No. 1
2024

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# MATHEMATICAL JOURNAL

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# *Some Geometric Inequalities*

**José Luis Díaz-Barrero and Mihály Bencze**

## 1 Introduction

Geometric inequalities are as old as geometry itself. In the first book of Euclid's Elements appears several theorems on inequalities involving the sides and angles of a triangle. The most well-known claims that *the sum of two sides of a triangle is greater than the third* (Proposition XX). In recent times many geometric inequalities have appeared. One of the most relevant is the one that relates the radii of the circles circumscribed and inscribed to a triangle $R \geq 2r$, published by Euler in $1765$.

Geometric inequalities have a wide range of applications, both within geometry itself and beyond the traditional fields of geometry and geometric applications. For example, in the theory of complex functions, in the calculus of variations and, more generally, in providing a priori estimates in various areas such as differential equations. In this paper some elementary plane geometry inequalities similar to the ones appeared in [1, 2, 3] are presented. Most of them deal with triangles and convex polygons.

## 2 Main results

Combining geometric ideas with algebra, the following results have been obtained. We begin with an inequality involving the elements of an acute triangle.

**Theorem 1.** *Let $ABC$ be an acute triangle. Then, with the usual notations, it holds:*

$$\sqrt{\frac{\sin A}{\sin B\sin C}}+\sqrt{\frac{\sin B}{\sin C\sin A}}+\sqrt{\frac{\sin C}{\sin A\sin B}}\geq\sqrt{\frac{abc}{rR^2}}.$$

*Proof.* From the Law of Sines, we have $b=a\,\dfrac{\sin B}{\sin A}$ and the area $[ABC]$ can be written as

$$[ABC]=\frac{1}{2}\,bc\,\sin C=\frac{1}{2}\,a^2\,\frac{\sin B\sin C}{\sin A}\quad\text{(cyclic)}.$$

Therefore,

$$a=\sqrt{\frac{2[ABC]\sin A}{\sin B\sin C}},\;\; b=\sqrt{\frac{2[ABC]\sin B}{\sin C\sin A}},\;\; c=\sqrt{\frac{2[ABC]\sin C}{\sin A\sin B}}.$$

Adding up the preceding expressions and, after rearranging terms, we get

$$\sqrt{\frac{\sin A}{\sin B\sin C}}+\sqrt{\frac{\sin B}{\sin C\sin A}}+\sqrt{\frac{\sin C}{\sin A\sin B}}=\frac{a+b+c}{\sqrt{2[ABC]}}.$$

Since

$$[ABC]=sr=\frac{(a+b+c)r}{2},$$

then $\sqrt{2[ABC]}=\sqrt{(a+b+c)r}$, and the RHS of the preceding expression, becomes

$$\frac{a+b+c}{\sqrt{2[ABC]}}=\sqrt{\frac{a+b+c}{r}}.$$

Since $R\geq 2r$ is equivalent to $s\geq\dfrac{2[ABC]}{R}$, then

$$\sqrt{\frac{a+b+c}{r}}\geq 2\sqrt{\frac{[ABC]}{rR}}=\sqrt{\frac{abc}{rR^2}}$$

account that $[ABC] = \dfrac{abc}{4R}$, as it is well-known. Equality holds when triangle $ABC$ is equilateral.

Using Cauchy's and mean inequalities, we have obtained

**Theorem 2.** *Let $P$ be any point inside the triangle $ABC$ and let denote by $A' = AP \cap BC, B' = BP \cap AC$ and $C' = CP \cap AB$, respectively. Then, it holds*

$$\frac{1}{AA'^2} + \frac{1}{BB'^2} + \frac{1}{CC'^2} \geq \frac{4\sqrt{3}}{3}\sqrt{\frac{1}{AP^4 + BP^4 + CP^4}}.$$

*Proof.* First, we observe that

$$\frac{AP}{AA'} = \frac{[ABP]}{[ABA']} = \frac{[ACP]}{[ACA']} = \frac{[ABP] + [ACP]}{[ABC]}.$$

Likewise, $\dfrac{BP}{BB'} = \dfrac{[BAP] + [BCP]}{[ABC]}$ and $\dfrac{CP}{CC'} = \dfrac{[CAP] + [CBP]}{[ABC]}$. Adding up the preceding expressions yields

$$\frac{AP}{AA'} + \frac{BP}{BB'} + \frac{CP}{CC'} = \frac{2[ABC]}{[ABC]} = 2 \tag{1}$$

Now, setting $\vec{u} = (AP, BP, CP)$ and $\vec{v} = \left(\dfrac{1}{AA'}, \dfrac{1}{BB'}, \dfrac{1}{CC'}\right)$ into CBS inequality, we have

$$\left(\frac{AP}{AA'} + \frac{BP}{BB'} + \frac{CP}{CC'}\right)^2 \leq (AP^2 + BP^2 + CP^2)\left(\frac{1}{AA'^2} + \frac{1}{BB'^2} + \frac{1}{CC'^2}\right)$$

and taking into account (1), we get

$$\frac{4}{AP^2 + BP^2 + CP^2} \leq \frac{1}{AA'^2} + \frac{1}{BB'^2} + \frac{1}{CC'^2} \tag{2}$$

Applying the AM-QM inequality, we have

$$\frac{AP^2 + BP^2 + CP^2}{3} \leq \sqrt{\frac{AP^4 + BP^4 + CP^4}{3}}$$

or equivalently,

$$\frac{4\sqrt{3}}{3}\sqrt{\frac{1}{AP^4+BP^4+CP^4}} \leq \frac{4}{AP^2+BP^2+CP^2} \tag{3}$$

Now, from (2) and (3) the statement immediately follows. Equality holds when Triangle $ABC$ is equilateral and $P$ coincides with its centroid.

Next, applying Jensen's inequality, we get

**Theorem 3.** *If $ABC$ is a triangle with circumradius $R$ and semi-perimeter $s$, then it holds*

$$\frac{(b+c)\cos A+(c+a)\cos B+(a+b)\cos C}{\sin A+\sin B+\sin C} \geq \frac{4}{9}s\sqrt{3}.$$

*Proof.* The given statement is equivalent to prove

$$2R = \frac{(b+c)\cos A+(c+a)\cos B+(a+b)\cos C}{\sin A+\sin B+\sin C} \geq \frac{4}{9}s\sqrt{3}.$$

Since $f(x) = \sin x$ is strictly concave in $[0, \pi]$ then, by Jensen's inequality, we have

$$\sin A+\sin B+\sin C = f(A)+f(B)+f(C)$$

$$\leq 3f\left(\frac{A+B+C}{3}\right) = 3\sin\left(\frac{A+B+C}{3}\right) = \frac{3\sqrt{3}}{2}.$$

On the other hand, from Sine's law, we get

$$2R = \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{a+b+c}{\sin A+\sin B+\sin C}$$

and

$$a+b+c = 2R(\sin A+\sin B+\sin C).$$

Finally, let $D$ be the intersection between the perpendicular to side $AC$ drawn by $A$ and the side $AC$. Then, in $\triangle ABC$, we have

$BD = c\cos B$ and in $\triangle ADC$, we get $DC = b\cos C$. Adding up the preceding expressions, yields

$$c\cos B + b\cos C = BD + DC = a.$$

In the same way, we obtain $a\cos B + b\cos A = c$ and $a\cos C + c\cos A = b$. Adding up the last three equalities, yields

$$(b+c)\cos A + (c+a)\cos B + (a+b)\cos C = a+b+c.$$

Now, combining the preceding results, the statement immediately follows. Equality holds only when $A = B = C = \pi/3$. That is, when $\triangle ABC$ is equilateral.

Note that we also have obtained that the diameter $2R$ of the circumcircle of triangle $ABC$ is given by

$$\frac{(b+c)\cos A + (c+a)\cos B + (a+b)\cos C}{\sin A + \sin B + \sin C}.$$

The concept of area combined with mean inequalities, allow us to obtain the following result.

**Theorem 4.** *Let $ABC$ be a triangle and let $P$ be any point inside it. The lines $AP, BP, CP$ mets the sides $BC, CA, AB$ at points $A_1, B_1, C_1$, respectively. If $x = \dfrac{AP}{PA_1}$, $y = \dfrac{BP}{PB_1}$, $z = \dfrac{CP}{PC_1}$, then it holds: (1) $xyz \geq 8$ and (2) $x+y+z \geq 6$.*

*Proof.* In triangle $ABC$, we have

$$\begin{aligned} x &= \frac{AP}{PA_1} = \frac{[PAB]}{[PA_1B]} = \frac{[PAC]}{[PA_1C]} \\ &= \frac{[PAB]+[PAC]}{[PA_1B]+[PA_1B]} = \frac{[PAB]+[PAC]}{[PBC]}. \end{aligned}$$

Likewise, we get $y = \dfrac{[PBC]+[PBA]}{[PCA]}$ and $z = \dfrac{[PCA]+[PCB]}{[PAB]}$.
On account of the identity

$$\frac{(a+b)(b+c)(c+a)}{abc} = \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b} + 2,$$

we obtain $x+y+z+2=xyz$.

(1) Putting $t=\sqrt[3]{xyz}$, taking into account the preceding identity and AM-GM inequality, we have $x+y+z+2=xyz\geq 2+\sqrt[3]{xyz}$ or $t^3\geq 2+3t$ or $(t-2)(t+1)^2\geq 0$. The last inequality holds only if $t\geq 2$ from which $xyz\geq 8$ follows. Equality holds when $x=y=z=2$. That is, when $\triangle ABC$ is equilateral and $P$ coincides with its centroid.

(2) Putting $r=x+y+z$, taking into account the preceding identity and AM-GM inequality, we have $x+y+z+2=xyz\leq\left(\frac{x+y+z}{3}\right)^3$ or $r^3\geq 27r+54$ or $(r-6)(r+3)^2\geq 0$. The last inequality holds only when $r\geq 6$ from which $x+y+z\geq 6$ follows. Equality holds when $x=y=z=2$. That is, when $\triangle ABC$ is equilateral and $P$ coincides with its centroid.

Next, Jensen's inequality is used to obtain a refinement of Euler's inequality. It is presented in the following

**Theorem 5.** *Let $p_1,p_2,p_3$ be positive numbers and $ABC$ be an acute triangle. With the usual notations, if*

$$\mathcal{R}=\sum_{cyc}\cos\left(\frac{p_1A+p_2B+p_3C}{p_1+p_2+p_3}\right),$$

*then it holds*

$$1+\frac{r}{R}\leq\mathcal{R}\leq\frac{3}{2}.$$

*Proof.* Putting $x_1=\dfrac{p_1A+p_2B+p_3C}{p_1+p_2+p_3}$, $x_2=\dfrac{p_2A+p_3B+p_1C}{p_1+p_2+p_3}$, $x_3=\dfrac{p_3A+p_1B+p_2C}{p_1+p_2+p_3}$ and applying Jensen's inequality to the concave function $f(x) = \cos x$ yields,

$$\begin{aligned}\cos\left(\frac{A+B+C}{3}\right) &= \cos\left(\frac{x_1+x_2+x_3}{3}\right)\\ &\geq \frac{\cos x_1+\cos x_2+\cos x_3}{3}\\ &\geq \frac{1}{3}\sum_{cyc}\frac{p_1\cos A+p_2\cos B+p_3\cos C}{p_1+p_2+p_3}\\ &= \frac{\cos A+\cos B+\cos C}{3} \geq \frac{1}{2}\end{aligned}$$

on account that $\dfrac{\cos A+\cos B+\cos C}{3} \geq \cos\left(\dfrac{A+B+C}{3}\right) = \dfrac{1}{2}$.
Equality holds when triangle $ABC$ is equilateral.

Finally, applying Cauchy's and mean inequalities, we get the following result involving the elements of a convex polygon.

**Theorem 6.** *Let $A_1, A_2, \cdots, A_n$ be the vertices of a convex polygon and let $P$ be any point inside it. If $P_1, P_2, \cdots, P_n$ are the projections of $P$ onto the sides $A_1A_2, A_2A_3, \ldots, A_nA_1$ respectively, then it holds*

$$\left(\sum_{k=1}^{n}\frac{1}{A_kA_{k+1}^2}\right)\left(\sum_{k=1}^{n}A_kP_k^2\right) \geq \frac{n^2}{4}$$

*where $A_{n+1} = A_1$.*

*Proof.* For $1 \leq k \leq n$ consider the triangles $PA_kP_k$ and $PA_{k+1}P_k$. Applying the theorem of Pythagoras we have

$$\begin{aligned}P_1P^2 = A_1P^2 - A_1P_1^2 &= A_2P^2 - A_2P_1^2\\ P_2P^2 = A_2P^2 - A_2P_2^2 &= A_3P^2 - A_3P_2^2\\ &\vdots\\ P_nP^2 = A_nP^2 - A_nP_n^2 &= A_1P^2 - A_1P_n^2\end{aligned}$$

Adding up the preceding equalities yields

$$\sum_{k=1}^{n}A_kP_k^2 = \sum_{k=1}^{n}A_{k+1}P_k^2$$

Setting $x_k = A_kP_k$ and $a_k = A_kA_{k+1}, 1 \leq k \leq n$ into the preceding expression, we obtain $x_1^2 + x_2^2 + \cdots + x_n^2 = (a_1 - x_1)^2 + (a_2 - x_2)^2 + \cdots + (a_n - x_n)^2$ or equivalently

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = \frac{1}{2}\big(a_1^2 + a_2^2 + \cdots + a_n^2\big)$$

Applying CBS inequality, we get

$$\left(\frac{1}{2}\big(a_1^2 + a_2^2 + \cdots + a_n^2\big)\right)^2 \leq (x_1^2 + x_2^2 + \cdots + x_n^2)(a_1^2 + a_2^2 + \cdots + a_n^2)$$

From the preceding, we get $\frac{1}{4}\big(a_1^2 + a_2^2 + \cdots + a_n^2\big) \leq x_1^2 + x_2^2 + \cdots + x_n^2$. Dividing by $n$ both sides of the preceding inequality and applying the AM-HM inequality, we have

$$\begin{aligned} \frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n} &\geq \frac{1}{4}\,\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} \\ &\geq \frac{n}{4\left(\dfrac{1}{a_1^2} + \dfrac{1}{a_2^2} + \cdots + \dfrac{1}{a_n^2}\right)} \end{aligned}$$

from which the statement follows. Equality holds when the polygon is regular, and $P$ coincides with $O$ the center of the regular polygon.

# References

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# *Counting and Adding Discretely*

**Joe Santmyer**

**Abstract**

This note presents two formulas, one that can be considered in the area of number theory or discrete mathematics and the other in classical analysis. Both results are finite in nature. One counts the size of a particular finite set, the other provides formulas for two finite sums.

## 1 Introduction

Counting and summation are two basic operations in mathematics. Throughout algebra we find counting arguments such as the proof of the well known theorem of Lagrange that states that the order of a subgroup in a finite group divides the order of the group. Here a formula is derived that counts a finite set associated with a group often discussed in an abstract algebra course. In [4] you find a multitude of trigonometric summation formulas. However, you will not find the two summation formulas discussed here.

## 2 A Counting Formula

A common exercise in an abstract algebra course, see p 62 in [2], is to show that $(G \setminus \{-1\}, *)$ is an abelian group where $G = \mathbb{Q}$ or

$G = \mathbb{R}$ and $*$ is the binary operation $a * b = a + b + ab$. If $n, a, b$ are integers with $n \geq 1$ and $a, b \geq 0$ consider the set

$$S_n = \{(a * b) \bmod n : \gcd(ab, n) = 1\}.$$

It is easy to see that $S_1 = \{0\}$ and $S_2 = \{1\}$.

**Theorem 1**: Let $|S_n|$ be the number of elements in $S_n$. Write $n$ as $n = 2^i 3^j m$ where $i, j \geq 0$ and $\gcd(6, m) = 1$. With the initial conditions $|S_1| = |S_2| = 1$ show that for $n > 2$

$$|S_n| = \begin{cases} n & \text{if } i = 0 \text{ and } j = 0 \\ \frac{n}{2} & \text{if } i = 1 \text{ and } j = 0 \\ \frac{n}{4} & \text{if } i > 1 \text{ and } j = 0 \\ \frac{2n}{3} & \text{if } i = 0 \text{ and } j > 0 \\ \frac{n}{3} & \text{if } i = 1 \text{ and } j > 0 \\ \frac{n}{6} & \text{if } i > 1 \text{ and } j > 0. \end{cases}$$

**Proof**: The formula is an easy consequence of the following result.

**Lemma**: For $n > 2$, $i \geq 1$ and $p$ a prime with $p > 3$

$$|S_n| = \begin{cases} \frac{\varphi(n)}{2} & \text{if } n = 2^i \\ \varphi(n) & \text{if } n = 3^i \\ n & \text{if } n = p^i \\ |S_k||S_m| & \text{if } n \text{ is composite, } n = km,\ k \neq m \\ & \text{and } \gcd(k, m) = 1 \end{cases}$$

where $\varphi$ is the Euler totient function.

**Proof**: Consider the case $n = p^i$ where $i \geq 1$, $p > 3$ and $p$ is a prime. Let $a = 1$ and $b = \frac{(j-1)(p^i+1)}{2}$ with $j \neq 1 \bmod p$. If $\gcd(b, p) = 1$ then

$$\begin{aligned} a * b &= 1 + \frac{(j-1)(p^i+1)}{2} + \frac{(j-1)(p^i+1)}{2} \\ &= 1 + (j-1)(p^i+1) \\ &= (j-1)p^i + j \\ &= j \bmod p^i. \end{aligned}$$

This produces all values mod $p^i$ except those 1 mod $p$. To get these values let $a = 2$ and $b = (p-1)x$ where $x$ is the inverse of 3 mod $p$. That is, $3x = 1 \bmod p$. Then

$$\begin{aligned} a * b &= 2 + (p-1)x + 2(p-1)x \\ &= 2 + (p-1)(3x) \\ &= 2 + (p-1)(1) \\ &= p + 1 \\ &= 1 \bmod p. \end{aligned}$$

Consider the case $n = 3^i$ where $i \geq 1$. Now $a, b = 0, 1, 2 \bmod 3$ but since $\gcd(ab, n) = 1$ we can eliminate $a, b = 0 \bmod 3$. If $a, b = 1, 1 \bmod 3$ then $a = 1 + 3x$, $b = 1 + 3y$ for integers $x, y$ and

$$\begin{aligned} a * b &= 1 + 3x + 1 + 3y + (1 + 3x)(1 + 3y) \\ &= 3(1 + 2x + 2y + 3xy) \\ &= 0 \bmod 3. \end{aligned}$$

If $a, b = 1, 2 \bmod 3$ then $a = 1 + 3x$, $b = 2 + 3y$ for integers $x, y$ and

$$\begin{aligned} a * b &= 1 + 3x + 2 + 3y + (1 + 3x)(2 + 3y) \\ &= 2 + 3(1 + 3x + 2y + 3xy) \\ &= 2 \bmod 3. \end{aligned}$$

If $a, b = 2, 2 \bmod 3$ then $a = 2 + 3x$, $b = 2 + 3y$ for integers $x, y$ and

$$\begin{aligned} a * b &= 2 + 3x + 2 + 3y + (2 + 3x)(2 + 3y) \\ &= 2 + 3(2 + 3x + 3y + 3xy) \\ &= 2 \bmod 3. \end{aligned}$$

This shows that the values $1 \bmod 3$ cannot be obtained. This leaves $3^i - 3^{i-1} = \varphi(3^i)$ values.

Consider the case $n = 2^i$ where $i > 1$. Now $a, b = 0, 1 \bmod 2$ but since $\gcd(ab, n) = 1$ we can eliminate $a, b = 0 \bmod 2$. If

$a, b = 1, 1 \bmod 2$ then $a = 1 + 2x$, $b = 1 + 2y$ for integers $x, y$ and

$$\begin{aligned} a * b &= 1 + 2x + 1 + 2y + (1 + 2x)(1 + 2y) \\ &= 3 + 4x + 4y + 4xy \\ &= 3 + 4(x + y + xy) \\ &= 3 \bmod 4. \end{aligned}$$

This shows that the values 0 mod 2 (which includes 0 mod 4 and 2 mod 4) and 1 mod 4 cannot be obtained. This leaves $2^i - 2^{i-1} - \frac{2^i - 2^{i-1}}{2} = \frac{2^i - 2^{i-1}}{2} = \frac{\varphi(2^i)}{2}$ values.

It remains to show that the function $f(n) = |S_n|$ is multiplicative. That is, $f(km) = f(k)f(m)$ when $\gcd(k, m) = 1$.

Consider the case $n = p^i q^k$ where $i, k \geq 1$, $p \neq q$, $p, q > 3$ and $p, q$ are prime. Let $a = 1$ and $b = \frac{(j-1)(p^i q^k + 1)}{2}$ with $j \neq 1 \bmod p$ and $j \neq 1 \bmod q$. If $\gcd(b, pq) = 1$ then

$$\begin{aligned} a * b &= 1 + \frac{(j-1)(p^i q^k + 1)}{2} + \frac{(j-1)(p^i q^k + 1)}{2} \\ &= 1 + (j-1)(p^i q^k + 1) \\ &= (j-1)p^i q^k + j \\ &= j \bmod p^i q^k. \end{aligned}$$

This produces all values mod $p^i q^k$ except those 1 mod $p$ or 1 mod $q$. To get these values let $a = 2$ and $b = (pq - 1)x$ where $x$ is the inverse of 3 mod $pq$. That is, $3x = 1 \bmod pq$. Then

$$\begin{aligned} a * b &= 2 + (pq - 1)x + 2(pq - 1)x \\ &= 2 + (pq - 1)(3x) \\ &= 2 + (pq - 1)(1) \\ &= pq + 1 \\ &= 1 \bmod pq. \end{aligned}$$

This shows that $f(p^i q^k) = p^i q^k$. By what was shown above we also know that $f(p^i) = p^i$ and $f(q^k) = q^k$. Hence, $f(p^i q^k) = f(p^i)f(q^k)$.

Consider the case $n = 3^i q^k$ where $i, k \geq 1$, $q > 3$ and $q$ is prime. As shown above, only the values 0 mod 3 and 2 mod 3 are possible.

So, of the $3^i q^k$ values we eliminate the $3^{i-1}q^k$ values that are 1 mod 3. This leaves $3^i q^k - 3^{i-1}q^k = (3^i - 3^{i-1})q^k = \varphi(3^i)q^k$ values. This shows that $f(3^i q^k) = \varphi(3^i)q^k$. By what was shown above, $f(3^i) = \varphi(3^i)$ and $f(q^k) = q^k$. Hence, $f(3^i q^k) = f(3^i)f(q^k)$.

Consider the case $n = 2^i q^k$ where $i > 1$, $k \geq 1$, $q > 3$ and $q$ is prime. As shown above, the values 0 mod 2 and 1 mod 4 are not possible. So, of the $2^i q^k$ values we eliminate $2^{i-1}q^k$ values 0 mod 2 and $\frac{2^i - 2^{i-1}}{2}$ values 1 mod 4. This leaves $2^i q^k - 2^{i-1}q^k - \frac{2^i - 2^{i-1}}{2}q^k = \frac{2^i - 2^{i-1}}{2}q^k = \frac{\varphi(2^i)}{2}q^k$ values. This shows that $f(2^i q^k) = \frac{\varphi(2^i)}{2}q^k$. By what was shown above, $f(2^i) = \frac{\varphi(2^i)}{2}$ and $f(q^k) = q^k$. Hence, $f(2^i q^k) = f(2^i)f(q^k)$.

The remaining cases are $n = 2 \cdot 3^k$ and $n = 2q^k$ for $k \geq 1$, $q > 3$ and $q$ a prime. These cases are straightforward. This proves the **Lemma**.

# 3 Two Finite Sums

**Theorem 2**: Show that

$$\sum_{k=1}^{n} \frac{1}{\cos(\theta) \pm \cos\left(\frac{2k\pi}{2n+1}\right)} = \begin{cases} \left[\left(n+\frac{1}{2}\right)\tan\left(\left(n+\frac{1}{2}\right)\theta\right) - \frac{1}{2}\tan\left(\frac{\theta}{2}\right)\right]\csc(\theta) \\ \left[\frac{1}{2}\cot\left(\frac{\theta}{2}\right) - \left(n+\frac{1}{2}\right)\cot\left(\left(n+\frac{1}{2}\right)\theta\right)\right]\csc(\theta) \end{cases}$$

$$\sum_{k=1}^{n} \frac{1}{\cos(\theta) \pm \cos\left(\frac{(2k-1)\pi}{2n+1}\right)} = \begin{cases} \left[\frac{1}{2}\cot\left(\frac{\theta}{2}\right) - \left(n+\frac{1}{2}\right)\cot\left(\left(n+\frac{1}{2}\right)\theta\right)\right]\csc(\theta) \\ \left[\left(n+\frac{1}{2}\right)\tan\left(\left(n+\frac{1}{2}\right)\theta\right) - \frac{1}{2}\tan\left(\frac{\theta}{2}\right)\right]\csc(\theta). \end{cases}$$

**Proof**: The formulas are established by applying material in chapters 5 and 14 of [1]. General references include [3], [4] and [5].

The $+$ case in each equation is justified. The $-$ case is similar and left to the reader. Consider the following formula (after correcting what seems to be a typing error) on p 43 in [1]

$$z^{2n+1} + 1 \;=\; (z+1)\prod_{k=1}^{n}\left[z^2 + 2z\cos\left(\frac{2k\pi}{2n+1}\right) + 1\right].$$

The equation is the result of the fact that the solutions to $z^{2n+1} + 1 = 0$ are $-1$ and $z_k = -e^{\frac{2\pi ik}{2n+1}}$ for $k \neq 0$ where the non-real solutions occur in conjugate pairs.

Divide both sides by $(z+1)(2z)^n$ to get

$$\frac{1}{2^n} \cdot \frac{z^{n+1} + z^{-n}}{z+1} = \prod_{k=1}^{n} \left[ \frac{z + z^{-1}}{2} + \cos\left( \frac{2k\pi}{2n+1} \right) \right].$$

Let $z = e^{i\theta}$ to get

$$\begin{aligned}
\frac{1}{2^n} \cdot \frac{e^{(n+1)i\theta} + e^{-ni\theta}}{e^{i\theta} + 1} &= \prod_{k=1}^{n} \left[ \frac{e^{i\theta} + e^{-i\theta}}{2} + \cos\left( \frac{2k\pi}{2n+1} \right) \right] \\
\frac{1}{2^n} \cdot \frac{e^{(n+1)i\theta} + e^{-ni\theta}}{e^{i\theta} + 1} &= \prod_{k=1}^{n} \left[ \cos(\theta) + \cos\left( \frac{2k\pi}{2n+1} \right) \right] \\
\frac{1}{2^n} \left[ \frac{\cos\left( \left(n + \frac{1}{2}\right)\theta \right)}{\cos\left(\frac{\theta}{2}\right)} \right] &= \prod_{k=1}^{n} \left[ \cos(\theta) + \cos\left( \frac{2k\pi}{2n+1} \right) \right] \\
\frac{\cos\left( \left(n + \frac{1}{2}\right)\theta \right)}{\cos\left(\frac{\theta}{2}\right)} &= 2^n \prod_{k=1}^{n} \left[ \cos(\theta) + \cos\left( \frac{2k\pi}{2n+1} \right) \right].
\end{aligned}$$

Take the logarithm and the derivative with respect to $\theta$ of both sides to get

$$\begin{aligned}
\ln\left[\cos\left(\left(n + \frac{1}{2}\right)x\right)\right] - \ln\left[\cos\left(\frac{\theta}{2}\right)\right] &= n\ln(2) + \sum_{k=1}^{n} \ln\left[\cos(\theta) + \cos\left(\frac{2k\pi}{2n+1}\right)\right] \\
-\frac{\left(n + \frac{1}{2}\right)\sin\left(\left(n + \frac{1}{2}\right)\theta\right)}{\cos\left(\left(n + \frac{1}{2}\right)\theta\right)} + \frac{\frac{1}{2}\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} &= \sum_{k=1}^{n} \frac{-\sin(\theta)}{\cos(\theta) + \cos\left(\frac{2k\pi}{2n+1}\right)} \\
\frac{\left(n + \frac{1}{2}\right)\sin\left(\left(n + \frac{1}{2}\right)\theta\right)}{\cos\left(\left(n + \frac{1}{2}\right)\theta\right)} - \frac{\sin\left(\frac{\theta}{2}\right)}{2\cos\left(\frac{\theta}{2}\right)} &= \sum_{k=1}^{n} \frac{\sin(\theta)}{\cos(\theta) + \cos\left(\frac{2k\pi}{2n+1}\right)} \\
\left(n + \frac{1}{2}\right)\tan\left(\left(n + \frac{1}{2}\right)\theta\right) - \frac{1}{2}\tan\left(\frac{\theta}{2}\right) &= \sum_{k=1}^{n} \frac{\sin(\theta)}{\cos(\theta) + \cos\left(\frac{2k\pi}{2n+1}\right)} \\
\left[\left(n + \frac{1}{2}\right)\tan\left(\left(n + \frac{1}{2}\right)\theta\right) - \frac{1}{2}\tan\left(\frac{\theta}{2}\right)\right]\csc(\theta) &= \sum_{k=1}^{n} \frac{1}{\cos(\theta) + \cos\left(\frac{2k\pi}{2n+1}\right)}. \qquad (1)
\end{aligned}$$

This justifies the $+$ case in the first equation.

A slight generalization of 5.15 on p 43 in [1] is

$$\frac{\sin(n\theta)}{\sin(\theta)} = 2^{n-1} \prod_{k=1}^{n-1} \left[ \cos(\theta) \pm \cos\left( \frac{k\pi}{n} \right) \right].$$

Take the logarithm and derivative to arrive at

$$[-n\cot(n\theta)+\cot(\theta)]\csc(\theta) \;=\; \sum_{k=1}^{n-1}\frac{1}{\cos(\theta)\pm\cos\left(\frac{k\pi}{m}\right)} \qquad (2)$$

which is a slight generalization of 14.2 on p 153 in [1].

Let

$$\begin{aligned} S &= \sum_{k=1}^{n}\frac{1}{\cos(\theta)+\cos\left(\frac{(2k-1)\pi}{2n+1}\right)} \\ T &= S+\sum_{k=1}^{n}\frac{1}{\cos(\theta)+\cos\left(\frac{2k\pi}{2n+1}\right)} \\ &= \sum_{k=1}^{2n+1-1}\frac{1}{\cos(\theta)+\cos\left(\frac{k\pi}{2n+1}\right)}. \end{aligned}$$

Apply (1) and (2) to get

$$\begin{aligned} S &= T-\sum_{k=1}^{n}\frac{1}{\cos(\theta)+\cos\left(\frac{2k\pi}{2n+1}\right)} \\ &= \sum_{k=1}^{2n+1-1}\frac{1}{\cos(\theta)+\cos\left(\frac{k\pi}{2n+1}\right)}-\sum_{k=1}^{n}\frac{1}{\cos(\theta)+\cos\left(\frac{2k\pi}{2n+1}\right)} \\ &= [-(2n+1)\cot((2n+1)\theta)+\cot(\theta)]\csc(\theta)-\left[\left(n+\frac{1}{2}\right)\tan\left(\left(n+\frac{1}{2}\right)\theta\right)-\frac{1}{2}\tan\left(\frac{\theta}{2}\right)\right]\csc(\theta) \\ &= \left[-2\left(n+\frac{1}{2}\right)\cot\left(2\left(n+\frac{1}{2}\right)\theta\right)+\cot(\theta)-\left(n+\frac{1}{2}\right)\tan\left(\left(n+\frac{1}{2}\right)\theta\right)+\frac{1}{2}\tan\left(\frac{\theta}{2}\right)\right]\csc(\theta) \\ &= \left[\frac{1}{2}\cot\left(\frac{\theta}{2}\right)-\left(n+\frac{1}{2}\right)\cot\left(\left(n+\frac{1}{2}\right)\theta\right)\right]\csc(\theta). \end{aligned}$$

This justifies the $+$ case in the second equation.

# 4 Final Remarks

Another common exercise in an abstract algebra course is to show that $(G\setminus\{1\},*)$ is an abelian group where $G=\mathbb{Q}$ or $G=\mathbb{R}$ and $*$ is the binary operation $a*b=a+b-ab$. If $n,a,b$ are integers with $n\geq 1$ and $a,b\geq 0$ consider the set

$$T_n=\{(a*b) \bmod n : \gcd(ab,n)=1\}.$$

Is $T_n = S_n$? If not, are $T_n$ and $S_n$ related in some way? Can you find a formula for $|T_n|$? How does it compare to the formula for $|S_n|$ in **theorem 1** above?

Use **theorem 2** to show that

$$\sum_{k=1}^{n} \frac{1}{\sin(\theta) \pm \cos\left(\frac{2k\pi}{2n+1}\right)}$$

$$= \begin{cases} -\frac{1}{2}\sec(\theta)\left(\cot\left(\frac{1}{4}(2\theta+\pi)\right) - (2n+1)\tan\left(\frac{1}{4}(2n+1)(\pi-2\theta)\right)\right) \\ \frac{1}{2}\sec(\theta)\left(\tan\left(\frac{1}{4}(2\theta+\pi)\right) - (2n+1)\cot\left(\frac{1}{4}(2n+1)(\pi-2\theta)\right)\right) \end{cases}$$

and

$$\sum_{k=1}^{n} \frac{1}{\sin(\theta) \pm \cos\left(\frac{(2k-1)\pi}{2n+1}\right)}$$

$$= \begin{cases} \frac{1}{2}\sec(\theta)\left(\tan\left(\frac{1}{4}(2\theta+\pi)\right) - (2n+1)\cot\left(\frac{1}{4}(2n+1)(\pi-2\theta)\right)\right) \\ -\frac{1}{2}\sec(\theta)\left(\cot\left(\frac{1}{4}(2\theta+\pi)\right) - (2n+1)\tan\left(\frac{1}{4}(2n+1)(\pi-2\theta)\right)\right). \end{cases}$$

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# *New class of integer solutions to ternary quadratic equation* $x^2 + y^2 = (a^2 + b^2)z^2$

**N. Thiruniraiselvi and M. A. Gopalan**

**Abstract**

This paper focuses on finding new class of integer solutions to the ternary quadratic diophantine equation given by $x^2 + y^2 = (a^2 + b^2)z^2$ through utilizing a typical factorization strategy involving sides of Pythagorean triangle.

## 1 Introduction

Diophantine equations, one of the areas in number theory occupy a pivotal role in the realm of mathematics and have a wealth of historical significance. It is well-known that quadratic Diophantine equations with three unknowns are rich in variety. One may refer [1-6] for problems on ternary quadratic Diophantine equations in which substitution strategies and factorization methods are employed in getting their solutions in integers. In this paper, a special factorization strategy involving sides of Pythagorean triangle is introduced in obtaining integer solutions to the polynomial equation of degree two with three unknowns represented by $x^2 + y^2 = (a^2 + b^2)z^2$.

## 2 Technical procedure

The ternary quadratic Diophantine equation to be solved is

$$x^2 + y^2 = (a^2 + b^2)z^2 \tag{1}$$

Assume

$$z = m^2 + n^2 \tag{2}$$

Let $p, q$ be any two non-zero distinct integers such that $p > q > 0$. Treating $p, q$ as the generators of Pythagorean triangle $T$, its sides are given by $p^2 - q^2, 2pq$, as it is well-known. Define

$$\begin{aligned} f(a,b,p,q) &= a(p^2 - q^2) + b(2pq), \\ g(a,b,p,q) &= a(2pq) + b(p^2 - q^2). \end{aligned}$$

After some algebra, it is obtained

$$\begin{aligned} &[f(a,b,p,q) + ig(a,b,p,q)] \cdot [f(a,b,p,q) - ig(a,b,p,q)] \\ &= \left(p^2 + q^2\right)^2 \left(a^2 + b^2\right). \end{aligned}$$

Thus, the complex factorization of integer $a^2 + b^2$ is given by

$$a^2 + b^2 = \frac{[f(a,b,p,q) + ig(a,b,p,q)] \cdot [f(a,b,p,q) - ig(a,b,p,q)}{\left(p^2 + q^2\right)^2} \tag{3}$$

Substituting (2) and (3) in (1) and applying the factorization strategy, (1) can be written as the system of double equations as shown below:

$$\begin{aligned} x + iy &= \frac{[f(a,b,p,q) + ig(a,b,p,q)](m + in)^2}{(p^2 + q^2)} \\ x - iy &= \frac{[f(a,b,p,q) - ig(a,b,p,q)](m - in)^2}{(p^2 + q^2)} \end{aligned}$$

On equating the real and imaginary parts in either of the above two expressions, we get

$$\begin{aligned} x &= \frac{[f(a,b,p,q)F(m,n) - G(m,n)g(a,b,p,q)]}{(p^2 + q^2)} \\ y &= \frac{[G(m,n)f(a,b,p,q) + F(m,n)g(a,b,p,q)]}{(p^2 + q^2)}, \end{aligned} \tag{4}$$

where $F(m,n) = m^2 - n^2$ and $G(m,n) = 2mn$. As our aim is to obtain solutions in integers, replacing $m$ by $(p^2+q^2)M$ and $n$ by $(p^2+q^2)N$, $M, N$ integers, in (2) and (4), the corresponding integer solutions to (1) are given by

$$x = [f(a,b,p,q)F(M,N) - G(M,N)g(a,b,p,q)]\left(p^2+q^2\right),$$
$$y = [G(M,N)f(a,b,p,q) + F(M,N)g(a,b,p,q)]\left(p^2+q^2\right),$$
$$z = \left(p^2+q^2\right)^2\left(M^2+N^2\right).$$

To illustrate the preceding, we give two examples:

- Putting $a = 6, b = 2, p = 2, q = 1, M = N$ in the preceding, we obtain $f(a,b,p,q) = 26$, $g(a,b,p,q) = 18$, $F(M,N) = 0$, $G(M,N) = 2N^2$ and the solutions are $x = -180N^2$, $y = 260N^2$, $z = 50N^2$.
- Putting $a = 3, b = 2, p = 3, q = 2, M = kN (k > 1)$ in the preceding, we obtain $f(a,b,p,q) = 39$, $g(a,b,p,q) = 26$, $F(M,N) = (k^2-1)N^2$, $G(M,N) = 2kN^2$ and the solutions are $x = (13N)^2[3k^2-4k-3]$, $y = (13N)^2[2k^2+6k-2]$, $z = (13N)^2(k^2+1)$.

**Remarkable observations**

(i) Suppose the given equation is $x^2+y^2 = (a^2+b^2)z^3$. In this case, the corresponding integer solutions are given by

$$x = [f(a,b,p,q)F(M,N) - G(M,N)g(a,b,p,q)]\left(p^2+q^2\right)^2,$$
$$y = [G(M,N)f(a,b,p,q) + F(M,N)g(a,b,p,q)]\left(p^2+q^2\right)^2,$$
$$z = \left(p^2+q^2\right)^2\left(M^2+N^2\right),$$

where $F(M,N) = M^3 - 3MN^2, G(M,N) = 3M^2N - N^3$.

(ii) Suppose the given equation is $x^2+y^2 = (a^2+b^2)z^4$. In this case, the corresponding integer solutions are given by

$$x = [f(a,b,p,q)F(M,N) - G(M,N)g(a,b,p,q)]\left(p^2+q^2\right)^3,$$
$$y = [G(M,N)f(a,b,p,q) + F(M,N)g(a,b,p,q)]\left(p^2+q^2\right)^3,$$
$$z = \left(p^2+q^2\right)^2\left(M^2+N^2\right),$$

where $F(M,N) = M^4 - 6M^2N^2 + N^4$ and $G(M,N) = 4M^3N - 4MN^3$.

Finally, as a conclusion one can obtain the integer solutions of the diophantine equation of degree $n$ with three unknowns given by $x^2 + y^2 = (a^2 + b^2)\, z^n$ by means of the factorization method presented above.

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# *A new inequality involving differences of means*

**Vasile Cîrtoaje**

## 1 Introduction

In [4] the following inequality involving differences of mean was published: *If* $a, b, c \geq 1$, *then*

$$\sqrt{\frac{a^2+b^2+c^2}{3}} - \frac{a+b+c}{3} \geq \sqrt{\frac{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}{3}} - \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}.$$

In this note a generalization of the previous result is presented.

## 2 Main result

Below is stated and proven the main result of this note.

**Theorem 1.** *Let* $a, b, c$ *be positive real numbers such that* $a = \min\{a, b, c\}$ *and* $a^{11/9}(a+b+c) \geq 3$, *and let*

$$F(a, b, c) = \sqrt{\frac{a^2+b^2+c^2}{3}} - \frac{a+b+c}{3}.$$

*Then* $F(a, b, c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ *holds.*

*Proof.* Assume that $a \le b \le c$, write the inequality as $E(a,b,c) \ge 0$, where

$$E(a,b,c) = \sqrt{a^2+b^2+c^2} - \frac{a+b+c}{\sqrt{3}} - \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} + \frac{1}{\sqrt{3}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

and show that

$$E(a,b,c) \ge E(a,x,x) \ge 0,$$

where

$$x = \frac{b+c}{2} \ge b \ge a, \quad a^{11/9}(a+2x) \ge 3, \quad x \ge 1.$$

Write the inequality $E(a,b,c) \ge E(a,x,x)$ as $A + B \ge C$, where

$$\begin{aligned} A &= \sqrt{a^2+b^2+c^2} - \sqrt{a^2+2x^2} \\ &= \frac{(b-c)^2}{2} \cdot \frac{1}{\sqrt{a^2+b^2+c^2}+\sqrt{a^2+2x^2}} \\ &\ge \frac{(b-c)^2}{2} \cdot \frac{1}{\sqrt{2b^2+c^2}+\sqrt{b^2+2x^2}}\,, \end{aligned}$$

$$B = \frac{1}{\sqrt{3}}\left(\frac{1}{b}+\frac{1}{c}-\frac{2}{x}\right) = \frac{(b-c)^2}{\sqrt{3}bc(b+c)}\,,$$

$$C = \sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}} - \sqrt{\frac{1}{a^2}+\frac{2}{x^2}}$$

$$= \frac{(b-c)^2(b^2+4bc+c^2)}{b^2c^2(b+c)^2} \cdot \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{a^2}+\frac{2}{x^2}}}$$

$$\le \frac{(b-c)^2(b^2+4bc+c^2)}{b^2c^2(b+c)^2} \cdot \frac{1}{\sqrt{\frac{2}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{b^2}+\frac{2}{x^2}}}\,.$$

Thus, we need to show that

$$\frac{1}{2} \cdot \frac{1}{\sqrt{2b^2+c^2}+\sqrt{b^2+2x^2}} + \frac{1}{\sqrt{3}bc(b+c)}$$

$$\ge \frac{b^2+4bc+c^2}{b^2c^2(b+c)^2} \cdot \frac{1}{\sqrt{\frac{2}{b^2}+\frac{1}{c^2}}+\sqrt{\frac{1}{b^2}+\frac{2}{x^2}}}.$$

Since $b^2 + 4bc + c^2 = 4bc + (b^2 + c^2)$, we get this inequality by summing the inequalities

$$\frac{1}{\sqrt{3}bc(b+c)} \geq \frac{4bc}{b^2c^2(b+c)^2} \cdot \frac{1}{\sqrt{\frac{2}{b^2}+\frac{1}{c^2}} + \sqrt{\frac{1}{b^2}+\frac{2}{x^2}}} \tag{1}$$

and

$$\frac{1}{2} \cdot \frac{1}{\sqrt{2b^2+c^2}+\sqrt{b^2+2x^2}} \geq \frac{b^2+c^2}{b^2c^2(b+c)^2} \cdot \frac{1}{\sqrt{\frac{2}{b^2}+\frac{1}{c^2}} + \sqrt{\frac{1}{b^2}+\frac{2}{x^2}}}. \tag{2}$$

Write (1) as

$$\sqrt{3\left(\frac{2}{b^2}+\frac{1}{c^2}\right)} + \sqrt{3\left(\frac{1}{b^2}+\frac{2}{x^2}\right)} \geq \frac{12}{b+c}.$$

Indeed,

$$\begin{aligned}
\sqrt{3\left(\frac{2}{b^2}+\frac{1}{c^2}\right)} + \sqrt{3\left(\frac{1}{b^2}+\frac{2}{x^2}\right)} &\geq \left(\frac{2}{b}+\frac{1}{c}\right) + \left(\frac{1}{b}+\frac{2}{x}\right) \\
&\geq \left(\frac{2}{b}+\frac{1}{c}\right) + \left(\frac{1}{c}+\frac{2}{x}\right) \\
&= 2\left(\frac{1}{b}+\frac{1}{c}+\frac{2}{b+c}\right) \\
&\geq 2\left(\frac{4}{b+c}+\frac{2}{b+c}\right) = \frac{12}{b+c}.
\end{aligned}$$

The inequality (2) can be obtained by multiplying the inequalities

$$\frac{1}{\sqrt{2b^2+c^2}+\sqrt{b^2+2x^2}} \geq \frac{1}{bc} \cdot \frac{1}{\sqrt{\frac{2}{b^2}+\frac{1}{c^2}} + \sqrt{\frac{1}{b^2}+\frac{2}{x^2}}}$$

and

$$bc(b+c)^2 \geq 2(b^2+c^2).$$

Write the first inequality as

$$\sqrt{b^2+2c^2} + \sqrt{c^2+\frac{2b^2c^2}{x^2}} \geq \sqrt{2b^2+c^2} + \sqrt{b^2+2x^2}.$$

Since $\sqrt{b^2+2c^2} \geq \sqrt{2b^2+c^2}$, it is sufficient to prove that

$$c^2 + \frac{2b^2c^2}{x^2} \geq b^2 + 2x^2,$$

that is

$$c^2 - b^2 \geq \frac{2(x^2-bc)(x^2+bc)}{x^2}.$$

Since $x^2 + bc \leq 2x^2$, it is true if

$$c^2 - b^2 \geq 4(x^2 - bc).$$

Indeed,

$$c^2 - b^2 - 4(x^2 - bc) = c^2 - b^2 - (c-b)^2 = 2b(c-b) \geq 0.$$

Since

$$1 \leq \frac{a^{11/9}(a+b+c)}{3} \leq \frac{b^{11/9}(2b+c)}{3}$$

to prove the second inequality it suffices to show that

$$bc(b+c)^2 \geq 2(b^2+c^2)\left[\frac{b^{11/9}(2b+c)}{3}\right]^{9/10}.$$

Due to homogeneity, we may set $b = 1$, which involves $c \geq 1$. The inequality becomes

$$c(1+c)^2 \geq 2(1+c^2)\left(\frac{2+c}{3}\right)^{9/10}.$$

By Bernoulli's inequality, we have

$$\left(\frac{2+c}{3}\right)^{9/10} = \left(1+\frac{c-1}{3}\right)^{9/10} \leq 1 + \frac{3(c-1)}{10} = \frac{3c+7}{10} \leq \frac{2c+3}{5}.$$

So, it suffices to prove that

$$5c(1+c)^2 \geq 2(1+c^2)(2c+3),$$

which is equivalent to the obvious inequality

$$(c-1)(c^2+5c+6) \geq 0.$$

Write now inequality $E(a,x,x)\geq 0$ as

$$\sqrt{3(a^2+2x^2)}-a-2x\geq\sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)}-\frac{1}{a}-\frac{2}{x},$$

where $x\geq a>0$ such that $a^{11/9}(a+2x)\geq 1$. Since both sides of the inequality are nonnegative, we only need to prove the homogeneous inequality

$$\sqrt{3(a^2+2x^2)}-a-2x\geq\left[\frac{a^{11/9}(a+2x)}{3}\right]^{9/10}\left[\sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)}-\frac{1}{a}-\frac{2}{x}\right]$$

for $x\geq a>0$. Due to homogeneity, we may set $a=1$. Thus, we need to show that $x\geq 1$ yields

$$x\left[\sqrt{3(1+2x^2)}-1-2x\right]\geq\left(\frac{1+2x}{3}\right)^{9/10}\left[\sqrt{3(x^2+2)}-x-2\right].$$

By Bernoulli's inequality, we have

$$\left(\frac{1+2x}{3}\right)^{9/10}=\left(1+\frac{2x-2}{3}\right)^{9/10}\leq 1+\frac{3(x-1)}{5}=\frac{3x+2}{5}.$$

So, it suffices to show that

$$5x\left[\sqrt{3(1+2x^2)}-1-2x\right]\geq(3x+2)\left[\sqrt{3(x^2+2)}-x-2\right].$$

Since

$$\sqrt{3(1+2x^2)}-1-2x=\frac{2(x-1)^2}{\sqrt{3(1+2x^2)}+1+2x}$$

and

$$\sqrt{3(x^2+2)}-x-2=\frac{2(x-1)^2}{\sqrt{3(x^2+2)}+x+2},$$

then the inequality is true if

$$\frac{5x}{\sqrt{3(1+2x^2)}+1+2x}\geq\frac{3x+2}{\sqrt{3(x^2+2)}+x+2},$$

or equivalently,

$$5x\sqrt{3(x^2+2)}-(3x+2)\sqrt{3(2x^2+1)}\geq(x-1)(x-2).$$

In addition, since

$$5x\sqrt{3(x^2+2)}-(3x+2)\sqrt{3(2x^2+1)} = \frac{\sqrt{3}(7x^4-24x^3+33x^2-12x-4)}{5x\sqrt{x^2+2}+(3x+2)\sqrt{2x^2+1}}$$

$$= \frac{\sqrt{3}(x-1)(7x^3-17x^2+16x+4)}{5x\sqrt{x^2+2}+(3x+2)\sqrt{2x^2+1}} \geq \frac{5}{3}\cdot\frac{(x-1)(7x^3-17x^2+16x+4)}{5x\sqrt{x^2+2}+(3x+2)\sqrt{2x^2+1}},$$

we need to show that $x \geq 1$ involves

$$\frac{5(7x^3-17x^2+16x+4)}{15x\sqrt{x^2+2}+3(3x+2)\sqrt{2x^2+1}} \geq x-2.$$

Since

$$7x^3-17x^2+16x+4 > 3x(2x^2-6x+5) > 0,$$

it suffices to consider the case $x \geq 2$, when

$$3\sqrt{x^2+2} < 3x+2, \qquad 3\sqrt{2x^2+1} < 5x,$$

Thus, it suffices to show that

$$\frac{5(7x^3-17x^2+16x+4)}{5x(3x+2)+5x(3x+2)} \geq x-2,$$

i.e.

$$7x^3-17x^2+16x+4 \geq 2x(3x+2)(x-2),$$

$$x^3-9x^2+24x+4 \geq 0.$$

Indeed,

$$x^3-9x^2+24x+4 > x^3-9x^2+24x-16 = (x-4)^2(x-1) \geq 0.$$

The proof is finished. The equality occurs for $a = b = c \geq 1$.

# 3 Other relevant results

The following similar inequalities were proved in [3].

• *Let* $a, b, c$ *be positive real numbers such that* $a = \min\{a, b, c\}$ *and* $a^4bc \geq 1$*, and let*

$$F(a,b,c) = \sqrt{\frac{a^2+b^2+c^2}{3}} - \frac{a+b+c}{3}.$$

*Then,*

$$F(a,b,c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

• *Let* $a, b, c$ *be positive real numbers such that* $a = \min\{a, b, c\}$ *and* $a^2(b+c) \geq 2$*, and let*

$$F(a,b,c) = \sqrt{\frac{a^2+b^2+c^2}{3}} - \frac{a+b+c}{3}.$$

*Then,*

$$F(a,b,c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

• *Let* $a, b, c$ *be positive real numbers such that* $a = \min\{a, b, c\}$ *and* $a^4(b^2+c^2) \geq 2$*, and let*

$$F(a,b,c) = \sqrt{\frac{a^2+b^2+c^2}{3}} - \frac{a+b+c}{3}.$$

*Then,*

$$F(a,b,c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

• *Let* $a, b, c$ *be positive real numbers such that* $a = \min\{a, b, c\}$ *and* $a^4b^7c^7 \geq 1$*, and let*

$$F(a,b,c) = \sqrt[3]{abc} - \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}.$$

*Then,*

$$F(a,b,c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

• *Let* $a, b, c, d$ *be positive real numbers such that* $ab \geq 1$ *and* $cd \geq 1$, *and let*

$$F(a,b,c,d) = \sqrt[4]{abcd} - \frac{4}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}}.$$

*Then,*

$$F(a,b,c,d) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right).$$

Also, the following inequalities involving differences of mean were published in [2].

• *Let* $a, b, c$ *be positive real numbers such that* $a = \min\{a,b,c\}$ *and* $a(b+c) \geq 2$, *and let* $F(a,b,c) = 3(a^2+b^2+c^2) - (a+b+c)^2$. *Then,*

$$F(a,b,c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

• *Let* $a, b, c$ *be positive real numbers such that* $\min\{a,b,c\} \geq \frac{1}{abc}$, *and let* $F(a,b,c) = a+b+c-3\sqrt[3]{abc}$. *Then,*

$$F(a,b,c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

• *Let* $a, b, c$ *be positive real numbers such that* $a = \min\{a,b,c\}$ *and* $a^2(b^2+c^2) \geq 2$, *and let* $F(a,b,c) = a+b+c-3\sqrt[3]{abc}$. *Then,*

$$F(a,b,c) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

• *Let $a, b, c, d$ be positive real numbers such that* $\min\{a^2, b^2, c^2, d^2\} \geq \frac{1}{abcd}$, *and let* $F(a, b, c, d) = a + b + c + d - 4\sqrt[4]{abcd}$. *Then,*

$$F(a, b, c, d) \geq F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right).$$

Moreover, the following inequalities involving differences of mean were published in [1].

• *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that* $a_1 = \min\{a_1, a_2, \ldots, a_n\}$ *and* $a_1(a_2 + a_3 + \cdots + a_n) \geq n - 1$, *and let* $F(a_1, a_2, \ldots, a_n) = a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n}$. *Then,*

$$F(a_1, a_2, \ldots, a_n) \geq F\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right).$$

• *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that* $a_1 = \min\{a_1, a_2, \ldots, a_n\}$ *and* $a_1^2(a_2^2 + a_3^2 + \cdots + a_n^2) \geq n - 1$, *and let*

$$F(a_1, a_2, \ldots, a_n) = n(a_1^2 + a_2^2 + \cdots + a_n^2) - (a_1 + a_2 + \cdots + a_n)^2.$$

*Then,*

$$F(a_1, a_2, \ldots, a_n) \geq F\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right).$$

• *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that* $a_1 = \min\{a_1, a_2, \ldots, a_n\}$ *and* $a_1^{n-1}(a_2 + a_3 + \cdots + a_n) \geq n - 1$, *and let*

$$F(a_1, a_2, \ldots, a_n) = \sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} - \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

*Then,*

$$F(a_1, a_2, \ldots, a_n) \geq F\left(\frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n}\right).$$

# References


[1] Cirtoaje, V. *Mathematical Inequalities-Extensions and Refinements of Jensen's Inequality*, Volume 4, LAP and Ed. Univ. Petrol-Gaze din Ploiesti, Ploisti, Romania, 2021. URL: http://ace.upg-ploiesti.ro/membri/vcirtoaje/vcirtoaje.php.

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# *Problems*

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.
2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to
`jose.luis.diaz@upc.edu`

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

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*Solutions to the problems stated in this issue should be posted before*

**October 30, 2024**

# Elementary Problems

***E–125.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* If $\alpha$ is an irrational number then prove that $\alpha^{17}$ and $\alpha^{19}$ cannot be both rational numbers.

***E–126.*** *Proposed by Michel Bataille, Rouen, France.* Let $n$ be a positive integer and $a$ a non-negative real number. Prove that

$$(1+a)^{n+1} \geq 1 + (n+1)a\sqrt{(1+a)^n}.$$

***E–127.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Points $D, E, F$ lie on the sides $BC, CA$ and $AB$ of triangle $ABC$ respectively, such that $BD = 3DC, CE = 3A$ and $AF = 3FB$. Point $P$ is the intersection point of $BE$ and $CF$, $Q$ is the intersection point of $CF$ and $AD$ and $R$ is the intersection point of $AD$ and $BE$. Determine $[PQR]/[ABC]$.

***E–128.*** *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* In how many different ways may you change an euro? That is, in how many different ways can you pay $100$ cents using six different kinds of coins, $1, 2, 5, 10, 20$ and $50$ cents, respectively.

***E–129.*** *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let $a, b, c$ be positive real numbers and let

$$x = \left(\frac{bc}{a}\right)^{lg\frac{b}{c}},\ y = \left(\frac{ca}{b}\right)^{lg\frac{c}{a}},\ z = \left(\frac{ab}{c}\right)^{lg\frac{a}{b}}.$$

If $x+y+z$ and $x^3+y^3+z^3$ are rational, then show that $x^{2024} + y^{2024} + z^{2024}$ is also rational.

***E–130.*** *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Find the largest positive integer $k$ for which $\dfrac{1000!}{10^k}$ is an integer number and determine the maximum power of $2$ that divides it.

# *Easy–Medium Problems*

***EM–125.*** *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let $ABC$ be a triangle with $AB < AC$. Let $M$ be the midpoint of the side $BC$, $I$ be its incenter, $G$ be its centroid and $D$ be the foot of the altitude drawn from $A$. If $MI \cap AD = \{X\}$ and $3\,AX = AD$, then show that $IG \parallel BC$.

***EM–126.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $n \geq 1$ be an integer. Find the first decimal figure of the real number $\sqrt{n^2 + 15n + 55}$.

***EM–127.*** *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let $ABC$ be a scalene triangle with incenter $I$ and centroid $G$. Let $G_a$ be the orthogonal projection of $G$ on $BC$. Let $M$ be the midpoint of $BC$ and $A_M$ be the reflection of $A$ in $M$. Denote with $P$ the second intersection point of the line $AI$ and the circumcircle of the triangle $ABC$. Knowing that the points $A_M, P, G_a$ are collinear, find the ratio $AI/IP$.

***EM–128.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $u, v, z, w$ be complex numbers. Prove that

$$2\text{Re}\Big(uz + vw\Big) \leq 2\Big(|u|^2 + |v|^2\Big) + \frac{1}{2}\Big(|z|^2 + |w|^2\Big).$$

***EM–129.*** *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let $\alpha \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Prove that every positive real number is the sum of nine numbers consisting only of $0$ and $\alpha$ in their digits and decimal part.

***EM–130.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Lisa and Bart play the following game. They first choose a positive integer $N$, and then they take turns writing numbers on a blackboard. Lisa starts by writing $1$. Thereafter, when one of them has written the number $n$, the next player writes down either $n + 1$ or

$2n$, provided the number is not greater than $N$. The player who writes $N$ on the blackboard wins.

**(a)** Determine which player has a winning strategy if $N = 2025$.
**(b)** Find the number of positive integers $N \leq 2025$ for which Bart has a winning strategy.

# *Medium–Hard Problems*

***MH–125.*** *Proposed by Michel Bataille, Rouen, France.* For a positive integer $x$, let $v(x)$ denote the greatest of the integers $r \geq 0$ such that $2^r$ divides $x$ and let $m, n$ be positive integers. Prove that $v(2023^m - 1) = v(2025^n - 1)$ if and only if $v(m) = v(n) \neq 0$.

***MH–126.*** *Proposed by Jordi Ferré García, CFIS, BarcelonaTech, Barcelona, Spain.* Let $ABC$ be a triangle with $AB < AC$, and $H$ be its orthocenter. Let $E$ and $F$ be the intersection of lines $BH$ and $CH$ with $AC$ and $AB$ respectively, and $H'$ be a point on line $EF$ such that $BH' \perp EF$. If we let $M$ be the midpoint of sid $BC$, and $T$ be the intersection of line $AM$ with the circumcircle of $ABC$, show that lines $MH'$ and $TH$ intersect on the circumcircle of triangle $MEF$.

***MH–127.*** *Proposed by Ruben Mason Carpenter, Yale University, New Haven, USA.* Let $p$ be a prime, and let $a_1, \ldots, a_p$ be positive integers, none of them divisible by $p$. Prove that, for every integer $n$, there is a nonempty subset $S \subset \{1, 2, \ldots, p\}$ such that

$$n - \sum_{s \in S} a_s$$

is divisible by $p$.

***MH–128.*** *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let $a, b, c$ be positive real numbers such that the sum of their inverses equals the inverse of their product. Find the maximum value of

$$\frac{a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}}{a^4 + b^4 + c^4}.$$

***MH–129.*** *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all divisors of a positive integer $n$. Find all $n$, such that $k \geq 6$ and

$$\frac{{d_6}^2 + 2024}{5d_4} = n.$$

***MH–130.*** *Proposed by Ander Lamaison Vidarte, Brno, Czech Republic.* An increasing sequence of positive integers $a_1 < a_2 < \cdots < a_n$ is **requenense** if for every $2 \leq k \leq n-1$ we have that $a_{k-1}a_{k+1}$ divides $a_k^4$.

- Prove that there exists a requenense sequence of length $10^6+1$ with $a_{10^6} = 6^{2024}$.
- Prove that there does not exist a requenense sequence of length $10^8+1$ with $a_{10^8} = 6^{2024}$.

# *Advanced Problems*

***A–125.*** *Proposed by José Luis Díaz Barrero, Barcelona, Spain.* Compute

$$\lim_{n\to\infty} \frac{1}{n^n} \prod_{k=1}^{n} \left( \frac{n\sqrt{n} + (n+1)\sqrt{k}}{\sqrt{n} + \sqrt{k}} \right).$$

***A–126.*** *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.* Consider in $\mathbb{R}^3$ a sphere and one of its equatorial planes. Find the geometric locus of the vertices of the cones circumscribed to the sphere and whose trace on the equatorial plane is a parabola.

***A–127.*** *Proposed by Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA.* Let the function $f(x)$ have a continuous second derivative on $[-a, a]$. Prove that if $f(0) = 0$, there exists $\xi \in (-a, a)$ such that $f''(\xi) = (1/a^2)[f(a) + f(-a)]$.

***A–128.*** *Proposed by Vasile Mircea Popa, Lucian Blaga University of Sibiu, Romania.* Prove that it holds:

$$\int_0^\infty \frac{|\cos(x)|}{1+x^2} dx = \frac{e^2+1}{e} \arctan\left(\frac{1}{e}\right).$$

***A–129.*** *Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploieşţi, Romania Romania.* Prove that $\frac{7}{6}$ is the least positive value of the constant $k$ such that

$$x_1^k + x_2^k + x_3^k + x_4^k + x_5^k \geq 5$$

for any nonnegative real numbers $x_i$ with at most one $x_i < 1$ and $x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 = 5$.

***A–130.*** *Proposed by Michel Bataille, Rouen, France.* For each positive integer $n$, let

$$I_n = \int_0^1 \frac{x(x^{2n}-1)\ln(x+1)}{x^2-1}\, dx.$$

Prove that there exist real numbers $a, b$ such that

$$\lim_{n\to\infty}(I_n - (a + b\ln n)) = 0.$$

# *Mathlessons*

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

# *Idempotent matrices and matrices $A$ such that rank$(A) \leq$ rank$(A^2)$*

**Michel Bataille**

**Abstract**

We present a mathlesson centered on idempotent matrices and extended to matrices $A$ such that $\operatorname{rank}(A) \leq \operatorname{rank}(A^2)$. Classical results are proved and illustrated by various problems, often extracted from problem corners.

## 1 Introduction

Unless mentioned otherwise, the matrices are $n$-by-$n$ matrices with complex entries. $O_n$ denotes the zero matrix and $I_n$ the unit matrix. Elements of $\mathbb{C}^n$ are considered as column vectors. Recall that for any matrix $A$,

$$\operatorname{range}(A) = \{AX : X \in \mathbb{C}^n\}, \quad \ker(A) = \{X \in \mathbb{C}^n : AX = 0\}$$

and $\operatorname{rank}(A) = \dim(\operatorname{range}(A))$. Note that $\operatorname{range}(A^2) \subset \operatorname{range}(A)$ for any $A$, hence the matrices such that $\operatorname{rank}(A) \leq \operatorname{rank}(A^2)$ are those satisfying $\operatorname{range}(A^2) = \operatorname{range}(A)$ (and $\operatorname{rank}(A) = \operatorname{rank}(A^2)$). Among them, of course there are the matrices such that $A^2 = A$, called idempotent matrices. Quite a number of their properties are offered in the next section. The final section will focus on an unexpected characterization of matrices such that $\operatorname{rank}(A) = \operatorname{rank}(A^2)$, with some applications.

# 2 Idempotent matrices

A cornerstone of linear algebra is the so-called rank-nullity theorem which asserts that $\text{rank}(A) + \dim(\ker(A)) = n$. This holds for any $A$, but is particularly obvious if

$$\text{range}(A) \oplus \ker(A) = \mathbb{C}^n. \qquad (*)$$

We show that $(*)$ holds if $A$ is idempotent. The proof below shows that it even holds as soon as $\text{range}(A^2) = \text{range}(A)$.

Suppose that $\text{range}(A^2) = \text{range}(A)$. From the rank-nullity theorem, $\dim(\ker(A)) = \dim(\ker(A^2)$ so that $\ker(A) = \ker(A^2)$ (since $\ker(A) \subset \ker(A^2)$ for any $A$).
Now, if $X \in \text{range}(A) \cap \ker(A)$, then $AX = 0$ and $X = AX_0$ for some $X_0$ and therefore $A^2X_0 = 0$. Thus, $X_0 \in \ker(A^2) = \ker(A)$ so that $X = AX_0 = 0$. This shows that $\text{range}(A) \cap \ker(A) = \{0\}$ and the rank-nullity theorem then ensures that $(*)$ holds.
Note that conversely, if $(*)$ holds and $Y = AX \in \text{range}(A)$, then $X = Y_1 + N$ with $Y_1 = AX_1 \in \text{range}(A)$ and $N \in \ker(A)$ so that $Y = A^2X_1$. This shows that $\text{range}(A) \subset \text{range}(A^2)$. Thus, $(*)$ is equivalent to $\text{rank}(A) \leq \text{rank}(A^2)$.

In addition to $(*)$, an idempotent matrix $A$ satisfies the following:

(a) $I_n - A$ is idempotent

(b) $\text{range}(A) = \ker(I_n - A),\ \ker(A) = \text{range}(I_n - A)$

(c) If $1 \leq r = \text{rank}(A) \leq n - 1$, then $A$ is similar to $J_r$ where $J_r$ contains $I_r$ in its north-west corner and $0$s elsewhere.

(d) The trace $\text{tr}(A)$ of $A$ is equal to $\text{rank}(A)$

*Proof.* (a) follows from $(I_n - A)^2 = I_n - 2A + A^2 = I_n - A$.

(b) If $X \in \ker(I_n - A)$, then $X = AX$, hence $X \in \text{range}(A)$. Conversely, if $Y = AX \in \text{range}(A)$, then $(I_n - A)Y = AX - A^2X = 0$, hence $Y \in \ker(I_n - A)$.
Applying the result to $I_n - A$ gives $\text{range}(I_n - A) = \ker(A)$.

(c) $A$ satisfies $(*)$, hence we may pick a basis $X_1, \dots X_n$ of $\mathbb{C}^n$ such that $X_1, \dots X_r$ is a basis of $\text{range}(A)$ and $X_{r+1}, \dots X_n$ is a basis of $\ker(A)$. Then, we have $AX_i = X_i$ for $i = 1, \ldots, r$ and $AX_i = 0$ for $i = r+1, \ldots, n$. The matrix of $A$ (considered as an endomorphism of $\mathbb{C}^n$) in this basis is $J_r$, hence $A$ is similar to $J_r$.

(d) If $1 \leq r = \operatorname{rank}(A) \leq n-1$, then $\operatorname{tr}(A) = \operatorname{tr}(J_r) = r$. If $r = 0$, then $A = O_n$ and $\operatorname{tr}(A) = 0 = \operatorname{rank}(A)$; if $r = n$, then $A$ is invertible and $A^2 = A$ yields $A = I_n$, hence $\operatorname{tr}(A) = n = \operatorname{rank}(A)$.

We proceed with our study of idempotent matrices by presenting a selection of problems in which they play the main part.

**Problem 1.** Let $A, B$ be two idempotent matrices. Prove that $A + B$ is idempotent if and only if $AB = BA = O_n$.

*Solution.* Since

$$(A+B)^2 = A^2 + AB + BA + B^2 = A + B + (AB + BA),$$

the matrix $A + B$ is idempotent if and only if $AB + BA = O_n$. This clearly holds if $AB = BA = O_n$. Conversely, $AB + BA = O_n$ implies $AB = A^2B = -ABA$ and $BA = BA^2 = -ABA$, hence $AB = BA$. From $2AB = AB + BA = O_n$, we then deduce $AB = BA = O_n$.

**Problem 2.** ([6]) Let $A$ be an $n \times n$ matrix with complex entries. Show that $A$ is idempotent if, and only if,

$$\operatorname{rank}(A) \leq \operatorname{tr}(A) \qquad \text{and} \qquad \operatorname{rank}(I_n - A) \leq \operatorname{tr}(I_n - A)$$

*Solution.* If $A$ is idempotent, then from properties (b) and (d) above, we have $\operatorname{rank}(A) = \operatorname{tr}(A)$ and $\operatorname{rank}(I_n - A) = \operatorname{tr}(I_n - A)$.
Conversely, if $\operatorname{rank}(A) \leq \operatorname{tr}(A)$ and $\operatorname{rank}(I_n - A) \leq \operatorname{tr}(I_n - A)$, then we have

$$\begin{aligned} n = \operatorname{rank}(A + I_n - A) &\leq \operatorname{rank}(A) + \operatorname{rank}(I_n - A) \\ &\leq \operatorname{tr}(A) + \operatorname{tr}(I_n - A) = \operatorname{tr}(I_n) = n \end{aligned}$$

and so $\operatorname{rank}(A) + \operatorname{rank}(I_n - A) = n$.
From the rank-nullity theorem, $\dim\ker(A) + \dim\ker(I_n - A) = n$ and since in addition $\ker(A) \cap \ker(I_n - A) = \{0\}$, we conclude: $\mathbb{C}^n = \ker(A) \oplus \ker(I_n - A)$.

Now, any vector $X$ of $\mathbb{C}^n$ can be written as $X_0 + X_1$ where $X_0$ is in

$\ker(A)$ and $X_1$ in $\ker(I_n - A)$. Hence, noticing that $A(I_n - A) = (I_n - A)A$,

$$A(I_n - A)X = (I_n - A)AX_0 + A(I_n - A)X_1 = (I_n - A)0 + A0 = 0.$$

Since $X$ is arbitrary, it follows that $A(I_n - A) = O$ and $A$ is idempotent.

**Problem 3.** (adapted from **[3]** and **[5]**). Let $A, B$ be idempotent matrices.
1/ Let $m$ be a complex number with $m \neq -1$. Prove that $I_n + mA$ is invertible.
2/ If $m \neq 0, -1$, prove that $\text{rank}(A + mB) = \text{rank}(A + B)$.
3/ Prove that $\text{rank}(A - B) \leq \text{rank}(A + B)$.

*Solution.* 1/ Note that $A^n = A$ for all integer $n \geq 2$. With an awful lack of rigour, we write

$$\begin{aligned}\frac{1}{I_n + mA} &= I_n - mA + m^2A^2 - m^3A^3 + \cdots \\ &= I_n - mA(1 - m + m^2 - \cdots) = I_n - \frac{m}{1+m}A.\end{aligned}$$

Although not an admissible proof, this suggests the following calculation

$$(I_n + mA)\Big(I_n - \frac{m}{1+m}A\Big).$$

The found result, $I_n$, does prove that $I_n + mA$ is invertible with

$$(I_n + mA)^{-1} = I_n - \frac{m}{1+m}A.$$

2/ From 1/, the matrices

$$U = I_n + \frac{1-m}{2m}A, \qquad V = \frac{2m}{1+m}\Big(I_n - \frac{1-m}{2}B\Big)$$

are invertible. A simple calculation shows that $U(A + B)V = A + mB$ and $\text{rank}(A + mB) = \text{rank}(A + B)$ follows.

3/ Let $r = \text{rank}(A - B)$. We may suppose that $r \geq 1$. Let $C$ be an $r$-by-$r$ submatrix of $A - B$ such that $\det(C) \neq 0$. For $n \in \mathbb{N}$, let $C_n$ be the $r$-by-$r$ submatrix of $A - \frac{n}{n+1}B$ obtained by selecting the same rows and columns as for $C$. As $n \to \infty$, $C_n$ converges to $C$, hence $\det(C_n)$ converges to $\det(C)$. It follows that for $n$ large enough we have $\det(C_n) \neq 0$ and therefore $\operatorname{rank}\Big(A - \frac{n}{n+1}B\Big) \geq r$. Since from 2/, $\operatorname{rank}\Big(A - \frac{n}{n+1}B\Big) = \operatorname{rank}(A+B)$, we are done.

**Problem 4.** ([4], slightly modified) Let $A, B$ be two matrices and suppose that $B$ is idempotent. Show that $AB = BA$ if and only if $\operatorname{range}(AB) = \operatorname{range}(BA)$ and $\operatorname{range}(A^TB^T) = \operatorname{range}(B^TA^T)$, where $C^T$ denotes the transpose of $C$.

*Solution.* If $AB = BA$, then clearly, $\operatorname{range}(AB) = \operatorname{range}(BA)$ and $\operatorname{range}(A^TB^T) = \operatorname{range}(B^TA^T)$ [since $B^TA^T = (AB)^T = (BA)^T = A^TB^T$].
Conversely, suppose that $\operatorname{range}(AB) = \operatorname{range}(BA)$ and $\operatorname{range}(A^TB^T) = \operatorname{range}(B^TA^T)$. As a lemma, we first show that

$$\ker(AB) = \ker(BA). \tag{1}$$

Suppose that $X \in \ker(AB)$. Then $ABX = 0$, or equivalently, $X^TB^TA^T = 0$ and so $X^TC_j = 0$ $(j = 1, \ldots, n)$ where $C_j$ denotes the $j$th column of $B^TA^T$. Since $\operatorname{range}(B^TA^T)$ is spanned by $C_1, C_2, \ldots, C_n$, it follows that $X^TY = 0$ for all $Y$ in $\operatorname{range}(B^TA^T) = \operatorname{range}(A^TB^T)$. As a result, $X^TA^TB^T = 0$, hence $BAX = 0$ that is, $X \in \ker(BA)$. Similarly, $\ker(BA) \subset \ker(AB)$ and (1) follows.

Let $X$ be any column vector. On the one hand, $X - BX \in \ker(AB)$ (since $B^2 = B$), hence $X - BX \in \ker(BA)$ and so

$$BAX = BABX. \tag{2}$$

On the other hand, $ABX \in \operatorname{range}(AB)$, hence $ABX \in \operatorname{range}(BA)$ and so $ABX = BAX'$ for some column vector $X'$. It follows that

$$BABX = B^2AX' = BAX' = ABX. \tag{3}$$

From (2) and (3), $ABX = BAX$ and the conclusion follows since $X$ is arbitrary.

Our last problem establishes a transition with our next section.

**Problem 5.** ([2]) Let $\mathcal{G} = \{A_1, A_2, \ldots, A_m\} \subset \mathcal{M}_n(\mathbb{R})$ such that

$(\mathcal{G}, \cdot)$ is a group. Prove that $\operatorname{tr}(A_1 + A_2 + \cdots + A_m)$ is an integer divisible by $m$.

*Solution.* Let $B = A_1 + A_2 + \cdots + A_m$. Since $(\mathcal{G}, \cdot)$ is a group, for any fixed $j \in \{1, 2, \ldots, m\}$ the mapping $A \mapsto AA_j$ is a bijection from $\mathcal{G}$ onto $\mathcal{G}$. It follows that $BA_j = B$, and since this is true for $j = 1, 2, \ldots, m$, we have

$$B^2 = B(A_1 + +A_2 + \cdots + A_m) = BA_1 + BA_2 + \cdots + BA_m = mB.$$

Now, the matrix $C = \frac{1}{m}B$ is idempotent since

$$C^2 = \frac{1}{m^2}B^2 = \frac{1}{m^2}(mB) = C,$$

hence $\operatorname{tr}(C) = \operatorname{rank}(C)$. Thus, $\operatorname{tr}(B) = m \cdot (\operatorname{rank}(C))$, a multiple of $m$.

# 3 The group inverse

In the wake of the previous problem, we consider any subset $\mathcal{G}$ of $\mathcal{M}_n(\mathbb{C})$, which, once equipped with the multiplication of matrices, is a group. Let us call *mat-group* such a group. Let $E$ be the neutral element of $\mathcal{G}$: for any $A \in \mathcal{G}$, we have $AE = EA = A$. In particular, $E^2 = E$, hence $E$ is idempotent, and $\operatorname{rank}(A) \leq \operatorname{rank}(E)$. Since $AA' = A'A = E$ for some $A' \in \mathcal{G}$, we also have $\operatorname{rank}(E) \leq \operatorname{rank}(A)$. Thus $\operatorname{rank}(A) = \operatorname{rank}(E)$ for all matrices $A$ of $\mathcal{G}$. It follows that $\operatorname{rank}(A^2) = \operatorname{rank}(A)$ and the matrices of $\mathcal{G}$ are fully in our subject. Conversely, let us show that any matrix $A$ satisfying $\operatorname{rank}(A^2) = \operatorname{rank}(A)$ belongs to some mat-group $\mathcal{G}$. The cases $A = O_n, A = I_n$ are obvious. Suppose now that $1 \leq r = \operatorname{rank}(A) \leq n - 1$. Since $A$ satisfies $(*)$, $A = P^{-1}U_rP$ where $P \in GL_n(\mathbb{C})$ and $U_r$ is formed by an $r$-by-$r$ invertible matrix $V$ in the north-west corner and 0s elsewhere. The result follows because when $V$ traverses $GL_r(\mathbb{C})$, the corresponding matrices $P^{-1}U_rP$ form a mat-group.
It is interesting to note that $A'$ is the unique matrix satisfying $AA'A = A$, $A'AA' = A'$, $AA' = A'A$. Indeed, if $A''$ satisfies

$AA''A = A$, $A''AA'' = A''$, $AA'' = A''A$, then $AA'' = A'A^2A'' = A'A$ and so $A'' = AA''^2 = A'AA'' = A'^2A = A'$. Thus, the matrix $A'$ is the inverse of $A$ in any mat-group containg $A$. It is called the group inverse of $A$.

Here is a couple of problems illustrating the results of this section.

**Problem 6.** Let $A, B$ be such that $AB = BA$ and $A^3B = A$. Show that $\text{rank}(A^2) = \text{rank}(A)$ and find the group inverse of $A$. Deduce that $A^{2n}B^n = A^2B$ for all positive integer $n$.

*Solution.* Since $A = A^2(AB)$, we have $\text{rank}(A) \leq \text{rank}(A^2)$, hence $A$ has a group inverse $A'$. Let us show that $A' = AB$. This follows from

$$A(AB)A = A^2BA = A^3B = A$$

and

$$(AB)A(AB) = BA^3B = BA = AB, \quad (AB)A = A^2B = A(AB).$$

Since $AA'$ is idempotent, we have $(AA')^n = AA'$ for all $n \geq 1$, that is, $(A^2B)^n = A^2B$ or $A^{2n}B^n = A^2B$ (since $A^2B = BA^2$).

**Problem 7.** ([1]) Let $A, B$ have the same rank and satisfy $A^2B = A$. Prove that $B^2A = B$.

*Solution.* First, we have

$$\text{rank}(A) = \text{rank}(A^2B) \leq \text{rank}(AB) \leq \text{rank}(B) = \text{rank}(A)$$

so that

$$\text{rank}(A) = \text{rank}(B) = \text{rank}(AB).$$

From the rank-nullity theorem applied to the restriction of the endomorphism $A$ to $\text{range}(B)$, we have

$$\text{rank}(B) = \text{rank}(AB) + \dim(\ker(A) \cap \text{range}(B)),$$

hence $\ker(A) \cap \text{range}(B) = \{0\}$.

On the other hand, since $\text{rank}(A) \leq \text{rank}(A^2)$, $A$ has a group inverse $A'$. Since

$$A'AB = A'(A^2B)B = (A'A^2)B^2 = AB^2$$

and

$$A'AB = A'(A'A^2)B = A'^2A^2B = A'^2A = A',$$

we see that $A' = AB^2$. In particular, $AB^2$ commutes with $A$: $AAB^2 = AB^2A$ or $AB = AB^2A$ (since $A^2B = A$).

Now, if $X \in \mathbb{C}^n$, then $(B - B^2A)X \in \ker(A) \cap \text{range}(B)$, hence $(B - B^2A)X = 0$ and since $X$ is arbitrary, $B^2A = B$.

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# *Contests*

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to find interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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# *Problems and solutions from the 11th edition of BarcelonaTech Mathcontest*

**Óscar Rivero Salgado and J. L. Díaz-Barrero**

## 1 Problems and solutions

Hereafter, we present the four problems that appeared in the paper given to the contestants of the BarcelonaTech Mathcontest 2024, as well as their official solutions.

***Problem 1.*** Prove that there is a unique digit $\alpha \in \{0,1,2,3,4,5,6,7,8,9\}$ for which there is a unique positive integer $n$ ending in $\alpha$ and with the property that the number $2n+1$ is the square of a prime number.

**Solution.** If the odd number $2n+1$ is the square of the prime number $p$, then $p$ is also odd. From the relation $p^2 = 2n+1$, it follows that $n = (p^2-1)/2 = (p-1)(p+1)/2$. Let us construct a table of the first few odd prime numbers $p$ and their corresponding numbers $n$.

| $p$ | 3 | 5 | 7 | 11 | 13 | 17 | 19 | 23 | 29 | 31 | 37 | 41 | 43 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $n$ | 4 | 12 | 24 | 60 | 84 | 144 | 180 | 264 | 420 | 480 | 684 | 840 | 924 |

The number $n$ is obviously even (as the table for several values of $p$ reveals) and it is divisible by four. This can be seen from the fact that the product $(p-1)(p+1)$ of two consecutive even numbers is always divisible by eight. Moreover, we observe from the table the numbers by which $n$ ends, resulting that numbers 0 and 4 occur several times, the number 2 only once, and the numbers 6 and 8 do not occur.

Let us see how the number $n$ ends in relation to the digit $a$ in which $p$ ends. If $p = 10k + a$, where $k$ is a non-negative integer and $a$ is an odd digit, then for each possible value of $a$, we have

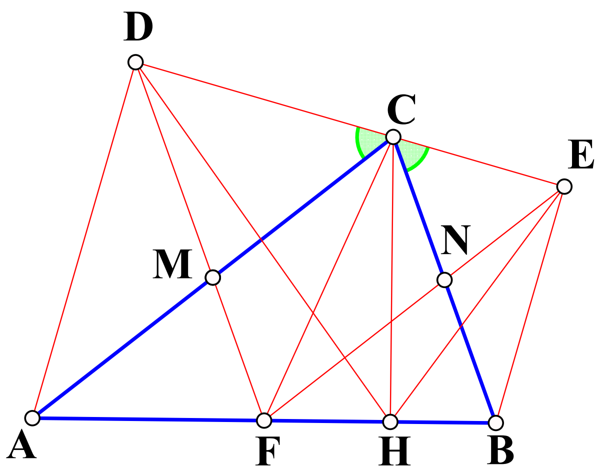
- If $a = 1$, then $n = 10k(5k + 1)$, so the number $n$ ends in $0$.
- If $a = 3$, then $n = 10k(5k + 4) + 4$, so the number $n$ ends in $4$.
- If $a = 5$, then $n = 10(5k^2 + 5k + 1) + 2$, so the number $n$ ends in $2$.
- If $a = 7$, then $n = 10(5k^2 + 7k + 2) + 4$, so the number $n$ ends in $4$.
- If $a = 9$, then $n = 10(k + 1)(5k + 4)$, so the number $n$ ends in $0$.

If $2n + 1$ is the square of an odd prime (odd number), then $n$ end only with the digits $0, 2, 4$. The only candidate for the number $\alpha$ to be found is $2$, since both 0 there are more than one option (e.g. 60 and 180) and the same occurs for 4 (e.g. 4 and 24). The digit $\alpha = 2$ works because if $2n + 1$ is the square of the prime factor and $n$ ends in $2$, the prime factor $p$ can be expressed as $10k + 5 = 5(2k + 1)$, so it is divisible by five. The only prime that is divisible by five is the number $5$. Thus, the desired number is $\alpha = 2$, for it there is a positive integer $n = 12$, which ends in $\alpha$, and $2n + 1 = 25$ is the square of a prime number.

***Problem 2.*** Let $ABC$ be a triangle and let $\ell$ be its external bisector through $C$. The points $D$ and $E$ are the projections of $A$ and $B$ onto $\ell$. The line $CH$ ($H \in AB$) is the altitude from $C$ to $AB$ and the point $F$ is the midpoint of $AB$. Prove that the points $D, E, H, F$ lie on the same circle.

**Solution.** Let $M$ and $N$ be the midpoints of $AC$ and $BC$, respectively. Then $FM \parallel BC$ and $FN \parallel AC$.

On the other hand, since $DM$ is a median to the hypotenuse in




Scheme for solving problem 2.

the right-angled $\triangle CDA$, then $M$ is the circumcenter of $\triangle CDA$. Thus, $\angle MDE = \angle MCD = \angle BCE$. Therefore, $DM \parallel BC$, hence the points $D, M$ and $F$ are collinear. Likewise, the points $E, N$ and $F$ are collinear. Also, $CMFN$ is a parallelogram, so we have $\angle EFD = \angle ACB$.

The quadrilaterals $AHCD$ and $BHCE$ are cyclic, so $\angle AHD = \angle ACD = \angle BCE = \angle BHE$. These give $\angle DHE = \angle ACB = \angle EFD$. Therefore, the quadrilateral $DEHF$ is cyclic, as desired.

***Problem 3.*** Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Prove that

$$\frac{1}{a_1} + \frac{2}{a_1 + a_2} + \ldots + \frac{n}{a_1 + \ldots + a_n} < e\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right).$$

**Solution.** Since $e > \left(1 + \frac{1}{n}\right)^n$ for all $n \geq 1$, then

$$e^k = \underbrace{e \cdot e \cdot e \cdots e}_{k} > \left(1 + \frac{1}{1}\right)^1 \left(1 + \frac{1}{2}\right)^2 \cdots \left(1 + \frac{1}{k}\right)^k$$

$$= 2 \cdot \left(\frac{3}{2}\right)^2 \cdot \left(\frac{4}{3}\right)^3 \cdot \cdots \cdot \left(\frac{k+1}{k}\right)^k = \frac{(k+1)^k}{k!}$$

holds for all integer $k \geq 1$. From the preceding result, it immediately follows that

$$k! > \left(\frac{k+1}{e}\right)^k \quad \text{for} \quad k \geq 1 \qquad (1)$$

For $1 \leq k \leq n$ let $x_k = \dfrac{k}{a_1 + \ldots + a_k}$. Using mean inequalities and (1), we have for $1 \leq k \leq n$:

$$\frac{1}{x_k} = \frac{a_1 + a_2 + \ldots + a_k}{k} \geq (a_1 \cdot a_2 \cdots a_k)^{1/k}$$

$$= (k!)^{1/k}\Big(a_1 \cdot \frac{a_2}{2} \cdot \frac{a_3}{3} \cdots \frac{a_k}{k}\Big)^{1/k} > \frac{k+1}{e} \cdot \frac{k}{\dfrac{1}{a_1} + \dfrac{2}{a_2} + \ldots + \dfrac{k}{a_k}},$$

and

$$x_k < \frac{e}{k(k+1)} \sum_{i=1}^{k} \frac{i}{a_i}.$$

Since $\dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$, then

$$\sum_{k=1}^{n} x_k < e \sum_{k=1}^{n} \frac{1}{k(k+1)} \sum_{i=1}^{k} \frac{i}{a_i}$$

$$= e \sum_{i=1}^{n} \frac{i}{a_i} \sum_{k=i}^{n} \Big(\frac{1}{k} - \frac{1}{k+1}\Big) < e \sum_{i=1}^{n} \frac{i}{a_i} \cdot \frac{1}{i} = e \sum_{i=1}^{n} \frac{1}{a_i},$$

and

$$\sum_{k=1}^{n} \frac{k}{a_1 + a_2 + \ldots + a_n} < e \sum_{i=1}^{n} \frac{1}{a_i}$$

holds.

***Problem 4.*** Fix an integer $n \geq 1$. Let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a monic polynomial of degree $n$ with integer coefficients. Over all possible choices of $P$, determine (as a function of $n$) the largest value that the quantity

$$\gcd(P(0), P(1), \ldots, P(n))$$

takes, or prove that it is unbounded.

**Solution.** The answer is that the quantity is indeed bounded, and it's maximum value is $n!$. The solution naturally splits into two parts.

**Claim 1.** *There exists a monic polynomial of degree $n$ with integer coefficients such that*

$$\gcd(P(0), P(1), \ldots, P(n)) = n!$$

**Proof.** Choose $P(x) = x(x-1)\cdots(x-(n-1))$, which is clearly a monic polynomial of degree $n$ with integer coefficients. Then, for $1 \leq i \leq n-1$, $P(i) = 0$, and addicionally $P(n) = n \cdot (n-1)\cdots 1 = n!$. Thus the greatest integer that divides all of $P(0), P(1), \ldots, P(n)$ is exactly $n!$.

Now, we must prove that $n!$ cannot be exceeded. This can be done by conjecturing and proving the following result.

**Lemma 1.** *For $n \geq 1$, let $P(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Then it must hold that*

$$\gcd(P(0), P(1), \ldots, P(n)) \mid |c_n| \cdot n!$$

**Proof.** We exhibit a straightforward proof by induction on $n$. The base case $n = 1$ is clear. If $P(x) = c_1 x + c_0$, then

$$\gcd(P(0), P(1)) = \gcd(c_0, c_1 + c_0) = \gcd(c_0, c_1) \mid |c_1| = |c_1| \cdot 1!$$

For the inductive step, let $P$ be as in the problem statement. The key is to consider the polynomial $\Delta(x) = P(x+1) - P(x)$.

This is relevant since, if $d = \gcd(P(0), P(1), \ldots, P(n))$, then it will be true that $d \mid \Delta(i) = P(i+1) - P(i)$ for every $0 \leq i \leq n-1$. We could apply the induction hypothesis to get information if $\Delta$ had degree $n-1$. But this is quite clear: Indeed, by Newton's binomial theorem

$$\Delta(x) = \sum_{i=0}^{n} c_i((x+1)^i - x^i) = \sum_{i=0}^{n} c_i \sum_{j=0}^{i-1} \binom{i}{j} x^j = na_n x^{n-1} + Q(x),$$

where $Q(x)$ is a polynomial of degree at most $n-2$.

Hence, we have the chain of divisibilities

$$\gcd(P(0),\ldots,P(n)) \mid \gcd(\Delta(0),\ldots,\Delta(n-1)) \mid |c_n n|\cdot(n-1)! = |a_n|\cdot n!,$$

completing the inductive step. Finally, the bound required from the problem follows by taking $c_n = 1$ in the above lemma.

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# Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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## Elementary Problems

***E–119.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Find all the prime numbers $p$ and integers $n$ such that $n^4 + n^2 + p = 2452$.

**Solution 1 by Michel Bataille, Rouen, France.** It is readily checked that $(2, 7)$ and $(2, -7)$ are solutions for the pair $(p, n)$. We show that there are no other solutions.
Let $(p, n)$ be a solution. The integer $n$ must be odd (otherwise $n^2, n^4$ would be multiple of $4$, as $2452$ is, a contradiction since $p$ is not) so we have $n^2 \equiv n^4 \equiv 1 \pmod 4$. It follows that $p \equiv 2 \pmod 4$ so that $p = 2$. Then $n^4 + n^2 - 2450 = 0$ gives $n^2 = 49$ and therefore $n = 7$ or $-7$. This completes the proof.

**Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA and Ioan Viorel Codreanu, Satulung, Maramures, Romania (same solution).** Let $n$ be an integer. Regardless of the parity of $n$, $n^4 + n^2$

is even, so $n^4 + n^2 + p = 2452$ requires that $p$ be even. As the only even prime is 2, $p$ must be equal to 2. It then follows that

$$n^4 + n^2 = 2450 \quad \text{or} \quad (n^2 - 49)(n^2 + 50) = 0;$$

hence, $n = \pm 7$. Thus, $n^4 + n^2 + p = 2452$ with $p$ prime and $n$ an integer if and only if $p = 2$ and $n = \pm 7$.

**Solution 3 by the proposer.** We distinguish two cases:

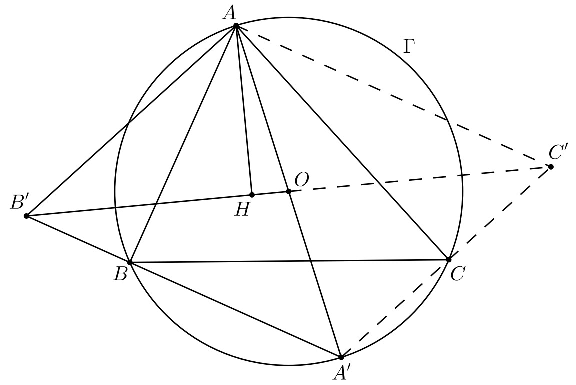
- If $n$ even, then $n^4 + n^2$ is even (use congruences $\pmod 2$) and from $n^4 + n^2 + p = 2452$, it follows $p = 2$.
- If $n$ odd, then $n^4 + n^2$ again is even (use congruences $\pmod 2$) and from $n^4 + n^2 + p = 2452$, it follows $p = 2$.

Thus, we have to find the solutions in positive integers of $n^4 + n^2 + 2 = 2452$ or $n^4 + n^2 - 2450 = 0$. Putting $t = n^2$ in the last equation, we get $t^2 + t - 2450 = 0$ or $(t - 49)(t + 50) = 0$ and $(n^2 - 49)(n^2 + 50) = 0$. It has the only integer solutions $n = \pm 7$ and the answer is $(p, n) = (2, -7)$ and $(p, n) = (2, 7)$.

**Also solved by** *José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain.*

***E–120.*** *Proposed by Michel Bataille, Rouen, France.* Let triangle $ABC$ (with no right angle) be inscribed in a circle with centre $O$ and let $A'$ be diametrically opposite to $A$. The perpendicular to $AC$ through $A$ intersects the line $A'B$ at $B'$. If $H$ is the orthogonal projection of $A$ onto the line $OB'$, prove that $B, H, O, C$ are concyclic.

**Solution 1 by the proposer.** Since $AA'$ is a diameter of the circumcircle $\Gamma$ of $\Delta ABC$, we have $A'B \perp BA$ and $A'C \perp CA$ (hence $CA'$ is parallel to $AB'$). Let $C'$ be the reflection of $B'$ about $O$. Then $AB'A'C'$ is a parallelogram ($O$ is the midpoint of $B'C'$ and $AA'$), hence $A'C'$ is parallel to $AB'$. It follows that $A', C, C'$ are collinear. Note that $AC' \perp AB$ (since $AC'$ is parallel to $A'B$).




Scheme for solving Problem E-120

Now, we have

$$\begin{aligned}\angle(HB, HC) &= \angle(HB, HB') + \angle(HC', HC)\\ &= \angle(AB, AB') + \angle(AC', AC) \qquad (*)\\ &= \angle(AB, AC) + \angle(AC, AB') + \angle(AC', AB) + \angle(AB, AC)\\ &= 2\angle(AB, AC) \qquad (**)\end{aligned}$$

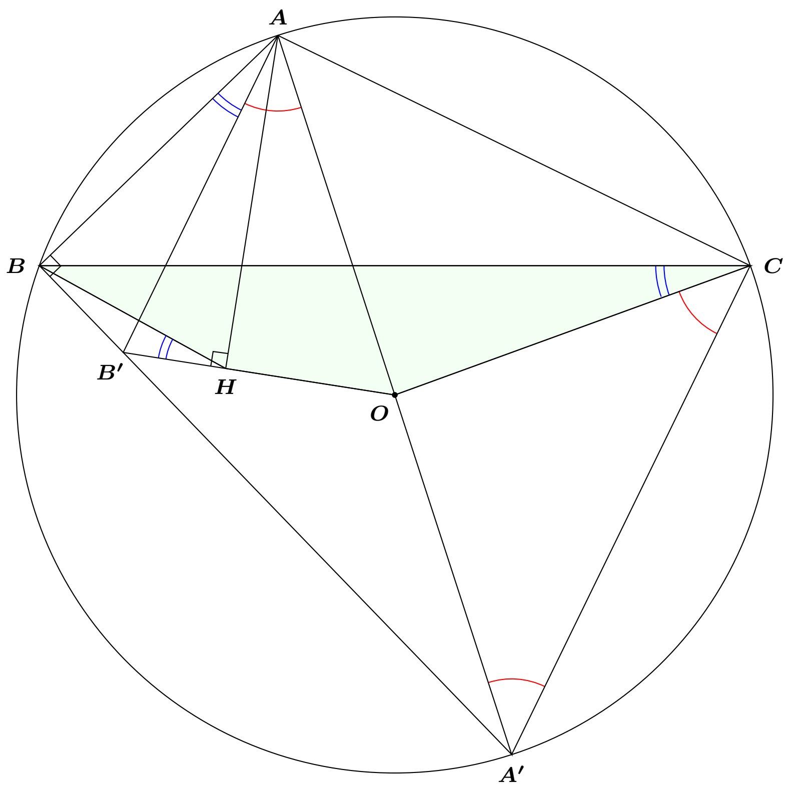
[$(*)$: since $A, H, B, B'$ (resp. $A, H, C, C'$) are on the circle with diameter $AB'$ (resp. $AC'$);
$(**)$: since $AC \perp AB'$ and $AB \perp AC'$.]
It follows that $\angle(HB, HC) = \angle(OB, OC)$ and therefore $H, O, B, C$ are concyclic (they are not collinear because of $\angle BAC \neq 90°$).

**Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.** The angle $ABA'$ is a right angle, since it is inscribed in a semicircle, as is $\angle A'CA$. The right angles at $B$ and $H$ make quadrilateral $ABB'H$ cyclic, and on chord $BB'$,

$$\angle B'HB = \angle B'AB. \tag{1}$$

In turn, $AB'$ and $CA'$, both perpendicular to $CA$, are parallel to each other, implying alternate angles $B'AA'$ and $AA'C$ are equal:

$$\angle B'AA' = AA'C. \tag{2}$$




Now, $ABA'C$ is cyclic, and on chord $BA'$,

$$\angle A'AB = \angle A'CB.$$

This may be written in the form

$$\angle A'AB' + \angle B'AB = \angle A'CO + \angle OCB$$

or, equivalently,

$$\angle AA'C + \angle B'HB = \angle OA'C + \angle OCB,$$

using (1), (2) and the fact that $\angle A'CO$ and $\angle OA'C$ are the base angles in isosceles triangle $OA'C$.

We subtract $\angle AA'C = \angle OA'C$ from both sides of the last equation, obtaining

$$\angle B'HB = \angle OCB,$$

that is, the exterior angle $B'HB$ of quadrilateral $BHOC$ is equal to the interior and opposite angle $C$, making $BHOC$ cyclic. This is equivalent to what we set out to prove.

**Also solved by** *José Luis Díaz-Barrero, Barcelona, Spain.*

***E–121.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Find the smallest side of a triangle $ABC$ knowing that the medians drawn by vertices $A$ and $B$ are perpendicular.

**Solution 1 by Michel Bataille, Rouen, France.** Let $BC = a, CA = b, AB = c$ and let $G$ be the centroid of $\Delta ABC$. Since $AG \perp GB$, we have $AB^2 = AG^2 + BG^2$, that is,

$$c^2 = \frac{2a^2 + 2c^2 - b^2}{9} + \frac{2b^2 + 2c^2 - a^2}{9} = \frac{a^2 + b^2 + 4c^2}{9},$$

from which we first deduce that $a^2 + b^2 = 5c^2$ and second that

$$\cos \angle ACB = \frac{a^2 + b^2 - c^2}{2ab} = \frac{2c^2}{ab}. \qquad (1)$$

Note also that $a \neq c$ since otherwise $b = 2c$ and $\frac{2c^2}{ab} = 1$, contradicting (1). Similarly, $b \neq c$.
Now, we calculate

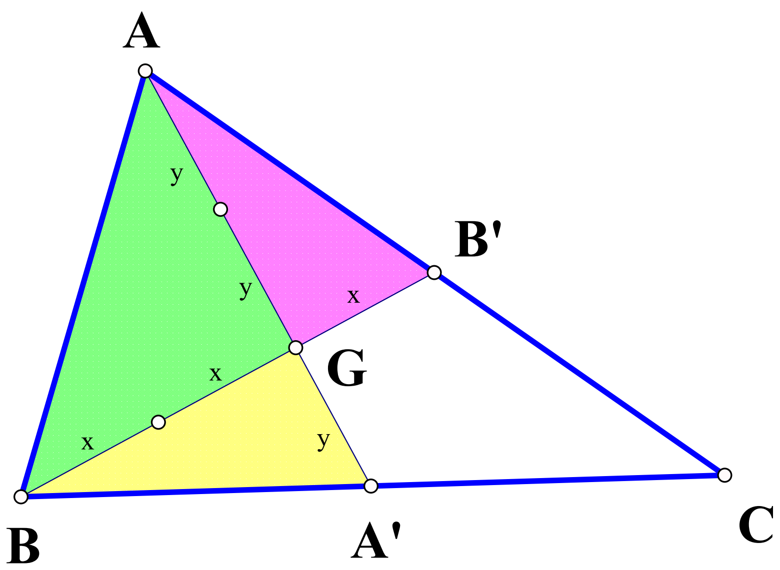
$$\cos \angle BAC = \frac{b^2 + c^2 - a^2}{2bc} = \frac{3c^2 - a^2}{bc}.$$

Now, from AM-GM and $a \neq c$, we have

$$2c^3 + a^3 = c^3 + c^3 + a^3 > 3(c^3 \cdot c^3 \cdot a^3)^{1/3} = 3c^2 a$$

and deduce that $\frac{3c^2 - a^2}{bc} < \frac{2c^2}{ab}$, that is, $\cos \angle BAC < \cos \angle ACB$. Similarly, $\cos \angle ABC < \cos \angle ACB$. It follows that $\angle ABC > \angle ACB$ and $\angle BAC > \angle ACB$ and therefore $b > c$ and $a > c$. Thus, the smallest side is $c = AB$.

**Solution 2 by the proposer.** Let $AB = c$, $BC = a$ and $AC = b$ be the length of the sides of $\triangle ABC$. In the figure below, we have three right triangles: $GAB, GAB'$and $GBA'$, respectively.




Scheme for solving problem E-121.

Applying Pithagoras theorem, yields $c^2 = 4x^2+4y^2$, $a^2/4 = 4x^2+y^2$ and $b^2/4 = x^2 + 4y^2$ from which we get

$$\begin{aligned} c^2 &= 4x^2 + 4y^2 < 16x^2 + 4y^2 = a^2 \Rightarrow c < a, \\ c^2 &= 4x^2 + 4y^2 < 16y^2 + 4x^2 = b^2 \Rightarrow c < b. \end{aligned}$$

Thus, the answer is that the shortest side of $\triangle ABC$ is $AB$.

**Also solved by** *José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain.*

***E–122.*** *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.* If $a, b, x, y$ are positive numbers then show that

$$\log_{\text{ab}}(a^x b^y)\ \log_{\text{ab}}\left(a^{1/\ x} b^{1/y}\right) \geq 1.$$

**Solution 1 by Michel Bataille, Rouen, France.** We recall that $\log_{ab}(u) = \frac{\ln(u)}{\ln(ab)}$ for positive $u$ (we assume that $ab \neq 1$). We have

$\log_{ab}(a^x b^y)\log_{ab}(a^{1/x}b^{1/y}) = \frac{N}{D}$ where $D = (\ln(a)+\ln(b))^2$ and

$$\begin{aligned} N &= (x\ln(a)+y\ln(b))\left(\frac{\ln(a)}{x}+\frac{\ln(b)}{y}\right) \\ &= (\ln(a))^2+(\ln(b))^2+\left(\frac{x}{y}+\frac{y}{x}\right)(\ln(a))(\ln(b)) \\ &= (\ln(a)+\ln(b))^2+\left(\frac{x}{y}+\frac{y}{x}-2\right)(\ln(a))(\ln(b)) \\ &= D+\left(\frac{x}{y}+\frac{y}{x}-2\right)(\ln(a))(\ln(b)). \end{aligned}$$

But $\frac{N}{D} \geq 1$ is equivalent to $N - D \geq 0$ (since $D > 0$) and $\frac{x}{y}+\frac{y}{x}-2 \geq 0$ (with equality only if $x = y = 1$), hence the required inequality holds if and only if $x = y = 1$ or $(\ln(a))(\ln(b)) \geq 0$.

**Solution 2 by G. C. Greubel, Newport News, VA, USA.** By changing base to the natural logarithm then the equation in question can be seen as:

$$\begin{gathered} \log_{ab}(a^x\, b^y)\ \log_{ab}(a^{1/x}\, b^{1/y}) \geq 1 \\ (x\ \ln a + y\ \ln b)\left(\frac{\ln a}{x}+\frac{\ln b}{y}\right) \geq (\ln(ab))^2 \\ ln^2 a + \left(\frac{x}{y}+\frac{y}{x}\right)\ln(ab) + \ln^2 b \geq (\ln a + \ln b)^2 \\ \left(\frac{x}{y}-2+\frac{y}{x}\right)\ \ln(a\underline{)} \geq 0 \\ \frac{1}{xy}\,(x-y)^2\ \ln(ab) \geq 0 \end{gathered}$$

or

$$(x-y)^2\ \ln(a^{1/x}\, b^{1/y}) \geq 0.$$

From this equation it can be noticed that equality occurs when either $x = y$, $a = 1$ or $b = 1$. Otherwise for $a, b \geq 1$ and $x, y \geq 0$ the equation holds.

**Solution 3 by Daniel Văcaru, Pitești, Romania.** We have

$$\log_{ab}(a^x b^y) = x\log_{ab} a + y\log_{ab} b \tag{1}$$

și

$$\log_{ab}\left(a^{1/x}b^{1/y}\right) = \frac{1}{x}\log_{ab} a + \frac{1}{y}\log_{ab} b \tag{2}$$

Then, we have

$$\log_{ab}(a^x b^y)\log_{ab}\left(a^{1/x}b^{1/y}\right) = (x\log_{ab} a + y\log_{ab} b)\cdot\left(\frac{1}{x}\log_{ab} a + \frac{1}{y}\log_{ab} b\right)$$

$$= \log^2_{ab} a + \frac{x}{y}\log_{ab} a\log_{ab} b + \frac{y}{x}\log_{ab} b\log_{ab} a + \log^2_{ab} b$$

$$= \log^2_{ab} a + \left(\frac{x}{y} + \frac{y}{x}\right)\log_{ab} a\log_{ab} b + \log^2_{ab} b$$

But $\frac{x}{y} + \frac{y}{x} \geq 2$ or $\frac{(x-y)^2}{xy} \geq 0$, for all $x, y > 0$. On account of the preceding, we obtain

$$\log_{ab}(a^x b^y)\log_{ab}\left(a^{1/x}b^{1/y}\right) \geq \log^2_{ab} a + 2\log_{ab} a\log_{ab} b + \log^2_{ab} b \tag{3}$$

We have

$$\log^2_{ab} a + 2\log_{ab} a\log_{ab} b + \log^2_{ab} b = (\log_{ab} a + \log_{ab} b)^2 = 1, \tag{4}$$

because we have $\log_{ab} a + \log_{ab} b = \log_{ab} ab = 1$. Using (1), (2),(3) and (4), we get

$$\log_{ab}(a^x b^y)\log_{ab}\left(a^{1/x}b^{1/y}\right) \geq 1.$$

**Solution 4 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.** The inequality may be written equivalently as

$$\begin{aligned}(x\ln a + y\ln b)\left(\frac{1}{x}\ln a + \frac{1}{y}\ln b\right) &\geq (\ln a + \ln b)^2\\ (\ln a)^2 + \ln a\ln b\left(\frac{x}{y} + \frac{y}{x}\right) + (\ln b)^2 &\geq (\ln a + \ln b)^2,\end{aligned}$$

which follows since by the GM-AM inequality $\frac{x}{y} + \frac{y}{x} \geq 2$.

**Solution 5 by Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA.** We have

$$\begin{aligned}\log_{ab}(a^x b^y)\ \log_{ab}(a^{1/x} b^{1/y}) &= (x\log_{ab} a + y\log_{ab} b)\,(\frac{1}{x}\log_{ab} a + \frac{1}{y}\log_{ab} b)\\ &= \log_{ab}^2 a + \log_{ab}^2 b + \left(\frac{x}{y} + \frac{y}{x}\right)\log_{ab} a\ \log_{ab} b\\ &\overset{\text{AGM}}{\geq} (\log_{ab} a + \log_{ab} b)^2 = 1.\end{aligned}$$

Equality holds if and only if $x = y$.

**Also solved by** *Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania and the proposers.*

***E–123.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ be a collection of subsets of the set $S = \{1, 2, \ldots, n\}$ satisfying the following conditions:

- Any two distinct sets from $\mathcal{F}$ have exactly one element in common,
- each element of $S$ is contained in exactly $k$ of the sets in $\mathcal{F}$.

Can $n$ be equal to 2024?

**Solution by the proposer.** Let $N$ count the number of triples $(\{A_i, A_j\}, t)$ where $1 \leq i < j \leq n$, $1 \leq t \leq n$ and $t \in A_i \cap A_j$. We compute $N$ through a double counting argument. From the perspective of any (unordered) pair $(A_i, A_j)$, they appear in exactly one such triple, yielding $N = \binom{n}{2}$. Likewise, for any $1 \leq t \leq n$, $t$ appears in exactly $\binom{k}{2}$ distinct pairs, forcing $N = n\binom{k}{2}$. Combining these, we find

$$\binom{n}{2} = N = n\binom{k}{2} \implies n - 1 = k(k-1).$$

As $n - 1$ is odd for $n = 2024$ whereas $k(k-1)$ is even, this is a contradiction. So the answer to the question asked is NOT.

***E–124.*** *Proposed by Mihaela Berindeanu, Bucharest, Romania.* If $x, y, z > 1$ and $xyz = 2$, then show that

$$\frac{(\log_2 x)^2 + \log_2 y}{\log_2 yz} + \frac{(\log_2 y)^2 + \log_2 z}{\log_2 zx} + \frac{(\log_2 z)^2 + \log_2 x}{\log_2 xy} \geq 2.$$

**Solution 1 by Michel Bataille, Rouen, France.** Let $a = \log_2 x, b = \log_2 y, c = \log_2 z$. Then $a, b, c > 0$ and $a + b + c = 1$ and the inequality to be proved becomes

$$\frac{a^2 + b}{1 - a} + \frac{b^2 + c}{1 - b} + \frac{c^2 + a}{1 - c} \geq 2$$

or, remarking that $\frac{u^2+v}{1-u} = \frac{1+v}{1-u} - (1 + u)$,

$$L := \frac{1 + b}{1 - a} + \frac{1 + c}{1 - b} + \frac{1 + a}{1 - c} \geq 6.$$

Let $m = ab + bc + ca, p = abc$. From AM-GM, we deduce that

$$L \geq 3\left(\frac{(1 + a)(1 + b)(1 + c)}{(1 - a)(1 - b)(1 - c)}\right)^{1/3} = 3\left(\frac{2 + m + p}{m - p}\right)^{1/3}.$$

Thus, it is sufficient to prove that $\frac{2+m+p}{m-p} \geq 8$, that is,

$$(1 - 3m) + (9p + 1 - 4m) \geq 0.$$

We are done because $1 \geq 3m$ and $9p + 1 \geq 4m$, inequalities which follow from $1 = (a + b + c)^2 = a^2 + b^2 + c^2 + 2m \geq m + 2m = 3m$ and from Schur's inequality $a(a - b)(a - c) + b(b - c)(b - a) + c(c - a)(c - b) \geq 0$, which rewrites as $a^3 + b^3 + c^3 + 3p \geq (a + b + c)(ab + bc + ca) - 3p = m - 3p$, combined with $a^3 + b^3 + c^3 - 3p = (a + b + c)((a^2 + b^2 + c^2) - (ab + bc + ca)) = 1 - 3m$.

**Solution 2 by Titu Zvonaru, Comănești, Romania.** We denote $a = \log_2 x, b = \log_2 y, c = \log_2 z$. Then, $a + b + c = \log_2 xyz = 1$ and we have to prove that

$$\frac{a^2 + b}{b + c} + \frac{b^2 + c}{c + a} + \frac{c^2 + a}{a + b} \geq 2.$$

The preceding inequality is known (it can be found on AoPS – Art of Problem Solving). Here is a proof:

$$\frac{a^2+b}{b+c}=\frac{a(1-b-c)+b}{b+c}=\frac{a+b}{b+c}-a$$

Using AM-GM inequality, it follows that

$$\frac{a^2+b}{b+c}+\frac{b^2+c}{c+a}+\frac{c^2+a}{a+b}=\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}-(a+b+c)\geq 2.$$

**Solution 3 by the proposer.** $\left.\begin{array}{l} x,y,z>1 \\ xyz=2 \end{array}\right\} \Rightarrow \log_2 x>0,\ \log_2 y>0,\ \log_2 z>0$ and $\log_2 x+\log_2 y+\log_2 z=\log_2 2 \Rightarrow$

$$\log_2 x+\log_2 y+\log_2 z=1$$

$$\frac{(\log_2 x)^2+\log_2 y}{\log_2 yz}+\frac{(\log_2 y)^2+\log_2 z}{\log_2 zx}+\frac{(\log_2 z)^2+\log_2 x}{\log_2 xy}\geq 2,$$

$$\frac{(\log_2 x)^2+\log_2 y}{\log_2 y+\log_2 z}+\frac{(\log_2 y)^2+\log_2 z}{\log_2 z+\log_2 x}+\frac{(\log_2 z)^2+\log_2 x}{\log_2 x+\log_2 y}\geq 2.$$

Add 1 to both sides of the inequality

$$\frac{(\log_2 x)^2+\log_2 y}{\log_2 y+\log_2 z}+\frac{(\log_2 y)^2+\log_2 z}{\log_2 z+\log_2 x}+\frac{(\log_2 z)^2+\log_2 x}{\log_2 x+\log_2 y}$$
$$+\underbrace{\log_2 x+\log_2 y+\log_2 z}_{1}\geq 3.$$

Grouping terms, we have

$$\left[\frac{(\log_2 x)^2+\log_2 y}{\log_2 y+\log_2 z}+\log_2 x\right]+\left[\frac{(\log_2 y)^2+\log_2 z}{\log_2 z+\log_2 x}+\log_2 y\right]$$
$$+\left[\frac{(\log_2 z)^2+\log_2 x}{\log_2 x+\log_2 y}+\log_2 z\right]\geq 3$$

$$\sum_{cyc}\frac{(\log_2 x)^2+\log_2 y}{\log_2 y+\log_2 z}+\log_2 x$$

$$= \sum_{cyc} \frac{(\log_2 x)^2 + \log_2 z + \log_2 x(\log_2 y + \log_2 z)}{log_2 x + \log_2 y}$$

$$= \sum_{cyc} \frac{\log_2 x(\log_2 x + \log_2 y + \log_2 z) + \log_2 z}{\log_2 x + \log_2 y} = \sum_{cyc} \frac{\log_2 x + \log_2 z}{\log_2 x + \log_2 y}$$

So, it must be shown that

$$\sum_{cyc} \frac{\log_2 x + \log_2 z}{\log_2 x + \log_2 y} \geq 3.$$

Applying mean inequalities, yields

$$\sum_{cyc} \frac{\log_2 x + \log_2 z}{\log_2 x + \log_2 y} \geq 3 \cdot \sqrt[3]{\frac{(\log_2 x + \log_2 z)(\log_2 y + \log_2 z)(\log_2 z + \log_2 x)}{(\log_2 x + \log_2 z)(\log_2 y + \log_2 z)(\log_2 z + \log_2 x)}}$$

After simplification results

$$\sum_{cyc} \frac{\log_2 x + \log_2 z}{\log_2 x + \log_2 y} \geq 3$$

with equality for $x = y = z = \sqrt[3]{2}$.

**Also solved by** *Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania.*

# Easy–Medium Problems

***EM–119.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Find all real solutions of the following the system of equations

$$\begin{aligned} z+\log\Big(x+\sqrt{x^2+1}\Big) &= y, \\ x+\log\Big(y+\sqrt{y^2+1}\Big) &= z, \\ y+\log\Big(z+\sqrt{z^2+1}\Big) &= x. \end{aligned}$$

**Solution 1 by Michel Bataille, Rouen, France.** The triple $(0,0,0)$ is obviously a solution for $(x,y,z)$. We show that there is no other solution.
Let $(x,y,z)$ be a solution. Since $\sinh(\log(u+\sqrt{u^2+1}))=u$ for all real $u$, we have

$$x=\sinh(y-z), \quad y=\sinh(z-x), \quad z=\sinh(x-y). \qquad (1)$$

Without loss of generality, we can suppose that $x=\max(x,y,z)$. Since $\sinh(u)-u$ has the same sign as $u$, we deduce from (1) that $y\le z-x$, $z\ge x-y$, hence $z-y\ge 2x-z-y$. Therefore $z\ge x$ and $z=x$ follows. From (1) and $\sinh(0)=0$, we see that $y=0$ and $\sinh(x)=-\sinh(x)$ so that $x=y=0$. Finally $x=y=z=0$ and the proof is complete.

**Solution 2 by José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain and the proposer.** Let $(a,b,c)=(x+\sqrt{x^2+1}, y+\sqrt{y^2+1}, z+\sqrt{z^2+1})$. If any of the unknowns is zero, then it is obvious that they all are, and so $(x,y,z)=(0,0,0)$ is a solution. Also note that if we add all three equations we get

$$\log\Big(x+\sqrt{x^2+1}\Big)+\log\Big(y+\sqrt{y^2+1}\Big)+\log\Big(z+\sqrt{z^2+1}\Big)=0,$$

or $\log(a\cdot b\cdot c)=0$, and therefore $a\cdot b\cdot c=1$.

Now we distinguish two cases:

- If $x > 0$, then $a > 1$, and $\log a > 0$ and therefore from the first equation, we get $y - z = \log a > 0$ or $y > z$. From the preceding we obtain $\lg b > \log c$ on account that $f(x) = \log x$ is increasing. Using the second and third equations, we get $z - x = \log b$ and $x - y = \log c$. Combining the preceding, yields $z - x = \log b > \log c = x - y$ from which $z + y > 2x > 0$ follows. Then, we have $y - z > 0$ and $y + z > 0$ from which $y > 0$ follows and $\log b > 0$ and $b > 1$. From $z - x = \log b > 0$ we get $z > x > 0$ and $\log c > 0$ from which $c > 1$ follows. Therefore $a \cdot b \cdot c > 1$, which contradicts the fact that $a \cdot b \cdot c = 1$.
- If $x < 0$, then $a < 1$, and $\log a < 0$, and therefore $y < 0$. From the first equation, we get $y - z = \log a < 0$ or $y < z$. From the preceding we obtain $\lg b < \log c$ on account that $f(x) = \log x$ is increasing. Using the second and third equations, we get $z - x = \log b$ and $x - y = \log c$. Combining the preceding, yields $z - x = \log b < \log c = x - y$ from which $z + y < 2x < 0$ follows. Then, we have $y - z < 0$ and $y + z < 0$ from which $y < 0$ follows and $\log b < 0$ and $b < 1$. From $z - x = \log b < 0$ we get $z < x < 0$ and $\log c < 0$ from which $c < 1$ follows. Therefore $a \cdot b \cdot c < 1$, which contradicts the fact that $a \cdot b \cdot c = 1$.

Thus, the only solution is $(x, y, z) = (0, 0, 0)$.

**Also solved by** *G. C. Greubel, Newport News, VA, USA.*

***EM–120.*** *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let $ABC$ be an equilateral triangle with $P$, an arbitrary point on side $BC$ and $X$, the midpoint of segment $AP$. If $BX \cap AC = \{M\}$ and $CX \cap AB = \{N\}$ show that the distance from the centroid of triangle $ABC$ to $MN$ does not depend on the choice of point $P$.

**Solution 1 by Michel Bataille, Rouen, France.** Let $P = tB + (1 - t)C$ where $t \in [0, 1]$. Then, $2X = A + tB + (1 - t)C$ so that $(2 - t)M = A + (1 - t)C$, $(1 + t)N = A + tB$. We readily deduce that

$$CM = \frac{a}{2 - t}, \quad BN = \frac{a}{1 + t}, \quad CM + BN = \frac{3a}{(1 + t)(2 - t)}$$

where $a$ denotes the side of $\Delta ABC$.
We also deduce that

$$\begin{aligned}(2-t)(1+t)\overrightarrow{MN} &= (2-t)(A+tB)-(1+t)(A+(1-t)C)\\ &= t(2-t)\overrightarrow{AB}-(1-t^2)\overrightarrow{AC}\end{aligned}$$

so that

$$\begin{aligned}(2-t)^2(1+t)^2MN^2 &= a^2\big(t^2(2-t)^2+(1-t^2)^2-t(1-t^2)(2-t)\big)\\ &= a^2(t^4-2t^3+3t^2-2t+1).\end{aligned}$$

A simple calculation then shows that $(CM+BN-a)^2 = MN^2$. Since $CM+MN+NB > BC$, we must have $CM+BN = BC+MN$, which implies that the convex quadrilateral $BNMC$ has an incircle. Clearly, this incircle must be the incircle of $\Delta ABC$, hence its radius is $\frac{a\sqrt{3}}{6}$. It follows that the distance from the incenter (that is, the centroid) of $\Delta ABC$ to the line $MN$ is $\frac{a\sqrt{3}}{6}$, independently of the location of $P$ on the side $BC$.

**Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.** Letting $\ell$ the length of the side of the given triangle and putting $\overline{BP} = x$, we have $\overline{PC} = \ell - x$.

Menelaus's theorem applied to $\triangle APC$ and transversal $BXM$ (FIGURE 1) asserts that

$$\frac{AX}{XP}\cdot\frac{PB}{BC}\cdot\frac{CM}{MA} = 1,$$

and therefore

$$\frac{1}{1}\cdot\frac{x}{\ell}\cdot\frac{CM}{MA} = 1,$$

yielding $\frac{CM}{MA} = \frac{\ell}{x}$. Since $CM + MA = \ell$, it follows that

$$CM = \frac{\ell^2}{\ell+x} \qquad MA = \frac{\ell x}{\ell+x}.$$

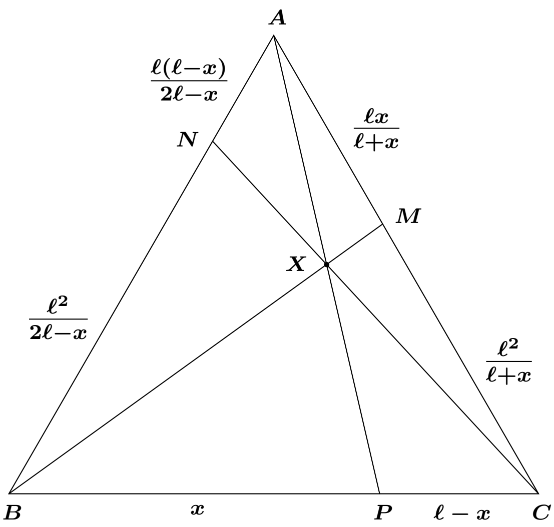



FIGURE 1

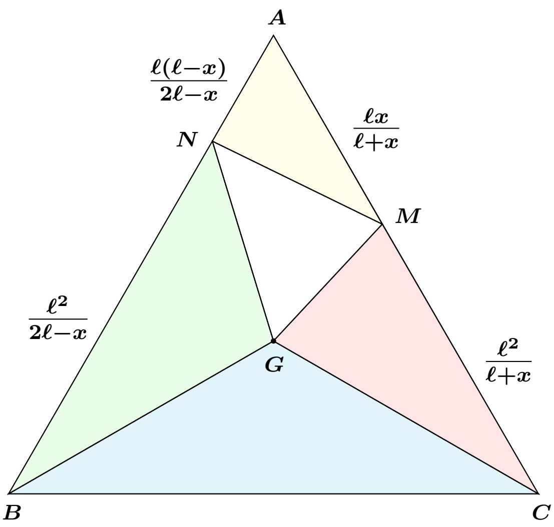



FIGURE 2

Now, by Ceva's theorem,

$$\frac{BP}{PC} \cdot \frac{CM}{MA} \cdot \frac{AN}{NB} = 1,$$

and therefore

$$\frac{x}{\ell - x} \cdot \frac{\ell}{x} \cdot \frac{AN}{NB} = 1,$$

yielding $\frac{AN}{NB} = \frac{\ell - x}{\ell}$. Since $AN + NB = \ell$, it follows that

$$AN = \frac{\ell(\ell - x)}{2\ell - x} \qquad NB = \frac{\ell^2}{2\ell - x}.$$

Since $\triangle ABC$ is equilateral, its centroid $G$ is actually the incentre of $\triangle ABC$ and therefore is equidistant from $BC$, $CA$ and $AB$. Hence $\triangle GBC$, $\triangle GCM$ and $\triangle GNB$ have equal altitudes from $G$ and taking in account that the areas of triangles with equal altitudes are proportional to the bases of the triangles, we have (with $[\ldots]$ denoting areas) (FIGURE 2)

$$[GBC] : [GCM] : [GNB] = BC : CM : NB$$

$$= \left(\ell : \frac{\ell^2}{\ell + x} = \frac{\ell^2}{2\ell - x}\right) = \frac{1}{\ell} : \frac{1}{\ell + x} : \frac{1}{2\ell - x},$$

whence

$$[GCM] = \frac{\ell}{\ell + x}[GBC] \qquad\qquad [GNB] = \frac{\ell}{2\ell - x}[GBC]$$

and since $[GBC] = \frac{1}{3}[ABC]$, we deduce

$$[GCM] = \frac{\ell}{3(\ell + x)}[ABC] \qquad\qquad [GNB] = \frac{\ell}{3(2\ell - x)}[ABC].$$

In turn,

$$\frac{[ANM]}{[ABC]} = \frac{AN \cdot AM}{AB \cdot AC} = \frac{\frac{\ell(\ell-x)}{2\ell-x} \cdot \frac{\ell x}{\ell+x}}{\ell \cdot \ell} = \frac{x(\ell - x)}{(2\ell - x)(\ell + x)}$$

and

$$[ANM] = \frac{x(\ell - x)}{(2\ell - x)(\ell + x)}[ABC].$$

Thus

$$\begin{aligned} [GMN] &= [ABC] - ([GBC] + [GCM] + [GNB] + [ANM]) \\ &= [ABC]\Big(1 - \tfrac{1}{3} - \tfrac{\ell}{3(\ell+x)} - \tfrac{\ell}{3(2\ell-x)} - \tfrac{x(\ell-x)}{(2\ell-x)(\ell+x)}\Big) \\ &= \tfrac{\ell^2-\ell x+x^2}{3(\ell+x)(2\ell-x)}[ABC] \end{aligned} \tag{1}$$

and the distance from $G$ to $MN$ equals the length of the altitude from $G$ in $\triangle GMN$ whic is

$$\frac{2[GMN]}{MN}. \tag{2}$$

But, from $\triangle AMN$ with $\angle A = 60^\circ$ opposite $MN$,

$$\begin{aligned} MN^2 &= AM^2 + AN^2 - AM \cdot AN \\ &= \Big(\tfrac{\ell x}{\ell+x}\Big)^2 + \Big(\tfrac{\ell(\ell-x)}{2\ell-x}\Big)^2 - \tfrac{\ell x}{\ell+x} \cdot \tfrac{\ell(\ell-x)}{2\ell-x} \\ &= \ell^2 \cdot \tfrac{\ell^4+x^4+3\ell^2x^2-2\ell x^3-2\ell^3x}{(\ell+x)^2(2\ell-x)^2} \\ &= \Big(\tfrac{\ell\big(\ell^2-\ell x+x^2\big)}{(\ell+x)(2\ell-x)}\Big)^2 \end{aligned}$$

and

$$MN = \frac{\ell(\ell^2 - \ell x + x^2)}{(\ell + x)(2\ell - x)}. \qquad (3)$$

By (1) and (3),

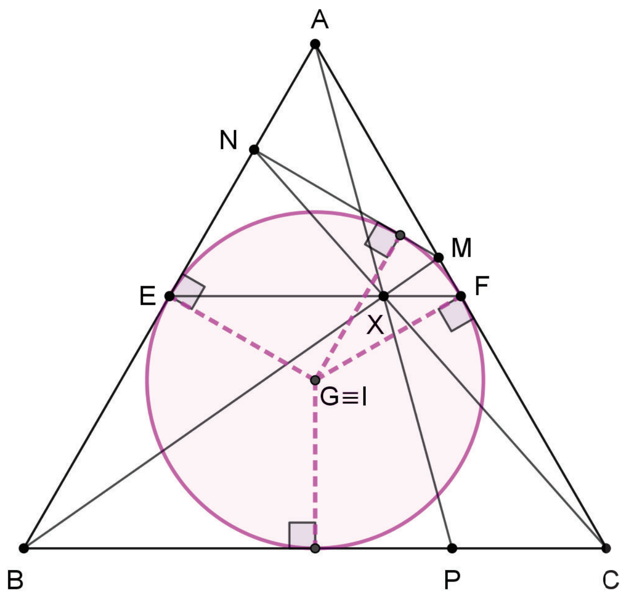
$$\frac{[GMN]}{MN} = \frac{[ABC]}{3\ell}.$$

When this is substituted into (2), we get

$$(\text{the distance from } G \text{ to } MN) = \frac{2[ABC]}{3\ell} = \frac{2}{3\ell} \cdot \frac{\sqrt{3}}{4}\ell^2 = \frac{\ell\sqrt{3}}{6},$$

which does not depend on the choice of point $P$.

**Solution 3 by the proposer.** Denote: $AB = BC = AC = a$, $E, F$ midpoints of $AB$, respectively $AC \Rightarrow E$, $X$, $F$ are collinear points; $\frac{EX}{XF} = k$




Scheme for solving Problem EM-120.

- Applying Menelaus' theorem in the $\triangle AEF$ cut by the transversal $BXM$, we have $\dfrac{BA}{BE}\cdot\dfrac{XE}{XF}\cdot\dfrac{MF}{MA}=1$ where $BA=2BE$ and $\dfrac{EX}{XF}=k \Rightarrow \dfrac{1}{2}\cdot\dfrac{MA}{MF}\cdot\dfrac{1}{k}=1\Rightarrow\dfrac{MA}{MF}=2k\Rightarrow\dfrac{MA}{AF}=\dfrac{2k}{2k+1}\Rightarrow MA=\dfrac{2k}{2k+1}\cdot\dfrac{a}{2}=\dfrac{ak}{2k+1}$.
- Applying Menelaus' theorem in the $\triangle AEF$ cut by the transversal $CXN$, we have $\dfrac{CF}{CA}\cdot\dfrac{NA}{NE}\cdot\dfrac{EX}{XF}=1$ where $CA=2CF$ and $\dfrac{EX}{XF}=k \Rightarrow \dfrac{1}{2}\cdot\dfrac{NA}{NE}\cdot k=1\Rightarrow\dfrac{NA}{NE}=\dfrac{2}{k}\Rightarrow\dfrac{NA}{AE}=\dfrac{2}{k+2}\Rightarrow NA=\dfrac{2}{k+2}\cdot\dfrac{a}{2}=\dfrac{a}{k+2}$.
- Calculate $MN$ using the cosine theorem in $\triangle AMN$

$$\begin{aligned}
MN^2 &= AN^2+AM^2-2AN\cdot AM\cos 60^\circ\\
&= \frac{a^2}{(k+2)^2}+\frac{a^2k^2}{(2k+1)^2}-\frac{a^2k}{(k+2)(2k+1)}\\
&= a^2\left[\frac{1}{(k+2)^2}+\frac{k^2}{(2k+1)^2}-\frac{k}{(k+2)(2k+1)}\right]\\
&= \frac{a^2(k^4+3k^3+3k^2+2k+1)}{(k+1)^2(2k+1)^2}=\frac{a^2(k^2+k+1)^2}{(k+2)^2(2k+1)^2}.
\end{aligned}$$

  So $MN=\dfrac{a(k^2+k+1)}{(k+2)(3k+1)}$.
- Show that $BCMN$ is a quadrilateral circumscribed is equivalent to show that $MN+BC=CM+BN$ where: $MN=\dfrac{a(k^2+k+1)}{(k+2)(2k+1)}$, $BC=a$, $CM=a-MA=a-\dfrac{ak}{2k+1}$, $BN=a-NA=a-\dfrac{a}{k+2}\Rightarrow$

$$\frac{a(k^2+k+1)}{(k+2)(2k+1)}+a=a-\frac{ak}{2k+1}+a-\frac{a}{k+2}$$

$$\Rightarrow\frac{k^2+k+1}{(k+2)(2k+1)}=1-\frac{1}{2k+1}-\frac{1}{k+2}\Rightarrow$$

$$\frac{2k^2+5k+2-2k-1-k^2-2k}{(k+2)(2k+1)}=\frac{k^2+k+1}{(k+2)(2k+1)}$$

The incircle of triangle $ABC$, with incenter $I$ (which coincides with the centroid $G$ of the equilateral triangle) and the inradius $r$, is tangent to the sides $AB, AC$ and $BC \Rightarrow MN$ is tangent to the incircle. So the distance from the centroid $G$ to $MN$ is the inradius $r$, regardless of the choice of point $P$. That is,

$$r=\frac{1}{3}\cdot\frac{a\sqrt{3}}{2}=\frac{a\sqrt{3}}{6}.$$

***EM–121.*** *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Find all possible values of the positive integers $x>1, y, z$ so that

$$\frac{x+1}{x-1}+\frac{y-1}{y+1}=\frac{z^2+1}{z}.$$

**Solution 1 by Titu Zvonaru, Comăneşti, Romania.** We have $\frac{z^2+1}{z}=z+\frac{1}{z}$. Suppose that $z\geq 4$, then

$$\frac{x+1}{x-1}+\frac{y-1}{y+1}>4,$$

which is equivalent to $(x+1)(y+1)+(x-1)(y-1)>4(x-1)(y+1) \Leftrightarrow xy+2x-2y-3<0 \Leftrightarrow (x-2)(y+2)+1<0$. Since $x\geq 2$, then $(x-2)(y+2)+1>0$ and we do not obtain solutions.

- Let $z=1$. The equation is $(x+1)(y+1)+(x-1)(y-1)=2(x-1)(y+1) \Leftrightarrow y=x-2$, hence the solutions $(x, x-2, 1)$, where $x$ is an positive integer, $x>1$.
- Let $z=2$. The equation is $2((x+1)(y+1)+(x-1)(y-1))=5(x-1)(y+1) \Leftrightarrow (x-5)(y+5)=-16$. Since $y+5\geq 5$, we obtain the solutions $(3,3,2)$ and $(4,11,2)$.
- Let $z=3$. The equation is $3((x+1)(y+1)+(x-1)(y-1))=10(x-1)(y+1) \Leftrightarrow (2x-5)(2y+5)=-9$. Since $2y+5\geq 5$, we obtain the solution $(2,2,3)$.

**Solution 2 by the proposer.** Subtracting 2, the condition is equivalent to

$$\begin{aligned} \frac{x+1}{x-1} - 1 + \frac{y-1}{y+1} - 1 &= \frac{z^2+1}{z} - 2 \\ \Leftrightarrow \quad \frac{2}{x-1} + \frac{-2}{y+1} &= \frac{(z-1)^2}{z} \\ \Leftrightarrow \quad \frac{2y-2x+4}{(x-1)(y+1)} &= \frac{(z-1)^2}{z} \\ \Leftrightarrow \quad 2z(y-x+2) &= (x-1)(y+1)(z-1)^2 \end{aligned}$$

Since $(x-1)(y+1)(z-1)^2 > 0$, hence $y-x+2 > 0$ and $x \leq y+2$.

$$x = \frac{1+y+2z+z^2+yz^2}{1+y-2yz+z^2+yz^2} = \frac{2z(y+1)}{y(z-1)^2+z^2+1} + 1$$

is a integer. Hence $y(z-1)^2+z^2+1 \mid 2z(y+1)$.

We have the following cases:

**1.** $z = 1$. Then

$$x = \frac{2(y+1)}{2} + 1 = y + 2$$

Any integer $t \geq 1$ give a solution $x = t+2, y = t, z = 1$.

Examples:

| $x = t+2$ | $y = t$ | $z = 1$ |
|---:|---:|---:|
| 3 | 1 | 1 |
| 4 | 2 | 1 |
| 5 | 3 | 1 |

**2.** $z = 2$. Then

$$x = \frac{4(y+1)}{y+5} + 1 = 5 - \frac{16}{y+5}$$

hence $y+5 \mid 16$.

If $y+5=8$, then $y=3$, $x=3$, $z=2$ and this is a solution.

If $y+5=16$, then $y=11$, $x=4$, $z=2$ and this is a solution.

**3.** $z=3$. Then

$$x=\frac{3(y+1)}{2y+5}+1=\frac{1}{2}\left(5-\frac{9}{2y+5}\right)$$

hence $2y+5 \mid 9$, or $2y+5=9, y=2$, $x=2$ and this is a solution.

**4.** $z \geq 4$. Then

$$x=\frac{2z(y+1)}{y(z-1)^2+z^2+1}+1=\frac{2zy+2z}{(z^2-2z+1)y+z^2+1}+1$$

and $(z^2-2z+1)y+z^2+1 \mid 2zy+2z$, $(z^2-2z+1)y+z^2+1 \leq 2zy+2z$.

Since $(z-1)^2>0$ or $z^2+1>2z$ for $z>1$ and since $(z^2-2z+1)y-2zy=z(z-4)y+y>0$ for $z\geq 4$, hence $(z^2-2z+1)y+z^2+1>2zy+2z$ and not solutions in this case.

**In summary**, the only solutions are: $x=3$, $y=3$, $z=2$; $x=4$, $y=11$, $z=2$; $x=2$, $y=2$, $z=3$ and $x=t+2$, $y=t$, $z=1$ for any positive integer $t$.

**Also solved by** *José Luis Díaz-Barrero and Josep Gibergans-Báguena both at BarcelonaTech, Barcelona, Spain.*

***EM–122.*** *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

1. Prove that every tetrahedron can be cut by a plane so that a parallelogram results in the section.
2. If the intersection of a regular tetrahedron and a plane is a rhombus, prove that the rhombus must be a square.

**Solution by the proposer.** 1. Let $M$ be a point on the edge $AB$ of the tetrahedron $ABCD$ (FIGURE 1 ). We construct a section of the tetrahedron by a plane $\pi$, say, passing through the point $M$ and parallel to the edges $AC$ and $BD$ (FIGURE 2 ).

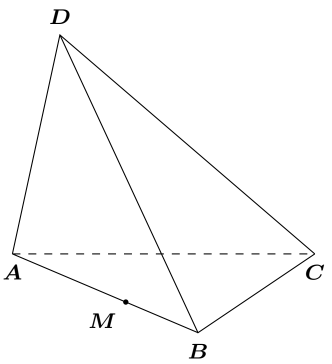



FIGURE 1

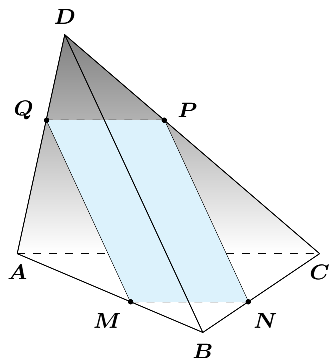


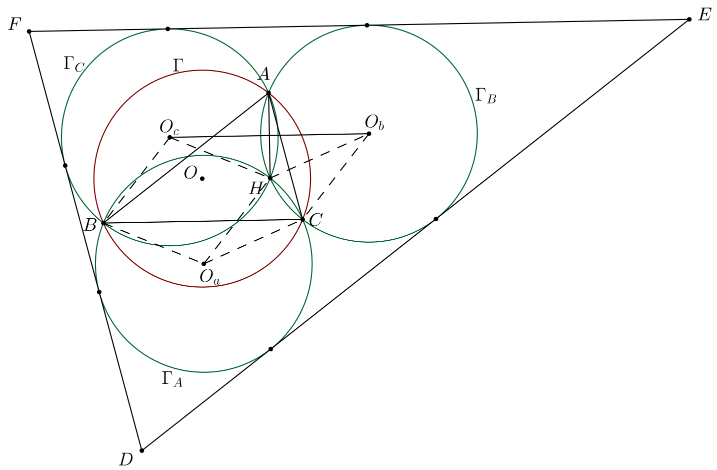

FIGURE 2

Since $\pi \parallel AC$, the line of intersection of planes $ABC$ and $\pi$ is parallel to $AC$. Let this line intersects $BC$ at $N$. Thus $MN \parallel AC$.
Since $\pi \parallel BD$, the lines of intersection of $\pi$ with the planes $ABD$ and $BCD$ are parallel to $BD$. Let $P$ the point of edge $CD$ such that $NP \parallel BD$ and let $Q$ the point of edge $DA$ such that $MQ \parallel BD$.
The fourth side $PQ$ of the section is parallel to the edge $AC$ and therefore the section is a parallelogram $MNPQ$.

2. Two opposite sides of the rhombus are parallel. The faces containing these sides meet on an edge of the tetrahedron; for example, $AB$, which in turn is parallel to the section plane. In the same way, the edge $CD$ is parallel to the section plane. Since the tetrahedron is *regular*, then $AB \perp CD$ and we get that two consecutive sides of the rhombus are perpendicular. This means that the rhombus is a square.

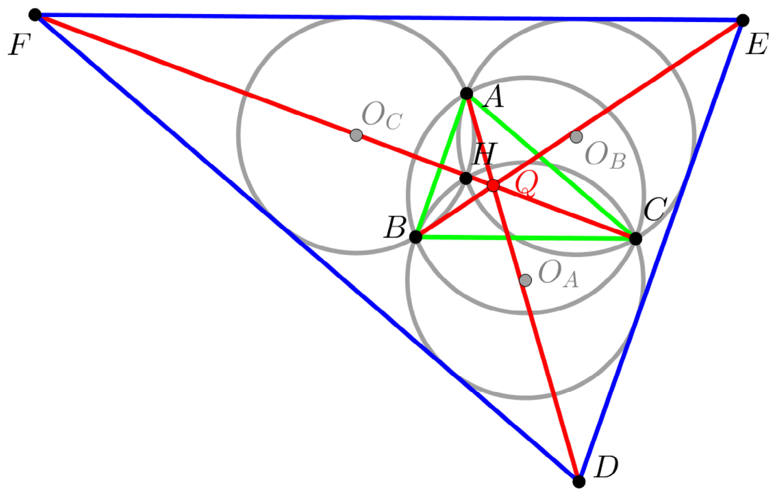
***EM–123.*** *Proposed by Alexandru Benescu, Romania.* Let $ABC$ be a triangle, $H$ its orthocenter and $\Gamma_A$, $\Gamma_B$, $\Gamma_C$ the circumscribed circles of $\triangle BHC$, $\triangle AHC$, $\triangle AHB$ respectively. Let $D$, $E$ and $F$ be points such that $DE$ is tangent to $\Gamma_A$ and $\Gamma_B$, $EF$ is tangent to $\Gamma_B$ and $\Gamma_C$, and $FD$ is tangent to $\Gamma_C$ and $\Gamma_A$, such that all 3 circles $\Gamma_A$, $\Gamma_B$ and $C$ lie inside $\triangle DEF$. Prove that lines $AD$, $BE$ and $CF$ are concurrent.

**Solution 1 by Michel Bataille, Rouen, France.** It is well-known that the reflections of $H$ in the sidelines of the triangle $ABC$ are on its circumcircle $\Gamma$. It follows that $\Gamma_A, \Gamma_B, \Gamma_C$ are the reflections of $\Gamma$ in $BC, CA, AB$, respectively. Their respective centers $O_a, O_b, O_c$ are the reflections of the circumcenter $O$, hence $HO_a = CO_a = CO_b = HO_b$. We deduce that $O_aCO_bH$ is a rhombus so that $\overrightarrow{CO_b} = \overrightarrow{O_aH}$. Similarly, $O_cBO_aH$ is a rhombus and $\overrightarrow{BO_c} = \overrightarrow{O_aH}$. From $\overrightarrow{CO_b} = \overrightarrow{BO_c}$ we obtain $\overrightarrow{BC} = \overrightarrow{O_cO_b}$, hence $BC \parallel O_cO_b$. But $\Gamma_B$ and $\Gamma_C$ are symmetric in the line $AH$, hence $EF$ and $O_bO_c$ are perpendicular to $AH$. It follows that $EF$ is parallel to $O_bO_c$, hence to $BC$. Similarly, $FD \parallel AC$ and $DE \parallel AB$. From Desargues' theorem, $AD, BE, CF$ are concurrent (at the center of the homothety transforming $A$ into $D$ and $B$ into $E$).




**Solution 2 by the proposer.** The circles $\Gamma_A$, $\Gamma_B$ and $\Gamma_C$ have the same radius. Indeed, since the reflection $H'$ of $H$ on $BC$ lies on the circumscribed circle $\Gamma$ of $ABC$, we have that the circumcircles $\Gamma_A$ of $\triangle BHC$ and $\Gamma$ of $\triangle BH'C$ are congruent because there is a symmetry that sends one to the other. Repeating the argument with the other sides of the triangle we see that $\Gamma_A$, $\Gamma_B$ and $\Gamma_C$ all have the same radius $R$ as $\Gamma$.

Let $O_B$ be the center of $\Gamma_B$ and $O_C$ be the center of $\Gamma_C$. The four sides of quadrilateral $O_BAO_CH$ are radii, so it is a rhombus. This implies $AH \perp O_BO_C$, but $O_BO_C$ is parallel to $EF$ because




the common tangent between two circles of the same radii are parallel to the line joining their centers. So, $EF$ is perpendicular to $AH$, which means it is parallel to $BC$. Similarly, $AB \parallel DE$ and $AC \parallel DF$, which implies that $\triangle ABC$ and $\triangle DEF$ have parallel sides. This implies that both triangles are either congruent or homothetic. We can discard the congruent case since the first one fits inside a circle $\Gamma$ of radius $R$ and the second one contains $\Gamma_A$, also of radius $R$. The lines $AD$, $BE$ and $CF$ concur at the center $Q$ of the homothety of the two triangles.

***EM–124.*** *Proposed by Goran Conar, Varaždin, Croatia.* Let $b > a > 1$ and $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1 + x_2 + \ldots + x_n = 1$. Prove that

$$\frac{a^{x_1}}{a^{x_1}+b} + \frac{a^{x_2}}{a^{x_2}+b} + \ldots + \frac{a^{x_n}}{a^{x_n}+b} \geq \frac{n\sqrt[n]{a}}{b+\sqrt[n]{a}}.$$

**Solution 1 by Michel Bataille, Rouen, France.** Since $\frac{a^x}{a^x+b} = 1 - \frac{b}{a^x+b}$, the left side $L$ is equal to $n - b(f(x_1) + f(x_2) + \cdots + f(x_n))$ where $f(x) = (a^x + b)^{-1}$ and the inequality is equivalent to

$$f(x_1) + f(x_2) + \cdots + f(x_n) \leq \frac{n}{\sqrt[n]{a}+b}. \tag{1}$$

An easy calculation gives $f''(x) = a^x(a^x+b)^{-3}(\ln a)^2(a^x-b)$ for $x \in (0,1)$. Since $\ln a > 0$, we have

$$a^x = e^{x\ln a} < e^{\ln a} = a < b,$$

hence $f''(x) < 0$ and $f$ is concave on $(0,1)$. From Jensen's inequality we obtain

$$f(x_1)+\cdots+f(x_n) \le nf\left(\frac{x_1+\cdots+x_n}{n}\right) = nf(1/n) = n(a^{1/n}+b)^{-1}$$

and (1) follows.

**Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA.** Because each $x_j$ is a positive real number and $x_1+x_2+\cdots+x_n = 1$, each $x_j \in (0,1)$. Consider the function

$$f(x) = \frac{a^x}{a^x+b}$$

for $x \in (0,1)$. Then

$$f''(x) = \frac{ba^x \ln^2 a(b-a^x)}{(a^x+b)^3}.$$

With $b > a > 1$ and $x \in (0,1)$, it follows that $a^x < a < b$ and $f''(x) > 0$. Hence, by Jensen's inequality,

$$\begin{aligned}\frac{a^{x_1}}{a^{x_1}+b} + \frac{a^{x_2}}{a^{x_2}+b} + \cdots + \frac{a^{x_n}}{a^{x_n}+b} &= \sum_{j=1}^n f(x_j)\\ &\ge nf\left(\frac{1}{n}\sum_{j=1}^n x_j\right) = nf\left(\frac{1}{n}\right) = \frac{n\sqrt[n]{a}}{b+\sqrt[n]{a}}.\end{aligned}$$

**Solution 3 by the proposer.** Define $f(x) := \frac{a^x}{a^x+b} = 1 - \frac{b}{a^x+b}$. First and second derivation is

$$f'(x) = \frac{b}{(a^x+b)^2}\cdot a^x \ln a, \quad f''(x) = \frac{b(a^x\cdot \ln a)^2}{(a^x+b)^3}\cdot\left(\frac{b}{a^x}-1\right).$$

It is satisfied $f'(x) > 0, \quad \forall x > 0, a, b > 1$ and $f''(x) > 0 \quad \Leftrightarrow$ $\frac{b}{a^x} > 1 \quad \Leftrightarrow \quad \frac{\ln b}{\ln a} > x,\ b, a > 1$. In other words $f$ is convex on $\left\langle 0, \frac{\ln b}{\ln a}\right\rangle$ and using Jensen's inequality we have

$$\frac{1}{n}\left[\frac{a^{x_1}}{a^{x_1}+b}+\frac{a^{x_2}}{a^{x_2}+b}+\ldots+\frac{a^{x_n}}{a^{x_n}+b}\right]=\frac{f(x_1)+f(x_2)+\ldots+f(x_n)}{n}\geq$$

$$\geq f\left(\frac{x_1+x_2+\ldots+x_n}{n}\right)=\frac{a^{\frac{x_1+x_2+\ldots+x_n}{n}}}{a^{\frac{x_1+x_2+\ldots+x_n}{n}}+b}.$$

Since $x_i, i \in \{1, 2, \ldots, n\}$, satisfies $0 < x_i < 1 < \frac{\ln b}{\ln a}$ they are all from interval where function $f$ is convex, so if we apply Jensen's inequality and use condition $x_1 + x_2 + \ldots + x_n = 1$ result follows immediately.

**Solution 4 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.** Note that if $f(x) = \dfrac{a^x}{a^x+b} = 1 - b(a^x + b)^{-1}$, with $b > a > 1$ and $x \in (0, 1)$, then $f'(x) = b(a^x + b)^{-2}a^x \ln a$, and

$$\begin{aligned} f''(x) &= -2b(a^x+b)^{-3}(a^x\ln a)^2 + b(a^x+b)^{-2}a^x(\ln a)^2 \\ &= b(a^x+b)^{-2}a^x(\ln a)^2\left(-2(a^x+b)^{-1}a^x+1\right) \\ &= \frac{ba^x}{(a^x+b)^2}\cdot\frac{a^x+b-2}{a^x+b} > 0. \end{aligned}$$

Since $f(x)$ is convex, by Jensen's inequality

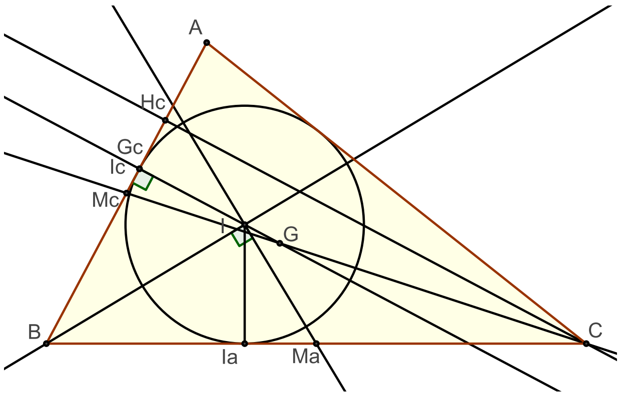
$$\frac{a^{x_1}}{a^{x_1}+b}+\frac{a^{x_2}}{a^{x_2}+b}+\ldots\frac{a^{x_n}}{a^{x_n}+b}\geq n\frac{a^{\frac{\sum x_i}{n}}}{a^{\frac{\sum x_i}{n}}+b}=\frac{n\sqrt[n]{a}}{b+\sqrt[n]{a}}.$$

**Also solved by** *Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA, Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania and Ioan Viorel Codreanu, Satulung, Maramures, Romania.*

# Medium–Hard Problems

***MH–119.*** *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let $ABC$ be a scalene triangle with incenter $I$ and centroid $G$. Let $M_a$ be the midpoint of $BC$, such that $BI$ and $IM_a$ are perpendicular. Prove that $IG$ is perpendicular to side line $AB$.

**Solution 1 by the proposer.** Let $a, b, c$ be the side lengths of triangle $ABC$. Let $I_a, I_c, G_c, H_c$ are the feet of perpendiculars of $I$ to $BC, AB$ and feet of perpendiculars of $G, C$ to $AB$, respectively. Let $M_c$ be the midpoint of $AB$.




Scheme for solving Problem MH-119

In right triangle $BIM_a$ we have $II_a{}^2 = BI_a \cdot I_aM_a$.

$$r^2 - BI_a \cdot I_aM_a = 0$$
$$\Leftrightarrow \frac{(a+b-c)(a-b+c)(-a+b+c)}{4(a+b+c)} -$$
$$- \frac{a-b+c}{2} \cdot \left(\frac{a}{2} - \frac{a-b+c}{2}\right) = 0$$
$$\Leftrightarrow \frac{(a+b-c)(a-b+c)(-a+b+c)}{4(a+b+c)} - \frac{(b-c)(a-b+c)}{4} = 0$$
$$\Leftrightarrow -\frac{a(a+b-3c)(a-b+c)}{4(a+b+c)} = 0$$
$$\Leftrightarrow a+b-3c = 0, \text{ since } \quad a > 0, a-b+c > 0, a+b+c > 0.$$

So $BI$ and $IM_a$ are perpendicular, if and only if

$$c = \frac{a+b}{3} \tag{1}$$

We will show that $I, I_c$ and $G$ are collinear.

Without loss of generality we may assume that $a \geq b$, hence $H_c, G_c, I_c$ are between $A$ and $M_c$.

$CH_c \perp AB, GG_c \perp AB$ hence $\triangle CH_cM_c \sim \triangle GG_cM_c$ and

$$\frac{G_cM_c}{H_cM_c} = \frac{GM_c}{CM_c} = \frac{1}{3}.$$

$$H_cM_c = H_cB - M_cB = a\cos B - \frac{c}{2} = \frac{a^2-b^2+c^2}{2c} - \frac{c^2}{2c} =$$

$$= \frac{a^2-b^2}{2c} = \frac{(a-b)(a+b)}{2c} = \frac{3(a-b)}{2}$$

$$G_cM_c = \frac{1}{3}H_cM_c = \frac{a-b}{2}$$

$$I_cM_c = BI_c - BM_c = BI_a - BM_c = \frac{a-b+c}{2} - \frac{c}{2} = \frac{a-b}{2}$$

$$G_cM_c = I_cM_c$$

Hence $G_c \equiv I_c$ and $I, I_c$ and $G$ are collinear and $IG \perp AB$ as required.

**Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.** We have (Figure 1)

$$90^\circ + \frac{\angle A}{2} = \text{(since } I \text{ is the incenter)} = \angle BIC = \angle BIM_a + \angle M_aIC$$

$$= 90^\circ + \angle M_aIC,$$

and $\angle M_aIC = \frac{\angle A}{2} = \angle CAI$.

The equal angles at $C$, ($\angle ACI = \dfrac{\angle C}{2} = \angle ICM_a$), make triangles $CAI$ and $CIM_a$ similar, so that

$$\frac{CI}{CA} = \frac{CM_a}{CI},$$

that is,

$$\frac{CI}{b} = \frac{a/2}{CI},$$

which is equivalent to

$$CI^2 = \frac{ab}{2}. \tag{2}$$

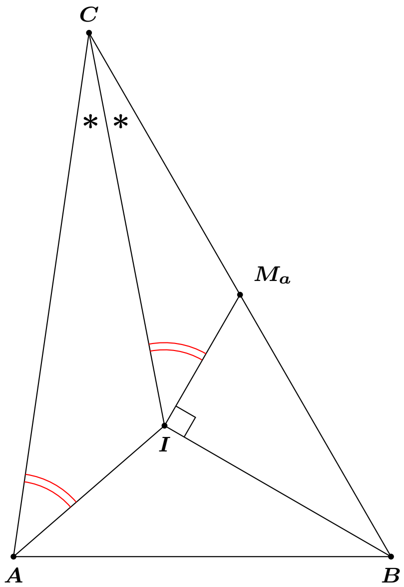



FIGURE 1

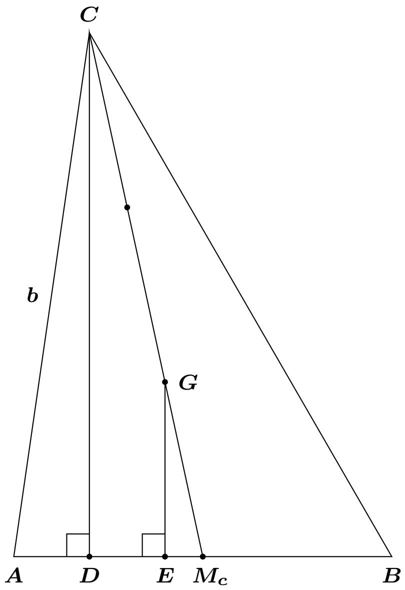



FIGURE 2

Let $r$ and $s$ denote the inradius and the semiperimeter of $\triangle ABC$, respectively. In (2) we substitute $\frac{r}{\sin \frac{C}{2}}$ for $CI$ and use Briggs's formulae to write $\sin^2 \frac{C}{2} = \frac{(s-a)(s-b)}{ab}$. This yields

$$2r^2 = (s-a)(s-b). \tag{3}$$

Since $r^2 s = (s-a)(s-b)(s-c)$ (Heron's formula), (3) becomes

$$2(s-c) = s,$$

which, in turn, is equivalent to

$$a + b = 3c. \tag{4}$$

Let $D$ and $E$ denote the feet of the perpendiculars from $C$ and $G$ to $AB$, respectively. Let $M_c$ the midpoint of $AB$.

We have (Figure 2)

$$\begin{aligned} AE &= AD + DE \\ &= AD + \tfrac{2}{3}DM_c \qquad \text{(since } G \text{ trisects } CM_c\text{)} \\ &= AD + \tfrac{2}{3}(AM_c - AD) \\ &= \tfrac{1}{3}(AD + AB) \qquad \text{(since } M_c \text{ bisects side } AB\text{)} \end{aligned}$$

Now,

$$AD = b\cos A = b \cdot \frac{b^2 + c^2 - a^2}{2bc} = \frac{c^2 + (b+a)(b-a)}{2c}$$

$$= \left( \begin{array}{c} \text{substituting} \\ 3c \text{ for } b+a \\ \text{from (4)} \end{array} \right) = \frac{-3a + 3b + c}{2},$$

and therefore

$$AE = \frac{1}{3}\left(\frac{-3a + 3b + c}{2} + c\right),$$

yielding

$$AE = s - a.$$

Since the tangents from $A$ to the incircle of $\triangle ABC$ are of length $s-a$, this implies that $E$ coincides with the point of tangency of the incircle with $AB$. We conclude that the orthogonals projections of the centroid and the incenter of $\triangle ABC$ onto the side $AB$ coincide. This is equivalent to what we set out to prove.

As a bonus, the following properties of such a triangle $ABC$ with $3c = a + b$ are listed in: Francisco Bellot Rosado, "Triángulos especiales (3)", Revista Escolar de la Olimpiada Iberoamericana de Matemáticas, nr. 19, May-June 2005, ISSN-1698-277X.

1. $\frac{1}{r_c} = \frac{1}{r_a} + \frac{1}{r_b}$, where $r_a$, $r_b$, $r_c$ are the exradii.
2. $s^2 = 2r_a r_b$.
3. $CI^2 = 4Rr$, where $R$ and $r$ denote the circumradius and the inradius, respectively.
4. $r = \frac{1}{2}r_c = \frac{AI \cdot BI}{CI}$.
5. $\cot\frac{C}{2} = \cot\frac{A}{2} + \cot\frac{B}{2}$.

6. $\tan\frac{A}{2}\tan\frac{B}{2}=\frac{1}{2}$.
7. $\sin\frac{C}{2}=\sin\frac{A}{2}\sin\frac{B}{2}=\frac{1}{3}\cos\frac{A-B}{2}=\sqrt{\frac{r}{4R}}$.
8. Area $\triangle ABC = cr_c$.
9. The circumcenter is equidistant from the incenter and the midpoint of $CI$.

**Solution 3 by Michel Bataille, Rouen, France.** Let $a=BC, b=CA, c=AB$ and $2s=a+b+c$, as usual. In barycentric coordinates relatively to $(A,B,C)$, we have

$$I=(a:b:c),\quad G=(1:1:1),\quad M_a=(0:1:1).$$

and deduce $2s\overrightarrow{BI}=a\overrightarrow{BA}+c\overrightarrow{BC}$, $\quad 2s\overrightarrow{IM_a}=-a\overrightarrow{BA}+(s-c)\overrightarrow{BC}$. Since the dot product $\overrightarrow{BI}\cdot\overrightarrow{IM_a}$ vanishes, we obtain

$$-a^2c^2+[a(s-c)-ca]\overrightarrow{BA}\cdot\overrightarrow{BC}+c(s-c)a^2=0$$

and recalling that $\overrightarrow{BA}\cdot\overrightarrow{BC}=\frac{c^2+a^2-b^2}{2}$, a short calculation gives

$$(a+b-3c)(a+c-b)(a+b+c)=0$$

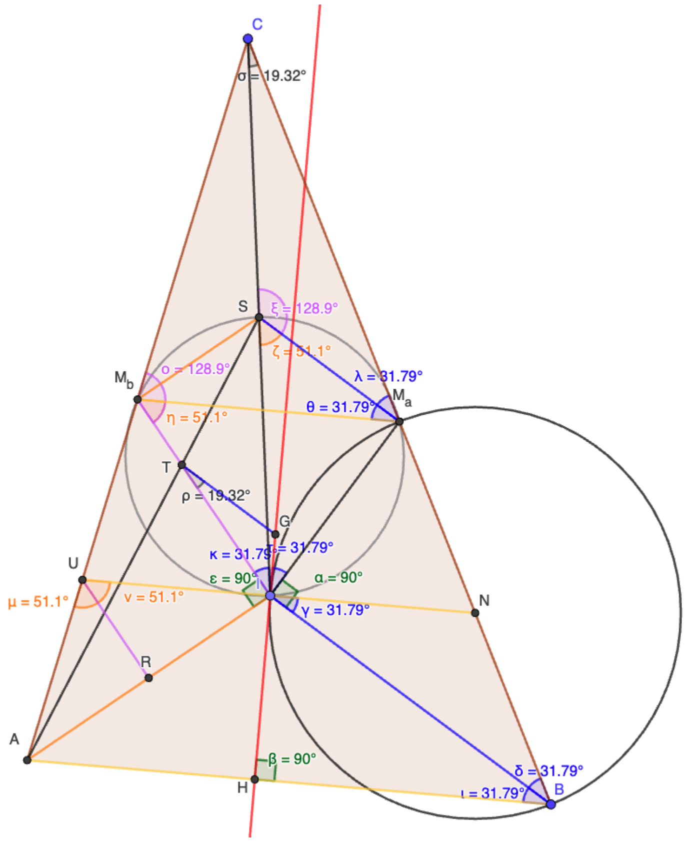
so that $a+b=3c$.
In a similar way, we have $6s\overrightarrow{IG}=(c+a-2b)\overrightarrow{AB}+(a+b-2c)\overrightarrow{AC}$, from which we get

$$12s\overrightarrow{IG}\cdot\overrightarrow{AB}=2(c+a-2b)c^2+(a+b-2c)(b^2+c^2-a^2)$$

or, using $a+b=3c$, $12s\overrightarrow{IG}\cdot\overrightarrow{AB}=2c^2(4c-3b)+c(6bc-8c^2)=0$. We conclude that $IG\perp AB$.

**Solution 4 by José Pérez Cano, Koh Young Technology, Granada, Spain (Alphageometry).** Below is the reference figure to be detailed.

Here $N$ is the midpoint of $M_aB$, $M_b$ is the midpoint of $AC$, $U$ is the midpoint of $M_bA$, $S$ is the midpoint of $CI$, $R$ is the midpoint $AI$ and $T$ is the intersection of $AS$ and $M_bI$. The proof has three main steps. First, prove that $I$, $U$ and $N$ are collineal. Then, show that $I$, $S$, $M_a$, $M_b$ are co-cyclic. And finally show that $TGI$ is similar to $CM_aI$. Using that and the fact that some




lines are parallel we can arrive at $\angle GIM_a = \angle NIB$ and therefore $\angle(GI - AB) = \angle GIN = 90°$.

Step 1

$M_bS \parallel AI$ because $M_b$ and $S$ are both midpoints. Similarly, $UR \parallel M_bI$, $M_aS \parallel IB$ and $UN \parallel AB \parallel M_aM_b$. This is visualised in the figure using the same colours for parallel lines. On the other hand, $\angle NIB = \angle NBI$ because $N$ is the circumcenter and $\angle NBI = \angle IBA$ because $I$ is the incenter. That implies $IN \parallel AB$ which means $I \in UN$.

Step 2

Since $UN \parallel M_aM_b$ we have $\frac{NC}{UC} = \frac{NM_a}{UM_b}$. Also, since $\angle UCI = \angle ICN$ by construction and $I$, $U$, $N$ are collineal, we have $\frac{IN}{IU} =$

$\frac{CN}{CU}$. Therefore $1 = \frac{IN}{NM_a} = \frac{IU}{UM_b}$ which means $\angle M_bIA = 90°$. Right angles are depicted in the figure in green. Now, $M_bS \parallel AI$ and $M_aS \parallel IB$ so $\angle SM_aI = 90° = \angle SM_bI$ which proves that $S$, $M_a$, $M_b$ and $I$ are co-cyclic.

Step 3

To prove that $TGI$ and $CM_aI$ are similar we are going to show $\frac{CM_a}{CI} = \frac{TG}{TI}$ and $\angle GTI = \angle M_aCI$. $M_bS \parallel AI$ implies $\frac{AT}{TS} = \frac{AI}{M_sS}$. $AB \parallel M_aM_b$ implies $\frac{AG}{M_aG} = \frac{AB}{M_aM_b}$. Also, $M_bS \parallel AI$ and $AB \parallel M_aM_b$ implies $\frac{AI}{M_bS} = \frac{AB}{M_aM_b}$. All together simplifies as $\frac{AT}{TS} = \frac{AG}{M_aG}$ which means $TG \parallel SM_a$, as shown in blue in the picture which in turn means $\frac{AT}{AS} = \frac{TG}{M_aS}$. Now, we also have $\frac{AT}{AS} = \frac{TI}{M_bI}$ so $\frac{TI}{M_BI} = \frac{TG}{M_aS}$ or equivalently $\frac{TG}{TI} = \frac{M_aS}{M_bI}$. Finally, $CM_aS$ is similar to $CIM_b$ so $\frac{CM_a}{CI} = \frac{M_aS}{M_bI}$ which concludes $\frac{CM_a}{CI} = \frac{TG}{TI}$.

For the angles, we have $\angle M_aSI = \angle M_aCS + \angle CM_aS$ and $\angle GTI = \angle IUR - \angle NIB$. Using the parallelism of the lines $M_bI$ and $UR$ and of $M_bM_a$ and $UN$ and the step 2 we get that $\angle M_aSI = \angle IUR$. On the other hand, $IN = NB$ and $SM_a$ is parallel to $IB$ so $\angle NIB = \angle NBI = \angle CM_aS$ which proves the equality of the angles $\angle GTI$ and $\angle M_aCI$.

Conclusion Having that $TGI$ and $CM_aI$ are similar we get that $\angle TGI = \angle CM_aI = \angle GIB$. Since $SM_a \parallel IB$ we have $\angle SM_aI = \angle M_aIB$. Thus, we end up with $\angle GIM_a = \angle CM_aS = \angle CBI = \angle NIB$. In the figure this angle in depicted in blue. With that we can conclude $\angle GIN = 90°$ and so $IG$ is perpendicular to $AB$.

**Also solved by** *Titu Zvonaru, Comăneşti, Romania.*

***MH–120.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $x, y, z$ be positive real numbers whose sum is $3$. Find the minimum value of

$$\frac{x^4 + y^4 + z^4}{(x^2 + y^2 + z^2)(x^3 + y^3 + z^3)}.$$

**Solution 1 by Michel Bataille, Rouen, France.** The required minimum value is $\frac{1}{3}$. This value being obtained when $x = y = z = 1$, we complete the proof by showing that

$$3(x^4 + y^4 + z^4) \geq (x^2 + y^2 + z^2)(x^3 + y^3 + z^3) \qquad (1)$$

whenever $x, y, z > 0$ and $x + y + z = 3$. Let $m = xy + yz + zx$ and $p = xyz$. We have $x^2 + y^2 + z^2 = (x + y + z)^2 - 2m = 9 - 2m$ and since $x, y, z$ are the roots of $X^3 - 3X^2 + mX - p$,

$$x^3 + y^3 + z^3 = 3(x^2 + y^2 + z^2) - 3m + 3p = 3(9 - 3m + p)$$

$$x^4+y^4+z^4 = 3(x^3+y^3+z^3)-m(x^2+y^2+z^2)+3p = 81-36m+12p+2m^2.$$

We easily see that (1) is equivalent to

$$9m + 3p + 2mp \geq 4m^2.$$

Now, $x(x - y)(x - z) + y(y - z)(y - x) + z(z - x)(z - y) \geq 0$ (Schur's inequality) leads to

$$(x + y + z)(xy + yz + zx) \leq x^3 + y^3 + z^3 + 6xyz,$$

that is, $4m \leq 3p + 9$. Also since $(x + y + z)^2 = 9$, the well-known inequality $xy + yz + zx \leq x^2 + y^2 + z^2$ gives $m \leq 3$. Thus,

$$9m + 3p + 2mp = m(3p + 9) + p(3 - m) \geq m \cdot 4m + p \cdot 0 = 4m^2$$

and we are done.

**Solution 2 by Titu Zvonaru, Comăneşti, Romania.** By Muirhead inequality we have $x^4y + xy^4 + y^4z + yz^4 + x^4z + xz^4 \geq x^3y^2 + x^2y^3 + y^3z^2 + y^2z^3 + x^3z^2 + x^2z^3$. Adding to both sides $x^5 + y^5 + z^5$, we obtain

$$(x + y + z)(x^4 + y^4 + z^4) \geq (x^2 + y^2 + z^2)(x^3 + y^3 + z^3),$$

that is

$$\frac{x^4 + y^4 + z^4}{(x^2 + y^2 + z^2)(x^3 + y^3 + z^3)} \geq \frac{1}{x + y + z} = \frac{1}{3}.$$

It results that the searched minimum is $1/3$, attained for $x = y = z = 1$.

**Solution 3 by the proposer.** Using Holder's inequality, we have

$$(x^4+y^4+z^4)^2(x+y+z) \geq (x^3+y^3+z^3)^3 \quad (1)$$

and

$$(x^4+y^4+z^4)(x+y+z)^2 \geq (x^2+y^2+z^2)^3 \quad (2)$$

Multiplying the inequalities $(1)$ and $(2)$, we obtain

$$(x^4+y^4+z^4)^3(x+y+z)^3 \geq (x^3+y^3+z^3)^3(x^2+y^2+z^2)^3$$

or equivalently,

$$x+y+z \geq \frac{(x^2+y^2+z^2)(x^3+y^3+z^3)}{x^4+y^4+z^4}.$$

Using the condition $x+y+z=3$, we obtain

$$3 \geq \frac{(x^2+y^2+z^2)(x^3+y^3+z^3)}{x^4+y^4+z^4}.$$

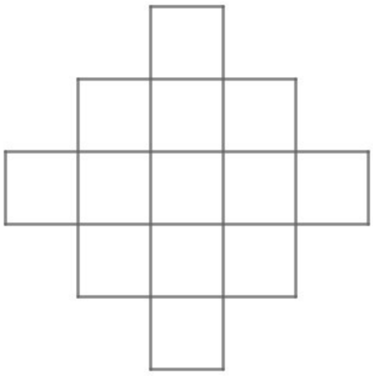
Therefore, maximum value of preceding the expression is $3$ from which we get

$$\frac{x^4+y^4+z^4}{(x^2+y^2+z^2)(x^3+y^3+z^3)} \geq \frac{1}{3}$$

with minimum value $1/3$ attained when $x=y=z=1$.

**Also solved by** *Ioan Viorel Codreanu, Satulung, Maramures, Romania and José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain.*

***MH–121.*** *Proposed by Alexandru Benescu, Romania.* For every odd natural number $n$ we consider a board in the shape of a rhombus with $n$ rows and $n$ columns as in the figure below. Let $d_n$ be the maximum number of queens that can be placed on the board, so that there are no two of them to attack each other (where a queen, as usual, can move freely on a row, column or diagonal). Find the minimum value $m$ such that $m - d_m > 2$, and compute $d_k$ for all odd natural number $k \leq m$.



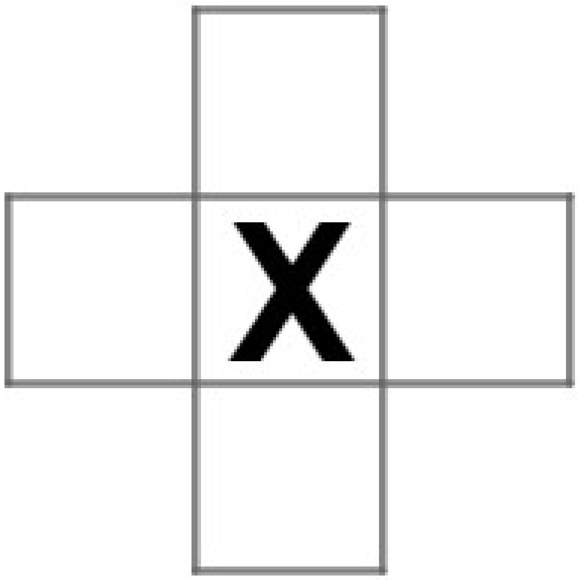
**Solution 1 by Jordi Ferré García, CFIS, BarcelonaTech, Barcelona, Spain.** We can easily see that $d_1 = 1$, so we can assume that $n \geq 3$.

We label the rows of the board from top to bottom with $R_i$ and the columns from left to right with $C_i$. According to this notation there is only 1 unit square in $R_1$ and $C_1$, there are 3 unit squares in $R_2$ and $C_2$, and so on.

We now begin to establish bounds for $d_n$. First, observe that if $d_n \geq n+1$, then by the Pigeonhole Principle, since the board has only $n$ rows, there must be at least two queens in the same row, resulting in them attacking each other. Additionally, the case $d_n = n$ can be easily dismissed, as it would imply placing a queen in every row. However, if there is a queen in row $R_1$ and another queen in row $R_n$, they would attack each other. We now work on the case $d_n = n-1$. By the previous argument we can assume that there is a queen on $R_1$ but not in $R_n$, and that the other $n-2$ queens are on $R_2$, $R_3$, ..., $R_{n-1}$. But in particular we find a contradiction when placing a queen on $R_2$, which we can not avoid attacking the queen on $R_1$. This already shows that $d_n \leq n-2$.

It is not difficult to show that for $n \in \{3,5,7\}$ this bound is sharp, as it can be shown in the following boards:

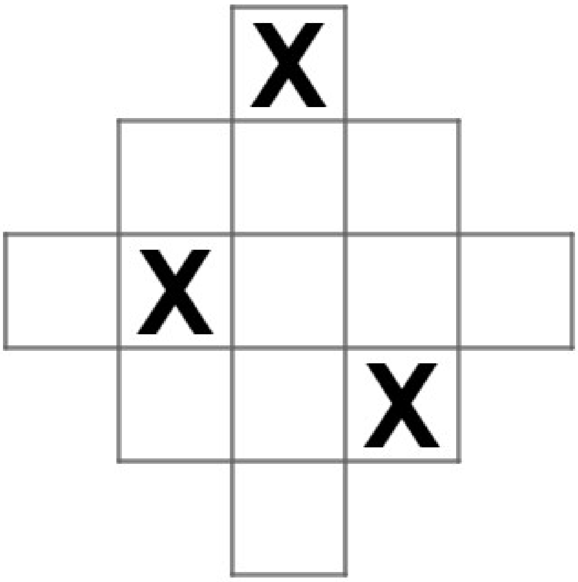
$$d_3 = 1$$




We will now prove that $d_9 = 6$. As $d_9 < 9 - 2 = 7$, this will tell us that $m = 9$, ending the problem. In order to obtain this we will show that it is impossible to place $7$ queens on a board with $9$ rows, so we assume the opposite and seek for a contradiction.

We start by splitting the board in $4$ Tetris T-piece blocks and one $5 \times 5$ square (see Figure). If there are no queens on $R_1$ or $R_2$ (which coincides with the top Tetris T-piece), then there must be one queen placed on each row starting from $R_3$ up to $R_9$. Precisely, this implies that there is one queen on $R_8$ and $R_9$, which is a contradiction as they will attack each other. So there must be at least one queen on the top Tetris T-piece, and by a similar argument there must also be a queen on the bottom, left and right Tetris T-piece. As it is clearly impossible to place two queens in a single Tetris T-piece, the remaining $3$ queens must be placed in the $5 \times 5$ square.

$$d_5 = 3$$



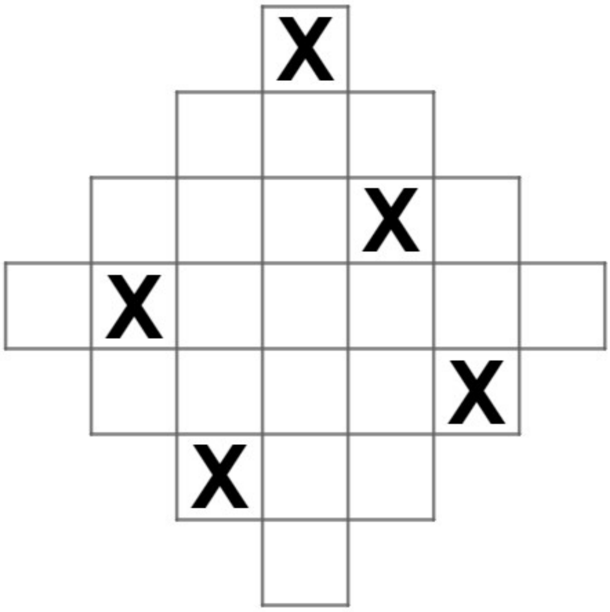

For each Tetris T-piece we distinguish its 4 squares in three cathegories as shown in the image in the next page.

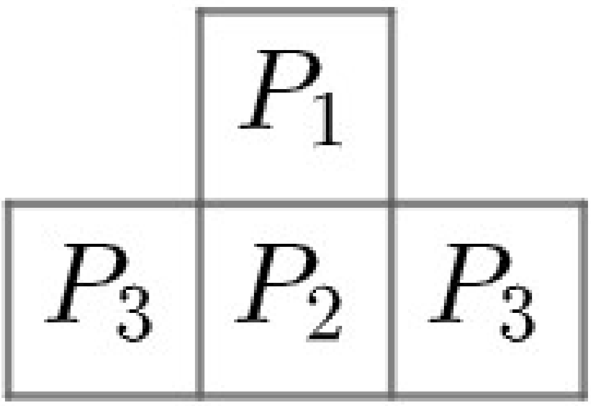
It's evident that at most one of the four $P_1$ squares can be occupied by a queen, and likewise, at most one one $P_2$ square (one can easily check that a queen on a $P_1$ square attacks all other $P_1$ squares, and that the same case holds for the $P_2$ squares). This observation now allows us to consider different cases.

**Case 1:** No $P_1$ or $P_2$ square is occupied. Then, the four queens will be placed on $P_3$ squares. Due to the condition of not attacking, all queens must be placed either on the right hand side $P_3$ squares or on the left hand side (which are analogous cases because they are symmetric).

This leaves us with 3 remaining queens to place on the $5 \times 5$ square. By inspecting the grid, one can check that rows $R_3, R_4, R_6$ and $R_7$ are being attacked by one of the queens, implying that all

$d_7 = 5$

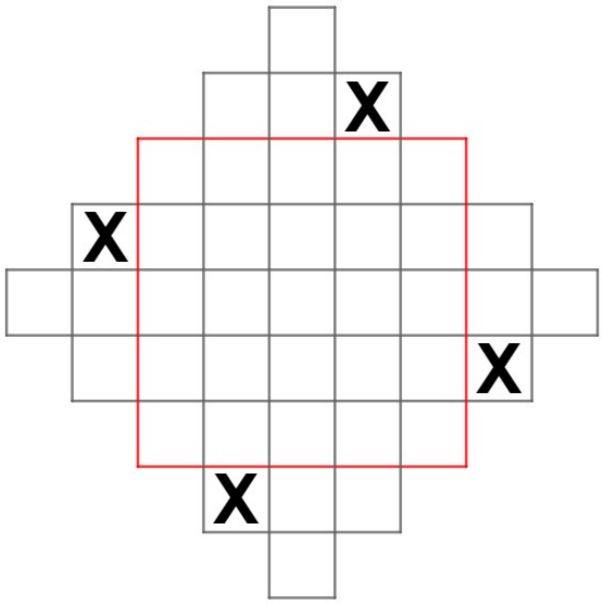


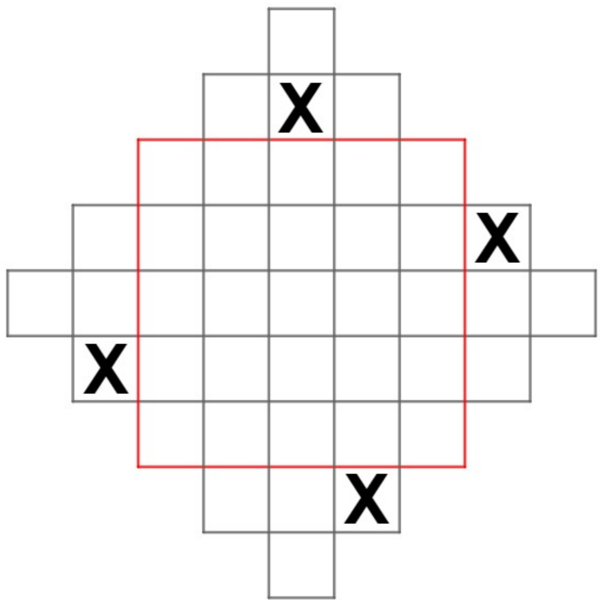

Scheme for labels in Tetris T-piece

3 queens must be placed on $R_5$, wihch is clearly impossible.

**Case 2:** No $P_1$ square is occupied, but a $P_2$ is. Then, the queens on the Tetris T-pieces are placed, necessary, as in the figure below



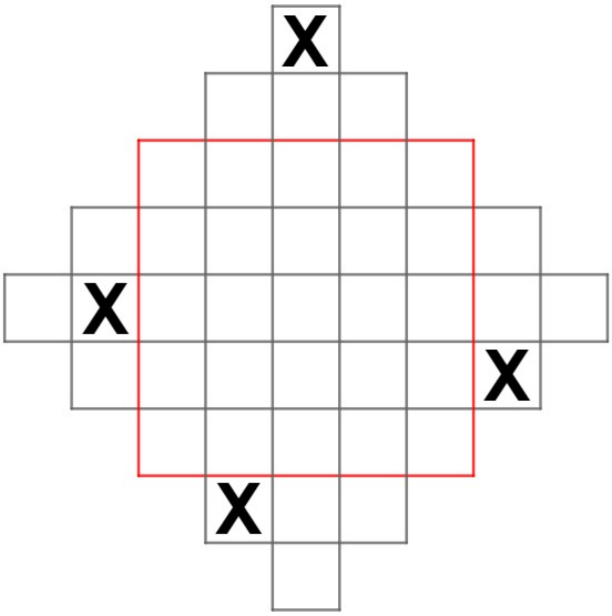
(only one chirality is shown).



Scheme for Case 2

But this leaves columns $C_5$, $C_6$ and $C_7$ attacked by some queen, so we are left with two columns, $C_3$ and $C_4$, to place $3$ queens, an impossibility.

**Case 3:** One $P_1$ is occupied. Then, the queens on the Tetris T-pieces are placed, necessary, as in the figure below (once again, only one chirality is shown).



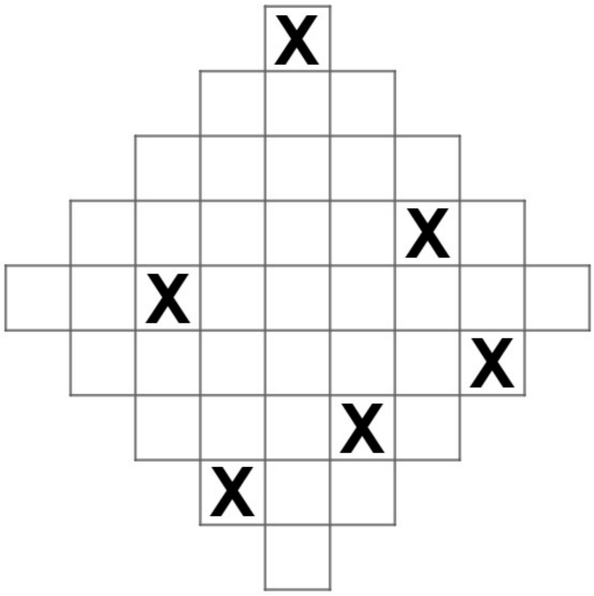

Scheme for Case 3

But then columns $C_3$, $C_4$ and $C_5$ are attacked, so we only have $C_6$ and $C_7$ left, which is once again a contradiction.

So we've established the desired bound of $d_9 \leq 6$. In fact we can get the equality $d_9 = 6$ by considering the figure below.

This ends the problem, from which we can conclude that $m = 9$ and $d_1 = 1$, $d_3 = 1$, $d_5 = 3$, $d_7 = 5$ and $d_9 = 6$.

**Also solved by** *the proposer.*




***MH–122.*** *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* In an euclidean plane a set of 2024 points are given in such a way that the distance between every two of these points is irrational. Will it be possible for every three points of the set to form a non degenerate triangle with rational area?

**Solution by the proposers.** Consider the points $P_i = (i, i^2)$, $1 \leq i \leq 2024$. They satisfy the following conditions:

1. For any distinct points $P_i, P_j$, we have

$$\begin{aligned} \overline{P_iP_j} &= \sqrt{(i-j)^2 + (i^2 - j^2)^2} \\ &= \sqrt{(i-j)^2 + (i-j)^2(i+j)^2} \\ &= |i-j|\sqrt{1+(i+j)^2}. \end{aligned}$$

   Assume that $\overline{P_iP_j}$ is rational, then there exists positive integers $p, q$ with $(p, q) = 1$ such that

$$\sqrt{1+(i+j)^2} = \frac{p}{q} \Leftrightarrow 1 + (i+j)^2 = \frac{p^2}{q^2}.$$

   From this we get that $p^2/q^2$ is a positive integer which means that $q^2 \mid p^2$ and $q \mid p$. Since $(p, q) = 1$ and $q \mid p$ then $q = 1$.

On account of the preceding $1 + (i + j)^2$ should be a perfect square. On the other hand,

$$(i + j)^2 < 1 + (i + j)^2 < 1 + (i + j)^2 + 2\,(i + j) = (i + j + 1)^2$$

and we get a contradiction because $1 + (i + j)^2$ is not a square. So, $\overline{P_i P_j}$ is irrational.

2. For any three distinct points $P_i, P_j, P_k$ with $i < j < k$, we have

$$A = \frac{1}{2} \left| \begin{vmatrix} 1 & i & i^2 \\ 1 & j & j^2 \\ 1 & k & k^2 \end{vmatrix} \right| = \frac{1}{2} \left| (j - i)(k^2 - i^2) - (k - i)(j^2 - i^2) \right|$$

which is rational.

So, the answer of the question posed in the statement is YES.

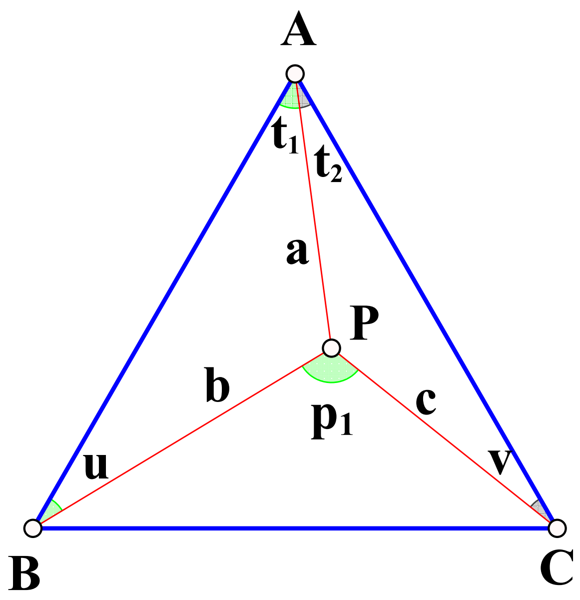
**Comment.** Since the vertices of $\triangle P_i P_j P_k$ are lattice points, then its area is also given by

$$A = I - 1 + \frac{B}{2}$$

on account of Pick's theorem. This is obviously rational.

***MH–123.*** *Proposed by Michel Bataille, Rouen, France.* Let $\mathcal{T}$ denote the interior of an equilateral triangle $ABC$ with side $s$. If $P \in \mathcal{T}$, let $a = PA, b = PB, c = PC$, $\alpha = \inf\{(a + b + c)^2 : P \in \mathcal{T}\}$ and $\beta = \sup\{a^2 + b^2 + c^2 + ab + bc + ca : P \in \mathcal{T}\}$. Prove that $\alpha = 3s^3 = \beta$.

**Solution by the proposer.** If $P$ is the center of $\Delta ABC$, then $a = b = c = \frac{s\sqrt{3}}{3}$, hence $(a + b + c)^2 = 3s^2$.
If $P$ approaches $C$, then $c$ approaches $0$ and $a, b$ approach $s$, so that $a^2 + b^2 + c^2 + ab + bc + ca$ approaches $3s^2$. It follows that it is sufficient to prove that $(a+b+c)^2 \geq 3s^2 \geq a^2+b^2+c^2+ab+bc+ca$ whenever $P \in \mathcal{T}$.
Let $P \in \mathcal{T}$ and let $p_1 = \angle BPC$, $t_1 = \angle BAP$, $t_2 = \angle CAP$, $u = \angle PBA$, $v = \angle PCA$ (see figure).




Scheme for solving Problem MH-121.

From $s = a\cos t_1 + b\cos u = a\cos t_2 + c\cos v$ and $a\sin t_1 = b\sin u$, $a\sin t_2 = c\sin v$, we deduce

$$\begin{aligned}\frac{a^2}{2} &= a^2\cos(t_1+t_2) = (a\cos t_1)(a\cos t_2) - (a\sin t_1)(a\sin t_2)\\ &= (s - b\cos u)(s - c\cos v) - bc\sin u\sin v\\ &= s^2 - bs\cos u - cs\cos v + bc\cos(u+v).\end{aligned}$$

Since $u + v = p_1 - \frac{\pi}{3}$ and $a^2 = s^2 + b^2 - 2bs\cos u = s^2 + c^2 - 2cs\cos v$ (so that $s^2 - bs\cos u - cs\cos v = a^2 - \frac{b^2+c^2}{2}$), it easily follows that

$$\frac{b^2+c^2-a^2}{2bc} = \cos\Big(p_1 - \frac{\pi}{3}\Big).$$

But we also have $\cos p_1 = \frac{b^2+c^2-s^2}{2bc}$, hence $\cos p_1 + \frac{s^2}{2bc} - \frac{a^2}{2bc} = \frac{1}{2}\cos p_1 + \frac{\sqrt{3}}{2}\sin p_1$ and therefore

$$\cos\Big(p_1 + \frac{\pi}{3}\Big) = \frac{a^2-s^2}{2bc}.$$

Since $\frac{\pi}{3} < p_1 < \pi$, we have $\frac{2\pi}{3} < p_1 + \frac{\pi}{3} < \frac{4\pi}{3}$, hence $-1 \leq \cos\left(p_1 + \frac{\pi}{3}\right) < -\frac{1}{2}$. As a result, $a^2+2bc \geq s^2 > a^2+bc$. In a similar way, we obtain $b^2+2ca \geq s^2 > b^2+ca$ and $c^2+2ab \geq s^2 > c^2+ab$; by addition, we get

$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \geq 3s^2 > a^2 + b^2 + c^2 + ab + bc + ca,$$

as desired.

***MH–124.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $n$ be a positive integer. Prove that

$$9 + 2\sum_{k=1}^{n} \left\{ \frac{L_{k+1}}{\log\left(1 + \dfrac{L_{k+1}}{L_k}\right)} \right\} < 4L_{n+1} + 5L_n,$$

where $L_n$ is the $n^{th}$ Lucas number defined by $L_1 = 1, L_2 = 3$ and $L_n = L_{n-1} + L_{n-2}$ for all $n \geq 3$.

**Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA.** Using the Lucas number recurrence relation, $L_{k+1} = L_{k+2} - L_k$, and

$$\log\left(1 + \frac{L_{k+1}}{L_k}\right) = \log\left(\frac{L_k + L_{k+1}}{L_k}\right) = \log\frac{L_{k+2}}{L_k} = \log L_{k+2} - \log L_k.$$

Thus,

$$\frac{L_{k+1}}{\log\left(1 + \frac{L_{k+1}}{L_k}\right)} = \frac{L_{k+2} - L_k}{\log L_{k+2} - \log L_k},$$

which is the logarithmic mean of $L_k$ and $L_{k+2}$. Because $L_k \neq L_{k+2}$ for any positive integer $k$, by the arithmetic mean - logarithmic mean inequality,

$$\frac{L_{k+1}}{\log\left(1 + \frac{L_{k+1}}{L_k}\right)} = \frac{L_{k+2} - L_k}{\log L_{k+2} - \log L_k} < \frac{L_{k+2} + L_k}{2}.$$

It then follows that

$$9 + 2\sum_{k=1}^{n} \left\{ \frac{L_{k+1}}{\log\left(1 + \frac{L_{k+1}}{L_k}\right)} \right\} < 9 + \sum_{k=1}^{n} (L_{k+2} + L_k).$$

But,

$$\begin{aligned}\sum_{k=1}^{n} L_k &= \sum_{k=1}^{n} (L_{k+1} - L_{k-1}) \\ &= L_{n+1} + L_n - L_1 - L_0 = L_{n+2} - 3,\end{aligned}$$

and

$$\sum_{k=1}^{n} L_{k+2} = \sum_{k=3}^{n+2} L_k = L_{n+4} - 3 - L_2 - L_1 = L_{n+4} - 7,$$

where we have used the fact that $L_0 = 2$. Finally,

$$\begin{aligned}9 + 2\sum_{k=1}^{n} \left\{ \frac{L_{k+1}}{\log\left(1 + \frac{L_{k+1}}{L_k}\right)} \right\} &< L_{n+4} + L_{n+2} - 1 \\ &= L_{n+3} + 2L_{n+2} - 1 \\ &= 3L_{n+2} + L_{n+1} - 1 \\ &= 4L_{n+1} + 3L_n - 1,\end{aligned}$$

which is a tighter bound than the desired result of $4L_{n+1} + 5L_n$.

**Solution 2 by G. C. Greubel, Newport News, VA, USA.** Using the expansion of the Bernoulli numbers of the second kind

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} b_n\, x^n = 1 + \frac{x}{2} - \frac{x^2}{12} + \frac{x^3}{24} - \frac{19\, x^4}{720} + \frac{3\, x^5}{160} + \cdots$$

then

$$\begin{aligned}S_n &= \sum_{k=1}^{n} \left\{ \frac{L_{k+1}}{\ln\left(1 + \frac{L_{k+1}}{L_k}\right)} \right\} \\ &= \sum_{k=1}^{n} L_k \left(1 + \frac{L_{k+1}}{2\, L_k} - \frac{1}{12}\left(\frac{L_{k+1}}{L_k}\right)^2 + \frac{1}{24}\left(\frac{L_{k+1}}{L_k}\right)^3 + \cdots\right) \\ &= \sum_{k=1}^{n} L_k + \frac{1}{2}\sum_{k=1}^{n} L_{k+1} - \frac{1}{12}\, f_n^{(2)} + \frac{1}{24}\, f_n^{(3)} + \cdots \\ &= \frac{1}{2}\,(L_{n+3} + 2\, L_{n+2} - 10) + \sum_{j=2}^{\infty} b_j\, f_n^{(j)} \\ &= \frac{1}{2}\,(4\, L_{n+1} + 3\, L_n - 10) + \sum_{j=2}^{\infty} b_j\, f_n^{(j)},\end{aligned}$$

where

$$f_n^{(j)} = \sum_{k=1}^{n} L_k \left(\frac{L_{k+1}}{L_k}\right)^j.$$

Now,

$$9 + 2\,S_n = 4\,L_{n+1} + 3\,L_n - 1 - \frac{1}{6}\,f_n^{(2)} + 2\,\sum_{j=3}^{\infty} b_j\,f_n^{(j)}$$

and using

$$\frac{1}{3}\,f_n^{(2)} > 2\,\sum_{j=3}^{\infty} b_j\,f_n^{(j)},$$

for $n \geq 4$, then

$$9 + 2\,S_n > 4\,L_{n+1} + 3\,L_n - \left(1 + \frac{1}{3}\,f_n^{(2)}\right).$$

Using

$$4\,L_{n+1} + 3\,L_n - \left\lceil 1 + \frac{1}{3}\,f_n^{(2)} \right\rceil \geq 4\,L_n + 5\,L_{n-1}$$

then

$$4\,L_{n+1} + 5\,L_n > 9 + 2\,S_n > 4\,L_{n+1} + 3\,L_n - \left\lceil 1 + \frac{1}{3}\,f_n^{(2)} \right\rceil$$

for $n \geq 4$. By considering the cases $n = \{0, 1, 2, 3\}$ it is demonstrated that the inequality holds.

**Solution 3 by Michel Bataille, Rouen, France.** Let $X_n$ denote the left side of the inequality. We show the better inequality

$$X_n < 4L_{n+1} + 3L_n - 1.$$

To this end, we use the inequality $\log(1+x) > \frac{2x}{2+x}$ for $x > 0$, which readily follows from a quick study of the function $f(x) = \log(1+x) - \frac{2x}{2+x}$. We deduce that $\log\left(1 + \frac{L_{k+1}}{L_k}\right) > \frac{2L_{k+1}}{2L_k + L_{k+1}}$ so that

$$\frac{2L_{k+1}}{\log\left(1 + \frac{L_{k+1}}{L_k}\right)} < 2L_k + L_{k+1}$$

for $k = 1, 2, \ldots, n$. As a result, we obtain

$$\begin{aligned} X_n &< 9 + \sum_{k=1}^{n} (2L_k + L_{k+1}) \\ &= 9 + L_{n+1} - L_1 + 3 \sum_{k=1}^{n} L_k \\ &= 8 + L_{n+1} + 3(L_{n+2} - 3) = 4L_{n+1} + 3L_n - 1 \end{aligned}$$

where we used $\sum_{k=1}^{n} L_k = L_{n+2} - 3$, easily proved by induction.

**Solution 4 by José Gibergnas-Báguna, BarcelonaTech, Terrassa, Spain and the proposer.** The statement follows from Polya's inequality

$$\frac{b-a}{\log b - \log a} < \frac{1}{3}\left(2\sqrt{ab} + \frac{a+b}{2}\right)$$

applied to the pair of Lucas numbers $(L_k, L_{k+2}), 1 \leq k \leq n$. Indeed, for $k = 1, 2, \ldots, n$, we have

$$\frac{L_{k+2} - L_k}{\log L_{k+2} - \log L_k} < \frac{1}{3}\left(2\sqrt{L_k L_{k+2}} + \frac{L_k + L_{k+2}}{2}\right)$$

or equivalently,

$$\frac{L_{k+1}}{\log\left(\frac{L_{k+2}}{L_k}\right)} < \frac{1}{3}\left(2\sqrt{L_k L_{k+2}} + \frac{L_k + L_{k+2}}{2}\right)$$

Using Lucas recursion and GM-AM inequality, we have

$$\frac{L_{k+1}}{\log\left(\frac{L_{k+2}}{L_k}\right)} = \frac{L_{k+1}}{\log\left(1 + \frac{L_{k+1}}{L_k}\right)} < \frac{1}{2}(L_k + L_{k+2}), \quad (1 \leq k \leq n).$$

Adding up the preceding inequalities and taking into account that $L_1 + L_2 + \cdots + L_n = L_{n+2} - 3$, we get

$$\begin{aligned} \sum_{k=1}^{n} \left\{ \frac{L_{k+1}}{\log\left(1 + \frac{L_{k+1}}{L_k}\right)} \right\} &< \frac{1}{2}[(L_1 + L_2 + \cdots + L_n) + (L_3 + \cdots + L_{n+2})] \\ &= -9 + L_{n+1} + 4L_{n+2} = -9 + 4L_{n+1} + 5L_n, \end{aligned}$$

from which the statement follows.

**Also solved by** *Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania.*

# *Advanced Problems*

***A–119.*** *Proposed by Vasile Mircea Popa, Affiliate Professor, "Lucian Blaga" University of Sibiu, Romania.* Calculate the integral

$$\int_0^\infty \frac{\sqrt{x}\arctan(x)\ln^2(x)}{x^3+x\sqrt{x}+1}dx.$$

**Solution 1 by Michel Bataille, Rouen, France.** Let $I$ denote the integral. The change of variable $x=\frac{1}{u}$ and the well-known relation $\arctan(1/u)=\frac{\pi}{2}-\arctan(u)$ for $u>0$ readily show that $I=\frac{\pi}{4}\cdot J$ where

$$J=\int_0^\infty \frac{\sqrt{u}\ln^2(u)}{u^3+u\sqrt{u}+1}\,du.$$

To calculate $J$, we first apply the substitution $u=w^{2/3}$, which yields

$$J=\frac{8}{27}\int_0^\infty \frac{(\ln(w))^2}{w^2+w+1}\,dw$$

and then refer to problem A-106 (see vol. 9, issue 2, solution 3) where it is proved that

$$\int_0^\infty \frac{(\ln(w))^2}{w^2+w+1}\,dw=\frac{16\pi^3}{81\sqrt{3}}.$$

We deduce that

$$I=\frac{\pi}{4}\cdot\frac{8}{27}\cdot\frac{16\pi^3}{81\sqrt{3}}=\frac{32\pi^4}{2187\sqrt{3}}.$$

**Solution 2 by G. C. Greubel, Newport News, VA, USA.** Let the integrand be given by

$$f(x)=\frac{\sqrt{x}\,\tan^{-1}(x)\,\ln^2(x)}{x^3+x\,\sqrt{x}+1}$$

then consider $f(1/x)$ for which

$$\begin{aligned} f\left(\frac{1}{x}\right) &= \frac{x^2\,\sqrt{x}\,\tan^{-1}(1/x)\,\ln^2(x)}{x^3+x\,\sqrt{x}+1} \\ &= \frac{\pi}{2}\,\frac{x^2\,\sqrt{x}\,\ln^2(x)}{x^3+x\,\sqrt{x}+1}-x^2\,f(x), \end{aligned}$$

where $\tan^{-1}(1/x) + \tan^{-1}(x) = \pi/2$ was used. Now, the integral in question becomes

$$\begin{aligned} I &= \int_0^\infty f(x)\,dx \\ &= \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx \\ &= \int_0^1 f(x)\,dx + \int_0^1 f\left(\frac{1}{x}\right)\frac{dx}{x^2} \\ &= \frac{\pi}{2}\int_0^1 \frac{\sqrt{x}\,\ln^2(x)}{x^3 + x\sqrt{x} + 1}\,dx. \end{aligned}$$

Let $x = t^2$ to obtain

$$I = 4\pi\int_0^1 \frac{t^2\,\ln^2(t)}{1 + t^3 + t^6}\,dt.$$

Using

$$\frac{1}{1+t^3+t^6} = \sum_{n=0}^{\infty} U_n\left(-\frac{1}{2}\right)t^{3n},$$

where $U_n(x)$ are the Chebyshev polynomials of the second kind, then

$$\begin{aligned} I &= 4\pi\sum_{n=0}^{\infty} U_n\left(-\frac{1}{2}\right)\int_0^1 t^{3n+2}\,\ln^2(t)\,dt \\ &= \frac{8\pi}{27}\sum_{n=1}^{\infty}\frac{1}{n^3}U_{n-1}\left(-\frac{1}{2}\right) \\ &= \frac{8\pi}{27}\,\frac{4\pi^3}{81\sqrt{3}} = \frac{32\pi^4}{2187\sqrt{3}}. \end{aligned}$$

This gives

$$\int_0^\infty \frac{\sqrt{x}\,\tan^{-1}(x)\,\ln^2(x)}{x^3 + x\sqrt{x} + 1}\,dx = \frac{32\pi^4}{2187\sqrt{3}}.$$

**Solution 3 by Moti Levy, Rehovot, Israel.** Let

$$I := \int_0^\infty \frac{\sqrt{x}\arctan(x)\ln^2(x)}{x^3 + x\sqrt{x} + 1}dx. \tag{1}$$

After change of variable $\sqrt{x} = y$,

$$I = 8\int_0^\infty \frac{y^2}{y^6+y^3+1}\ln^2(y)\arctan\left(y^2\right)dy.$$

Since $y^6+y^3+1 = \frac{1-y^9}{1-y^3}$, then

$$I = 8\int_0^\infty \frac{y^2}{1-y^9}\ln^2(y)\arctan\left(y^2\right)dy - 8\int_0^\infty \frac{y^5}{1-y^9}\ln^2(y)\arctan\left(y^2\right)dy. \tag{2}$$

After change of variable $\frac{1}{u} = y$,

$$\int_0^\infty \frac{y^2}{1-y^9}\ln^2(y)\arctan\left(y^2\right)dy = -\int_0^\infty u^5\frac{\ln^2(u)}{1-u^9}\arctan\left(\frac{1}{u^2}\right)du. \tag{3}$$

$$\begin{aligned} I &= 8\int_0^\infty \frac{y^2}{1-y^9}\ln^2(y)\arctan\left(y^2\right)dy + 8\int_0^\infty y^2\frac{\ln^2(y)}{1-y^9}\arctan\left(\frac{1}{y^2}\right)dy \\ &= 8\int_0^\infty \frac{y^2}{1-y^9}\ln^2(y)\left(\arctan\left(y^2\right)+\arctan\left(\frac{1}{y^2}\right)\right)dy, \\ I &= 4\pi\int_0^\infty \frac{y^2}{1-y^9}\ln^2(y)dy. \end{aligned} \tag{4}$$

$$\begin{aligned} I &= 4\pi\int_1^\infty \frac{y^2}{1-y^9}\ln^2(y)dy + 4\pi\int_0^1 \frac{y^2}{1-y^9}\ln^2(y)dy \\ &= 4\pi\int_0^1 \frac{-y^5}{1-y^9}\ln^2(y)dy + 4\pi\int_0^1 \frac{y^2}{1-y^9}\ln^2(y)dy \\ &= -4\pi\int_0^1 \sum_{k=0}^\infty y^{9k+5}\ln^2(y)dy + 4\pi\int_0^1 \sum_{k=0}^\infty y^{9k+2}\ln^2(y)dy \\ &= -4\pi\sum_{k=0}^\infty \int_0^1 y^{9k+5}\ln^2(y)dy + 4\pi\sum_{k=0}^\infty \int_0^1 y^{9k+2}\ln^2(y)dy. \end{aligned} \tag{5}$$

The following integral is known

$$\int_0^1 y^k\ln^2(y)dy = \frac{2}{(k+1)^3}, \quad k > -1 \tag{6}$$

$$\begin{aligned}
I &= -8\pi \sum_{k=0}^{\infty} \frac{1}{(9k+6)^3} + 8\pi \sum_{k=0}^{\infty} \frac{1}{(9k+3)^3} \\
&= \frac{8\pi}{9^3}\left(\sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{1}{3}\right)^3} - \sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{2}{3}\right)^3}\right) \\
&= \frac{8\pi}{9^3}\left(\sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{1}{3}\right)^3} - \sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{2}{3}\right)^3}\right)
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{2}{3}\right)^3} &= -\frac{1}{2}\psi^{(2)}\left(\frac{2}{3}\right) \\
\sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{1}{3}\right)^3} &= -\frac{1}{2}\psi^{(2)}\left(\frac{1}{3}\right) \\
\sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{1}{3}\right)^3} - \sum_{k=0}^{\infty} \frac{1}{\left(k+\frac{2}{3}\right)^3} &= -\frac{1}{2}\left(\psi^{(2)}\left(\frac{1}{3}\right) - \psi^{(2)}\left(\frac{2}{3}\right)\right),
\end{aligned}$$

where $\psi^{(2)}(z)$ is the polygamma function of order 2. The reflection relation for $\psi^{(2)}(z)$ is

$$\psi^{(2)}(1-z) - \psi^{(2)}(z) = \pi \frac{d^2}{dz^2}(\cot(\pi z)) = 2\pi^3(\cot(\pi z))\left(\cot^2(\pi z)+1\right).$$

Hence,

$$\begin{aligned}
-\frac{1}{2}\left(\psi^{(2)}\left(\frac{1}{3}\right) - \psi^{(2)}\left(\frac{1}{3}\right)\right) &= -\frac{1}{2}\left(2\pi^3\left(\cot\left(\frac{2\pi}{3}\right)\right)\left(\cot^2\left(\frac{2\pi}{3}\right)+1\right)\right) \\
&= \frac{4}{9}\sqrt{3}\pi^3.
\end{aligned}$$

We conclude that the integral is

$$I = \frac{32\sqrt{3}\pi^4}{9^4} \cong 0.822\,89.$$

**Solution 4 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA.** Let

$$I = \int_0^{\infty} \frac{\sqrt{x}\arctan x \ln^2 x}{x^3 + x\sqrt{x} + 1}\, dx,$$

and write

$$I = \int_0^1 \frac{\sqrt{x}\arctan x \ln^2 x}{x^3 + x\sqrt{x} + 1}\, dx + \int_1^\infty \frac{\sqrt{x}\arctan x \ln^2 x}{x^3 + x\sqrt{x} + 1}\, dx.$$

In the second integral on the right side, make the change of variables $x \to \frac{1}{x}$ to obtain

$$\begin{aligned}\int_1^\infty \frac{\sqrt{x}\arctan x \ln^2 x}{x^3 + x\sqrt{x} + 1}\, dx &= \int_0^1 \frac{\sqrt{x}\arctan \frac{1}{x} \ln^2 x}{x^3 + x\sqrt{x} + 1}\, dx \\ &= \int_0^1 \frac{\sqrt{x}\left(\frac{\pi}{2} - \arctan x\right) \ln^2 x}{x^3 + x\sqrt{x} + 1}\, dx.\end{aligned}$$

It follows that

$$I = \frac{\pi}{2}\int_0^1 \frac{\sqrt{x}\ln^2 x}{x^3 + x\sqrt{x} + 1}\, dx.$$

Now, multiplying the numerator and denominator of the integrand by $1 - x^{3/2}$ and using the geometric series

$$\frac{1}{1 - x^{9/2}} = \sum_{n=0}^{\infty} x^{9n/2}$$

yields

$$\begin{aligned}I &= \frac{\pi}{2}\int_0^1 \frac{(1 - x^{3/2})x^{1/2}\ln^2 x}{1 - x^{9/2}}\, dx \\ &= \frac{\pi}{2}\int_0^1 \sum_{n=0}^{\infty}\left(x^{(9n+1)/2} - x^{(9n+4)/2}\right)\ln^2 x\, dx \\ &= \frac{\pi}{2}\sum_{n=0}^{\infty}\int_0^1\left(x^{(9n+1)/2} - x^{(9n+4)/2}\right)\ln^2 x\, dx.\end{aligned}$$

After two integration by parts

$$\int_0^1 x^k \ln^2 x\, dx = \frac{2}{(k+1)^3},$$

so

$$\begin{aligned}I &= \frac{\pi}{2}\sum_{n=0}^{\infty}\left(\frac{16}{(9n+3)^3} - \frac{16}{(9n+6)^3}\right) \\ &= \frac{8\pi}{729}\sum_{n=0}^{\infty}\left(\frac{1}{(n+1/3)^3} - \frac{1}{(n+2/3)^3}\right) \\ &= \frac{4\pi}{729}\left(\psi_2\left(\frac{2}{3}\right) - \psi_2\left(\frac{1}{3}\right)\right),\end{aligned}$$

where $\psi_2(x)$ is the tetragamma function. By the reflection formula for the tetragamma function,

$$\psi_2\left(\frac{2}{3}\right)-\psi_2\left(\frac{1}{3}\right) = \pi\frac{d^2}{dx^2}\cot(\pi x)\Big|_{x=1/3} = 2\pi^3\csc^2\left(\frac{\pi}{3}\right)\cot\left(\frac{\pi}{3}\right) = \frac{8\pi^3}{3\sqrt{3}}.$$

Thus,

$$I = \int_0^\infty \frac{\sqrt{x}\arctan x\ln^2 x}{x^3+x\sqrt{x}+1}\,dx = \frac{32\pi^4}{2187\sqrt{3}}.$$

**Solution 5 by the proposer.** Let us denote:

$$I = \int_0^\infty \frac{\sqrt{x}\arctan(x)\ln^2(x)}{x^3+x\sqrt{x}+1}dx;\ J = \int_0^\infty \frac{\sqrt{x}\,\mathrm{arccot}(x)\ln^2(x)}{x^3+x\sqrt{x}+1}dx$$

We have:

$$I+J = \frac{\pi}{2}\int_0^\infty \frac{\sqrt{x}\ln^2(x)}{x^3+x\sqrt{x}+1}dx.$$

In the integral $J$ we make variable change $y = \frac{1}{x}$. We get immediately $J = I$. So:

$$I = \frac{\pi}{4}\int_0^\infty \frac{\sqrt{x}\ln^2(x)}{x^3+x\sqrt{x}+1}dx.$$

We calculate:

$$K = \int_0^\infty \frac{\sqrt{x}\ln^2(x)}{x^3+x\sqrt{x}+1}dx;$$

$$K = \int_0^1 \frac{\sqrt{x}\ln^2(x)}{x^3+x\sqrt{x}+1}dx + \int_1^\infty \frac{\sqrt{x}\ln^2(x)}{x^3+x\sqrt{x}+1}dx.$$

Let us denote:

$$A = \int_0^1 \frac{\sqrt{x}\ln^2(x)}{x^3+x\sqrt{x}+1}dx;\ B = \int_1^\infty \frac{\sqrt{x}\ln^2(x)}{x^3+x\sqrt{x}+1}dx.$$

We consider the integral A. In this integral we make the variable change: $x = y^{\frac{2}{3}}$. We obtain:

$$A = \frac{8}{27}\int_0^1 \frac{\ln^2(y)}{y^2+y+1}dy.$$

We have, successively:

$$A = \frac{8}{27}\int_0^1 \frac{\ln^2(y)}{y^2+y+1}dy = \frac{8}{27}\int_0^1 \frac{(1-y)\ln^2(y)}{1-y^3}dy$$
$$= \frac{8}{27}\int_0^1 \frac{\ln^2(y)}{1-y^3}dy - \frac{8}{27}\int_0^1 \frac{y\ln^2(y)}{1-y^3}dy.$$

$$A = \frac{8}{27}\int_0^1 \sum_{n=0}^{\infty} y^{3n}\ln^2(y)dy - \frac{8}{27}\int_0^1 \sum_{n=0}^{\infty} y^{3n+1}\ln^2(y)dy$$
$$A = \frac{8}{27}\sum_{n=0}^{\infty}\left(\int_0^1 y^{3n}\ln^2(y)dy - \int_0^1 y^{3n+1}\ln^2(y)dy\right).$$

We will use the following relationship

$$\int_0^1 x^a \ln^2(x)dx = \frac{2}{(a+1)^3}, \text{where } \ a\in\mathbb{R},\ a\geq 0.$$

We obtain

$$\begin{aligned} A &= \frac{8}{27}\sum_{n=0}^{\infty}\left[\frac{2}{(3n+1)^3} - \frac{2}{(3n+2)^3}\right] \\ &= \frac{8}{27}\sum_{n=0}^{\infty}\left[\frac{\frac{2}{27}}{\left(n+\frac{1}{3}\right)^3} - \frac{\frac{2}{27}}{\left(n+\frac{2}{3}\right)^3}\right]. \end{aligned}$$

We now use the following relationship

$$\psi_2(x) = -\sum_{n=0}^{\infty}\frac{2}{(x+n)^3}.$$

where $\psi_2$ is the tetragamma function. We obtained the value of the integral $A$:

$$A = \frac{8}{729}\left[-\psi_2\left(\frac{1}{3}\right) + \psi_2\left(\frac{2}{3}\right)\right]$$

We consider the integral $B$. We make the variable change

$$x = \frac{1}{y},\ y = \frac{1}{x}.$$

We obtain immediately: $B = A$ Result:

$$K = 2A = \frac{16}{729}\left[-\psi_2\left(\frac{1}{3}\right) + \psi_2\left(\frac{2}{3}\right)\right].$$

We use the reflection formula for the tetragamma function

$$\psi_2(p) - \psi_2(1-p) = f(p), \text{ where: } f(x) = -\pi\frac{d^2}{dx^2}\cot(\pi x).$$

We obtain

$$\psi_2\left(\frac{2}{3}\right) - \psi_2\left(\frac{1}{3}\right) = \frac{8\pi^3\sqrt{3}}{9}.$$

Result

$$K = \frac{16}{729}\cdot\frac{8\pi^3\sqrt{3}}{9} = \frac{128\pi^3\sqrt{3}}{6561}; I = \frac{\pi}{4}K.$$

$$I = \frac{32\pi^4\sqrt{3}}{6561}.$$

Thus, the problem is solved.

**Also solved by** *Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania and Ankush Kumar Parcha, India.*

***A–120.*** *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let $m, n$ be positive integers, such that

$$a = \frac{(m+3)^n + 1}{4m}$$

is an integer. Prove that 3 divides $a + 2^{n+1}$.

**Solution by the proposer.** Since $a$ is an integer number, then $4m$ divides $(m+3)^n + 1$ and $m$ is even.

We consider the following cases:

**(1)** $m \equiv 2 \pmod 4$, then

$$(m+3)^n + 1 \equiv (2+3)^n + 1 \equiv 1 + 1 = 2 \pmod 4$$

but $4m \equiv 0 \pmod 4$ and so $a$ is not an integer.

**(2)** $m \equiv 0 \pmod 4$. Since $a$ is an integer number, then

$$0 \equiv (m+3)^n + 1 \equiv 3^n + 1 \pmod 4$$

so $3^n \equiv -1 \pmod 4$ and $n = 2n_1 + 1$ is odd for some integer $n_1 \geq 0$.

**(2.1)** $m \equiv 0 \pmod 8$, then

$$(m+3)^n+1 \equiv 3^n+1 = 3^{2n_1+1}+1 = 3 \cdot 9^{n_1}+1 \equiv 3 \cdot 1^{n_1}+1 = 4 \pmod 8$$

but $4m \equiv 0 \pmod 8$ and so $a$ is not an integer.

**(2.2)** $m \equiv 4 \pmod 8$, then

$$(m+3)^n + 1 \equiv (4+3)^n + 1 \equiv (-1)^{2n_1+1} + 1 \equiv 0 \pmod 8$$

and $4m \equiv 16 \equiv 0 \pmod 8$.

**(2.2.1)** $m \equiv 0 \pmod 3$, then $m = 24m_1 + 12 = 12(2m_1+1)$, for some integer $m_1 \geq 0$. Therefore $(m+3)^n + 1 \equiv 1 \pmod 3$, but $4m \equiv 0 \pmod 3$ and so $a$ is not an integer.

**(2.2.2)** $m \equiv 1 \pmod 3$, then $m = 24m_1 + 4 = 4(6m_1+1)$, for some integer $m_1 \geq 0$.

$$(m+3)^n+1 \equiv 4^n+1 = 4^{2n_1+1}+1 \equiv 1^{2n_1+1}+1 \equiv 2 \equiv -1 \pmod 3$$

and $4m \equiv 4 \equiv 1 \pmod 3$.

Therefore in this case, if $a$ is an integer, $a \equiv -1 \pmod 3$.

**(2.2.3)** $m \equiv 2 \equiv -1 \pmod 3$, then $m = 24m_1 + 20 = 4(6m_1+5)$, for some integer $m_1 \geq 0$. There exists an odd prime $p$ such that $p \equiv -1 \pmod 3$ and $p \mid m$. Since $a$ is an integer,

$$0 \equiv (m+3)^n + 1 \equiv 3^{2n_1+1} + 1 \pmod m$$

and $3^{2n_1+1} \equiv -1 \pmod p$.

Let $r$ be a primitive root modulo $p$; let $s$ be a positive integer, such that $3 \equiv r^s \pmod{p}$. Thus $r^{(2n_1+1)s} \equiv 3^{(2n_1+1)} \equiv -1 \pmod{p}$. Note that

$$\left(\frac{p}{3}\right) = \left(\frac{-1}{3}\right) = -1$$

We consider the following cases:

**(2.2.3.1)** $p \equiv 1 \pmod{4}$. From the quadratic reciprocity law, $(\frac{-1}{p}) = 1$, so $r^{2t} \equiv -1 \equiv r^{(2n_1+1)s} \pmod{p}$ for some positive integer $t$. Therefore $s$ is even and $(\frac{3}{p}) = 1$. Again, from the quadratic reciprocity law,

$$-1 = \left(\frac{3}{p}\right)\left(\frac{p}{3}\right) = (-1)^{\frac{(3-1)(p-1)}{4}} = 1$$

a contradiction.

**(2.2.3.2)** $p \equiv 3 \pmod{4}$. From the quadratic reciprocity law, $(\frac{-1}{p}) = -1$, so $r^{2t+1} \equiv -1 \equiv r^{(2n_1+1)s} \pmod{p}$ for some positive integer $t$. Therefore $s$ is odd and $(\frac{3}{p}) = -1$. Again, from the quadratic reciprocity law,

$$1 = \left(\frac{3}{p}\right)\left(\frac{p}{3}\right) = (-1)^{\frac{(3-1)(p-1)}{4}} = -1$$

a contradiction.

Thus in this case $a$ is not an integer.

Now

$$2^{n+1} \equiv 2^{2(n_1+1)} \equiv (-1)^{2(n_1+1)} \equiv 1 \pmod{3}$$

From the above, we see that if $a$ is an integer, $a+2^{n+1} \equiv -1+1 \equiv 0 \pmod{3}$.

Examples:

| $m$ | $n$ | $a + 2^{n+1}$ |
|---|---|---|
| 28 | 3 | $282 = 3 \cdot 94$ |
| 28 | 9 | $236068056030 = 3 \cdot 78689352010$ |
| 76 | 9 | $394248670996479 = 3 \cdot 131416223665493$ |

***A–121.*** *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.*
Solve the equation

$$\begin{vmatrix} \frac{x+7}{5(x+2)} & \frac{x+11}{7(x+4)} & \frac{x+15}{9(x+6)} & \cdots & \frac{x+4n+3}{(2n+3)(x+2n)} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \cdots & \frac{1}{2n+3} \\ \frac{1}{7} & \frac{1}{9} & \frac{1}{11} & \cdots & \frac{1}{2n+5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n+1} & \frac{1}{2n+3} & \frac{1}{2n+5} & \cdots & \frac{1}{4n-1} \end{vmatrix} = 0.$$

**Solution 1 by the proposer.** Subtracting the second row from the first one, we obtain the equivalent equation

$$\begin{vmatrix} \frac{1}{x+2} & \frac{1}{x+4} & \frac{1}{x+6} & \cdots & \frac{1}{x+2n} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \cdots & \frac{1}{2n+3} \\ \frac{1}{7} & \frac{1}{9} & \frac{1}{11} & \cdots & \frac{1}{2n+5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n+1} & \frac{1}{2n+3} & \frac{1}{2n+5} & \cdots & \frac{1}{4n-1} \end{vmatrix} = 0.$$

By inspection, we see that $x \in \{3, 5, 7, \ldots, 2n-1\}$ are solutions of the equation. Next, we prove that there are no more. Indeed, the last determinant is Cauchy's type where $(x_1, x_2, \ldots, x_n) = (x, 3, 5, \ldots, 2n-1)$ and $(y_1, y_2, \ldots, y_n) = (2, 4, 6, \ldots, 2n)$, respectively. Next we compute the general Cauchys's determinant for ease of reference.

*Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be complex numbers such that $x_i + y_j \neq 0$ for $1 \leq i, j \leq n$. Compute*

$$C(n)=\begin{vmatrix}\frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n}\\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \cdots & \frac{1}{x_2+y_n}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{x_n+y_1} & \frac{1}{x_n+y_2} & \cdots & \frac{1}{x_n+y_n}\end{vmatrix}$$

For $n=2$, we have

$$C(2)=\begin{vmatrix}\frac{1}{x_1+y_1} & \frac{1}{x_1+y_2}\\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2}\end{vmatrix}=\frac{1}{x_1+y_1}\cdot\frac{1}{x_2+y_2}-\frac{1}{x_2+y_1}\cdot\frac{1}{x_1+y_2}$$

$$=\frac{(x_2-x_1)(y_2-y_1)}{(x_1+y_1)(x_1+y_2)(x_2+y_1)(x_2+y_2)}=\frac{V(x_1,x_2)V(y_1,y_2)}{\prod\limits_{1\le i,j\le 2}(x_i+y_j)},$$

where $V(a,b)$ is the Vandermonde's determinant of order $2$.

Next, we compute

$$C(3)=\begin{vmatrix}\frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \frac{1}{x_1+y_3}\\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \frac{1}{x_2+y_3}\\ \frac{1}{x_3+y_1} & \frac{1}{x_3+y_2} & \frac{1}{x_3+y_3}\end{vmatrix}=\frac{1}{A}\begin{vmatrix}\frac{x_3+y_1}{x_1+y_1} & \frac{x_3+y_2}{x_1+y_2} & \frac{x_3+y_3}{x_1+y_3}\\ \frac{x_3+y_1}{x_2+y_1} & \frac{x_3+y_2}{x_2+y_2} & \frac{x_3+y_3}{x_2+y_3}\\ 1 & 1 & 1\end{vmatrix},$$

where $A=\prod\limits_{j=1}^{3}(x_3+y_j)$. Since $\dfrac{x_3+y_j}{x_i+y_j}=1+\dfrac{x_3-x_i}{x_i+y_j}$, then

$$C(3)=\frac{1}{A}\begin{vmatrix}1+\frac{x_3-x_1}{x_1+y_1} & 1+\frac{x_3-x_1}{x_1+y_2} & 1+\frac{x_3-x_1}{x_1+y_3}\\ 1+\frac{x_3-x_2}{x_2+y_1} & 1+\frac{x_3-x_2}{x_2+y_2} & 1+\frac{x_3-x_2}{x_2+y_3}\\ 1 & 1 & 1\end{vmatrix}$$

$$= \frac{1}{A}\begin{vmatrix} \dfrac{x_3-x_1}{x_1+y_1} & \dfrac{x_3-x_1}{x_1+y_2} & \dfrac{x_3-x_1}{x_1+y_3} \\ \dfrac{x_3-x_2}{x_2+y_1} & \dfrac{x_3-x_2}{x_2+y_2} & \dfrac{x_3-x_2}{x_2+y_3} \\ 1 & 1 & 1 \end{vmatrix}$$

after subtracting the third row from row 1 and row 2, respectively. We also have,

$$C(3) = \frac{(x_3-x_1)(x_3-x_2)}{A}\begin{vmatrix} \dfrac{1}{x_1+y_1} & \dfrac{1}{x_1+y_2} & \dfrac{1}{x_1+y_3} \\ \dfrac{1}{x_2+y_1} & \dfrac{1}{x_2+y_2} & \dfrac{1}{x_2+y_3} \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{(x_3-x_1)(x_3-x_2)}{A}\begin{vmatrix} \dfrac{y_3-y_1}{(x_1+y_1)(x_1+y_3)} & \dfrac{y_3-y_2}{(x_1+y_2)(x_1+y_3)} & \dfrac{1}{x_1+y_3} \\ \dfrac{y_3-y_1}{(x_2+y_1)(x_2+y_3)} & \dfrac{y_3-y_2}{(x_2+y_2)(x_2+y_3)} & \dfrac{1}{x_2+y_3} \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \frac{(x_3-x_1)(x_3-x_2)(y_3-y_1)(y_3-y_2)}{A(x_1+y_3)(x_2+y_3)}\begin{vmatrix} \dfrac{1}{x_1+y_1} & \dfrac{1}{x_1+y_2} \\ \dfrac{1}{x_2+y_1} & \dfrac{1}{x_2+y_2} \end{vmatrix}$$

$$= \frac{\prod\limits_{1\le i<3}(x_3-x_i)(y_3-y_i)}{(x_3+y_3)\prod\limits_{1\le i<3}(x_3+y_i)(y_3+x_i)}\,C(2)$$

$$= \frac{(x_3-x_1)(x_3-x_2)(x_2-x_1)(y_3-y_1)(y_3-y_2)(y_2-y_1)}{(x_3+y_1)(x_3+y_2)(x_3+y_3)(x_1+y_3)(x_2+y_3)(x_1+y_1)(x_1+y_2)(x_2+y_1)(x_2+y_2)}$$

$$= \frac{V(x_1,x_2,x_3)V(y_1,y_2,y_3)}{\prod\limits_{1\le i,j\le 3}(x_i+y_j)}.$$

In the general case carrying out the same procedure, we get

$$C(n) = \frac{\prod\limits_{1\le i<n}(x_n-x_i)(y_n-y_i)}{(x_n+y_n)\prod\limits_{1\le i<n}(x_n+y_i)(y_n+x_i)}\,C(n-1)$$

$$= \frac{V(x_1, x_2, \dots, x_n)V(y_1, y_2, \dots, y_n)}{\prod\limits_{1 \le i,j \le n} (x_i + y_j)}.$$

Therefore the only roots of the equation are the ones of $V(x_1, x_2, \dots, x_n)$ and $V(y_1, y_2, \dots, y_n)$. Clearly $V(y_1, y_2, \dots, y_n) = V(2, 4, \dots, 2n)$ doesn't have any, and $V(x_1, x_2, \dots, x_n) = V(x, 3, 5, \dots, 2n-1)$ has roots $x = 3, 5, \dots, 2n-1$, which is precisely what we wanted to show.

**Solution 2 by G. C. Greubel, Newport News, VA, USA.** A few examples of the determinant are given by the following for $n \ge 1$. For $n = 1$,

$$D_1(x) = \left| \tfrac{x+7}{5\,(x+2)} \right| = 0$$

gives the equation $x + 7 = 0$ or $x = -7$. For $n = 2$ then

$$D_2(x) = \begin{vmatrix} \frac{x+7}{5\,(x+2)} & \frac{x+11}{7\,(x+4)} \\ \frac{1}{5} & \frac{1}{7} \end{vmatrix} = 0$$

gives

$$D_2(x) = -\frac{2\,(x-3)}{5 \cdot 7\,(x+2)(x+4)} = 0$$

and leads to $x = 3$. The next few determinants are given as

$$D_3(x) = \frac{2^5}{5 \cdot 7^2 \cdot 9^2 \cdot 11} \, \frac{(x-3)(x-5)}{(x+2)(x+4)(x+6)}$$

$$D_4(x) = \frac{2^{12}}{7^2 \cdot 9^2 \cdot 11^3 \cdot 13^2 \cdot 15^2} \, \frac{(x-3)(x-5)(x-7)}{(x+2)(x+4)(x+6)(x+8)}$$

and has the general pattern

$$D_n(x) = \frac{2^{a_n}}{b_n} \, \frac{q_{n-1}(x)}{p_n(x)} = \frac{2^{a_n}}{b_n} \, \frac{\prod\limits_{j=1}^{n-1} (x - (2j+1))}{\prod\limits_{j=1}^{n} (x + 2j)},$$

where $a_n \in \{1, 5, 12, 23, 38, 56, 77, \cdots\}_{n \ge 2}$ and $b_n$ is the product of factors of $\{5, 7, 9, 11, 13, 15, 17, 19, \cdots\}$. The zeroes of $D_n(x) = 0$

are seen to be those of

$$\prod_{j=1}^{n-1}(x-2j-1)=0$$

and are in the set $x\in\{3,5,7,9,\cdots,2n-1\}_{n\geq 2}$. For example the roots of $D_3(x)=0$ are $x\in\{3,5\}$ and $D_4(x)=0$ are $x\in\{3,5,7\}$.

**Solution 3 by Michel Bataille, Rouen, France.** We observe that

$$\frac{x+4k+3}{(2k+3)(x+2k)}-\frac{1}{2k+3}=\frac{1}{x+2k}.$$

It follows that subtracting row 2 from row 1 does not change the value of the determinant but change the first row into

$$\frac{1}{x+2} \quad \frac{1}{x+4} \quad \cdots \quad \frac{1}{x+2n}.$$

As a result, two rows are identical (and the determinant vanishes) when we substitute one of the numbers $3,5,\ldots,2n-1$ for $x$. Therefore, the $n-1$ integers $3,5,\ldots,2n-1$ are solutions to the equation.

On the other hand, expanding along the (new) first row, we see that the determinant can be written as

$$\frac{\delta_1}{x+2}+\frac{\delta_2}{x+4}+\cdots+\frac{\delta_n}{x+2n}=\frac{P(x)}{(x+2)(x+4)\cdots(x+2n)}$$

for some real numbers $\delta_1,\delta_2,\ldots,\delta_n$ and where $P(x)$ is a polynomial whose degree is $\leq n-1$. Since $P(x)=0$ when $x=3,5,\ldots,2n-1$, $P(x)$ has at least $n-1$ roots and therefore its degree is exactly $n-1$ and its roots are $3,5,\ldots,2n-1$. We can now conclude that the required solutions of the equation are $3,5,\ldots,2n-1$.

**Solution 4 by Moti Levy, Rehovot, Israel.** The equation is rewritten as follows,

$$\begin{vmatrix} \frac{1}{5}+\frac{1}{x+2} & \frac{1}{7}+\frac{1}{x+4} & \frac{1}{9}+\frac{1}{x+6} & \cdots & \frac{1}{2n+3}+\frac{1}{x+2n} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \cdots & \frac{1}{2n+3} \\ \frac{1}{7} & \frac{1}{9} & \frac{1}{11} & \cdots & \frac{1}{2n+5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n+1} & \frac{1}{2n+3} & \frac{1}{2n+5} & \cdots & \frac{1}{4n-1} \end{vmatrix}=0.$$

Subtracting the second row from the first row results in

$$\begin{vmatrix} \frac{1}{x+2} & \frac{1}{x+4} & \frac{1}{x+6} & \cdots & \frac{1}{x+2n} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \cdots & \frac{1}{2n+3} \\ \frac{1}{7} & \frac{1}{9} & \frac{1}{11} & \cdots & \frac{1}{2n+5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n+1} & \frac{1}{2n+3} & \frac{1}{2n+5} & \cdots & \frac{1}{4n-1} \end{vmatrix} = 0.$$

It follows that the equation is of the form

$$\sum_{k=1}^{n} \frac{a_k}{x+2k} = 0,$$

where $(a_k)$ are rational numbers.

Let

$$P_n(x) := \sum_{k=1}^{n} \left( a_k \prod_{j=1, j\neq k}^{n} (x+2j) \right),$$

then the equation is

$$\frac{P_n(x)}{\prod_{j=1}^{n} (x+2j)} = 0.$$

The degree of the polynomial $P_n(x)$ is $n-1$, hence the equation has up to $n-1$ solutions.

Claim: The solutions are the $n-1$ integers: $3, 5, \ldots, 2n-1$.

Proof of Claim: Substitute $x = 2k-1$ , $2 \leq k \leq n$ in the first row of the determinant, then the result is equal to the $k$-th row of the matrix. The determinant of a matrix with two equal rows is zero.

***A–122.*** *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let $n \geq 1$ be an integer. Compute

$$\lim_{n\to\infty} \frac{\binom{n+1}{2}}{2^{n-1}} \sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k}.$$

**Solution 1 by Moti Levy, Rehovot, Israel.** Let

$$F_n(z) := \sum_{k=0}^{n} \binom{n}{k} z^k = (1+z)^n.$$

$$\int_0^1 F_n(t)dt = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \int_0^1 (1+t)^n dt = \frac{2^{n+1}-1}{n+1}. \quad (1)$$

$$\int_0^1 tF_n(t)dt = \sum_{k=0}^{n} \frac{1}{k+2}\binom{n}{k} = \int_0^1 t(1+t)^n dt$$
$$= \frac{2^{n+1}n+1}{(n+1)(n+2)} \quad (2)$$

$$\int_0^1 t^2F_n(t)dt = \sum_{k=0}^{n} \frac{1}{k+3}\binom{n}{k} = \int_0^1 t^2(1+t)^n dt$$
$$= \frac{2^{n+1}(2+n+n^2)}{(n+1)(n+2)(n+3)} \quad (3)$$

$$\frac{k+4}{(k+1)(k+2)(k+3)} = \frac{3}{2(k+1)} - \frac{2}{k+2} + \frac{1}{2(k+3)} \quad (4)$$

From (1), (2), (3) and (4), we obtain

$$\sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)}\binom{n}{k}$$
$$= \int_0^1 \left(\frac{1}{2}t^2 - 2t + \frac{3}{2}\right)F_n(t)dt = \frac{-19n + 8\cdot 2^n n + 40\cdot 2^n - 3n^2 - 30}{2(n+3)(n+2)(n+1)}.$$

Finally,

$$\lim_{n\to\infty} \frac{\binom{n+1}{2}}{2^{n-1}} \sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)}\binom{n}{k}$$
$$= \lim_{n\to\infty} \frac{-19n + 8\cdot 2^n n + 40\cdot 2^n - 3n^2 - 30}{2(n+3)(n+2)(n+1)} \frac{n(n+1)}{2^n}$$
$$= \lim_{n\to\infty} \frac{8\cdot 2^n n + 40\cdot 2^n}{2(n+3)(n+2)} \frac{n}{2^n} = \lim_{n\to\infty} \frac{8n^2}{2(n+3)(n+2)} = 4.$$

**Solution 2 by G. C. Greubel, Newport News, VA, USA.** Starting with

$$\sum_{k=0}^{n} \binom{n}{k} t^k = (1+t)^n$$

then by repeated integration

$$\sum_{k=0}^{n} \binom{n}{k} \frac{t^{k+3}}{(k+1)(k+2)(k+3)}$$

$$= \frac{k!}{(k+3)!} \left( (1+t)^{k+3} - 1 - (n+3)\,t - \binom{n+3}{2} t^2 \right).$$

Multiplying by $t$ and differentiating leads to the series

$$S_n = \sum_{k=0}^{n} \binom{n}{k} \frac{(k+4)\,t^{k+3}}{(k+1)(k+2)(k+3)} = \frac{n!}{(n+3)!}$$
$$\times \left( (n+4)\,(1+t)^{n+3} - (n+3)\,(1+t)^{n+2} - 1 - 2\,(n+3)\,t - 3\binom{n+3}{2} t^2 \right).$$

Letting $t = 1$ gives the desired series required. From this, then

$$\begin{aligned} L &= \lim_{n\to\infty} \frac{\binom{n+1}{2}}{2^{n-1}} \sum_{k=0}^{n} \binom{n}{k} \frac{k!\,(k+4)}{(k+3)!} \\ &= \lim_{n\to\infty} \frac{n}{2\,(n+2)(n+3)} \left( 8\,(n+5) - \frac{3\,n^2 + 19\,n + 32}{2^n} \right) \\ &= 4 \lim_{n\to\infty} \frac{1 + \frac{5}{n}}{\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)} - \lim_{n\to\infty} \frac{n}{2^{n+1}} \frac{3 + \frac{19}{n} + \frac{32}{n^2}}{\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)} \\ &= 4. \end{aligned}$$

**Solution 3 by Michel Bataille, Rouen, France.** Let $S_n$ denote the sum $\sum\limits_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k}$. Since

$$\frac{k+4}{(k+1)(k+2)(k+3)} = \frac{3}{2} \cdot \frac{1}{k+1} - \frac{2}{k+2} + \frac{1}{2} \cdot \frac{1}{k+3}$$

we have $S_n = \frac{3}{2} U_n - 2V_n + \frac{1}{2} W_n$ where

$$U_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+1}, \quad V_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+2}, \quad W_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{k+3}.$$

We calculate

$$U_n = \int_0^1 \left(\sum_{k=0}^{n} \binom{n}{k} x^k\right) dx = \int_0^1 (1+x)^n \, dx = \frac{2^{n+1}}{n+1} - \frac{1}{n+1}$$

$$\begin{aligned} V_n &= \int_0^1 x(1+x)^n \, dx = \left[\frac{x(1+x)^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{(1+x)^{n+1}}{n+1} dx \\ &= \frac{2^{n+1}}{n+1} - \frac{U_{n+1}}{n+1} \end{aligned}$$

$$\begin{aligned} W_n &= \int_0^1 x^2(1+x)^n \, dx = \left[\frac{x^2(1+x)^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{2x(1+x)^{n+1}}{n+1} dx \\ &= \frac{2^{n+1}}{n+1} - \frac{2V_{n+1}}{n+1}. \end{aligned}$$

We deduce that

$$S_n = -\frac{3}{2(n+1)} + \frac{2U_{n+1}}{n+1} - \frac{V_{n+1}}{n+1}.$$

Observing that for $\alpha > 0$,

$$\lim_{n\to\infty} \frac{n(n+1)}{2^n} \cdot \frac{1}{n^\alpha} = \lim_{n\to\infty} \frac{n(n+1)}{2^n} \cdot \frac{2^n}{n^{2+\alpha}} = 0,$$

we easily obtain

$$\begin{aligned} \lim_{n\to\infty} \frac{\binom{n+1}{2}}{2^{n-1}} S_n &= \lim_{n\to\infty} \frac{n(n+1)}{2^n} S_n \\ &= \lim_{n\to\infty} \frac{n(n+1)}{2^n} \left(\frac{2^{n+3}}{(n+1)(n+2)} - \frac{2^{n+2}}{(n+1)(n+2)}\right) \\ &= \lim_{n\to\infty} \frac{4n}{n+2} = 4. \end{aligned}$$

**Solution 4 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.** Note that

$$\sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k}$$

$$= \frac{1}{(n+3)(n+2)(n+1)} \sum_{k=0}^{n} (k+4)\binom{n+3}{k+3},$$

so the proposed limit may be written as

$$L = \lim_{n\to\infty} \frac{\binom{n+1}{2}}{2^{n-1}(n+3)(n+2)(n+1)} \sum_{k=0}^{n} (k+4)\binom{n+3}{k+3}.$$

Let $f(x) = \sum_{k=0}^{n} \binom{n+3}{k+3} x^{k+3} = (1+x)^{n+3} - \left(1 + (n+3)x + \binom{n+3}{2}x^2\right)$. Then, $\sum_{k=0}^{n} (k+4)\binom{n+3}{k+3} = g(1)$, wehre $g(x) = f'(x) + f(x)$. Therefore,

$$L = \lim_{n\to\infty} \frac{(n+3)2^{n+2} + 2^{n+3} - (n^2+8n+16)}{2^n(n+3)} = 4.$$

**Solution 5 by Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA.** The limit equals 4. To prove this, we start with the following identity:

$$\sum_{k=0}^{n} \binom{n}{k} \left( \frac{(x_3^{k+2} - x_2^{k+2})y^{n-k}}{(k+1)(k+2)} - \frac{(x_3 - x_2)x_1^{k+1}y^{n-k}}{k+1} \right)$$
$$= \frac{(x_3+y)^{n+2} - (x_2+y)^{n+2}}{(n+1)(n+2)} - \frac{(x_3-x_2)(x_1+y)^{n+1}}{n+1}, \quad (5)$$

the proof of which follows from the easily established identity

$$\int_{x_2}^{x_3} \sum_{k=0}^{n} \binom{n}{k} \frac{(x^{k+1} - x_1^{k+1})y^{n-k}}{k+1}\, dx = \int_{x_2}^{x_3} \frac{(x+y)^{n+1} - (x_1+y)^{n+1}}{n+1}\, dx.$$

Now we have

$$\sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)}\binom{n}{k} = \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)}\left(1 + \frac{1}{k+3}\right)\binom{n}{k}$$

$$= \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)}\binom{n}{k} + \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)(k+3)}\binom{n}{k}.$$

Letting $x_1 = x_2 = 0$ and $x_3 = y = 1$ in (1), we see that

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)} \binom{n}{k} = \frac{2^{n+2}-1}{(n+1)(n+2)} - \frac{1}{n+1}. \tag{6}$$

Letting $x_1 = x_2 = 0$ and $y = 1$ in (1) and then integrating the resulting function in $x_3$ from 0 to $x$ and setting $x = 1$, we obtain

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)(k+3)} \binom{n}{k}$$

$$= \frac{2^{n+3}-1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+1)(n+2)} - \frac{1}{2(n+1)}.$$

Therefore, substituting (3) and (4) in (2) yields

$$\begin{aligned} &\sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k} \\ &= \frac{2^{n+2}-1}{(n+1)(n+2)} - \frac{1}{n+1} + \frac{2^{n+3}-1}{(n+1)(n+2)(n+3)} \\ &- \frac{1}{(n+1)(n+2)} - \frac{1}{2(n+1)} \\ &= \frac{2^{n+3}(n+5) - 3n^2 - 19n - 32}{2(n+1)(n+2)(n+3)}. \end{aligned}$$

Finally, denoting by $S$ the expression whose limit we must find, we have

$$\begin{aligned} S &= \frac{\binom{n+1}{2}}{2^{n-1}} \cdot \frac{2^{n+3}(n+5) - 3n^2 - 19n - 32}{2(n+1)(n+2)(n+3)} \\ &= \frac{n(n+1)}{2^n} \cdot \left( \frac{2^{n+2}(n+5)}{(n+1)(n+2)(n+3)} - \frac{3n^2+19n+32}{2(n+1)(n+2)(n+3)} \right) \\ &= 4 \cdot \frac{n(n+5)}{(n+2)(n+3)} - \frac{3n^3+19n^2+32n}{2^{n+1}(n+2)(n+3)} \\ &\to 4 \cdot 1 - 0 = 4 \text{ as } n \to \infty. \end{aligned}$$

**Solution 6 by the proposers.** We have

$$\begin{aligned}\sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)}\binom{n}{k} &= \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)}\left(1+\frac{1}{k+3}\right)\binom{n}{k} \\ &= \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)}\binom{n}{k} \\ &+ \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)(k+3)}\binom{n}{k}\end{aligned}$$

To compute the preceding sums we consider the function $f:\mathbb{R}\to\mathbb{R}$ defined by $f(x)=(1+x)^n$. Since $f$ is continuous in $\mathbb{R}$, then $f$ is integrable in any interval $[a,b]$ with $a\le b$. On account of the Binomial theorem, we have that for $u>0$, holds

$$\int_0^u (1+x)^n\,dx = \int_0^u \left(\sum_{k=0}^{n}\binom{n}{k}x^k\right)dx$$

and

$$\frac{(1+u)^{n+1}-1}{n+1} = \sum_{k=0}^{n}\frac{u^{k+1}}{k+1}\binom{n}{k}$$

Integrating the above identity, we obtain for $v>0$,

$$\int_0^v \frac{(1+u)^{n+1}-1}{n+1}\,du = \int_0^v \left(\sum_{k=0}^{n}\frac{u^{k+1}}{k+1}\binom{n}{k}\right)du$$

or

$$\frac{(1+u)^{n+2}}{(n+1)(n+2)} - \frac{u}{n+1}\Bigg|_0^v = \sum_{k=0}^{n}\frac{u^{k+2}}{(k+1)(k+2)}\binom{n}{k}\Bigg|_0^v$$

Applying Barrow's rule, yields

$$\sum_{k=0}^{n}\frac{v^{k+2}}{(k+1)(k+2)}\binom{n}{k} = \frac{(1+v)^{n+2}-(n+2)v-1}{(n+1)(n+2)}$$

from which follows, after putting $v=1$,

$$\sum_{k=0}^{n}\frac{1}{(k+1)(k+2)}\binom{n}{k} = \frac{2^{n+2}-n-3}{(n+1)(n+2)}$$

Now integrating again in $[0, 1]$, we get

$$\int_0^1 \sum_{k=0}^{n} \frac{v^{k+2}}{(k+1)(k+2)} \binom{n}{k} dv = \int_0^1 \left( \frac{(1+v)^{n+2} - (n+2)v - 1}{(n+1)(n+2)} \right) dv$$

and

$$\sum_{k=0}^{n} \frac{v^{k+3}}{(k+1)(k+2)(k+3)} \binom{n}{k} \Bigg|_0^1$$

$$= \frac{1}{(n+1)(n+2)} \left( \frac{(1+v)^{n+3}}{n+3} - \frac{(n+2)v^2}{2} - v \right) \Bigg|_0^1$$

from which follows

$$\sum_{k=0}^{n} \frac{1}{(k+1)(k+2)(k+3)} \binom{n}{k} = \frac{2^{n+4} - n(n+7) - 14}{2(n+1)(n+2)(n+3)}$$

Therefore, on account of the preceding, we have

$$\sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k} = \frac{2^{n+3}(n+5) - 3n^2 - 19n - 32}{2(n+1)(n+2)(n+3)}.$$

Therefore,

$$\lim_{n\to\infty} \frac{\binom{n+1}{2}}{2^{n-1}} \sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)} \binom{n}{k}$$

$$= \lim_{n\to\infty} \frac{n^2+n}{2^n} \left( \frac{2^{n+3}(n+5) - 3n^2 - 19n - 32}{2(n+1)(n+2)(n+3)} \right) = 4,$$

and we are done.

**Also solved by** *Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA, USA, Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania and José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain.*

***A–123.*** *Proposed by Michel Bataille, Rouen, France.*

Let $m$ and $n$ be nonnegative integers. Prove that

$$\sum_{k=0}^{n}\binom{m+k}{k}\binom{m+n+1}{n-k}-\sum_{k=0}^{m}(-1)^k 2^{k+n+1}\binom{m+n+1}{m-k}=(-1)^{m+1}.$$

**Solution 1 by G. C. Greubel, Newport News, VA, USA.** Let the series in question be given by $S_n^m$,

$$S_n^m=\sum_{k=0}^{n}\binom{m+k}{k}\binom{m+n+1}{n-k}-\sum_{k=0}^{m}\binom{m+n+1}{m-k}(-1)^k\,2^{k+n+1}.$$

Consider the generating of this series in two parts. The first being by

$$\begin{aligned}
S_1 &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{n}\binom{m+k}{k}\binom{m+n+1}{n-k}\right)x^n\,t^m\\
&= \sum_{m=0}^{\infty}t^m\,\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{m+k}{k}\binom{m+n+k+1}{n}x^{n+k}\\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\binom{m+k}{k}t^m\,x^k\,(1-x)^{-m-k-2}\\
&= (1-x)^{-2}\,\sum_{m=0}^{\infty}\left(\frac{t}{1-x}\right)^m\sum_{k=0}^{\infty}\binom{m+k}{k}\left(\frac{x}{1-x}\right)^k\\
&= (1-x)^{-2}\,\sum_{m=0}^{\infty}\left(\frac{t}{1-x}\right)^m\left(1-\frac{x}{1-x}\right)^{-m-1}\\
&= \frac{1}{(1-x)(1-2x)}\,\sum_{m=0}^{\infty}\left(\frac{t}{1-2x}\right)^m\\
&= \frac{1}{(1-x)(1-t-2x)}.
\end{aligned}$$

The second is, in a similar pattern as the first,

$$\begin{aligned}
S_2 &= 2\,\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left(\sum_{k=0}^{m}\binom{m+n+1}{m-k}(-2)^k\right)(2x)^n\,t^m\\
&= \frac{1}{(1+t)(1-t-2x)}.
\end{aligned}$$

With these generating functions then

$$\begin{aligned} S = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} S_n^m\, x^n\, t^m &= \frac{1}{1-t-2x}\left(\frac{1}{1-x}-\frac{2}{1+t}\right) \\ &= -\frac{1}{(1-x)(1+t)} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{m+1}\, x^n\, t^m. \end{aligned}$$

From this it can be seen that $S_n^m = (-1)^{m+1}$ and can be stated as

$$\sum_{k=0}^{n}\binom{m+k}{k}\binom{m+n+1}{n-k} - \sum_{k=0}^{m}\binom{m+n+1}{m-k}(-1)^k\, 2^{k+n+1} = (-1)^{m+1}.$$

**Solution 2 by the proposer.** Let

$$S = \sum_{k=0}^{n}\binom{m+k}{k}\binom{m+n+1}{n-k}.$$

The well-known formula $\binom{q+1}{p+1} = \sum_{j=p}^{q}\binom{j}{p}$ (that holds for integers $p, q$ with $0 \le p \le q$) leads to

$$S = \sum_{k=0}^{n}\binom{m+k}{k}\binom{m+n+1}{m+k+1} = \sum_{k=0}^{n}\binom{m+k}{k}\sum_{j=k}^{n}\binom{m+j}{m+k}.$$

Remarking first that $\binom{m+k}{k}\binom{m+j}{m+k} = \binom{m+j}{j}\binom{j}{k}$ and then changing the order of summation give

$$S = \sum_{j=0}^{n}\sum_{k=0}^{j}\binom{m+j}{j}\binom{j}{k} = \sum_{j=0}^{n}\binom{m+j}{j}2^j = \frac{1}{m!}f^{(m)}(2)$$

where $f^{(m)}(x)$ denotes the $m$th derivative of the function $f(x) = \sum_{j=0}^{n} x^{m+j}$.

Since $f(x) = x^m(1-x)^{-1} - x^{m+n+1}(1-x)^{-1}$ and $[(1-x)^{-1}]^{(j)} = j!(1-x)^{-1-j}$, Leibniz's formula gives

$$f^{(m)}(x) = \sum_{j=0}^{m}\binom{m}{j}\big([x^m]^{(m-j)} - [x^{m+n+1}]^{(m-j)}\big)[(1-x)^{-1}]^{(j)}$$

$$= \sum_{j=0}^{m} \binom{m}{j}(m-j)!\left(\binom{m}{m-j}x^j - \binom{m+n+1}{m-j}x^{n+j+1}\right)j!(1-x)^{-1-j}$$

$$= m!\sum_{j=0}^{m} \binom{m}{m-j}x^j(1-x)^{-1-j} - m!\sum_{j=0}^{m} \binom{m+n+1}{m-j}x^{n+j+1}(1-x)^{-1-j}$$

and therefore

$$\frac{1}{m!}f^{(m)}(2) = \sum_{j=0}^{m} \binom{m}{j}2^j(-1)^{j+1} - \sum_{j=0}^{m} \binom{m+n+1}{m-j}2^{n+j+1}(-1)^{j+1}.$$

Since $\sum_{j=0}^{m} \binom{m}{j}(-2)^j = (-1)^m$, we finally obtain

$$S = -(-1)^m + \sum_{j=0}^{m} \binom{m+n+1}{m-j}2^{n+j+1}(-1)^j$$

and the required identity follows.

**Also solved by** *Moti Levy, Rehovot, Israel.*

***A–124.*** *Proposed by José Luis Díaz Barrero, Barcelona, Spain and Mihály Bencze, Braşov, Romnia.* For each integer $n \geq 0$ let $a_n = (n^2 + n + 1)\, 2^n$. Given the power series

$$f(x) = \sum_{n\geq 0} a_n x^n,$$

show that there is a relation of the form $a_n + pa_{n+1} + qa_{n+2} + ra_{n+3} = 0$, in which $p, q, r$ are constants independent of $n$. Find these constants and the sum of the power series.

**Solution 1 by Michel Bataille, Rouen, France.** The sequence $(a_n)$ satisfies a linear recursion with characteristic equation $(x-2)^3 = 0$. Therefore $a_{n+3} = 6a_{n+2} - 12a_{n+1} + 8a_n$, that is,

$$a_n - \frac{3}{2}\,a_{n+1} + \frac{3}{4}\,a_{n+2} - \frac{1}{8}\,a_{n+3} = 0.$$

Thus, $p = -\frac{3}{2},\ q = \frac{3}{4},\ r = -\frac{1}{8}$.

Note that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$, hence the radius of convergence of the power series is $\frac{1}{2}$. Since $\lim\limits_{n\to\infty} \frac{|a_n|}{2^n} = \infty$, the power series is divergent when $|x| = \frac{1}{2}$. From now on, we suppose that $|x| < \frac{1}{2}$. Since for all $n \geq 0$

$$x^3(a_n x^n) + px^2(a_{n+1}x^{n+1}) + qx(a_{n+2}x^{n+2}) + ra_{n+3}x^{n+3} = 0$$

we have

$$x^3 f(x) + px^2(f(x) - a_0) + qx(f(x) - a_0 - a_1 x) \\ + r(f(x) - a_0 - a_1 x - a_2 x^2) = 0,$$

that is,

$$f(x)(x^3 + px^2 + qx + r) = ra_0 + (ra_1 + qa_0)x + (ra_2 + qa_1 + pa_0)x^2.$$

With the values of $p, q, r$ found earlier, we readily obtain

$$f(x) = \frac{1 + 4x^2}{(1 - 2x)^3}.$$

**Solution 2 by G. C. Greubel, Newport News, VA, USA.** Since $a_n = 2^n\,(n^2 + n + 1)$ then

$$\begin{aligned} S &= \sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=0}^{\infty} (n^2 + n + 1)\,(2x)^n \\ &= \left(x^2\, D^2 + 2\, x\, D + 1\right) \frac{1}{1 - 2x} \\ &= \frac{1 + 4x^2}{(1 - 2x)^3} = \frac{1 + 4x^2}{1 - 6x + 12x^2 - 8x^3}, \end{aligned}$$

where $D = \dfrac{d}{dx}$. Based on the denominator of the generating function it is proposed that the recurrence equation is

$$a_{n+3} = 6\, a_{n+2} - 12\, a_{n+1} + 8\, a_n.$$

To demonstrate this consider finding the generating function by using the recurrence equation. This is seen by the following.

$$\begin{aligned} \sum_{n=3}^{\infty} a_n\, t^{n-3} &= 6 \sum_{n=2}^{\infty} a_n\, t^{n-2} - 12 \sum_{n=1}^{\infty} a_n\, t^{n-1} + 8 \sum_{n=0}^{\infty} a_n\, t^n \\ A - (a_0 + a_1\, t + a_2\, t^2) &= 6\, t\,(A - a_0 - a_1\, t) - 12\, t^2\,(A - a_0) + 8\, t^3\, A \\ (1 - 2t)^3\, A &= a_0 + (a_1 - 6a_0)\, t + (a_2 - 6a_1 + 12a_0)\, t^2, \end{aligned}$$

or

$$A(t)=\sum_{n=0}^{\infty} a_n\, t^n = \frac{a_0+(a_1-6a_0)\,t+(a_2-6a_1+12a_0)\,t^2}{1-2t)^3}.$$

Since $a_0 = 1$, $a_1 = 6$, and $a_2 = 28$, then

$$A(t)=\sum_{n=0}^{\infty} a_n\, t^n = \frac{1+4\,t^2}{(1-2t)^3}.$$

This is the same as the generating function obtained by the function given for $a_n$. This gives

$$\begin{aligned} a_n &= 2^n\,(n^2+n+1)\\ a_{n+3} &= 6\,a_{n+2}-12\,a_{n+1}+8\,a_n\\ \sum_{n=0}^{\infty} a_n\, t^n &= \frac{1+4\,t^2}{(1-2t)^3}. \end{aligned}$$

**Solution 3 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.** Constants $p, q, r$ may be found by using that $a_n = (n^2+n+1)2^n$ and $a_n + pa_{n+1} + qa_{n+2} + ra_{n+3} = 0$. That is, $(n^2+n+1)+2p(n^2+3n+3)+4q(n^2+5n+7)+8r(n^2+7n+13) = 0$, from where $p = -3/2$, $q = 3/4$ and $r = -1/8$.

In order to find the sum of the power series, we use that $\sum_{n\geq 0} 2^n x^n = \frac{1}{1-2x}$, and if $f(x) = \frac{1}{1-2x}$, then $\sum_{n\geq 0} n2^n x^n = xf'(x) = \frac{2x}{(1-2x)^2}$, and $\sum_{n\geq 0} n^2 2^n x^n = x(xf'(x))' = \frac{2x(1+2x)}{(1-2x)^3}$. Therefore

$$\sum_{n\geq 0} a_n x^n 0 \sum_{n\geq 0} (n^2+n+1)2^n x^n = \frac{1+4x^2}{(1-2x)^3}.$$

**Solution 4 by the proposers.** The relation we are searching is

$$(n^2+n+1)\,2^n + p((n+1)^2+n+2)\,2^{n+1}$$

$$+q((n+2)^2+n+3)\,2^{n+2} + r((n+3)^2+n+4)\,2^{n+3} = 0,$$

which is equivalent to

$$(2p+4q+8r+1)\,n^2+(6p+20q+56r+1)\,n+6p+28q+104r+1 = 0.$$

The preceding equation holds for all $n$ if and only if

$$\begin{aligned} 2p + 4q + 8r + 1 &= 0, \\ 6p + 20q + 56r + 1 &= 0, \\ 6p + 28q + 104r + 1 &= 0. \end{aligned}$$

These linear equations have the solution $p = -3/2$, $q = 3/4$, $r = -1/8$, so

$$a_n - \frac{3}{2}a_{n+1} + \frac{3}{4}a_{n+2} - \frac{1}{8}a_{n+3} = 0.$$

Proceeding formally, we have

$$\begin{aligned} x^3 f(x) &= a_0x^3 + a_1x^4 + \dots \\ px^2 f(x) &= pa_0x^2 + pa_1x^3 + pa_2x^4 + \dots \\ qxf(x) &= qa_0x + qa_1x^2 + qa_2x^3 + qa_3x^4 + \dots \\ rf(x) &= ra_0 + ra_1x + ra_2x^2 + ra_3x^3 + ra_4x^4 + \dots \end{aligned}$$

Adding up the preceding and taking into account the relation between the $a_n$'s, yields

$$f(x)[x^3 + px^2 + qx + r] = ra_0 + (qa_0 + ra_1)x + (pa_0 + qa_1 + ra_2)x^2.$$

In our particular case, we have

$$f(x)[x^3 - \frac{3}{2}x^2 + \frac{3}{4}x - \frac{1}{8}] = -\frac{1}{8}(1 + 4x^2)$$

from which

$$f(x) = \frac{1 + 4x^2}{1 - 36x + 12x^2 - 8x^3}$$

follows. Using the ratio test we conclude that the series converges for $|x| < 1/2$. Hence the formal manipulations above are valid for these values of $x$.

**Also solved by** *Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA.*

# *Arhimede Mathematical Journal*

Volume 11, No. 1 Spring 2024

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Edited in Barcelona by the Arhimede Association

ISSN 2462-537X