1)^{k-1} \frac{H_k}{k} + \sum_{k=2n+1}^{\infty} (-1)^{k-1} \frac{H_k}{k} - \left(\frac{1}{2}H_n^2 + \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k^2}\right)\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\left(\ln(2n) + \gamma + O\left(\frac{1}{n}\right)\right)^2 + \frac{\pi^2}{12} + O\left(\frac{1}{n}\right) - \left(\frac{\pi^2}{12} - \frac{1}{2}\ln^2 2\right) + O\left(\frac{\ln(2n+1)}{2n+1}\right)\right.$$

$$\left. - \left(\frac{1}{2}\left(\ln(n) + \gamma + O\left(\frac{1}{n}\right)\right)^2 + \frac{\pi^2}{12} + O\left(\frac{1}{n}\right)\right)\right)$$

$$= \frac{\ln 2}{2}\ln n - \frac{\pi^2}{24} + \frac{1}{2}\gamma \ln 2 + \frac{\ln^2 2}{2} + O\left(\frac{\ln n}{n}\right)$$

and so

$$a = -\frac{\pi^2}{24} + \frac{1}{2}\gamma \ln 2 + \frac{\ln^2 2}{2} \approx 0.029040695597809973.$$

Solution 2 by the proposer. Since $\frac{x(x^{2n}-1)}{x^2-1} = \sum\limits_{k=1}^{n} x^{2k-1}$, we have $I_n = \sum\limits_{k=1}^{n} \int_0^1 x^{2k-1} \ln(1+x)\, dx.$

Using $\ln(1+x) = \sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$, we calculate

$$\begin{aligned}\int_0^1 x^{2k-1}\ln(1+x)\,dx &= \int_0^1 \left(\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{n+2k-1}}{n}\right)dx \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^1 x^{n+2k-1}\,dx \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+2k)} = \frac{1}{2k}\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{1}{n}-\frac{1}{n+2k}\right) \\ &= \frac{1}{2k}\left(\ln 2 - \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+2k}\right) \\ &= \frac{1}{2k}\big(\ln 2 - (\ln 2 - \overline{H}_{2k})\big) = \frac{\overline{H}_{2k}}{2k}\end{aligned}$$

where $\overline{H}_m = \sum_{j=1}^{m} \frac{(-1)^{j+1}}{j}$ denotes the mth skew-harmonic number.
(the interchange $\sum / \int$ is possible since

$$\sum_{n=1}^{\infty}\int_0^1\left|(-1)^{n+1}\frac{x^{n+2k-1}}{n}\right|dx = \sum_{n=1}^{\infty}\frac{1}{n(n+2k)} < \infty).$$

From $\frac{\overline{H}_{2k}}{2k} \sim \frac{\ln 2}{2k}$ as $k \to \infty$ and the Stolz-Cesaro Theorem, we deduce that

$$I_n = \sum_{k=1}^{n}\frac{\overline{H}_{2k}}{2k} \sim \frac{\ln 2}{2}\sum_{k=1}^{n}\frac{1}{k} \sim \frac{\ln 2}{2}\cdot \ln n \qquad \text{as } n\to\infty.$$

Now, let $U_n = I_n - \frac{\ln 2}{2}\ln n = \left(\sum\limits_{k=1}^{n}\frac{\overline{H}_{2k}}{2k}\right) - \frac{\ln 2}{2}\ln n$. We have

$$U_{n+1} - U_n = \frac{\overline{H}_{2n+2}}{2(n+1)} - \frac{\ln 2}{2}\ln\left(1+\frac{1}{n}\right).$$

It is well-known that $\overline{H}_m = \ln 2 + (-1)^{m+1}\int_0^1 \frac{x^m}{1+x}\,dx$ $(m \geq 1)$ and it follows that

$$U_{n+1}-U_n = \frac{\ln 2}{2}\left(\frac{1}{n}\left(1+\frac{1}{n}\right)^{-1} - \ln\left(1+\frac{1}{n}\right)\right) - \frac{1}{2(n+1)}\int_0^1 \frac{x^{2n+2}}{1+x}\,dx.$$

As $n \to \infty$, we have

$$\frac{1}{n}\left(1+\frac{1}{n}\right)^{-1} - \ln\left(1+\frac{1}{n}\right) = \frac{1}{n}\left(1-\frac{1}{n}+o(1/n)\right) - \left(\frac{1}{n}-\frac{1}{2n^2}+o(1/n^2)\right)$$
$$= -\frac{1}{2n^2} + o(1/n^2)$$

and

$$\int_0^1 \frac{x^{2n+2}}{1+x}\,dx = \frac{1}{4n} + o(1/n)$$

(since $\lim\limits_{n\to\infty}(2n+2)\int_0^1 \frac{x^{2n+2}}{1+x}\,dx = \left[\frac{1}{1+x}\right]_{x=1} = \frac{1}{2}$) and therefore

$$U_{n+1} - U_n \;\sim\; \frac{k}{n^2}$$

where $k = -\dfrac{1+\ln 4}{8}$. Thus, the series $\sum\limits_{n\geq 1}(U_{n+1}-U_n)$ is convergent and $\lim\limits_{n\to\infty} U_n$ is a real number a.
We can now conclude: If $b = \frac{\ln 2}{2}$, then $\lim\limits_{n\to\infty}(I_n - b\ln n) = a$, and the result follows.

Solution 3 by Moti Levy, Rehovot, Israel.

$$\frac{x(x^{2n}-1)}{x^2-1} = \sum_{k=1}^{n} x^{2k-1}$$

$$I_n = \int_0^1 \sum_{k=1}^{n} x^{2k-1}\ln(x+1)dx = \sum_{k=1}^{n}\int_0^1 x^{2k-1}\ln(x+1)dx$$

$$\int_0^1 x^{2k-1}\ln(x+1)dx = \frac{H_{2k}-H_k}{2k}$$

$$I_n = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}(H_{2k}-H_k)$$

Now we apply the Abel's summation formula with

$$b_k := \frac{1}{k}, \quad a_k := H_{2k} - H_k.$$

$$\begin{aligned}I_n &= \frac{1}{2}(H_{2n}-H_n)H_n - \frac{1}{2}\sum_{k=1}^{n-1} H_k\left(\frac{1}{2k+2}+\frac{1}{2k+1}-\frac{1}{k+1}\right)\\ &= \frac{1}{2}(H_{2n}-H_n)H_n - \frac{1}{2}\sum_{k=1}^{n-1}\frac{H_k}{(2k+1)(2k+2)}.\end{aligned}$$

$$\sum_{k=1}^{\infty}\frac{H_k}{(2k+1)(2k+2)} = \frac{\pi^2}{12} - \ln^2(2).$$

The asymptotic expression for H_n is

$$H_n \sim \ln(n) + \gamma + O(\frac{1}{n}).$$

The asymptotic expression for H_{2n} is

$$H_{2n} \sim \ln(n) + \gamma + \ln(2) + O(\frac{1}{n})$$

It follows that the asymptotic expression for $H_{2n} - H_n$ is

$$H_{2n} - H_n \sim \ln(2) + O(\frac{1}{n})$$

The asymptotic expression for $(H_{2n} - H_n)H_n$ is

$$\begin{aligned}(H_{2n}-H_n)H_n &\sim \left(\ln(2) + O(\frac{1}{n})\right)\left(\ln(n) + \gamma + O(\frac{1}{n})\right)\\ &= \ln(2)\ln(n) + \gamma\ln(2) + O(\frac{\ln(n)}{n})\end{aligned}$$

$$I_n \sim \frac{\ln(2)}{2}\ln(n) + \gamma\frac{\ln(2)}{2} - \frac{\pi^2}{24} + \frac{\ln^2(2)}{2} + O(\frac{\ln(n)}{n})$$

We conclude that indeed $\lim_{n\to\infty}(I_n - (a + b\ln(n))) = 0$, where

$$\begin{aligned}a &= \gamma\frac{\ln(2)}{2} - \frac{\pi^2}{24} + \frac{\ln^2(2)}{2},\\ b &= \frac{\ln(2)}{2}.\end{aligned}$$

Arhimede Mathematical Journal

Volume 11, No. 2 — Autumn 2024

Aim and Scope

The goal of *Arhimede Mathematical Journal* is to provide a means of publication of useful materials to train students for Mathematical Contests at all levels. Potential contributions include any work involving fresh ideas and techniques, problems and lessons helpful to train contestants, all written in a clear and elegant mathematical style. All areas of mathematics, including algebra, combinatorics, geometry, number theory and real and complex analysis, are considered appropriate for the journal.

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Edited in Barcelona by the Arhimede Association

ISSN 2462-537X