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Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

On Goddyn's iterated circumcenters conjecture

Roger Lidón Ardanuy

1 Introduction

In this paper we prove a conjecture proposed by Goddyn [1], which was known to the author through its page at the Open Problem Garden [2]. The conjecture, as shown in [2], states the following:

Theorem 1. *Let $P_1, P_2, P_3, \dots$ be a sequence of points in $\mathbb{R}^n$ such that for every $i \geq n+2$, the points $P_{i-1}, P_{i-2}, \dots P_{i-n-1}$ are distinct, lie on a unique sphere, and furthermore, P_i is the center of this sphere. If this sequence is periodic, then its period is exactly $2n+4$.*

Before going into the proof, we need to take care of the corner case $n = 1$. Indeed, if $n = 1$, then P_i is the midpoint of P_{i-1} and P_{i-2}. Therefore, we have that

$$|\overrightarrow{P_{i-1}P_i}| = 2|\overrightarrow{P_{i-2}P_{i-1}}|.$$

The above relation easily implies that the sequence of points must converge, and in particular, can not be periodic. So, from now on, let us assume that $n \geq 2$.

2 Proof preliminaries

To prove theorem 1, we are going to study a more general type of sequences and prove results stronger than Goddyn's problem itself, then use those infer the original problem. Before jumping in, let us define the following generalization of the perpendicular bisector hyperplane of two points.

Definition 1. *Given $A, B \in \mathbb{R}^n$, define $m(A, B, \lambda)$ as the hyperplane through point $\lambda A + (1-\lambda)B$ orthogonal to line AB. Therefore, $m(A, B, \lambda)$ is given by the equation*

$$\left(\overrightarrow{X} - \lambda\overrightarrow{A} - (1-\lambda)\overrightarrow{B}\right) \cdot \left(\overrightarrow{B} - \overrightarrow{A}\right) = 0.$$

The following generalized kind of circumcenter sequence will be the focus of this article.

Definition 2 (Pseudo-circumcenter sequence). *Let $\lambda_1, \lambda_2, \ldots, \lambda_{n+2}$ be nonzero real numbers, where we look at the indices modulo $n+2$ (hence $\lambda_i = \lambda_{i+n+2}$ for every $i \in \mathbb{Z}^+$). We say that a sequence $(P_i)_{i\geq 1}$ of points of $\mathbb{R}^n$ is a pseudo-circumcenter sequence with coefficients $\lambda_1, \lambda_2, \ldots, \lambda_{n+2}$ if for every positive integer k*

- *P_{k+n+2} lies on $m(P_i, P_{i+1}, \lambda_i)$ for $k+1 \leq i \leq k+n$,*
- *P_k is different to each of the points $P_{k+1}, P_{k+2}, \ldots, P_{k+n}$ for each $k \in \mathbb{Z}^+$, and*
- *the vectors*

  $$\overrightarrow{P_kP_{k+1}},\ \overrightarrow{P_{k+1}, P_{k+2}},\ \overrightarrow{P_{k+2}, P_{k+3}},\ \ldots,\ \overrightarrow{P_{k+n-2}P_{k+n-1}}$$

  are linearly independent.

The gist of this definition is that if $P_1, P_2, \ldots$ is defined by the circumcenters as in the original problem, then $P_n, P_{n+1}, P_{n+2}, \ldots$ is a pseudo-circumcenter sequence with $\lambda_i = \frac{1}{2}$ for all i. We will formally prove this fact towards the end of the article.

3 Study of pseudo-circumcenter sequences

From this point on, let us fix the dimension $n \geq 2$ and assume that $P_1, P_2, \ldots$ is a pseudo-circumcenter sequence with coefficients

$\lambda_1, \lambda_2, \ldots, \lambda_{n+2}$. Firstly, recall that the pseudo-circumcenter condition implies that

$$\left(\overrightarrow{P_{k+n+2}} - \lambda_i \overrightarrow{P_i} - (1-\lambda_i)\overrightarrow{P_{i+1}}\right) \cdot \left(\overrightarrow{P_{i+1}} - \overrightarrow{P_i}\right) = 0$$

for any $i, k \in \mathbb{Z}^+$ such that $k+1 \leq i \leq k+n$. We might rearrange this equality as

$$\overrightarrow{P_{k+n+2}} \cdot \left(\overrightarrow{P_{i+1}} - \overrightarrow{P_i}\right) = \left(\lambda_i \overrightarrow{P_i} + (1-\lambda_i)\overrightarrow{P_{i+1}}\right) \cdot \left(\overrightarrow{P_{i+1}} - \overrightarrow{P_i}\right).$$

Note that as the right hand side of this equation depends solely on i, we can define

$$a_i \coloneqq \left(\lambda_i \overrightarrow{P_i} + (1-\lambda_i)\overrightarrow{P_{i+1}}\right) \cdot \left(\overrightarrow{P_{i+1}} - \overrightarrow{P_i}\right).$$

Lemma 1. *Vectors $\overrightarrow{P_k P_{k+1}}$ and $\overrightarrow{P_{k+n+2} P_{k+n+3}}$ are parallel.*

Proof. We know that for all $i \in \{k+2, k+3, \ldots, k+n\}$,

$$\overrightarrow{P_{k+n+2}} \cdot \left(\overrightarrow{P_{i+1}} - \overrightarrow{P_i}\right) = a_i = \overrightarrow{P_{k+n+3}} \cdot \left(\overrightarrow{P_{i+1}} - \overrightarrow{P_i}\right)$$

and therefore

$$\left(\overrightarrow{P_{k+n+3}} - \overrightarrow{P_{n+k+2}}\right) \cdot \left(\overrightarrow{P_{i+1}} - \overrightarrow{P_i}\right) = 0.$$

This implies that $\overrightarrow{P_{n+k+2} P_{n+k+3}}$ is perpendicular to the vectors

$$\overrightarrow{P_{k+2} P_{k+3}}, \overrightarrow{P_{k+3}, P_{k+4}}, \ldots, \overrightarrow{P_{k+n} P_{k+n+1}}.$$

Let $\mathcal{S}$ denote the set of the above $n-1$ vectors. We can also prove that $\overrightarrow{P_k P_{k+1}}$ is orthogonal to every vector in $\mathcal{S}$. Notice that for every $i \in \{k+2, k+3, \ldots, k+n+1\}$, we have that

$$\overrightarrow{P_i} \cdot \left(\overrightarrow{P_{k+1}} - \overrightarrow{P_k}\right) = a_k.$$

By subtracting these equalities it follows that for every $i \in \{k+2, k+3, \ldots, k+n\}$,

$$\left(\overrightarrow{P_{i+1}} - \overrightarrow{P_i}\right) \cdot \left(\overrightarrow{P_{k+1}} - \overrightarrow{P_k}\right) = 0.$$

This directly implies that $\overrightarrow{P_kP_{k+1}}$ is orthogonal to every vector in $\mathcal{S}$.

By the definition of pseudo-circumcenter sequence, the vectors of $\mathcal{S}$ are linearly independent. Hence as $\mathcal{S}$ consists of $n-1$ linearly independent vectors, it follows that all vectors $\overrightarrow{u}$ such that $\overrightarrow{u} \perp \overrightarrow{s}$ for all $s \in \mathcal{S}$ must lie on a single direction. Thus it follows that $\overrightarrow{P_kP_{k+1}}$ and $\overrightarrow{P_{n+k+2}P_{n+k+3}}$ are parallel. $\square$

Lemma 2. *There is a unique nonzero real number r such that*

$$\overrightarrow{P_{k+n+2}P_{k+n+3}} = r \cdot \overrightarrow{P_kP_{k+1}}$$

for every positive integer k.

Proof. Fix a positive integer k. By lemma 1 there are real numbers a and b such that

$$\begin{aligned}\overrightarrow{P_{k+n+2}P_{k+n+3}} &= a \cdot \overrightarrow{P_kP_{k+1}},\\ \overrightarrow{P_{k+n+3}P_{k+n+4}} &= b \cdot \overrightarrow{P_{k+1}P_{k+2}}.\end{aligned}$$

Furthermore, a and b are nonzero since no two consecutive points of a pseudo-circumcenter sequence can be the same. We are going to prove that $a = b$, which implies the lemma. We know that, by the pseudo-circumcenter condition,

$$\overrightarrow{P_{k+2}} \cdot \left(\overrightarrow{P_{k+1}} - \overrightarrow{P_k}\right) = a_k = \left(\lambda_k \overrightarrow{P_k} + (1-\lambda_k)\overrightarrow{P_{k+1}}\right) \cdot \left(\overrightarrow{P_{k+1}} - \overrightarrow{P_k}\right).$$

This equality can be rearranged into

$$\begin{aligned}&\overrightarrow{P_{k+2}}\cdot\left(\overrightarrow{P_{k+1}} - \overrightarrow{P_k}\right) = \lambda_k\left(\overrightarrow{P_k} - \overrightarrow{P_{k+1}}\right)\cdot\left(\overrightarrow{P_{k+1}} - \overrightarrow{P_k}\right) + \overrightarrow{P_{k+1}}\cdot\left(\overrightarrow{P_{k+1}} - \overrightarrow{P_k}\right)\\ \iff &\left(\overrightarrow{P_{k+2}} - \overrightarrow{P_{k+1}}\right) \cdot \left(\overrightarrow{P_{k+1}} - \overrightarrow{P_k}\right) = -\lambda_k \cdot \left|\overrightarrow{P_{k+1}} - \overrightarrow{P_k}\right|^2\\ \iff &\lambda_k = \frac{\left(\overrightarrow{P_{k+1}} - \overrightarrow{P_{k+2}}\right) \cdot \left(\overrightarrow{P_{k+1}} - \overrightarrow{P_k}\right)}{\left|\overrightarrow{P_{k+1}} - \overrightarrow{P_k}\right|^2} = \frac{\overrightarrow{P_{k+2}P_{k+1}} \cdot \overrightarrow{P_kP_{k+1}}}{\left|\overrightarrow{P_kP_{k+1}}\right|^2}.\end{aligned}$$

Analogously it follows that

$$\lambda_{k+n+2} = \frac{\overrightarrow{P_{k+n+4}P_{k+n+3}} \cdot \overrightarrow{P_{k+n+2}P_{k+n+3}}}{\left|\overrightarrow{P_{k+n+2}P_{k+n+3}}\right|^2}.$$

Therefore, we have that

$$\frac{\overrightarrow{P_{k+2}P_{k+1}} \cdot \overrightarrow{P_kP_{k+1}}}{\left|\overrightarrow{P_kP_{k+1}}\right|^2} = \lambda_k = \lambda_{k+n+2}$$
$$= \frac{\overrightarrow{P_{k+n+4}P_{k+n+3}} \cdot \overrightarrow{P_{k+n+2}P_{k+n+3}}}{\left|\overrightarrow{P_{k+n+2}P_{k+n+3}}\right|^2}$$
$$= \frac{\left(b \cdot \overrightarrow{P_{k+2}P_{k+1}}\right) \cdot \left(a \cdot \overrightarrow{P_kP_{k+1}}\right)}{a^2 \cdot \left|\overrightarrow{P_kP_{k+1}}\right|^2}$$

where we used that $\lambda_k = \lambda_{k+n+2}$ by definition. Finally, as $\lambda_k \neq 0$ by the definition of pseudo-circumcenter sequence, we have that $\overrightarrow{P_{k+2}P_{k+1}} \cdot \overrightarrow{P_kP_{k+1}} \neq 0$ and $\overrightarrow{P_{k+n+4}P_{k+n+3}} \cdot \overrightarrow{P_{k+n+2}P_{k+n+3}} \neq 0$. Therefore the above equality reduces to

$$1 = \frac{b \cdot a}{a^2}$$

which instantly gives that $a = b$. $\square$

This is the only point in the article in which we use that the λ_k are nonzero. However, it is likely that extending the above proof to cases where $\lambda_k = 0$ is possible by setting first

$$\overrightarrow{P_{k+n+2}P_{k+n+3}} = a \cdot \overrightarrow{P_kP_{k+1}},$$
$$\overrightarrow{P_{k+n+4}P_{k+n+5}} = b \cdot \overrightarrow{P_{k+2}P_{k+3}}$$

and later

$$\overrightarrow{P_{k+n+2}P_{k+n+3}} = a \cdot \overrightarrow{P_kP_{k+1}},$$
$$\overrightarrow{P_{k+n+5}P_{k+n+6}} = b \cdot \overrightarrow{P_{k+3}P_{k+4}}$$

and proving that $a = b$ in both cases, with a method similar to the proof of lemma 2. However, this is not necessary at all for solving Goddyn's original problem, so we might as well overlook this edge case.

Using lemma 2, the following key result follows readily.

Theorem 2. *There exists a point $O \in \mathbb{R}^n$ and a nonzero real number r such that*

$$\overrightarrow{OP_{n+k+2}} = r \cdot \overrightarrow{OP_k}$$

for every positive integer k.

Proof. Let r be as in lemma 2 and let O be the unique point such that $\overrightarrow{OP_{n+3}} = r \cdot \overrightarrow{OP_1}$. We will prove by induction on k that $\overrightarrow{OP_{k+n+2}} = r \cdot \overrightarrow{OP_k}$ for every positive integer k, starting from $k = 1$. Indeed, if $\overrightarrow{OP_{k+n+2}} = r \cdot \overrightarrow{OP_k}$, then

$$\begin{aligned} \overrightarrow{OP_{k+n+3}} &= \overrightarrow{OP_{k+n+2}} + \overrightarrow{P_{k+n+2}P_{k+n+3}} \\ &= r \cdot \overrightarrow{OP_k} + r \cdot \overrightarrow{P_kP_{k+1}} = r \cdot OP_{k+1} \end{aligned}$$

implying the result by induction. □

This theorem roughly tells us that any pseudo-circumcenter sequence follows a distorted logarithmic spiral that either converges towards O if $|r| < 1$, goes away from O if $|r| > 1$ or stays fixed if $|r| = 1$. Figure 1 illustrates the first vertices of a pseudo-circumcenter sequence in which $n = 3$ and $r \approx -0.87$.

Observe that if $(P_i)_{i \geq 1}$ were periodic, then r would be either 1 or -1. This is clear, as if $|r|$ is not 1, then the sequence $P_{n+2}, P_{2(n+2)}, P_{3(n+2)}, \dots$ would contain infinitely many different points, making impossible that $(P_i)_{i \geq 1}$ is periodic.

From now on, suppose that $(P_i)_{i \geq 1}$ is periodic with minimal period d, hence $r \in \{-1, 1\}$. As $r^2 = 1$, it follows that

$$\overrightarrow{OP_{k+2n+4}} = r \cdot \overrightarrow{OP_{k+n+2}} = r^2 \cdot \overrightarrow{OP_k} = \overrightarrow{OP_k}$$

implying that P_{k+2n+4} and P_k are the same point, hence $(P_i)_{i \geq 1}$ is periodic with period $2n + 4$. Therefore, d must divide $2n + 4$. It is also of note that $d \geq n + 1$. This follows from recalling that P_k is different from $P_{k+1}, P_{k+2}, \dots, P_{k+n}$ by the definition of pseudo-circumcenter sequence. However, if $d = n + 1$, we would have that $n + 1 \mid 2n + 4$, implying that $n + 1 \mid 2$, which is impossible for $n \geq 2$. Therefore $d \geq n + 2$. From these observations, the following result follows instantly.

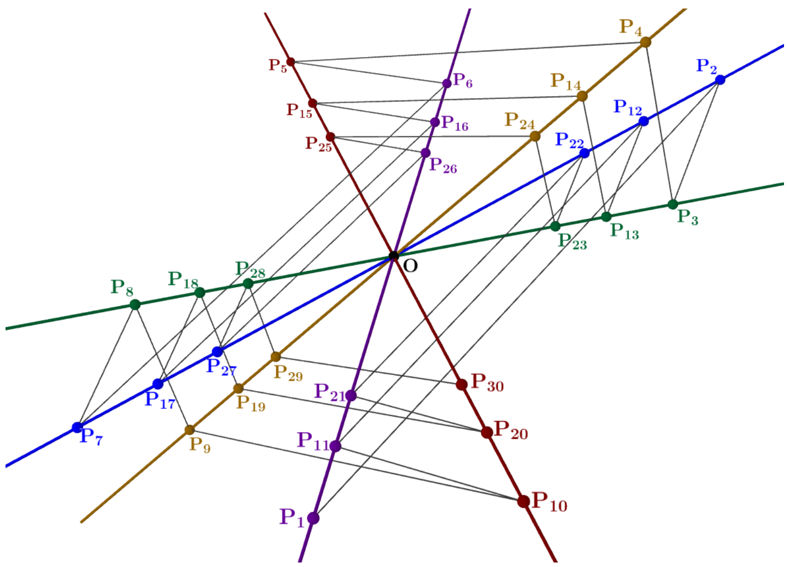

Figure 1: Example of a pseudo-circumcenter sequence for $n = 3$ (projected onto the paper).

Theorem 3. *If a pseudo-circumcenter sequence in n dimensions is periodic, then its period is either $2n + 4$ or $n + 2$.*

This is actually the best result we can prove for general pseudo-circumcenter sequences, as there do exist pseudo-circumcenter sequences with period $n + 2$. For instance, consider the following construction in $n = 2$ dimensions:

$$P_1 = (0, 0)$$
$$P_2 = (1, 0)$$
$$P_3 = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$
$$P_4 = \left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$$
$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \left(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{3}{2}\right).$$

In this case, P_4 is the center of equilateral triangle $P_1P_2P_3$. It is straightforward to check that the construction is valid and $P_5 = P_1$. More generally, it is not hard to check that if $n = 2$ and P_1, P_2, P_3, P_4 form an orthocentric system then there exist $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ giving a pseudo-circumcenter sequence of period 4.

4 Return to the original problem

Now we return to the kind of sequence proposed in the original problem. Call a sequence of points $P_1, P_2, P_3, \cdots \in \mathbb{R}^n$ a circumcenter sequence if it satisfies the conditions in the original problem, i.e. that P_i is the center of the unique n-sphere through the distinct points $P_{i-1}, P_{i-2}, \ldots, P_{i-n-1}$ for every positive integer $i \geq n+2$.

Lemma 3. *Let $P_1, P_2, P_3, \ldots$ be a circumcenter sequence. The sequence $(P_i)_{i \geq n+2}$ is a pseudo-circumcenter sequence with $\lambda_1 = \lambda_2 = \cdots = \lambda_{n+2} = \frac{1}{2}$.*

Proof. First observe that P_k is different from $P_{k+1}, P_{k+2}, \ldots, P_{k+n}$ by definition. On the other hand, notice that if $k \geq n+2$, then $\left|\overrightarrow{P_k P_i}\right| = \left|\overrightarrow{P_k P_{i+1}}\right|$ for every $i \in \{k-n-1, \ldots, k-2\}$. However, this equality is equivalent to

$$\begin{aligned}
&\left|\overrightarrow{P_k P_i}\right|^2 = \left|\overrightarrow{P_k P_{i+1}}\right|^2 \\
\iff &\overrightarrow{P_k P_i} \cdot \overrightarrow{P_k P_i} = \overrightarrow{P_k P_{i+1}} \cdot \overrightarrow{P_k P_{k+1}} \\
\iff &(\overrightarrow{P_i} - \overrightarrow{P_k}) \cdot (\overrightarrow{P_i} - \overrightarrow{P_k}) = (\overrightarrow{P_{i+1}} - \overrightarrow{P_k}) \cdot (\overrightarrow{P_{i+1}} - \overrightarrow{P_k}) \\
\iff &\overrightarrow{P_i} \cdot \overrightarrow{P_i} - 2\overrightarrow{P_i} \cdot \overrightarrow{P_k} + \overrightarrow{P_k} \cdot \overrightarrow{P_k} = \overrightarrow{P_{i+1}} \cdot \overrightarrow{P_{i+1}} - 2\overrightarrow{P_{i+1}} \cdot \overrightarrow{P_k} + \overrightarrow{P_k} \cdot \overrightarrow{P_k} \\
\iff &\overrightarrow{P_i} \cdot \overrightarrow{P_i} - \overrightarrow{P_{i+1}} \cdot \overrightarrow{P_{i+1}} - 2\overrightarrow{P_k} \cdot \left(\overrightarrow{P_i} - \overrightarrow{P_{i+1}}\right) = 0 \\
\iff &(P_i + P_{i+1} - 2P_k) \cdot \left(\overrightarrow{P_i} - \overrightarrow{P_{i+1}}\right) = 0.
\end{aligned}$$

The last equality is equivalent to P_k lying on $m(P_i, P_{i+1}, \frac{1}{2})$, which will hold for any positive integer i such that $k - n - 1 \leq i \leq k - 2$.

Finally, we need to check that vectors

$$\overrightarrow{P_k P_{k+1}}, \overrightarrow{P_{k+1}, P_{k+2}}, \ldots, \overrightarrow{P_{k+n-2} P_{k+n-1}}$$

are linearly independent for every positive integer k. This follows readily from the fact that there is a unique point equidistant from $P_k, P_{k+1}, \ldots, P_{k+n}$. Recall that for any $A \in \mathbb{R}^n$,

$$\left|\overrightarrow{AP_i}\right| = \left|\overrightarrow{AP_j}\right| \iff \overrightarrow{A} \cdot \overrightarrow{P_iP_j} = \frac{1}{2}\left(\overrightarrow{P_i} + \overrightarrow{P_j}\right) \cdot \overrightarrow{P_iP_j}.$$

Therefore, A will be the circumcenter of the $n+1$ points P_k, $P_{k+1}, \ldots, P_{k+n}$ if and only if

$$\overrightarrow{A} \cdot \overrightarrow{P_iP_{i+1}} = \frac{1}{2}\left(\overrightarrow{P_i} + \overrightarrow{P_{i+1}}\right) \cdot \overrightarrow{P_iP_{i+1}}$$

for every positive integer i such that $k \leq i \leq k+n-1$. This, however, is nothing else than a system of equations with n variables (the components of $\overrightarrow{A}$) and over n equations (one for each choice of i). As this system must have exactly one solution, it follows that the n coefficient vectors must be linearly independent. As $\overrightarrow{P_kP_{k+1}}, \overrightarrow{P_{k+1}P_{k+2}}, \ldots, \overrightarrow{P_{k+n-2}P_{k+n-1}}$ are all among the coefficient vectors, the result follows. $\square$

Therefore by theorem 3 it follows that any periodic circumcenter sequence in n dimensions must have period either $n+2$ or $2n+4$. It only remains to prove that no circumcenter sequence can have period $n+2$. Note that if $P_1, P_2, P_3, \ldots$ were to form a circumcenter sequence with period $n+2$, then any point in $\{P_1, P_2, P_3, \ldots, P_{n+2}\}$ would be the circumcenter of the unique sphere going through the other $n+1$ points. Therefore, it follows that distances P_iP_j are the same for any different $i, j \in \{1, 2, \ldots, n+2\}$.

Lemma 4. *There do not exist $n+2$ different points $P_1, \ldots, P_{n+2} \in \mathbb{R}^n$ such that distances P_iP_j with $1 \leq i < j \leq n+2$ are all the same.*

Proof. While there are multiple known proofs of this fact available online, we present one for the sake of completeness. By shifting and scaling, we can assume without loss of generality that P_{n+2} is the origin, and that $P_iP_j = 1$ for all $i \neq j$. Therefore, we have that for every $a \in \{1, 2, \ldots, n+1\}$,

$$1 = P_{n+2}P_a = \left|\overrightarrow{P_a}\right| \implies 1 = \overrightarrow{P_a} \cdot \overrightarrow{P_a}.$$

Furthermore, if $a, b \in \{1, 2, \ldots, n+1\}$ are different, then since $P_aP_b = 1$ we have that

$$\begin{aligned}1 &= \left|\overrightarrow{P_aP_b}\right|^2 = (\overrightarrow{P_a} - \overrightarrow{P_b}) \cdot (\overrightarrow{P_a} - \overrightarrow{P_b}) \\ &= \overrightarrow{P_a} \cdot \overrightarrow{P_a} + \overrightarrow{P_b} \cdot \overrightarrow{P_b} - 2\overrightarrow{P_a} \cdot \overrightarrow{P_b} \\ &= 2 - 2\overrightarrow{P_a} \cdot \overrightarrow{P_b}\end{aligned}$$

which reduces to $\overrightarrow{P_a} \cdot \overrightarrow{P_b} = \frac{1}{2}$. However, as $\overrightarrow{P_1}, \overrightarrow{P_2}, \ldots, \overrightarrow{P_{n+1}}$ are $n+1$ vectors in $\mathbb{R}^n$, they must be linearly dependent. Therefore, we can write some P_i as a combination of other points (which, without loss of generality we can assume is P_{n+1}):

$$\overrightarrow{P_{n+1}} = \sum_{i=1}^{n} \mu_i \overrightarrow{P_i}$$

for some $\mu_1, \mu_2, \ldots, \mu_n \in \mathbb{R}$ not all zero. We will then have that for any $a \in \{1, 2, \ldots, n\}$

$$\frac{1}{2} = \overrightarrow{P_a} \cdot \overrightarrow{P_{n+1}} = \overrightarrow{P_a} \cdot \sum_{i=1}^{n} \mu_i \overrightarrow{P_i} = \sum_{i=1}^{n} \left(\mu_i \overrightarrow{P_a} \cdot \overrightarrow{P_i}\right).$$

However, as $\overrightarrow{P_a} \cdot \overrightarrow{P_i} = \frac{1}{2}$ for all $i \neq a$ and $\overrightarrow{P_a} \cdot \overrightarrow{P_a} = 1$, the above equality reduces to

$$\frac{1}{2} = \frac{\mu_a}{2} + \sum_{i=1}^{n} \frac{\mu_i}{2}$$

which will hold for every $a \in \{1, 2, \ldots, n\}$. Therefore

$$\mu_1 = \mu_2 = \cdots = \mu_n = 1 - \sum_{i=1}^{n} \mu_i$$

quickly implying that

$$\mu_1 = \mu_2 = \cdots = \mu_n = \frac{1}{n+1}.$$

This, however, leads to a contradiction, as we can compute

$$\begin{aligned}
\overrightarrow{P_{n+1}} \cdot \overrightarrow{P_{n+1}} &= \left(\sum_{i=1}^{n} \frac{\overrightarrow{P_i}}{n+1}\right) \cdot \left(\sum_{i=1}^{n} \frac{\overrightarrow{P_i}}{n+1}\right) \\
&= \frac{1}{(n+1)^2} \left(\sum_{i=1}^{n} \overrightarrow{P_i}\right) \cdot \left(\sum_{i=1}^{n} \overrightarrow{P_i}\right) \\
&= \frac{1}{(n+1)^2} \left(\sum_{i=1}^{n} \overrightarrow{P_i} \cdot \overrightarrow{P_i} + 2 \sum_{1 \le i < j \le n} \overrightarrow{P_i} \cdot \overrightarrow{P_j}\right) \\
&= \frac{1}{(n+1)^2} \left(n + \binom{n}{2}\right) = \frac{n + \frac{1}{2}n(n-1)}{(n+1)^2} \\
&= \frac{n^2+n}{2(n+1)^2} = \frac{n}{2(n+1)} < \frac{1}{2},
\end{aligned}$$

contradicting the fact that $\overrightarrow{P_{n+1}} \cdot \overrightarrow{P_{n+1}} = 1$. □

With this lemma proven, we can discard the option of a circumcenter sequence having period $n+2$, therefore establishing that any periodic circumcenter sequence must have period $2n+4$, just as the problem asked.

References

[1] Goddyn, L. *Iterated circumcenters*. URL: `https://www.sfu.ca/~goddyn/Circles/`.

[2] *Open problem garden. A conjecture on iterated circumcenters*. URL: `http://www.openproblemgarden.org/op/a_conjecture_on_iterated_circumcentres`.

Roger Lidón Ardanuy
BarcelonaTech (CFIS),
Barcelona, Spain
roger.lidon@estudiantat.upc.edu

Constrain Inequalities

Mihály Bencze and José Luis Díaz-Barrero

1 Introduction

In [2] the following problem was posed: *Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that*

$$(ab + bc + ca)\left(\frac{a}{b^2 + b} + \frac{b}{c^2 + c} + \frac{c}{a^2 + a}\right) \geq \frac{3}{4}.$$

A solution to the preceding proposal and some related results appeared in [1]. Our aim in this paper is to generalize it and to give some of its applications.

2 Main results

Applying Cauchy's inequality, we get the following result.

Theorem 1. *Let x and $a_k, b_k, (1 \leq k \leq n)$ be positive real numbers. Then, it holds:*

$$\left(\sum_{k=1}^{n} \frac{a_k}{(x + b_k)^2}\right)\left(\sum_{k=1}^{n} a_k(x + b_k)\right)^2 \geq \left(\sum_{k=1}^{n} a_k\right)^3.$$

Proof. Setting

$$\vec{u} = \left(\frac{\sqrt{a_1}}{x + b_1}, \frac{\sqrt{a_2}}{x + b_2}, \ldots, \frac{\sqrt{a_n}}{x + b_n}\right)$$

and

$$\vec{v} = \left(\sqrt{a_1}, \sqrt{a_2}, \ldots, \sqrt{a_n}\right)$$

into the CBS inequality, we get

$$\left(\sum_{k=1}^{n} \frac{a_k}{x+b_k}\right)^2 \le \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{a_k}{(x+b_k)^2}\right).$$

On the other hand, setting

$$\vec{u} = \left(\sqrt{\frac{a_1}{x+b_1}}, \sqrt{\frac{a_2}{x+b_2}}, \ldots, \sqrt{\frac{a_n}{x+b_n}}\right)$$

and

$$\vec{v} = \left(\sqrt{a_1(x+b_1)}, \sqrt{a_2(x+b_2)}, \ldots, \sqrt{a_n(x+b_n)}\right)$$

into the CBS inequality again, we obtain

$$\left(\sum_{k=1}^{n} a_k\right)^2 \le \left(\sum_{k=1}^{n} \frac{a_k}{x+b_k}\right)\left(\sum_{k=1}^{n} a_k(x+b_k)\right)$$

from which

$$\left(\sum_{k=1}^{n} \frac{a_k}{x+b_k}\right)^2 \ge \left(\sum_{k=1}^{n} a_k\right)^4 \Big/ \left(\sum_{k=1}^{n} a_k(x+b_k)\right)^2$$

follows. Combining the preceding, yields

$$\begin{aligned}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{a_k}{(x+b_k)^2}\right) &\ge \left(\sum_{k=1}^{n} \frac{a_k}{x+b_k}\right)^2 \\ &\ge \left(\sum_{k=1}^{n} a_k\right)^4 \Big/ \left(\sum_{k=1}^{n} a_k(x+b_k)\right)^2,\end{aligned}$$

and

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{a_k}{(x+b_k)^2}\right) \ge \left(\sum_{k=1}^{n} a_k\right)^4 \Big/ \left(\sum_{k=1}^{n} a_k(x+b_k)\right)^2$$

holds. After dividing by $\sum\limits_{k=1}^{n} a_k$ and rearranging terms, the statement follows. □

An inequality that can be obtain immediately from the preceding result is given in

Corollary 1. *Let b_k $(1 \leq k \leq n)$ be positive real numbers. Then,*

$$\left(\frac{1}{n}\sum_{k=1}^{n}\frac{k}{(1+b_k)^2}\right)\left(\frac{1}{n}\sum_{k=1}^{n}k(1+b_k)\right)^2 \geq \left(\frac{n+1}{2}\right)^3.$$

Proof. Set $a_k = k$ $(1 \leq k \leq n)$ and $x = 1$ in Theorem 1. □

Putting $\sum_{k=1}^{n} a_k = 1$ and using the preceding, we get

Corollary 2. *Let a_k, b_k $(1 \leq k \leq n)$ be positive real numbers such that $a_1 + a_2 + \ldots + a_n = 1$. Then,*

$$\left(\sum_{k=1}^{n}\frac{a_k}{(x+b_k)^2}\right)\left(\sum_{k=1}^{n}a_k(x+b_k)\right)^2 \geq 1.$$

Integrating the above results, we obtain

Theorem 2. *Let $0 \leq y < z$ and $a_k, b_k, (1 \leq k \leq n)$ be positive real numbers. Then,*

$$\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}\frac{a_k}{(y+b_k)(z+b_k)}\right)$$

$$\geq \sum_{k=1}^{n}\frac{a_k^2}{(y+b_k)(z+b_k)} + \frac{1}{z-y}\log\prod_{1\leq i<j\leq n}\left(\frac{(y+b_j)(z+b_i)}{(y+b_i)(z+b_j)}\right)^{\frac{2a_ia_j}{b_j-b_i}}$$

$$\geq \frac{\left(\sum_{k=1}^{n}a_k\right)^4}{\left(y\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}a_kb_k\right)\left(z\sum_{k=1}^{n}a_k+\sum_{k=1}^{n}a_kb_k\right)}$$

Proof. From the preceding, we have

$$\int_y^z\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}\frac{a_k}{(x+b_k)^2}\right)dx \geq \int_y^z\left(\sum_{k=1}^{n}\frac{a_k}{x+b_k}\right)^2 dx$$

$$\geq \int_y^z\frac{\left(\sum_{k=1}^{n}a_k\right)^4}{\left(\sum_{k=1}^{n}a_k(x+b_k)\right)^2}\,dx,$$

and with a little straightforward algebra, the statement follows after dividing by $z-y$. $\square$

Corollary 3. *Let $0 \le y < z$ and a_k, b_k $(1 \le k \le n)$ be positive real numbers. Then, there exists $c \in (y,z)$ such that*

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} \frac{a_k}{(y+b_k)(z+b_k)}\right) \ge \left(\sum_{k=1}^{n} \frac{a_k}{c+b_k}\right)^2$$

$$\ge \frac{\left(\sum_{k=1}^{n} a_k\right)^4}{\left(y\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k b_k\right)\left(z\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k b_k\right)}$$

Proof. Applying Lagrange's Mean Value Theorem to the function

$$f(x) = \int_0^x \left(\sum_{k=1}^{n} \frac{a_k}{t+b_k}\right)^2 dt$$

yields

$$\int_y^z \left(\sum_{k=1}^{n} \frac{a_k}{t+b_k}\right)^2 dt = f(z) - f(y) = (z-y)\left(\sum_{k=1}^{n} \frac{a_k}{c+b_k}\right)^2.$$

Putting this in Theorem 2 the statement follows and this completes the proof. $\square$

Applying again Theorem 2 we get the following inequalities.

Corollary 4. *Let $a_k, b_k, (1 \le k \le n)$ be positive real numbers such that $a_1 + a_2 + \ldots + a_n = 1$. Then,*

$$\sum_{k=1}^{n} \frac{a_k}{b_k(1+b_k)} \ge \sum_{k=1}^{n} \frac{a_k^2}{b_k(1+b_k)} + \log \prod_{1\le i<j\le n} \left(\frac{b_j(1+b_i)}{b_i(1+b_j)}\right)^{\frac{2a_i a_j}{b_j - b_i}}$$

$$\ge \frac{1}{\left(\sum_{k=1}^{n} a_k b_k\right)\left(1 + \sum_{k=1}^{n} a_k b_k\right)}.$$

Corollary 5. *Let a_k $(1 \le k \le n)$ be positive real numbers such that $a_1 + a_2 + \ldots + a_n = 1$. Then*

$$\sum_{cyc} \frac{a_1}{a_2(1+a_2)} \ge \sum_{cyc} \frac{a_1^2}{a_2(1+a_2)} + \log \prod_{1 \le i < j \le n} \left(\frac{a_{j+1}(1+a_{i+1})}{a_{i+1}(1+a_{j+1})} \right)^{\frac{2a_i a_j}{a_{j+1}-a_{i+1}}}$$

$$\ge \frac{1}{\left(\sum_{cyc} a_1 a_2\right)\left(1 + \sum_{cyc} a_1 a_2\right)}.$$

Proof. Setting $b_k = a_{k+1}$ $(1 \le k \le n)$ and $a_{n+1} = a_1$ into the preceding corollary the statement follows. □

Notice that this result is a generalization and refinement of the inequality posed in [2]. Indeed, for $n = 3$, we have

Corollary 6. *Let a, b, c be positive numbers of sum one. Then,*

$$\frac{a}{b(1+b)} + \frac{b}{c(1+c)} + \frac{c}{a(1+a)} \ge \frac{a^2}{b(1+b)} + \frac{b^2}{c(1+c)} + \frac{c^2}{a(1+a)}$$

$$+ \log\left(\left(\frac{a(1+c)}{c(1+a)}\right)^{\frac{2bc}{a-c}} \left(\frac{b(1+a)}{a(1+b)}\right)^{\frac{2ca}{b-a}} \left(\frac{c(1+b)}{b(1+c)}\right)^{\frac{2ab}{c-b}} \right) \ge \frac{9}{4}.$$

Proof. Taking into account that for all positive numbers a, b, c with sum one is $ab + bc + ca \le \frac{1}{3}(a+b+c)^2 \le \frac{1}{3}$ and corollary 5, we get

$$\frac{a}{b(1+b)} + \frac{b}{c(1+c)} + \frac{c}{a(1+a)} \ge \frac{a^2}{b(1+b)} + \frac{b^2}{c(1+c)} + \frac{c^2}{a(1+a)}$$

$$+ \log\left(\left(\frac{a(1+c)}{c(1+a)}\right)^{\frac{2bc}{a-c}} \left(\frac{b(1+a)}{a(1+b)}\right)^{\frac{2ca}{b-a}} \left(\frac{c(1+b)}{b(1+c)}\right)^{\frac{2ab}{c-b}} \right)$$

$$\ge \frac{1}{(ab+bc+ca)(1+ab+bc+ca)} \ge \frac{9}{4}.$$ □

If we leave the $(ab+bc+ca)$ factor as is, we recover the inequality in [2].

Combining the inequality posed by Dospinescu in [2] and the inequality posed in [1] by Janous, namely, if x, y, z are positive reals such that $x+y+z=1$, then

$$(xy+yz+zx)\left(\frac{x}{1+y^2}+\frac{y}{1+z^2}+\frac{z}{1+x^2}\right) \leq \frac{3}{4},$$

the following application is obtained.

Problem 1. *Let a, b, c be positive real numbers. Prove that*

$$\sum_{cyc} \frac{a}{b(a+2b+c)} \geq \frac{3(a+b+c)}{4(ab+bc+ca)} \geq \sum_{cyc} \frac{a}{b^2+(a+b+c)^2}.$$

Solution. Putting $x = \frac{a}{a+b+c}$, $y = \frac{b}{a+b+c}$ and $x = \frac{c}{a+b+c}$ into

$$\left(\sum_{cyc} xy\right)\left(\sum_{cyc} \frac{x}{y(1+y)}\right) \geq \frac{3}{4} \geq \left(\sum_{cyc} xy\right)\left(\sum_{cyc} \frac{x}{1+y^2}\right)$$

the statement follows. $\square$

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Mihály Bencze
Str. Harmanului 6,
505600 Sacele, Jud. Braşov, Braşov
Romania
benczemihaly@gmail.com

José Luis Díaz-Barrero
Civil and Environmental Engineering
BarcelonaTech
Barcelona, Spain
jose.luis.diaz@upc.edu

Derivative Polynomials for Trigonometric and Hyperbolic Functions

Joe Santmyer

Abstract

This note was motivated by properties discussed in [1] and [3] for the coefficients of what are referred to as *derivative polynomials*. The note presents a different set of properties for the coefficients of these kind of polynomials.

1 Introduction

As the author mentions in [3] sometimes problems naturally occur in pairs. Consider the set of function pairs

$$P = \{(sec, tan), (csc, cot), (sech, tanh), (csch, coth)\}.$$

Let $f^{(n)}$ be the n^{th} derivative of f where $n = 0$ is f itself. For each $(f, g) \in P$ it is not difficult to see that $f^{(n)}(x) = f(x)Q_n(u)$ where $u = g(x)$ and

$$Q_n(u) = \sum_{k=0}^{n} S_{n,k}u^k$$

is a polynomial. The polynomials Q_n are associated with what are called *derivative* polynomials in [3].

2 Coefficient Properties

Consider the coefficients $S_{n,k}$. By the way Q_n is defined $S_{n,k} = 0$ if $k < 0$ or $k > n$. Otherwise, if $0 \le k \le n$ the following properties for $S_{n,k}$ will be established.

a. If n and k have different parity, then $S_{n,k} = 0$.
b. If $(f, g) = (sec, tan)$, then $S_{n,k} = kS_{n-1,k-1} + (k+1)S_{n-1,k+1}$.
c. If $(f, g) = (csc, cot)$, then $S_{n,k} = -[kS_{n-1,k-1}+(k+1)S_{n-1,k+1}]$.
d. If $(f, g) = (sech, tanh)$ or $(f, g) = (csch, coth)$, then $S_{n,k} = -[kS_{n-1,k-1} - (k+1)S_{n-1,k+1}]$.

For b and c it will be shown that $S_{n,0}$ is the sequence $1, 1, 5, 61$, $1385, 50521, \ldots$ of Euler numbers A000364 in OEIS. For d it will be shown that $S_{n,0}$ is the sequence $1, -1, 5, -61, 1385, -50521, \ldots$ of alternating Euler numbers A028296 in OEIS.

Technology is used to produce a 10×9 table of values for the numbers $S_{n,k}$ where $0 \le n \le 9$ and $0 \le k \le 8$ in part d.

3 Justifying the Properties

Part a is relatively easy to prove and is left to the reader. Consider part b. Since $f^2 - g^2 = 1$ and $(f', g') = (fg, f^2)$ we have

$$\begin{aligned}
f^{(n-1)} &= fQ_{n-1} \\
f^{(n)} &= fQ_n \\
f^{(n)} &= f'Q_{n-1} + fQ'_{n-1} \\
fQ_n &= fgQ_{n-1} + fQ'_{n-1} \\
fQ_n &= f[gQ_{n-1} + Q'_{n-1}] \\
Q_n &= gQ_{n-1} + Q'_{n-1}
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{n} S_{n,k}g^k &= g\sum_{k=0}^{n-1} S_{n-1,k}g^k + \sum_{k=1}^{n-1} kS_{n-1,k}g^{k-1}g' \\
&= g\sum_{k=0}^{n-1} S_{n-1,k}g^k + \sum_{k=1}^{n-1} kS_{n-1,k}g^{k-1}f^2 \\
&= \sum_{k=0}^{n-1} S_{n-1,k}g^{k+1} + \sum_{k=1}^{n-1} kS_{n-1,k}g^{k-1}(1+g^2) \\
&= \sum_{k=0}^{n-1} S_{n-1,k}g^{k+1} + \sum_{k=1}^{n-1} kS_{n-1,k}g^{k-1} + \sum_{k=1}^{n-1} kS_{n-1,k}g^{k+1}.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
\sum_{k=0}^{n-1} S_{n-1,k}g^{k+1} &= \sum_{k=1}^{n} S_{n-1,k-1}g^k \\
\sum_{k=1}^{n-1} kS_{n-1,k}g^{k-1} &= \sum_{k=0}^{n-2} (k+1)S_{n-1,k+1}g^k \\
\sum_{k=1}^{n-1} kS_{n-1,k}g^{k+1} &= \sum_{k=2}^{n} (k-1)S_{n-1,k-1}g^k.
\end{aligned}$$

Consequently

$$\sum_{k=0}^{n} S_{n,k}g^k = \sum_{k=1}^{n} S_{n-1,k-1}g^k + \sum_{k=0}^{n-2}(k+1)S_{n-1,k+1}g^k + \sum_{k=2}^{n}(k-1)S_{n-1,k-1}g^k.$$

For $2 \le k \le n-2$ we have

$$\begin{aligned}
S_{n,k}g^k &= S_{n-1,k-1}g^k + (k+1)S_{n-1,k+1}g^k + (k-1)S_{n-1,k-1}g^k \\
S_{n,k} &= S_{n-1,k-1} + (k+1)S_{n-1,k+1} + (k-1)S_{n-1,k-1} \\
&= kS_{n-1,k-1} + (k+1)S_{n-1,k+1}.
\end{aligned}$$

If $k = 0$ the left hand side (LHS) is $S_{n,0}$. The right hand side (RHS) is $S_{n-1,1} = (k+1)S_{n-1,k+1}$. Also, $kS_{n-1,k-1} = 0 \cdot 0 = 0$. Hence, $S_{n,k} = kS_{n-1,k-1} + (k+1)S_{n-1,k+1}$.

If $k = 1$ the LHS is $S_{n,k}g = S_{n,1}g$ and the RHS is $[S_{n-1,0} + 2S_{n-1,2}]g = [kS_{n-1,k-1}+(k+1)S_{n-1,k+1}]g$. Hence, $S_{n,k} = kS_{n-1,k-1}+(k+1)S_{n-1,k+1}$.

If $k = n-1$ the LHS is $S_{n,k}g^k = S_{n,n-1}g^{n-1}$ and the RHS is $[S_{n-1,n-2} + (n-2)S_{n-1,n-2}]g^{n-1} = (n-1)S_{n-1,n-2}g^{n-1} = kS_{n-1,k-1}g^{n-1}$. Since $S_{n-1,k+1} = S_{n-1,n} = 0$ the RHS is $[kS_{n-1,k-1} + (k+1)S_{n-1,k+1}]g^{n-1}$. Hence, $S_{n,k} = kS_{n-1,k-1} + (k+1)S_{n-1,k+1}$.

If $k = n$ the LHS is $S_{n,k}g^k = S_{n,n}g^n$ and the RHS is $[S_{n-1,n-1}+(n-1)S_{n-1,n-1}]g^n = nS_{n-1,n-1}g^n = kS_{n-1,k-1}g^n$. Since $S_{n-1,n+1} = S_{n-1,k+1} = 0$ the RHS is $[kS_{n-1,k-1} + (k+1)S_{n-1,k+1}]g^n$. Hence, $S_{n,k} = kS_{n-1,k-1} + (k+1)S_{n-1,k+1}$.

Summarizing, for $0 \le k \le n$ we have $S_{n,k} = kS_{n-1,k-1} + (k+1)S_{n-1,k+1}$. This establishes the recurrence formula in part b.

Since $(f,g) = (sec, tan)$ we have

$$\begin{aligned} f^{(n)}(0) &= f(0)Q_n(g(0)) \\ sec^{(n)}(0) &= sec(0)Q_n(tan(0)) \\ sec^{(n)}(0) &= S_{n,0}. \end{aligned}$$

The Taylor series is

$$\begin{aligned} sec(x) &= \sum_{n=0}^{\infty} \frac{sec^{(n)}(0)}{n!}x^n \\ &= \sum_{n=0}^{\infty} \frac{S_{n,0}}{n!}x^n. \end{aligned}$$

Hence, by 20.25 on p 111 in [4] we know that $S_{n,0}$ is the sequence of Euler numbers.

Consider part c. It is easy to establish $Q_n(-u) = (-1)^n Q_n(u)$ and is left to the reader. Now

$$csc^{(n)}(x) = csc(x)P_n(cot(x))$$

where $P_n(cot(x)) = \sum_{k=0}^{n} R_{n,k}cot^k(x)$. Since $tan\left(x + \frac{\pi}{2}\right) = -cot(x)$ and $sec\left(x + \frac{\pi}{2}\right) = -csc(x)$ from $sec^{(n)}(x) = sec(x)Q_n(tan(x))$

and $Q_n(-u) = (-1)^n Q_n(u)$ it follows that

$$\begin{aligned}
-csc^{(n)}(x) &= sec^{(n)}\Big(x + \frac{\pi}{2}\Big) \\
-csc^{(n)}(x) &= sec\Big(x + \frac{\pi}{2}\Big)Q_n\Big(tan\Big(x + \frac{\pi}{2}\Big)\Big) \\
-csc^{(n)}(x) &= -csc(x)Q_n(-cot(x)) \\
csc^{(n)}(x) &= csc(x)(-1)^n Q_n(\cot(x)) \\
csc^{(n)}(x) &= csc(x)\sum_{k=0}^{n}(-1)^n S_{n,k} cot^k(x).
\end{aligned}$$

Hence, $R_{n,k} = (-1)^n S_{n,k}$. Consequently

$$\begin{aligned}
R_{n,k} &= (-1)^n S_{n,k} \\
&= (-1)^n[kS_{n-1,k-1} + (k+1)S_{n-1,k+1}] \\
&= -[k(-1)^{n-1}S_{n-1,k-1} + (k+1)(-1)^{n-1}S_{n-1,k+1}] \\
&= -[kR_{n-1,k-1} + (k+1)R_{n-1,k+1}]
\end{aligned}$$

and the coefficients satisfy the desired recurrence formula. Since $R_{n,k} = (-1)^n S_{n,k}$ we have the same initial conditions $S_{n,0}$ as in part b.

Part d can be justified by an argument similar to the one used in part b. If $(f, g) = (sech, tanh)$ then

$$\begin{aligned}
f^{(n)}(0) &= f(0)Q_n(g(0)) \\
sech^{(n)}(0) &= sech(0)Q_n(tanh(0)) \\
sech^{(n)}(0) &= S_{n,0}.
\end{aligned}$$

The Taylor series is

$$\begin{aligned}
sech(x) &= \sum_{n=0}^{\infty} \frac{sech^{(n)}(0)}{n!}x^n \\
&= \sum_{n=0}^{\infty} \frac{S_{n,0}}{n!}x^n.
\end{aligned}$$

Hence, by 20.37 on p 112 in [4] we know that $S_{n,0}$ is the sequence of alternating Euler numbers.

To show that the recurrence formula also holds for $(csch, coth)$ use $tanh\big(x + \frac{\pi}{2}i\big) = coth(x)$ and $sech\big(x + \frac{\pi}{2}i\big) = -i\,csch(x)$. From $sech^{(n)}(x) = sech(x)Q_n(tanh(x))$ we get

$$\begin{aligned} -i\,csch^{(n)}(x) &= sech^{(n)}\Big(x + \frac{\pi}{2}i\Big) \\ &= sech\Big(x + \frac{\pi}{2}i\Big)Q_n\Big(tanh\Big(x + \frac{\pi}{2}i\Big)\Big) \\ &= -icsch(x)Q_n(coth(x)) \\ csch^{(n)}(x) &= csch(x)Q_n(\coth(x)). \end{aligned}$$

Hence, the same recurrence holds for $(csch, coth)$ with the same initial conditions for $S_{n,0}$.

A `python` program produced table 1 which contains values $S_{n,k}$ for part d.

	k								
n	0	1	2	3	4	5	6	7	8
0	1								
1	0	-1							
2	-1	0	2						
3	0	5	0	-6					
4	5	0	-28	0	24				
5	0	-61	0	180	0	-120			
6	-61	0	662	0	-1320	0	720		
7	0	1385	0	-7266	0	10920	0	-5040	
8	1385	0	-24568	0	83664	0	-100800	0	
9	0	-50521	0	408360	0	-1023120	0	1028160	0

Table 1: Values $S_{n,k}$ for part d

4 Final Remark

Here is a challenge left for the reader to resolve. A polynomial with real coefficients is said to be *hyperbolic* if all of its roots are real. *Mathematica* shows that the polynomials Q_n in property d mentioned above are hyperbolic for $1 \le n \le 9$. Prove or disprove: for $n \ge 1$ the polynomials Q_n are hyperbolic. A technique used in [2] to show that polynomials whose coefficents are Stirling numbers of the second kind are hyperbolic might help. The author was unable to prove or disprove this observation.

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[1] Chen, H. "Another extension of Lobachevsky's formula". *European Mathematical Society, Elemente der Mathematik* (2022). DOI: 10.4171/EM/494.

[2] Harper, L. H. "Stirling behavior is asymptotically normal". *The Annuals of Mathematical Statistics* 38.2 (1967), pp. 410–414.

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Joe Santmyer
Las Cruces, NM 88011,
santmyerjoe@yahoo.com

Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.
2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

April 30, 2024

Elementary Problems

E–119. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Find all the prime numbers p and integers n such that $n^4 + n^2 + p = 2452$.

E–120. *Proposed by Michel Bataille, Rouen, France.* Let triangle ABC (with no right angle) be inscribed in a circle with centre O and let A' be diametrically opposite to A. The perpendicular to AC through A intersects the line $A'B$ at B'. If H is the orthogonal projection of A onto the line OB', prove that B, H, O, C are concyclic.

E–121. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Find the smallest side of a triangle ABC knowing that the medians drawn by vertices A and B are perpendicular.

E–122. *Proposed by Toyesh Prakash Sharma, Agra College, Agra, India.* If a, b, x, y are positive numbers then show that

$$\log_{\mathrm{ab}}(a^x b^y)\ \log_{\mathrm{ab}}\left(a^{1/\ x} b^{1/y}\right) \geq 1.$$

E–123. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $\mathcal{F} = \{A_1, A_2, \ldots, A_n\}$ be a collection of subsets of the set $S = \{1, 2, \ldots, n\}$ satisfying the following conditions:

- Any two distinct sets from $\mathcal{F}$ have exactly one element in common,
- each element of S is contained in exactly k of the sets in $\mathcal{F}$.

Can n be equal to 2024?

E–124. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* If $x, y, z > 1$ and $xyz = 2$, then show that

$$\frac{(\log_2 x)^2 + \log_2 y}{\log_2 yz} + \frac{(\log_2 y)^2 + \log_2 z}{\log_2 zx} + \frac{(\log_2 z)^2 + \log_2 x}{\log_2 xy} \geq 2.$$

Easy–Medium Problems

EM–119. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Find all real solutions of the following the system of equations

$$\begin{aligned} z+\log\left(x+\sqrt{x^2+1}\right) &= y,\\ x+\log\left(y+\sqrt{y^2+1}\right) &= z,\\ y+\log\left(z+\sqrt{z^2+1}\right) &= x. \end{aligned}$$

EM–120. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let ABC be an equilateral triangle with P, an arbitrary point on side BC and X, the midpoint of segment AP. If $BX\cap AC=\{M\}$ and $CX\cap AB=\{N\}$ show that the distance from the centroid of triangle ABC to MN does not depend on the choice of point P.

EM–121. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Find all possible values of the positive integers $x>1, y, z$ so that

$$\frac{x+1}{x-1}+\frac{y-1}{y+1}=\frac{z^2+1}{z}.$$

EM–122. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.*

1. Prove that every tetrahedron can be cut by a plane so that a parallelogram results in the section.
2. If the intersection of a regular tetrahedron and a plane is a rhombus, prove that the rhombus must be a square.

EM–123. *Proposed by Alexandru Benescu, Romania.* Let ABC be a triangle, H its orthocenter and Γ_A, Γ_B, Γ_C the circumscribed circles of $\triangle BHC$, $\triangle AHC$, $\triangle AHB$ respectively. Let D, E and F be points such that DE is tangent to Γ_A and Γ_B, EF is tangent to Γ_B and Γ_C, and FD is tangent to Γ_C and Γ_A, such that all 3 circles Γ_A, Γ_B and C lie inside $\triangle DEF$. Prove that lines AD, BE and CF are concurrent.

EM–124. *Proposed by Goran Conar, Varaždin, Croatia.* Let $b > a > 1$ and $x_1, x_2, \ldots, x_n$ be positive real numbers such that $x_1 + x_2 + \ldots + x_n = 1$. Prove that

$$\frac{a^{x_1}}{a^{x_1} + b} + \frac{a^{x_2}}{a^{x_2} + b} + \ldots + \frac{a^{x_n}}{a^{x_n} + b} \geq \frac{n\sqrt[n]{a}}{b + \sqrt[n]{a}}.$$

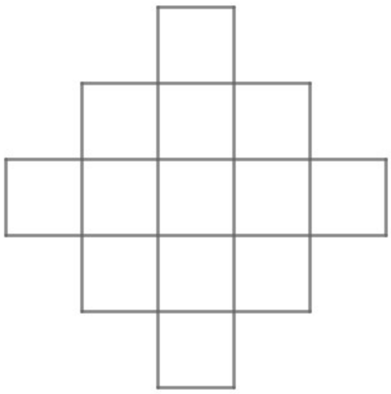
Medium–Hard Problems

MH–119. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a scalene triangle with incenter I and centroid G. Let M_a be the midpoint of BC, such that BI and IM_a are perpendicular. Prove that IG is perpendicular to side line AB.

MH–120. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let x, y, z be positive real numbers whose sum is 3. Find the minimum value of

$$\frac{(x^4 + y^4 + z^4)}{(x^2 + y^2 + z^2)(x^3 + y^3 + z^3)}.$$

MH–121. *Proposed by Alexandru Benescu, Romania.* Let n be a positive integer. We consider a diamond-shaped board with n rows and n columns as in the figure below. Let d_n be the maximum number of queens that can be placed on the board, so that there are no two of them to attack each other (on row, column or diagonals). Find the minimum value of m such that $m - d_m > 2$, being d_k an odd positive integer for all $k \leq m$.



MH–122. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* In an euclidian plane a set of 2024 points are given in such a way that the distance between every two of these points is irrational. Will it be possible for every three points of the set to form a non-degenerate triangle with rational area?

MH–123. *Proposed by Michel Bataille, Rouen, France.* Let $\mathcal{T}$ denote the interior of an equilateral triangle ABC with side s. If $P \in \mathcal{T}$, let $a = PA$, $b = PB$, $c = PC$, $\alpha = \inf\{(a+b+c)^2 : P \in \mathcal{T}\}$ and $\beta = \sup\{a^2 + b^2 + c^2 + ab + bc + ca : P \in \mathcal{T}\}$. Prove that $\alpha = 3s^2 = \beta$.

MH–124. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let n be a positive integer. Prove that

$$9 + 2\sum_{k=1}^{n} \left\{ \frac{L_{k+1}}{\log\left(1 + \dfrac{L_{k+1}}{L_k}\right)} \right\} < 4L_{n+1} + 5L_n,$$

where L_n is the n^{th} Lucas number defined by $L_1 = 1, L_2 = 3$ and $L_n = L_{n-1} + L_{n-2}$ for all $n \geq 3$.

Advanced Problems

A–119. *Proposed by Vasile Mircea Popa, Affiliate Professor, "Lucian Blaga" University of Sibiu, Romania.* Calculate the integral

$$\int_0^\infty \frac{\sqrt{x}\arctan(x)\ln^2(x)}{x^3+x\sqrt{x}+1}dx.$$

A–120. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let m, n be positive integers, such that

$$a=\frac{(m+3)^n+1}{4m}$$

is an integer. Prove that 3 divides $a+2^{n+1}$.

A–121. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Solve the equation

$$\begin{vmatrix} \frac{x+7}{5(x+2)} & \frac{x+11}{7(x+4)} & \frac{x+15}{9(x+6)} & \cdots & \frac{x+4n+3}{(2n+3)(x+2n)} \\ \frac{1}{5} & \frac{1}{7} & \frac{1}{9} & \cdots & \frac{1}{2n+3} \\ \frac{1}{7} & \frac{1}{9} & \frac{1}{11} & \cdots & \frac{1}{2n+5} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2n+1} & \frac{1}{2n+3} & \frac{1}{2n+5} & \cdots & \frac{1}{4n-1} \end{vmatrix} = 0.$$

A–122. *Proposed by Óscar Rivero Salgado, Santiago de Compostela, Spain and José Luis Díaz-Barrero, Barcelona, Spain.* Let $n \geq 1$ be an integer. Compute

$$\lim_{n\to\infty} \frac{\binom{n+1}{2}}{2^{n-1}} \sum_{k=0}^{n} \frac{k+4}{(k+1)(k+2)(k+3)}\binom{n}{k}.$$

A–123. *Proposed by Michel Bataille, Rouen, France.* Let m and n be nonnegative integers. Prove that

$$\sum_{k=0}^{n}\binom{m+k}{k}\binom{m+n+1}{n-k} - \sum_{k=0}^{m}(-1)^k 2^{k+n+1}\binom{m+n+1}{m-k} = (-1)^{m+1}.$$

A–124. *Proposed by José Luis Díaz Barrero, Barcelona, Spain and Mihály Bencze, Braşov, Romnia.* For each integer $n \geq 0$ let $a_n = (n^2 + n + 1)\,2^n$. Given the power series

$$f(x) = \sum_{n\geq 0} a_n x^n,$$

show that there is a relation of the form $a_n + pa_{n+1} + qa_{n+2} + ra_{n+3} = 0$, in which p, q, r are constants independent of n. Find these constants and the sum of the power series.

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

Inequalities involving differences of means

Vasile Mircea Popa

1 Introduction

Within the field of algebraic inequalities involving differences of means can be found in [1, 3, 2, 4]. These inequalities are interesting and their proofs are not easy or immediate. In this mathematical note we state and prove three inequalities including differences of means.

2 An inequality including arithmetic and geometric means

Below we state and prove our first result.

Theorem 1. *If $a, b, c \geq 1$, then*

$$\frac{a+b+c}{3} - \sqrt[3]{abc} \geq \frac{1}{3}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - \sqrt[3]{\frac{1}{abc}}.$$

Proof. Without loss of generality, we may assume that $a \geq b \geq c \geq 1$ and we write inequality claimed in the form $E(a, b, c) \geq 0$ where

$$E(a, b, c) = a + b + c - 3\sqrt[3]{abc} - \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + \frac{3}{\sqrt[3]{abc}}.$$

Next, we will prove that $E(a,b,c) \geq E(a,x,x) \geq 0$, where

$$E(a,x,x) = a + 2x - 3\sqrt[3]{ax^2} - \left(\frac{1}{a} + \frac{2}{x}\right) + \frac{3}{\sqrt[3]{ax^2}}$$

and $x = \sqrt{bc} \leq a$ in several steps.

a) First, we will prove that $E(a,b,c) \geq E(a,x,x)$. To do that we write the inequality as follows:

$$A - B \geq C - D,$$

where

$$A = a + b + c - (a + 2x) = b + c - 2x = \left(\sqrt{b} - \sqrt{c}\right)^2,$$

$$B = 3\sqrt[3]{abc} - 3\sqrt[3]{ax^2} = 0,$$

$$C = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - \left(\frac{1}{a} + \frac{2}{x}\right) = \frac{1}{b} + \frac{1}{c} - \frac{2}{x} = \frac{\left(\sqrt{b} - \sqrt{c}\right)^2}{x^2},$$

and

$$D = \frac{3}{\sqrt[3]{abc}} - \frac{3}{\sqrt[3]{ax^2}} = 0.$$

So, to prove the inequality $E(a,b,c) \geq E(a,x,x)$ it will be sufficient to prove that

$$\left(\sqrt{b} - \sqrt{c}\right)^2 \geq \frac{\left(\sqrt{b} - \sqrt{c}\right)^2}{x^2}.$$

Indeed, for $b = c$ we have case of equality. For $b \neq c$ it remains to show that $1 \geq \frac{1}{x^2}$, which is obviously true. Thus, the inequality $E(a,b,c) \geq E(a,x,x)$ is proven.

b) Now, we will prove that $E(a,x,x) \geq 0$. We have to show that

$$a + 2x - 3\sqrt[3]{ax^2} \geq \frac{1}{a} + \frac{2}{x} - \frac{3}{\sqrt[3]{ax^2}}.$$

With the substitutions: $z = \sqrt[3]{a}$, $y = \sqrt[3]{x}$, $z \geq y \geq 1$ the inequality can be written successively as

$$z^3 + 2y^3 - 3zy^2 \geq \frac{1}{z^3} + \frac{2}{y^3} - \frac{3}{zy^2},$$

$$z^3 + 2y^3 - 3zy^2 \geq \frac{y^3 + 2z^3 - 3z^2y}{z^3y^3},$$
$$(z-y)^2(2y+z) \geq \frac{(z-y)^2(y+2z)}{z^3y^3}.$$

It remains to show that:

$$2y + z \geq \frac{y+2z}{z^3y^3}, \quad \text{or} \quad z^3y^3(2y+z) \geq y + 2z.$$

Since, $z^3y^3 \geq zy$, it suffices to prove that $zy(2y+z) \geq y+2z$. But, $zy(2y+z) - y - 2z = 2y^2z + yz^2 - y - 2z = 2z(y^2-1) + y(z^2-1) \geq 0$.

Thus, the inequality $E(a,x,x) \geq 0$ is proved and the inequality in the statement $E(a,b,c) \geq 0$ is also proven. $\square$

3 An inequality including geometric and harmonic means

Our second result is presented in

Theorem 2. *If $a, b, c \geq 1$, then*

$$\sqrt[3]{abc} - \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \geq \sqrt[3]{\frac{1}{abc}} - \frac{3}{a+b+c}.$$

Proof. WLOG we may assume $a \geq b \geq c \geq 1$. We write the inequality in the form $E(a,b,c) \geq 0$, where

$$E(a,b,c) = \sqrt[3]{abc} - \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} - \frac{1}{\sqrt[3]{abc}} + \frac{3}{a+b+c}.$$

We will prove that $E(a,b,c) \geq E(a,x,x) \geq 0$, where

$$E(a,x,x) = \sqrt[3]{ax^2} - \frac{3}{\frac{1}{a} + \frac{2}{x}} - \frac{1}{\sqrt[3]{ax^2}} + \frac{3}{a+2x}$$

and $x = \sqrt{bc} \leq a$ in the following steps:

a) We begin by proving that $E(a,b,c) \geq E(a,x,x)$. Indeed, this inequality can be written as

$$A - B \geq C - D,$$

where

$$A = \sqrt[3]{abc} - \sqrt[3]{ax^2} = 0,$$

$$B = \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} - \frac{3}{\frac{1}{a}+\frac{2}{x}}; \quad B = -\frac{3\left(\sqrt{b}-\sqrt{c}\right)^2}{x^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{a}+\frac{2}{x}\right)}$$

$$C = \frac{1}{\sqrt[3]{abc}} - \frac{1}{\sqrt[3]{ax^2}} = 0$$

$$D = \frac{3}{a+b+c} - \frac{3}{a+2x}; \quad D = -\frac{3\left(\sqrt{b}-\sqrt{c}\right)^2}{(a+b+c)(a+2x)}$$

So, to prove the inequality $E(a,b,c) \geq E(a,x,x)$ we have to prove that:

$$\frac{3\left(\sqrt{b}-\sqrt{c}\right)^2}{x^2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{a}+\frac{2}{x}\right)} \geq \frac{3\left(\sqrt{b}-\sqrt{c}\right)^2}{(a+b+c)(a+2x)}$$

For $b = c$ this relationship is true (case of equality).
For $b \neq c$ it remain to show that:

$$(a+b+c)\left(a+2\sqrt{bc}\right) \geq \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{bc}{a}+2\sqrt{bc}\right)$$

This inequality is true. We compare the corresponding terms in the two members.
Thus, the inequality $E(a,b,c) \geq E(a,x,x)$ is proved.

b) Now, we will prove the inequality: $E(a,x,x) \geq 0$.
We have to prove that:

$$\sqrt[3]{ax^2} - \frac{3}{\frac{1}{a}+\frac{2}{x}} \geq \frac{1}{\sqrt[3]{ax^2}} - \frac{3}{a+2x}$$

With the substitutions: $z = \sqrt[3]{a}$, $y = \sqrt[3]{x}$, $z \geq y \geq 1$, the inequality is written successively:

$$zy^2 - \frac{3}{\frac{1}{z^3}+\frac{2}{y^3}} \geq \frac{1}{zy^2} - \frac{3}{z^3+2y^3};$$

$$\frac{zy^2(2z^3+y^3-3z^2y)}{2z^3+y^3} \geq \frac{z^3+2y^3-3zy^2}{zy^2(z^3+2y^3)};$$

$$\frac{zy^2(z-y)^2(2z+y)}{2z^3+y^3} \geq \frac{(z-y)^2(z+2y)}{zy^2(z^3+2y^3)}.$$

It remains to show that:

$$\frac{zy^2(2z+y)}{2z^3+y^3} \geq \frac{z+2y}{zy^2(z^3+2y^3)},$$

or, equivalently:

$$z^2y^4(2z+y)\left(z^3+2y^3\right) \geq (z+2y)\left(2z^3+y^3\right).$$

Because:

$$z^2y^4 \geq z;\ 2z+y \geq z+2y,$$

it suffices to prove that:

$$z\left(z^3+2y^3\right) \geq 2z^3+y^3.$$

We have:

$$z\left(z^3+2y^3\right)-2z^3-y^3 = z^4+(2z-1)y^3-2z^3 \geq$$
$$\geq z^4+(2z-1)-2z^3 = (z+1)(z-1)^3 \geq 0$$

Thus, the inequality $E(a,x,x) \geq 0$ is proved.

So, we proved the inequality in the statement: $E(a,b,c) \geq 0$. □

4 An Inequality including quadratic and arithmetic means

Our third results is given in the following

Theorem 3. *If $a,b,c \geq 1$, then*

$$\sqrt{\frac{a^2+b^2+c^2}{3}}-\frac{a+b+c}{3} \geq \sqrt{\frac{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}{3}}-\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}.$$

Proof. Assume, without loss of generality, that $c \geq b \geq a \geq 1$. We write inequality in the form: $E(a,b,c) \geq 0$, where

$$E(a,b,c) = \sqrt{3(a^2+b^2+c^2)} - (a+b+c)$$
$$-\sqrt{3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)} + \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

We will prove that $E(a,b,c) \geq E(a,x,x) \geq 0$, where

$$E(a,x,x) = \sqrt{3(a^2+2x^2)} - (a+2x) - \sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)} + \left(\frac{1}{a}+\frac{2}{x}\right)$$

and $x = \sqrt{bc} \geq a$ in the following steps:

a) We will prove the inequality: $E(a,b,c) \geq E(a,x,x)$. This inequality can be written as follows

$$A - B \geq C - D,$$

where

$$A = \sqrt{3(a^2+b^2+c^2)} - \sqrt{3(a^2+2x^2)}$$
$$= \frac{3(b-c)^2}{\sqrt{3(a^2+b^2+c^2)} + \sqrt{3(a^2+2x^2)}}$$

and

$$A \geq \frac{3(b-c)^2}{\sqrt{3(x^2+b^2+c^2)} + 3x}.$$

$$B = a+b+c-(a+2x) = b+c-2x = \left(\sqrt{b}-\sqrt{c}\right)^2.$$

$$C = \sqrt{3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)} - \sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)}$$
$$= \frac{3(b-c)^2}{x^4\left(\sqrt{3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)} + \sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)}\right)}$$

and

$$C \leq \frac{3(b-c)^2}{x^4\left(\sqrt{3\left(\frac{1}{x^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)} + \frac{3}{x}\right)} = \frac{3(b-c)^2}{x^2\left(\sqrt{3(x^2+b^2+c^2)} + 3x\right)}.$$

$$D=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\left(\frac{1}{a}+\frac{2}{x}\right)=\frac{1}{b}+\frac{1}{c}-\frac{2}{x}=\frac{\left(\sqrt{b}-\sqrt{c}\right)^2}{x^2}$$

To prove the inequality $E(a,b,c)\geq E(a,x,x)$ it is sufficient to show that

$$\frac{3(b-c)^2}{\sqrt{3(x^2+b^2+c^2)}+3x}-\left(\sqrt{b}-\sqrt{c}\right)^2$$

$$\geq\frac{3(b-c)^2}{x^2\left(\sqrt{3(x^2+b^2+c^2)}+3x\right)}-\frac{\left(\sqrt{b}-\sqrt{c}\right)^2}{x^2}$$

or

$$\left(\sqrt{b}-\sqrt{c}\right)^2\left(\frac{3\left(\sqrt{b}+\sqrt{c}\right)^2}{\sqrt{3(x^2+b^2+c^2)}+3x}-1\right)$$

$$\geq\left(\sqrt{b}-\sqrt{c}\right)^2\left(\frac{3\left(\sqrt{b}+\sqrt{c}\right)^2}{x\left(\sqrt{3(x^2+b^2+c^2)}+3x\right)}-\frac{1}{x^2}\right)$$

For $b=c$ this relationship holds (equality case). For $b\neq c$ it remains to show that

$$\frac{3\left(\sqrt{b}+\sqrt{c}\right)^2}{\sqrt{3(x^2+b^2+c^2)}+3x}\left(1-\frac{1}{x^2}\right)\geq 1-\frac{1}{x^2}$$

or

$$3\left(\sqrt{b}+\sqrt{c}\right)^2\geq\sqrt{3(x^2+b^2+c^2)}+3x,$$

$$3\left(\sqrt{b}+\sqrt{c}\right)^2\geq\sqrt{3(bc+b^2+c^2)}+3\sqrt{bc}.$$

Let us denote:

$$y=\frac{b}{c},\ \ 0<y\leq 1.$$

The previous inequality may be written in the equivalent form

$$3(\sqrt{y}+1)^2\geq\sqrt{3(y+y^2+1)}+3\sqrt{y}$$

But, we have for any y, $0 < y \le 1$ that

$$3\left(y + y^2 + 1\right) \le \left(\frac{6}{5}y + \frac{9}{5}\right)^2.$$

Indeed, we calculate

$$\left(\frac{6}{5}y + \frac{9}{5}\right)^2 - 3\left(y + y^2 + 1\right) = \frac{3}{25}(1-y)(13y+2) \ge 0.$$

So, it is enough to prove the inequality

$$3\left(\sqrt{y}+1\right)^2 \ge \frac{6}{5}y + \frac{9}{5} + 3\sqrt{y}$$

or

$$5\left(\sqrt{y}+1\right)^2 \ge 2y + 3 + 5\sqrt{y}$$

$$5\left(\sqrt{y}+1\right)^2 - 2y - 3 - 5\sqrt{y} = 3y + 5\sqrt{y} + 2 \ge 0,$$

then, the inequality $E(a,b,c) \ge E(a,x,x)$ is proved.

b) Now, we will prove the inequality $E(a,x,x) \ge 0$. We have to show that

$$\sqrt{3(a^2+2x^2)} - (a+2x) \ge \sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)} - \left(\frac{1}{a}+\frac{2}{x}\right).$$

The inequality is successively written in the following equivalent forms

$$\frac{3(a^2+2x^2)-(a+2x)^2}{\sqrt{3(a^2+2x^2)}+a+2x} \ge \frac{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)-\left(\frac{1}{a}+\frac{2}{x}\right)^2}{\sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)}+\left(\frac{1}{a}+\frac{2}{x}\right)}$$

$$\frac{(a-x)^2}{\sqrt{3(a^2+2x^2)}+a+2x} \ge \frac{(a-x)^2}{a^2x^2\left(\sqrt{3\left(\frac{1}{a^2}+\frac{2}{x^2}\right)}+\left(\frac{1}{a}+\frac{2}{x}\right)\right)}$$

$$\frac{(a-x)^2}{\sqrt{3(a^2+2x^2)}+a+2x} \ge \frac{(a-x)^2}{\sqrt{3(a^2x^4+2a^4x^2)}+(ax^2+2a^2x)}$$

The last inequality holds because

$a^2x^4+2a^4x^2 \geq a^2+2x^2,$ equivalent to: $a^2\left(x^4-1\right)+2x^2\left(a^4-1\right) \geq 0$

$ax^2+2a^2x \geq a+2x,$ equivalent to: $a\left(x^2-1\right)+2x\left(a^2-1\right) \geq 0.$

Thus, the inequality $E(a,x,x) \geq 0$ is proved.

So, the inequality in the statement $E(a,b,c) \geq 0$ is also proved. □

Finally, we want to point out that other inequalities involving differences of means have been published in ([1] [3] [2] [4]). The reader is invited to check and study these inequalities.

References

[1] Cirtoaje, V. *Mathematical Inequalities, Vol. 1-5*. LAP LAMBERT Academic Publishing, 2021.

[2] Cirtoaje, V. and Popa, V. M. "Problem U639". *Mathematical Reflections* 5 (2023), p. 3.

[3] Popa, V. M. *An Interesting Inequality*. Art of Problem Solving, 2021.

[4] Popa, V. M. "Problem SP.531". *Mathematical Reflections* 36-RMM (2025), p. 4.

Vasile Mircea Popa
Affiliate Professor, "Lucian Blaga" University of Sibiu
Sibiu, Romania
popavm@yahoo.com

On Second Degree Polynomial

Navid Safaei

1 Introduction

In several instances, we have to deal with some formulations akin to the second-degree polynomial. Indeed, the substantial part of the proof would take its root in that topic. In this article, we present some applications of second-degree polynomials in solving inequalities, number theory problems, and functional equations.

2 Some basic observations

By the *quadratic form* we mean any polynomial $ax^2+bxy+cy^2$ for some real numbers a, b, c. Of special interest is the **second-degree polynomial** that is sometimes called **quadratic trinomial**. That is, the polynomial of the form $P(x) = ax^2 + bx + c$.

There are several identities in relation to quadratic polynomials, one of them is the identity of completing the square, that is

$$4aP(x) = (2ax + b)^2 + 4ac - b^2.$$

3 Second-degree polynomial and inequalities

Several inequalities problems can be solved by applying known facts about second-degree polynomials. This section helps readers becoming be more aware of such nice applications. In so doing we will prepare the following proposition about the extremal values of a second-degree polynomial in a certain interval.

Proposition 1. *Here are some facts about the quadratic polynomial $ax^2 + bx + c$.*

- *If $a > 0$ then the function is strictly decreasing on the interval $(-\infty, \frac{-b}{2a})$ and is strictly increasing in the interval $(\frac{-b}{2a}, +\infty)$.* Indeed, in order to find the maxima and minima of such a function on an interval $[A, B]$ we should compare A, B with $\frac{-b}{2a}$. The maximum value would be $\max\{P(A), P(B)\}$, but the minimum value would depend on the relative location of A, B with respect to $\frac{-b}{2a}$. Indeed, if $A < B < \frac{-b}{2a}$ or $\frac{-b}{2a} < A < B$ then the minimum occurs at B or A. Finally, if $A < \frac{-b}{2a} < B$ then the minimum occurs at $\frac{-b}{2a}$.
- *If $a < 0$ exchange every maximum by a minimum in the above lines.*

We shall then provide the following note for our upcoming problems.

Note. *Sometimes in order to show that some inequality such as $P(x_1, \ldots, x_n) \geq 0$ is true, we can use what we have learned about quadratic polynomials. It suffices to ensure that the polynomial is in degree 2 with respect to at least one of $x_i, i = 1, \ldots, n$. Then, we can rewrite it as $A(x_2, \ldots, x_n)x_1^2 + B(x_2, \ldots, x_n)x_1 + C(x_2, \ldots, x_n) \geq 0$ and try to use the discriminant for proving it.*

Problem 1. *Let n be a positive integer and $a_1 \leq \cdots \leq a_{2n}$ be real numbers. Prove that*

$$a_1a_{n+1} + a_2a_{n+2} + \cdots + a_na_{2n} \leq \frac{1}{4n}(a_1 + \cdots + a_{2n})^2.$$

Solution. Let $P_i(x) = (x - a_i)(x - a_{i+n}), i = 1, 2, \ldots, n$ and

$$P(x) = P_1(x) + \cdots + P_n(x)$$
$$= nx^2 - (a_1 + \cdots + a_{2n})x + a_1 a_{n+1} + \cdots + a_n a_{2n}.$$

Since $a_n \leq x \leq a_{n+1}$ we have $P_i(x) \leq 0$ for each i. Thus, $P(x) \leq 0$. Hence, the discriminant must be positive. Thus,

$$(a_1 + \cdots + a_{2n})^2 - 4n(a_1 a_{n+1} + \cdots + a_n a_{2n}) \geq 0. \qquad \square$$

The next problem is related to the presented proposition.

Problem 2. *Let x, y, z be real numbers such that $x - y + z - 1 = 0$ and $xy + 2z^2 + 6z + 1 = 0$ find the minimum of $(x-1)^2 + (y+1)^2$*

Solution. Note that $y = x + z - 1$ and

$$x(x + z - 1) + 2z^2 - 6z + 1 = x^2 + (z-1)x + 2z^2 - 6z + 1.$$

The discriminant is $-7z^2 + 22z - 3 \geq 0$. Hence, $z \in [\frac{1}{7}.3]$. Thus,

$$(x-1)^2 + (y+1)^2 = (y-x)^2 + 2xy + 2(y-x) + 2$$
$$= (z-1)^2 + 2(-2z^2 + 6z - 1) + 2(z-1) + 2 = -3z^2 + 12z - 1.$$

So, we need to find the minimal value of $P(z) = -3z^2 + 12z - 1$ in the interval $[\frac{1}{7}, 3]$. According to the above proposition, we should compare $P(\frac{1}{7})$ and $P(3)$. The answer is $\frac{22}{49}$ occurs at $z = \frac{1}{7}$. $\square$

Next problem is a bit more tricky.

Problem 3. *Let a, b, c be non-negative real numbers such that $a + b + c = 3$. Find the maximum of $a + ab + bc + ca$ and $a + ab + bc$.*

Solution. We have

$$a(1 + b + c) + bc \leq a(1 + b + c) + \frac{(b+c)^2}{4}$$
$$= a(4 - a) + \frac{(3-a)^2}{4} = \frac{1}{4}(10a + 9 - 3a^2).$$

According to what presented in the proposition, $10a+9-3a^2 \leq \frac{52}{3}$ the maximum occurs at $a = \frac{5}{3}$. On the other hand, $a+b \leq a+b+c=3$ hence

$$\begin{aligned} a+ab+bc &= a+b(a+c) = a+b(3-b) \\ &\leq 3-b+b(3-b) = (b+1)(3-b) \\ &= 4-(b-1)^2 \leq 4. \end{aligned}$$

□

The next problem contains a nice application of this topic.

Problem 4. *Find the smallest k such that*

$$k(x^2+y^2+z^2-xy-yz-zx)+(xyz-x-y-z+2) \geq 0$$

for all $x,y,z \geq 0$.

Solution. Taking $x=y=2, z=0$ then $k \geq \frac{1}{2}$. We prove that $k=\frac{1}{2}$ works. There are two of x,y,z, say y,z such that $x(y-1)(z-1) \geq 0$. Yielding

$$xyz \geq xy+xz-x$$

Hence,

$$\begin{aligned} 2P &\geq (x^2+y^2+z^2-xy-yz-zx)+2(xy+xz-x-x-y-z+2) \\ &= x^2+(y+z-4)x+(y^2+z^2-yz-2y-2z+4). \end{aligned}$$

The discriminant is

$$(y+z-4)^2-4(y^2+z^2-yz-2y-2z+4) = -3(y-z)^2 \leq 0.$$

□

Remark. *As an alternative proof for the last part, we can also write*

$$(x+\frac{y}{2}+\frac{z}{2}-2)^2+\frac{3}{4}(y-z)^2 \geq 0.$$

The next problem is the last problem of this section. In order to bring about insightful proof, we added plenty of subtleties to the proof.

Problem 5. *Let $a, b, c \in [0,1], a+b+c=2$ find the maximum value of*

$$a^4+b^4+c^4+\frac{11}{2}abc$$

Solution. We want to find real numbers A, B, C such that $x^4 \leq Ax^2+Bx+C$, for all $x \in [0,1]$. In so doing, we need to have $x^4-Ax^2-Bx-C \leq 0$ for all $x \in [0,1]$. We also want to make this partial inequality consistent with our original inequality. That is, we want to plug $x=a, x=b, x=c$ and add up them to have

$$\begin{aligned} a^4+b^4+c^4 &\leq A(a^2+b^2+c^2)+B(a+b+c)+3C \\ &= A(4-2(ab+ac+bc))+2B+3C. \end{aligned}$$

That is,

$$a^4+b^4+c^4 \leq 4A+2B+3C-2A(ab+ac+bc).$$

So, the equality case of our partial inequality must be consistent with our original inequality. We now need an educated guess about the maximum of $a^4+b^4+c^4+\frac{11}{2}abc$. If we put $c=1$ for example then we should maximize $a^4+b^4+\frac{11}{2}ab$ under the condition $a+b=1$. We can see that this can happen at $(a,b)=(\frac{1}{2},\frac{1}{2})$. Thus, in order to determine A, B, C, we can impose equality cases $x=\frac{1}{2}, 1$ to our inequality. Since we opt that our inequality holds true for all $x \in [0,1]$ we should have a double root at $\frac{1}{2}$, otherwise, we would have a change of sign around $\frac{1}{2}$. So, three out of the four roots of the polynomial x^4-Ax^2-Bx-C must be $1, \frac{1}{2}, \frac{1}{2}$. Since in this polynomial the sum of the roots is zero, it follows that -2 would be the remaining root. That is,

$$x^4-Ax^2-Bx-C=(x+2)(x-\frac{1}{2})^2(x-1)$$

It is clear that the right side would always be non-positive in $[0,1]$. So, comparing the coefficients of x^2, x^1, x^0, it follows that $(A,B,C)=(\frac{11}{4},-\frac{9}{4},\frac{1}{2})$. Indeed, we can state the following lemma.

Lemma 1. $x^4 \leq \frac{1}{4}(11x^2-9x+2)$, *for all* $x \in [0,1]$

Proof. Notice that the desired inequality is equivalent to

$$(4x^2 - 4x + 1)(x^2 + x - 2) = (2x - 1)^2(x - 1)(x + 2) \le 0$$

Hence,

$$\begin{aligned} a^4 + b^4 + c^4 &\le \frac{11(a^2 + b^2 + c^2) - 9(a + b + c) + 6}{4} \\ &= 8 - \frac{11}{2}(ab + ac + bc). \end{aligned}$$

Hence,

$$a^4 + b^4 + c^4 + \frac{11}{2}abc \le 8 - \frac{11}{2}(ab + ac + bc - abc).$$

Notice that

$$\begin{aligned} ab + ac + bc - abc &= (1 - a)(1 - b)(1 - c) + a + b + c - 1 \\ &= (1 - a)(1 - b)(1 - c) + 1 \ge 1. \end{aligned}$$

That is,

$$a^4 + b^4 + c^4 + \frac{11}{2}abc \le 8 - \frac{11}{2} = \frac{5}{2}$$

Equality occurs whenever $(a, b, c) = (1, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, 1)$. □
□

4 Second-degree polynomial and Number Theory

In this section, we present several problems that either need direct or indirect application of second-degree polynomials. In our course, we leap frequently between algebra and number theory. The very first problem of this section only needs completing the square.

Problem 6. *Let x, y, z be positive integers such that*

$$2(x^2 + y^2 + z^2 + 2) = (x + y + z)^2$$

prove that $xy + x + y - z, yz + y + z - x, zx + z + x - y$ are perfect square.

Solution. Notice that $(x+y-z)^2 = 4(xy-1)$. Hence, $xy-1 = a^2$ thus, $x+y-z = 2a$ or $-2a$. Then,

$$xy + x + y - z = (a \pm 1)^2$$

□

For the next three problems, we need the following important corollary.

Corollary 1. *Let a, b, c be rational numbers such that $a \neq 0$ and the polynomial ax^2+bx+c has a rational root, then its discriminant, $D = b^2 - 4ac$ must be square of a rational number.*

We also need the following key note.

Note. *In the course of investigation that the discriminant $D = b^2 - 4ac$ is a perfect square, comparing it with some consecutive perfect squares of integers can bring lots of fruition.*

Problem 7. *Find all a such that there are infinitely many pairs (k, n) of integers such that*

$$kn^2 + (2k+1)n + k^2 = a.$$

Solution. Rewrite it as $k^2 + k(2n + n^2) + 2n - a = 0$. The discriminant is $D = (n^2 + 2n)^2 - 4(n - a)$ which must be a perfect square, but $D - (n^2 + 2n + 1)^2 = -2n^2 - 8n + 1 + 4a$ and $D - (n^2 + 2n - 1)^2 = 2n^2 + 4a + 1$. Thus, the leading coefficient of $-2n^2 - 8n + 1 + 4a$ is negative, so there would be a positive integer A such that $-2n^2 - 8n + 1 + 4a < 0$ for all $|n| \geq A$. On the other hand, since the leading coefficient of the quadratic trinomial $2n^2 + 4a + 1$ is positive, there would be a positive integer B such that $2n^2 + 4a + 1 > 0$ for all $|n| \geq B$. Let Thus, for a fixed a we have

$$(n^2 + 2n - 1)^2 < D < (n^2 + 2n)^2$$

for all $|n| \geq \max\{A, a\}$. Hence, D can not be a perfect square for all but finitely many positive integers n. Since n is bounded, the possibilities for k would also be bounded. Because for a certain n there would at most be two possibilities for k, that is, $k = \frac{-2n-n^2\pm\sqrt{D}}{2n^2}$, the total choices. □

Problem 8. *Find all positive integers a, b such that $a + b^2$ is divisible by $a^2b - 2$.*

Solution. If $a = 1$ then $b \in \{1, 3, 7\}$. If $a = 2$ then $b \in \{1, 2, 5\}$. If $a = 3$ then there would be no b. Assume now, $a \geq 4$ then writing $a + b^2 = c(a^2b - 2)$ or $b^2 - ca^2b + a + 2c = 0$. Hence,

$$D = (ca^2)^2 - 4(a + 2c) < (ca^2)^2.$$

Further,

$$\begin{aligned} D - (ca^2 - 1)^2 &= 2c(a^2 - 4) - 4a - 1 \\ &\geq 2(a^2 - 4) - 4a - 1 = 2a(a - 2) - 9 > 0. \quad \square \end{aligned}$$

The next problem has more number theoretic substance. That is, we should also use this well-know fact that for every prime number p of the form $4k + 1$, there are unique positive integers x, y such that $p = x^2 + y^2$. Moreover, if a prime number p of the form $4k + 3$ divides $x^2 + y^2$ it must divide x and y.

Problem 9. *Find all prime numbers p such that there is a positive integer n and positive integers k, m such that*

$$\frac{(mk^2 + 2)p - (m^2 + 2k^2)}{mp + 2} = n^2.$$

Solution. Rewrite it as $p(mk^2 - mn^2 + 2) = 2n^2 + m^2 + 2k^2$ that is,

$$p = \frac{2n^2 + m^2 + 2k^2}{2 + m(k - n)(k + n)} = \frac{(k + n)^2 + (k - n)^2 + m^2}{2 + m(k - n)(k + n)}.$$

Let $(a, b, c) = (k + n, k - n, m)$ then p can be written in the form $\frac{a^2+b^2+c^2}{2+abc}$. That is, we want to find which primes can be written in this form for some integers a, b, c. Let $p \equiv 1 \pmod 4$ then there are positive integers x, y such that $p = x^2 + y^2$ then taking $a = x + y, b = x - y$ and $c = 0$ we find that $p = \frac{1}{2}(2x^2 + 2y^2) = x^2 + y^2$. For $p = 2$ then taking $a = 2, b = c = 0$. Assume now

$p \equiv 3 \pmod 4$. Then, we prove that there is no such a, b, c. We can also assume that $a \geq b \geq c$. Then if $0 > b \geq c$ then replace (b, c) with $(-b, -c)$. If $c < 0$ since $2 + abc > 0$ we find that $(a, b, c) = (1, 1, -1)$. Thus, $p = 3$. So, assume that $c > 0$. Rewrite it as

$$a^2 - (pbc)a + b^2 + c^2 - 2 = 0$$

Consider the solution (a, b, c) with $a \geq b \geq c > 0$ such that $a+b+c$ has the minimal value. According to the choice of (a, b, c), it follows that $a = \frac{pbc - \sqrt{(pbc)^2 - 4b^2 - 4c^2 + 8p}}{2}$. Therefore,

$$\frac{pbc - \sqrt{(pbc)^2 - 4b^2 - 4c^2 + 8p}}{2} \geq b$$

then,

$$pbc - 2b \geq \sqrt{(pbc)^2 - 4b^2 - 4c^2 + 8p}.$$

That is,

$$2b^2 + c^2 \geq 2p + pb^2c$$

Since $b \geq c \geq 0$ then $3b^2 \geq 2b^2 + c^2 \geq 2p + pb^2c > pb^2c$. That is $pc < 3$ which is not possible. So, the only possible solutions are $p = 2, p = 3, p \equiv 1 (\text{mod} 4)$. $\square$

Remark. *Considering the minimal solution (a, b, c) with respect to the sum $a + b + c$ has a root in a well-known approach called Vieta's jumping. There were several problems in mathematical competitions that needed the adoption of such techniques. However, we have not recently had good problems in mathematical competitions about that.*

In the next two examples, the reader must use some properties of quadratic expressions inline with number theoretic facts about primes dividing sum of two squares of integers.

Problem 10. *Let p, q be prime numbers such that $p^3 - p^2 - q^2$ is a perfect square. Prove that $p = q$.*

Solution. Assume $p^3 - p^2 - q^2 = n^2$. If $p \equiv 1 (\text{mod} 4)$ then 4 divides $q^2 + n^2$. Then $q = 2$. Thus, $n^2 = p^3 - p^2 - 4 = (p - 2)(p^2 + p + 2)$. Since $\gcd(p - 2, p^2 + p + 2) = 1$ we find that $p^2 + p + 2$ must be a perfect square. Absurd. If $p \equiv 3 \pmod 4$ then p divides $q^2 + n^2$. This implies that $p \mid q$. Hence, $p = q$. $\square$

Problem 11. *Find all positive integers m, n such that $\frac{m^4-m^2+1}{2n^3-2n+11}$ is an integer.*

Solution. Let $N = 2n^3-2n+11 = 2n(n-1)(n+1)+11$ then $N \equiv 3 \pmod 4$. Hence, it has a prime divisor of that form. Thus, there is a prime $p \equiv 3 \pmod 4$ dividing $m^4 - m^2 + 1 = (m^2-1)^2 + m^2$. Impossible. □

This problem indirectly depends on quadratic expressions, restrictions between consecutive squares of integers, and some algebraic calculation.

Problem 12. *Let $x > y > 2022$ and $xy+x+y$ is a perfect square. Prove that there is a positive integer z such that $z \in [x+3y+1, 3x+y+1]$ and $x+y+z, x^2+xy+y^2$ are coprime.*

Solution. Let $t = \sqrt{xy+x+y}$ and $z = x+y+2t+1$ then note that

$$y < \sqrt{y^2+y} < \sqrt{xy+x+y} = t < \sqrt{x^2+2x} < x+1.$$

That is, $y < t \le x$. Then, $x+3y+1 \le z \le 3x+y+1$. Moreover,

$$xy+yz+zx = (x+y+t)^2,$$
$$xy+yz+zx+z+y+x = (x+y+t+1)^2.$$

If there is a prime number p such that p divides $x+y+z$ and x^2+xy+y^2 then $y+x \equiv -z \pmod p$ hence,

$$\begin{aligned} x^2+y^2+xy = (x+y)^2 - xy &\equiv -z(x+y) - xy \\ &\equiv -(xy+zx+yz) \pmod p. \end{aligned}$$

Then, p divides $xy+zx+yz$ and $x+y+z$. Thus, p divides $(x+y+t)^2$ and $(x+y+t+1)^2$. □

Remark. *The key idea of this problem is borrowed from the following problem that was originally proposed as Kvant 1979.*

Problem. *Let x, y be positive integers such that $xy+x+y$ is a perfect square, prove that there is a positive integer z such that $xy+z, yz+x, zx+y, yz+y+z, xz+z+x, xy+yz+zx, xy+yz+zx+x+y+z$ are all perfect square.*

Solution. That is, taking $t = \sqrt{xy + y + x}$ then taking $z = x + y + 2t + 1$.

$$\begin{aligned} yz + y + z &= (y + t + 1)^2, \\ xz + z + x &= (z + t + 1)^2, \\ xy + z &= (t + 1)^2, \\ yz + x &= (y + t)^2, \\ zx + y &= (x + t)^2, \\ xy + yz + zx &= (x + y + t)^2, \\ xy + yz + zx + x + y + z &= (x + y + t + 1)^2. \end{aligned} \qquad \square$$

We finish this section with a nice problem that needs almost all of the techniques we presented as well as the adoption of the *Chinese Remainder Theorem.*

Problem 13. *Let n be a positive integer and $P_1, \ldots, P_n$ be quadratic polynomials with integer coefficients. Prove that there is a positive integer k such that none of the equations $P_1(x) = k, \ldots, P_n(x) = k$ has no integer solutions.*

Solution. Let $P_i(x) = a_i x^2 + b_i x + c_i$ then we should find some integer k such that $a_i x^2 + b_i x + c_i - k = 0$ has no integer solutions. That is, $b_i^2 - 4a_i(c_i - k)$ must not be perfect square.

Taking primes $p_1, \ldots, p_n$ large enough then there would be a positive integer y_i such that $4a_i y_i \equiv 1 \pmod{p_i^2}$. Letting

$$\begin{aligned} k &\equiv (4a_1c_1 - b_1^2 + p_1)y_1 \pmod{p_1^2}, \\ &\ldots, \\ k &\equiv (4a_nc_n - b_n^2 + p_n)y_n \pmod{p_n^2}. \end{aligned}$$

Then,

$$4a_i k \equiv 4a_i y_i (4a_i c_i - b_i^2 + p_i) \equiv 4a_i c_i - b_i^2 + p_i.$$

That is, $D_i \equiv b_i^2 - 4a_i c_i + 4a_i k \equiv p_i \pmod{p_i^2}$. $\square$

Problem 14. *Find all integers m, n such that*

$$(2n^2 + +5m - 5n - mn)^2 = m^3 n.$$

Solution. Rewrite it as $(2n^2+5m-5n-mn)^2-n^4 = m^3n-n^4$. It follows that $(n^2+5m-5n-mn)(3n^2+5m-5n-mn) = n(m^3-n^3)$ thus, $(m-n)(5-n)(3n^2+5m-5n-mn) = n(m-n)(m^2+mn+n^2)$ that is, $(5-n)(3n^2-5n)+(n-5)^2m = nm^2+n^2m+n^3$ therefore, $nm^2+5(2n-5)m+n(2n-5)^2 = 0$. Now, if $n=0$ then $m=0$. Otherwise, according to the quadratic formula, it follows that $m = \frac{5(5-2n)\pm\sqrt{25(n-5)^2-4n^2(2n-5)^2}}{2n} = \frac{5(5-2n)\pm(2n-5)\sqrt{25-4n^2}}{2n}$. Thus, $25-4n^2 = k^2$, for some positive integer k. It follows that $n=\pm 2$ and therefore, $m = 2, -18$. The only solutions are $(m,n) = (0,0), (-18,-2)$. □

5 Second-degree polynomial and Functional Equations

Sometimes using what we know about the behaviour and the range of a second-degree polynomial in a certain interval can bring immense insight to solve a problem in the functional equation. We present one challenging problem that could have hardly been solved without these observations.

Problem 15. *Find all* $f : \mathbb{R}^+ \to \mathbb{R}^+ \cup \{0\}$ *such that for all* $x, y > 0$

$$f(x) - f(x+y) = f(x + x^2 f(y)).$$

Solution 1. It is easy to verify that $f(x) \geq f(x+y)$ for all $x, y > 0$. If $f(r) = f(s)$ for some $0 < r < s$ putting $(x,y) = (r, s-r)$ it follows that $f(r + s^2 f(s-r)) = 0$. Thus, there is a positive c such that $f(c) = 0$. then for all $x \in [c, +\infty)$ we have $f(x) = 0$. We then study the interval $[0, c)$. Indeed, fix $z > 0$ and let $0 < \epsilon < z$ taking $(x,y) = (c-\epsilon, z)$ to find

$$0 < f(c-\epsilon) = f(c-\epsilon) - f(c-\epsilon+z) = f((c-\epsilon)^2 f(z) + c - \epsilon).$$

Thus, $(c-\epsilon)^2 f(z) + c - \epsilon < c$ or $(c-\epsilon)^2 f(z) < \epsilon$ but if we take ϵ a sufficiently small one it follows that $(c-\epsilon)^2 f(z)$ tends to $c^2 f(z)$. That is if we take $\epsilon < c^2 f(z)$ then the inequality can not be true. That is, f is either zero of f is injective. Now, setting $(x, y) = (t, 1), (t, t^2 f(1))$ it follows that

$$f(t+1) = f(t) - f(t^2 f(1) + t) = f(t^2 f(t^2 f(1)) + t).$$

Hence, $1 = t^2 f(t^2 f(1))$. That is, $f(x) = \frac{b}{x}$ for all $x > 0$. $\square$

Solution 2. Again in the course of injectivity proof, we can assume that there is a positive $c > 0$ such that $f(c) = 0$, and hence for all $x > c, f(x) = 0$ and $f(x) > 0$ for all $0 < x < c$. Choose an arbitrary $y < c$ we prove that there is a $x < c$ such that $x^2 f(y) + x > c, x + y > c$. Indeed, for the latter, we only need to have $x > c - y$ and $x < c$. For the former, notice that $x^2 f(y) + x$ is a quadratic function in terms of x. Choosing $x > \frac{c}{2}$ to obtain that $x^2 f(y) > \frac{c^2}{4} f(y)$. Then, if we choose $x > c - \frac{c^2}{4} f(y) > c - x^2 f(y)$ we find that $x^2 f(y) + x > c$. So, we need to have $c > x > \max\{\frac{c}{2}, c - y, c - \frac{c^2}{4} f(y)\}$.

Then, by choosing such x for that fixed $y < c$ we find that $f(x + y) = f(x^2 f(y) + x) = 0$. Hence, $f(x) = 0$. This is absurd. So, the function is injective. Finally, putting $(x, y) = (1, t), (1, f(t))$ it follows that

$$f(1) = f(1+t) + f(1 + f(t)) = f(f(t) + 1) + f(1 + f(f(t))).$$

Hence, $f(f(t)) = t$. Thus, plugging $(x, y) = (x, f(\frac{y}{x^2}))$ it follows that

$$f(x) = f(y + x) + f(x + f(\frac{y}{x^2})) = f(y + x) + f(x + x^2 f(y)).$$

Yielding

$$f(\frac{y}{x^2}) = x^2 f(y).$$

Plugging $y = 1$, we find that $f(x) = \frac{b}{x}$. $\square$

6 Second-degree polynomial and higher degree polynomials

In our last section, we show that our learned content concerning a second-degree polynomial can be applied in higher-degree polynomials. Indeed, let $P(x) = Ax^2 + Bx + C$ with $A \neq 0$ then $P(x) = P(-\frac{B}{A} - x)$ for each x. This yields to the fact that $P(x)$ can be written as $A(x + \frac{B}{2A})^2 + \frac{4AC-B^2}{4A}$. In short, we can say that $P(x) = Q(x + \frac{B}{2A})$, where $Q(x) = Ax^2 + \frac{4AC-B^2}{4A}$. This can also be applied in the general case, that is; if for a certain polynomial $P(x)$ we have $P(x) = P(C - x)$ for some constant C we can then prove that $P(x) = Q(x^2 + Cx)$, for some polynomial $Q(x)$. In the next problem, we shall use different facts about quadratic expressions.

Problem 16. *Find all polynomials $P(x)$ such that $P(x + 3y) + P(3x - y)$ is constant for all x, y such that $x^2 + y^2 = 1$.*

Solution. The transformation $(x, y) \to (x + 3y, 3x - y)$ converts the unit circle to $x^2 + y^2 = 10$. The transformation is invertible via $(a, b) \to (\frac{a+3b}{10}, \frac{3a-b}{10})$. Then, $P(x) + P(y)$ is constant on $x^2 + y^2 = 10$.

Let us denote the unit circle by A then (x, y) and $(x, -y)$ are both in A. Hence, $P(x) + P(-y) = P(x) + P(y)$. That is, for infinitely many y we have $P(y) = P(-y)$. Hence, $P(x) = Q(x^2)$. Thus,

$$C = P(x) + P(y) = Q(x^2) + Q(y^2) = Q(x^2) + Q(10 - x^2).$$

That is,

$$Q(x) + Q(10 - x) = C.$$

Hence, $Q(5 + x) + Q(5 - x) = C$. Then, $Q(5 + x) - \frac{C}{2}$ is an odd polynomial. That is,

$$Q(5 + x) - \frac{C}{2} = xR(x^2)$$

Hence,

$$P(x) = (x^2 - 5)R((x^2 - 5)^2) + \frac{C}{2}$$

and all polynomials of this form work. □

Navid Safaei
Institute of Mathematics and Informatics.
Bulgarian Academy of Science
Bulgaria
navid@math.bas.bg

Contests

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to find interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

jose.luis.diaz@upc.edu

Problems and solutions from the 64th edition of the International Mathematical Olympiad (IMO)

Marc Felipe i Alsina

1 Problems and solutions

Below, we present now the five problems which were solved at least by some Spaniard contestant (problems 1, 2, 3, 4 and 5), and include the solutions given to them by our team (in some case slightly modified by the deputy leader). In all the cases, the solutions follow the ideas presented by the contestants, but we have done some little modifications to ease the exposition.

Problem (1 IMO 2023). *Determine all composite integers $n > 1$ that satisfy the following property: if $d_1, d_2, \ldots, d_k$ are all the positive divisors of n with $1 = d_1 < d_2 < \cdots < d_k = n$ then d_i divides $d_{i+1} + d_{i+2}$ for every $1 \le i \le k-2$.*

Solution by Guillem Beltrán. First, we see that, since $n > 1$ is composite, it has at least three divisors.

Then, notice that for any prime number p, $n = p^m$ works for any integer $m \ge 2$ as its positive divisors are $d_1 = 1 < p < p^2 < \cdots < p^{m-1} < p^m = d_k$ and $p^i \mid p^{i+1} + p^{i+2}$ because p^i divides both added numbers for every $1 \le i \le k-2$.

Finally we want to prove that the problem statement is false whether two or more different primes divide n, because if not, n would be of the form p^m.

Suppose that $p < q$ are the lowest primes dividing n. Then, the biggest positive divisors of n will be $d_k = \frac{n}{p^0} > \frac{n}{p^1} > \cdots > \frac{n}{p^i} > \frac{n}{q}$ for some integer $i \geq 1$ by the assumption that p is the lowest prime dividing n. Then the problem statement would imply that

$$\frac{n}{q} \mid \frac{n}{p^i} + \frac{n}{p^{i-1}} \Rightarrow np^i \mid nq + npq \Rightarrow p^i \mid q(1+p)$$

Which is a contradiction because p divides p^i for $i \geq 1$ but does not divide neither q nor $p+1$. $\square$

Problem (2 IMO 2023). *Let ABC be an acute-angled triangle with $AB < AC$. Let Ω be the circumcircle of ABC. Let S be the midpoint of the arc CB of Ω containing A. The perpendicular from A to BC meets BS at D and meets Ω again at $E \neq A$. The line through D parallel to BC meets line BE at L. Denote the circumcircle of triangle BDL by ω. Let ω meet Ω again at $P \neq B$. Prove that the line tangent to ω at P meets line BS on the internal angle bisector of $\angle BAC$.*

Solution by Jordi Ferré. We start by showing the following claim:

Claim 1. $\measuredangle DPA = 90°$.

Proof. Notice how

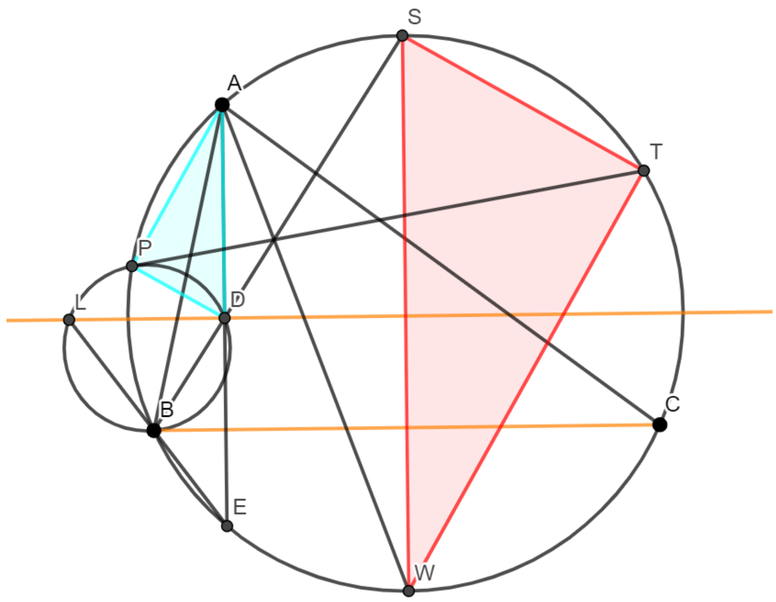
$$\measuredangle PDL = \measuredangle PBL = 180° - \measuredangle PBE = \measuredangle PAD.$$

Thus we get that

$$\measuredangle PAD + \measuredangle ADP = \measuredangle PDL + \measuredangle ADP.$$

But now notice how $BC \perp AD$ and $LD \parallel BC$ clearly implies that $LD \perp AD$, which shows that $\measuredangle PDL + \measuredangle ADP = \measuredangle ADL = 90°$. So by looking at triangle $\triangle APD$, we get that

$$\measuredangle DPA = 180° - \measuredangle PDL - \measuredangle APL = 90°. \qquad \square$$



Now define W as the midpoint of arc BC not containing A, and T as the point on Ω for which TP is tangent to ω.

Claim 2. $TS \parallel PD$

Proof. Just notice that

$$\measuredangle TPD = \measuredangle DBP = \measuredangle SBP = \measuredangle STP,$$

implying the desired result. □

Now we notice how triangles $\triangle APD$ and $\triangle WTS$ are homothetic, as $PD \parallel TS$, $AD \parallel SW$ and $\measuredangle DPA = \measuredangle STW = 90°$. Thus, they must have a center of homothety, which will be exactly the intersection of AW, PT and DS, implying that these three lines concur. But as AW is the internal bisector of $\angle BAC$ we are already done. □

Problem (3 IMO 2023). *For each integer $k \geq 2$, determine all infinite sequences of positive integers $a_1, a_2, \ldots$ for which there exists a polynomial P of the form $P(x) = x^k + c_{k-1}x^{k-1} + \cdots + c_1x + c_0$,* *where $c_0, c_1, \ldots, c_{k-1}$ are non-negative integers, such that*

$$P(a_n) = a_{n+1}a_{n+2}\cdots a_{n+k}$$

for every integer $n \geq 1$.

Solution. The answer is all arithmetic sequences $a_i = a + (i-1)d$, with $d \geq 0$ and $a \in \mathbb{Z}$, $a > 0$. The solution is split into three parts.

Part I (Construction). We provide a construction for arithmetic sequences as described above: take $P(x) = (x+d)\cdots(x+kd)$. With this choice

$$\begin{aligned} P(a_n) &= P(a + (n-1)d) \\ &= (a+nd)\cdots(a+(n-1+k)d) \\ &= a_{n+1}\cdots a_{n+k}, \end{aligned}$$

so the condition indeed works, as desired.

Part II (Analysis). Now, we begin our proof that the any sequence satisfying the condition must be of this form. In this part we make two crucial observations about the behaviour of the sequence.

Claim 1. *The sequence $(a_i)_{i\geq 1}$ is nondecreasing.*

Proof. Assume for the sake of contradiction that this is not true. Choose m such that $a_m = \min\{a_n \mid n \geq 2 \text{ and } a_{n-1} > a_n\}$, which must exist since this set is nonempty by assumption, and the sequence $(a_i)_{i\geq 1}$ is bounded below (by 0).

First I claim that in fact $a_m = \min\{a_n \mid n \geq m\}$. If not, there would be some $m' > m$ with $a_{m'} < a_m$. In order to not contradict the definition of m, it must happen that $a_{m'-1} \leq a_{m'}$, so $a_{m'-1} < a_m$. Repeat the argument with $m'-1$ instead of m', and do it as many times as necessary until obtaining the contradiction $a_m < a_m$, so the equality $a_m = \min\{a_n \mid n \geq m\}$ holds.

Now, since P has nonnegative coefficients, P is increasing. Hence from $a_m < a_{m-1}$ we obtain

$$1 > \frac{P(a_m)}{P(a_{m-1})} = \frac{a_{m+1}\cdots a_{m+k}}{a_m\cdots a_{m+k-1}} = \frac{a_{m+k}}{a_m},$$

so $a_{m+k} < a_m$. But this contradicts the minimality $a_m = \min\{a_n \mid n \geq m\}$. $\square$

Claim 2. *There exists a constant C such that $0 \leq a_{n+1} - a_n \leq C$ for every $n \geq 1$.*

Proof. First, we show that there is an integer $C > 0$ such that $x^{k-1}(x + C) \geq P(x)$ for every integer $x > 0$. This is clear; just take $C = c_{k-1} + c_{k-2} + \cdots + c_0$. Then

$$\begin{aligned} P(x) &= x^k + c_{k-1}x^{k-1} + \cdots c_1 x + c_0 \\ &\leq x^k + c_{k-1}x^k + \cdots c_1 x^k + c_0 x^k \\ &= x^{k-1}(x + C) \end{aligned}$$

Now let $n \geq 1$ be any integer. Then, using Claim 1, we have

$$a_n^{k-1}(a_n + C) \geq P(a_n) = a_{n+1} \cdots a_{n+k} \geq a_{n+1}a_n^{k-1},$$

so, since a_n is nonnegative, $a_{n+1} \leq a_n + C$, and hence $a_{n+1} - a_n \leq C$. The other inequality follows immediately from Claim 1. $\square$

Finally, note that Claim 1 implies that if the sequence is not eventually constant, then it has limit infinity. If the sequence is eventually constant (to some integer a), then it holds that $a^k = P(a) = a^k + c_{k-1}a^{k-1} + \cdots + a_0 \geq a^k$, so $P(x) = x^k$ identically. By backwards induction $(a_i)_{i \geq 1}$ will be constant equal to a. With this case out of the way, assume henceforth that the sequence has limit infinity.

Part III (Answer extraction). We now use the results of Part II to solve the problem. By Claim 2, for every n it is true that

$$S_n = (a_{n+1} - a_n, a_{n+2} - a_n, \ldots, a_{n+k} - a_n) \in \{0, 1, \ldots, C\}^k.$$

Since there is only a finite amount of possibilities for S_n, by infinite pigeonhole principle, there is at least one k-tuple $S = (s_1, \ldots, s_k)$ such that $S_n = S$ for infinitely many n. In this case

$$P(a_n) = a_{n+1}a_{n+2} \cdots a_{n+k} = (a_n + s_1)(a_n + s_2) \cdots (a_n + s_k)$$

holds for infinitely many values a_n (since we are working on the case where the sequence has limit infinity), so in fact $P(x) =$

$(x+s_1)(x+s_2)\cdots(x+s_k)$ holds identically. Notice that $s_1 \le s_2 \le \cdots \le s_k$ because of Claim 1.

Suppose that there exists another tuple $\hat{S} = (\hat{s}_1, \hat{s}_2, \ldots, \hat{s}_k) \neq S$ such that $\hat{S} = S_n$ for infinitely many S. We would then conclude, using the reasoning above, that the polynomial in the statement must be $\hat{P}(x) = (x+\hat{s}_1)(x+\hat{s}_2)\cdots(x+\hat{s}_k)$. Since only one polynomial can satisfy the hypotheses of the statement, and a polynomial determines its roots, we see that $P = \hat{P}$ and $S = \hat{S}$, a contradiction. Therefore, all tuples different from S eventually die out and we have $S_n = S$ for every n past some constant N, that is, $a_{n+i} - a_n = s_i$ for $i = 1, 2, \ldots, k$ whenever $n \ge N$.

Comparing consecutive values of n, it follows by a quick induction on i that $s_i = is_1$ for every $1 \le i \le k$. Hence the tail sequence $(a_i)_{i \ge N}$ is an arithmetic sequence with common difference s_1, and $P(x) = (x+s_1)(x+2s_1)\cdots(x+ks_1)$. Finally, by backwards induction, and monotonicity of P, we conclude that $(a_i)_{i\ge 1}$ must be an arithmetic sequence. This concludes the solution. $\square$

Problem (4 IMO 2023). *Let $x_1, x_2, \ldots, x_{2023}$ be pairwise different positive real numbers such that*

$$a_n = \sqrt{(x_1+x_2+\cdots+x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\cdots+\frac{1}{x_n}\right)}$$

is an integer for every $n = 1, 2, \ldots, 2023$. Prove that $a_{2023} \geqslant 3034$.

Solution. We claim that $a_{2n+1} \ge 3n+1 \quad \forall n \in \mathbb{N}_0$. We proceed by induction, starting with $n = 0$, for which the proposition is true since $a_1 = \sqrt{x_1 \frac{1}{x_1}} = 1$.

The given equation clearly implies $a_i > a_{i-1}$, which is the same as $a_i \ge a_{i-1} + 1$, so it suffices to prove that at least one of $a_{2n} \ge a_{2n-1} + 2$ or $a_{2n+1} \ge a_{2n} + 2$ holds. Therefore, we will arrive to a contradiction supposing $a_{2n} = a_{2n-1} + 1$ and $a_{2n+1} = a_{2n} + 1$.

Let $b_n = \sum_{i=1}^{n} x_i$, $c_i = \sum_{i=1}^{n} \frac{1}{x_i}$, so that $b_i c_i = a_i^2$. Now consider we have that:

$$\begin{aligned}(a_{2n-1}+1)^2 &= a_{2n}^2 = b_{2n}c_{2n}\\ &= (b_{2n-1}+x_{2n})(c_{2n-1}+x_{2n}^{-1})\\ &= b_{2n-1}c_{2n-1}+x_{2n}x_{2n}^{-1}+(b_{2n-1}x_{2n}^{-1}+c_{2n-1}x_{2n})\\ &\geq a_{2n-1}^2+1+2\sqrt{b_{2n-1}c_{2n-1}}\\ &= a_{2n-1}^2+1+2a_{2n-1}\\ &= (a_{2n-1}+1)^2,\end{aligned}$$

where we used the AM-GM inequality. Since the left and right sides of the inequality are equal, equality must occur where we used AM-GM. Therefore it must happen that

$$b_{2n-1}x_{2n}^{-1} = c_{2n-1}x_{2n} \implies x_{2n} = \sqrt{\frac{b_{2n-1}}{c_{2n-1}}}.$$

But one can deduce in a similar fashion that

$$\begin{aligned}x_{2n+1} &= \sqrt{\frac{b_{2n}}{c_{2n}}} = \sqrt{\frac{b_{2n-1}+x_{2n}}{c_{2n-1}+x_{2n}^{-1}}} = \sqrt{\frac{b_{2n-1}+x_{2n}^2x_{2n}^{-1}}{c_{2n-1}+x_{2n}^{-1}}}\\ &= \sqrt{\frac{b_{2n-1}+\frac{b_{2n-1}}{c_{2n-1}}x_{2n}^{-1}}{c_{2n-1}+x_{2n}^{-1}}} = \sqrt{\frac{b_{2n-1}c_{2n-1}+b_{2n-1}x_{2n}^{-1}}{c_{2n-1}(c_{2n-1}+x_{2n}^{-1})}}\\ &= \sqrt{\frac{b_{2n-1}}{c_{2n-1}}} = x_{2n},\end{aligned}$$

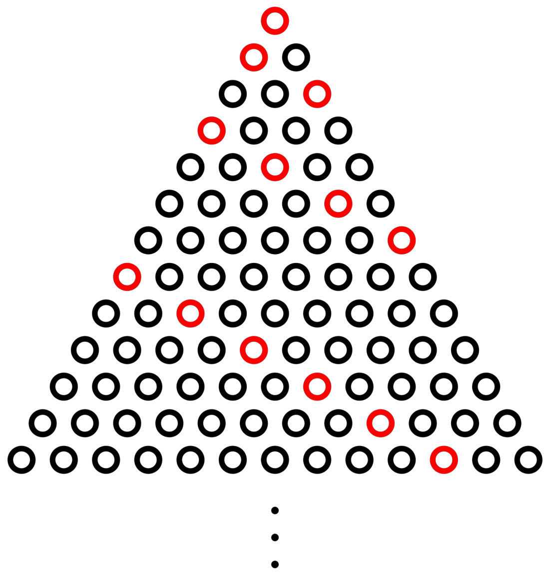
contradicting the fact that all the x_i are pairwise different. Therefore we have arrived to a contradiction and the induction is complete. □

Problem (5 IMO 2023). *Let n be a positive integer. A Japanese triangle consists of $1+2+\cdots+n$ circles arranged in an equilateral triangular shape such that for each $i = 1, 2, \ldots, n$, the i^{th} row contains exactly i circles, exactly one of which is coloured red. A ninja path in a Japanese triangle is a sequence of n circles obtained by starting in the top row, then repeatedly going from a circle to one* *of the two circles immediately below it and finishing in the bottom row.*

In terms of n, find the greatest k such that in each Japanese triangle there is a ninja path containing at least k red circles.

Solution by Roger Lidón. We claim the answer is $k = \lfloor \log_2 n \rfloor + 1$.

First, we construct a japanese triangle with n rows such such that every ninja path achieves at most $\lfloor \log_2 n \rfloor + 1$ red circles. Number the rows of the triangle $1, 2, \ldots, n$ and split them in $\lfloor \log_2 n \rfloor + 1$ blocks, namely $\{1\}, \{2, 3\}, \{4, 5, 6, 7\}, \ldots$ So, the i-th block contains only rows between 2^{i-1} and $2^i - 1$. Now, if a row is the k-th of its respective block, we paint its $2k - 1$-th circle red. In practice, we are taking the first n rows of the following infinite pattern:



Now it is easy to see the ninja cannot pass through more than $\lfloor \log_2 n \rfloor + 1$ red circles, as there are $\lfloor \log_2 n \rfloor + 1$ blocks in the triangle and the path cannot pass through two red circles of the same block.

Now, we need to prove that there always exists a ninja path passing through $\lfloor \log_2 n \rfloor + 1$ red circles. We will prove this using strong induction, but we need to define a couple notions first. Let the *score* of a circle be the maximum possible number of red circles in a path going from the uppermost circle to the circle itself, both included if red. So, the problem amounts to proving there exists a circle with score at least $\lfloor \log_2 n \rfloor + 1$. Recall that we numbered the rows $1, 2, \ldots, n$ from top to bottom, and for all $k = 1, 2, \ldots, n$ define $s(k)$ as the sum of the scores of the circles in row k and $m(k)$ as the greatest score among the circles in row k.

Our objective is therefore to prove $m(k) \geq \lfloor \log_2 n \rfloor + 1$ The key claim of the solution is the following:

Claim 1. $s(k+1) \geq s(k) + m(k) + 1$ *for all* $k = 1, 2, \ldots, n-1$.

Proof. Let $x_1, x_2, \ldots, x_k$ be the scores of the circles in row k, and let $y_1, y_2, \ldots, y_{k+1}$ be the scores of the circles in row $k+1$, in both cases from left to right. By convention, define $x_0 = x_{k+1} = 0$. Now observe that

$$y_r = \max\{x_{r-1}, x_r\}$$

for each $r = 1, 2, \ldots, k+1$, with the exception of the (exactly one) case where the circle in position r is red, for which

$$y_r = \max\{x_{r-1}, x_r\} + 1$$

holds instead. Let $p \in \{1, 2, \ldots, k\}$ be the index such that x_p is a maximum, that is $x_p = m(k)$, and notice that we also have $y_p = m(k)$ (or $y_p = m(k) + 1$ if red). Now, observe that

$$\begin{aligned}
s(k+1) &= \sum_{i=1}^{k+1} y_i = \sum_{i=1}^{p-1} y_i + y_p + \sum_{i=p+1}^{k+1} y_i = \\
&= \sum_{i=1}^{p-1} \max\{x_i, x_{i-1}\} + m(k) + \sum_{i=p+1}^{k+1} \max\{x_i, x_{i-1}\} + 1 \\
&\geq \sum_{i=1}^{p-1} x_i + m(k) + \sum_{i=p+1}^{k+1} x_{i-1} + 1 = \sum_{i=1}^{k} x_i + m(k) + 1 \\
&= s(k) + m(k) + 1
\end{aligned}$$

so the claim is proven. $\square$

Now we can take on the induction. As induction hypothesis, suppose that $m(k) \geq \lfloor \log_2 k \rfloor + 1$ for every $k = 1, 2, \ldots, n-1$. The base case $n = 1$ is trivial, as $s(1) = m(1) = 1$ if $n = 1$. Notice that we just need to prove that $m(n) \geq \lfloor \log_2 n \rfloor + 1$, hence the trivial bound

$$m(n) \geq m(n-1) = \lfloor \log_2 (n-1) \rfloor + 1 = \lfloor \log_2 n \rfloor + 1$$

works whenever n is not a power of two. So, from now on, suppose $n = 2^a$. Observe that by the claim

$$\begin{aligned}
s(n) = s(1) + \sum_{i=2}^{n} (s(i) - s(i-1)) &\geq 1 + \sum_{i=2}^{n} (m(i-1) + 1) \\
&= n + m(1) + m(2) + \ldots + m(n-1) \\
&\geq n + \sum_{i=1}^{n-1} \lfloor \log_2 i + 1 \rfloor = 2^a + \sum_{i=1}^{2^a-1} \lfloor \log_2 i + 1 \rfloor \\
&= 2^a + (1) + (2+2) + (3+3+3+3) + \cdots = 2^a + \sum_{i=0}^{a-1} 2^i (i+1) \\
&= 2^a + \sum_{i=0}^{a-1} \sum_{j=0}^{i} 2^i = 2^a + \sum_{j=0}^{a-1} \sum_{i=j}^{a-1} 2^i = 2^a + \sum_{j=0}^{a-1} \frac{2^a - 2^j}{2-1} \\
&= 2^a + \sum_{j=0}^{a-1} (2^a - 2^j) = a2^a + 2^a - \sum_{j=0}^{a-1} 2^j = 2^a a + 1
\end{aligned}$$

and we therefore obtain that

$$m(n) \geq \frac{s(n)}{n} = \frac{2^a a + 1}{2^a} > a \implies m(n) \geq a + 1 = \lfloor \log_2 n \rfloor + 1$$

as we wanted to prove. The induction is complete. □

Marc Felipe i Alsina
BarcelonaTech
Barcelona, Spain
marc.felipe.alsina@gmail.com

Problems and solutions from the 2023 Barcelona Spring Matholympiad

O. Rivero Salgado and J. L. Díaz-Barrero

1 Problems and solutions

Hereafter, we present the four problems that appeared in the paper given to the contestants of the Barcelona Spring Matholympiad 2023 (Category II), as well as their official solutions.

Problem 1. Compute the value of the following sum:

$$\sum \frac{1}{j_1 j_2 \cdots j_k},$$

where the summation is taken over all nonempty subsets $\{j_1, j_2, \ldots, j_k\}$ of the set $\{1, 2, \ldots, 2023\}$.

Solution. We have,

$$\sum \frac{1}{j_1 j_2 \cdots j_k}$$

$$= 1+\frac{1}{2}+\ldots+\frac{1}{2023}+\frac{1}{1\cdot 2}+\ldots+\frac{1}{2022\cdot 2023}+\ldots+\frac{1}{1\cdot 2\cdots 2022\cdot 2023}.$$

Let

$$P = \left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\ldots\left(1+\frac{1}{2023}\right).$$

Multiplying out, we obtain the sum of 2^{2023} terms, one of which equals 1 and the others which constitute exactly the sum we wish to evaluate. But,

$$P = \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{2024}{2023} = 2024.$$

So,

$$\sum \frac{1}{j_1 j_2 \cdots j_k} = 2023,$$

and we are done.

Problem 2. Let ABC be an acute and scalene triangle and let H_A, H_B, and H_C be the feet of its altitudes. If side BC meets H_BH_C at A', side CA meets H_AH_C at B' and side AB meets H_AH_B at C', then prove that points A', B', C' are collinear.

Solution. Applying Menelaus theorem to triangle ABC with transversal $A'H_CH_B$ we get

$$\frac{A'B}{A'C} \cdot \frac{H_BC}{H_BA} \cdot \frac{H_CA}{H_CB} = 1.$$

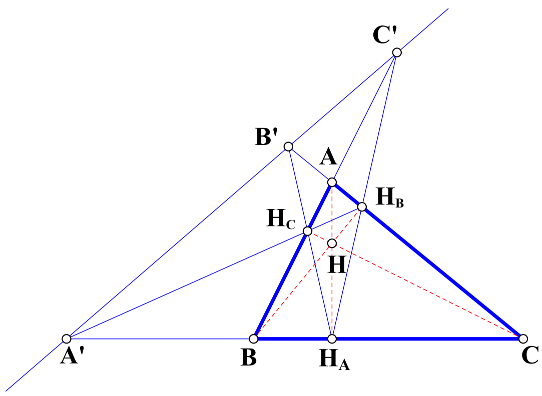


Figure 1: Scheme for solving problem 2.

Likewise, using transversal $B'H_CH_A$ we get

$$\frac{B'C}{B'A} \cdot \frac{H_CA}{H_CB} \cdot \frac{H_AB}{H_AC} = 1,$$

and using transversal $C'H_BH_A$ we obtain

$$\frac{C'A}{C'B} \cdot \frac{H_BC}{H_BA} \cdot \frac{H_AB}{H_AC} = 1.$$

Multiplying up the preceding expressions yields

$$\left(\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B}\right) \cdot \left(\frac{H_AB}{H_AC} \cdot \frac{H_BC}{H_BA} \cdot \frac{H_CA}{H_CB}\right)^2 = 1.$$

Since AH_A, BH_B, and CH_C are cevians that met at the orthocenter H, then on account of Ceva's Theorem, we have

$$\frac{H_AB}{H_AC} \cdot \frac{H_BC}{H_BA} \cdot \frac{H_CA}{H_CB} = 1$$

Substituting this expression in the above, we get

$$\frac{A'B}{A'C} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1.$$

Using the reciprocal of Menelaus theorem, we conclude that points A', B', C' are collinear.

Problem 3. Suppose that 2023 distinct points are chosen in the plane and the distances between them are measured. Show that the total number of distances among the given points is at least 32.

Solution. Let V be a set of n points on the Euclidean plane and ℓ_1, ℓ_2, $\dots$, ℓ_k all distinct distances between the points. For each $v \in V$, define $d_i(v)$ to be the number of points that are ℓ_i away from v. For each $i = 1, 2, \dots, k$, denote by T_i the set of pairs $(v, \{a, b\}) \in V \times \binom{V}{2}$ such that both a and b are at distance ℓ_i away from v. Obviously, for each $v \in V$, there are exactly $\binom{d_i(v)}{2}$ unordered pairs $\{a, b\} \in \binom{V}{2}$ that make $(v, \{a, b\}) \in T_i$. For each

$\{a, b\} \in \binom{V}{2}$, there are at most two such v's. Therefore, we have the inequality

$$\sum_{v \in V} \binom{d_i(v)}{2} = |T_i| \leq 2\binom{n}{2}.$$

Finally, sum over all $i = 1, 2, \ldots, k$ to obtain

$$\sum_{v \in V} k \binom{\frac{1}{k} \sum_{i=1}^{k} d_i(v)}{2} \leq \sum_{v \in V} \sum_{i=1}^{k} \binom{d_i(v)}{2} \leq 2k\binom{n}{2}.$$

Since $\sum_{i=1}^{k} d_i(v)$ is the total number of points apart from v itself, then we get

$$nk\binom{(n-1)/k}{2} = \sum_{v \in V} k \binom{\frac{1}{k} \sum_{i=1}^{k} d_i(v)}{2} \leq 2k\binom{n}{2}$$

from which it follows

$$k \geq \frac{\sqrt{8n-7}-1}{4}.$$

In particular, if $n = 2023$ we obtain $k \geq 32$, and the number of distinct distances among the given points is at least 32.

Problem 4. Let m, n be integers greater or equal than 2. Show that there exist positive integers $a_1 < a_2 < \ldots < a_m$, such that for any integer $1 \leq i < j \leq m$ the number $\dfrac{a_j}{a_j - a_i} \equiv 0 \pmod{n}$.

Solution. An increasing sequence of integers $a_1, a_2, \ldots, a_t$ (not necessarily positive) is called a **good** sequence if for every $1 \leq i < j \leq t$ the number $\dfrac{a_j}{a_j - a_i}$ is integer and divisible by n. Suppose the sequence $a_1, a_2, \ldots, a_t$ is good. We claim:

1. If $a_t < 0$, then the sequence $a_1, a_2, \ldots, a_t, 0$ is good.
2. The sequence $a_1 + x, a_2 + x, \ldots, a_t + x$ is good as long as the integer x is divided by $n \cdot \prod_{1 \leq i < j \leq t} (a_j - a_i)$.

Part (1) of the observation is obvious. Part (2) follows from the fact that if

$$x = yn \cdot \prod_{1 \le i < j \le t} (a_j - a_i)$$

for some integer y, then for any $1 \le i < j \le t$ the number

$$\frac{a_j + x}{(a_j + x) - (a_i + x)} = \frac{a_j}{a_j - a_i} + yn \cdot \prod_{\substack{1 \le r < s \le t \\ (r,s) \neq (i,j)}} (a_s - a_r)$$

is integer and divisible by n.

Let us to solve the problem. We argue by induction on m. For $m = 2$ it is enough to take $a_1 = n - 1$ and $a_2 = n$. Induction step. Suppose that the sequence $a_1 < a_2 < \ldots < a_m$ is good and $a_1 > 0$. Construct a good sequence of length $m + 1$ positive terms. Let x be a multiple of $n \cdot \prod_{1 \le i < j \le m} (a_j - a_i)$ so large that $x > a_m$. Then the sequence $a_1 - x, \ldots, a_2 - x, a_m - x$ is good and has negative terms. Adding 0 to the end of this sequence we obtain a good sequence $a_1 - x, \ldots, a_2 - x, a_m - x, 0$ of length $m + 1$. Let

$$y = n \cdot \prod_{1 \le i < j \le m} (a_j - a_i) \cdot \prod_{k=1}^{m} (x - a_k).$$

Then $y > x - a_1$. Therefore, the sequence $a_1 - x + y, \ldots, a_2 - x + y, a_m - x + y, y$ is good, has positive terms and is of length $m + 1$. This concludes the inductive step and the problem is solved.

Óscar Rivero Salgado
University of Santiago de Compostela
Spain
riverosalgado@gmail.com

José Luis Díaz-Barrero
Civil and Environmental Engineering
BarcelonaTech
Barcelona, Spain
jose.luis.diaz@upc.edu

Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

jose.luis.diaz@upc.edu

Elementary Problems

E–113. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* For every positive integer n we define a_n as the last digit of the sum of the first n positive integers. Compute $a_1 + a_2 + \cdots + a_{2023}$.

Solution 1 by Henry Ricardo, Westchester Area Math Circle, New York, USA. The sum is 7080, as we shall show. We note that a_n is the last digit of $T_n = n(n+1)/2$, the n-th triangular number. Furthermore, the integers a_n form a sequence of period 20:

$$\begin{aligned} a_n = a_{n+20} &\iff T_n \equiv T_{n+20} \pmod{20} \\ &\iff n(n+1) \equiv (n+20)(n+21) \pmod{20} \\ &\iff 0 \equiv 40n + 420 \pmod{20}, \end{aligned}$$

which is clearly true.

We can easily check that $\sum_{n=1}^{20} a_n = 70$ and, since $2023 = 101(20)+3$, we have

$$\sum_{n=1}^{2023} a_n = \sum_{k=0}^{100} \sum_{n=20k+1}^{20(k+1)} a_n + a_{2021} + a_{2022} + a_{2023}$$

$$= 101 \cdot 70 + a_1 + a_2 + a_3 = 7080.$$

Solution 2 by Michel Bataille, Rouen, France. Let $T_n = \sum\limits_{k=0}^{n} k = \frac{n(n+1)}{2}$. We calculate $T_0 = 0, T_1 = 1, T_2 = 3, T_3 = 6, T_4 = 10, T_5 = 15, T_6 = 21, T_7 = 28, T_8 = 36, T_9 = 45$ so that $a_0+a_1+\cdots+a_9 = 35$. For $0 \le n \le 19$, we have

$$\begin{aligned} T_n - T_{19-n} &= \frac{n(n+1) - (19-n)(20-n)}{2} \\ &= 20n - 190 \equiv 0 \pmod{10}, \end{aligned}$$

hence $a_{19-n} = a_n$ and therefore $a_0 + a_1 + \cdots + a_{19} = 70$.
We also have

$$\begin{aligned} T_{n+20} - T_n &= \frac{(n+20)(n+21) - n(n+1)}{2} \\ &= 20n + 210 \equiv 0 \pmod{10}, \end{aligned}$$

hence $a_{n+20} = a_n$ for all nonnegative integers n. We deduce that

$$\begin{aligned} \sum_{k=1}^{2023} a_k &= \left(\sum_{k=0}^{100} \sum_{j=0}^{19} a_{j+20k} \right) + a_{2020} + a_{2021} + a_{2022} + a_{2023} \\ &= 101 \cdot \sum_{j=0}^{19} a_j + a_0 + a_1 + a_2 + a_3 = 101 \cdot 70 + 0 + 1 + 3 + 6 \end{aligned}$$

and conclude: $a_1 + a_2 + \cdots + a_{2023} = 7080$.

Solution 3 by Ioan Viorel Codreanu, Satulung, Maramures, Romania and the proposer. We compute the first values of a_n, and we have

$$\{a_n\} = \{a_1, a_2, \ldots, a_{20}, a_{21}, a_{22}, a_{23}, \ldots, a_{38}, a_{39}, a_{40}, a_{41}, a_{42}, a_{43}, \ldots\}$$

$$= \{1, 3, 6, 0, 5, 1, 8, 6, 5, 5, 6, 8, 1, 5, 0, 6, 3, 1, 0, 0, 1, 3, 6, \ldots, 1, 0, 0, 1, 3, 6, \ldots\},$$

so $a_i = a_{20+i}$. Since $a_1 + a_2 + \ldots + a_{20} = 70$, then

$$\sum_{i=1}^{2023} a_i = \sum_{i=1}^{2020} a_i + a_{2021} + a_{2022} + a_{2023} = 70 \cdot 101 + 1 + 3 + 6 = 7070 + 10 = 7080.$$

Also solved by *Alberto Espuny Díaz, Universität Heidelberg, Heidelberg, Germany.*

E–114. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.* Let P, Q, R be points on the sides of a triangle ABC which trisect the perimeter of $\triangle ABC$. Suppose that P, Q lie on side AB. Prove that

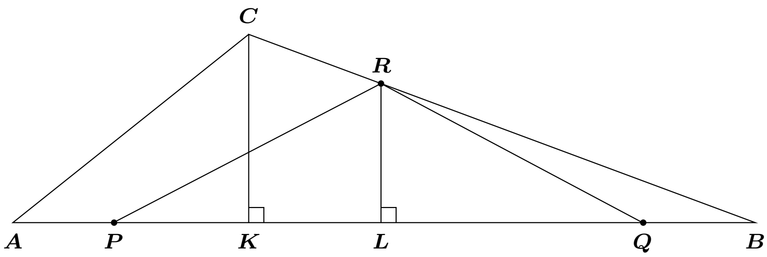
$$\frac{\text{Area }(\triangle PQR)}{\text{Area }(\triangle ABC)} > \frac{2}{9}.$$

Solution by the proposer. Let K, L be the feet of the altitudes from C, R in $\triangle ABC$, $\triangle PQR$, respectively.

We have

$$\frac{\text{Area }(\triangle PQR)}{\text{Area }(\triangle ABC)} = \frac{\frac{1}{2} PQ \cdot RL}{\frac{1}{2} AB \cdot CK} = \frac{PQ}{AB} \cdot \frac{RL}{CK}. \tag{1}$$

Without lost of generality, suppose that $\triangle ABC$ has perimeter 1.



Since

$$PQ = \frac{1}{3}(\text{the perimeter of } \triangle ABC) = \frac{1}{3} \cdot 1 = \frac{1}{3}$$

and

$$AB < (\text{by the triangle inequality}) < \text{the semiperimeter of } \triangle ABC = \frac{1}{2},$$

then,

$$\frac{PQ}{AB} > \frac{2}{3}. \tag{2}$$

We also have

$$QB \leq QB + AP = AB - PQ < \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

and

$$QB + BR = \frac{1}{3}(\text{the perimeter of } \triangle ABC) = \frac{1}{3}.$$

Therefore

$$BR = \frac{1}{3} - QB > \frac{1}{3} - \frac{1}{6} = \frac{1}{6}.$$

Thus

$$\frac{RL}{CK} = \left(\begin{array}{c} \text{from similar right-angled} \\ \text{triangles } LBR \text{ and } KBC \end{array} \right) = \frac{BR}{BC} > \frac{1/6}{BC} > \frac{1/6}{1/2} = \frac{1}{3}, \tag{3}$$

where the last inequality holds because a side of a triangle is less than its semiperimeter. From (1), (2) and (3), then,

$$\frac{\text{Area }(\triangle PQR)}{\text{Area }(\triangle ABC)} > \frac{2}{3} \cdot \frac{1}{3} = \frac{2}{9}.$$

E–115. *Proposed by Goran Conar, Varaždin, Croatia.* Let d_a, d_b, d_c be distances from center of circumcircle to the sides of triangle ABC and let r be the radius of its incircle. Prove that for any real $p > 1$, it holds

$$d_a^p + d_b^p + d_c^p \geq 3r^p.$$

Solution 1 by Michel Bataille, Rouen, France. Let O denote the circumcenter. If $A = \angle BAC$ is not obtuse, then $\angle BOC = 2A$ and $\angle OBC = \angle OCB = 90^\circ - A$. Otherwise, $\angle BOC = 2(180^\circ - A)$ and $\angle OBC = \angle OCB = A - 90^\circ$. In any case, we have $d_a = R\cos A$ where R is the circumradius. We deduce that

$$d_a + d_b + d_c = R(\cos A + \cos B + \cos C) = R\Big(1 + \frac{r}{R}\Big) = R + r \geq 3r$$

where the inequality follows from Euler's inequality $R \geq 2r$. Since the function $x \mapsto x^p$ is convex on $(0, \infty)$, it follows that

$$d_a^p + d_b^p + d_c^p \geq 3\left(\frac{d_a + d_b + d_c}{3}\right)^p \geq 3r^p.$$

Solution 2 by Arkady Alt, San Jose, California, USA. Let R be circumradius of $\triangle ABC$. Since $(d_a, d_b, d_c) = R(\cos A, \cos B, \cos C)$ and $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$ then $d_a + d_b + d_c = R\left(1 + \frac{r}{R}\right) = R + r \geq 3r$ because $R \geq 2r$ (Euler's inequality). Also, by PM-AM inequality we have $\left(\frac{d_a^p + d_b^p + d_c^p}{3}\right)^{1/p} \geq \frac{d_a + d_b + d_c}{3} \iff$ $d_a^p + d_b^p + d_c^p \geq \frac{(d_a + d_b + d_c)^p}{3^{p-1}} \geq \frac{(3r)^p}{3^{p-1}} = 3r^p$.

Solution 3 by the proposer. For every triangle hold the following equality

$$d_a + d_b + d_c = R + r$$

(this is Carnot's theorem) where R is its circumradius. Now we use inequality $R \geq 2r$ to get $d_a + d_b + d_c \geq 3r$. Finally, from power mean inequalities between powers 1 and p we get

$$\sqrt[p]{\frac{d_a^p + d_b^p + d_c^p}{3}} \geq \frac{d_a + d_b + d_c}{3} = \frac{R + r}{3} \geq \frac{3r}{3} = r$$

from which

$$d_a^p + d_b^p + d_c^p \geq 3r^p$$

follows.

Also solved by *Ioan Viorel Codreanu, Satulung, Maramures, Romania, and José Luis Díaz-Barrero, Barcelona, Spain.*

E–116. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $n \geq 0$ be an integer number. Prove that $N = 10^{n^3+3n^2+2n+2}$ can be written as a sum of four perfect cubes.

Solution 1 by Henry Ricardo, Westchester Area Math Circle, New York, USA and Ioan Viorel Codreanu, Satulung, Maramures, Romania. We proceed by induction. For $n = 0$ we have $N = 10^2 = 100 = 1^3 + 2^3 + 3^3 + 4^3$. Now assume that the result holds for some integer $M \geq 0$: $10^{M^3+3M^2+2M+2} = r_1^3 + r_2^3 + r_3^3 + r_4^3$ $r_i \in \mathbb{Z}, i = 1, 2, 3, 4$. Then

$$\begin{aligned} N &= 10^{(M+1)^3+3(M+1)^2+2(M+1)+2} \\ &= \left(10^{M^3+3M^2+2M+2}\right) \cdot \left(10^{3M^2+9M+6}\right) \\ &= (r_1^3 + r_2^3 + r_3^3 + r_4^3) \cdot \overbrace{\left(10^{M^2+3M+2}\right)}^{R}{}^3 \\ &= (r_1 R)^3 + (r_2 R)^3 + (r_3 R)^3 + (r_4 R)^3 \end{aligned}$$

and the proof is finished.

Solution 2 by Henry Ricardo, Westchester Area Math Circle, New York, USA. Since $n^3 + 3n^2 + 2n = n(n+1)(n+2)$, then $3 \mid n^3 + 3n^2 + 2n$ and we have

$$10^{n^3+3n^2+3n+2} = 100 \cdot \left(10^{\frac{n^3+3n^2+2n}{3}}\right)^3.$$

On account that $1^3 + 2^3 + 3^3 + 4^3 = 100$, then

$$\begin{aligned} N &= (1^3 + 2^3 + 3^3 + 4^3) \cdot \left(10^{\frac{n^3+3n^2+2n}{3}}\right)^3 \\ &= \left(1 \cdot 10^{\frac{n^3+3n^2+2n}{3}}\right)^3 + \left(2 \cdot 10^{\frac{n^3+3n^2+2n}{3}}\right)^3 \\ &+ \left(3 \cdot 10^{\frac{n^3+3n^2+2n}{3}}\right)^3 + \left(4 \cdot 10^{\frac{n^3+3n^2+2n}{3}}\right)^3, \end{aligned}$$

and we are done.

Solution 3 by Michel Bataille, Rouen, France. The proof is by induction. Let $p(n) = n^3 + 3n^2 + 2n + 2$. Then $p(0) = 2$ and $10^{p(0)} = 100 = 1^3 + 2^3 + 3^3 + 4^3$.

Now, assume that for some nonnegative integer n, we have $10^{p(n)} = a^3 + b^3 + c^3 + d^3$ where a, b, c, d are positive integers. Let

$$a' = a \cdot 10^{n^2+3n+2}, \quad b' = b \cdot 10^{n^2+3n+2}, \quad c' = c \cdot 10^{n^2+3n+2}$$

and $d' = d \cdot 10^{n^2+3n+2}$.

Since $p(n) + 3(n^2 + 3n + 2) = p(n + 1)$ (easily checked), we obtain

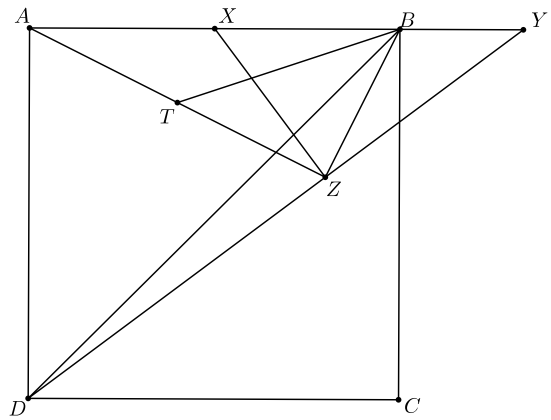
$$a'^3+b'^3+c'^3+d'^3 = 10^{3(n^2+3n+2)}(a^3+b^3+c^3+d^3) = 10^{p(n)+3(n^2+3n+2)},$$

that is, $a'^3 + b'^3 + c'^3 + d'^3 = 10^{p(n+1)}$. This completes the induction step and the proof.

Also solved by *the proposer.*

E–117. *Proposed by Mihaela Berindeanu, Bucharest.* Let $ABCD$ be a square. If X is the midpoint of the side AB, Y is taken on the extension of side AB, so that $BY = AB/3$, Z is the foot of the perpendicular drawn from X to DY and T is the midpoint of AZ, then show that $\angle TBA = \angle DBZ$.

Solution 1 by Michel Bataille, Rouen, France. Let a be the side of the square $ABCD$. Since $XA \perp AD$ and $XZ \perp ZD$, the points A and Z are on the circle with diameter XD.



Scheme for solving Problem E-117

It follows that $YZ \cdot YD = YX \cdot YA = \frac{5a}{6} \cdot \frac{4a}{3} = \frac{10a^2}{9}$ and therefore $YZ = \frac{2a}{3}$ (since the relation $YD^2 = YA^2 + AD^2$ provides $YD = \frac{5a}{3}$). From $XZ^2 = XY^2 - YZ^2$, we then deduce that $XZ = \frac{a}{2}$. Note that $ZD = YD - YZ = a$ and that $XZ = XA = XB$ so that the triangle AZB is right-angled at Z.

We also have $ZB = \frac{a}{\sqrt{5}}$ (readily obtained from Stewart's relation

$$BY \cdot ZX^2 - YX \cdot ZB^2 + XB \cdot ZY^2 - BY \cdot YX \cdot XB = 0),$$

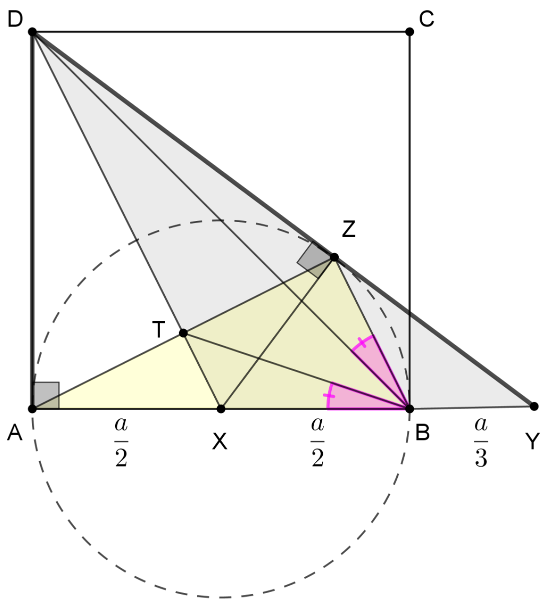
$AZ = \sqrt{AB^2 - BZ^2} = \frac{2a}{\sqrt{5}}$ and $BT = a\sqrt{\frac{2}{5}}$ (since the median BT satisfies $4BT^2 = 2\left(a^2 + \frac{a^2}{5}\right) - \frac{4a^2}{5} = \frac{8a^2}{5}$). The result now follows from $\cos\angle TBA = \cos\angle DBZ$:

$$\cos\angle TBA = \frac{AB^2 + BT^2 - AT^2}{2AB \cdot BT} = \frac{a^2 + (2a^2/5) - (a^2/5)}{2a \cdot a\sqrt{2/5}} = \frac{3}{\sqrt{10}}$$

and

$$\cos\angle DBZ = \frac{BD^2 + BZ^2 - ZD^2}{2BD \cdot BZ} = \frac{2a^2 + (a^2/5) - a^2}{2a\sqrt{2} \cdot (a/\sqrt{5})} = \frac{3}{\sqrt{10}}.$$

Solution 2 by the proposer. Denote $AB = BC = CD = AD = a \Rightarrow XY = \dfrac{a}{2} + \dfrac{a}{3} = \dfrac{5a}{6}$ and $AY = a + \dfrac{a}{3} = \dfrac{4a}{3}$



Scheme for solving Problem E-117

Applying the Pythagorean Theorem in $\triangle DAY$:

$$DY^2 = a^2 + \left(a + \frac{a}{3}\right)^2 = a^2 + \frac{16a^2}{9} = \frac{25a^2}{9} \Rightarrow DY = \frac{5a}{3}$$

Calculate the area $\triangle DXY$ in two ways to find the length XZ:

$$\begin{cases} A(DXY) = \dfrac{XY \cdot AD}{2} = \dfrac{5a^2}{6} \\ \\ A(DXY) = \dfrac{DY \cdot XZ}{2} = \dfrac{5a}{6} \cdot XZ \end{cases} \Rightarrow \frac{5a}{6} \cdot XZ = \frac{5a^2}{12} \Rightarrow XZ = \frac{a}{2} \Rightarrow XZ = AX = XB = \frac{a}{2}$$

So, X is the circumcenter of the right triangle ABZ.

$$\begin{cases} AD \perp AB \\ \\ DZ \perp XZ \end{cases} \Rightarrow AD \text{ and } DZ \text{ are tangents to the circumcircle of } \triangle ABZ$$

In $\triangle ABZ$, BT is the median from B, BD is the symedian from $B \Rightarrow BT$, BD are isogonal lines, so

$$\angle TBA = \angle DBZ$$

Solution 3 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Since BT is the median from B in triangle ABZ, the equality to be proved is equivalent to establish the fact que BD is the $B-$symmedian in this triangle.

Let BD intersects AZ, XZ at U, V, respectively (FIGURE 1).

We apply the Menelaus's theorem to the two triads of points UVB, DVB on the sides of the two triangles AXZ, XYZ obtaining

$$\frac{AB}{BX} \cdot \frac{XV}{VZ} \cdot \frac{ZU}{UA} = 1, \qquad \frac{XB}{BY} \cdot \frac{YD}{DZ} \cdot \frac{ZV}{VX} = 1 \tag{1}$$

where $\frac{AB}{BX} = 2$ and $\frac{XB}{BY} = \frac{XB}{AB} \cdot \frac{AB}{BY} = \frac{1}{2} \times 3 = \frac{3}{2}$.

After multiplying the expressions (1) together and doing a modest amount of cancellation, we are left with

$$\frac{ZU}{UA} = \frac{1}{3} \cdot \frac{DZ}{DY}. \tag{2}$$

By the Pythagorean theorem, applied to $\triangle DAY$, in which $\frac{AD}{AY} = \frac{AB}{AY} = \frac{3}{4}$, we obtain

$$DY : YA : AD = 5 : 4 : 3.$$

Now, as $\triangle DAY$ and $\triangle XYZ$ are similar, we have $DY : YA : AD = XY : YZ : ZX$ and then

$$XY : YZ : ZX = 5 : 4 : 3,$$

giving $\frac{ZX}{XY} = \frac{AY}{DY} = \frac{4}{5}$ and

$$\frac{DZ}{DY} = 1 - \frac{ZY}{DY} = 1 - \frac{ZY}{XY} \cdot \frac{XY}{AY} \cdot \frac{AY}{DY} = 1 - \frac{4}{5} \cdot \frac{5}{8} \cdot \frac{4}{5} = \frac{3}{5}.$$

Hence, from equation (2), we get

$$\frac{ZU}{UA} = \frac{1}{5}. \tag{3}$$

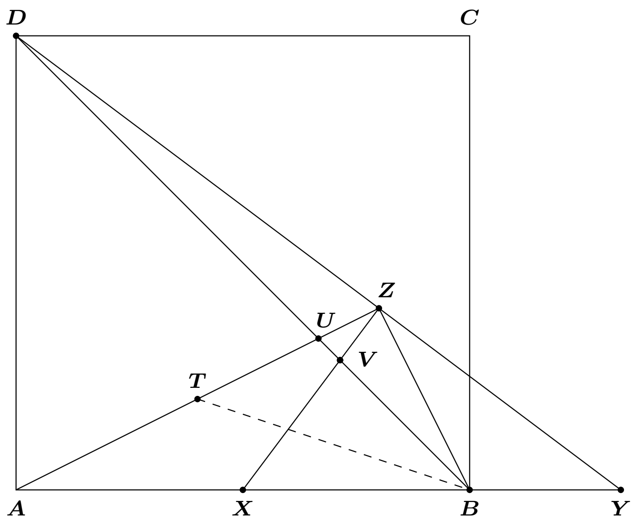


FIGURE 1.

On the other hand (FIGURE 2), $\triangle ZXB$ is isosceles with

$$ZX = \left(\frac{3}{5} XY\right) = XB.$$

Let M be the midpoint of BZ and P the foot of the perpendicular from Z to XB.

From similar right-angled triangles ZPB and XMB,

$$\frac{XB}{BM} = \frac{BZ}{PB},$$

from which we obtain (since $BM = \frac{1}{2}BZ$ and $XB = \frac{1}{2}AB$),

$$BZ^2 = AB \cdot PB$$

and

$$\left(\frac{BZ}{AB}\right)^2 = \frac{PB}{AB} = \frac{PY - BY}{AB} = \frac{\frac{(YZ)^2}{XY} - BY}{AB} = \left(\frac{YZ}{XY}\right)^2 \cdot \frac{XY}{AB} - \frac{BY}{AB}$$
$$= \left(\frac{4}{5}\right)^2 \cdot \frac{5}{6} - \frac{1}{3} = \frac{1}{5}. \tag{4}$$

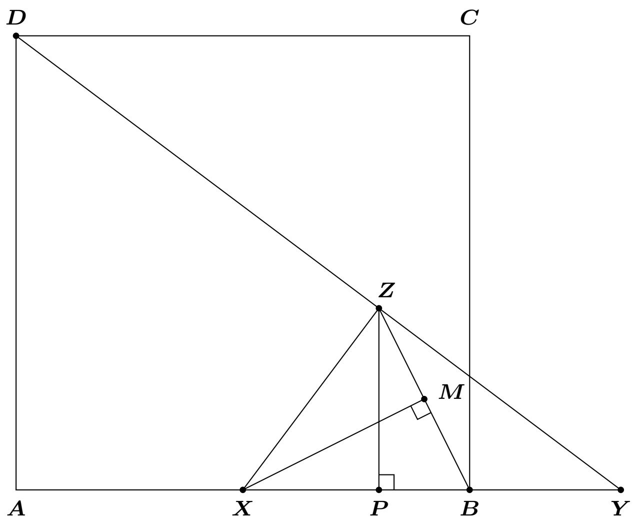


FIGURE 2.

From (3) and (4),

$$\left(\frac{BZ}{AB}\right)^2 = \frac{ZU}{UA},$$

i.e., BU divides divides the opposite side ZA in the ratio of the squares of the adjacent sides, making BU the B-symmedian of $\triangle ABZ$. Since BT is the B-median in this triangle, then BT and BU are equally inclined to the arms of $\angle B$. The conclusion follows.

Solution 4 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. We consider a Cartesian rectangular coordinate system with the unity of measure the same along both coordinate axes.

We place the vertices A, B, C of the given square at convenient points, A at $(0,0)$, B at $(1,0)$, C at $(1,1)$.

The coordinates of D then are $(0,1)$, the coordinates of X are $\left(\frac{1}{2},0\right)$, the coordinates of Y are $\left(\frac{4}{3},0\right\}$ and those of Z are $\left(\frac{4}{5},\frac{2}{5}\right)$.

Let the two lines BD and AZ intersect at U. By solving simultaneously the equations

$$BD : x + y = 1 \qquad AZ : x - 2y = 0,$$

we find $U\left(\frac{2}{3},\frac{1}{3}\right)$.

According to the formula for the distance between two points, we get $\overline{AU} = \frac{\sqrt{5}}{3}$, $\overline{UZ} = \frac{\sqrt{5}}{15}$, $\overline{BZ} = \frac{1}{\sqrt{5}}$.

Hence

$$\frac{AU}{UZ} = 5 \qquad \text{and} \qquad \frac{AB}{BZ} = (\text{since } AB = 1) = \sqrt{5}.$$

Thus

$$\frac{AU}{UZ} = \frac{AB^2}{BZ^2},$$

i.e., BU divides divides the opposite side ZA in the ratio of the squares of the adjacent sides, making BU the B-symmedian of $\triangle ABZ$. Since BT is the B-median in this triangle, then BT and BU are equally inclined to the arms of $\angle B$. The conclusion follows.

E–118. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Determine the integers $n \geq 0$ for which there exists a real number $a > 0$ such that

$$(a+11)^n + (a+13)^n + (a+17)^n = (a+12)^n + (a+14)^n + (a+15)^n.$$

Solution 1 by Michel Bataille, Rouen, France. Obviously, $n = 0$ and $n = 1$ are solutions. We show that there are no other solutions.

Suppose that $n \geq 2$ and assume that for some $a > 0$, we have

$$a_7^n - a_5^n - (a_2^n - a_1^n) - (a_4^n - a_3^n) = 0$$

where, for short, $a_i = a + 10 + i \ \ (i = 1, 2, 3, 4, 5, 7)$.

Then, since $x^n - y^n = (x - y) \sum_{k=0}^{n-1} x^{n-1-k} y^k$, we obtain

$$2 \sum_{k=0}^{n-1} a_7^{n-1-k} a_5^k - \sum_{k=0}^{n-1} a_2^{n-1-k} a_1^k - \sum_{k=0}^{n-1} a_4^{n-1-k} a_3^k = 0,$$

that is,

$$\sum_{k=0}^{n-1} \left(a_7^{n-1-k} a_5^k - a_2^{n-1-k} a_1^k\right) + \sum_{k=0}^{n-1} \left(a_7^{n-1-k} a_5^k - a_4^{n-1-k} a_3^k\right) = 0. \quad (1)$$

However, since $a_7 > a_2 > 0, a_5 > a_1 > 0$ and $n \geq 2$, we have $a_7^{n-1-k} a_5^k > a_2^{n-1-k} a_1^k$ for $k = 0, 1, \ldots, n-1$ and the first sum in (1) is a positive number. Similarly, the second sum is positive and we have reached a contradiction. This completes the proof.

Solution 2 by the proposer. For $n = 0$ we have $3 = 3$ and for $n = 1$ yields $3a + 41 = 3a + 41$. For $n \geq 2$, we consider the increasing convex function $f : [0, +\infty) \to \mathbb{R}$ defined by $f(x) = x^n$. Applying Jensen's inequality for all $x_1, x_2 \in [0, +\infty)$ with $x_1 \neq x_2$, we have

$$\frac{f(x_1) + f(x_2)}{2} > f\left(\frac{x_1 + x_2}{2}\right).$$

Since $a > 0$, then

$$\begin{aligned}
\frac{(a+11)^n+(a+13)^n}{2} &> \left(\frac{a+11+a+13}{2}\right)^n=(a+12)^n,\\
\frac{(a+13)^n+(a+17)^n}{2} &> \left(\frac{a+13+a+17}{2}\right)^n=(a+15)^n,\\
\frac{(a+11)^n+(a+17)^n}{2} &> \left(\frac{a+11+a+17}{2}\right)^n=(a+14)^n.
\end{aligned}$$

Adding up, we get

$$(a+11)^n+(a+13)^n+(a+17)^n > (a+12)^n+(a+14)^n+(a+15)^n.$$

So, the answer is $n = 0$ and $n = 1$.

Also solved by *Arkady Alt, San Jose, California, USA.*

Easy–Medium Problems

EM–113. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* The equation $x^3 + Ax - B = 0$ has three real roots a, b, c. Determine the integers A and B with $AB < 0$ for which $a^6 + b^6 + c^6 = 277$.

Solution 1 by Michel Bataille, Rouen, France. Suppose that a, b, c are the roots of $x^3 + Ax - B = 0$ where A, B are integers such that $AB < 0$ and that $a^6 + b^6 + c^6 = 277$.

We have $a+b+c = 0, ab+bc+ca = A, abc = B$, hence $a^2+b^2+c^2 = (a + b + c)^2 - 2(ab + bc + ca) = -2A$ and

$$\begin{aligned} a^6 + b^6 + c^6 &= (B - aA)^2 + (B - bA)^2 + (B - cA)^2 \\ &= 3B^2 + A^2(a^2 + b^2 + c^2) - 2AB(a + b + c) \\ &= 3B^2 - 2A^3. \end{aligned}$$

Thus, the relation $277 + 2A^3 = 3B^2$ holds. Also we must have $A < 0$ since otherwise the function $x \mapsto x^3 + Ax - B$ would be strictly increasing and the given equation would have non-real roots. From $-2A^3 = 277 - 3B^2 < 277$, we then deduce that $A \in \{-1, -2, -3, -4, -5\}$. However, it is readily checked that $277 + 2A^3$ cannot be written as $3B^2$ for some integer B if $A \in \{-1, -2, -3, -4\}$, while $277 + 2(-5)^3 = 3 \cdot 3^2$. It follows that the only candidate for (A, B) is $(-5, 3)$.

Conversely, a quick study of $p(x) = x^3 - 5x - 3$ shows that the equation $p(x) = 0$ has three real roots a, b, c and, since $a+b+c = 0, ab+bc+ca = -5, abc = 3$, we have $a^6+b^6+c^6 = 3\cdot9+2\cdot125 = 277$.

We conclude that $(-5, 3)$ is the only solution for (A, B).

Solution 2 by the proposer. Since a, b, and c are the real roots of $x^3 + Ax - B = 0$, we construct a sequence $d_n = a^n + b^n + c^n$ for each nonnegative integer n. Since $a^3 = -A \cdot a + B$, then multiplying by a^n, we get $a^{n+3} = -Aa^{n+1} + Ba^n$ and $d_{n+3} =$

$-Ad_{n+1} + Bd_n$ for each $n \geq 0$. Using $(x-a)(x-b)(x-c) = 0$, we have $d_1 = a+b+c = 0$, $ab+bc+ca = A$, and $abc = B$. Since $d_0 = 3$, we may compute

$$d_2 = a^2 + b^2 + c^2, \quad d_3 = 3B, \quad d_4 = -Ad_2,$$

$$d_5 = -3AB + Bd_2, \quad \text{and} \quad d_6 = 140 = A^2 d_2 + 3B^2.$$

Note that in the computation of d_3 we have used the well-known identity

$$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

We also have $d_1^2 = d_2 + 2A = 0$, so $d_2 = -2A \geq 0$ from which $A \leq 0$ follows. But, from $AB < 0$ we conclude that $A < 0$ and $B > 0$. Thus $277 = -2A^3 + 3B^2$ with integers $A < 0$ and $B > 0$. Solving the quadratic equation, we get $B = \pm\sqrt{6A^3 + 861}/3$. The possible integer values for A are $-1, -2, -3, -4, -5$ and only for $A = -5$ we get the integers $B = \pm 3$. Since $B > 0$, then the unique solution is $(A, B) = (-5, 3)$.

Also solved by *Henry Ricardo, Westchester Area Math Circle, Purchase, New York, USA.*

EM–114. *Proposed by Michel Bataille, Rouen, France.* Let P be a point on the circumcircle of the triangle ABC and let A', B', C' be its orthogonal projections onto the lines BC, CA, AB, respectively. Prove that

$$\frac{B'C'^2 \cot A + C'A'^2 \cot B + A'B'^2 \cot C}{BC^2 \cot A + CA^2 \cot B + AB^2 \cot C} = \frac{1}{2}.$$

Solution by the proposer. Let $a = BC, b = CA, c = AB$ and let F be the area of ΔABC and R its circumradius. We have

$$\begin{aligned} F = \frac{abc}{4R} &= \frac{8R^3 \sin A \sin B \sin C}{4R} \\ &= \frac{R^2(\sin 2A + \sin 2B + \sin 2C)}{2} \\ &= \frac{R(a\cos A + b\cos B + c\cos C)}{2}. \end{aligned}$$

We readily deduce that $a^2 \cot A + b^2 \cot B + c^2 \cot C = 4F$ (note that $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$.)

On the other hand, B', C' being on the circle with diameter PA, we have $PA = \frac{B'C'}{\sin(\angle B'AC')} = \frac{B'C'}{\sin A}$ (since $\angle B'AC' = \angle BAC$ or $180^\circ - \angle BAC$). It follows that $B'C'^2 \cot A = PA^2 \sin^2 A \cdot \frac{\cos A}{\sin A} = \frac{1}{2}(PA^2 \sin 2A)$. Similar results hold for $C'A'^2$ and $A'B'^2$ so that

$$B'C'^2 \cot A + C'A'^2 \cot B + A'B'^2 \cot C = \frac{X}{2}$$

where $X = PA^2 \sin 2A + PB^2 \sin 2B + PC^2 \sin 2C$. The problem now amounts to proving that $X = 4F$.

The circumcenter O of the triangle is known to be the center of masses of A, B, C with respective masses $\sin 2A, \sin 2B, \sin 2C$. Using Leibniz's formula, it follows that

$$X = mPO^2 + OA^2 \sin 2A + OB^2 \sin 2B + OC^2 \sin 2C$$

(where $m = \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$) and therefore $X = 8R^2 \sin A \sin B \sin C = 4F$, as desired.

EM–115. *Proposed by Toyesh Prakash Sharma (Student) Agra College, Agra, India.* Show that for any $n \geq 1$, it holds that

$$F_n^{\frac{1}{F_n}}\left(\frac{1}{F_n}\right)^{F_n} + L_n^{\frac{1}{L_n}}\left(\frac{1}{L_n}\right)^{L_n} \geq 2F_{n+1}^{\frac{1}{F_{n+1}}}\left(\frac{1}{F_{n+1}}\right)^{F_{n+1}},$$

where F_n and L_n are the n^{th} Fibonacci and Lucas number, respectively.

Solution by Michel Bataille, Rouen, France. The inequality is readily checked for $n = 1, 2, 3$. If $n \geq 4$, then F_n and L_n are in $[e, \infty)$ and the function $f(x) = x^{\frac{1}{x} - x}$ is convex on this interval (see proof below). Using $F_n + L_n = 2F_{n+1}$ for all n (easily proved by induction), it follows that

$$f(F_n) + f(L_n) \geq 2f\left(\frac{F_n + L_n}{2}\right) = 2f(F_{n+1}),$$

which is the required inequality. To prove that f is convex on $[e, \infty)$, we calculate $f''(x)$ and show that $f''(x) \geq 0$ for $x \geq e$. Indeed, the computation gives

$$f'(x) = \left(\frac{1}{x^2} - \frac{\ln x}{x^2} - \ln x - 1\right) f(x)$$

$$f''(x) = \left((\ln x)^2 \left(1 + \frac{1}{x^2}\right)^2 + \frac{(2\ln x)(x^4 + x - 1)}{x^4} + \frac{p(x)}{x^4}\right) f(x)$$

where $p(x) = x^4 - x^3 - 2x^2 - 3x + 1$. Since $p(x) = (x-1)^4 + 3x^2\left(x - \frac{8}{3}\right) + x > 0$ for $x \geq e$ (note that $e > \frac{8}{3}$), we see that $f''(x) > 0$ for $x \geq e$.

Also solved by *Henry Ricardo, Westchester Area Math Circle, NY, USA; José Luis Díaz-Barrero Barcelona, Spain, and the proposer.*

EM–116. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all divisors of a positive integer n. Find all n, such that $k \geq 6$ and

$$45({d_4}^2 + {d_6}^2) = 2n^2.$$

Solution by the proposer. First note that $45 = 3^2 \cdot 5$ is a canonical representation of 45. It is clear that 45 divide n^2, so $3 \mid n$ and $5 \mid n$. Since n is divisible by 3 and 5, it is also divisible by 15, hence $d_i = 15$, for some $i \geq 3$.

Case 1: $2 \mid n$. Then $d_2 = 2, d_3 = 3$. Since n is divisible by 2 and 3, it is also divisible by 6.

It is clear that $8 \mid 2n^2 = 45({d_4}^2 + {d_6}^2)$, hence d_4 and d_6 are even numbers. Since $d_4 \leq 5$ and $2 \mid d_4$, hence $d_4 = 4, d_5 = 5, d_6 = 6$ and $2n^2 = 45({d_4}^2 + {d_6}^2) = 45(4^2 + 6^2) = 2340$, $n^2 = 1170$, which is not a perfect square.

Case 2: $2 \nmid n$. Then $d_2 = 3, d_3 = 5$.

The possible values of (d_4, d_5) are $(7, 9)$, $(9, 15)$, $(9, p)$, $(p, 15)$, $(15, 25)$, $(15, p)$, or (p, q), where p, q are prime numbers.

If $d_6 = p$, where p is a prime number, $2n^2 = 45({d_4}^2 + p^2)$. Since $p \mid 2n^2 = 45({d_4}^2 + p^2)$, then $p \mid 45{d_4}^2$, which is impossible for prime $p > d_4 > 5$, hence d_6 is not a prime number.

(1). If $d_4 = 7, d_5 = 9 = 3^2$ and d_6 not a prime, then $d_6 = 15$. Thus $n^2 = 45({d_4}^2 + {d_6}^2)/2 = 45(7^2 + 15^2)/2 = 6165$, which is not a perfect square and this is not a solution.

(2). $d_4 = 9 = 3^2, d_5 = 15 = 3 \cdot 5$. The possible values of d_6 are $25 = 5^2$, $27 = 3^3$.

If $d_6 = 25$, $n^2 = 45({d_4}^2 + {d_6}^2)/2 = 45(9^2 + 25^2)/2 = 6165$, which is not a perfect square and this is not a solution.

If $d_6 = 27$, $n^2 = 45({d_4}^2 + {d_6}^2)/2 = 45(9^2 + 27^2)/2 = 18225 = 135^2$ and direct verification shows that $n = 135$ is indeed a solution.

(3). $d_4 = 9, d_5 = p$, where p is a prime number. The possible values of d_6 are $15 = 3 \cdot 5$, q, where q is a prime number.

If $d_6 = 15$, $n^2 = 45(9^2 + 15^2)/2 = 6885$, which is not a perfect square and this is not a solution.

(4). $d_4 = p, d_5 = 15$, where p is a prime number, $5 < p < 15$. The possible values of d_6 are 25, $3p$.

If $d_6 = 25$, then $2n^2 = 45({d_4}^2 + {d_6}^2) = 45(p^2 + 25^2)$. Since $p \mid 2n^2 = 45(p^2 + 25^2)$, then $p \mid 45 \cdot 25^2$, which is impossible for prime $p > 5$.

If $d_6 = 3p$, where $p = d_4$ is a prime number, $5 < p < 15$, then $2n^2 = 45({d_4}^2 + {d_6}^2) = 45(p^2 + 9p^2) = 450p^2 = 2(15p)^2$, so $n = 15p$, where $p \in \{7, 11, 13\}$. Direct verification shows that $n = 105 = 7 \cdot 15, n = 165 = 11 \cdot 15, n = 195 = 13 \cdot 15$ are indeed solutions.

(5). $d_4 = 15, d_5 = 25$. The possible values of d_6 are $75 = 3 \cdot 25$, or q, where q is a prime number.

If $d_6 = 75$, $n^2 = 45(15^2 + 75^2)/2 = 131625$, which is not a perfect square and this is not a solution.

(6). $d_4 = 15, d_5 = p$, where p is a prime number, $p > 15$. The possible values of d_6 are 25, $3p$, or q, where q is a prime number.

If $d_6 = 25$, $n^2 = 45({d_4}^2 + {d_6}^2)/2 = 45(15^2 + 25^2)/2 = 19125$, which is not a perfect square and this is not a solution.

If $d_6 = 3p$, then $2n^2 = 45({d_4}^2 + {d_6}^2) = 45(15^2 + 9p^2) = 405(25 + p^2)$. Since $p \mid 2n^2 = 405(25 + p^2)$, then $p \mid 405 \cdot 25$, which is impossible for prime $p > 15$.

(7). $d_4 = p, d_5 = q$, where p, q are prime numbers. The only possible values of d_6 is 15.

If $d_6 = 15$, $n^2 = 45({d_4}^2 + {d_6}^2)/2 = 45(p^2 + 15^2)/2$. Since $p \mid 2n^2 = 45(p^2 + 15^2)$, then $p \mid 45 \cdot 15^2$, which is impossible for prime $p > 5$.

In summary, the only solutions are $n = 105$, $n = 135$, $n = 165$ and $n = 195$.

EM–117. *Proposed by José Luis Díaz–Barrero, Barcelona, Spain.* Let F_n be the n^{th} Fibonacci number defined by $F_1 = 1, F_2 = 1$, and for all $n \geq 3$, $F_{n+1} = F_n + F_{n-1}$. Prove that for each positive integer n there is a Fibonacci number ending in at least n zeros.

Solution 1 by Michel Bataille, Rouen, France. For $n \geq 1$, let $s(n) = 15 \cdot 10^{n-1}$. We show that $F_{s(n)}$ is a multiple of 10^n so that $F_{s(n)}$ ends with at least n zeros. The proof is by induction. First, we have $F_{s(1)} = F_{15} = 610$, a multiple of 10. Then, assume that $F_{s(n)} \equiv 0 \pmod{10^n}$ holds for some positive integer n. We shall use the following known result: for each positive integer m, $\frac{F_{5m}}{5F_m}$ is an integer congruent to 1 modulo 10 (see **[1]**).

With $m = 2 \cdot 15 \cdot 10^{n-1}$, we deduce that $F_{s(n+1)} = 5F_{2s(n)}(1 + 10k)$ for some positive integer k. Since $F_{2m} = L_m F_m$ where L_m denote the mth Lucas number, we obtain

$$F_{s(n+1)} = 5L_{s(n)}F_{s(n)}(1 + 10k) = 5L_{s(n)} \cdot \ell \cdot 10^n(1 + 10k)$$

where $\ell \in \mathbb{N}$. Now, $L_0 = 2$ and for $m \geq 0$, $L_{3m+3} = L_{3m+2} + L_{3m+1} = L_{3m} + 2L_{3m+1} \equiv L_{3m} \pmod 2$, hence any Lucas number of the form L_{3m} is even. As a result, $F_{s(n+1)} = 10r \cdot \ell \cdot 10^n(1 + 10k)$ for some integer r and $F_{s(n+1)}$ is a multiple of 10^{n+1}. This completes the induction step and the proof.

[1] Solution to Problem 11968, *The American Mathematical Monthly*, Vol. 126, No 1, January 2019, p. 85

Solution 2 by the proposer. Since we have to see that a positive integer (Fibonacci number) ends in at least n zeros, then seems to be appropriate to work $\pmod{10^n}$. We consider the pairs $(F_k, F_{k+1}) \pmod{10^n}$. Observe that there can only be 10^{2n} distinct ones, Thus, by the Pigeon Hole Principle, among the first $10^{2n} + 1$ there must be two which coincide. Suppose they are (F_i, F_{i+1}) and (F_j, F_{j+1}) with $i < j$, then $F_i \equiv F_j \equiv a \pmod{10^n}$ and $F_{i+1} \equiv F_{j+1} \equiv b \pmod{10^n}$ with $0 \leq a, b < 10^n$. By the recursive definition of Fibonacci numbers, we have $F_{i-1} = F_{i+1} - F_i$ and $F_{j-1} = F_{j+1} - F_j$. Thus, $F_{i-1} = F_{i+1} - F_i \equiv b - a \pmod{10^n}$ and $F_{j-1} = F_{j+1} - F_j \equiv b - a \pmod{10^n}$ from which we conclude that $(F_{i-1}, F_i) \equiv (b - a, a) \pmod{10^n}$ and $(F_{j-1}, F_j) \equiv (b - a, a) \pmod{10^n}$ and the pairs (F_{i-1}, F_i) and (F_{j-1}, F_j) also coincide. We can repeat this argument until the first one is $(F_1, F_2) = (1, 1)$ and the second one is (F_r, F_{r+1}) with $r > 1$. Thus F_r and F_{r+1} are both congruent to 1 modulo 10^n. Then the number $F_{r-1} = F_{r+1} - F_r \equiv 0 \pmod{10^n}$ and it ends in at least n zeros.

Comment. For $n = 1$ pairs $(F_1, F_2) = (1, 1)$ and $(F_{61}, F_{62}) = (2504730781961, 4052739537881)$ coincide $\pmod{10}$ and $F_{60} \equiv 0 \pmod{10}$.

EM–118. *Proposed by Goran Conar, Varaždin, Croatia.* The inradius of triangle ABC is $r = 1$. Prove that

$$\sum_{cyclic} \frac{1}{r_a + r_b}\left(1 + \frac{r_b}{r_c}\right)\left(1 + \frac{r_a}{r_c}\right) \geq 2,$$

where r_a, r_b, r_c are their exradii. When does equality occur?

Solution 1 by Michel Bataille, Rouen, France. With the usual notations, we have

$$\tan\frac{A}{2}=\frac{r}{s-a}=\frac{r_a}{s},\ \tan\frac{B}{2}=\frac{r}{s-b}=\frac{r_b}{s},\ \tan\frac{C}{2}=\frac{r}{s-c}=\frac{r_c}{s},$$

hence $r_ar_br_c=\dfrac{r^3s^3}{(s-a)(s-b)(s-c)}=rs^2$ and

$$r_a+r_b=rs\Big(\frac{1}{s-a}+\frac{1}{s-b}\Big)=\frac{rsc}{(s-a)(s-b)}=\frac{cs}{r_c}$$

(since $rs=\sqrt{s(s-a)(s-b)(s-c)}$ by Heron's formula). It follows that

$$\begin{aligned}\frac{1}{r_a+r_b}\Big(1+\frac{r_b}{r_c}\Big)\Big(1+\frac{r_a}{r_c}\Big)&=\frac{(r_a+r_b)(r_b+r_c)(r_c+r_a)}{[r_c(r_a+r_b)]^2}\\&=\frac{abcs^3}{r_ar_br_cc^2s^2}=\frac{abc}{rsc^2}=\frac{4R}{c^2}\end{aligned}$$

(using $abc=4rRs$). From the general inequality $x^2+y^2+z^2\geq xy+yz+zx$ (with equality if and only if $x=y=z$), we obtain

$$\sum_{\text{cyclic}}\frac{1}{r_a+r_b}\Big(1+\frac{r_b}{r_c}\Big)\Big(1+\frac{r_a}{r_c}\Big)\geq\Big(\frac{4R}{ab}+\frac{4R}{bc}+\frac{4R}{ca}\Big)=\frac{4R\cdot 2s}{4rRs}=\frac{2}{r}.$$

The required inequality follows since $r=1$. Equality holds if and only if $\frac{1}{a}=\frac{1}{b}=\frac{1}{c}$ and $r=1$ that is, if and only if ABC is equilateral with side $2\sqrt{3}$.

Solution 2 by the proposer. Let us prove the following lemma:

Lemma. *Let $x,y,z>0$ be real numbers. Then,*

$$\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\geq x+y+z\,.$$

Proof. Without loss of generality we can assume $x\geq y\geq z$ (because of the symmetry). Let's calculate

$$\frac{yz}{x}+\frac{zx}{y}+\frac{xy}{z}\geq x+y+z\quad\Leftrightarrow\quad z\Big(\frac{x}{y}-1\Big)+x\Big(\frac{y}{z}-1\Big)+y\Big(\frac{z}{x}-1\Big)\geq 0$$

$$\Leftrightarrow \quad \frac{z}{y}(x-y)+\frac{x}{z}(y-z)+\frac{y}{x}(z-x)\geq 0$$

$$\Leftrightarrow \quad \frac{z}{y}(x-y)+\frac{x}{z}(y-z)\geq \frac{y}{x}((x-y)+(y-z))\,. \quad (*)$$

While $z\leq y\leq x$ it is $\frac{y}{x}\leq\frac{y}{z}$ and $\frac{y}{x}\leq\frac{y}{y}$ so

$$\frac{y}{x}((x-y)+(y-z))\leq\frac{y}{z}(y-z)+\frac{y}{y}(x-y)\,.$$

But

$$\frac{y}{z}(y-z)+\frac{y}{y}(x-y)\leq\frac{z}{y}(x-y)+\frac{x}{z}(y-z) \quad \Leftrightarrow$$

$$0\leq\frac{1}{z}(x-y)(y-z)-\frac{1}{y}(x-y)(y-z)$$

$$\Leftrightarrow \quad 0\leq\frac{1}{yz}(x-y)(y-z)^2\,.$$

Hence $(*)$ is true, and also inequality from lemma is true (because they are equivalent). This proves lemma. □

We use well known identity $\frac{1}{r}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}$ where r denotes inradius of triangle. Here is $r=1$ so it holds

$$1=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c} \quad \Leftrightarrow \quad r_ar_br_c=r_ar_b+r_br_c+r_cr_a\,.$$

Applying lemma on numbers $x=r_a(r_b+r_c), y=r_b(r_c+r_a), z=r_c(r_a+r_b)$ we get

$$\sum_{cyc}\frac{r_br_c(r_c+r_a)(r_a+r_b)}{r_a(r_b+r_c)}\geq 2(r_ar_b+r_br_c+r_cr_a)=2r_ar_br_c$$

$$\Leftrightarrow \quad (r_ar_br_c)\sum_{cyc}\frac{(r_c+r_a)(r_a+r_b)}{r_a^2(r_b+r_c)}\geq 2r_ar_br_c \quad \Leftrightarrow$$

$$\sum_{cyc}\frac{1}{r_b+r_c}\left(1+\frac{r_c}{r_a}\right)\left(1+\frac{r_b}{r_a}\right)\geq 2\,,$$

which we have to prove. Equality holds if and only if equality in lemma holds, which in our case implies

$$r_a(r_b + r_c) = r_b(r_c + r_a) = r_c(r_a + r_b) \quad \Leftrightarrow \quad r_a = r_b = r_c\,.$$

For our triangle that means it has all side lengths equal, because of known identites:

$$r_a = \frac{P}{s-a}\,, r_b = \frac{P}{s-b}\,, r_c = \frac{P}{s-c}\,,$$

where P denotes area, $s = \frac{a+b+c}{2}$ and a, b, c are side lengths. Hence, equality holds if and only if is triangle equilateral.

Also solved by *Arkady Alt, San Jose, California, USA.*

Medium–Hard Problems

MH–113. *Proposed by José Luis Díaz–Barrero, Barcelona, Spain.* Let M be a subset of $\{1, 2, 3, \ldots, 2023\}$ such that for any three elements (not necessarily distinct) a, b, c of M we have $|a+b-c| > 12$. Determine the largest possible number of elements of M.

Solution by Álvaro De Irízar Larrauri, CFIS, BarcelonaTech (Student), Barcelona, Spain. First, we will prove that

$$|a+b-c| > 12 \quad \forall a, b, c \in M \iff |a - \mathrm{d}(b, c)| > 12 \quad \forall a, b, c \in M,$$

where $\mathrm{d}(b, c) = |b - c|$ is the distance between the elements b and c.

$\Rightarrow$) The first condition implies $|a + c - b| > 12$ $\forall a, b, c \in M$ since we can choose a, b, c in any order. Then,

$$|a - \mathrm{d}(b, c)| = \begin{cases} |a + b - c| > 12 & \text{if } c \geq b \\ |a + c - b| > 12 & \text{if } b > c \end{cases}$$

$\Leftarrow$) Choosing $b = c$ implies $a > 12$ $\forall a \in M$. Then

$$|a + b - c| = \begin{cases} |a - \mathrm{d}(b, c)| > 12 & \text{if } c \geq b \\ |a + \mathrm{d}(b, c)| = a + \mathrm{d}(b, c) \geq a > 12 & \text{if } b > c \end{cases}$$

Therefore, the condition can be interpreted as "any element a in M has to be more than 12 units away from any possible distance between two elements of M". We will define the following sets:

- $A = \{1, 2, 3, \ldots, 2023\}$
- $D = \{d \in A | d = \mathrm{d}(b, c) \quad b, c \in M\}$, the set of possible distances between elements of M
- $N = \{n \in A | \exists d \in D \text{ tq } |n - d| \leq 12\}$, the set of elements in A made "unavailable" by being 12 or less units away from an element in D.

Notice $D \subset N$. Furthermore, since any element in M must be bigger than 12, the maximum distance between any two elements in M can be at most $2023 - 13 = 2010$. Let $p = \max(D)$. Since $p \le 2010$, $[p, p + 12] \subset A$ and, therefore, $[p, p + 12] \subset N$. Because p is the maximum possible distance, $[p + 1, p + 12] \cap D = \emptyset$. Thus, $|N| \ge |D| + 12$.

Suppose $|M| = m$. If we take the distance between the smallest element in M and the rest, we get $m - 1$ distinct distances, so $|D| \ge m - 1$ and $|N| \ge m - 1 + 12 = m + 11$. By construction, $M \cap N = \emptyset$, so $2023 \ge |M| + |N| \ge 2m + 11 \implies m \le 1006$. We have found an upper bound of 1006. We will now find an example where this upper bound is attained. Let $M = \{1018, 1019, \dots, 2023\}$, formed by the last 1006 elements in A. The biggest distance is $2023 - 1018 = 1005$ and the smallest element is 1018. Therefore,

$$|a - \mathrm{d}(b, c)| = a - \mathrm{d}(b, c) \ge 1018 - 1005 = 13 > 12 \quad \forall a, b, c \in M$$

Solution by the proposer. The set $M = \{1018, 1019, \dots, 2023\}$ has 1006 elements and satisfy the required property, since $a, b, c \in M \Rightarrow a + b - c \ge 1018 + 1018 - 2023 = 13$. We will show that this is optimal. Indeed, suppose that M satisfies the conditions of the problem. Let k be the minimal element of M. Then $k = |k + k - k| > 12 \Rightarrow k \ge 13$. Note that for every integer m, the numbers $m, m + k - 12$ cannot both belong to M, since

$$m + k - (m + k - 12) = 12.$$

Claim 1: M contains at most $k - 12$ out of $2k - 24$ consecutive integers. Indeed, we can partition the set $\{m, m + 1, \dots, m + 2k - 23\}$ into $k - 12$ pairs as follows:

$$\{m, m+k-12\}, \{m+1, m+k-11\}, \dots, \{m+k-13, m+2k-25\}.$$

It remains to note that M contains at most one element to each pair.

Claim 2: M contains at most

$$\left\lfloor \frac{t + k - 12}{2} \right\rfloor$$

out of t consecutive integers. Indeed, write $t = q(2k - 24) + r$ with $r \in \{0, 1, 2, \dots, 2k - 25\}$. From the set of the first $q(2k - 24)$

integers, by Claim 1, at most $q(k-12)$ can belong to M. Also by Claim 1, it follows that from the last r integers, at most $\min\{r, k-12\}$ can belong to M. Thus,

- If $r \le k-12$, then at most
$$q(k-12)+r = \frac{t+r}{2} \le \frac{t+k-12}{2}$$
integers belong to M.
- If $r > k-12$, then at most
$$q(k-12)+k-12 = \frac{t-r+2(k-12)}{2} \le \frac{t+k-12}{2}$$
integers belong to M.

By Claim 2, the number of elements of M amongst $k+1, k+2, \ldots, 2023$ is at most

$$\left\lfloor \frac{(2023-k)+(k-12)}{2} \right\rfloor = \left\lfloor \frac{2011}{2} \right\rfloor = 1005.$$

Since amongst $\{1, 2, \ldots, k\}$ only k belongs to M, we conclude that M has at most $1005+1 = 1006$ elements as we initially claimed.

MH–114. *Proposed by Michel Bataille, Rouen, France.* Let r, s be positive integers with $r \le s$. Prove that

$$\sum_{k=r}^{s} \binom{r+s}{k}^2 \le 4 \sum_{k=r}^{s} \binom{r+s-1}{k}^2.$$

Solution by the proposer. We first show that

$$\sum_{k=r}^{s} \binom{r+s}{k}^2 = 2 \sum_{k=r}^{s} \binom{r+s}{k} \binom{r+s-1}{k}. \tag{1}$$

Let

$$\Delta = \sum_{k=r}^{s} \binom{r+s}{k}^2 - \sum_{k=r}^{s} \binom{r+s}{k} \binom{r+s-1}{k-1}.$$

The following calculation

$$\begin{aligned}\Delta &= \sum_{k=r}^{s}\binom{r+s}{k}\left(\binom{r+s}{k}-\binom{r+s-1}{k-1}\right)\\ &= \sum_{k=r}^{n}\binom{r+s}{k}\binom{r+s-1}{k}\\ &= \sum_{k=r}^{s}\binom{r+s}{r+s-k}\binom{r+s-1}{r+s-k-1}\\ &= \sum_{j=r}^{s}\binom{r+s}{j}\binom{r+s-1}{j-1}\end{aligned}$$

shows that $\sum\limits_{k=r}^{s}\binom{r+s}{k}\binom{r+s-1}{k} = \sum\limits_{k=r}^{s}\binom{r+s}{k}\binom{r+s-1}{k-1} = \Delta$ and (1) is obtained.

Now, the Cauchy-Schwarz inequality gives

$$\sum_{k=r}^{s}\binom{r+s}{k}\binom{r+s-1}{k} \le \left(\sum_{k=r}^{s}\binom{r+s}{k}^2\right)^{1/2}\left(\sum_{k=r}^{s}\binom{r+s-1}{k}^2\right)^{1/2}$$

so that (1) leads to

$$\left(\sum_{k=r}^{s}\binom{r+s}{k}^2\right)^{1/2} \le 2\left(\sum_{k=r}^{s}\binom{r+s-1}{k}^2\right)^{1/2}.$$

Squaring yields the desired inequality.

MH–115. *Proposed by José Luis Díaz–Barrero, Barcelona, Spain.* Find a function $f:\mathbb{R}-\{0,\pm1\}\to\mathbb{R}$ that is continuous everywhere and satisfies the equation

$$\frac{1}{x+1}\,f\left(\frac{x}{x+1}\right)+\frac{2}{x+1}\,f(x+1)=1.$$

Solution 1 by Michel Bataille, Rouen, France. Extending the problem, we show that there exists a unique function $f:\mathbb{R}-\{0,1\}\to\mathbb{R}$ such that

$$\frac{1}{x+1}f\left(\frac{x}{x+1}\right)+\frac{2}{x+1}f(x+1)=1$$

for any $x \in \mathbb{R} - \{0, -1\}$. The function $t \mapsto f_0(t) = \frac{1}{9}\left(\frac{2}{t} + \frac{1}{1-t} + 4t - 2\right)$ is the unique solution. A simple calculation gives

$$f_0\left(\frac{x}{x+1}\right) + 2f_0(x+1) = \frac{1}{9}\left[\left(\frac{4x}{x+1} + \frac{2}{x} + x + 1\right) + 2\left(\frac{2}{x+1} - \frac{1}{x} + 4x + 2\right)\right]$$
$$= \frac{1}{9}(9x+9) = x+1,$$

for any $x \in \mathbb{R} - \{0, -1\}$, hence f_0 is a solution.
Conversely, let f be any solution and let $t \neq 0, 1$. Note that $\frac{1}{1-t}$ and $1 - \frac{1}{t}$ are also different from 0 and 1. Set $x = \frac{t}{1-t}$. Then $x \neq 0, -1$ and the equation provides

$$f(t) + 2f\left(\frac{1}{1-t}\right) = \frac{1}{1-t} \tag{1}$$

which holds for any $t \neq 0, 1$. Therefore, we also have

$$f\left(\frac{1}{1-t}\right) + 2f\left(\frac{1}{1 - \frac{1}{1-t}}\right) = \frac{1}{1 - \frac{1}{1-t}},$$

that is,

$$f\left(\frac{1}{1-t}\right) + 2f\left(1 - \frac{1}{t}\right) = 1 - \frac{1}{t}. \tag{2}$$

But (1) gives

$$f\left(\frac{1}{1-t}\right) = \frac{1}{2}\left(\frac{1}{1-t} - f(t)\right) \quad \text{and} \quad f\left(1 - \frac{1}{t}\right) = t - 2f(t)$$

that we carry into (2) to obtain

$$\frac{1}{2}\left(\frac{1}{1-t} - f(t)\right) + 2t - 4f(t) = 1 - \frac{1}{t}.$$

Solving for $f(t)$ shows that $f(t) = f_0(t)$, as desired.

Solution 2 by Arkady Alt, San Jose, California, USA. We have $\frac{1}{x+1} f\left(\frac{x}{x+1}\right) + \frac{2}{x+1} f(x+1) = 1 \iff f\left(\frac{x}{x+1}\right) + 2f(x+1) = x+1$

$h(x) = \dfrac{x}{x+1}, h(h(x)) = \dfrac{\frac{x}{x+1}}{\frac{x}{x+1}+1} = \dfrac{x}{2x+1}, h_3(x) = \dfrac{\frac{x}{2x+1}}{\frac{x}{2x+1}+1} = \dfrac{x}{3x+1}$. Let $t = x+1$ then $x = t-1$ and $f\left(\dfrac{x}{x+1}\right) + 2f(x+1) = x+1$ becomes

(1) $\quad f\left(\dfrac{t-1}{t}\right) + 2f(t) = t$

Let $h(t) := \dfrac{t-1}{t}, h_0(t) := t, h_n = h \circ h_{n-1}, n \in \mathbb{N}$.Then $h_2(t) = (h \circ h_1)(t) =$

$\dfrac{\frac{t-1}{t}-1}{\frac{t-1}{t}} = \dfrac{1}{1-t}$ $, h_3(t) := (h \circ h_2)(t) = \dfrac{\frac{1}{1-t}-1}{\frac{1}{1-t}} = t = h_0(t)$. Thus, **(1)** $\Longleftrightarrow$

(2) $\quad f(h(t)) + 2f(t) = t \iff f \circ h_1 + 2f \circ h_0 = h_0$ and by replacing t with $h_1(t), h_2(t)$ we

obtain $f \circ h_2 + 2f \circ h_1 = h_1, \ , f \circ h_3 + 2f \circ h_2 = h_2 \iff f + 2f \circ h_2 = h_2$.

By exclusion h_2, h_3 in the system $\left\{\begin{array}{c} f \circ h_1 + 2f = h_0 \\ f \circ h_2 + 2f \circ h_1 = h_1 \\ f + 2f \circ h_2 = h_2 \end{array}\right.$ we obtain

$f = \dfrac{1}{9}(4h_0 - 2h_1 \ + h_2)$,that is $f(t) = \dfrac{1}{9}\left(4t - 2 \cdot \dfrac{t-1}{t} + \dfrac{1}{1-t}\right) = \dfrac{4t^3 - 6t^2 + 3t - 2}{9t(t-1)}$.

Thus, $\; f(x) = \dfrac{4x^3 - 6x^2 + 3x - 2}{9x(x-1)}$.

Solution 3 by Álvaro De Irízar Larrauri, CFIS, BarcelonaTech (Student), Barcelona, Spain. For simplicity, we will multiply both sides of the equation by $(x+1)$ to obtain:

$$f\left(\frac{x}{x+1}\right)+2f(x+1)=x+1$$

We will now do the change of variable $x=-\frac{y+1}{y}$, valid for any real $x\neq -1$. We get

$$f\left(\frac{-\frac{y+1}{y}}{-\frac{y+1}{y}+1}\right)+2f\left(-\frac{y+1}{y}+1\right)=-\frac{y+1}{y}+1 \iff$$

$$f(y+1)+2f\left(-\frac{1}{y}\right)=-\frac{1}{y}$$

Doing a similar change of variable $y=-\frac{z+1}{z}$, where $y\neq -1$ (so $x\neq 0$), we obtain

$$f\left(-\frac{z+1}{z}+1\right)+2f\left(-\frac{1}{-\frac{z+1}{z}}\right)=-\frac{1}{-\frac{z+1}{z}} \iff$$

$$f\left(-\frac{1}{z}\right)+2f\left(\frac{z}{z+1}\right)=\frac{z}{z+1}$$

We can express all three equations in terms of the same variable as long as we restrict the domain to not take the values 0 and -1. The equations are also invalid for those values of x which cancel denominators or make the argument of f be -1, 0, or 1. Therefore, for $x\in\mathbb{R}\setminus\{-2,-1,-\frac{1}{2},0,1\}$, we have

$$\begin{cases} f\left(\frac{x}{x+1}\right) & + & 2f(x+1) & = & x+1 \\ f(x+1) & + & 2f\left(-\frac{1}{x}\right) & = & -\frac{1}{x} \\ f\left(-\frac{1}{x}\right) & + & 2f\left(\frac{x}{x+1}\right) & = & \frac{x}{x+1} \end{cases}$$

We get a system of linear equations. We can solve for $f(x+1)$. Substracting twice the third equation from the second one we get

$$f(x+1)-4f\left(\frac{x}{x+1}\right)=-\frac{1}{x}-\frac{2x}{x+1}=-\frac{2x^2+x+1}{x^2+x}$$

And adding 4 times the first one

$$9f(x+1)=-\frac{2x^2+x+1}{x^2+x}+4x+4=\frac{4x^3+6x^2+3x-1}{x^2+x}$$

$$\implies f(x+1) = \frac{4x^3+6x^2+3x-1}{9x^2+9x}$$

Which means the final answer is

$$f(x) = \frac{4(x-1)^3+6(x-1)^2+3(x-1)-1}{9(x-1)^2+9(x-1)} = \frac{4x^3-6x^2+3x-2}{9x^2-9x}$$

Being a quotient of polynomials, it's clearly continuous in $\mathbb{R}\setminus\{0,1\}$ and, therefore, it remains continuous when we restrict its domain to $\mathbb{R}\setminus\{-1,0,1\}$. By construction, it must also satisfy the original equation for $x \in \mathbb{R}\setminus\{-2,-1,-\frac{1}{2},0,1\}$. Since this equation is not valid for -2, -1, $-\frac{1}{2}$, or 0 (because the argument of the function takes a value outside its domain), we only have to check its validity for $x = 1$. We can do so using an argument of continuity:

$$f\left(\frac{1}{2}\right) + 2f(2) = \lim_{x\to 1}\left[f\left(\frac{x}{x+1}\right) + 2f(x+1)\right] = \lim_{x\to 1}(x+1) = 2$$

Where the first equality is justified by the continuity of f and the second one is due to the fact that the equation is valid for any value of x in a sufficiently small entourage of 1.

Solution 4 by the proposer. Multiplying both sides of the given equation by $x+1$, we get the equivalent equation

$$f\left(\frac{x}{x+1}\right) + 2\,f(x+1) = x+1.$$

Putting $t = x+1$, we obtain

$$f\left(\frac{t-1}{t}\right) + 2\,f(t) = t.$$

If we denote by $g(t) = \dfrac{t-1}{t}$, then we have $g(g(t)) = g\left(\dfrac{t-1}{t}\right) = -\dfrac{1}{t-1}$ and $g(g(g(t))) = t$, as can be easily checked. By subsequently substituting these relations into the equation, we obtain the following system

$$\begin{aligned} f\left(\frac{t-1}{t}\right)+2\,f(t) &= t, \\ f\left(\frac{-1}{t-1}\right)+2\,f\left(\frac{t-1}{t}\right) &= \frac{t-1}{t}, \\ f(t)+2\,f\left(\frac{t-1}{t}\right) &= -\frac{1}{t-1}. \end{aligned}$$

Solving it, we find $5(t-2f(t))+f(t)=t+\dfrac{1}{t+1}+\dfrac{2(t-1)}{t}$ from which

$$f(t)=\frac{1}{9}\left(4t-\frac{1}{t-1}-\frac{2(t-1)}{t}\right)=\frac{4t^3-6t^2+3t-2}{9t(t-1)}$$

follows.

MH–116. *Proposed by Andrés Sáez Schwedt, Universidad de León, León, Spain.* Let $ABCD$ be a cyclic quadrilateral such that the segments AC and BD intersect at point E, and the lines AB and CD intersect at point F. The circumcircle of triangle BCE meets the line EF again at point $G \neq E$. Prove that

$$\frac{GB}{GC}=\frac{FB}{FC}.$$

Solution 1 by the proposer. Denote by α the equal angles $\angle DCA = \angle DBA$. Similarly, let $\beta = \angle GBC = \angle GEC$ and $\gamma = \angle BCG = \angle BEG$. Multiple applications of the law of sines allows us to compare the ratios $\frac{FB}{FC}$ and $\frac{GB}{GC}$.

$$\begin{aligned} \frac{GB}{GC} &= \frac{\sin(\angle GCB)}{\sin(\angle GBC)}=\frac{\sin(\gamma)}{\sin(\beta)}, \\ \frac{FB}{FC} &= \frac{FB}{FE}\cdot\frac{FE}{FC}=\frac{\sin(\angle FEB)}{\sin(\angle FBE)}\cdot\frac{\sin(\angle FCE)}{\sin(\angle FEC)} \\ &= \frac{\sin(\gamma)}{\sin(180^o-\alpha)}\cdot\frac{\sin(180^o-\alpha)}{\sin(\beta)}, \end{aligned}$$

both ratios are equal, and we are finished.

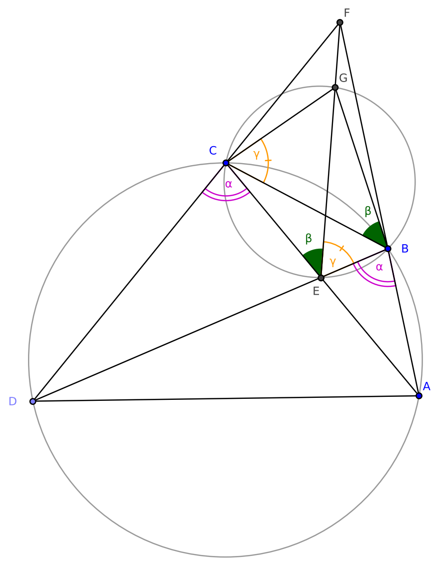


Figure 1: Scheme for solving problem MH-116

Solution 2 by Michel Bataille, Rouen, France. Let Γ be the circle through A, B, C, D and γ be the circumcircle of ΔBCE. Let $\mathbf{I}$ be the inversion with center E such that $\mathbf{I}(C) = A$. Then $\mathbf{I}(B) = D$ and therefore $\mathbf{I}(\gamma)$ is the line AD. It follows that $\mathbf{I}(G)$ is the point H at which AD meets the line EF. Also, if p denotes the power of E with respect to Γ, we have

$$GB = \frac{|p| \cdot HD}{EH \cdot ED} \qquad \text{and} \qquad GC = \frac{|p| \cdot HA}{EH \cdot EA}$$

so that

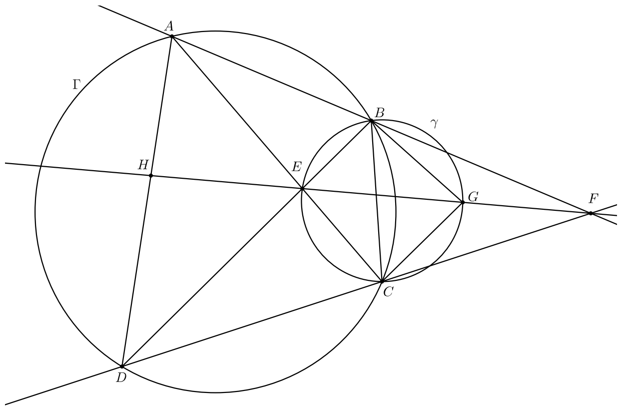
$$\frac{GB}{GC} = \frac{HD}{HA} \cdot \frac{EA}{ED}. \tag{1}$$

Now, we apply Menelaus's theorem, first to ΔADB and then to ΔACD with the same transversal EF and get

$$\frac{HA}{HD} \cdot \frac{ED}{EB} \cdot \frac{FB}{FA} = 1 = \frac{HA}{HD} \cdot \frac{FD}{FC} \cdot \frac{EC}{EA}.$$

By multiplication, we obtain

$$\frac{HA^2}{HD^2} \cdot \frac{ED}{EB} \cdot \frac{EC}{EA} \cdot \frac{FD}{FC} \cdot \frac{FB}{FA} = 1. \tag{2}$$



Scheme for solving problem MH-116.

But we have $ED \cdot EB = EA \cdot EC$, hence $\frac{ED}{EB}\frac{EC}{EA} = \frac{EC^2}{EB^2}$ and $FD \cdot FC = FA \cdot FB$, hence $\frac{FD}{FC}\frac{FB}{FA} = \frac{FB^2}{FC^2}$. With (2), we deduce that $\frac{HA}{HD} \cdot \frac{EC}{EB} \cdot \frac{FB}{FC} = 1$ and therefore

$$\frac{FB}{FC} = \frac{EB}{EC} \cdot \frac{HD}{HA} = \frac{EA}{ED} \cdot \frac{HD}{HA} = \frac{GB}{GC}$$

(using (1)).

MH–117. *Proposed by José Luis Díaz–Barrero, Barcelona, Spain.* Suppose that 2023 distinct points are chosen in the plane and the distances between them are measured. Show that the total number of distances among the given points is at least 32.

Solution 1 by Alberto Espuny Díaz, Universität Heidelberg, Heidelberg, Germany. Let us assume, for a contradiction, that there is a choice of 2023 distinct points in the plane such that the number of distances among the points is $k \leq 31$. Fix one of the points v and partition all other points into k sets based on their distance to v. Since $2022/k > 65$, by the pigeonhole principle, one of these sets must contain at least 66 points; let us call this set P. Moreover, observe that, by definition, all points of P are placed on a circle around v. Now fix any $u \in P$ and, since P is contained in a circle, observe that, for each distance, there cannot be more than 2 points of P at that distance from u. Again by the pigeonhole principle, this means that the points of P themselves create at least 33 distinct distances (in fact, this is true just for their distances to u), which contradicts the fact that $k \leq 31$.

Solution 2 by the proposer. Let V be a set of n points on the Euclidean plane and $\ell_1, \ell_2, \ldots, \ell_k$ all distinct distances between the points. For each $v \in V$, define $d_i(v)$ to be the number of points that are ℓ_i away from v. For each $i = 1, 2, \ldots, k$, denote by T_i the set of pairs $(v, \{a, b\}) \in V \times \binom{V}{2}$ such that both a and b are at distance ℓ_i away from v. Obviously, for each $v \in V$, there are exactly $\binom{d_i(v)}{2}$ unordered pairs $\{a, b\} \in \binom{V}{2}$ that make $(v, \{a, b\}) \in T_i$. For each $\{a, b\} \in \binom{V}{2}$, there are at most two such v's. Therefore, we have the inequality

$$\sum_{v \in V} \binom{d_i(v)}{2} = |T_i| \leq 2\binom{n}{2}.$$

Finally, sum over all $i = 1, 2, \ldots, k$ to obtain

$$\sum_{v \in V} k\binom{\frac{1}{k}\sum_{i=1}^{k} d_i(v)}{2} \leq \sum_{v \in V}\sum_{i=1}^{k}\binom{d_i(v)}{2} \leq 2k\binom{n}{2}.$$

Since $\sum_{i=1}^{k} d_i(v)$ is the total number of points apart from v itself, then we get

$$nk\binom{(n-1)/k}{2} = \sum_{v \in V} k\binom{\frac{1}{k}\sum_{i=1}^{k} d_i(v)}{2} \leq 2k\binom{n}{2},$$

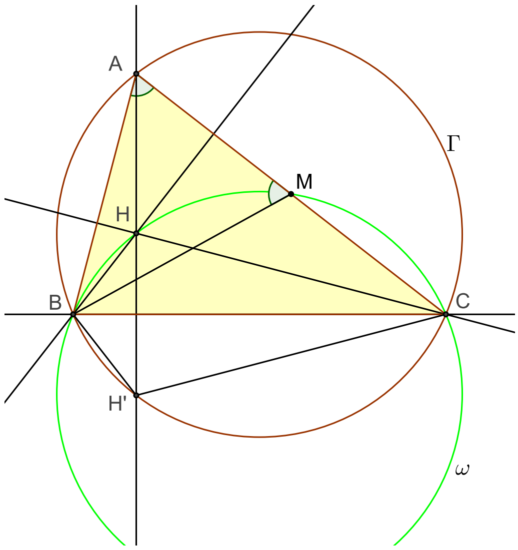
from which it follows that

$$k \geq \frac{\sqrt{8n-7}-1}{4}.$$

In particular, if $n = 2023$ we obtain $k \geq 32$, and the number of distinct distances among the given points is at least 32.

MH–118. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let $BC = a, CA = b, AB = c$ are the side lengths of integer sided non-degenerate triangle ABC with orthocenter H. Let M be the midpoint of AC. Knowing that B, C, H, M are concyclic, find all primitive triples (a, b, c) of positive integers, with the additional property that a, b, c have no positive common divisor other than unity.

Solution by the proposer. Let ω be the circumcircle of triangle BHC and Γ — circumcircle of triangle ABC.



Scheme for solving problem MH-113.

Let H' be the symmetric point of H with respect to BC. It is well known that $H' \in \Gamma$, hence ω is symmetric of Γ with respect to BC.

It is clear that $\angle BAC < 90^\circ$ and $\angle ACB < 90^\circ$ since $M \in \omega$.

$$\angle BHC = \angle BH'C = \begin{cases} 180^\circ - \angle BAC, & \text{if } \angle CBA \leq 90^\circ; \\ \angle BAC, & \text{if } \angle CBA > 90^\circ \end{cases}$$

$$\angle BMC = \left.\begin{cases} \angle BHC, & \text{if } \angle CBA \leq 90^\circ \\ 180^\circ - \angle BHC, & \text{if } \angle CBA > 90^\circ \end{cases}\right\} = 180^\circ - \angle BAC$$

Hence $\angle AMB = 180^\circ - \angle BMC = \angle BAC = \angle BAM$ so $BM = BA = c$.

It is well known that BM is a median in triangle ABC and

$$4BM^2 = 2a^2 - b^2 + 2c^2; \quad 4c^2 = 2a^2 - b^2 + 2c^2.$$

So B, C, H, M are concyclic, if and only if

$$2a^2 - b^2 - 2c^2 = 0. \tag{1}$$

Now we will find all primitive triples (a, b, c) of positive integers with the property (1).

$$b = \sqrt{2(a-c)(a+c)}; \ a > c \geq 1$$

It is clear that $(a-c)(a+c)$ is even and hence b is even, i.e. a, c have the same parity. But if a, c are even, $2 \mid \{a, b, c\}$ and triples (a, b, c) are not primitive. Hence $a = 2a_1 - 1, c = 2c_1 - 1$ for some positive integers a_1, c_1,

$$b = 2\sqrt{2(a_1 - c_1)(a_1 + c_1 - 1)}.$$

Now $2(a_1 - c_1)(a_1 + c_1 - 1)$ must be perfect square, so either $a_1 - c_1 = 2km^2$; $a_1 + c_1 - 1 = kn^2$ or $a_1 - c_1 = km^2$; $a_1 + c_1 - 1 = 2kn^2$ for some positive integers k, m, n.

$$b = 4kmn$$

Case 1. $a_1 - c_1 = 2km^2$; $a_1 + c_1 - 1 = kn^2$

$$\begin{aligned} a_1 &= c_1 + 2km^2 \\ 2c_1 &= 1 + kn^2 - 2km^2 \\ c &= 2c_1 - 1 = k(n^2 - 2m^2) \\ a &= 2a_1 - 1 = k(n^2 + 2m^2) \end{aligned}$$

Hence $k \mid \{a, b, c\}$ and the triples (a, b, c) are primitive, if and only if $k = 1$, n is odd and n, m are relatively prime. From triangle inequality $a + c > b$ hence $2kn^2 > 4kmn$; $n > 2m$.

Examples

n	m	a	b	c	$2a^2 - b^2 - 2c^2$
3	1	11	12	7	0
5	1	27	20	23	0
5	2	33	40	17	0
7	1	51	28	47	0
7	2	57	56	41	0
7	3	67	84	31	0

Case 2. $a_1 - c_1 = km^2;\ a_1 + c_1 - 1 = 2kn^2$

$$\begin{aligned} a_1 &= c_1 + km^2 \\ 2c_1 &= 1 + 2kn^2 - km^2 \\ c &= 2c_1 - 1 = k(2n^2 - m^2) \\ a &= 2a_1 - 1 = k(2n^2 + m^2) \end{aligned}$$

Hence $k \mid \{a, b, c\}$ and the triples (a, b, c) are primitive, if and only if $k = 1$, m is odd and n, m are relatively prime. From triangle inequality $a + c > b$ hence $4kn^2 > 4kmn$; $n > m$.

Examples

n	m	a	b	c	$2a^2 - b^2 - 2c^2$
2	1	9	8	7	0
3	1	19	12	17	0
4	1	33	16	31	0
4	3	41	48	23	0
5	1	51	20	49	0
5	3	59	60	41	0

In conclusion, all primitive triples (a, b, c) of positive integers are $(n^2+2m^2, 4mn, n^2-2m^2)$, $m \geq 1, n > 2m$, n is odd and n, m are relatively prime and $(2n^2 + m^2, 4mn, 2n^2 - m^2)$, $m \geq 1, n > m$, m is odd and n, m are relatively prime.

Advanced Problems

A–113. *Proposed by Marian Ursărescu and Florică Anastase, Romania.* Let $A \in M_2(\mathbb{C})$ such that $\det A = 1$. For all $B \in M_2(\mathbb{C})$ prove that $A^2B - BA^2 = BA^{-2} - A^{-2}B$.

Solution 1 by Michel Bataille, Rouen, France. From the Hamilton-Cayley Theorem and the hypothesis $\det A = 1$, we deduce that A satisfies $A^2 - tA + I_2 = O_2$ (where t is the trace of A). This leads to

$$A^4 + I_2 = (tA - I_2)^2 + I_2 = t^2A^2 - 2tA + 2I_2 = t^2A^2 - 2(tA - I_2)$$

so that $A^4 + I_2 = (t^2 - 2)A^2$.

By multiplication by A^{-2}, we obtain $A^2 + A^{-2} = (t^2 - 2)I_2$ and therefore $(A^2 + A^{-2})B = B(A^2 + A^{-2})$. The required equality follows at once.

Solution 2 by Álvaro De Irízar Larrauri, CFIS, BarcelonaTech (Student), Barcelona, Spain.

$$\text{Let } A^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a, b, c, d \in \mathbb{C}$$

We know that $\det(A^2) = (\det A)^2 = 1^2 = 1$. Now, using Cramer's Rule for inverse matrices we get:

$$A^{-2} = (A^2)^{-1} = \frac{1}{\det(A^2)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

It follows that

$$A^2 + A^{-2} = \begin{pmatrix} a+d & 0 \\ 0 & a+d \end{pmatrix} = (a+d)\mathrm{Id}$$

Since any multiple of the identity matrix clearly commutes with any other matrix, we can conclude

$$(A^2 + A^{-2})B = B(A^2 + A^{-2}) \iff A^2B + A^{-2}B = BA^2 + BA^{-2}$$
$$\iff A^2B - BA^2 = BA^{-2} - A^{-2}B$$

Solution 3 by Moti Levy, Rehovot, Israel. $\det A = 1$ implies that matrix A is similar to $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$, that is

$$A = P\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}P^{-1},$$
$$A^2 = P\begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{bmatrix}P^{-1},$$

where P is invertible matrix.

The problem statement is equivalent to

$$A^2B + A^{-2}B = BA^{-2} + BA^2,$$

or to

$$\left(A^2 + A^{-2}\right)B = B\left(A^2 + A^{-2}\right).$$

Now,

$$\begin{aligned} A^2 + A^{-2} &= P\begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{bmatrix}P^{-1} + P\begin{bmatrix} \lambda^{-2} & 0 \\ 0 & \lambda^2 \end{bmatrix}P^{-1} \\ &= P\begin{bmatrix} \lambda^2 + \lambda^{-2} & 0 \\ 0 & \lambda^2 + \lambda^{-2} \end{bmatrix}P^{-1} = \left(\lambda^2 + \lambda^{-2}\right)I_2. \end{aligned}$$

Hence $A^2 + A^{-2}$ is diagonal matrix which commutes with any matrix $B \in M_2(\mathbb{C})$.

Also solved by *the proposer.*

A–114. *Proposed by Gonzalo Gómez Abejón, Madrid, Spain.* We have an urn with N balls of different colors. Until they are all of the same color, we repeat the following step:

- Select two balls at random, of different colours (if they are the same color we put them back and draw another two until they are of different color).
- Then paint the first ball of the color of the second one, then put them back.

Prove that given an initial set of balls, the average number of steps needed is always an integer, and in particular if we start with N balls of N different colors, it will take an average of $\frac{N(N-1)}{2}$ steps.

Solution 1 by Moti Levy, Rehovot, Israel. Let us assign a number from 1 to n to each one of the n colors.

A set of the balls in the urn may be represented by n-tuple $(N_1, N_2, \cdots, N_n)$, where N_i is the number of balls of color i.

At each step, if the colors of the balls are the same, then they are returned to the urn and this step is not counted. If the colors are different, then the first ball is painted by the color of the second one, and then they are returned to the urn.

Let V_k be the state of the urn, that is V_k is the n-tuple at step k. Thus the initial state is $V_0 = \left(\underbrace{1, 1, \ldots, 1}_{n\ -\text{times}}\right)$, an intermediate state at step t is $V_t = \left(N_1^t, \ldots N_n^t\right)$, and the final state is any permutation of $V_T = \left(n, \underbrace{0, 0, \ldots, 0}_{n-1\ -\text{times}}\right)$.

The following solution is after reference [1].

Now we define a discrete-time stochastic process Φ_t, $0 \leq t \leq T$,

$$\Phi_t := \sum_{k=1}^{n} \left(N_k^t\right)^2, \quad \text{for } V_t = \left(N_1^t, \ldots N_n^t\right). \tag{1}$$

Clearly

$$\Phi_0 = n, \qquad \Phi_T = n^2. \tag{2}$$

The essence of the proof is to show that the discrete-time stochastic process Y_t

$$Y_t := \Phi_t - 2t \tag{3}$$

is a martingale, that is, to show that

$$E[Y_{t+1} | V_t, V_{t-1}, \ldots, V_r] = Y_t, \qquad t \geq r \geq 0. \tag{4}$$

It follows from the urn model that

$$E[Y_{t+1}|V_t, V_{t-1}, \ldots, V_r] = E[Y_{t+1}|V_t],$$

hence we will show that

$$E[Y_{t+1}|V_t] = Y_t, \qquad t \geq r \geq 0 \tag{5}$$

Claim: If V_t is not the final state then

$$E[\Phi_{t+1}|V_t] = \Phi_t + 2. \tag{6}$$

Proof of Claim. Suppose colour i is drawn first and then colour j is drawn ($i \neq j$), then

$$\begin{aligned}\Phi_{t+1} &= \Phi_t + \left(N_i^t - 1\right)^2 - \left(N_i^t\right)^2 + \left(N_j^t + 1\right)^2 - \left(N_j^t\right)^2 \\ &= \Phi_t - 2N_i^t + 2N_j^t + 2.\end{aligned} \tag{7}$$

Now suppose colour j is drawn first and then color i is drawn, then

$$\begin{aligned}\Phi_{t+1} &= \Phi_t + \left(N_j^t - 1\right)^2 - \left(N_j^t\right)^2 + \left(N_i^t + 1\right)^2 - \left(N_i^t\right)^2 \\ &= \Phi_t - 2N_j^t + 2N_i^t + 2.\end{aligned} \tag{8}$$

The probability of the event colour i was drawn first and then colour j was drawn is $\frac{N_i^t}{n}\frac{N_j^t}{n-1}$ and the probability of the event colour j was drawn first and then colour i is $\frac{N_j^t}{n}\frac{N_i^t}{n-1}$.

Hence the two probabilities are equal. Therefore

$$\begin{aligned}&E\left[\Phi_{t+1}|V_t, C^{\{i.j\}}\right] \\ &= E\left[\Phi_{t+1}|V_t, C^{\{i.j\}}\right] \Pr(\text{colour } i \text{ was drawn first}) \\ &+ E[\Phi_{t+1}|V_t] \Pr\left(\text{colour } j \text{ was drawn first}, C^{\{i.j\}}\right) \\ &= \left(\Phi_t - 2N_i^t + 2N_j^t + 2\right)\left(\frac{1}{2}\right) + \left(\Phi_t - 2N_j^t + 2N_i^t + 2\right)\left(\frac{1}{2}\right) \\ &= \Phi_t + 2,\end{aligned}$$

where $C^{\{i.j\}}$ is the event that the colours chosen at step $t+1$ are $\{i.j\}$ regardless which colour was chosen first.

We see that $E\big[\Phi_{t+1}|V_t, C^{\{i,j\}}\big]$ is the same for all $\{i,j\}$, hence

$$E\big[\Phi_{t+1}|V_t, C^{\{i,j\}}\big] = E[\Phi_{t+1}|V_t] = \Phi_t + 2.$$

Claim: The discrete-time stochastic process $Y_t = \Phi_t - 2t$ is a martingale.

Proof of Claim. We have

$$\begin{aligned} E[Y_{t+1}|V_t] &= E[\Phi_{t+1} - 2(t+1)|V_t] \\ &= E[\Phi_{t+1}|V_t] - 2t - 2. \end{aligned} \tag{9}$$

Plugging (6) into (9) we get

$$\begin{aligned} E[Y_{t+1}|V_t] &= (\Phi_t + 2) - 2t - 2 \\ &= \Phi_t - 2t = Y_t, \end{aligned}$$

which shows that the process Y_t is martingale.

Suppose the initial state Y_r of urn is known then $E[Y_r] = \Phi_r - 2r$.

Now we use the property of martingale (see [2] chapter 6, page 239)

$$E[Y_t] = E[Y_r] = \Phi_r - 2r, \quad t \geq r \geq 0. \tag{10}$$

In our case, T is the stopping time, it follows that

$$E[Y_T] = E[\Phi_T - 2T] = \Phi_r - 2r, \quad T > r \geq 0. \tag{11}$$

Equation (11) implies that given an initial set of balls, the average number of steps needed is always an integer since both Φ_r and r are integers.

For $r = 0$, we have

$$E[\Phi_T - 2T] = \Phi_0 = n,$$

or

$$E[\Phi_T] - 2T = n.$$

but

$$E[\Phi_T] = \Phi_T = n^2,$$

hence

$$n^2 - 2T = n,$$

which implies that

$$T = \frac{n(n-1)}{2}.$$

References:

[1] S. Hart, R. Perez, B. Weiss, "*A Colourful Urn*", June 2009.

[2] S. Karlin, H. M. Taylor, "*A First Course in Stochastic Processes*", second edition , Academic Press.

Solution 2 by Gonzalo Gómez Abejón, Madrid, Spain. Let $x_1(t)$, $x_2(t), \ldots, x_m(t)$ be the number of balls of color 1, color 2, and the rest of the m colors at step t. We can prove that $x_1^2 + x_2^2 + \ldots + x_m^2$ is expected to increase by 2 each step. Lets say the balls selected at step t are of colors A and B. Since it is just as likely that we have drawn A first and B second as the other way around, after one step we have:

$$E[x_A(t+1)^2 + x_B(t+1)^2] = \frac{1}{2}((x_A(t)+1)^2 + (x_B(t)-1)^2)$$

$$+\frac{1}{2}((x_A(t)-1)^2 + (x_B(t)+1)^2)$$

$$= \frac{x_A(t)^2 - 2x_A(t) + 1 + x_B(t)^2 + 2x_B(t) + 1}{2}$$

$$+ \frac{x_A(t)^2 + 2x_A(t) + 1 + x_B(t)^2 - 2x_B(t) + 1}{2}$$

$$= \frac{2x_A(t)^2 + 2x_B(t)^2 + 4}{2} = x_A(t)^2 + x_B(t)^2 + 2.$$

Since the number of balls of colors other than A and B are unchanged, this means that at any step t, the sum of squares is expected to increase by 2:

$$E[\sum_{i=1}^{m} x_i(t+1)^2] = \sum_{i=1}^{m} x_i(t)^2 + 2.$$

Therefore if we define

$$F(t) = \sum_{i=1}^{m} x_i(t)^2 - 2t$$

we always have $E[F(t+1)] = F(t)$, unless by step t the balls were all of the same color (that is, $F(t)$ is a martingale). If the final step is $t = T$, we have:

$$E[F(T)] = E[F(0)] = F(0) = x_1(0)^2 + \ldots + x_m(0)^2.$$

On the other hand, at the final step all our $x_i(T)$ are zero except for the surviving color j for which $x_j(T) = N$, so

$$E[F(T)] = E[N^2 - 2T] = N^2 - 2E[T].$$

In other words, $E[T] = \frac{N^2 - F(0)}{2}$, which is always an integer since N^2 has the same oddness than $N = x_1 + \ldots + x_m$ and than $x_1^2 + x_2^2 + \ldots + x_m^2 = F(0)$ (since $a^2 - a = a(a-1)$ is even).

Finally, if $x_1(0) = \ldots = x_m(0) = 1$, $F(0) = N$ and we are left with $E[T] = \frac{N(N-1)}{2}$.

A–115. *Proposed by Henry Ricardo, Westchester Area Math Circle, New York, USA.* Let p be a prime number. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

Solution 1 by Michel Bataille, Rouen, France. We have

$$\binom{2}{0}\binom{2}{0}+\binom{2}{1}\binom{3}{1}+\binom{2}{2}\binom{4}{2} = 1+6+6 = 13 \equiv 5 = 2^2+1 \pmod{2^2},$$

hence the equality holds for $p = 2$. If p is odd, then the problem is problem B4 of the 1991 Putnam Mathematical Competition. Two solutions can be found in *The American Mathematical Monthly*, Vol. 99, No 8, Oct. 1992, p. 723. and a third solution in *Mathematics Magazine* Vol. 65, No 2, April 1992, p. 142-3.

Solution 2 by Moti Levy, Rehovot, Israel. Let n be a positive integer. We begin with showing that

$$\sum_{j=0}^{n}\binom{n}{j}\binom{n+j}{j}=\sum_{j=0}^{n}\binom{n}{j}^{2}2^{n-j}. \tag{1}$$

The following lemma explains how to express a binomial sum as a hypergeometric function:

Lemma: Let $(\alpha_k)_{k\geq 0}$ satisfies the following conditions:

$$\begin{aligned}\alpha_0 &= 1,\\ \frac{\alpha_{k+1}}{\alpha_k} &= \frac{1}{k+1}\frac{(k+a)(k+b)}{(k+c)}z.\end{aligned}$$

Then

$$\sum_{k=0}^{\infty}\alpha_k = {}_2F_1(a,b;c;z),$$

where ${}_2F_1(a,b;c;z)$ is Gauss hypergeometric function.

Let $(\alpha_k)_{k\geq 0}$ be the sequence $\alpha_k=\binom{n}{k}\binom{n+k}{k}$. Then $\alpha_0=1$ and

$$\frac{\alpha_{k+1}}{\alpha_k}=\frac{\binom{n}{k+1}\binom{n+k+1}{k+1}}{\binom{n}{k}\binom{n+k}{k}}=\frac{1}{k+1}\frac{(k-n)(k+n+1)}{k+1}(-1),$$

hence

$$\sum_{j=0}^{n}\binom{n}{j}\binom{n+j}{j}={}_2F_1(-n,n+1;1;-1). \tag{2}$$

Similarly, let $(\beta_k)_{k\geq 0}$ be the sequence $\beta_k=\frac{1}{2^n}\binom{n}{k}^2 2^{n-k}$. Then $\beta_0=1$ and

$$\frac{\beta_{k+1}}{\beta_k}=\frac{\binom{n}{k+1}^2 2^{n-k-1}}{\binom{n}{k}^2 2^{n-k}}=\frac{1}{k+1}\frac{(k-n)^2}{k+1}\frac{1}{2}$$

hence

$$\sum_{j=0}^{n}\binom{n}{j}^{2}2^{n-j}=2^{n}{}_2F_1\left(-n,-n;1;\frac{1}{2}\right). \tag{3}$$

The Pfaff transformation is (see Wikipedia entry: Hypergeometric function).

$$ {}_2F_1[a, b; c; y] = (1-y)^{-a} {}_2F_1[a, c-b; 1; \frac{y}{y-1}]. \tag{4} $$

With $a = -n$, $b = n+1$ and $c = 1$, the Pfaff transformation implies that

$$ {}_2F_1[-n, n+1, 1, -1] = 2^n {}_2F_1[-n, -n; 1; \frac{1}{2}]. \tag{5} $$

It follows from (2), (3) and (5) that indeed

$$ \sum_{j=0}^{n} \binom{n}{j} \binom{n+j}{j} = \sum_{j=0}^{n} \binom{n}{j}^2 2^{n-j} = 1 + 2^n + \sum_{j=1}^{n-1} \binom{n}{j}^2 2^{n-j}. \tag{6} $$

Now, if $n = p$, a prime number, then

$$ p^2 \text{ divides } \sum_{j=1}^{p-1} \binom{p}{j}^2 2^{p-j}. \tag{7} $$

We conclude from (6) and (7) that

$$ p^2 \text{ divides } \sum_{j=0}^{p} \binom{p}{j} \binom{p+j}{j} - 2^p - 1. $$

Remark: The origin of this problem is the theory of Legendre polynomials $P_n(x)$.

It is known that

$$ P_n(x) = \sum_{j=0}^{n} \binom{n}{j} \binom{n+j}{j} \left(\frac{x-1}{2} \right)^j, \tag{8} $$

and that

$$ P_n(x) = \frac{1}{2^n} \sum_{j=0}^{n} \binom{n}{j}^2 (x-1)^{n-j} (x+1)^j. \tag{9} $$

Setting $x = 3$ in (8) and (9) gives the identity

$$ \sum_{j=0}^{n} \binom{n}{j} \binom{n+j}{j} = \sum_{j=0}^{n} \binom{n}{j}^2 2^j = \sum_{j=0}^{n} \binom{n}{j}^2 2^{n-j}. $$

Solution by the proposer. First we use a combinatorial argument to reduce the sum to one that makes it easier to see the truth of the problem's claim:

$$\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j} = \sum_{j=0}^{p}2^j\binom{p}{j}^2.$$

Let $S=\{1,2,\ldots,p\}$ and $T=\{p+1,p+2,\ldots,2p\}$. We count the number of ordered pairs (X,Y) of subsets X of S and Y of $X\cup T$, Y having p elements. First of all, this number is $\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j}$ because for each j, $0\le j\le p$, we can choose a subset with j elements X of S in $\binom{p}{j}$ ways; and once this X is selected, we can choose Y (a subset with p elements of $X\cup T$) in $\binom{p+j}{p}=\binom{p+j}{j}$ ways.

On the other hand, we can first choose Y as a subset of $S\cup T$. More precisely, we first choose $Y\cap T$ as a subset of T that can have any number $j\le p$ of elements from the p elements of T, and this can be done in $\binom{p}{j}$ ways. The remaining $p-j$ elements of Y can be chosen from the p elements of S in $\binom{p}{p-j}=\binom{p}{j}$ ways, and for each of these choices, X can be completed with some of the other j elements of S (other than those already put into Y) in 2^j ways. Thus a pair (X,Y) of sets $X\subseteq S$ and $Y\subseteq X\cup T$ with $|Y|=p$ can also be chosen in $\sum_{j=0}^{p}2^j\binom{p}{j}^2$ ways, and the equality of the two sums is proved.

$$\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j} = \sum_{j=0}^{p}2^j\binom{p}{j}^2 = 1+2^p+\sum_{j=1}^{p-1}2^j\binom{p}{j}^2$$

and will show that $\sum_{\mathrm{j=1}}^{\mathrm{p-1}}2^{\mathrm{j}}\binom{\mathrm{p}}{\mathrm{j}}^2\equiv 0 \pmod{\mathrm{p}^2}$. To see this, we write (for $0<j<p$)

$$\binom{p}{j} = \frac{p(p-1)(p-2)\cdots(p-j+1)}{j!}$$

$$= \frac{p(p-1)(p-2)\cdots(p-j+1)}{j(j-1)(j-2)\cdots 2\cdot 1} = \frac{A}{B} = N,$$

which is an integer. Now $p|A$ and $p \nmid B$ (since p is prime and $j < p$) imply that $p|N$. This, in turn, implies that $p^2|N^2$, or $p^2|\binom{p}{j}^2$. Therefore,

$$\begin{aligned}\sum_{j=0}^{p}\binom{p}{j}\binom{p+j}{j} &= \sum_{j=0}^{p}2^j\binom{p}{j}^2 \\ = 1+2^p+\sum_{j=1}^{p-1}2^j\binom{p}{j}^2 &\equiv 1+2^p+0 \pmod{p^2}.\end{aligned}$$

Also solved by *José Luis Díaz-Barrero, Barcelona, Spain.*

A–116. *Proposed by Marian Ursărescu and Florică Anastase, Romania.* Let $(a_n)_{n\geq 1}$ be the sequence defined by $a_1 = e$, $a_n = e^n \cdot a_{n-1}^n$ and $(b_n)_{n\geq 1}$ such that

$$\left(1+\frac{1}{n}\right)^{n+b_n} = \prod_{k=1}^{n}\left(1+\frac{1}{\log a_k}\right).$$

Compute $\lim\limits_{n\to\infty} b_n$.

Solution by Michel Bataille, Rouen, France. Let $R_n = \sum\limits_{k=n+1}^{\infty}\frac{1}{k!}$. We first show (by induction) that for all positive integer n, we have

$$\log(a_n) = n!(e - R_{n-1}) \qquad (E_n).$$

Since $\log(a_1) = 1$ and $R_0 = e-1$, the equality (E_1) holds. Assume that (E_n) holds for some positive integer n. Then, we have

$$\begin{aligned}\log(a_{n+1}) &= n+1+(n+1)\log(a_n) \\ &= (n+1)!\left(\frac{1}{n!}+e-R_{n-1}\right) = (n+1)!(e-R_n),\end{aligned}$$

hence (E_{n+1}) holds, completing the induction step.

Note that $1+\log(a_n) = n!\left(\frac{1}{n!}+e-R_{n-1}\right) = n!(e-R_n)$ so that $\frac{1+\log(a_n)}{\log(a_n)} = \frac{e-R_n}{e-R_{n-1}}$. It follows that

$$\prod_{k=1}^{n}\left(1+\frac{1}{\log a_k}\right) = \prod_{k=1}^{n}\frac{e-R_k}{e-R_{k-1}} = e - R_n$$

so that $b_n = [\log(1+1/n)]^{-1}[1+\log(1-R_n/e) - n\log(1+1/n)]$. Now, as $n \to \infty$ we have $[\log(1+1/n)]^{-1} \sim n$; also $R_n \sim \frac{1}{(n+1)!}$ (since $1 \leq (n+1)!R_n \leq \sum\limits_{k=0}^{\infty} \frac{1}{(n+2)^k} = \frac{n+2}{n+1}$) and we deduce that $\log(1-R_n/e) \sim -\frac{1}{e(n+1)!} = o(1/n)$. Since

$$1 - n\log(1+1/n) = 1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + o(1/n^2)\right) = \frac{1}{2n} + o(1/n),$$

we finally obtain that $b_n \sim n \cdot \frac{1}{2n}$ as $n \to \infty$ and we conclude:

$$\lim_{n\to\infty} b_n = \frac{1}{2}.$$

Also solved by *the proposers.*

Editor's comment. Moti Levy, Rehovot, Israel, wrote: I suspect that there is a typo error in the problem statement, so I took the liberty to correct it, as follows:

Let $(a_n)_{n\geq 1}$ be the sequence defined by $a_1 = e$, $a_n = e^n \cdot a_{n-1}^n$ and $(b_n)_{n\geq 1}$ such that

$$\left(1+\frac{1}{n}\right)^{nb_n} = \prod_{k=1}^{n}\left(1+\frac{1}{\log a_k}\right).$$

Compute $\lim\limits_{n\to\infty} b_n$.

Solution by Moti Levi. Let $f(n) = \ln(a_n)$, then $a_1 = e$, $a_n = e^n a_{n-1}^n$ imply the recurrence for $f(n)$,

$$f(n) = nf(n-1) + n, \; f(1) = 1, \tag{1}$$

or,

$$\frac{f(n)}{n!} = \frac{f(n-1)}{(n-1)!} + \frac{1}{(n-1)!}. \tag{2}$$

Let $g(n) := \frac{f(n)}{n!}$, $g(1) = 1$, then it follows from (2) that the recurrence for $g(n)$ is

$$g(n) = g(n-1) + \frac{1}{(n-1)!}.$$

$$g(n) = 1 + \sum_{k=1}^{n-1} \frac{1}{k!}$$

$$\lim_{n\to\infty} g(n) = e,$$

or,

$$\lim_{n\to\infty} \frac{f(n)}{n!} = e. \tag{3}$$

Taking logarithm of both sides of (1) we get the recurrence,

$$\ln(f(k)) = \ln(k) + \ln(f(k-1)+1). \tag{4}$$

Taking logarithm of both sides of $\left(1+\frac{1}{n}\right)^{nb_n} = \prod_{k=1}^{n}\left(1+\frac{1}{\ln(a_k)}\right)$, we get

$$(nb_n)\ln\left(1+\frac{1}{n}\right) = \sum_{k=1}^{n} \ln\left(1+\frac{1}{f(k)}\right). \tag{5}$$

After telescoping of the sum in the right hand side of (5) (using the recurrence (4)), we get

$$\begin{aligned}
(nb_n)\ln\left(1+\frac{1}{n}\right) &= \sum_{k=1}^{n} \ln(f(k)+1) - \ln(f(k)) \\
&= \ln(f(1)+1) - \ln(f(1)) + \sum_{k=2}^{n} \ln(f(k)+1) - \ln(f(k)) \\
&= \ln(2) + \sum_{k=2}^{n} \ln(f(k)+1) - \ln(f(k)) \\
&= \ln(2) + \sum_{k=2}^{n} \ln(f(k)+1) - \ln(f(k-1)+1) - \ln(k) \\
&= \ln\left(\frac{f(n)+1}{n!}\right) \qquad (6)
\end{aligned}$$

Taking the limit of the left hand of (6) gives

$$\lim_{n\to\infty} b_n \ln\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} b_n \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} b_n. \tag{7}$$

Taking the limit of the right hand of (6) gives

$$\lim_{n\to\infty} \ln\left(\frac{f(n)+1}{n!}\right) = \lim_{n\to\infty} \ln\left(\frac{f(n)}{n!}\right) = \ln \lim_{n\to\infty}\left(\frac{f(n)}{n!}\right) = \ln(e) = 1. \tag{8}$$

Equating (7) and (8) we get

$$\lim_{n\to\infty} b_n = 1.$$

A–117. *Proposed by Vasile Mircea Popa, Affiliate Professor, "Lucian Blaga" University of Sibiu, Romania.* Show that

$$\int_0^\infty \frac{x\ln x}{x^3+x\sqrt{x}+1}\,dx = \frac{32}{81}\pi^2\sin\frac{\pi}{18}.$$

Solution 1 by Michel Bataille, Rouen, France. Let I denote the integral. The change of variables $x=u^2$ gives $I=4J$ where

$$J=\int_0^\infty \frac{u^3\ln u}{u^6+u^3+1}\,du=\int_0^1 \frac{u^3\ln u}{u^6+u^3+1}\,du+\int_1^\infty \frac{u^3\ln u}{u^6+u^3+1}\,du$$
$$=\int_0^1 \frac{u^3\ln u}{u^6+u^3+1}\,du-\int_0^1 \frac{v\ln v}{v^6+v^3+1}\,dv$$

(from the change of variables $u=\frac{1}{v}$ in the second integral).
Since $u^6+u^3+1=\frac{1-u^9}{1-u^3}$, we first obtain

$$J=\int_0^1 \frac{u^3(1-u^3)\ln u}{1-u^9}\,du-\int_0^1 \frac{u(1-u^3)\ln u}{1-u^9}\,du$$

and then *via* the substitution $u=t^{1/9}$,

$$81J=\int_0^1 \frac{t^{\frac{-5}{9}}\ln t}{1-t}\,dt-\int_0^1 \frac{t^{\frac{-2}{9}}\ln t}{1-t}\,dt-\int_0^1 \frac{t^{\frac{-7}{9}}\ln t}{1-t}\,dt+\int_0^1 \frac{t^{\frac{-4}{9}}\ln t}{1-t}\,dt$$
$$=-\psi_1(4/9)+\psi_1(7/9)+\psi_1(2/9)-\psi_1(5/9)$$

where ψ_1 denotes the trigamma function defined by

$$\psi_1(z)=-\int_0^1 \frac{t^{z-1}\ln t}{1-t}\,dt.$$

From the known relation $\psi_1(z)+\psi_1(1-z)=\frac{\pi^2}{\sin^2(\pi z)}$, we obtain

$$I=4J=\frac{4\pi^2}{81}\left(\frac{1}{\sin^2(2\pi/9)}-\frac{1}{\sin^2(4\pi/9)}\right).$$

Thus, the problem now amounts to showing that

$$\frac{1}{\sin^2(2\pi/9)} - \frac{1}{\sin^2(4\pi/9)} = 8\sin\frac{\pi}{18}. \tag{1}$$

Using the formulas $\sin^2 x - \sin^2 y = \sin(x-y)\sin(x+y)$ and $\sin x = \cos\left(\frac{\pi}{2} - x\right)$, the left-hand side of (1) is equal to

$$\frac{\sqrt{3}/2}{\cos^2(\pi/18)\cos(5\pi/18)}$$

and (1) reduces to

$$\sin\frac{\pi}{18}\cos\frac{5\pi}{18}\cos^2\frac{\pi}{18} = \frac{\sqrt{3}}{16}.$$

We are done since

$$\begin{aligned}\sin\frac{\pi}{18}\cos\frac{5\pi}{18}\cos^2\frac{\pi}{18} &= \frac{1}{2}\sin\frac{\pi}{9}\cos\frac{5\pi}{18}\cos\frac{\pi}{18}\\ &= \frac{1}{4}\sin\frac{\pi}{9}\left(\cos\frac{\pi}{3} + \cos\frac{2\pi}{9}\right)\\ &= \frac{1}{8}\left(\sin\frac{\pi}{9} + \sin\frac{\pi}{3} - \sin\frac{\pi}{9}\right) = \frac{1}{8}\cdot\frac{\sqrt{3}}{2}\end{aligned}$$

(since $2\sin\frac{\pi}{9}\cos\frac{2\pi}{9} = \sin\frac{\pi}{3} - \sin\frac{\pi}{9}$).

Solution 2 by the proposer. Let us denote:

$$I = \int_0^\infty \frac{x\ln x}{x^3 + x\sqrt{x} + 1}dx, \quad A = \int_0^1 \frac{x\ln x}{x^3 + x\sqrt{x} + 1}dx,$$

$$B = \int_1^\infty \frac{x\ln x}{x^3 + x\sqrt{x} + 1}dx.$$

We consider the integral A. We make the variable change: $x = y^{\frac{2}{3}}$. We have, successively:

$$A = \frac{4}{9}\int_0^1 \frac{(1-y)y^{\frac{1}{3}}\ln y}{1-y^3}dy = \frac{4}{9}\left(\int_0^1 \frac{y^{\frac{1}{3}}\ln y}{1-y^3}dy - \int_0^1 \frac{y^{\frac{4}{3}}\ln y}{1-y^3}dy\right)$$

$$A=\frac{4}{9}\left(\int_0^1\sum_{k=0}^{\infty}y^{3k+\frac{1}{3}}\ln y dy-\int_0^1\sum_{k=0}^{\infty}y^{3k+\frac{4}{3}}\ln y dy\right);$$
$$A=\frac{4}{9}\sum_{k=0}^{\infty}\left(\int_0^1 y^{3k+\frac{1}{3}}\ln y dy-\int_0^1 y^{3k+\frac{4}{3}}\ln y dy\right)$$

We will use the following relationship:

$$\int_0^1 x^a\ln x dx=-\frac{1}{(a+1)^2}$$

where $a\in\mathbb{R}$ and $a\geq 0$. We obtain

$$A=\frac{4}{9}\sum_{k=0}^{\infty}\left[\frac{1}{\left(3k+\frac{7}{3}\right)^2}-\frac{1}{\left(3k+\frac{4}{3}\right)^2}\right],\ A=\frac{4}{9}\sum_{k=0}^{\infty}\left[\frac{\frac{1}{9}}{\left(k+\frac{7}{9}\right)^2}-\frac{\frac{1}{9}}{\left(k+\frac{4}{9}\right)^2}\right].$$

We now use the following relationship:

$$\psi_1(x)=\sum_{n=0}^{\infty}\frac{1}{(x+n)^2}$$

where $\psi_1(x)$ is the trigamma function. We obtain the value of the integral A:

$$A=\frac{4}{81}\left[-\psi_1\left(\frac{4}{9}\right)+\psi_1\left(\frac{7}{9}\right)\right].$$

We consider the integral B. We make the variable change: $x=\frac{1}{y}$. Then, by preceeding to the integral A, we obtain:

$$B=\frac{4}{81}\left[\psi_1\left(\frac{2}{9}\right)-\psi_1\left(\frac{5}{9}\right)\right].$$

Result:

$$I=A+B=\frac{4}{81}\left[\psi_1\left(\frac{2}{9}\right)-\psi_1\left(\frac{4}{9}\right)-\psi_1\left(\frac{5}{9}\right)+\psi_1\left(\frac{7}{9}\right)\right].$$

We use the reflection formula:

$$\psi_1(x)+\psi_1(1-x)=\frac{\pi^2}{\sin^2(\pi x)}$$

to obtain

$$\psi_1\left(\frac{2}{9}\right)+\psi_1\left(\frac{7}{9}\right)=\frac{\pi^2}{\sin^2\frac{2\pi}{9}};\psi_1\left(\frac{4}{9}\right)+\psi_1\left(\frac{5}{9}\right)=\frac{\pi^2}{\sin^2\frac{4\pi}{9}}.$$

Result:

$$I=\frac{4}{81}\pi^2\left(\frac{1}{\sin^2\frac{2\pi}{9}}-\frac{1}{\sin^2\frac{4\pi}{9}}\right).$$

We have

$$\frac{1}{\sin^2\frac{2\pi}{9}}-\frac{1}{\sin^2\frac{4\pi}{9}}=8\sin\frac{\pi}{18}.$$

We will prove this equality. We use the relationship:

$$\sin 3a=\sin a(1+2\cos 2a).$$

We consider:

$$E=\frac{1}{\sin^2\frac{2\pi}{9}}-\frac{1}{\sin^2\frac{4\pi}{9}}=\frac{\left(1+2\cos\frac{4\pi}{9}\right)^2}{\sin^2\frac{2\pi}{3}}-\frac{\left(1+2\cos\frac{8\pi}{9}\right)^2}{\sin^2\frac{4\pi}{3}}$$

$$\begin{aligned}E&=\frac{16}{3}\left(1+\cos\frac{4\pi}{9}+\cos\frac{8\pi}{9}\right)\left(\cos\frac{4\pi}{9}-\cos\frac{8\pi}{9}\right)\\&=\frac{16}{3}\left(1+\cos\frac{4\pi}{9}+\cos\frac{8\pi}{9}\right)2\sin\frac{2\pi}{3}\sin\frac{2\pi}{9}\end{aligned}$$

$$E=\frac{16\sqrt{3}}{3}\left(1+\cos\frac{4\pi}{9}-\cos\frac{\pi}{9}\right)\sin\frac{2\pi}{9}=\frac{8\sqrt{3}}{3}\left(\sin\frac{2\pi}{9}-\sin\frac{\pi}{9}\right).$$

So, $E=8\sin\dfrac{\pi}{18}$.

Finally,

$$I=\frac{32}{81}\pi^2\sin\frac{\pi}{18},$$

and the problem is solved.

Also solved by *Moti Levy, Rehovot, Israel.*

A–118. *Proposed by Michel Bataille, Rouen, France.* For $n \in \mathbb{N}$, let $S(n) = \sum\limits_{k=1}^{n} \frac{(-1)^{k+1}}{k}$. Prove that the series

$$\sum_{n=2}^{\infty} (-1)^n \frac{S(\lfloor n/2 \rfloor)}{n+1}$$

is convergent and evaluate its sum.

Solution 1 by the proposer. Let $I(n) = \int_0^1 \frac{x^n}{1+x}\,dx$. It is well-known that $\ln 2 = S(n) + (-1)^n I(n)$ and that

$$\lim_{n\to\infty} nI(n) = \left[\frac{1}{1+x}\right]_{x=1} = \frac{1}{2}.$$

Now, let

$$a_n = (-1)^n \frac{S(\lfloor n/2 \rfloor)}{n+1} = \ln(2)\frac{(-1)^n}{n+1} + \frac{(-1)^n(-1)^{\lfloor n/2 \rfloor + 1} I(\lfloor n/2 \rfloor)}{n+1}.$$

$\sum\limits_{n=2}^{\infty} \frac{(-1)^n}{n+1}$ is convergent; also, the series $\sum\limits_{n=2}^{\infty} \frac{(-1)^n(-1)^{\lfloor n/2 \rfloor+1} I(\lfloor n/2 \rfloor)}{n+1}$ is absolutely convergent since as $n \to \infty$

$$\frac{I(\lfloor n/2 \rfloor)}{n+1} \sim \frac{1}{n+1} \cdot \frac{1}{2\lfloor n/2 \rfloor} \sim \frac{1}{n^2}.$$

It follows that $\sum\limits_{n=2}^{\infty} a_n$ is convergent. Let S be its sum.

We have $\ln(2) \sum\limits_{n=2}^{\infty} \dfrac{(-1)^n}{n+1} = (\ln(2))\left(\ln(2) - 1 + \dfrac{1}{2}\right) = (\ln(2))^2 - \dfrac{\ln(2)}{2}$.

Let $\quad T = \sum\limits_{n=2}^{\infty} \dfrac{(-1)^n(-1)^{\lfloor n/2 \rfloor+1} I(\lfloor n/2 \rfloor)}{n+1}$.

Then

$$\begin{aligned} T &= \sum_{m=1}^{\infty}\left(\frac{(-1)^{2m}(-1)^{m+1}I(m)}{2m+1} + \frac{(-1)^{2m+1}(-1)^{m+1}I(m)}{2m+2}\right) \\ &= \sum_{m=1}^{\infty}\int_0^1 \frac{(-1)^{m+1}u^m}{(2m+1)(2m+2)(u+1)}\,du. \end{aligned}$$

Since $\sum\limits_{m=1}^{\infty}\displaystyle\int_0^1 \left|\frac{(-1)^{m+1}u^m}{(2m+1)(2m+2)(u+1)}\right| du \le \sum\limits_{m=1}^{\infty} \frac{1}{(2m+1)(2m+2)} < \infty$, we can interchange $\sum$ and $\int$ and obtain

$$T = \int_0^1 \left(\sum_{m=1}^{\infty} \frac{(-1)^{m+1} u^n}{(2m+1)(2m+2)} \right) \frac{du}{1+u}$$

where for $u \in [0,1]$,

$$\begin{aligned}\sum_{m=1}^{\infty} \frac{(-1)^{m+1} u^m}{(2m+1)(2m+2)} &= \sum_{m=1}^{\infty} (-1)^{m+1} u^m \left(\frac{1}{2m+1} - \frac{1}{2m+2} \right) \\ &= \sum_{m=1}^{\infty} \frac{(-1)^{m+1} u^m}{2m+1} - \frac{1}{2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1} u^m}{m+1}.\end{aligned}$$

From classical power series expansion, we deduce

$$\sum_{m=1}^{\infty} \frac{(-1)^{m+1} u^m}{(2m+1)(2m+2)} = \left(1 - \frac{\arctan(\sqrt{u})}{\sqrt{u}} \right) - \frac{1}{2} \left(1 - \frac{\ln(1+u)}{u} \right)$$

(the function on the right being extended by continuity at 0). In consequence,

$$T = \frac{1}{2} \int_0^1 \frac{du}{1+u} - \int_0^1 \frac{\arctan(\sqrt{u})}{\sqrt{u}(1+u)}\, du + \frac{1}{2} \int_0^1 \frac{\ln(1+u)}{u(1+u)}\, du$$

with

$$\int_0^1 \frac{du}{1+u} = \ln 2, \qquad \int_0^1 \frac{\arctan(\sqrt{u})}{\sqrt{u}(1+u)}\, du = 2 \int_0^1 \frac{\arctan x}{1+x^2}\, dx = \frac{\pi^2}{16}$$

and

$$\int_0^1 \frac{\ln(1+u)}{u(1+u)}\, du = \int_0^1 \frac{\ln(1+u)}{u}\, du - \int_0^1 \frac{\ln(1+u)}{1+u}\, du = \frac{\pi^2}{12} - \frac{(\ln 2)^2}{2}.$$

Gathering the results, we readily obtain

$$S = \frac{3(\ln(2))^2}{4} - \frac{\pi^2}{48}.$$

Solution 2 by Álvaro De Irízar Larrauri, CFIS, BarcelonaTech (Student), Barcelona, Spain. First, we will pair consecutive terms of the sum. Setting $n = 2m$ for even terms and $n = 2m + 1$ for odd terms, we get

$$\sum_{m=1}^{\infty}\left((-1)^{2m}\frac{S(\lfloor 2m/2\rfloor)}{2m+1} + (-1)^{2m+1}\frac{S(\lfloor (2m+1)/2\rfloor)}{2m+2}\right)$$
$$= \sum_{m=1}^{\infty}\left(\frac{S(m)}{2m+1} - \frac{S(m)}{2m+2}\right) = \sum_{m=1}^{\infty}\frac{S(m)}{(2m+1)(2m+2)}$$

Now, let's recall the Maclaurin series for $\log(x+1)$:

$$\log(x+1) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}x^k \quad \forall x \in (-1, 1]$$

This means that $\lim_{m\to\infty} S(m) = \log(2)$. Thus, $\forall\varepsilon > 0, \exists M \in \mathbb{N}$ such that $\forall m \geq M, |S(m) - \log(2)| < \varepsilon$. Let's take $\varepsilon < 1 - \log(2)$ and its correspondent value of M. For $m > M, 0 < S(m) < 1$. We will use this information to prove the series convergence.

$$\sum_{m=1}^{\infty}\frac{S(m)}{(2m+1)(2m+2)} = \sum_{m=1}^{M-1}\frac{S(m)}{(2m+1)(2m+2)}$$
$$+ \sum_{m=M}^{\infty}\frac{S(m)}{(2m+1)(2m+2)}$$

The first sumatory is a finite sum and thus takes a finite value. As for the second one, since all the terms are positive, we get

$$0 < \sum_{m=M}^{\infty}\frac{S(m)}{(2m+1)(2m+2)} < \sum_{m=M}^{\infty}\frac{1}{(2m+1)(2m+2)}$$
$$< \sum_{m=M}^{\infty}\frac{1}{(2m+1)^2} < \sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{6}$$

Since the series is bounded and all its terms are positive (therefore the sequence of partial sums is strictly increasing), by the Monotone Convergence Theorem it must converge. In fact, it's absolutely convergent. Once we have proven the convergence, let's go back to the sum and rewrite it in terms of an integral:

$$\sum_{m=1}^{\infty} \frac{S(m)}{(2m+1)(2m+2)} = \sum_{m=1}^{\infty} S(m) \int_0^1 (1-x)x^{2m}\, dx$$

We have already proven the absolute convergence of the series, which is why we can interchange the summatory and integral. We will also expand $S(m)$ using its definition:

$$\int_0^1 \sum_{m=1}^{\infty} (1-x)x^{2m} \sum_{k=1}^{m} \frac{(-1)^{k+1}}{k} = \int_0^1 \sum_{m=1}^{\infty} \sum_{k=1}^{m} (1-x)x^{2m} \frac{(-1)^{k+1}}{k}\, dx$$

Again, thanks to absolute convergence, we can interchange the summatories:

$$\int_0^1 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(1-x) \sum_{m=k}^{\infty} x^{2m}\, dx = \int_0^1 \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}(1-x)\frac{x^{2k}}{1-x^2}\, dx$$

$$= \int_0^1 \frac{1}{1+x} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^{2k}\, dx = \int_0^1 \frac{\log(1+x^2)}{1+x}\, dx$$

We will evaluate this integral using Feynman's technique. Let's define

$$I(a) = \int_0^1 \frac{\log(1+ax^2)}{1+x}\, dx \implies I'(a) = \int_0^1 \frac{x^2}{(1+ax^2)(1+x)}\, dx$$

$$= \frac{1}{1+a}\left[\int_0^1 \frac{1}{1+x}\, dx + \int_0^1 \frac{x}{1+ax^2}\, dx - \int_0^1 \frac{1}{1+ax^2}\, dx\right]$$

Which we get by applying partial fraction decomposition. All of these integral are direct:

$$I'(a) = \frac{\log(2)}{1+a} + \frac{\log(1+a)}{2a(1+a)} - \frac{\arctan(\sqrt{a})}{\sqrt{a}(1+a)}$$

Now we can integrate both sides of the equation with respect to a. The first term integrates directly, and the third term is of the form $\int 2uu'\,dx$ for $u = \arctan(\sqrt{a})$. We will apply partial fraction decomposition again to the second term:

$$\frac{1}{2}\left[\int \frac{\log(1+a)}{a}\,da - \int \frac{\log(1+a)}{1+a}\,da\right]$$

The second integral is direct. For the first one we will use the substitution $u = -a$:

$$\int \frac{\log(1+a)}{a}\,da = -\left(-\int \frac{\log(1-u)}{u}\,du\right) = -Li_2(u) = -Li_2(-a)$$

The integral we got was by definition the dilogarithm, $Li_2(x)$, which has the property that its Maclaurin series, valid for $|x| \leq 1$, is $Li_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$. This can be easily proven by substituting the logarithm by its Maclaurin series and interchanging the summatory and integral. Finally, we get

$$I(a) = \log(2)\log(1+a) - \frac{Li_2(-a)}{2} - \frac{\log^2(1+a)}{4} - \arctan^2(\sqrt{a}) + C$$

Where C is a constant of integration. Notice that $I(0) = C$ in this equation, but using the definition of $I(a)$, we get $I(0) = 0$. Thus, $C = 0$. The final result we want is $I(1)$. Using its Maclaurin series expansion, $Li_2(-1)$ is the alternated sum of the inverses of squares, a well-known series that yields $-\frac{\pi^2}{12}$. Evaluating the expression, we get

$$\begin{aligned} I(1) &= \log^2(2) - \frac{Li_2(-1)}{2} - \frac{\log^2(2)}{4} - \arctan^2(1) \\ &= \frac{3}{4}\log^2(2) + \frac{\pi^2}{24} - \frac{\pi^2}{16} = \frac{3}{4}\log^2(2) - \frac{\pi^2}{48} \end{aligned}$$

Also solved by *Moti Levy, Rehovot, Israel.*

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