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A class of combinatorial identities

Joe Santmyer

Abstract

The proof of a result in complex variables becomes the source of a class of combinatorial identities. In this case the emergence of an identity was the result of asking a different question. A preliminary result leading to the Weierstrass Factorization Theorem needs coefficients of a Taylor expansion to be non-negative. A question one might ask is exactly what do the coefficients look like. The answer results in an infinite class of combinatorial identities.

1 Question

One of the steps to prove the Weierstrass Factorization Theorem is to establish the inequality

$$|1 - E_m(z)| \leq |z|^{m+1} \qquad \text{if } |z| \leq 1$$

where

$$E_m(z) = \begin{cases} 1 - z & \text{if } m = 0 \\ (1-z)e^{z + \frac{z^2}{m} + \cdots + \frac{z^m}{m}} & \text{if } m = 1, 2, \ldots \end{cases}$$

The function $f(z) = 1 - E_m(z)$ is an entire function and hence has a Taylor expansion

$$f(z) \;=\; \sum_{n=0}^{\infty} a_n z^n.$$

A proof of the above inequality in [2] establishes the easy equation

$$\sum_{n=0}^{\infty} a_n = 1 \qquad (1)$$

since $f(1) = 1 - E_m(1) = 1$. That proof only needs the fact that $a_n \geq 0$ for $n \geq 0$. But can a formula for the a_n be derived?

1.1 Brief Answer

The Weierstrass Factorization Theorem is a result in complex analysis, an area of mathematics that deals with the continuous and infinity. What is developed below is of a discrete nature and illustrates the interplay between continuous and discrete mathematics. Many other interesting examples of this can be found in [4].

We begin by considering

$$\begin{aligned} f'(z) &= -E'_m(z) \\ &= z^m e^{\left(z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\cdots+\frac{z^m}{m}\right)} \\ &= z^m e^{h(z)} \end{aligned} \qquad (2)$$

where $h(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \cdots + \frac{z^m}{m}$. Two series for $f'(z)$ are

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1} \qquad (3)$$

$$f'(z) = \sum_{n=0}^{\infty} \frac{z^m (h(z))^n}{n!}. \qquad (4)$$

If $m = 1$ then $h(z) = z$. If c_n represents the coefficents in (4) then $c_0 = 0$. Equating coefficients in (3) and (4), it is easy to see that $a_0 = 0$ and

$$a_{n+1} = \frac{c_n}{n+1} \qquad (5)$$

for $n = 0, 1, 2, \ldots$. Series (4) in this case is

$$f'(z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n!}.$$

Using this together with (5) it is straightforward to show that $a_1 = 0$ and

$$a_n = \frac{1}{n(n-2)!}$$

for $n = 2, 3, \ldots$. Since $\sum_{n=0}^{\infty} a_n = 1$ then

$$\sum_{n=2}^{\infty} \frac{1}{n(n-2)!} = 1. \tag{6}$$

Of course (6) can be obtained directly. A formula for the a_n and an identity for their sum have been derived when $m = 1$.

Starting from an equation such as (2) and equating coefficents is often done to obtain properties of coefficients as in [1] for the Bernoulli polynomials and numbers, in [3] for series solutions to differential equations, *etc.*

2 Complete Answer

Consider the case $m \geq 2$. Let $k = \left\lceil \frac{n+m(m-1)}{m} \right\rceil$. Let $\mathbf{a}, \mathbf{b}, \mathbf{n} \in \mathbb{N}^m$ with $\mathbf{a} = (1, 1, \ldots, 1)$, $\mathbf{b} = (0, 1, \ldots, m-1)$ and $\mathbf{n} = (n_1, n_2, \ldots, n_m)$ where $\mathbf{a} \cdot \mathbf{n} = k - m + j$ and $\mathbf{b} \cdot \mathbf{n} = n - k - j$ for $j = 0, 1, \ldots, n - k$. Then

$$c_n = \sum_{j=0}^{n-k} \frac{1}{(k-m+j)!} \sum_{\substack{\mathbf{a}\cdot\mathbf{n}=k-m+j \\ \mathbf{b}\cdot\mathbf{n}=n-k-j}} \binom{k-m+j}{n_1\ n_2 \ldots n_m} \prod_{i=1}^{m} \left(\frac{1}{i}\right)^{n_i}.$$

The expression for c_n is a straightforward application of the multinomial formula. Based on the initial conditions $a_0 = a_1 = 0$ when $m = 1$, in general we have $a_0 = a_1 = \cdots = a_m = 0$. From (1) and (5) it follows that

$$1 = \sum_{n=m}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n-k} \frac{1}{(k-m+j)!} \sum_{\substack{\mathbf{a}\cdot\mathbf{n}=k-m+j \\ \mathbf{b}\cdot\mathbf{n}=n-k-j}} \binom{k-m+j}{n_1\ n_2 \ldots n_m} \prod_{i=1}^{m} \left(\frac{1}{i}\right)^{n_i}.$$

The above equation represents an infinite class of combinatorial identities, one for each $m \geq 2$.

2.1 Applications

Example 1: Lets apply the formula to calculate c_n when $m = 2$. In this case $k = \lceil \frac{n+2}{2} \rceil$, $\mathbf{a} = (1, 1)$, $\mathbf{b} = (0, 1)$, $\mathbf{n} = (n_1, n_2)$. For $j = 0, 1, \ldots, n-k$ the system of equations has the solution

$$\mathbf{n} = (n_1, n_2) = (2k + 2j - n - 2, n - k - j).$$

Consequently

$$\begin{aligned} c_n &= \sum_{j=0}^{n-k} \frac{1}{(k-2+j)!} \sum_{\substack{\mathbf{a}\cdot\mathbf{n}=k-2+j \\ \mathbf{b}\cdot\mathbf{n}=n-k-j}} \binom{k-2+j}{n_1, n_2} \prod_{i=1}^{2} \left(\frac{1}{i}\right)^{n_i} \\ &= \sum_{j=0}^{n-k} \frac{1}{(k-2+j)!} \binom{k-2+j}{2k+2j-n-2 \quad n-k-j} \left(\frac{1}{1}\right)^{2k+2j-n-2} \left(\frac{1}{2}\right)^{n-k-j} \\ &= \sum_{j=0}^{n-k} \frac{1}{2^{n-k-j}(n-k-j)!(2k+2j-n-2)!}. \end{aligned}$$

For $m = 2$ we have $a_0 = a_1 = a_2 = 0$. Apply (1) and (5) to get

$$\begin{aligned} 1 &= 0 + 0 + 0 + \sum_{n=2}^{\infty} \frac{c_n}{n+1} \\ 1 &= \sum_{n=2}^{\infty} \frac{1}{n+1} \sum_{j=0}^{n-k} \frac{1}{2^{n-k-j}(n-k-j)!(2k+2j-n-2)!}. \end{aligned}$$

The formula can be simplified with the substitution $n-k-j = i-1$. Then

$$\begin{aligned} n - i + 1 &= k + j \\ 2n - 2i + 2 &= 2k + 2j \\ 2n - 2i + 2 - n - 2 &= 2k + 2j - n - 2 \\ n - 2i &= 2k + 2j - n - 2. \end{aligned}$$

This shows that

$$2^{n-k-j}(n-k-j)!(2k+2j-n-2)! \;=\; 2^{n-i}(n-i)!(n-2i)!$$

As j ranges from 0 to $n-k$ the value of i ranges from 1 to $\lfloor \frac{n}{2} \rfloor$. Of course, we must also ensure that the number of terms are equal in both cases. To show that the number of terms are equal it must be shown that $n-k+1 = \lfloor \frac{n}{2} \rfloor$ where $k = \lceil \frac{n+2}{2} \rceil$. The verification of this is straightforward and left to the reader.

Since the number of terms and the terms themselves are equal the two sums are equal. That is

$$\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^{i-1}(i-1)!(n-2i)!} = \sum_{j=0}^{n-k} \frac{1}{2^{n-k-j}(n-k-j)!(2k+2j-n-2)!}.$$

Consequently

$$1 = \sum_{n=2}^{\infty} \frac{1}{n+1} \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^{i-1}(i-1)!(n-2i)!}.$$

This is problem 12276 in [6].

Example 2: Consider a sequence of fractions defined as $b_1 = b_2 = 1$ and $b_n = \frac{b_{n-1}+b_{n-2}}{n-1}$. Show that $\lim_{n\to\infty} b_n = 0$.

Mathematica can be used to generate the partial sums for the series in **example 1** and it produces the sequence

$$\{s_n\} = \left\{\frac{1}{3}, \frac{7}{12}, \frac{47}{60}, \frac{161}{180}, \frac{601}{630}, \frac{9889}{10080}, \frac{18013}{18144}, \frac{452413}{453600}, \frac{2492569}{2494800}, \frac{59857681}{59875200}, \ldots\right\}.$$

The sequence

$$\{d_n\} = \{3, 12, 60, 180, 630, 10080, 18144, 453600, 2494800, 59875200, \ldots\}$$

of denominators is essentially sequence OEIS A069944 in the Online Encyclopedia Integer Sequences. These are denominators of fractions defined by the recurrence relation $b_n = \frac{b_{n-1}+b_{n-2}}{n-1}$ with initial conditions $b_1 = b_2 = 1$ which is the given sequence in the example. The sequence can be generated with *Mathematica* to produce

$$\{b_n\} = \left\{1, 1, 1, \frac{2}{3}, \frac{5}{12}, \frac{13}{60}, \frac{19}{180}, \frac{29}{630}, \frac{191}{10080}, \frac{131}{18144}, \frac{1187}{453600}, \frac{2231}{2494800}, \frac{17519}{59875200}, \ldots\right\}.$$

The sequence $c_n = 1 - b_n$ is

$$\{c_n\} = \left\{0, 0, 0, \frac{1}{3}, \frac{7}{12}, \frac{47}{60}, \frac{161}{180}, \frac{601}{630}, \frac{9889}{10080}, \frac{18013}{18144}, \frac{452413}{453600}, \frac{2492569}{2494800}, \frac{59857681}{59875200}, \ldots\right\}.$$

Sequences $\{c_n\}$ and $\{s_n\}$ are essentially the same and behave the same when $n \to \infty$. By **Example 1** we know that $\lim_{n\to\infty} s_n = 1$ so we can conclude that

$$\begin{aligned} 1 &= \lim_{n\to\infty} s_n \\ &= \lim_{n\to\infty} c_n \\ &= \lim_{n\to\infty} (1 - b_n) \\ &= 1 - \lim_{n\to\infty} b_n \\ \lim_{n\to\infty} b_n &= 0. \end{aligned}$$

3 A Second Class of Combinatorial Identities

Taking the first derivative $f'(z)$ and equating coefficients produced the above class of combinatorial identities. Lets repeat the process with $f''(z)$ to get

$$\begin{aligned} f''(z) &= \sum_{n=2}^{\infty} n(n-1)a_n z^{n-2} \\ f''(z) &= [mz^{m-1} + z^m + z^{m+1} + z^{m+2} + \cdots + z^{2m-1}] \sum_{n=0}^{\infty} \frac{(h(z))^n}{n!}. \end{aligned}$$

Since $a_0 = a_1 = 0$ write the first series as

$$f''(z) = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} z^n$$

and let c_n represent the coefficient of z^n in the second series. Equate coefficients to get

$$a_{n+2} = \frac{c_n}{(n+2)(n+1)}. \tag{7}$$

Use the multinomial formula. In this case, let $k_r = \left\lceil \frac{n+(m-1)(m+r-1)}{m} \right\rceil$ for $r = 0, 1, 2, \ldots, m$. Let $d_0 = m$ and $d_r = 1$ for $r = 1, 2, \ldots, m$.

Vectors $\mathbf{a}, \mathbf{b}, \mathbf{n} \in \mathbb{N}^m$ are the same as before. Then

$$c_n = \sum_{r=0}^{m} \sum_{j=0}^{n-k_r} \frac{d_r}{(k_r - m - r + 1 + j)!} \times$$

$$\sum_{\substack{\mathbf{a}\cdot\mathbf{n}=k_r-m-r+1+j \\ \mathbf{b}\cdot\mathbf{n}=n-k_r-j}} \binom{k_r - m - r + 1 + j}{n_1\; n_2 \ldots n_m} \prod_{i=1}^{m} \left(\frac{1}{i}\right)^{n_i}.$$

The initial conditions are the same, namely $a_0 = a_1 = a_2 = \cdots = a_m = 0$. Since $1 = \sum\limits_{n=0}^{\infty} a_n$ we get

$$1 = a_{m+1} + a_{m+2} + \sum_{n=m+1}^{\infty} \frac{1}{(n+2)(n+1)} \sum_{r=0}^{m} \sum_{j=0}^{n-k_r} \frac{d_r}{(k_r - m - r + 1 + j)!} \times$$

$$\sum_{\substack{\mathbf{a}\cdot\mathbf{n}=k_r-m-r+1+j \\ \mathbf{b}\cdot\mathbf{n}=n-k_r-j}} \binom{k_r - m - r + 1 + j}{n_1\; n_2 \ldots n_m} \prod_{i=1}^{m} \left(\frac{1}{i}\right)^{n_i}.$$

That is

$$1 - a_{m+1} - a_{m+2} = \sum_{n=m+1}^{\infty} \frac{1}{(n+2)(n+1)} \sum_{r=0}^{m} \sum_{j=0}^{n-k_r} \frac{d_r}{(k_r - m - r + 1 + j)!} \times$$

$$\sum_{\substack{\mathbf{a}\cdot\mathbf{n}=k_r-m+1-r+j \\ \mathbf{b}\cdot\mathbf{n}=n-k_r-j}} \binom{k_r - m - r + 1 + j}{n_1\; n_2 \ldots n_m} \prod_{i=1}^{m} \left(\frac{1}{i}\right)^{n_i}.$$

The above equation represents a class of identities, one for each $m \geq 2$.

3.1 Applications

Example 3: Lets apply the formula in the case $m = 2$. We have $k_r = \lceil \frac{n+r+1}{2} \rceil$ for $r = 0, 1, 2$. Also, $d_0 = 2$ and $d_1 = d_2 = 1$. The system of equations

$$\begin{aligned} \mathbf{a} \cdot \mathbf{n} &= k_r - 1 - r + j \\ \mathbf{b} \cdot \mathbf{n} &= n - k_r - j \end{aligned}$$

has the solution $\mathbf{n} = (2k_r + 2j - n - r - 1, n - k_r - j)$. Consequently

$$\begin{aligned} c_n &= \sum_{r=0}^{2} \sum_{j=0}^{n-k_r} \frac{d_r}{(k_r - 1 - r + j)!} \binom{k_r - 1 - r + j}{2k_r + 2j - n - r - 1 \quad n - k_r - j} \left(\frac{1}{2}\right)^{n-k_r-j} \\ &= \sum_{r=0}^{2} \sum_{j=0}^{n-k_r} \frac{d_r}{2^{n-k_r-j}(n - k_r - j)!(2k_r + 2j - n - r - 1)!}. \end{aligned}$$

It is easy to see that $c_0 = 0$, $c_1 = 2$, $c_2 = 3$. For $n \geq 3$ the value of c_n is calculated using the above equation. Equating coefficients we have $a_3 = \frac{1}{3}$ and $a_4 = \frac{1}{4}$. For $n \geq 5$ use (7) to calculate a_n. Consequently, $1 - \frac{1}{3} - \frac{1}{4} = \frac{5}{12}$ and we get

$$\frac{5}{12} = \sum_{n=3}^{\infty} \frac{1}{(n+2)(n+1)} \left[\sum_{r=0}^{2} \sum_{j=0}^{n-k_r} \frac{d_r}{2^{n-k_r-j}(n - k_r - j)!(2k_r + 2j - n - r - 1)!} \right].$$

Of course, we can begin the series at $n = 1$ to get

$$1 = \sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} \left[\sum_{r=0}^{2} \sum_{j=0}^{n-k_r} \frac{d_r}{2^{n-k_r-j}(n - k_r - j)!(2k_r + 2j - n - r - 1)!} \right].$$

The formula can be simplified with the substitution $i - 1 = n - k_r - j$ to get

$$1 = \sum_{n=1}^{\infty} \frac{1}{(n+2)(n+1)} \left[\sum_{r=0}^{2} \sum_{i=1}^{n-k_r+1} \frac{d_r}{2^{i-1}(i-1)!(n - 2i - r + 1)!} \right].$$

Example 4: Show that for $N \geq 2$ we have

$$\sum_{n=2}^{N} \frac{1}{n+1} \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^{i-1}(i-1)!(n-2i)!}$$

$$= \sum_{n=1}^{N-1} \frac{1}{(n+2)(n+1)} \left[\sum_{r=0}^{2} \sum_{i=1}^{n-k_r+1} \frac{d_r}{2^{i-1}(i-1)!(n - 2i - r + 1)!} \right]$$

where k_r and d_r are defined as in **Example 3**.

Mathematica can be used to generate the partial sums for the series in **example 3** and it produces the sequence

$$s_n = \left\{ \frac{1}{3}, \frac{7}{12}, \frac{47}{60}, \frac{161}{180}, \frac{601}{630}, \frac{9889}{10080}, \frac{18013}{18144}, \frac{452413}{453600}, \frac{2492569}{2494800}, \frac{59857681}{59875200}, \ldots \right\}.$$

This is *exactly* the same sequence of partial sums for the series in **example 1** and the equation follows.

4 Closing Remarks

What do the identities look like for $m = 3$? The author attempted this case but could not obtain a simple closed form solution. **Example 4** shows that the second class of identities based on the second derivative $f''(z)$ produced that same class using the first derivative $f'(z)$. Consequently, taking higher order derivatives does not produce anything new. What it does is represent the identities in a different form. A search in [5] did not find the sums and series that appear here.

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Some relations between the tangent lengths of a bicentric quadrilateral

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Abstract

In this paper new identities and inequalities for a bicentric quadrilateral are given.

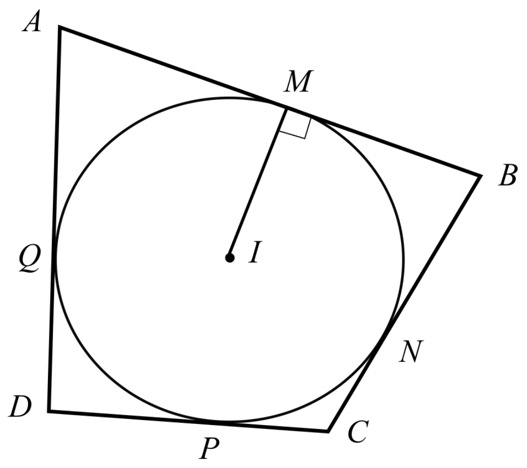
1 Introduction

The purpose of this work is to give some new identities and inequalities involving the sides a, b, c, d, the inradius r and circumradius R, the semiperimeter s and the tangent lengths t_1, t_2, t_3, t_4 of a bicentric quadrilateral. A good source where many inequalities (in particular geometric inequalities) are published is *Octogon Mathematical Magazine (2000-2022).*

In what follows we will call the distances from the vertices of quadrilateral $ABCD$ to the points of the tangency of the sides with $\mathcal{C}(I, r)$ the tangent lengths. Let us denote by M, N, P, Q the points where the sides AB, BC, CD, DA touch the circle $\mathcal{C}(I, r)$. Also, we denote the tangent lengths with $AM = t_1$, $BN = t_2$, $CP = t_3$, $DQ = t_4$.

We begin with the following

Lemma 1. *In every bicentric quadrilateral $ABCD$, with the usual notations, the following identities hold:*



(i) $t_1 + t_2 + t_3 + t_4 = s.$

(ii) $t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 = 2r\left(\sqrt{4R^2 + r^2} - r\right).$

(iii) $\sum\limits_{1\le i<j\le 4} t_it_j = 2r\sqrt{4R^2 + r^2}.$

(iv) $\sum\limits_{1\le i<j<k\le 4} t_it_jt_k = r^2s.$

(v) $t_1t_2t_3t_4 = r^4.$

Proof. Next, we give the proof of the identities claimed.

(i) We have $t_1+t_2 = a$, $t_2+t_3 = b$, $t_3+t_4 = c$, $t_4+t_1 = d$. Adding up these equalities the statement follows.

(ii) Using $\tan\dfrac{A}{2} = \dfrac{MI}{AM} = \sqrt{\dfrac{ad}{bc}}$ we obtain $\dfrac{r}{t_1} = \sqrt{\dfrac{ad}{bc}}$ or

$$t_1 = r\sqrt{\frac{bc}{ad}},\ t_2 = r\sqrt{\frac{cd}{ba}},\ t_3 = r\sqrt{\frac{ad}{bc}},\ t_4 = r\sqrt{\frac{ab}{cd}}.$$

So

$$t_1t_2 + t_2t_3 + t_3t_4 + t_4t_1 = r^2\left(\frac{a}{c} + \frac{b}{d} + \frac{c}{a} + \frac{d}{b}\right)$$
$$= r^2\left(\frac{(a+c)^2}{ac} - 2 + \frac{(b+d)^2}{bd} - 2\right) = r^2\left(s^2\frac{ac+bd}{abcd} - 4\right)$$
$$= r^2\left(\frac{ac+bd}{r^2} - 4\right) = r^2\left(\frac{ac+bd-4r^2}{r^2}\right)$$
$$= ac + bd - 4r^2 = 2r\sqrt{4R^2+r^2} - 2r^2.$$

(iii) We have $t_1t_3 = t_2t_4 = r^2$. So from (ii) we have

$$\sum_{1\le i<j\le 4} t_it_j = 2r\sqrt{4R^2+r^2} - 2r^2 + 2r^2 = 2r\sqrt{4R^2+r^2}.$$

(iv) We have

$$\sum_{1\le i<j\le 4} t_it_jt_k = r^2\left(\sqrt{\frac{ab}{cd}} + \sqrt{\frac{cb}{da}} + \sqrt{\frac{dc}{ab}} + \sqrt{\frac{ad}{bc}}\right)$$
$$= r^3\left(\frac{ab+dc}{\sqrt{abcd}} + \frac{bc+ad}{\sqrt{abcd}}\right) = r^3\frac{ab+dc+bc+ad}{rs} = r^2s.$$

(v) Since $t_1t_3 = t_2t_4 = r^2$ then $t_1t_2t_3t_4 = r^4$.

2 Main results

We now present the main results of this work.

Theorem 1. *Let $ABCD$ be a bicentric quadrilateral. Then t_1, t_2, t_3, t_4 are the roots of a four degree equation with the coefficient depending only on a, b, c, d.*

Proof. Recall that t_1, t_2, t_3, t_4 are the roots of the quartic equation $t^4 - \sigma_1 t^3 + \sigma_2 t^2 - \sigma_3 t + \sigma_4 = 0$, where according to the Lemma $\sigma_1 = s$ and $\sigma_2 = 2r\sqrt{4R^2+r^2}$. Since $ac+bd = 2r\sqrt{4R^2+r^2}+2r^2$ then $\sigma_2 = ac + bd - 2r^2 - ac + bd - \dfrac{2abcd}{s^2}$, $\sigma_3 = r^2 s = \dfrac{abcd}{s}$, $\sigma_4 = 4r^4 = \dfrac{4a^2b^2c^2d^2}{s^4}$, and therefore $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ depend only on a, b, c, d. Then, we may write the equation as

$$t^4 - \frac{a+b+c+d}{2}t^3 + \left(ac + bd - \frac{8abcd}{(a+b+c+d)^2}\right)t^2$$
$$-\frac{2abcd}{a+b+c+d}t + \frac{16a^2b^2c^2d^2}{(a+b+c+d)^4} = 0$$

which roots are t_1, t_2, t_3, t_4.

Theorem 2. *Let $ABCD$ be a bicentric quadrilateral. Then t_1, t_2, t_3, t_4 are the roots of a four degree equation with coefficients depending only on R, r and s.*

Proof. From lemma we have

$$t^4 - st^3 + 2r\sqrt{4R^2 + r^2t^2} - r^2st + r^4 = 0$$

which roots are t_1, t_2, t_3, t_4.

Corollary 1. *Let $ABCD$ be a bicentric quadrilateral. Then t_1, t_3 are the roots of the equation $st^2 - x_1t + sr^2 = 0$ and t_2, t_4 are the roots of the equation $st^2 - x_2t + sr^2 = 0$.*

Proof. We have $\dfrac{t_1^2}{r^2} = \dfrac{bc}{ad}$ or $\dfrac{t_1^2 + r^2}{r^2} = \dfrac{x_1}{ad}$. Also $\dfrac{t_1^2 + r^2}{t_1^2} = \dfrac{x_1}{bc}$. Multiplying up these two equalities, we obtain

$$\left(\frac{t_1^2 + r^2}{rt_1}\right)^2 = \left(\frac{x_1}{sr}\right)^2 \Leftrightarrow (t_1^2 + r^2)s = x_1t \Leftrightarrow st_1^2 - x_1t_1 + r^2s = 0.$$

Likewise, we get $st_3^2 - x_1t_3 + r^2s = 0$, $st_2^2 - x_2t_2 + r^2s = 0$, and $st_4^2 - x_2t_4 + r^2s = 0$.

Corollary 2. *In every bicentric quadrilateral the following equality holds:*

$$\begin{aligned}&t^4 - st^3 + 2r\sqrt{4R^2 + r^2}\,t^2 - r^2st + r^4\\ =&\Big(t^2 - \frac{x_1}{s}\,t + r^2\Big)\Big(t^2 - \frac{x_2}{s}\,t + r^2\Big).\end{aligned}$$

Proof. We have

$$\begin{aligned}&\Big(t^2 - \frac{x_1}{s}\,t + r^2\Big)\Big(t^2 - \frac{x_2}{s}\,t + r^2\Big)\\ &= t^4 - \left(\frac{x_1 + x_2}{s}\right)t^3 + \Big(2r^2 + \frac{x_1x_2}{s^2}\Big)t^2 - \frac{r^2}{s}(x_1 + x_2)t + r^4\\ &= t^4 - st^3 + \left(2r^2 + \frac{16R^2r^2}{2r\big(\sqrt{4R^2 + r^2} + r\big)}\right)t^2 - r^2st + r^4\\ &= t^4 - st^3 + 2r\sqrt{4R^2 + r^2}\,t^2 - r^2st + r^4.\end{aligned}$$

Solving the equations in Corollary 1 we may write t_1, t_2, t_3, t_4 in terms of x_1, x_2, r.

Corollary 3. *In every bicentric quadrilateral we have*

$$\{t_1, t_3\} = \frac{x_1 \pm \sqrt{x_1^2 - 4r^2s^2}}{2s} \quad \textit{and} \quad \{t_1, t_3\} = \left\{\frac{x_2 \pm \sqrt{x_2^2 - 4r^2s^2}}{2s}\right\}.$$

Proof. It follows from Corollary 1.

Note that, since $x_1 = ab + cd$, $x_2 = ac + bd$, then t_1, t_2, t_3, t_4 can be expressed in terms of a, b, c, d.

Corollary 4. *In every bicentric quadrilateral it holds:*

$$\begin{aligned}&(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2\\ &= (t_1-t_3)^4(t_2-t_4)^4\big[(t_1-t_3)^2-(t_2-t_4)^2\big].\end{aligned}$$

Proof. Since $a = t_1 + t_2$, $b = t_2 + t_3$, $c = t_3 + t_4$, $d = t_4 + t_1$, then

$$\begin{aligned}&(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2\\ =&(t_1-t_3)^2(t_1+t_2-t_3-t_4)(t_2-t_4)^2\\ &(t_2-t_4)^2(t_2+t_3-t_1-t_4)^2(t_3-t_1)^2\\ =&(t_1-t_3)^4(t_2-t_4)^4\big[(t_1-t_3)^2-(t_2-t_4)^2\big]^2.\end{aligned}$$

In the following we will write t_1, t_2, t_3, t_4 using only a, b, c, d.

Corollary 5. *In every bicentric quadrilateral the following holds:*

$$t_1 = \frac{bc}{s}, \; t_2 = \frac{cd}{s}, \; t_3 = \frac{da}{s}, \; t_4 = \frac{ab}{s}.$$

Proof. We have

$$t_1 = r\sqrt{\frac{bc}{ad}} = \frac{F}{s}\sqrt{\frac{bc}{ad}} = \frac{\sqrt{abcd}}{s}\sqrt{\frac{bc}{ad}} = \frac{bc}{s}.$$

In the same way we get t_2, t_3 and t_4.

In [3] was published Blundon-Eddy's inequality for bicentric quadrilaterals. It also was proven in [1] and [2] using algebraic and geometric arguments. This inequality claims that in any bicentric quadrilateral $ABCD$ it holds that $s_1 \leq s \leq s_2$, where $s_1 = \sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}$ and $s_2 = \sqrt{4R^2+r^2}+r$. In the next corollary we give a new prove of this result. Other sources of this kind of results are [5] and [4].

Corollary 6. *In every bicentric quadrilateral the following identity holds:*

$$\begin{aligned}&(a-b)^2(a-c)^2(a-d)^2(b-c)^2(b-d)^2(c-d)^2\\&=16s^2r^4\Big[\Big(\sqrt{4R^2+r^2}\Big)^2-s^2\Big]\Big[s^2-8r\Big(\sqrt{4R^2+r^2}-r\Big)\Big].\end{aligned}$$

Proof. We compute

$$\begin{aligned}&s^8(t_1-t_3)^4(t_2-t_3)^4=(ad-bc)^4(ab-dc)^4\\&=\Big[(ad+bc)^2-4abcd\Big]^2\Big[(ab+dc)^2-4abcd\Big]^2\\&=\Big[(x_1^2-4F)^2(x_2^2-4F)^2\Big]^2=\Big[(x_1x_2)^2-4F^2(x_1^2+x_2^2)+16F^4\Big]^2\\&=\Big[4r^2\Big(\sqrt{4R^2+r^2}-r\Big)^2s^4-4s^6r^2+16s^4r^3\Big(\sqrt{4R^2+r^2}-r\Big)+16s^4r^4\Big]^2\\&=4^2s^8r^4\Big[\Big(\sqrt{4R^2+r^2}-r\Big)^2-s^2+4r\Big(\sqrt{4R^2+r^2}-r\Big)+4r^2\Big]\\&=4^2r^4s^8\Big[\Big(\sqrt{4R^2+r^2}+r\Big)^2-s^2\Big]^2.\end{aligned}$$

Also

$$\begin{aligned}&s^4\Big[(t_1-t_3)^2-(t_2-t_4)^2\Big]^2=\Big[(ad-bc)^2-(ab-cd)^2\Big]^2\\&=\Big[(ad+bc)^2-4abcd-(ab+cd)^2+4abcd\Big]^2=(x_1^2-x_2^2)^2\\&=(x_1^2+x_2^2)^2-4x_1^2x_2^2=\Big[(x_1+x_2)^2-2x_1x_2\Big]^2-4x_1^2x_2^2\\&=(s^4-2x_1x_2)^2-4x_1^2x_2^2=s^8-4s^4x_1x_2\\&=s^4\Big[s^4-8r\Big(\sqrt{4R^2+r^2}-r\Big)s^2\Big]\\&=s^6\Big[s^2-8r\Big(\sqrt{4R^2+r^2}-r\Big)\Big].\end{aligned} \tag{1}$$

We obtain

$$(t_1-t_3)^4(t_2-t_4)^4=16r^4\Big[\Big(\sqrt{4R^2+r^2}+r\Big)^2-s^2\Big]^2$$

and

$$\left[(t_1 - t_3)^2 - (t_2 - t_4)^2\right] = s^2\left[s^2 - 8r\left(\sqrt{4R^2 + r^2} - r\right)\right].$$

From (1) and Corollary 4 we obtain the statement.

Let $M = AB \cap \mathcal{C}(I, r)$. We have

Lemma 2. *Let $ABCD$ be a bicentric quadrilateral. Then*

$$MO = \sqrt{R^2 - \frac{bc^2d}{s^2}}.$$

Proof. We have $R^2 - MO^2 = t_1t_2$ or $MO = \sqrt{R^2 - \dfrac{bc^2d}{s^2}}$.

Corollary 7. *In every bicentric quadrilateral it holds:*

$$MO^2 + NO^2 + PO^2 + QO^2 = \left(\sqrt{4R^2 + r^2} - r\right)^2.$$

Proof From the preceding, we have

$$MO^2 + NO^2 + PO^2 + QO^2 = 4R^2 - \left(\frac{a^2bd}{s^2} + \frac{ab^2c}{s^2} + \frac{bc^2d}{s^2} + \frac{ad^2c}{s^2}\right)$$

$$= 4R^2 - \frac{ac(b^2+d^2) + bd(a^2+c^2)}{s^2} = 4R^2 - \frac{ac(s^2-2bd) + bd(s^2-2ac)}{s^2}$$

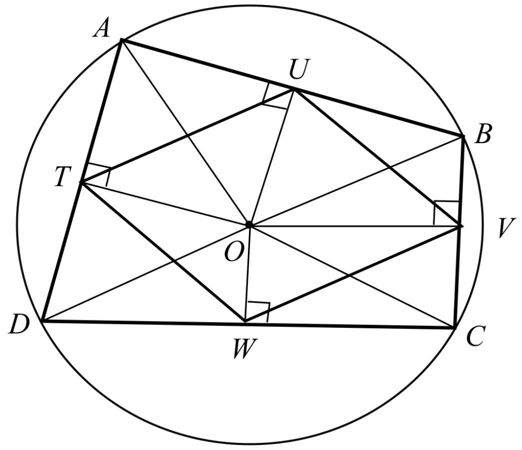
$$= 4R^2 - \frac{s^2(ac+bd) - 4abcd}{s^2} = 4R^2 - ac - bd + 4r^2$$

$$= 4R^2 - 2r\sqrt{4R^2 + r^2} - 2r^2 + 4r^2 = \left(\sqrt{4R^2 + r^2} - r\right)^2.$$

To prove the next theorem we need the following lemma:

Lemma 3. *In every bicentric quadrilateral it holds:*

$$\sqrt{4R^2 - a^2} + \sqrt{4R^2 - b^2} + \sqrt{4R^2 - c^2} + \sqrt{4R^2 - d^2} = 2\sqrt{4R^2 + r^2} + 2r$$



Proof. Let $OU \perp AB$, $OV \perp BC$, $OW \perp DC$, $OT \perp AD$.

We denote $OU = x$, $OV = y$, $OW = z$, $OT = t$, $BD = d_1$, $AC = d_2$, $UT = VW = \frac{d_1}{2}$, $UV = TW = \frac{d_2}{2}$, $OA = OB = OC = OD = R$. From Ptolemy's theorem applied to cyclic quadrilaterals $AUOT$, $BUOV$, $CVOW$, $OTDW$, we have $\frac{xd}{2} + \frac{at}{2} = \frac{d_1 R}{2}$, $\frac{xb}{2} + \frac{ya}{2} = \frac{d_2 R}{2}$, $\frac{zb}{2} + \frac{yc}{2} = \frac{d_1 R}{2}$, $\frac{zd}{2} + \frac{ct}{2} = \frac{d_2 R}{2}$. Adding these equalities we obtain

$$x(b+d) + y(a+c) + z(b+d) + t(a+c) = 2R(d_1 + d_2)$$

or $x + y + z + t = \frac{2R(d_1 + d_2)}{s}$. Let us denote by $\alpha = d_1 + d_2$. Then from Ptolemy theorem, we have $\frac{d_1}{d_2} = \frac{x_2}{x_1}$ and $d_1 d_2 = x_3 = 2r\left(\sqrt{4R^2 + r^2} + r\right)$. So $\frac{d_1}{\alpha} = \frac{x_2}{s^2}$ and $\frac{d_2}{\alpha} = \frac{x_1}{s^2}$.

By multiplying we obtain $\frac{d_1 d_2}{\alpha^2} = \frac{x_1 x_2}{s^4}$. But $x_1 x_2 = \frac{16R^2 r^2 s^2}{x_3}$.

We obtain $\frac{x_3}{\alpha^2} = \frac{16R^2 r^2}{x_3 s^2}$ or $\alpha = \frac{x_3 s}{4Rr}$.

So we obtain $x + y + z + t = \sqrt{4R^2 + r^2} + r$.

But $x = \frac{1}{2}\sqrt{4R^2 - a^2}$. So $\frac{1}{2}\sum_{cyc}\sqrt{4R^2 - a^2} = \sqrt{4R^2 + r} + r$ or

$$\sum_{cyc}\sqrt{4R^2 - a^2} = 2\sqrt{4R^2 + r^2} + 2r. \quad (2)$$

This completes the proof.

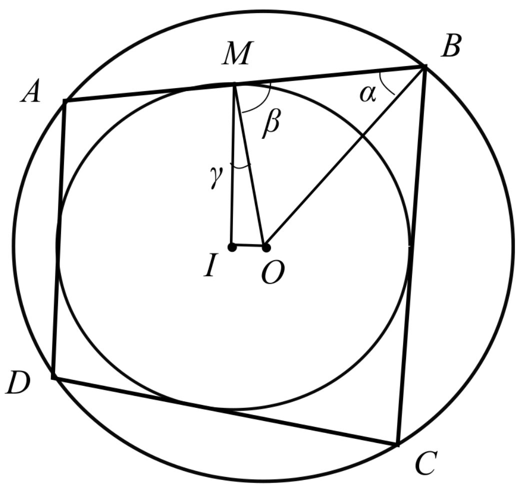
Theorem 3 (Fuss). *In every bicentric quadrilateral the following equality holds:* $\overline{d}^2 = R^2 + r^2 - r\sqrt{4R^2 + r^2}$.

Proof. On account to Sine Law applied to triangle MOA, we have

$$\frac{R}{\sin\alpha} = \frac{a}{\sin(\pi - 2\alpha)} \text{ or } \cos\alpha = \frac{a}{2R}.$$

From Sine Law applied to triangle MOB we have $\frac{BO}{\sin\beta} = \frac{MO}{\sin\alpha}$ or

$$\sin\beta = \frac{BO}{MO}\sin\alpha = \frac{R}{MO}\sqrt{1 - \frac{a^2}{4R^2}} = \frac{1}{2MO}\sqrt{4R^2 - a^2}.$$



We have

$$\cos\gamma = \cos\left(\frac{\pi}{2} - \beta\right) = \sin\beta = \frac{1}{2MO}\sqrt{4R^2 - a^2}.$$

From Cosine Law applied to triangle MIO, we get

$$\overline{d}^2 = MI^2 + MO^2 - 2MI\,MO\cos\gamma = r^2 + R^2 - \frac{bc^2d}{s^2} - r\sqrt{4R^2 - a^2}.$$

In the same way, we obtain

$$\overline{d}^2 = R^2 + r^2 - \frac{ad^2c}{s^2} - r\sqrt{4R^2 - b^2}$$
$$\overline{d}^2 = R^2 + r^2 - \frac{ba^2d}{s^2} - r\sqrt{4R^2 - c^2} \quad \text{and}$$
$$\overline{d}^2 = R^2 + r^2 - \frac{cd^2a}{s^2} - r\sqrt{4R^2 - d^2}$$

Adding up the preceding, yields

$$4\overline{d}^2 = 4R^2 + 4r^2 - \frac{1}{s^2}\left(a^2bd + bc^2d + ad^2c + ab^2c\right)$$
$$-r\left(\sqrt{4R^2 - a^2} + \sqrt{4R^2 - b^2} + \sqrt{4R^2 - c^2} + \sqrt{4R^2 - d^2}\right) \qquad (3)$$

We have

$$\frac{1}{s^2}\left(a^2bd + bc^2d + ad^2c + ab^2d\right) = \frac{1}{s^2}\left[(a^2 + c^2)bd + (b^2 + d^2)ac\right]$$
$$= \frac{1}{s^2}\left(s^2(ac + bd) - 4s^2r^2\right) = ac + bd - 4r^2 \qquad (4)$$

On account of the previous lemma, we have

$$4\overline{d}^2 = 4R^2 + 4r^2 - 2r\sqrt{4R^2 + r^2} - 2r^2 + 4r^2 - 2r\sqrt{4R^2 + r^2} - 2r^2 \quad \text{or}$$
$$4\overline{d}^2 = 4R^2 + 4r^2 - 4r\sqrt{4R^2 + r^2}$$

from which

$$\overline{d}^2 = R^2 + r^2 - r\sqrt{4R^2 + r^2}$$

follows.

Corollary 8. *Let M, N, P, Q be the points where AB, BC, CD, DA cut $\mathcal{C}(I, r)$. Then*

$$MN = 2r\sqrt{\frac{t_2}{t_2 + t_4}}, \quad NP = 2r\sqrt{\frac{t_3}{t_1 + t_3}},$$
$$PQ = 2r\sqrt{\frac{t_4}{t_2 + t_4}}, \quad MQ = 2r\sqrt{\frac{t_1}{t_1 + t_3}}.$$

Proof. We have $AI = \dfrac{r}{\sin\frac{A}{2}}$. As $AMIQ$ is cyclic, by Ptolemy's theorem we get $AM \cdot QI + MI \cdot AQ = MQ \cdot AI$ or $MQ \cdot AI = 2t_1 r$, or

$$MQ = \frac{2t_1 r}{r/\sin\frac{A}{2}} = 2t_1 \sin\frac{A}{2} = 2t_1\sqrt{\frac{ad}{ad+bc}}$$
$$= 2t_1\sqrt{\frac{t_3}{t_1+t_3}} = 2\sqrt{t_1t_3}\sqrt{\frac{t_1}{t_1+t_3}} = 2r\sqrt{\frac{t_1}{t_1+t_3}}$$

Corollary 9. *In every bicentric quadrilateral, with the notation from above it holds* $MP \perp QN$.

Proof. From Corollary 8 we have $MQ^2 + NP^2 = MN^2 + QP^2$. So $MNPQ$ is orthodiagonal.

Corollary 10. *In every bicentric quadrilateral, it holds:*

$$MN \cdot NP \cdot PQ \cdot QM = \frac{2r^5\left(\sqrt{4R^2+r^2}+r\right)}{R^2}.$$

Proof. From Corollary 8, we have

$$MN \cdot NP \cdot PQ \cdot QN = \frac{16\sqrt{t_1t_2t_3t_4}\,r^4}{(t_1+t_3)(t_2+t_4)} = \frac{16r^5}{2r\left(\sqrt{4R^2+r^2}-r\right)}$$
$$= \frac{2r^5}{R^2}\left(\sqrt{4R^2+r^2}+r\right).$$

Corollary 11. *In every bicentric quadrilateral the following inequality holds:*

$$4r\sqrt{1+\sqrt{\frac{2r}{\sqrt{4R^2+r^2}-r}}} \le MN + NP + PQ + QM \le$$
$$\le \frac{r\left(\sqrt{4R^2+r^2}+2\sqrt{2}R+r\right)}{2} \le 4\sqrt{2}\,r.$$

Proof. According to Corollary 8 we have $f : [s_1, s_2] \to \mathbb{R}$,

$$f(s) = MN + NP + PQ + QM$$

$$= 2r\left(\sqrt{\frac{t_2}{t_2+t_4}} + \sqrt{\frac{t_4}{t_2+t_4}} + \sqrt{\frac{t_1}{t_1+t_3}} + \sqrt{\frac{t_3}{t_1+t_3}}\right)$$

$$= 2r\left(\frac{\sqrt{ab}+\sqrt{cd}}{\sqrt{ab+cd}} + \frac{\sqrt{ad}+\sqrt{bc}}{\sqrt{ad+bc}}\right) = 2r\left(\sqrt{\frac{x_1+2sr}{x_1}} + \sqrt{\frac{x_2+2sr}{x_2}}\right)$$

$$= 2r\sqrt{2 + \frac{2sr(x_1+x_2)}{x_1x_2} + 2\sqrt{\frac{x_1x_2 + 2sr(x_1+x_2) + 4s^2r^2}{x_1x_2}}}$$

$$= 2r\sqrt{2 + \frac{2sr\ s^2\ x_3}{16R^2r^2s^2} + 2\sqrt{1 + \frac{2sr\ s^2\ x_3}{16R^2r^2s^2} + \frac{4s^2r^2x_3}{16R^2r^2s^2}}}$$

$$= 2r\sqrt{2 + \frac{sx_3}{8R^2r} + 2\sqrt{1 + \frac{sx_3}{8R^2r} + \frac{x_3}{4R^2}}},$$

which is increasing in s. So, $f(s_1) \le f(s) \le f(s_2)$ or

$$\begin{aligned} M_1N_1 + N_1P_1 + P_1Q_1 + Q_1M_1 &\le MN + NP + PQ + QN \\ &\le M_2N_2 + N_2P_2 + P_2Q_2 + Q_2M_2, \end{aligned}$$

where M_1, N_1, P_1, Q_1 are the intersection points of the incircle with the sides of $A_1B_1C_1D_1$ and M_2, N_2, P_2, Q_2 the intersection points of the sides $A_2B_2C_2D_2$ with incircle $\mathcal{C}(I, r)$ on account of Blundon theorem. We have

$$M_1N_1+N_1P_1+P_1Q_1+Q_1M_1 = 2r\left(\frac{\sqrt{a_1b_1}+\sqrt{c_1d_1}}{\sqrt{a_1b_1+c_1d_1}} + \frac{\sqrt{a_1d_1}+\sqrt{b_1c_1}}{\sqrt{a_1d_1+b_1c_1}}\right)$$

From Blundon-Eddy theorem we have

$$\begin{aligned} a_1 = c_1 &= \sqrt{R^2-(r-d)^2} + \sqrt{R^2-(r+d)^2} \\ b_1 &= 2\sqrt{R^2-(r-d)^2}, \quad d_1 = 2\sqrt{R^2-(r+d)^2} \end{aligned}$$

So

$$M_1N_1 + N_1P_1 + P_1Q_1 + Q_1M_1 = 2\frac{\sqrt{b_1}+\sqrt{d_1}}{\sqrt{b_1+d_1}} = 2\sqrt{\frac{b_1+d_1+2\sqrt{b_1d_1}}{b_1+d_1}}$$

$$= 2\sqrt{1+\frac{2\sqrt{4r^2}}{s_1}} = 2\sqrt{1+\frac{4r}{\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)}}} = 2\sqrt{1+\sqrt{\frac{2r}{\sqrt{4R^2+r^2}}}}$$

Also

$$M_2N_2+N_2P_2+P_2Q_2+Q_2M_2 = 2r\left(\frac{\sqrt{a_2b_2}+\sqrt{c_2d_2}}{\sqrt{a_2b_2+c_2d_2}}+\frac{\sqrt{a_2d_2}+\sqrt{b_2c_2}}{\sqrt{a_2d_2+b_2c_2}}\right).$$

From Blundon theorem we have

$$a_2 = b_2 = \frac{2R}{R+d}\sqrt{(R+d)^2-r^2}\,, \quad c_2 = d_2 = \frac{2R}{R-d}\sqrt{(R-d)^2-r^2}$$

So

$$\begin{gathered} M_2N_2+N_2P_2+P_2Q_2+Q_2M_2 \\ = \frac{a_2+c_2}{\sqrt{a_2^2+c_2^2}}+\sqrt{2} = \frac{a_2+c_2}{\sqrt{(a_2+c_2)^2-2a_2c_2}} = \frac{s_2}{\sqrt{s_2^2-\frac{16R^2r^2}{R^2-d^2}}}. \end{gathered}$$

We have

$$\begin{aligned} a_2c_2 &= \frac{4R^2}{R^2-d^2}\sqrt{[(R+d)^2-r^2][(R-d)^2-r^2]} \\ &= \frac{4R^2}{R^2-d^2}\sqrt{[(R-r)^2-d^2][(R+r)^2-d^2]} \\ &= \frac{4R^2}{R^2-d^2}\sqrt{\left(r\sqrt{4R^2+r^2}-2Rr\right)\left(r\sqrt{4R^2+r^2}+2Rr\right)} = \frac{4R^2r^2}{R^2-d^2}. \end{aligned}$$

So

$$M_2N_2+N_2P_2+P_2Q_2+Q_2M_2 = \frac{\sqrt{4R^2+r^2}+r}{\sqrt{\left(\sqrt{4R^2+r^2}+r\right)^2-\frac{8R^2r^2}{R^2-d^2}}}+\sqrt{2}.$$

We have

$$\begin{gathered} \left(\sqrt{4R^2+r^2}+r\right)^2-\frac{8^2r^2}{R^2-d^2} \\ = 4R^2+2r^2+2r\sqrt{4R^2+r^2}-\frac{8R^2r^2}{r\left(\sqrt{4R^2+r^2}-r\right)} \\ = 4R^2+2r^2+2r\sqrt{4R^2+r^2}-2r\left(\sqrt{4R^2+r^2}+r\right) = 4R^2. \end{gathered}$$

We obtain

$$M_2N_2+N_2P_2+P_2Q_2+Q_2M_2 = \frac{\left(\sqrt{4R^2+r^2}+r+2\sqrt{2}R\right)r}{R}\,.$$

Corollary 12. *In every bicentric quadrilateral the following inequality is true:*

$$MN + NP + PQ + QM \geq \frac{8r^2}{R}$$

Proof. From Corollary 11, if we denote $\frac{R}{r} = x \geq \sqrt{x}$, it will be sufficient to prove that

$$\sqrt{1 + \sqrt{\frac{2}{\sqrt{4x^2+1}-1}}} \geq \frac{2}{x} \quad \text{or} \quad x^2 + x^2\sqrt{\frac{2}{\sqrt{4x^2+1}-1}} \geq 4,$$

or

$$x^2\sqrt{\frac{2}{\sqrt{4x^2+1}-1}} \geq 4 - x^2. \tag{5}$$

If $x \geq \sqrt{2}$ the inequality is true.
We will prove that inequality (5) is true for each $\sqrt{2} \leq x \leq 2$.
If we denote $x^2 = y$, we will prove that $y\sqrt{\frac{2}{\sqrt{4y+1}-1}} \geq 4 - y$, for all $y \in [2, 4]$ or $2y^2 \geq (4-y)^2\sqrt{4y+1} - (4-y)^2$, for all $y \in [2, 4]$, or $3y^2 - 8y + 16)^2 \geq (4-y)^4(4y+1)$, for all $y \in [2, 4]$, or $(y-2)(y^3 - 16y^2 + 72y - 128) \leq 0$, for all $y \in [2, 4]$, or $y^3 - 16y^2 + 72y - 128 \leq 0$, for all $y \in [2, 4]$. Since $y^3 - 16y^2 + 72y - 128 = 0$ has only the real root $y_0 \simeq 10,148$ and $f : \mathbb{R} \to \mathbb{R}$ is an increasing function, it follows that $f(y) \leq 0$, for all $y \in [2, 4]$.

Corollary 13. *In every bicentric quadrilateral the following inequality is true:*

$$\frac{32r^4\left(\sqrt{4R^2+r^2}-2r\right)}{\sqrt{4R^2+r^2}-r} \geq MN^4 + NP^4 + PQ^4 + QM^4 \geq$$
$$\geq \frac{r^4\left(24R^4 - 8R^2r^2 - 4r^4 - 4r^3\sqrt{4R^2+r^2}\right)}{R^4}$$

Proof. From Corollary 8 we have $f : [s_1, s_2] \to \mathbb{R}$

$$f(s) = MN^4 + NP^4 + PQ^4 + QM^4 = 16r^4\left[\frac{t_1^2 + t_3^2}{(t_1 + t_3)^2} + \frac{t_2^2 + t_4^2}{(t_2 + t_4)^2}\right]$$
$$= 16r^4\left[\frac{\frac{ad}{bc} + \frac{bc}{ad}}{\frac{ad}{bc} + \frac{bc}{ad} + 2} + \frac{\frac{ab}{cd} + \frac{cd}{ab}}{\frac{ab}{cd} + \frac{cd}{ab} + 2}\right]$$
$$= 16r^4\left[\frac{(ad + bc)^2 - 2abcd}{(ad + bc)^2} + \frac{(ab + cd)^2 - 2abcd}{(ab + dc)^2}\right]$$
$$= 16r^4\left(\frac{x_1^2 - 2r^2s^2}{x_1^2} + \frac{x_2^2 - 2r^2s^2}{x_2^2}\right) = 16r^4\left(2 - \frac{2r^2s^2(x_1^2 + x_2^2)}{x_1^2x_2^2}\right)$$
$$= 16r^4\left[2 - \frac{2r^2s^2(s^4 - 2x_1x_2)}{x_1^2x_2^2}\right] = 16r^4\left[2 - \frac{2r^2s^2\left(s^4 - \frac{32R^2r^2s^2}{x_3}\right)}{\frac{256R^2r^2s^2}{x_3^2}}\right]$$
$$= 16r^4\left[2 - \frac{x_3(x_3s^4 - 32R^2r^2s^2)}{128R^2}\right].$$

From Blundon-Eddy inequality we have

$$x_3s^2 \geq x_3 \cdot 8r\left(\sqrt{4R^2 + r^2} - r\right)$$
$$= 16r^2\left(\sqrt{4R^2 + r^2} + r\right)\left(\sqrt{4R^2 + r^2} - r\right) = 64R^2r^2 \geq 32R^2r^2,$$

or $x_3s^2 > 32R^2r^2$.
So, if we consider the function $g : [s_1, s_2] \to \mathbb{R}$, defined by

$$g(s) = x_3s^4 - 32R^2r^2s^2 = s^2(x_3s^2 - 32R^2r^2),$$

then g is an increasing function because it is a product of two positive increasing functions.
It follows that $f : [s_1, s_2] \to \mathbb{R}$, $f(s) = 16r^4\left(2 - \frac{x_3g(s)}{128R^2}\right)$ is a decreasing function on $[s_1, s_2]$, or $f(s_2) \leq f(s) \leq f(s_1)$, for all $s \in [s_1, s_2]$, or

$$M_1N_1^4 + N_1P_1^4 + P_1Q_1^4 + Q_1M_1^4 \geq MN^4 + NP^4 + PQ^4 + QM^4 \geq$$
$$\geq M_2N_2^4 + N_2P_2^4 + P_2Q_2^4 + Q_2M_2^4.$$

We have

$$\begin{aligned}&M_1N_1^4+N_1P_1^4+P_1Q_1^4+Q_1M_1^4\\&=16r^4\left(\frac{a_1^2d_1^2+b_1^2c_1^2}{(a_1d_1+b_1c_1)^2}+\frac{a_1^2b_1^2+c_1^2d_1^2}{(a_1b_1+c_1d_1)^2}\right)\end{aligned}$$

$$\begin{aligned}&32r^4\frac{b_1^2+d_1^2}{(b_1+d_1)^2}\pm 32r^4\frac{(b_1+d_1)^2-2b_1d_1}{(b_1+d_1)^2}=32r^4\,\frac{s_1^2-8r^2}{s_1^2}\\&=\frac{32r^4\left(8r\left(\sqrt{4R^2+r^2}-r\right)-8r^2\right)}{8r\left(\sqrt{4R^2+r^2}-r\right)}=\frac{32r^4\left(\sqrt{4R^2+r^2}-2r\right)}{\sqrt{4R^2+r^2}-r}\,.\end{aligned}$$

Also

$$\begin{aligned}&M_1N_1^4+N_1P_1^4+P_1Q_1^4+Q_1M_1^4\\&=16r^4\left(\frac{a_2^2d_2^2+b_2^2c_2^2}{(a_2d_2+b_2c_2)^2}+\frac{a_2^2b_2^2+c_2^2d_2^2}{(a_2b_2+c_2d_2)^2}\right)\\&=16r^4\left(\frac{2a_2^2c_2^2}{4a_2^2c_2^2}+\frac{a_2^4+c_2^4}{(a_2^2+c_2^2)^2}\right)=16r^4\left(\frac{1}{2}+\frac{a_2^4+c_2^4}{(a_2^2+c_2^2)^2}\right).\end{aligned}$$

We have $a_2+c_2=s_2=\sqrt{4R^2+r^2}+r$. Also

$$a_2c_2=\frac{4R^2r^2}{R^2-d^2}=r\left(\sqrt{4R^2+r^2}+r\right).$$

We have

$$\begin{aligned}a_2^2+c_2^2&=(a_2+c_2)^2-2a_2c_2\\&=\left(\sqrt{4R^2+r^2}+r\right)^2-2r\left(\sqrt{4R^2+r^2}+r\right)\\&=\left(\sqrt{4R^2+r^2}+r\right)\left(\sqrt{4R^2+r^2}-r\right)=4R^2.\end{aligned}$$

Also

$$\begin{aligned}a_2^4+c_2^4&=(a_2^2+c_2^2)^2-2a_2^2c_2^2=16R^4-2r^2\left(\sqrt{4R^2+r^2}+r\right)^2\\&=16R^4-8R^2r^2-4r^4-4r^3\sqrt{4R^2+r^2}.\end{aligned}$$

So

$$\begin{aligned}&M_2N_2^4+N_2P_2^4+P_2Q_2^4+Q_2M_2^4\\&=16r^4\left(\frac{1}{2}+\frac{16R^4-8R^2r^2-4r^4-4r^3\sqrt{4R^2+r^2}}{16R^4}\right)\\&=r^4\frac{24R^4-8R^2r^2-4r^4-4r^3\sqrt{4R^2+r^2}}{R^4},\end{aligned}$$

from which we obtain the statement.

Corollary 14. *In every bicentric quadrilateral the following inequality is true:*

$$16r^4\leq MN^4+NP^4+PQ^4+QM^4\leq 8R^2r^2$$

Proof. From Corollary 13 it results that we have to prove

$$MN^4+NP^4+PQ^4+QM^4\geq 16r^4.$$

if we denote $x=\dfrac{R}{r}\geq\sqrt{2}$. It will be sufficient to prove that,

$$\frac{24x^4-8x^2-4\sqrt{4x^2+1}}{x^4}\geq 16,\ \text{for all } x\geq\sqrt{2}$$

or $(2x^4-2x^2-1)^2\geq 4x^2+1$, for all $x\geq\sqrt{2}$, or $4x^6(x^2-2)\geq 0$, which is true.
The RHS of the inequality of the statement is equivalent, using Corollary 13, to $32\left(\sqrt{4x^2+1}-2\right)\leq 8x^2\left(\sqrt{4x^2+1}-1\right)$ or

$$(x^2-4)\sqrt{4x^2+1}\geq x^2-8,\ \text{for all } x\geq\sqrt{2}.$$

If $4\leq x^2\leq 8$, the inequality is true. If $2\leq x^2\leq 4$, the inequality is equivalent (if we denote $x^2=y$) to
$(8-y)^2\geq(4-y)^2(4y+1)$, for all $y\in[2,4]$, or $(y-2)(y^2-6y+6)\leq 0$, for all $y\in[2,4]$, or $y^2-6y+6\leq 0$, or $y\in\left[3-\sqrt{3},3+\sqrt{3}\right]$. If $x^2\geq 8$ the inequality is equivalent to $(y-4)^2(4x^2+1)\geq(y-8)^2$, for all $y\geq 8$, or $(y-2)(y^2-6y+6)\geq 0$, which is true since $y\geq 8$. Now, we refine RHS of the previous inequality.

Corollary 15. *In every bicentric quadrilateral the following inequality is true:*

$$MN^4 + NP^4 + PQ^4 + QM^4 \le \sqrt[4]{2^{13}}\sqrt{R^3}\sqrt{r}.$$

Proof. In the same way as in Corollary 14, to prove the inequality from statement, it will be sufficient to prove that

$$32\left(\sqrt{4x^2+1}-2\right) \le \sqrt[4]{2^{13}}\sqrt{x^3}\left(\sqrt{4x^2+1}-1\right), \text{ for all } x \ge \sqrt{2}\,,$$

which can be verified using a Computer System Algebra like Wolphram Alpha.

We finish the paper with a proof of th efollowing result.

Theorem 4 (Blundon-Eddy). *In every bicentric quadrilateral the following inequality holds:*

$$\sqrt{8r\left(\sqrt{4R^2+r^2}-r\right)} \le s \le \sqrt{4R^2+r^2}+r.$$

Proof. On account of Lemma 1, the LHS of the inequality using the equalities $s = t_1 + t_2 + t_3 + t_4$ and $(t_1 + t_3)(t_2 + t_4) = 2r\left(\sqrt{4R^2+r^2}-r\right)$, is equivalent to $4(t_1 + t_3)(t_2 + t_4) \le (t_1 + t_3 + t_2 + t_4)$ which is true according to AM-GM inequality. Also

$$\sqrt{4R^2+r^2}-r = \frac{(t_1+t_3)(t_2+t_4)}{2r}$$

or

$$\sqrt{4R^2+r^2}+r = \frac{(t_1+t_3)(t_2+t_4)}{2r}+2r$$

or, since $r = \sqrt[4]{t_1t_2t_3t_4}$, we have

$$\sqrt{4R^2+r^2}+r = \frac{(t_1+t_3)(t_2+t_4)}{2\sqrt[4]{t_1t_2t_3t_4}}+2\sqrt[4]{t_1t_2t_3t_4}.$$

So the RHS of the inequality is equivalent to

$$2\sqrt[4]{t_1t_2t_3t_4}(t_1+t_2+t_3+t_4) \le (t_1+t_3)(t_2+t_4)+4\sqrt{t_1t_2t_3t_4}$$

or, since $t_1t_3 = t_2t_4$,

$$2\sqrt{t_1t_3}\left(t_1 + t_2 + t_3 + \frac{t_1t_3}{t_2}\right) \leq (t_1 + t_3)\left(t_3 + \frac{t_1t_3}{t_2}\right) + 4t_1t_3$$

or

$$2\sqrt{t_1t_3}\left[t_1t_3 + t_2^2 + t_2(t_1 + t_3)\right] \leq (t_1 + t_3)(t_2^2 + t_1t_3) + 4t_1t_3t_2.$$

If we denote $x = t_2$, $y = t_1 + t_3$, $z = \sqrt{t_1t_3}$, we obtain

$$2z(x^2 + xy + z^2) \leq y(x^2 + z^2) + 4z^2x$$

or $(y - 2z)(x^2 + z^2 - xz) \geq 0$, which is true because on account of AM-GM inequality $t_1 + t_3 \geq 2\sqrt{t_1t_3}$ and $y \geq 2z$.

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Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.
2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

October 30, 2023

Elementary Problems

E–113. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* For every positive integer n we define a_n as the last digit of the sum of the first n positive integers. Compute $a_1 + a_2 + \cdots + a_{2023}$.

E–114. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.* Let P, Q, R be points on the sides of a triangle ABC which trisect the perimeter of $\triangle ABC$. Suppose that P, Q lie on side AB. Prove that

$$\frac{\text{Area }(\triangle PQR)}{\text{Area }(\triangle ABC)} > \frac{2}{9}.$$

E–115. *Proposed by Goran Conar, Varaždin, Croatia.* Let d_a, d_b, d_c be distances from center of circumcircle to the sides of triangle ABC and let r be radius of its incircle. Prove that for any real $p > 1$, it holds

$$d_a^p + d_b^p + d_c^p \geq 3r^p.$$

E–116. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Let $n \geq 0$ be an integer number. Prove that $N = 10^{n^3+3n^2+2n+2}$ can be written as a sum of four perfect cubes.

E–117. *Proposed by Mihaela Berindeanu, Bucharest.* Let $ABCD$ be a square. If X is the midpoint of the side AB, Y is taken on the extension of side AB, so that $BY = AB/3$, Z is the foot of the perpendicular drawn from X to DY and T is the midpoint of AZ, then show that $\angle TBA = \angle DBZ$.

E–118. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* Determine the integers $n \geq 0$ for which there exists a real number $a > 0$ such that

$$(a+11)^n + (a+13)^n + (a+17)^n = (a+12)^n + (a+14)^n + (a+15)^n.$$

Easy–Medium Problems

EM–113. *Proposed by José Luis Díaz-Barrero, Barcelona, Spain.* The equation $x^3 + Ax - B = 0$ has three real roots a, b, c. Determine the integers A and B with $AB < 0$ for which $a^6 + b^6 + c^6 = 277$.

EM–114. *Proposed by Michel Bataille, Rouen, France.* Let P be a point on the circumcircle of the triangle ABC and let A', B', C' be its orthogonal projections onto the lines BC, CA, AB, respectively. Prove that

$$\frac{B'C'^2 \cot A + C'A'^2 \cot B + A'B'^2 \cot C}{BC^2 \cot A + CA^2 \cot B + AB^2 \cot C} = \frac{1}{2}.$$

EM–115. *Proposed by Toyesh Prakash Sharma (Student) Agra College, Agra, India.* Show that for any $n \geq 1$, it holds that

$$F_n{}^{\frac{1}{F_n}} \left(\frac{1}{F_n}\right)^{F_n} + L_n{}^{\frac{1}{L_n}} \left(\frac{1}{L_n}\right)^{L_n} \geq 2F_{n+1}{}^{\frac{1}{F_{n+1}}} \left(\frac{1}{F_{n+1}}\right)^{F_{n+1}},$$

where F_n and L_n are the n^{th} Fibonacci and Lucas number, respectively.

EM–116. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let $1 = d_1 < d_2 < \cdots < d_k = n$ be all divisors of a positive integer n. Find all n, such that $k \geq 6$ and

$$45({d_4}^2 + {d_6}^2) = 2n^2.$$

EM–117. *Proposed by José Luis Díaz–Barrero, Barcelona, Spain.* Let F_n be the n^{th} Fibonacci number defined by $F_1 = 1, F_2 = 1$, and for all $n \geq 3$, $F_{n+1} = F_n + F_{n-1}$. Prove that for each positive integer n there is a Fibonacci number ending in at least n zeros.

EM–118. *Proposed by Goran Conar, Varaždin, Croatia.* The inradius of triangle ABC is $r = 1$. Prove that

$$\sum_{cyclic} \frac{1}{r_a + r_b}\left(1 + \frac{r_b}{r_c}\right)\left(1 + \frac{r_a}{r_c}\right) \geq 2,$$

where r_a, r_b, r_c are their exradii. When does equality occur?

Medium–Hard Problems

MH–113. *Proposed by José Luis Díaz–Barrero, Barcelona, Spain.* Let M be a subset of $\{1, 2, 3, \ldots, 2023\}$ such that for any three elements (not necessarily distinct) a, b, c of M we have $|a+b-c| > 12$. Determine the largest possible number of elements of M.

MH–114. *Proposed by Michel Bataille, Rouen, France.* Let r, s be positive integers with $r \leq s$. Prove that

$$\sum_{k=r}^{s} \binom{r+s}{k}^2 \leq 4 \sum_{k=r}^{s} \binom{r+s-1}{k}^2.$$

MH–115. *Proposed by José Luis Díaz–Barrero, Barcelona, Spain.* Find a function $f : \mathbb{R} - \{0, \pm 1\} \to \mathbb{R}$ that is continuous everywhere and satisfies the equation

$$\frac{1}{x+1} f\left(\frac{x}{x+1}\right) + \frac{2}{x+1} f(x+1) = 1.$$

MH–116. *Proposed by Andrés Sáez Schwedt, Universidad de León, León, Spain.* Let $ABCD$ be a cyclic quadrilateral such that the segments AC and BD intersect at point E, and the lines AB and CD intersect at point F. The circumcircle of triangle BCE meets the line EF again at point $G \neq E$. Prove that

$$\frac{GB}{GC} = \frac{FB}{FC}.$$

MH–117. *Proposed by José Luis Díaz–Barrero, Barcelona, Spain.* Suppose that 2023 distinct points are chosen in the plane and the distances between them are measured. Show that the total number of distances among the given points is at least 32.

MH–118. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let $BC = a, CA = b, AB = c$ are the side lengths of integer sided non-degenerate triangle ABC with orthocenter H. Let M be the midpoint of AC. Knowing that B, C, H, M are concyclic, find all primitive triples (a, b, c) of positive integers, with the additional property that a, b, c have no positive common divisor other than unity.

Advanced Problems

A–113. *Proposed by Marian Ursărescu and Florică Anastase, Romania.* Let $A \in M_2(\mathbb{C})$ such that $\det A = 1$. For all $B \in M_2(\mathbb{C})$ prove that $A^2B - BA^2 = BA^{-2} - A^{-2}B$.

A–114. *Proposed by Gonzalo Gómez Abejón, Madrid, Spain.* We have an urn with N balls of different colors. Until they are all of the same color, we repeat the following step:

- Select two balls at random, of different colors (if they are the same color we put them back and draw another two until they are of different colors).
- Then paint the first ball of the color of the second one, then put them back.

Prove that given an initial set of balls, the average number of steps needed is always an integer, and in particular if we start with N balls of N different colors, it will take an average of $\frac{N(N-1)}{2}$ steps.

A–115. *Proposed by Henry Ricardo, Westchester Area Math Circle, New York, USA.* Let p be a prime number. Prove that

$$\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.$$

A–116. *Proposed by Marian Ursărescu and Florică Anastase, Romania.* Let $(a_n)_{n\geq 1}$ be the sequence defined by $a_1 = e$, $a_n = e^n \cdot a_{n-1}^n$ and $(b_n)_{n\geq 1}$ such that

$$\left(1+\frac{1}{n}\right)^{n+b_n} = \prod_{k=1}^{n}\left(1+\frac{1}{\log a_k}\right).$$

Compute $\lim\limits_{n\to\infty} b_n$.

A–117. *Proposed by Vasile Mircea Popa, Affiliate Professor, "Lucian Blaga" University of Sibiu, Romania.* Show that

$$\int_0^\infty \frac{x\ln x}{x^3 + x\sqrt{x} + 1}\,dx = \frac{32}{81}\pi^2 \sin\frac{\pi}{18}.$$

A–118. *Proposed by Michel Bataille, Rouen, France.* For $n \in \mathbb{N}$, let $S(n) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$. Prove that the series

$$\sum_{n=2}^{\infty} (-1)^n \frac{S(\lfloor n/2 \rfloor)}{n+1}$$

is convergent and evaluate its sum.

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

Muirhead via repeated refinements

Marc Felipe Alsina

1 Introduction

Muirhead's inequality ([2], [1]) is a very famous (and sometimes infamous) inequality between symmetric polynomial expressions. Here, we will state and prove this inequality and introduce some notation that will help us along the way. Finally, to illustrate the theoretical results some of its applications are also given.

Definition 1. *Let $a_1 \geq a_2 \geq \cdots \geq a_n$ and $b_1 \geq b_2 \geq \cdots \geq b_n$ be non-negative real numbers. We say that $(a_1, a_2, \ldots, a_n)$ majorizes $(b_1, b_2, \ldots, b_n)$, and write $(a_1, a_2, \ldots, a_n) \succ (b_1, b_2, \ldots, b_n)$, if the following holds:*

$$\begin{aligned} a_1 &\geq b_1 \\ a_1 + a_2 &\geq b_1 + b_2 \\ &\vdots \\ a_1 + a_2 + \cdots + a_{n-1} &\geq b_1 + b_2 + \cdots + b_{n-1} \\ a_1 + a_2 + \cdots + a_{n-1} + a_n &= b_1 + b_2 + \cdots + b_{n-1} + b_n \end{aligned}$$

Note that the last one is an equality.

Definition 2. *Let $x_1, x_2, \ldots, x_n$ be some fixed positive real numbers and $a_1 \geq a_2 \geq \cdots \geq a_n$ be non-negative real numbers. Then, we denote by $[a_1, a_2, \ldots, a_n]$ the expression:*

$$[a_1, a_2, \ldots, a_n] := \frac{1}{n!} \sum_{sym} \prod_{k=1}^{n} x_k^{a_k} = \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \prod_{k=1}^{n} x_{\pi(k)}^{a_k}$$

$\mathfrak{S}_n$ represents the group of permutations π of n elements, so this is the arithmetic mean of all the possible ways to raise the fixed numbers to the given exponents in some order. For example, if $x_1 = x$, $x_2 = y$ and $x_3 = z$, we have that $[1, 1, 1] = xyz$, that $[2, 1, 0] = \frac{1}{6}(x^2y + y^2x + x^2z + z^2x + y^2z + z^2y)$ and that $[1, 1, 0] = \frac{1}{3}(xy + yz + zx)$.

With these definitions, we can now state Muirhead's theorem as follows:

Theorem 1 (Muirhead's inequality). *Let $x_1, x_2, \ldots, x_n$ be some fixed positive real numbers. Suppose that we have two sequences $(a_1, a_2, \ldots, a_n) \succ (b_1, b_2, \ldots, b_n)$. Then,*

$$[a_1, a_2, \ldots, a_n] \geq [b_1, b_2, \ldots, b_n]$$

with equality if and only if either $x_1 = x_2 = \cdots = x_n$ or $a_k = b_k$ for $k = 1, 2, \ldots, n$.

We will first prove a particular case of Muirhead's inequality, that will help us prove the general case.

Lemma 1 (Muirhead's inequality for almost-equal exponents). *Let $x_1, x_2, \ldots, x_n$ be some fixed non-negative real numbers. Suppose that $(a_1, a_2, \ldots, a_n) \succ (b_1, b_2, \ldots, b_n)$. Additionally, suppose that there exist two indices i and j, with $i < j$, such that $a_k = b_k$ if $k \neq i, j$. Then,*

$$[a_1, a_2, \ldots, a_n] \geq [b_1, b_2, \ldots, b_n]$$

with equality if and only if either $x_1 = x_2 = \cdots = x_n$ or we have $a_i = b_i$ and $a_j = b_j$.

Proof. First of all, notice that from the majorization condition, more precisely from $a_1 + a_2 + \cdots + a_i \geq b_1 + b_2 + \cdots + b_i$, we obtain $a_i \geq b_i$. We write $a_i = b_i + \lambda$ for some non-negative λ, and from the equality condition, we then obtain $a_j - \lambda = b_j$. Since $i < j$ implies $b_i \geq b_j$, we have $a_i = b_i + \lambda \geq b_i \geq b_j \geq b_j - \lambda = a_j$.

We want to prove $[a_1, a_2, \ldots, a_n] \geq [b_1, b_2, \ldots, b_n]$, or

$$\frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \prod_{k=1}^{n} x_{\pi(k)}^{a_k} \geq \frac{1}{n!} \sum_{\pi \in \mathfrak{S}_n} \prod_{k=1}^{n} x_{\pi(k)}^{b_k}$$

Once we cancel the normalization constants on both sides, the idea of the proof is to group the terms of each member of the inequality in pairs, so that the sum of the left-hand pair is greater than the sum of the corresponding right-hand pair. To do that, we partition the set of all permutations into pairs (σ, τ) where τ is obtained from σ (and vice versa) by pre-composing by the transposition that swaps i and j. This is $\tau(k) = \sigma(k)$ if $k \neq i, j$ and $\tau(i) = \sigma(j)$, $\tau(j) = \sigma(i)$.

Therefore, the sum over all permutations becomes the sum over all such pairs (σ, τ), so the inequality becomes:

$$\sum_{\pi \in \mathfrak{S}_n} \prod_{k=1}^{n} x_{\pi(k)}^{a_k} = \sum_{(\sigma,\tau)} \left(\prod_{k=1}^{n} x_{\sigma(k)}^{a_k} + \prod_{k=1}^{n} x_{\tau(k)}^{a_k} \right)$$

$$\geq \sum_{(\sigma,\tau)} \left(\prod_{k=1}^{n} x_{\sigma(k)}^{b_k} + \prod_{k=1}^{n} x_{\tau(k)}^{b_k} \right) = \sum_{\pi \in \mathfrak{S}_n} \prod_{k=1}^{n} x_{\pi(k)}^{b_k}$$

We want to show the following, which we later rewrite only in terms of σ and expand:

$$\prod_{k=1}^{n} x_{\sigma(k)}^{a_k} + \prod_{k=1}^{n} x_{\tau(k)}^{a_k} \geq \prod_{k=1}^{n} x_{\sigma(k)}^{b_k} + \prod_{k=1}^{n} x_{\tau(k)}^{b_k}$$

$$x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} \cdot \ldots \cdot x_{\sigma(i)}^{a_i} \cdot \ldots \cdot x_{\sigma(j)}^{a_j} \cdot \ldots \cdot x_{\sigma(n)}^{a_n} +$$

$$x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} \cdot \ldots \cdot x_{\sigma(j)}^{a_i} \cdot \ldots \cdot x_{\sigma(i)}^{a_j} \cdot \ldots \cdot x_{\sigma(n)}^{a_n} \geq$$

$$x_{\sigma(1)}^{b_1} x_{\sigma(2)}^{b_2} \cdot \ldots \cdot x_{\sigma(i)}^{b_i} \cdot \ldots \cdot x_{\sigma(j)}^{b_j} \cdot \ldots \cdot x_{\sigma(n)}^{b_n} +$$

$$x_{\sigma(1)}^{b_1} x_{\sigma(2)}^{b_2} \cdot \ldots \cdot x_{\sigma(j)}^{b_i} \cdot \ldots \cdot x_{\sigma(i)}^{b_j} \cdot \ldots \cdot x_{\sigma(n)}^{b_n}$$

By dividing everything by $\prod_{k \neq i,j} x_{\sigma(k)}^{a_k} = \prod_{k \neq i,j} x_{\sigma(k)}^{b_k}$, it is equivalent to

$$x_{\sigma(i)}^{a_i} x_{\sigma(j)}^{a_j} + x_{\sigma(j)}^{a_i} x_{\sigma(i)}^{a_j} \geq x_{\sigma(i)}^{b_i} x_{\sigma(j)}^{b_j} + x_{\sigma(j)}^{b_i} x_{\sigma(i)}^{b_j}$$

and, to avoid cumbersome notation, we make the change of variables $y := x_{\sigma(i)}$, $z := x_{\sigma(j)}$:

$$y^{a_i}z^{a_j} + y^{a_j}z^{a_i} \geq y^{b_i}z^{b_j} + y^{b_j}z^{b_i}$$

The lowest exponent here is a_j, so we divide everything by $y^{a_j}z^{a_j}$ to obtain:

$$y^{a_i-a_j} + z^{a_i-a_j} \geq y^{b_i-a_j}z^{b_j-a_j} + y^{b_j-a_j}z^{b_i-a_j}$$

$$y^{b_i-a_j+\lambda} + z^{b_i-a_j+\lambda} \geq y^{b_i-a_j}z^{\lambda} + y^{\lambda}z^{b_i-a_j}$$

$$y^{b_i-a_j+\lambda} - y^{b_i-a_j}z^{\lambda} - y^{\lambda}z^{b_i-a_j} + z^{b_i-a_j+\lambda} \geq 0$$

$$(y^{b_i-a_j} - z^{b_i-a_j})(y^{\lambda} - z^{\lambda}) \geq 0$$

The last inequality is true because $b_i - a_j \geq 0$ and consequently the factors have the same sign. Therefore, the inequality has been proven. Equality is reached when $y = z$ or when $\lambda = 0$ (since $b_i - a_j = 0$ also implies $\lambda = 0$). The first condition translates to $x_{\sigma(i)} = x_{\sigma(j)}$, while the last condition means that $a_i = b_i$ and $a_j = b_j$. If the latter doesn't happen, then the values $x_{\sigma(i)}$ and $x_{\sigma(j)}$ must be equal for any choice of $\sigma \in \mathfrak{S}_n$, which means that all of $x_1, x_2, \dots, x_n$ must be equal.

The proof ends here, but in order to make the proof more accessible, we repeat the steps of the proof on a concrete example: $[7, 4, 1] \geq [5, 4, 3]$. If we use x, y and z instead of x_1, x_2, x_3, the inequality becomes:

$$\frac{1}{6}\big(x^7y^4z + x^7z^4y + y^7x^4z + y^7x^4z + z^7x^4y + z^7y^4x\big)$$
$$\geq \frac{1}{6}\big(x^5y^4z^3 + x^5z^4y^3 + y^5x^4z^3 + y^5x^4z^3 + z^5x^4y^3 + z^5y^4x^3\big)$$

We multiply by 6 and group the terms in pairs, marked by different types of brackets:

$$\big(x^7y^4z + z^7y^4x\big) + \big[x^7z^4y + y^7x^4z\big] + \big\{y^7x^4z + z^7x^4y\big\}$$
$$\geq \big(x^5y^4z^3 + z^5y^4x^3\big) + \big[x^5z^4y^3 + y^5x^4z^3\big] + \big\{y^5x^4z^3 + z^5x^4y^3\big\}$$

Finally, we compare each pair with the corresponding pair on the other side:

$$x^7y^4z + z^7y^4x \geq x^5y^4z^3 + z^5y^4x^3 \iff$$
$$\iff x^6 + z^6 \geq x^4z^2 + z^4x^2 \iff$$
$$\iff (x^4 - z^4)(x^2 - z^2) \geq 0$$

which holds, and similarly for the other pairs. □

Now, we can prove the general case. The idea is to make a finite number of refinements between the left-hand side and the right-hand side such that each step can be proven using the particular case that we now know how to prove.

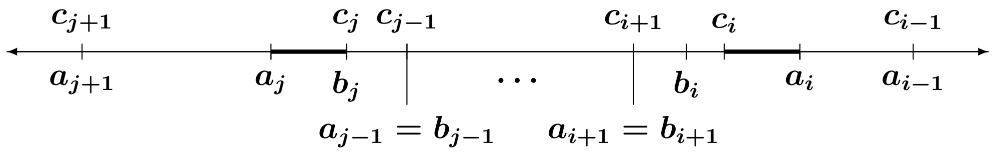
Proof of Muirhead's inequality. Suppose that $(a_1, a_2, \ldots, a_n) \succ (b_1, b_2, \ldots, b_n)$. If $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$, then both sides of the inequality are the same and we have equality, so suppose that this doesn't happen.

If l is the first index with $a_l \neq b_l$, it must happen that $a_l > b_l$, as $a_1 + a_2 + \cdots + a_l \geq b_1 + b_2 + \cdots + b_l$. Since the total sum is equal, we must have some index m, greater than l, with $a_m < b_m$. In between, there must exist two indices i and j ($l \leq i < j \leq m$) such that $a_i > b_i$, $a_j < b_j$ and $a_k = b_k$ for $i < k < j$ (this last condition is empty if $j = i + 1$). We denote by λ and μ the positive quantities $a_i - b_i$ and $b_j - a_j$, respectively. From $a_1 + a_2 + \cdots + a_j \geq b_1 + b_2 + \cdots + b_j$, we can cancel all the terms not involving i nor j, and the result can be reordered into $\lambda \geq \mu$.

Now, we will construct the sequence $c_1, c_2, \ldots, c_n$ as follows: $c_k = a_k$ if $k \neq i, j$ and $c_i = a_i - \mu$, $c_j = b_j = a_j + \mu$. We claim that this sequence is non-increasing and satisfies

$$(a_1, a_2, \ldots, a_n) \succ (c_1, c_2, \ldots, c_n) \succ (b_1, b_2, \ldots, b_n)$$

The fact that the sequence is non-increasing is easier to see with a picture than with equations, so we present the relevant numbers on the real number line and leave to the reader the task of writing down the appropriate inequalities. The same is true for $(a_1, a_2, \dots, a_n) \succ (c_1, c_2, \dots, c_n)$.



As for the other majorization, since we know $(a_1, a_2, \dots, a_n) \succ (b_1, b_2, \dots, b_n)$, the only place we need to check is where the partial sums $a_1 + a_2 + \dots + a_k$ and $c_1 + c_2 + \dots + c_k$ differ, which happens only when $i \leq k < j$ and, indeed, the inequality holds:

$$(c_1 + c_2 + \dots + c_{i-1}) + c_i + (c_{i+1} + \dots + c_k) =$$
$$(a_1 + a_2 + \dots + a_{i-1}) + c_i + (b_{i+1} + \dots + b_k) \geq$$
$$(b_1 + b_2 + \dots + b_{i-1}) + b_i + (b_{i+1} + \dots + b_k)$$

Now that we have $(a_1, a_2, \dots, a_n) \succ (c_1, c_2, \dots, c_n) \succ (b_1, b_2, \dots, b_n)$, we want to prove

$$[a_1, a_2, \dots, a_n] \geq [c_1, c_2, \dots, c_n] \geq [b_1, b_2, \dots, b_n]$$

For the first equality, we can make use of the particular case because the sequences only differ on two terms. In order to prove the second one, we repeat the process and find a new refinement, and we keep doing this until we reach the right-hand side. Since in the set of indices k with $c_k = b_k$ increases by at least one in each refinement, after some time we will be able to use the lemma also on the right equality.

As an example, we present the series of refinements we do to prove that $[7, 5, 3, 3, 0, 0] \geq [5, 5, 4, 2, 1, 1]$:

$$[7, 5, 3, 3, 0, 0] \geq [6, 5, 4, 3, 0, 0] \geq [6, 5, 4, 2, 1, 0] \geq [5, 5, 4, 2, 1, 1]$$

In order to complete the proof, we need to discuss the equality cases. By construction, each refinement is strict unless all the variables are equal (the other equality condition does not hold). So, the only way that we can mantain equality is either by doing no refinements (so $a_k = b_k$ for all k) or by having $x_1 = x_2 = \cdots = x_n$. □

We end by presenting some problems that can be solved, directly or indirectly, using Muirhead's inequality:

Problem 1 (Arithmetic Mean - Geometric Mean). *Given positive real numbers $x_1, x_2, \dots, x_n$, prove*

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \dots x_n}$$

Solution. This is a direct application of Muirhead's theorem. The left-hand side is $[1, 0, 0, \dots, 0]$ and the right-hand side is $\left[\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right]$. Since $(1, 0, 0, \dots, 0) \succ \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$, the inequality follows. □

Problem 2 (Nesbitt inequality). *Given positive real numbers a, b, c, prove*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

Solution. In order to apply Muirhead's inequality, we clear denominators.

$$2a(a+b)(c+a)+2b(b+c)(a+b)+2c(c+a)(b+c) \geq 3(a+b)(b+c)(c+a)$$

After expanding and cancelling terms, we end up with

$$2(a^3 + b^3 + c^3) \geq a^2b + a^2c + b^2a + b^2c + c^2a + c^2b$$

This is $6[3, 0, 0] \geq 6[2, 1, 0]$, which holds by Muirhead. □

Problem 3. *Let x, y and z be positive real numbers such that $xyz = 1$ and let $n \geq 0$ be an integer. Prove that*

$$x^{n+1} + y^{n+1} + z^{n+1} \geq x^n + y^n + z^n$$

Solution. We multiply the right-hand side by $\sqrt[3]{xyz}$, which is equal to 1, in order to make both sides have the same degree. Then, the left-hand side equals $3[n+1,0,0]$ while the right-hand side equals $3\left[n+\frac{1}{3},\frac{1}{3},\frac{1}{3}\right]$. The first sequence majorizes the second one, so the inequality holds. $\square$

Problem 4. *Let $P(x) = x^3 - Ax^2 + Bx - C$ be a polynomial with three positive real roots. Prove that $B^2 \geq 3AC$.*

Solution. By Cardano-Viète, if α, β, γ are the roots of P, then $A = \alpha+\beta+\gamma$, $B = \alpha\beta+\beta\gamma+\gamma\alpha$ and $C = \alpha\beta\gamma$. The inequality becomes $(\alpha\beta+\beta\gamma+\gamma\alpha)^2 \geq 3(\alpha+\beta+\gamma)\alpha\beta\gamma$. After expanding and simplifying:

$$\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 \geq \alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2$$

or $3[2,2,0] \geq 3[2,1,1]$, which hold by Muirhead. $\square$

Problem 5 (IMC 2012, Problem 4). *Let $n \geq 3$ and let $x_1, x_2, \ldots, x_n$ be nonnegative real numbers. Define $A = \sum\limits_{i=1}^{n} x_i$, $B = \sum\limits_{i=1}^{n} x_i^2$, $C = \sum\limits_{i=1}^{n} x_i^3$. Prove that*

$$(n+1)A^2B + (n-2)B^2 \geq A^4 + (2n-2)AC$$

Solution. First, let $x_1, x_2, \ldots, x_n$ be positive. We compute A^2B, B^2, A^4 and AC:

$$A^2B = \left(\sum_{i=1}^{n} x_i\right)^2 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i^4 + \sum_{i\neq j} x_i^2x_j^2 + 2\sum_{i\neq j} x_i^3x_j + 2\sum_{\substack{i\neq j\\ i\neq k\\ j<k}} x_i^2x_jx_k$$

$$B^2 = \left(\sum_{i=1}^{n} x_i^2\right)^2 = \sum_{i=1}^{n} x_i^4 + 2\sum_{i<j} x_i^2x_j^2$$

$$A^4 = \left(\sum_{i=1}^{n} x_i\right)^4 = \sum_{i=1}^{n} x_i^4 + 4\sum_{i\neq j} x_i^3x_j + 6\sum_{i<j} x_i^2x_j^2 + \\ + 12\sum_{\substack{i\neq j\\ i\neq k\\ j<k}} x_i^2x_jx_k + 24\sum_{i<j<k<l} x_ix_jx_kx_l$$

$$AC = \sum_{i=1}^{n} x_i \sum_{i=1}^{n} x_i^3 = \sum_{i=1}^{n} x_i^4 + 2\sum_{i\neq j} x_i^3x_j$$

In Muirhead's notation,

$$\begin{aligned}
A^2B &= n[4,0,\dots,0] + n(n-1)[2,2,0\dots,0] + \\
&\quad + 2n(n-1)[3,1,0,\dots,0] + 2\frac{n(n-1)(n-2)}{2}[2,1,1,0,\dots,0] \\
B^2 &= n[4,0,\dots,0] + 2\frac{n(n-1)}{2}[2,2,0\dots,0] \\
A^4 &= n[4,0,\dots,0] + 4n(n-1)[3,1,0,\dots,0] + \\
&\quad + 6\frac{n(n-1)}{2}[2,2,0,\dots,0] + 12n\frac{(n-1)(n-2)}{2}[2,1,1,0,\dots,0] + \\
&\quad + 24\frac{n(n-1)(n-2)(n-3)}{24}[1,1,1,1,0,\dots,0] \\
AC &= n[4,0,\dots,0] + 2n(n-1)[3,1,0,\dots,0]
\end{aligned}$$

After substituting that in, we need to prove:

$$2n(n-1)(n-2)[2,2,0\dots,0] + n(n-1)(n-2)(n-5)[2,1,1,0\dots,0] \geq$$

$$\geq n(n-1)(n-2)(n-3)[1,1,1,1,0\dots,0]$$

which holds by Muirhead.

Thus, the inequality holds when $x_1, x_2, \dots, x_n$ are positive. We use continuity to extend the domain to the nonnegative reals. □

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Bounds of zeros from the beginning to Cauchy

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1 Introduction

The purpose of this note is to give some bounds for the zeros z_k of the polynomial with complex coefficients $A(z) = a_0 + a_1 z + \ldots + a_{n-1}z^{n-1} + a_n z^n$, as functions of all or part of its coefficients. That is, we want to express $z_k = z_k(a_0, a_1, \ldots, a_n)$. This process consists in determine regions of the complex plane $\Gamma = \Gamma(a_0, a_1, \ldots, a_n)$ that enclose in their interior all the zeros of $A(z)$. For example, we try to find circles or annulus of smallest radius having in their interior all the zeros of $A(z)$.

Historically speaking, our subject dates from about the time when the geometric representation of complex numbers was introduced into mathematics. This problem first received attention in the seventeenth century. The French mathematician and successor of Descartes, Florimond de Beaune (1601-1652), in his *De Limitibus Aequationum*, tried to show that the bounds of positive zeros might be found from the coefficients in polynomials up to the fourth degree. Unfortunately, he only considered specific cases and did not dared to solve the problem for the nth-degree. In 1683, John Wallis, in commenting on de Beaune's rules for finding bounds of the zeros, stated that this "subject is yet capable of further development".

The first general methods for obtaining limits for the moduli of the zeros date from Newton (1622) and MacLaurin (1748). Since then, researches of many mathematicians have resulted in numerous methods for obtaining more useful bounds. In part this development resulted from the efforts to extend from the real domain to complex domain the classical results of Rolle, Descartes and Sturm. Also contributed to, the development of the theory of functions of complex variables with results such as the Principle of Argument and its corollary Rouché's theorem which are fundamental as basic tools in the study of polynomial from an analytic view point. Among the researchers that have faced the problem of finding bounds for the zeros, deserves a special mention the contribution to the subject made by Lagrange, Gauss and Cauchy. Incidental to his proof of the Fundamental Theorem of Algebra Gauss showed that polynomial

$$A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$$

has no zeros outside of the circle $|z| = r$. In the case that all the coefficients a_k are reals, he showed in 1799 that $r = \max\{1, 2^{1/2}S\}$ where S is the sum of the positive a_k, and he also showed in 1816 that

$$r = \max_{0 \le k \le n-1} \{2^{1/2} n |a_k|)^{1/n-k}\}.$$

2 Bounds for the zeros

Bounds for the zeros of a polynomial can be classified into two categories: explicit bounds and implicit bounds. Explicit bounds, usually are functions of all or part of the coefficients of the polynomial, meanwhile implicit bounds usually involve the solution of an algebraic equation derived from the polynomial. In general, implicit bounds are sharper than explicit bounds.

We begin with classical results. An upper bound for all the positive zeros of real polynomials involving the derivative was first established by Newton [6].

Theorem (Newton). *Let $A(z)$ be a polynomial with real coefficients defined by*

$$A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k.$$

If for $z_0 > 0$, $A(z_0), A'(z_0), A''(z_0), \ldots, A^{(n)}(z_0)$ are non–negative, then any positive zero of $A(z)$ is greater than z_0.

Proof. Taylor's formula allows us to write

$$A(z) = \sum_{k=0}^{n} \frac{A^{(k)}(z_0)}{k!}(z - z_0)^k.$$

Since, for $z > z_0$ we have $A(z) > 0$, then the polynomial can not have zeros greater than z_0.

Among the simpler test for bounds of the positive zeros of real polynomials we have the formula of MacLaurin [3].

Theorem (MacLaurin). *Let $A(z)$ be a polynomial with real coefficients defined by*

$$A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k.$$

If G is the absolute value of its greatest negative coefficient, then all its positive zeros are bounded by $1 + G$.

Lagrange also contributed to the subject establishing the following bound for the positive zeros of a real polynomial.

Theorem (Lagrange). *Given the monic polynomial*

$$A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$$

in which all the coefficients are real and some are negative. Let a_r be the first negative coefficient, that is, $a_r < 0$ and $a_{r+1} \geq 0$, $a_{r+2} \geq 0, \ldots, a_{n-1} \geq 0$, and let G be the greatest absolute value of all negative coefficients. Then an upper bound for the positive zeros of $A(z)$ is $M = 1 + \sqrt[n-r]{G}$.

Proof. We have to prove that $A(z) > 0$ for any real number $z \geq M$. From the definition of G and r it follows

$$\begin{aligned} A(z) &= z^n + \sum_{k=0}^{n-1} a_k z^k \\ &\geq z^n + \sum_{k=0}^{r} a_k z^k \\ &\geq z^n - G(z^r + z^{r-1} + \ldots + z + 1) \\ &= z^n - G\,\frac{z^{r+1} - 1}{z - 1}. \end{aligned}$$

Let z be such that $z \geq 1 + \sqrt[n-r]{G}$. Since $A(z)$ has some negative coefficient we conclude that $0 < G \leq (z-1)^{n-r}$ and $z < 1$. Then

$$\begin{aligned} G(z^{r+1} - 1) < Gz^{r+1} &\leq z^{r+1}(z-1)^{n-r} \\ &= z^{r+1}(z-1)(z-1)^{n-r-1} \\ &< z^{r+1}(z-1)z^{n-r-1} \\ &= z^n(z-1). \end{aligned}$$

So, we have that if $z \geq 1 + \sqrt[n-r]{G}$, then $G(z^{r+1} - 1) < z^n(z-1)$. This completes the proof.

Another result concerning on upper limits for the zeros of real polynomials and easy to remember and applying is the following due to Jean J. Bret. In many cases, however, this is a very high limit.

Theorem (Bret). *If, in a polynomial with real coefficients in which the coefficient of the highest power of the variable is positive, the absolute value of each negative coefficient be divided by the sum of all the positive coefficients which precede it, the greatest quotient so obtained increased by unity is an upper bound for the positive zeros.*

Proof. In order to fix our ideas, we regard, for example, the fourth coefficient as negative, and we consider also a negative coefficient in general, say $-a_r$. So, we can write $A(z)$ in the form

$$\begin{aligned} A(z) &= a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} - a_{n-3}z^{n-3} \\ &+ \ldots - a_r z^r + \ldots + a_1 z + a_0. \end{aligned}$$

The identity

$$\frac{z^k - 1}{z - 1} = z^{k-1} + z^{k-2} + \ldots + z + 1$$

allows us to write

$$a_k z^k = a_k(z-1)(z^{k-1} + z^{k-2} + \ldots + z + 1) + a_k.$$

In the expression of $A(z)$ we expand each term with positive coefficient to obtain

$$\begin{aligned} A(z) &= a_n(z-1)z^{n-1} + \ldots + a_n(z-1)z^{n-r} + \ldots + a_n \\ &+ a_{n-1}(z-1)z^{n-2} + \ldots + a_{n-1}(z-1)z^{n-r} + \ldots + a_{n-1} \\ &+ a_{n-2}(z-1)z^{n-3} + \ldots + a_{n-2}(z-1)z^{n-r} + \ldots + a_{n-2} \\ &- a_{n-3}z^{n-3} + \ldots - a_{n-r}z^{n-r} + \ldots + a_0. \end{aligned}$$

Grouping terms of the same power of the variable we get the successive coefficients of $z^{n-1}, z^{n-2}, \cdots$, being

$$a_n(z-1), (a_n + a_{n-1})(z-1), (a_n + a_{n-1} + a_{n-2})(z-1) - a_{n-3}, \cdots,$$

$$(a_n + a_{n-1} + \cdots + a_{n-r+1})(z-1) - a_{n-r}, \cdots$$

Observe that any value of $z > 1$ is sufficient to make positive every term in which no negative coefficients occurs. To make the latter terms positive, we must have

$$\begin{gathered} (a_n + a_{n-1} + a_{n-2})(z-1) > a_{n-3} \\ \cdots\cdots\cdots\cdots\cdots \\ (a_n + a_{n-1} + \cdots + a_{r-1})(z-1) > a_{n-r} \\ \cdots\cdots\cdots\cdots\cdots \end{gathered}$$

Hence

$$z > \frac{a_{n-3}}{a_n + a_{n-1} + a_{n-2}} + 1,$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$z > \frac{a_{n-r}}{a_n + a_{n-1} + \cdots + a_{n-r}} + 1,$$
$$\cdots\cdots\cdots\cdots\cdots$$

To ensure every term being made positive, we must take the value of the greatest of the quantities found in this way. Such a value z, therefore, is a bound for the positive zeros of $A_n(z)$.

In 1849 Gauss established the following implicit bound for the zeros

Theorem (Gauss). *Let $A(z) = z^n + \sum_{k=0}^{n-1} a_k z^k$ be a monic polynomial with complex coefficients. If r is the unique positive root of equation*

$$G_n(z) = z^n - 2^{1/2}\Big(|a_{n-1}|z^{n-1} + \ldots + |a_1|z + |a_0|\Big) = 0,$$

then all the zeros of $A(z)$ lie in the disk $\mathcal{C} = \{z \in \mathbb{C} : |z| \leq r\}$.

Proof. Let $M = \max_{0\leq k\leq n-1}\Big\{|a_k|\Big\}$, then $\sqrt{2}\sum_{k=0}^{n-1}|a_k|r^k \leq M\sqrt{2}\sum_{k=0}^{n-1} r^k$. Hence,

$$r^n - \sqrt{2}\sum_{k=0}^{n-1}|a_k|r^k \geq r^n - M\sqrt{2}\sum_{k=0}^{n-1} r^k$$
$$= r^n\Big[1 - M\sqrt{2}\Big(\frac{1}{r} + \frac{1}{r^2} + \ldots + \frac{1}{r^n}\Big)\Big].$$

For $r > 1$ is $\sum_{k=1}^{n}\frac{1}{r^k} < \sum_{k=1}^{\infty}\frac{1}{r^k} = \frac{1}{r-1}$, and the preceding is greater than

$$r^n\Big[1 - \frac{M\sqrt{2}}{r-1}\Big] = r^n\Big[\frac{r - 1 - M\sqrt{2}}{r-1}\Big] > 0$$

when $r > 1 + M\sqrt{2}$. Therefore, taking $r = 1 + M\sqrt{2}$ the theorem is proved.

Note that when $A(z) = G_n(z)$ at least one zero lie on $|z| = r$.

In 1829 Cauchy [1] contributed to with two important results sharpening the bounds previously stated by Gauss. The first, is the following explicit bound

Theorem (Cauchy). *Let $A_n(z) = \sum_{k=0}^{n} a_k z^k$ be a polynomial of degree n with complex coefficients. Then all the zeros of $A_n(z)$ lie in the disk*

$$\mathcal{C} = \left\{ z \in \mathbb{C} \,:\, |z| < 1 + \frac{K}{|a_n|} \right\},$$

where $K = \max_{1 \le k \le n} \{|a_{n-k}|\}$.

Proof. Assume that $|z| > 1$. Then, from

$$|A_n(z)| \ge |a_n|\,|z|^n - (|a_{n-1}|\,|z|^{n-1} + \ldots + |a_1||z| + |a_0|),$$

it follows

$$\begin{aligned} |A_n(z)| &\ge |a_n|\,|z|^n - K(|z|^{n-1} + \ldots + |z| + 1) \\ &= |a_n|\,|z|^n \Big(1 - \frac{K}{|a_n|} \sum_{k=1}^{n} |z|^{-k}\Big) \\ &> |a_n|\,|z|^n \Big(1 - \frac{K}{|a_n|} \sum_{k=1}^{\infty} |z|^{-k}\Big) \\ &= |a_n|\,|z|^n \frac{|a_n|\,|z| - (|a_n| + K)}{|z| - 1}. \end{aligned}$$

Hence, if $|z| \ge 1 + K/|a_n|$, then $|A_n(z)| > 0$. Hence, the only zeros in $|z| > 1$ are those satisfying $|z| < 1 + \dfrac{K}{|a_n|}$. But, all the zeros in $|z| \le 1$ also satisfy the foregoing inequality and the theorem is fully proven.

To get a lower bound for the moduli of the zeros first we observe how affects an inversion in the variable to the zeros. If an inversion is performed, we must change z by $1/\overline{z}$. Then, the transformed polynomial is

$$\begin{aligned} A_n(1/\overline{z}) &= \frac{1}{\overline{z}^n} + \frac{a_{n-1}}{\overline{z}^{n-1}} + \ldots + \frac{a_1}{\overline{z}} + a_0 \\ &= \frac{1}{\overline{z}^n}(1-\overline{z}z_1)(1-\overline{z}z_2)\ldots(1-\overline{z}z_n) \\ &= \left[\frac{(-1)^n}{\overline{z}^n}\right]\Big(\overline{z}-\frac{1}{z_1}\Big)\Big(\overline{z}-\frac{1}{z_2}\Big)\ldots\Big(\overline{z}-\frac{1}{z_n}\Big)\prod_{k=1}^{n} z_k. \end{aligned}$$

Taking conjugates and multiplying by z^n, we have

$$\begin{aligned} z^n\overline{A_n(1/\overline{z})} &= \overline{a}_0 z^n + \overline{a}_1 z^{n-1} + \ldots + \overline{a}_{n-1}z + 1 \\ &= \overline{a}_0 \prod_{k=1}^{n}\Big(z - \frac{1}{\overline{z}_k}\Big). \end{aligned}$$

Thus, after carrying out an inversion, the zeros of the transformed polynomial are the inverses and conjugates of the zeros of the original polynomial. For a polynomial whose coefficients are the conjugates in reverse order of $A_n(z)$ coefficients, we say that it is its *reciprocal or inverse.* We denote it by

$$A_n^*(z) = z^n\overline{A_n(1/\overline{z})} = \sum_{k=0}^{n} \overline{a}_{n-k} z^k.$$

From Cardan-Viète formulas [2] we have

$$\begin{aligned} \frac{\overline{a}_1}{\overline{a}_0} &= \frac{(-1)^{n-1}[\overline{z_1\ldots z_{n-1}} + \ldots + \overline{z_2\ldots z_n}]}{(-1)^n z_1 z_2 \ldots z_n} \\ &= (-1)^1\left[\frac{1}{\overline{z}_1} + \frac{1}{\overline{z}_2} + \ldots + \frac{1}{\overline{z}_n}\right] \\ &= (-1)^1\sum_i \frac{1}{\overline{z}_i}. \end{aligned}$$

The next coefficient is

$$\begin{aligned} \frac{\overline{a}_2}{\overline{a}_0} &= (-1)^2\left[\frac{1}{\overline{z_1 z_2}} + \frac{1}{\overline{z_1 z_3}} + \ldots + \frac{1}{\overline{z_{n-1}z_n}}\right] \\ &= (-1)^2\sum_{i<j}\frac{1}{\overline{z_i z_j}}, \end{aligned}$$

and continuing in the same way we get finally

$$\begin{aligned}\frac{1}{\overline{a_0}} &= (-1)^n \frac{1}{\overline{z_1 z_2 \dots z_n}} \\ &= (-1)^n \left[\frac{1}{\overline{z_1}} \frac{1}{\overline{z_2}} \dots \frac{1}{\overline{z_1}}\right].\end{aligned}$$

The preceding expressions are the Cardan-Viète formulas for $A_n^*(z)$ and they show again that its zeros are the inverses and conjugates of those of $A_n(z)$. From a geometric view point it does mean that the zeros of a polynomial and its reverse are symmetric in the unit circle. If $A_n(z) = uA_n^*(z)$ with $|u| = 1$, then $A_n(z)$ is called *self-inversive* or *self-reciprocal.*

By applying the last theorem to $A_n^*(z)$, we get

$$\begin{aligned}\left|\frac{1}{z_k}\right| &= \left|\frac{1}{\overline{z}_k}\right| < 1 + \max_{1\le k\le n}\left\{\left|\frac{\overline{a}_k}{\overline{a}_0}\right|\right\} \\ &= \max\left\{1 + \left|\frac{\overline{a}_n}{\overline{a}_0}\right|, 1 + \left|\frac{\overline{a}_{n-1}}{\overline{a}_0}\right|, \cdots, 1 + \left|\frac{\overline{a}_1}{\overline{a}_0}\right|\right\} \\ &= \max\left\{\frac{|\overline{a}_0| + |\overline{a}_n|}{|\overline{a}_0|}, \frac{|\overline{a}_0| + |\overline{a}_{n-1}|}{|\overline{a}_0|}, \cdots, \frac{|\overline{a}_0| + |\overline{a}_1|}{|\overline{a}_0|}\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}|z_k| &= |\overline{z}_k| \ge \frac{1}{\max\left\{\frac{|\overline{a}_0| + |\overline{a}_n|}{|\overline{a}_0|}, \frac{|\overline{a}_0| + |\overline{a}_{n-1}|}{|\overline{a}_0|}, \cdots, \frac{|\overline{a}_0| + |\overline{a}_1|}{|\overline{a}_0|}\right\}} \\ &\ge \min\left\{\frac{|\overline{a}_0|}{|\overline{a}_0| + |\overline{a}_n|}, \frac{|\overline{a}_0|}{|\overline{a}_0| + |\overline{a}_{n-1}|}, \cdots, \frac{|\overline{a}_0|}{|\overline{a}_0| + |\overline{a}_1|}\right\} \\ &= \min_{1\le k\le n}\left\{\frac{|\overline{a}_0|}{|\overline{a}_0| + |\overline{a}_k|}\right\} = \min_{1\le k\le n}\left\{\frac{|a_0|}{|a_0| + |a_k|}\right\}.\end{aligned}$$

Now, calling $r_1 = \min_{1\le k\le n}\left\{\frac{|a_0|}{|a_0| + |a_k|}\right\}$ and $r_2 = 1 + \max_{0\le k\le n-1}\left\{\left|\frac{a_k}{a_n}\right|\right\} = \max_{0\le k\le n-1}\left\{\frac{|a_k| + |a_n|}{|a_n|}\right\}$, we have obtained a more complete version of Cauchy's explicit bound. It can be stated as follows

Theorem. *Let* $A_n(z) = \sum_{k=0}^{n} a_k z^k$ *be a complex polynomial. Then, all its zeros lie on the annulus* $\mathcal{A} = \{z \in \mathbb{C} \,:\, r_1 \leq |z| < r_2\}$, *where*

$$r_1 = \min_{1\leq k\leq n}\left\{\frac{|a_0|}{|a_0| + |a_k|}\right\},$$

and

$$r_2 = 1 + \max_{0\leq k\leq n-1}\left\{\left|\frac{a_k}{a_n}\right|\right\} = \max_{0\leq k\leq n-1}\left\{\frac{|a_k| + |a_n|}{|a_n|}\right\}.$$

A second result, due also to Cauchy is the following implicit bound.

Theorem (Cauchy). *All the zeros of* $A_n(z)$ *lie in the disk* $\mathcal{C} = \{z \in \mathbb{C} \,:\, |z| \leq r\}$, *where* r *is the unique positive root of the equation*

$$|a_n|z^n = |a_0| + |a_1|z + \ldots + |a_{n-1}|z^{n-1}.$$

Proof. We consider the function

$$C(z) \equiv |a_n|z^n - (|a_0| + |a_1|z + \ldots + |a_{n-1}|z^{n-1}) = 0$$

If $|z| > r$, from the preceding it follows that $C(|z|) > 0$. Since

$$|A_n(z)| \geq |a_n|\,|z|^n - (|a_{n-1}|\,|z|^{n-1} + \ldots + |a_0|) = C(|z|),$$

we conclude that $|A_n(z)| > 0$. That is, $A_n(z) \neq 0$, for $|z| > r$. Hence, all zeros of $A_n(z)$ must lie in the disk $|z| \leq r$. Observe that the limit is attained when $A(z) = |a_n|z^n - (|a_0| + |a_1|z + \ldots + |a_{n-1}|z^{n-1})$.

A basic tools in the study of polynomial from an analytic view point are results of the theory of functions of complex variables such as the Principle of Argument and its corollary Rouché's theorem [5].

Theorem (Rouché). *If two functions* $f(z)$ *and* $g(z)$ *are analytic on and inside the close path* γ, $f(z) \neq 0$ *on* γ, *and* $|f(z)| > |g(z)|$ *on* γ, *then* $f(z)$ *and* $f(z) + g(z)$ *have the same number of zeros inside* γ.

Applying Rouché's theorem, Pellet [4] generalized the preceding result of Cauchy as we state in the following

Theorem (Pellet). *If polynomial*

$$A_n^j(z) = |a_0| + \ldots + |a_{j-1}|z^j - |a_j|z^j + |a_{j+1}|z^{j+1} + \ldots + |a_n|z^n$$

has two positive zeros r_1 and r_2, $(r_1 < r_2)$, then polynomial $A_n(z)$ has exactly j zeros inside the disk $\mathcal{C} = \{z \in \mathbb{C} : |z| \leq r_1\}$, and no zeros in the ring $\mathcal{R} = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$.

Finally, we have obtained a sufficient condition for a self–inversive polynomial to have all its zeros on the unit circle. It is stated in the following

Theorem. *Let $A_n(z)$ be a self–inversive monic complex polynomial. If its implicit bound of Cauchy is one, then all its zeros lie on the unit circle.*

Proof. Let $A_n(z) = z^n + \sum\limits_{k=0}^{n-1} a_k z^k, |a_0| = 1$. As we have seen before, its implicit bound of Cauchy is the unique positive root of the equation

$$|a_n|z^n = |a_{n-1}|z^{n-1} + \ldots + |a_1|z + 1.$$

If $z = 1$ is the unique positive solution of the previous equation, then

$$1 = |a_{n-1}| + |a_{n-2}| + \ldots + |a_1| + 1,$$

and therefore,

$$\sum_{k=1}^{n-1} |a_{n-k}| = 0.$$

Since, $|a_{n-k}| \geq 0, k = 1, 2, \cdots, n-1$, then from the preceding it follows $|a_{n-k}| = 0$ and $a_{n-k} = 0$ for $k = 1, 2, \ldots, n-1$. Therefore, $A_n(z) = z^n + a_0 = z^n + 1$, and the theorem is proved.

Remark. The reciprocal, in general does not hold as we show in the following example. Polynomial $A_4(z) = z^4 + z^3 + z^2 + z + 1$ has all its zeros on $|z| = 1$, but its implicit bound of Cauchy is $r \simeq 1.93$.

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Contests

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to find interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Problems and solutions from the 10th edition of BarcelonaTech Mathcontest

O. Rivero Salgado and J. L. Díaz-Barrero

1 Problems and solutions

Hereafter, we present the four problems that appeared in the paper given to the contestants of the BarcelonaTech Mathcontest 2023, as well as their official solutions.

Problem 1. Let a and b be two positive perfect squares such that $a+b$ is also a perfect square. Show that $16a-9b$ cannot be a positive prime number.

Solution. Firstly, writing $a=x^2$, $b=y^2$, we have that

$$16a-9b=16x^2-9y^2=(4x+3y)(4x-3y),$$

so the unique option for this quantity being prime occurs when $4x+3y$ is prime and $4x-3y=1$. However, writing $a+b=z^2$, we have that

$$16a-9b=16x^2+16y^2-25y^2=16z^2-25y^2=(4z+5y)(4z-5y)$$

is also prime, so again it must occur $4z-5y=1$. If $4x-3y=4z-5y$, we obtain that $z=\frac{2x+y}{2}$. Since $z^2=x^2+y^2$, it occurs that

$$z^2=x^2+xy+\frac{y^2}{4}=x^2+y^2.$$

Therefore, $x=\frac{3y}{4}$, $z=x+\frac{y}{2}=\frac{5y}{4}$. However, this means that $4x-3y=0$, which contradicts the fact that $4x-3y=1$.

Problem 2. Three ants lie at different vertices of a rectangle, and they can move in the plane of the rectangle according to the following rule: in each turn only one of them moves in the direction parallel to the straight line determined by the other two ants. Show that the three ants cannot lie simultaneously at the three midpoints of the sides of the rectangle.

Solution. Let A, B, C be the three ants that are moving on the rectangle $ABCD$ of the figure. At the beginning the three ants are on the vertex corresponding to its name.

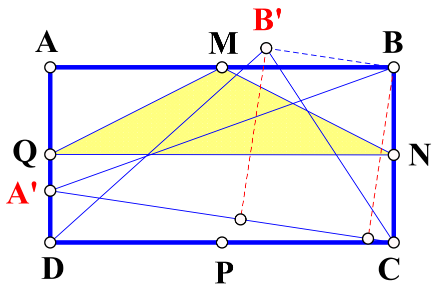

Figure 1: Scheme for solving problem 2.

Let $AB = a$ and $BC = b$. Then the area of triangle ABC is $ab/2$. Note that if one of the three moves parallel to the segment formed by the other two, the area of the triangle formed by them will remain constant, since the base of the triangle (segment formed by the two that do not move) is maintained and the length of the altitude of the triangle does not change, because it is moving parallel. Then the area of the triangle formed by A, B, C in each move will always be $ab/2$ (invariant).

Now, if they were located at the midpoints (M, N, Q in the figure), the area of the triangle formed by them would be $ab/4$. Therefore the three ants cannot be located at three of the midpoints of the sides of the rectangle simultaneously.

Note that the same result is obtained when the ants start the game at vertices: ABD, ACD, BCA, respectively.

Problem 3. Let n be the number of 9-tuples of positive integers of the form $(a_1, a_2, \ldots, a_9)$ satisfying

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_9} = 1.$$

Determine the remainder of n upon division by 6.

Solution. We start by finding the parity of n. Given a 9-tuple, assume that we have the numbers $b_1, \ldots, b_r$ ($r \leq 9$) a total number of $m_1, \ldots, m_r$ times, with $m_1 \geq \ldots \geq m_9$. Then, the different permutations of that solution give a total of $\frac{9!}{m_1! \cdots m_r!}$ solutions. We claim that the unique way in which such quotient is even occurs when $r = 1$ (and therefore $m_1 = 9$) or when $r = 2$, $m_1 = 8$ and $m_2 = 1$. Indeed, $2^7 \mid 9!$, so it must also divide the product of the m_i.

- If $r = 1$, we have $m_1 = 9$ and the quotient is 1.
- If $r = 2$, we see that 128 does not divide any of the numbers $5! \cdot 4!$, $6! \cdot 3!$ and $7! \cdot 2!$. However, if $m_1 = 8$ and $m_2 = 1$, the quotient is 9.
- If $r = 3$, $m_1 \leq 7$. If $6 \leq m_1 \leq 7$, the maximum power of two dividing the product of the m_i is at most $2^{5+1} = 64$. If $4 \leq m_1 \leq 5$, then the maximum power of two is either 2^5 or 2^6. Finally, if $m_1 = 3$, then $m_2 = m_3 = 3$ and the quotient is even.
- If $r \geq 4$, the unique way in which $m_1 = 6$ is $(6, 1, 1, 1)$, which gives an even quotient. The cases where $m_1 \geq 4$ are the following ones: $(5, 1, 1, 1, 1)$, $(5, 2, 1, 1)$, $(4, 1, 1, 1, 1, 1)$, $(4, 2, 1, 1, 1)$ and $(4, 2, 2, 1)$. All of them give an even quotient. Let a, b, c be the number of times that we have the numbers $1, 2, 3$, respectively. Then, $b + c \geq 7$, but $a + 2b + 3c = 9$. From here, we conclude that

  $$9 = a + c + 2(b + c) \geq a + c + 14 \geq 14,$$

  which is a contradiction.

We must count the number of 9-tuples where all of them are equal or where a_1 is different from the other eight, and at the same time these are equal among them. The first case corresponds to $a_1 = \ldots = a_9 = 9$. For the second case, consider

$$\frac{1}{a} + \frac{8}{b} = 1, \quad a \neq b.$$

Observe that either $a \leq 2$ or $b \leq 16$. In the first case, $a = 2, b = 16$. For the second one, we must also consider $a = 3, b = 12$; and $a = 5, b = 10$.

This gives 4 solutions with an odd number of permutations, so n must be even.

Arguing in the same way and keeping the same notations, we can determine the remainder upon division by 3. Since $81 \mid 9!$, 81 also divides the product of the factorials. We distinguish three cases:

- If $r = 1$, the quotient is 1, which is not a multiple of 3.
- If $6 \leq m_1 \leq 8$, the greatest power of 3 dividing the product of the factorials is at most 3^{2+1}, so the quotient is a multiple of 3.
- If $m_1 < 6$, then each factorial gives at most a factor of 3, and it is not possible to have more than 3 of the m_i greater or equal than 3. Hence, the greatest power of 3 dividing the product of factorials is again 3^3, so the quotient is a multiple of 3.

Hence, n must be 1 modulo 3, since the only solution not giving a multiple of 3 of options corresponds to $a_1 = a_2 = \ldots = a_9 = 9$.

Combining the two results, we have that n gives a remainder of 4 upon division by 6.

Problem 4. Let $\mathbb{R}^+$ denote the set of real positive numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$, such that $f(x + y) + f(x) \cdot f(y) = f(xy) + f(x) + f(y)$.

Solution. Putting $x = y = 2$, we get $f(4) + f^2(2) = f(4) + 2f(2)$. Hence $f(2) = 0$. Putting $x = y = 1$, we get $f(2) + f^2(1) = 3f(1)$, that is $f^2(1) - 3f(1) + 2 = 0$. This is a quadratic equation in $f(1)$ that has two roots 1 and 2. We distinguish the two cases:

- Case 1: $f(1) = 1$. We shall prove that in this case the function f is additive as well as multiplicative. That is, $f(u+v) = f(u) + f(v)$ and $f(uv) = f(u)f(v)$ for all $u, v \in \mathbb{R}^+$. It follows from the statement that it is enough to prove one of the two properties. Let us prove the additive property. Substituting $y = 1$ in the equation, yields $f(x+1) + f(x) \cdot f(1) = 2f(x) + f(1)$, that is, $f(x+1) = f(x) + 1$ for all positive x. Now let u and v be two arbitrary positive numbers. Putting in the statement the pairs $x = u, y = v/u$ and $x = u, y = v/u + 1$, yields

$$f(u + v/u) + f(u) \cdot f(v/u) = f(v) + f(u) + f(v/u),$$

and

$$\begin{aligned} &f(u + v/u + 1) + f(u) \cdot f(v/u + 1) \\ &= f(u \cdot (v/u + 1)) + f(u) + f(v/u + 1), \end{aligned}$$

or

$$f(u+v/u)+1+f(u)(f(v/u)+1) = f(u+v)+f(u)+f(v/u)+1$$

from which $f(u+v) = f(u)+f(v)$ is obtained after subtracting the two equations. It follows from the additive property that $f(n) = n$ for all positive integers n, and it follows from the multiplicative property that $f(k/n) = f(k)/f(n) = k/n$ for all integers k, n. Thus $f(q) = q$ for all positive rational numbers q. Additivity also implies that the function is monotonic. That is, $f(x) < f(x) + f(y - x) = f(y)$ for $x < y$. So, the only such function in this case is the identity $f(x) = x$.
- Case 2: $f(1) = 2$. We shall show that the function is $f(u) = 2$ (constant) for all positive u. Indeed, putting $y = 1$ into the equation again, we get $f(x+1) + 2f(x) = 2f(x) + 2$, thus $f(x+1) = 2$. That proves the statement for $u > 1$. Now let u be an arbitrary real number. If v is a number such that v, uv and $u + v$ are all greater than 1. The equation states that $f(u+v) + f(u) \cdot f(v) = f(uv) + f(u) + f(v)$, that is, $2 + f(u) \cdot 2 = 2 + f(u) + 2$. Hence $f(u) = 2$.

Finally, we conclude that the statement is satisfied by two functions $f(x) = x$ and $f(x) = 2$, as can be easily checked.

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Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

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Elementary Problems

E–107. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Suppose the polynomial with complex coefficients $A(z) = z^5 + a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0$ has distinct zeros and that any three of them are noncollinear. Show that we can always choose four of these zeros as vertices of a convex quadrilateral.

Solution 1 by Michel Bataille, Rouen, France. If M, N, P are three noncollinear points in the plane, let $\widehat{MNP}$ denote the angle with vertex N limited by the rays $[NM)$ and $[NP)$.
Let A, B, C, D, E be the points whose affixes are the roots of the polynomial.
As the first case, we suppose that one of these points is interior to a triangle whose vertices are three of the other points. Say D is interior to $\triangle ABC$. The point E is in one of the angles $\widehat{BDC}, \widehat{CDA}, \widehat{ADB}$, say $E \in \widehat{BDC}$. If E is exterior to $\triangle ABC$, then $DBEC$ is convex (since the segments $[BC]$ and $[DE]$ are

concurrent); otherwise, E is interior to $\widehat{BDC}$ while A is in $\widehat{BED}$ or $\widehat{CED}$, say the former. Then $ADEB$ is convex.
Now, suppose that D, E are both exterior to $\triangle ABC$. If $D \in \widehat{BAC}$ (say), then $ABDC$ is convex. If $D \notin \widehat{BAC}, D \notin \widehat{CBA}, D \notin \widehat{ACB}$, say $D \in \widehat{B'AC'}$ where B' and C' are the reflections of B and C about A, then A is interior to $\triangle DBC$ and we are back to the first case. This completes the proof.

Solution 2 by the proposer. The convex hull of the five zeros of $A(z)$ may have the form of a pentagon, a quadrilateral or a triangle. If it is a pentagon or a quadrilateral the statement is obvious. In the last case, we observe that the triangle has two interior points. Draw a line by these two points. It divides the triangles in two parts. On account of Pigeonhole principle, in one of the parts lie two vertices of the triangle that jointly with the two interior points form a convex quadrilateral.

Solution 3 by Henry Ricardo, Westchester Area Math Circle, New York, USA. First, if the convex hull C of the five given points in the plane is a convex pentagon or a convex quadrilateral, we are done. (Given a convex pentagon, any four of the points can be connected to form a convex quadrilateral.) If C is a triangle $A_0A_1A_2$, then two of the five points, say P and Q, lie in the interior of C. The line ℓ joining P and Q intersects two sides of C in interior points. If A_1 and A_2 are two vertices of C lying on the same side of ℓ, then A_1, A_2, P, Q are the vertices of a convex quadrilateral.

Comment. We note that the polynomial information is irrelevant: Given any five points in the plane, no three of which are collinear, we can choose four of them as vertices of a convex quadrilateral.

Also solved by *José Gibergans-Báguena, BarcelonaTech, Terrassa, Spain.*

E–108. *Proposed by Rovsen Pirkuliyev, Sumgait City, Azerbaijan.*
Solve in $\mathbb{R}$ the following equation

$$2022\sin^{2022}x+\cos^4 x-3\sin 2x\cos x-13\cos^2 x+11\sin x+2022=0.$$

Solution 1 by Michel Bataille, Rouen, France. From $\cos^2 x = 1-\sin^2 x$, $\cos^4 x = (1-\sin^2 x)^2$, $3\sin 2x \cos x = 6\sin x(1-\sin^2 x)$, we deduce that the given equation writes as $P(\sin x) = 0$ where P is the polynomial defined by

$$P(X) = 2022X^{2022} + X^4 + 6X^3 + 11X^2 + 5X + 2010.$$

Clearly, we have $P(X) > 0$ when $X > 0$. In addition, if $-1 \le X \le 0$, then $-11 \le 6X^3 + 5X \le 0$ so that $2010 + 6X^3 + 5X > 1999$. Thus, $P(X) \neq 0$ when $-1 \le X \le 1$ and we conclude that the given equation has no real solution.

Solution 2 by the proposer. We have that the given equation can be written as

$$2022\sin^{2022} x + (1-\cos^2 x)^2 + (1-\cos x)(6\sin x + 9)$$
$$+2(1-\cos^2 x) + 5\sin x + 2010 = 0,$$

or

$$2022\sin^{2022} x + (1-\cos^2 x)(1-\cos^2 x + 6\sin x + 9)$$
$$+2\sin^2 x + 5\sin x + 2010 = 0,$$

or

$$2022\sin^{2022} x + \sin^4 x + 6\sin^3 x + 11\sin^2 x + 5\sin x + 2010 = 0.$$

Putting $t = \sin x$, we obtain

$$2022t^{2022} + t^4 + 6t^3 + 9t^2 + 2t^2 + 5t + 2010 = 0$$

or

$$2022t^{2022} + t^2(t+3)^2 + 2t^2 + 5t + 2010 = 0.$$

Since $2022t^{2022} \ge 0$, $t^2(t+3)^2 \ge 0$ and $2t^2+5t+2010 > 0$ (because its discriminant is negative), then LHS is positive and RHS is zero, and the equation has no solutions in $\mathbb{R}$.

Also solved by *José Luis Díaz-Barrero, Barcelona, Spain.*

E–109. *Proposed by Michel Bataille, Rouen, France.* Let $f(x) = \dfrac{\cos(x/2)}{1-\sin(x/2)}$ and let α, β, γ be the angles of a triangle. Prove that

$$f(\alpha) + f(\beta) + f(\gamma) = f(\alpha) \cdot f(\beta) \cdot f(\gamma).$$

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Writing $f(x)$ in the form $f(x) = \dfrac{1+\sin(x/2)}{\cos(x/2)}$, the equality to be proved becomes equivalent to

$$\left(1+\sin\frac{\alpha}{2}\right)\cos\frac{\beta}{2}\cos\frac{\gamma}{2}+\left(1+\sin\frac{\beta}{2}\right)\cos\frac{\gamma}{2}\cos\frac{\alpha}{2}+\left(1+\sin\frac{\gamma}{2}\right)\cos\frac{\alpha}{2}\cos\frac{\beta}{2}$$
$$=\left(1+\sin\frac{\alpha}{2}\right)\left(1+\sin\frac{\beta}{2}\right)\left(1+\sin\frac{\gamma}{2}\right).$$

Expanding out,

$$\sum_{\text{cyclic}}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\sum_{\text{cyclic}}\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$$
$$=1+\sum_{\text{cyclic}}\sin\frac{\alpha}{2}+\sum_{\text{cyclic}}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \qquad (1)$$

Now,

$$\sin\frac{\alpha}{2}=\cos\left(90^\circ-\frac{\alpha}{2}\right)=\cos\left(\frac{\beta}{2}+\frac{\gamma}{2}\right)=\cos\frac{\beta}{2}\cos\frac{\gamma}{2}-\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$$

With this and two similar results, we find

$$\sum_{\text{cyclic}}\sin\frac{\alpha}{2}=\sum_{\text{cyclic}}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}-\sum_{\text{cyclic}}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}.$$

Thus, in order to prove (1) it suffices to prove that

$$\sum_{\text{cyclic}}\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}=1+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}. \qquad (2)$$

But this is virtually immediate, for

$$\begin{aligned}
&\sum_{\text{cyclic}} \sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} - \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\\
&= \sin\frac{\alpha}{2}\left(\cos\frac{\beta}{2}\cos\frac{\gamma}{2} - \sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right)\\
&\quad + \cos\frac{\alpha}{2}\left(\sin\frac{\beta}{2}\cos\frac{\gamma}{2} + \cos\frac{\beta}{2}\sin\frac{\gamma}{2}\right)\\
&= \sin\frac{\alpha}{2}\cos\left(\frac{\beta}{2}+\frac{\gamma}{2}\right) + \cos\frac{\alpha}{2}\sin\left(\frac{\beta}{2}+\frac{\gamma}{2}\right)\\
&= \sin\frac{\alpha}{2}\cos\left(90^\circ - \frac{\alpha}{2}\right) + \cos\frac{\alpha}{2}\sin\left(90^\circ - \frac{\alpha}{2}\right)\\
&= \sin^2\frac{\alpha}{2} + \cos^2\frac{\alpha}{2} = 1,
\end{aligned}$$

which is equivalent to (2).

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Observe that

$$\tan(x/2) = \frac{\sin(x/2)}{\cos(x/2)} = \frac{2\sin(x/2)\cos(x/2)}{2\cos^2(x/2)} = \frac{\sin x}{1+\cos x},$$

from which it follows that

$$\tan\left(\frac{\pi+x}{4}\right) = \frac{\sin\left(\frac{\pi+x}{2}\right)}{1+\cos\left(\frac{\pi+x}{2}\right)} = \frac{\cos(x/2)}{1-\sin(x/2)} = f(x).$$

Moreover, if A, B, and C are any angles satisfying $A+B+C=\pi$, then

$$-\tan C = -\tan(\pi - A - B) = \tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B},$$

or

$$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C.$$

Because α, β, γ are the angles of a triangle,

$$\frac{\pi+\alpha}{4} + \frac{\pi+\beta}{4} + \frac{\pi+\gamma}{4} = \frac{3\pi+\alpha+\beta+\gamma}{4} = \pi.$$

Therefore,

$$\begin{aligned} f(\alpha)+f(\beta)+f(\gamma) &= \tan\left(\frac{\pi+\alpha}{4}\right)+\tan\left(\frac{\pi+\beta}{4}\right)+\tan\left(\frac{\pi+\gamma}{4}\right) \\ &= \tan\left(\frac{\pi+\alpha}{4}\right)\cdot\tan\left(\frac{\pi+\beta}{4}\right)\cdot\tan\left(\frac{\pi+\gamma}{4}\right) \\ &= f(\alpha)\cdot f(\beta)\cdot f(\gamma). \end{aligned}$$

Solution 3 by the proposer. We remark that

$$\frac{\cos(\alpha/2)}{1-\sin(\alpha/2)} = \frac{1+\sin(\alpha/2)}{\cos(\alpha/2)} = \frac{1+\cos((\pi-\alpha)/2)}{\sin((\pi-\alpha)/2)}.$$

Now, let $u=\frac{\pi-\alpha}{2}$, $v=\frac{\pi-\beta}{2}$, $w=\frac{\pi-\gamma}{2}$. Then, $u+v+w=\pi$ (so that u,v,w are the angles of a triangle) and we are reduced to proving that

$$\frac{1+\cos u}{\sin u}+\frac{1+\cos v}{\sin v}+\frac{1+\cos w}{\sin w} = \frac{1+\cos u}{\sin u}\cdot\frac{1+\cos v}{\sin v}\cdot\frac{1+\cos w}{\sin w}.$$

Since $\frac{1+\cos u}{\sin u} = \frac{2\cos^2(u/2)}{2\sin(u/2)\cos(u/2)} = \frac{1}{\tan(u/2)}$ the latter is equivalent to

$$\frac{1}{\tan(u/2)}+\frac{1}{\tan(v/2)}+\frac{1}{\tan(w/2)} = \frac{1}{\tan(u/2)\tan(v/2)\tan(w/2)}.$$

We are done since this is equivalent to the well-known relation

$$\tan(u/2)\tan(v/2)+\tan(v/2)\tan(w/2)+\tan(w/2)\tan(u/2)=1$$

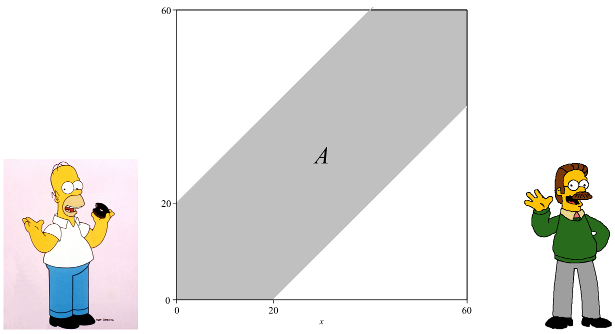
which immediately results from $\frac{w}{2}=\frac{\pi-u-v}{2}$ and

$$\tan(w/2) = \frac{1}{\tan((u+v)/2)} = \frac{1-\tan(u/2)\tan(v/2)}{\tan(u/2)+\tan(v/2)}.$$

Also solved by *Rovsen Pirkuliyev, Sumgait City, Azerbaija, Vishwesh Ravi Shrimali, Jaipur, India and Daniel Văcaru, Pitești, Romania.*

E–110. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Homer and Flanders have an appointment at a certain time, and each will arrive at the meeting place with a delay between 0 and 1 hour, with all pairs of delays equally likely. The first to arrive will wait for 20 minutes and will leave if the other has not yet arrived. What is the probability that they will meet?

Solution 1 by Henry Ricardo, Westchester Area Math Circle, New York, USA. If we represent Homer's arrival time after the appointment time by x and Flanders' arrival time by y, then x and y are independent random variables. Furthermore, the square of side 60 (minutes) given by $\{(x, y) : 0 \leq x \leq 60, 0 \leq y \leq 60\}$ represents all the equally likely possibilities of the arrivals of Homer and Flanders at the meeting place.



Scheme for solving problem E-110 (Solution 1)

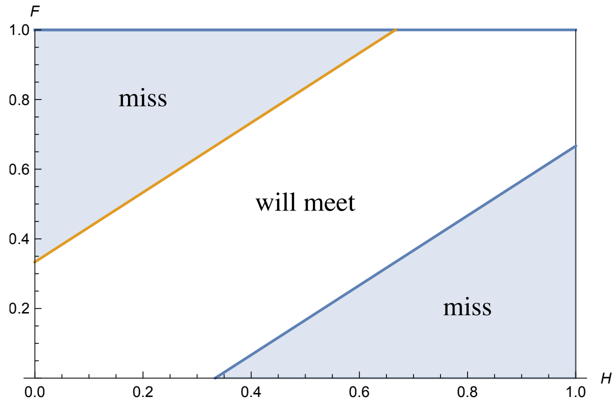
The area A is bounded by the two lines $y = x + 20$ and $y = x - 20$, so that inside A we have $|x - y| \leq 20$. It follows that the two will meet only if their arrival times x and y lie in region A. Thus the probability of their meeting is given by the ratio of the area A to the area of the square:

$$\frac{3600 - \left(\frac{40 \cdot 40}{2} + \frac{40 \cdot 40}{2}\right)}{60 \cdot 60} = \frac{5}{9}.$$

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. The figure below depicts the pair of delays, with Homer's delay indicated along the horizontal axis and Flanders' delay indicated along the vertical axis.

The two will meet if the delay pair falls within the center diagonal section (the non-shaded section). Because all pairs of delays are equally likely, the probability the two will meet is given by the ratio of the area of the unshaded region to the area of the full square. Thus, the probability the two will meet is

$$1 - 2 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{2}{3} = 1 - \frac{4}{9} = \frac{5}{9}.$$



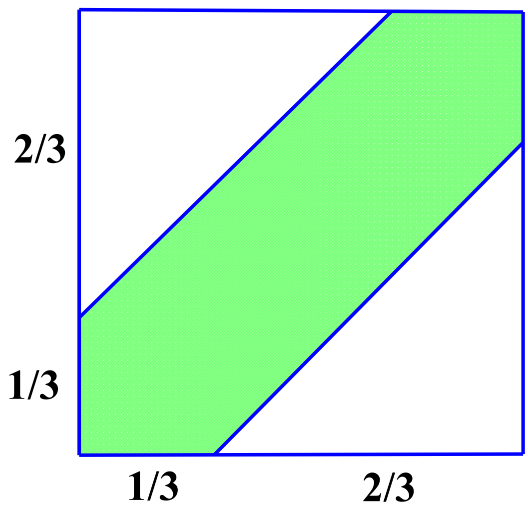
Scheme for solving problem E-110 (solution 2)

Solution 3 by José Gibergans-Báguena and the proposer, both at BarcelonaTech, Barcelona, Spain. Let us use as a sample space the unit square, whose elements are the possible pairs of delays for the two of them. Our interpretation of equally likely pairs of delays is to let the probability of a subset of the sample space E equal to its area. This probability law satisfies the three probability axioms. The event that Homer and Flanders will meet is the shaded region in the figure and its probability is

$$p = \frac{area(F)}{area(E)} = 1 - \frac{4}{9} = \frac{5}{9}.$$

E–111. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Prove that if n is a nonnegative integer, then 2023 divides

$$68^{2n+1} + 197 \cdot 51^{2n+1}.$$



Scheme for solving problem E-110 (Solution 3)

Solution 1 by José Gibergans-Báguena, BarcelonaTech, Terrasa, Spain. We have to see that there exists an integer m such that:

$$68^{2n+1} + 197 \cdot 51^{2n+1} = 2023 \cdot m$$

We will prove this using mathematical induction. Indeed,

- For $n = 1$ we have

$$68^3 + 197 \cdot 51^3 = 2023 \cdot 13073$$

- For $n = k$ we assume that is true. Then

$$68^{2k+1} + 197 \cdot 51^{2k+1} = 2023 \cdot m$$

  and

$$197 \cdot 51^{2k+1} = 2023 \cdot m - 68^{2k+1} \qquad (1)$$

- Now we have prove that is true for $n = k + 1$

$$68^{2(k+1)+1} + 197 \cdot 51^{2(k+1)+1} = 68^{2k+3} + 197 \cdot 51^{2k+1} \cdot 51^2$$

  and substituting (1) into the right hand of the above equality, we get

$$68^{2k+3} + \left(2023 \cdot m - 68^{2k+1}\right) \cdot 51^2 = 2023 \cdot 51^2 \cdot m + 68^{2k+1}\left(68^2 - 51^2\right)$$

$$= 2023 \cdot 51^2 \cdot m + 68^{2k+1} \cdot 2023 = 2023 \cdot \left(51^2 \cdot m + 68^{2k+1}\right),$$

which is divisible by 2023.

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. For $n = 0$, the given sum adds up to 10115, a multiple of 2023.

Suppose $n \geq 1$. We have

$$\begin{aligned} 68^{2n+1} + 197 \cdot 51^{2n+1} &= (4 \cdot 17)^{2n+1} + 197 \cdot (3 \cdot 17)^{2n+1} \\ &= 17^{2n+1}(4^{2n+1} + (1 + 196) \cdot 3^{2n+1}) \\ &= 17^2 \cdot 17^{2n-1}((4^{2n+1} + 3^{2n+1}) + 196 \cdot 3^{2n+1}), \qquad (1) \end{aligned}$$

where the exponent $2n - 1$ in (1) is a positive integer (being $n \geq 1$).

Since $4^{2n+1} + 3^{2n+1} = (4+3) \sum_{j=0}^{2n} 4^{2n}(-3)^j = 7 \sum_{j=0}^{2n} 4^{2n}(-3)^j$ and $196 = 7 \times 28$, we rewrite (1) in the form

$$17^2 \cdot 7 \left[17^{2n-1} \left(\sum_{j=0}^{2n} 4^{2n}(-3)^j + 28 \cdot 3^{2n+1} \right) \right],$$

so that

$$68^{2n+1} + 197 \cdot 51^{2n+1} = 2023 \left[17^{2n-1} \left(\sum_{j=0}^{2n} 4^{2n}(-3)^j + 28 \cdot 3^{2n+1} \right) \right],$$

a multiple of 2023. This completes the proof.

Solution 3 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. If $n = 0$, then

$$68 + 197 \cdot 51 = 17(4 + 3 \cdot 197) = 17^2 \cdot 7 \cdot 5,$$

which is divisible by $2023 = 17^2 \cdot 7$. For $n \geq 1$,

$$68^{2n+1} + 197 \cdot 51^{2n+1} = 17^{2n+1}(4^{2n+1} + 197 \cdot 3^{2n+1}),$$

which is clearly divisible by 17^2. It therefore suffices to show that $4^{2n+1} + 197 \cdot 3^{2n+1}$ is divisible by 7. But

$$197 \equiv 1 \pmod 7, \quad \text{and} \quad 4 \equiv -3 \pmod 7,$$

so

$$4^{2n+1} + 197 \cdot 3^{2n+1} \equiv \left((-3)^{2n+1} + 3^{2n+1}\right) \pmod 7 \equiv 0 \pmod 7.$$

Solution 4 by Henry Ricardo, Westchester Area Math Circle, New York, USA and Rovsen Pirguliyev, Sumgait City, Azerbaijan (same solution). First we note that $2023 = 17 \cdot 119 = 17(68 + 51)$. Now we have

$$\begin{aligned} 68^{2n+1} + 197 \cdot 51^{2n+1} &= 68^{2n}(68 + 197 \cdot 51) - 197 \cdot 68^{2n} \cdot 51 \\ &\quad + 51^{2n}(197 \cdot 51) \\ &= 68^{2n}(5 \cdot 2023) - 51 \cdot 197(68^{2n} - 51^{2n}) \\ &= 5 \cdot 68^{2n} \cdot 2023 - 3 \cdot 17 \cdot 197(68 + 51) \cdot K \\ &= 2023\,(5 \cdot 68^{2n} - 3 \cdot 197 \cdot K), \end{aligned}$$

where we have used the identity

$$68^{2n} - 51^{2n} = (68 + 51) \cdot \sum_{j=0}^{2n-1} (-1)^j \, 68^{2n-1-j} \cdot 51^j = 119 \cdot K, \; K \in \mathbb{N}.$$

Solution 5 by Henry Ricardo, Westchester Area Math Circle, New York, USA; Daniel Văcaru, Pitești, Romania and Ioan Viorel Codreanu, Satulung, Maramures, Romania (same solution). We proceed by induction. Indeed, for $n = 0$, we see that $68^1 + 197 \cdot 51^1 = 10,115 = 5 \cdot 2023$. Assuming that $68^{2n+1} + 197 \cdot 51^{2n+1} = 2023K$, $K \in \mathbb{N}$, for some positive value of n, we see that

$$\begin{aligned} 68^{2(n+1)+1} + 197 \cdot 51^{2(n+1)+1} &= 68^2(68^{2n+1} + 197 \cdot 51^{2n+1}) \\ &\quad + 197 \cdot 51^{2n+1}(51^2 - 68^2) \\ &= 68^2 \cdot 2023K - 197 \cdot 51^{2n+1}(2023) \\ &= 2023\,(68^2 \dot{K} - 197 \cdot 51^{2n+1}), \end{aligned}$$

and our inductive proof is complete.

Solution 6 by Michel Bataille, Rouen, France. Let $X_n = 68^{2n+1} + 197 \cdot 51^{2n+1}$. Since $2023 = 7 \times 17^2$, we have to show that 7 and 17^2 divide X_n for every $n \geq 0$.
Modulo 7, we have $68 \equiv 5 \equiv -2$, $197 \equiv 1$, $51 \equiv 2$, hence $X_n \equiv (-2)^{2n+1} + 2^{2n+1} = 0$. Thus 7 divides X_n.
Since $68 = 4 \times 17$ and $51 = 3 \times 17$, we see that

$$X_n = 17^{2n+1}(4^{2n+1} + 197 \cdot 3^{2n+1})$$

and therefore 17^2 certainly divides X_n if $n \geq 1$. In addition, it is readily checked that $X_0 = 17(4 + 3 \cdot 197) = 17^2 \cdot 35$ and 17^2 also divides X_0. The conclusion follows.

Solution 7 by the proposer. Let

$$F(n) = 68^{2n+1} + 197 \cdot 51^{2n+1};\ n \geq 0.$$

For $n = 0$, $F(0) = 68 + 197 \cdot 51 = 10115 = 5 \cdot 2023$, so $2023 \mid F(0)$.

Let $n > 0$.

$$\begin{aligned} F(n) &= 68^{2n+1} + 197 \cdot 51^{2n+1} = \\ &= 68(68^{2n} - 51^{2n}) + (68 + 197 \cdot 51)51^{2n} = \\ &= 68(68^{2n} - 51^{2n}) + 5 \cdot 2023 \cdot 51^{2n} \end{aligned}$$

From identity

$$a^{2n} - b^{2n} = (a^2 - b^2) \cdot (a^{2n-2} + a^{2n-4}b^2 + \cdots + a^2b^{2n-4} + b^{2n-2})$$

hence $68^2 - 51^2 = 2023$ divides $(68^{2n} - 51^{2n})$. It follows that $2023 \mid F(n)$ for all integers $n \geq 0$.

Also solved by *Vishwesh Ravi Shrimali, Jaipur, India and José Luis Díaz-Barrero, Barcelona, Spain.*

E–112. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* In the acute triangle ABC, F is the foot of the altitude from A and D, E are the projections of F on AB, respectively AC. The tangents to the circumcircle $\triangle ADE$ in D and E intersect in X. If $AX \cap BC = \{Y\}$, prove that triangle ABY and ACY have the same area.

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Since a median bisects the area of a triangle, it suffices to show that $BY = YC$.

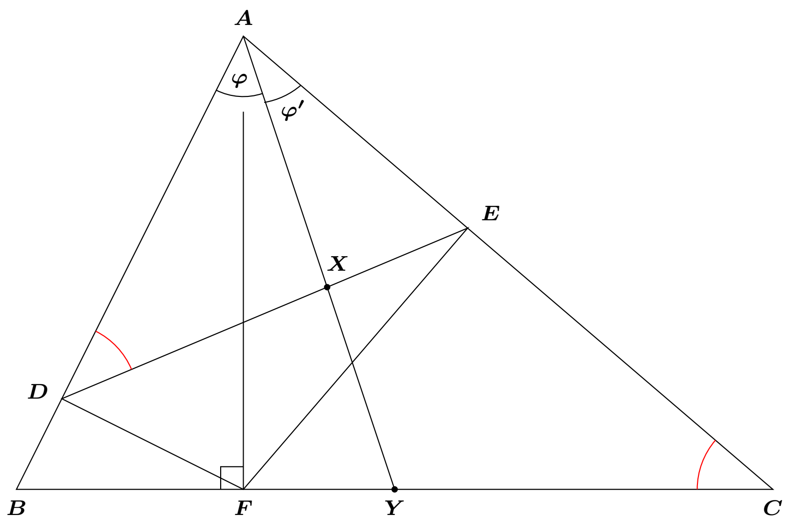
By a standard mean proportion[1], applied to right-angled triangles ABF and AFC,

$$AB \cdot AD = AF^2 = AC \cdot AE.$$

Thus

$$\frac{AB}{AC} = \frac{AE}{AD}, \tag{1}$$

making triangles ABC and AED similar (S-A-S) with $\angle ABC = \angle DEA$ and $\angle BCA = \angle ADE$.



[1] Leg's theorem

Since the tangents to the circumcircle of a triangle at two of its vertices meet on the symmedian from the third vertex, AY is the symmedian of $\triangle ADE$ at A. And because AY intersects DE at X, then

$$\frac{XE}{DX} = \frac{AE^2}{AD^2} = \text{(from (1))} = \frac{AB^2}{AC^2}. \tag{2}$$

We put $\angle BAY = \varphi$ and $\angle YAC = \varphi'$. By the law of sines, applied to the triads $\triangle ABY$, $\triangle AYC$, $\triangle ABC$ and $\triangle ADX$, $\triangle AXE$, $\triangle ABC$, we get

$$\frac{BY}{YC} = \frac{BY/AY}{YC/AY} = \frac{\frac{\sin\varphi}{\sin(\angle ABY)}}{\frac{\sin\varphi'}{\sin(\angle YCA)}} = \frac{\frac{\sin\varphi}{\sin(\angle ABC)}}{\frac{\sin\varphi'}{\sin(\angle BCA)}} = \frac{\sin\varphi}{\sin\varphi'}\cdot\frac{AB}{AC}$$

and

$$\frac{XE}{DX} = \frac{XE/AX}{DX/AX} = \frac{\frac{\sin\varphi'}{\sin(\angle XEA)}}{\frac{\sin\varphi}{\sin(\angle ADX)}} = \frac{\frac{\sin\varphi'}{\sin(\angle DEA)}}{\frac{\sin\varphi}{\sin(\angle ADE)}} = \frac{\frac{\sin\varphi'}{\sin(\angle ABC)}}{\frac{\sin\varphi}{\sin(\angle BCA)}} = \frac{\sin\varphi'}{\sin\varphi}\cdot\frac{AB}{AC}.$$

Multiplying up the preceding, we obtain

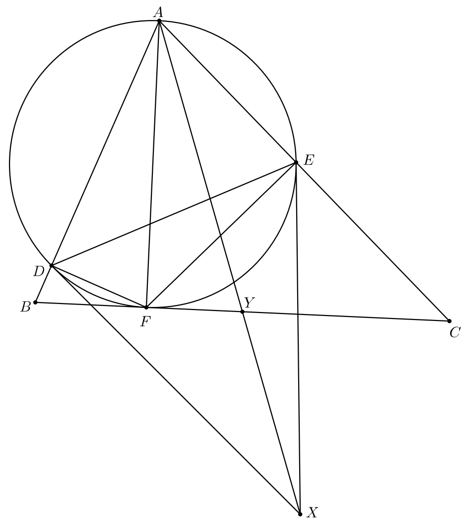
$$\frac{BY}{YC}\cdot\frac{XE}{DX} = \frac{AB^2}{AC^2},$$

which implies, by (2), that $BY = YC$ and we are done.

Solution 2 by Michel Bataille, Rouen, France. Since the circle with diameter AF passes through D and E, we have $\angle(DE, DA) = \angle(FE, FA)$. Since $FE \perp AC$ and $FA \perp BC$, it follows that $\angle(DE, DA) = \angle(CA, CB)$. As a result, BC is antiparallel to DE in the angle $\angle BAC = \angle DAE$ (see **[1]** p. 145). Observing that the line AX is the symmedian from A in $\triangle DAE$ (see **[1]** p. 146), we deduce that AX bisects BC. This means that Y is the midpoint of BC and consequently

$$\begin{aligned}\text{Area}(ABY) &= YB\cdot YA\sin(\angle BYA)\\ &= YC\cdot YA\sin(\angle CYA) = \text{Area}(ACY).\end{aligned}$$

[1] M. Bataille, Characterizing a Symmedian, *Crux Mathematicorum*, Vol. 43(4), April 2017, pp. 145-150



Scheme for solving problem E-112.

Solution 3 by the proposer. Denote $AX \cap DE = \{A\}$

$\triangle ABY$ and $\triangle ACY$ have the same altitude (AF), so to prove that $\sigma(ABY) = \sigma(ACY)$ it is enough to show that $BY = YC$.

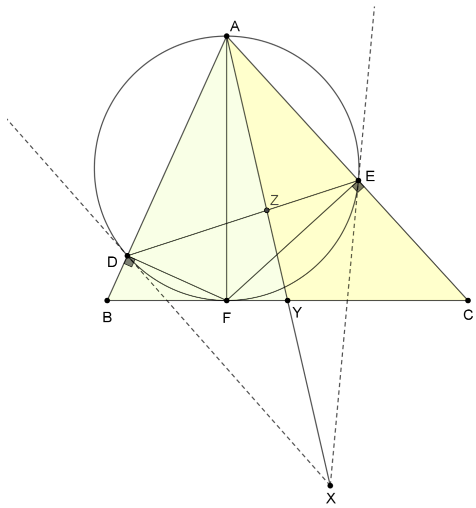
$$\frac{BY}{YC} = \frac{\sigma(ABY)}{\sigma(ACY)} = \frac{AB \cdot AY \cdot \sin(BAY) \cdot \frac{1}{2}}{AC \cdot AY \cdot \sin(CAY) \cdot \frac{1}{2}} = \frac{AB \cdot \sin(BAY)}{AC \cdot \sin(CAY)}$$

- Prove that $\dfrac{AD}{AE} = \dfrac{AC}{AB}$. Indeed, XD and XE are tangents to circumcircle of $\triangle ADE \Rightarrow XA$ is symmedian and

$$\frac{DZ}{DE} = \frac{AD^2}{AE^2}$$

$$\frac{AD^2}{AE^2} = \frac{\sigma(DAZ)}{\sigma(EAZ)} = \frac{AD \cdot AZ \cdot \sin(DAZ)}{AE \cdot AZ \cdot \sin(EAZ)} \Rightarrow \frac{\sin(DAZ)}{\sin(EAZ)} = \frac{AD}{AE}$$

According to the Leg Rule $\left.\begin{array}{l} AF^2 = AD \cdot AB \\ AF^2 = AE \cdot AC \end{array}\right\} \Rightarrow \dfrac{AD}{AE} = \dfrac{AC}{AB}$



Scheme for solving problem E-112

- Prove that ABY and ACY are equivalent triangles.
  From $\dfrac{AD}{AE} = \dfrac{AC}{AB} \Rightarrow \dfrac{\sin(DAZ)}{\sin(EAZ)} = \dfrac{\sin(BAY)}{\sin(CAY)} = \dfrac{AC}{AB} \Rightarrow$ $\sigma(ABY) = \sigma(ACY)$, so $\triangle ABY$ and $\triangle ACY$ are equivalent triangles.

Also solved by *José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.*

Easy–Medium Problems

EM–107. *Proposed by Mihály Bencze, Braşov, Romania.* Prove that

$$\sum_{0\le i<j\le n}(-1)^{i+j-1}(i+j)\binom{n}{i}\binom{n}{j}=n\binom{2n-1}{n}.$$

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. For $n=0$, the indicated expression reduces to $0=0$, while for $n=1$, the expression reduces to $1=1$. Now, let $n>1$. On the one hand,

$$\begin{aligned}\sum_{0\le i,j\le n}&(-1)^{i+j-1}(i+j)\binom{n}{i}\binom{n}{j}\\ &= 2\sum_{0\le i<j\le n}(-1)^{i+j-1}(i+j)\binom{n}{i}\binom{n}{j}+\sum_{j=0}^{n}(-1)^{2j-1}\cdot 2j\binom{n}{j}^2\\ &= 2\left(\sum_{0\le i<j\le n}(-1)^{i+j-1}(i+j)\binom{n}{i}\binom{n}{j}-\sum_{j=0}^{n}j\binom{n}{j}^2\right).\end{aligned}$$

On the other hand,

$$\sum_{0\le i,j\le n}(-1)^{i+j-1}(i+j)\binom{n}{i}\binom{n}{j}=\sum_{i=0}^{n}(-1)^i\binom{n}{i}\sum_{j=0}^{n}(-1)^{j-1}(i+j)\binom{n}{j}.$$

By the binomial theorem,

$$\sum_{j=0}^{n}\binom{n}{j}x^j=(1+x)^n \quad\text{and}\quad \sum_{j=0}^{n}j\binom{n}{j}x^{j-1}=n(1+x)^{n-1}.$$

It follows that for $n>1$,

$$\sum_{j=0}^{n}(-1)^j\binom{n}{j}=\sum_{j=0}^{n}(-1)^{j-1}j\binom{n}{j}=0.$$

Thus,

$$\sum_{0\le i,j\le n}(-1)^{i+j-1}(i+j)\binom{n}{i}\binom{n}{j}=0$$

and

$$\sum_{0\le i<j\le n}(-1)^{i+j-1}(i+j)\binom{n}{i}\binom{n}{j}=\sum_{j=0}^{n} j\binom{n}{j}^2.$$

But,

$$\sum_{j=0}^{n} j\binom{n}{j}^2=\sum_{j=0}^{n} j\binom{n}{j}\binom{n}{n-j}$$

is the coefficient of x^{n-1} in the product $n(1+x)^{n-1}(1+x)^n = n(1+x)^{2n-1}$, so

$$\sum_{0\le i<j\le n}(-1)^{i+j-1}(i+j)\binom{n}{i}\binom{n}{j}=n\binom{2n-1}{n-1}=n\binom{2n-1}{n}.$$

Solution 2 by G. C. Greubel, Newport News, VA, USA. This sum is the same as

$$S_n=\sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^{i+j}\,(i+j)\,\binom{n}{i}\,\binom{n}{j}$$

or

$$S_n=\sum_{j=0}^{n}\sum_{i=0}^{n}(-1)^{j}\,(j+2i)\,\binom{n}{i}\,\binom{n}{j+i}.$$

Considering the summation with the index j then

$$\begin{aligned}S_1&=\sum_{j=0}^{n}(-1)^{j}\,(j+2i)\,\binom{n}{j+i}\\&=(-1)^n\left(\frac{i(i+1)}{(n+i)(n+i+1)}+\frac{n(n+1)}{n+i+1}\right)\binom{n-2}{-i-1}-\binom{n-2}{i-1}\\&\quad+\frac{2\,i^2}{n}\binom{n}{i}+2\,(-1)^n\,i\binom{n-1}{-i-1}\end{aligned}$$

From this it can be seen that if $i\ge 0$ then

$$S_1=\frac{2\,i^2}{n}\binom{n}{i}-\binom{n-2}{i-1}.$$

This leads to

$$\begin{aligned}
S_n &= \sum_{i=0}^{j} \binom{n}{i} \left(\frac{2\, i^2}{n} \binom{n}{i} - \binom{n-2}{i-1} \right) \\
&= \frac{2}{n} \sum_{i=0}^{n} i^2 \binom{n}{i}^2 - \sum_{i=0}^{n} \binom{n}{i} \binom{n-2}{i-1} \\
&= \frac{2}{n} \sum_{i=0}^{n} i^2 \binom{n}{i}^2 - \sum_{i=0}^{n-1} \binom{n}{i+1} \binom{n-2}{i} \\
&= \frac{2}{n}\, n^2 \binom{2n}{n} - \frac{n}{2\,(2n-1)} \binom{2n}{n} \\
&= \frac{n}{2} \binom{2n}{n}
\end{aligned}$$

and

$$\sum_{j=0}^{n} \sum_{i=0}^{j} (-1)^{i+j}\, (i+j) \binom{n}{i} \binom{n}{j} = \frac{n}{2} \binom{2n}{n}.$$

This result is the same as that proposed in the problem.

Solution 3 by the proposer. Setting $a_k = \binom{n}{k} x^k$ in the identity

$$2 \sum_{0 \le i < j \le n} a_i a_j = \left(\sum_{k=0}^{n} a_k \right)^2 - \sum_{k=0}^{n} a_k^2$$

yields

$$2 \sum_{0 \le i < j \le n} x^{i+j} \binom{n}{i} \binom{n}{j} = \left(\sum_{k=0}^{n} \binom{n}{k} x^k \right)^2 - \sum_{k=0}^{n} \left(\binom{n}{k} x^k \right)^2$$

$$= (1+x)^{2n} - \sum_{k=0}^{n} x^{2k} \binom{n}{k}^2.$$

Differentiating and rearranging terms, we get

$$\sum_{0 \le i < j \le n} (i+j) x^{i+j-1} \binom{n}{i} \binom{n}{j} = n(1+x)^{2n-1} - \sum_{k=0}^{n} k x^{2k-1} \binom{n}{k}^2.$$

Finally, putting $x = -1$ in the preceding expressions, we obtain

$$\sum_{0 \le i < j \le n} (-1)^{i+j-1}(i+j)\binom{n}{i}\binom{n}{j} = -\sum_{k=0}^{n} k(-1)^{2k-1}\binom{n}{k}^2$$

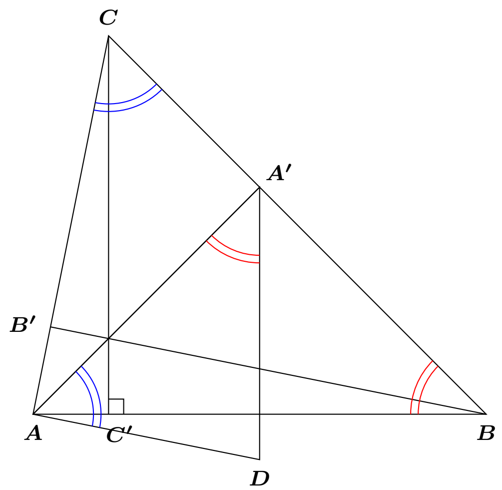
$$= \sum_{k=0}^{n} k\binom{n}{k}^2 = \frac{n}{2}\binom{2n}{n} = n\binom{2n-1}{n}.$$

Also solved by *Michel Bataille, Rouen, France.*

EM–108. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.* Triangles are erected on the altitudes AA', BB', CC' of a triangle ABC, whose other sides are parallel to the other altitudes. Let S_a, S_b, S_c denote the areas of these triangles and let S be the area of $\triangle ABC$. Prove that

$$\frac{4}{S} = 2\sum_{\text{cyclic}} \frac{1}{\sqrt{S_a S_b}} - \sum_{\text{cyclic}} \frac{1}{S_a}.$$

Solution 1 by the proposer. Let a, b, c respectively denote the lengths of the sides BC, CA, AB and let h be the altitude from A to BC.



Let the parallel to BB' through A and the parallel to CC' through A' intersect at D. Since CC' is perpendicular to AB, as is altitude, it follows that $A'D$ is perpendicular to AB. Thus altitude AA' and $A'D$ form an angle whose sides are perpendicular to those of $\angle ABC$, making

$$\angle AA'D = \angle ABC.$$

Similarly, $\angle A'AD = \angle BCA$. Therefore triangles $AA'D$ and ABC are similar (A-A-A) yielding

$$\frac{S_a}{S} = \left(\frac{AA'}{BC}\right)^2 = \left(\frac{h}{a}\right)^2,$$

where S_a denotes the area of $\triangle AA'D$. Hence

$$\frac{1}{S_a} = \frac{a^2}{h^2 S} = (\text{substituting for } h \text{ from } ah = 2S) = \frac{a^4}{4S^3},$$

and cyclically.

Consequently,

$$\frac{1}{\sqrt{S_a S_b}} = \frac{a^2 b^2}{4S^3},$$

and cyclically.

Thus

$$2\sum_{\text{cyclic}} \frac{1}{\sqrt{S_a S_b}} - \sum_{\text{cyclic}} \frac{1}{S_a} = \frac{1}{4S^3}\left(2\sum_{\text{cyclic}} a^2 b^2 - \sum_{\text{cyclic}} a^4\right)$$
$$= \frac{2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)}{4S^3},$$

in which we use the Heron's formula

$$16S^2 = ((a+b+c)(-a+b+c)(a-b+c)(a+b-c))$$
$$= 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - \left(a^4 + b^4 + c^4\right)$$

to write

$$2\sum_{\text{cyclic}} \frac{1}{\sqrt{S_a S_b}} - \sum_{\text{cyclic}} \frac{1}{S_a} = \frac{16S^2}{4S^3} = \frac{4}{S},$$

as desired.

Solution 2 by Michel Bataille, Rouen, France. Let A'' be the third vertex of the triangle constructed on the altitude AA'. Since $A'A''$ is parallel to BB', we have $A'A'' \perp CA$. We also have $AA' \perp BC$ so that $\angle(A'A'', A'A) = \angle(CA, CB)$.
Similarly, $\angle(AA', AA'') = \angle(BC, BA)$ and it follows that the triangles $AA'A''$ and BCA are similar. We deduce that

$$\frac{S}{S_a} = \left(\frac{BC}{AA'}\right)^2 = \left(\frac{a}{h_a}\right)^2.$$

[Here and in what follows, we use the familiar notations for the elements of $\triangle ABC$.]
In the same way, we obtain $\frac{S}{S_b} = \left(\frac{b}{h_b}\right)^2$, $\frac{S}{S_c} = \left(\frac{c}{h_c}\right)^2$ and therefore

$$\begin{aligned} S\left(2\sum_{\text{cyclic}} \frac{1}{\sqrt{S_aS_b}} - \sum_{\text{cyclic}} \frac{1}{S_a}\right) &= 2\sum_{\text{cyclic}} \frac{ab}{h_ah_b} - \sum_{\text{cyclic}} \frac{a^2}{h_a^2} \\ &= 2\left(\frac{a^2b^2}{4S^2} + \frac{b^2c^2}{4S^2} + \frac{c^2a^2}{4S^2}\right) - \frac{a^4+b^4+c^4}{4S^2} \end{aligned}$$

(since $ah_a = bh_b = ch_c = 2S$).
Finally, we obtain

$$\begin{aligned} S\left(2\sum_{\text{cyclic}} \frac{1}{\sqrt{S_aS_b}} - \sum_{\text{cyclic}} \frac{1}{S_a}\right) &= \frac{2(a^2b^2+b^2c^2+c^2a^2) - (a^4+b^4+c^4)}{4S^2} \\ &= \frac{16S^2}{4S^2} = 4 \end{aligned}$$

and the required result follows.

EM–109. *Proposed by Michel Bataille, Rouen, France.* **(Correction.)** Let p be an odd prime and let r be an integer with $0 \le r < p$. Prove that

$$\sum_{k=r}^{p-1} (-1)^{k-r} \binom{k}{r} \equiv 2^{-r} \pmod{p}.$$

Solution by the proposer. Let S_r denote the sum on the left. Changing the index of summation and because $\binom{k}{r} = \binom{k}{k-r}$, we have

$$S_r = \sum_{j=0}^{p-1-r} (-1)^j \binom{r+j}{j}.$$

We remark that for $0 \le j \le p-1-r$,

$$j!\binom{r+j}{j} = (r+j)(r+j-1)\cdots(r+1)$$
$$\equiv (-1)^j(p-r-j)(p-r-(j-1))\cdots(p-r-1) \pmod{p},$$

that is,

$$j!\binom{r+j}{j} \equiv (-1)^j j!\binom{p-r-1}{j} \pmod{p}.$$

Since $j!$ is coprime with p, it follows that

$$\binom{r+j}{j} \equiv (-1)^j\binom{p-r-1}{j} \pmod{p}$$

and therefore

$$S_r \equiv \sum_{j=0}^{p-1-r}\binom{p-r-1}{j} = 2^{p-1-r} \pmod{p}.$$

Finally, since we have $2^{p-1} \equiv 1 \pmod{p}$ (by the Fermat Little Theorem), we see that $S_r \equiv 2^{-r} \pmod{p}$.

EM–110. *Proposed by Goran Conar, Varaždin, Croatia.* Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be angles of a convex polygon with $n \ge 3$ vertices. Prove that the following inequality holds

$$\sin\Big(\frac{\alpha_1}{\pi}\Big)\sin\Big(\frac{\alpha_2}{\pi}\Big)\ldots\sin\Big(\frac{\alpha_n}{\pi}\Big) + \cos\Big(\frac{\alpha_1}{\pi}\Big)\cos\Big(\frac{\alpha_2}{\pi}\Big)\ldots\cos\Big(\frac{\alpha_n}{\pi}\Big) < 1.$$

Solution 1 by the proposer. We use second Minkowski's inequality

$$\left(\prod_{i=1}^{n}(a_i+b_i)^{w_i}\right)^{\frac{1}{W_n}} \ge \left(\prod_{i=1}^{n}a_i^{w_i}\right)^{\frac{1}{W_n}} + \left(\prod_{i=1}^{n}b_i^{w_i}\right)^{\frac{1}{W_n}},$$

where $a_i, b_i > 0$, $i \in \{1,2,\ldots,n\}$, and $w_i > 0$, $i \in \{1,2,\ldots,n\}$, $W_n = w_1 + w_2 + \ldots + w_n$. Specially, for $w_1 = w_2 = \ldots = w_n = \frac{1}{n}$ we have $W_n = 1$ and hence

$$\sqrt[n]{\prod_{i=1}^{n}(a_i+b_i)} \ge \sqrt[n]{\prod_{i=1}^{n}a_i} + \sqrt[n]{\prod_{i=1}^{n}b_i}. \quad (1)$$

Putting $a_i = \sin^2\left(\frac{\alpha_i}{\pi}\right), b_i = \cos^2\left(\frac{\alpha_i}{\pi}\right) = 1 - a_i$, $i \in \{1, 2, \ldots, n\}$ in (1) we get

$$1 \geq \sqrt[n]{\prod_{i=1}^{n} \sin^2\left(\frac{\alpha_i}{\pi}\right)} + \sqrt[n]{\prod_{i=1}^{n} \cos^2\left(\frac{\alpha_i}{\pi}\right)} > \sqrt[n]{\prod_{i=1}^{n} \sin^n\left(\frac{\alpha_i}{\pi}\right)} + \sqrt[n]{\prod_{i=1}^{n} \cos^n\left(\frac{\alpha_i}{\pi}\right)}$$
$$= \prod_{i=1}^{n} \sin\left(\frac{\alpha_i}{\pi}\right) + \prod_{i=1}^{n} \cos\left(\frac{\alpha_i}{\pi}\right).$$

While polygon is convex it is satisfied $0 < \alpha_i < \pi$, $\forall i \in \{1, 2, \ldots, n\}$, i.e. $\frac{\alpha_i}{\pi} \in (0, 1) \subset \left(0, \frac{\pi}{2}\right)$, $\forall i \in \{1, 2, \ldots, n\}$ so $\sin\left(\frac{\alpha_i}{\pi}\right), \cos\left(\frac{\alpha_i}{\pi}\right) \in (0, 1)$, $i \in \{1, 2, \ldots, n\}$. This we have used in last inequality.

Solution 2 by Michel Bataille, Rouen, France. Let f and g be the functions defined on $(0, \frac{\pi}{2})$ by $f(x) = \ln(\sin x)$, $g(x) = \ln(\cos x)$. Since their second derivatives $f''(x) = \frac{-1}{\sin^2 x}$, $g''(x) = \frac{-1}{\cos^2 x}$ are negative, the functions f and g are concave. It follows that

$$\sum_{k=1}^{n} \ln(\sin(\alpha_k/\pi)) \leq n \ln\left(\sin \frac{\alpha_1 + \cdots + \alpha_n}{n\pi}\right) = n \ln\left(\sin\left(\frac{n-2}{n}\right)\right)$$
$$\sum_{k=1}^{n} \ln(\cos(\alpha_k/\pi)) \leq n \ln\left(\cos \frac{\alpha_1 + \cdots + \alpha_n}{n\pi}\right) = n \ln\left(\cos\left(\frac{n-2}{n}\right)\right)$$

and in consequence

$$\prod_{k=1}^{n} \sin(\alpha_k/\pi) \leq \left(\sin\left(1 - \frac{2}{n}\right)\right)^n, \quad \prod_{k=1}^{n} \cos(\alpha_k/\pi) \leq \left(\cos\left(1 - \frac{2}{n}\right)\right)^n.$$

Note that we have $\frac{1}{3} \leq 1 - \frac{2}{n} \leq 1$. It follows that $\sin\left(1 - \frac{2}{n}\right) \leq 1 - \frac{2}{n}$ and therefore $\left(\sin\left(1 - \frac{2}{n}\right)\right)^n \leq \left(1 - \frac{2}{n}\right)^n \leq e^{-2}$ (using the known fact that the sequence $\left(\left(1 - \frac{2}{n}\right)^n\right)_{n \geq 3}$ is increasing with limit e^{-2} as $n \to \infty$). Also, we have $\cos\left(1 - \frac{2}{n}\right) \leq \cos\frac{1}{3} < 1$, hence $\left(\cos\left(1 - \frac{2}{n}\right)\right)^n \leq \left(\cos\frac{1}{3}\right)^n \leq \left(\cos\frac{1}{3}\right)^3$. Thus we obtain

$$\prod_{k=1}^{n} \sin(\alpha_k/\pi) + \prod_{k=1}^{n} \cos(\alpha_k/\pi) \leq e^{-2} + \left(\cos\frac{1}{3}\right)^3$$

and the required result follows since $e^{-2} + \left(\cos\frac{1}{3}\right)^3 < 1$.

Also solved by *José Gibergans-Báguena and José luis Díaz-Barrero both at BarcelonaTech, Barcelona, Spain.*

EM–111. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* How many are the different pairs of integers x and y from 1 to 2023, for which $(x^2+y^2)/49$ is an integer? (The pairs (x,y) and (y,x) are considered equal).

Solution 1 by Michel Bataille, Rouen, France. First, we show that x^2+y^2 is divisible by 49 if and only if x and y are both divisible by 7.
Clearly, if 7 divides x and y, then 49 divides x^2 and y^2, hence also x^2+y^2. Conversely, suppose that 49 divides x^2+y^2. Assume that 7 does not divide y. Then, y is invertible modulo 7, meaning that there exists and integer a such that $ay \equiv 1 \pmod 7$. We deduce that

$$0 \equiv a^2(x^2+y^2) \equiv (ax)^2+1 \pmod 7$$

and therefore -1 is a square modulo 7, which is false (the squares modulo 7 are $0,1,2,4$). Thus, 7 divides y and similarly, 7 divides x.
Back to the problem, since $2023 = 7\times 289$, there are 289 multiples of 7 between 1 and 2023 (included), hence there are 289×289 ordered pairs (x,y) such that 49 divides x^2+y^2. Each of these pairs (x,y) with $x\neq y$ can be coupled with the pair (y,x) and there are 289 pairs of the form (a,a). Thus, the number of non-ordered pairs is $\frac{289\times 288}{2}+289 = 41905$.

Solution 2 by the proposer. We claim that $x^2+y^2\equiv 0 \pmod{49}$ if and only if $x\equiv 0 \pmod 7$ and $y\equiv 0 \pmod 7$. Indeed,

(⇐) Obvious.
(⇒) Assume that $x^2+y^2\equiv 0 \pmod{49}$. To see that $x\equiv 0 \pmod 7$ and $y\equiv 0 \pmod 7$ we write $x=7x_1+\alpha$ and $y=7y_1+\beta$, where x_1,y_1 are nonnegative integers and $\alpha,\beta\in\{0,\pm1,\pm2,\pm3\}$. Then,

$$\begin{aligned} x^2+y^2 &= (7x_1+\alpha)^2+(7y_1+\beta)^2 \\ &= 49(x_1^2+y_1^2)+14(\alpha x_1+\beta y_1)+\alpha^2+\beta^2. \end{aligned}$$

Since $x^2+y^2\equiv 0 \pmod{49}$, then $x^2+y^2\equiv 0 \pmod 7$ and $\alpha^2+\beta^2\equiv 0 \pmod 7$ on account of the preceding expression.

The maximum value that can be taken by $\alpha^2+\beta^2$ is $(\pm 3)^2+(\pm 3)^2=18$. The only integers between 0 and 18 divisible by 7 are $0,7,14$. The values of α^2 and β^2 are $0,1,4,9$ and the possible multiple of 7 is $\alpha^2+\beta^2=0$. So, the claim is proven.

Let n be the number of positive integers smaller or equal than 2023 that are multiple of 7. That is,

$$n=\left\lfloor\frac{2023}{7}\right\rfloor=289.$$

In this case the number of different pairs of integers x or y $(x\neq y)$ for which $x^2+y^2\equiv 0 \pmod{49}$ is $\binom{289}{2}=41616$. Furthermore, there are 289 pairs of the form (x,x). So, the number of different pairs is equal to $41616+289=41905$

EM–112. *Proposed by Toma-Ioan Dumitrescu, Bucharest, Romania.* Find all surjective functions $f:(0,+\infty)\to(0,+\infty)$ such that for every positive reals x,y, satisfy

$$f\left(\frac{xf(x)}{f(y)}\right)=f^4(y)f\left(\frac{f(y)}{x}\right).$$

Solution 1 by Marc Felipe i Alsina, BarcelonaTech, Bescanó, Girona. Since f is surjective, we may take y such that $f(y)=x$, to obtain

$$f(f(x))=x^4\cdot f(1).$$

Alternatively, we can take y such that $f(y)=xf(x)$ instead:

$$f(1)=(xf(x))^4\cdot f(f(x)).$$

Substituting the expression for $f(f(x))$, we obtain

$$f(1)=x^4f^4(x)\cdot x^4\cdot f(1)$$

from which we can isolate $f(x)$:

$$f(x)=\frac{1}{x^2}.$$

Solution 2 by Michel Bataille, Rouen, France. It is easily checked that the function $x \mapsto \frac{1}{x^2}$ is a solution. We show that there is no other solution.
Let f be a solution and let $a = f(1)$. With $x = y = 1$, the given equation yields $a = a^4 f(a)$, hence

$$f(a) = \frac{1}{a^3}. \tag{1}$$

Let $z \in (0, \infty)$. Since f is surjective, there exists $y \in (0, \infty)$ such that $z = f(y)$. We deduce that for $x, z > 0$

$$f\left(\frac{xf(x)}{z}\right) = z^4 f\left(\frac{z}{x}\right). \tag{2}$$

With $z = x$, (2) gives $f(f(x)) = ax^4$, which, with $x = 1$, provides $f(a) = a$. Recalling (1), it follows that $a = 1$, from which we deduce that $f(f(x)) = x^4$ for all $x > 0$. As a result, f is injective (for $u, v > 0$, $f(u) = f(v)$ implies $f(f(u)) = f(f(v))$, hence $u^4 = v^4$, hence $u = v$). Taking $z = 1$ in (2), we see that for all $x > 0$ we have $f(xf(x)) = f(1/x)$, hence $xf(x) = \frac{1}{x}$ and $f(x) = \frac{1}{x^2}$ follows.

Also solved by *the proposer.*

Medium–Hard Problems

MH–107. *Proposed by Michel Bataille, Rouen, France.* Let a, b be real numbers such that the polynomial $x^4 - ax^3 + (b-a)x^2 + bx - 1$ has three positive real roots. Prove that

$$ab \geq 4 + |a-b|.$$

Solution by the proposer. Let $P(x) = x^4 - ax^3 + (b-a)x^2 + bx - 1$. From the hypothesis and $P(-1) = 0$, we deduce that the roots of $P(x)$ are $-1, u, v, w$ where u, v, w are positive real numbers. Vieta's formulas then give

$$u + v + w = a + 1, \quad uvw = 1, \quad uv + vw + wu = b + 1$$

and the required inequality can be written as $(u+v+w)(uv+vw+wu) \geq 5 + a + b + |a-b|$, that is, setting $S = u^2v + uv^2 + v^2w + vw^2 + w^2u + wu^2$,

$$S \geq 2\max(u+v+w, uv+vw+wu). \tag{1}$$

(Note that $a+1+b+1+|(a+1)-(b+1)| = 2\max(a+1, b+1)$.)
First, suppose that $uv + vw + wu \geq u + v + w$ and let $T = u^2v + v^2w + w^2u + u + v + w$. Using the arithmetic mean-geometric mean inequality and $uvw = 1$, we obtain

$$\begin{aligned} T &= (u^2v + uv^2w) + (v^2w + uvw^2) + (w^2u + u^2vw) \\ &\geq 2\sqrt{u^3v^3w} + 2\sqrt{uv^3w^3} + 2\sqrt{u^3vw^3} = 2(uv + vw + wu), \end{aligned}$$

hence $u^2v + v^2w + w^2u + u + v + w \geq uv + vw + wu + u + v + w$ and therefore $u^2v + v^2w + w^2u \geq uv + vw + wu$.
Exchanging u and v gives $uv^2 + vw^2 + wu^2 \geq uv + vw + wu$ and by addition

$$S \geq 2(uv + vw + wu) = 2\max(u+v+w, uv+vw+wu).$$

If $uv + vw + wu \leq u + v + w$, let $m = \frac{1}{u}, n = \frac{1}{v}, p = \frac{1}{w}$. Then $m, n, p > 0$ with $mnp = 1$ and $mn + np + pm \geq m + n + p$. From the previous case, we have $m^2n + n^2p + p^2m + mn^2 + np^2 + pm^2 \geq 2\max(mn + np + pm, m + n + p)$. Returning to u, v, w yields (1) again and we are done.

MH–108. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all positive integers n for which the equation

$$4\left(a_1^4 + a_2^4 + \ldots + a_n^4\right) = 7\left(a_1^3 + a_2^3 + \ldots + a_n^3\right)$$

has solution in the set of positive integers and determine them.

Solution by the proposer. Suppose that a_k, $1 \le k \le n$ is a solution. Then

$$\sum_{k=1}^{n}\left(a_k^4 - \frac{7}{4}\,a_k^3\right) = 0.$$

Observe that

$$\begin{aligned}\sum_{k=1}^{n}(a_k-1)^4 &= \sum_{k=1}^{n}(a_k^4 - 4a_k^3 + 6a_k^2 - 4a_k + 1)\\ &= \sum_{k=1}^{n}\left(a_k^4 - \frac{7}{4}a_k^3 - \frac{9}{4}a_k^3 + 6a_k^2 - 4a_k + 1\right)\\ &= \sum_{k=1}^{n}\left(-\frac{9}{4}a_k^3 + 6a_k^2 - 4a_k + 1\right).\end{aligned}$$

Since

$$-\frac{9}{4}a_k^3 + 6a_k^2 - 4a_k = -a_k\left(\frac{3}{2}\,a_k - 2\right)^2 \le 0,$$

then

$$\sum_{k=1}^{n}(a_k-1)^4 \le n.$$

This inequality implies that each a_k is equal to either 1 or 2. Now, suppose x of the numbers $a_1, a_2, \ldots, a_n$ are equal to 1 and y of them are equal to 2. Then $x+y=n$ and the original equation gives $4(x+16y) = 7(x+8y)$. Solving this system, we get $x = 8n/11$ and $y = 3n/11$. So, the possible solutions are those for which $n \equiv 0 \pmod{11}$. That is, if $n = 11j$, $(j \ge 1)$ then the solutions are

$$\underbrace{1, 1, 1, \ldots, 1, 1}_{8j}, \underbrace{2, \ldots, 2}_{3j},$$

and all their permutations.

Also solved by *Michel Bataille, Rouen, France; and Silvano Rossetto, centro Morin, Paderno del Grappa, TV, Italy, and Giovanni Vincenzi, dept. Math. University of Salerno, Italy.*

MH–109. *Proposed by Ruben Carpenter, Barcelona, Spain.* Given a triangle $\triangle ABC$, let A_1, B_1 and C_1 be the feet of the internal angle bisectors from A, B and C, respectively. Further, let the circumscribed circle ω of $\triangle A_1B_1C_1$ intersect AB and AC at $C_2 \neq C_1$ and $B_2 \neq B_1$, respectively. Assume that circle ω is tangent to BC. Show that either $|AB| = |AC|$ or $|AB_1| = |AC_2|$.

Solution 1 by the proposer. We will assume $|AB| \neq |AC|$ and hence prove that $|AB_1| = |AC_2|$.

To solve the problem we rely on projective geometry. The key point of our solution is the following technical result.

Lemma 1. *Let A_2 be the other intersection point of A_1A and ω. The cyclic quadrilateral formed by points A_1, B_1, A_2, C_1 is harmonic.*

Proof. Consider the points

$$X = BC \cap B_1C_1 \text{ and } Y = AA_1 \cap B_1C_1.$$

Clearly X is not at infinity because the assumption that $AB \neq AC$ means that BC and B_1C_1 are not parallel. In $\triangle AB_1C_1$, the cevians AY, B_1B and C_1C concurr at the incenter of $\triangle ABC$. Thus by Ceva's Theorem,

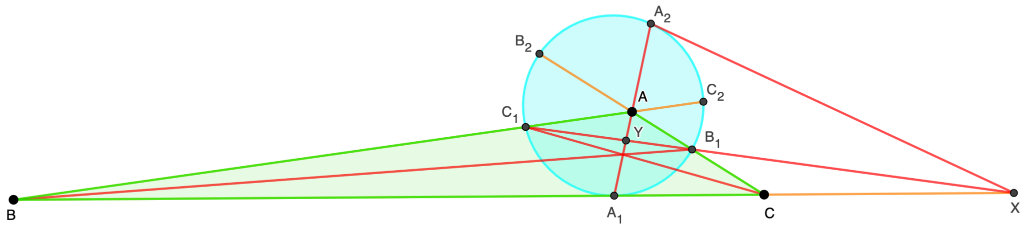
$$\frac{C_1Y}{YB_1} \cdot \frac{B_1C}{CA} \cdot \frac{AB}{BC_1} = 1. \tag{1}$$

Also, since points $X \in B_1C_1$, $C \in AB_1$ and $B \in AC_1$ are collinear, Menelaus' Theorem yields

$$\frac{C_1X}{XB_1} \cdot \frac{B_1C}{CA} \cdot \frac{AB}{BC_1} = -1, \tag{2}$$

where lengths are directed in both Eq. 1 and Eq. 2. Dividing Eq. 1 by Eq. 2,

$$-1 = \frac{C_1Y}{YB_1} \cdot \frac{XB_1}{C_1X} = (C_1B_1; YX). \tag{3}$$



Scheme for solving problem MH-109

A perspectivity centered at $A_1 \in \omega$ onto ω is a projective transformation, and hence preserves cross ratios.

Thus, from point A_1, we project the harmonic bundle in line B_1C_1 (displayed in Eq. 3) onto ω:

$$\begin{aligned} -1 &= (C_1, B_1; Y, X) \\ &= (A_1C_1 \cap \omega, A_1B_1 \cap \omega; A_1Y \cap \omega, A_1X \cap \omega) \\ &= (C_1, B_1; A_2, A_1). \end{aligned}$$

In this step we have used the tangency condition between ω and BC to show that the image of X is A_1. Then, points A_1, A_2, B_1, C_1 form a harmonic quadrilateral on circle ω. □

By the definition of a cyclic harmonic quadrilateral, Lemma 1 is equivalent to the concurrence of the following three lines: the tangent to ω through A_1, the tangent to ω through A_2 and line B_1C_1. Hence XA_2 is tangent to ω.

Let O be the center of ω. We finish with the following claim.

Claim 1. *The points O, A and X are collinear, and this line bisects $\angle C_1AB_1$*

Proof. Firstly, equation (3) implies that the pencil (AC_1, AB_1, AY, AX) is harmonic. AY is the internal bisector of $\angle B_1AC_1$, so this condition implies that AX must be the external bisector of $\angle B_1AC_1$. Hence $AX \perp A_1A_2$.

On the other hand, since XA_1 and XA_2 are both tangent to ω, then OX must be the perpendicular bisector of segment A_1A_2.

Together, the above observations complete this proof. □

Finally, notice that from Claim 1 we deduce that

$$\angle OAB_1 = \angle OAC_2.$$

We also know that $|OB_1| = |OC_2|$, and both triangles share a side of length $|OA|$. Hence, we have shown that $\triangle OAB_1$ and $\triangle OAC_2$ are oppositely congruent triangles.

We then deduce that $|AB_1| = |AC_2|$, as desired.

Solution 2 by Michel Bataille, Rouen, France. Let $a = |BC|, b = |CA|, c = |AB|$, as usual. In barycentric coordinates relatively to (A, B, C), we have $A_1 = (0 : b : c), B_1 = (a : 0 : c), C_1 = (a : b : 0)$ and the equation of ω is of the form $a^2yz + b^2zx + c^2xy = (x + y + z)(\alpha x + \beta y + \gamma z)$. Expressing that A_1, B_1, C_1 are on this circle gives

$$\alpha = \frac{bc}{2}(-u + v + w), \beta = \frac{ca}{2}(u - v + w), \gamma = \frac{ab}{2}(u + v - w)$$

where here and in what follows $u = \frac{a}{b+c}, v = \frac{b}{c+a}, w = \frac{c}{a+b}$.
Since ω is tangent to BC $(x = 0)$, $(y, z) = (b, c)$ is a double solution to $a^2yz = (y + z)(\beta y + \gamma z)$, that is, $\beta y^2 + yz(\beta + \gamma -$

$a^2) + \gamma z^2 = 0$. We deduce that $\frac{b^2}{c^2} = \frac{\gamma}{\beta}$, which easily writes as $c(u+v-w) = b(u-v+w)$ or, after a simple calculation,

$$(b-c)((b+c)^3 - a(a^2-b^2-c^2+ab+ca-bc)) = 0. \qquad (1)$$

Similarly, C_1 and C_2 are obtained as the intersections of ω and the line $z=0$, which leads to the equation $\alpha x^2 + xy(\alpha+\beta-c^2)+\beta y^2 = 0$. One of the solution is $(x,y) = (a,b)$ and the other one is $(\beta b, \alpha a)$ so that $C_2 = (u-v+w : -u+v+w : 0)$. We deduce that $2w\overrightarrow{AC_2} = (-u+v+w)\overrightarrow{AB}$ and $|AC_2| = \frac{c}{2}\left|1+\frac{v-u}{w}\right|$.
To complete the solution it remains to show that if $(b+c)^3 - a(a^2 - b^2 - c^2 + ab + ca - bc) = 0$, then $|AC_2| = |AB_1|$. Now, the latter writes as $\frac{2b}{a+c} = \left|1+\frac{v-u}{w}\right|$ or

$$2bc(b+c) = |c(c+a)(c+b) + (b^2-a^2)(a+b+c)|.$$

But using $(b+c)^3 - a(a^2-b^2-c^2+ab+ca-bc) = 0$, we easily obtain $c(c+a)(c+b) - (b^2-a^2)(a+b+c) = -2bc(c+b)$, so we are done.

MH–110. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let x_{ij}, $(1 \le i \le m, 1 \le j \le n)$ be nonnegative real numbers. Find the minimum value of

$$\prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m} \frac{\sqrt[3]{x_{ij}}}{1+2\sqrt[3]{x_{ij}}}\right) + \prod_{i=1}^{m}\left(1 - \prod_{j=1}^{n} \frac{1+\sqrt[3]{x_{ij}}}{1+2\sqrt[3]{x_{ij}}}\right).$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY, USA. In the October, 2022 issue of *Octogon Mathematical Magazine* (p. 831), the proposer proves the following result using a clever probabilistic interpretation:

Let m, n be positive integers and let $a_{ij} \in [0,1]$ for $1 \le i \le m$ and $1 \le j \le n$. Then

$$\prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m} a_{ij}\right) + \prod_{i=1}^{m}\left(1 - \prod_{j=1}^{n}(1-a_{ij})\right) \ge 1.$$

Setting $a_{ij} = \sqrt[3]{x_{ij}}/(1 + 2\sqrt[3]{x_{ij}})$ in this theorem, we see that a lower bound of the sum of products is 1. If $a_{ij} = 0$ or 1 $\forall i, j$ in the proposer's theorem, then the sum attains its minimum value. In our problem, equality holds if $x_{ij} = 0$ $\forall i, j$.

Solution 2 by Michel Bataille, Rouen, France. Let $X_{m,n}$ denote the given expression. Clearly, $X_{m,n} = 1$ when $x_{ij} = 0$ for all i, j. We show that 1 is the minimum value of $X_{m,n}$ by showing that $X_{m,n} \geq 1$ for all $x_{ij} \geq 0$.
We use the result obtained in the article J. L. Díaz-Barrero, Classical inequalities and applications, *Arhimede Mathematical Journal*, Vol. 8, Issue 2, Problem 8, p. 200:

$$\prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m} y_{ij}\right) + \prod_{i=1}^{m}\left(1 - \prod_{j=1}^{n}(1 - y_{ij})\right) \geq 1$$

when $0 \leq y_{ij} \leq 1$.
Taking $y_{ij} = \frac{\sqrt[3]{x_{ij}}}{1+2\sqrt[3]{x_{ij}}}$ directly leads to $X_{m,n} \geq 1$.

Solution 3 by the proposer. Putting $x_{ij} = 0$ in the given expression we obtain the value one. We claim that it is larger or equal than one. Indeed, Putting $a_{ij} = \dfrac{\sqrt[3]{x_{ij}}}{1 + 2\sqrt[3]{x_{ij}}}$ in the inequality

$$\prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m} \frac{\sqrt[3]{x_{ij}}}{1 + 2\sqrt[3]{x_{ij}}}\right) + \prod_{i=1}^{m}\left(1 - \prod_{j=1}^{n} \frac{1 + \sqrt[3]{x_{ij}}}{1 + 2\sqrt[3]{x_{ij}}}\right) \geq 1,$$

we get

$$\prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m} a_{ij}\right) + \prod_{i=1}^{m}\left(1 - \prod_{j=1}^{n}(1 - a_{ij})\right) \geq 1.$$

Since $0 \leq a_{ij} < 1$, then we can assume that $a_{ij} = p[A_{ij}]$ where A_{ij} are independent events in a probability space $(E, \mathcal{P}(E), p)$. It is well known that

$$\bigcup_{j=1}^{n}\left(\bigcap_{i=1}^{m} A_{ij}\right) \subseteq \bigcap_{i=1}^{m}\left(\bigcup_{j=1}^{n} A_{ij}\right).$$

Taking into account Morgan's Law, yields

$$\bigcup_{j=1}^{n}\left(\bigcap_{i=1}^{m} A_{ij}\right) = \overline{\bigcap_{j=1}^{n}\overline{\left(\bigcap_{i=1}^{m} A_{ij}\right)}} \subseteq \bigcap_{i=1}^{m}\overline{\left(\bigcap_{j=1}^{n} \overline{A}_{ij}\right)} = \bigcap_{i=1}^{m}\left(\bigcup_{j=1}^{n} A_{ij}\right).$$

Thus,

$$p\left[\overline{\bigcap_{j=1}^{n}\overline{\left(\bigcap_{i=1}^{m} A_{ij}\right)}}\right] \le p\left[\bigcap_{i=1}^{m}\overline{\left(\bigcap_{j=1}^{n} \overline{A}_{ij}\right)}\right]$$

or equivalently

$$1 - \prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m} a_{ij}\right) \le \prod_{i=1}^{m}\left(1 - \prod_{j=1}^{n}(1 - a_{ij})\right).$$

Equality holds when all the $x_{ij} = 0$ and this is the minimum value, as claimed.

Also solved by *José Gibergnas-Báguena, BarcelonaTech, Barcelona, Spain.*

MH–111. *Proposed by Félix Moreno Peñarrubia, Charles University, Prague, Czech Republic.* Let n be a positive integer, and let $\mathbb{Q}_n = \{\frac{a}{b} \in \mathbb{Q} : 1 \le a, b \le n\}$. Let S be a subset of $\mathbb{Q}_n$ with $|S| = n$. Prove that there exist two distinct (not necessarily nonempty) subsets $A, B \subseteq S$ such that $\prod_{x\in A} x = \prod_{y\in B} y$.

Solution by the proposer. Consider an undirected graph on n vertices labeled with numbers $1, \ldots, n$ and for each $\frac{a}{b} \in S$ draw an edge between vertices a and b (there may be loops or more than one edge between the same pair of vertices). Since the graph has n vertices and n edges, it must have a cycle. Let $v_1, \ldots, v_n$ be the vertices of the cycle in order. Place the fraction corresponding to edge $v_i v_{i+1}$ (indices taken cyclically) in subset A if the original number was $\frac{v_i}{v_{i+1}}$, place it in subset B if the original number was $\frac{v_{i+1}}{v_i}$. Now, note that all the terms get cancelled.

MH–112. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* For an integer $n \geq 3$ we consider a circle containing n vertices. To each vertex we assign a positive integer, and these integers do not necessarily have to be distinct. Such an assignment of integers is called stable if the product of any three adjacent integers is n. For how many values of n with $3 \leq n \leq 2023$ does there exist a stable assignment?

Solution by the proposer. Suppose n is not a multiple of 3 and that we have a stable assignment of the numbers $a_1, a_2, \ldots, a_n$, in that order on the circle. Then we have $a_i a_{i+1} a_{i+2} = n$ for all i, where the indices are considered modulo n. Hence,

$$a_{i+1}a_{i+2}a_{i+3} = n = a_i a_{i+1} a_{i+2},$$

which yields $a_{i+3} = a_i$ (as all numbers are positive). Through induction, we find that $a_{3k+1} = a_1$ for all integers $k \geq 0$. Because n is not a multiple of 3, the numbers $3k+1$ for $k \geq 0$ take on all values modulo n. Indeed, 3 has a multiplicative inverse modulo n, hence $k \equiv 3^{-1} \cdot (b-1)$ implies $3k+1 \equiv b \pmod{n}$ for all b. We conclude that all numbers on the circle must equal a_1. Hence, we have $a_1^3 = n$, where a_1 is a positive integer. Hence, if n is not a multiple of 3, then n must be a cube.

If n is a multiple of 3, then we put the numbers $1, 1, n, 1, 1, n, \ldots$ in that order on the circle. In that case, the product of three adjacent numbers always equals $1 \cdot 1 \cdot n = n$. If n is a cube, say $n = m^3$, then we put the numbers $m, m, m, \ldots$ on the circle. In that case, the product of three adjacent numbers always equals $m^3 = n$.

We conclude that a stable assignment exists if and only if n is a multiple of 3, or a cube. Now we have the count the number of such n. The multiples of 3 with $3 \leq n \leq 2023$ are $36, 9, \ldots, 2019, 2022$, these are $\lfloor 2023/3 \rfloor = 674$ numbers. The cubes with $3 \leq n \leq 2023$ are $2^3, 3^3, \ldots, 12^3$, because $12^3 = 1728 < 2023$ and $13^3 = 2197 > 2023$. These are 11 cubes, of which 4 are divisible by 3, hence there are 7 cubes which are not a multiple of 3. Altogether, there are $674 + 7 = 681$ values of n satisfying the conditions.

Also solved by *Shamil Abbasov. Azerbaijan, Baku.*

Advanced Problems

A–107. *Proposed by Joseph Santmyer, US Federal Government (retired), Las Crues, New Mexico, USA.* Let us denote by A and B the following integrals

$$\begin{aligned} A &= \int_0^{2\pi} \cos(t)\sin[e^{\cos(t)}\cos(\sin(t)-t)]\cosh[e^{\cos(t)}\sin(\sin(t)-t)]dt, \\ B &= \int_0^{2\pi} \sin(t)\cos[e^{\cos(t)}\cos(\sin(t)-t)]\sinh[e^{\cos(t)}\sin(\sin(t)-t)]dt. \end{aligned}$$

Prove that

$$2\pi\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^{2n}}{(2n+1)!(2n)!} = A - B.$$

(Correction of Problem A-103)

Solution 1 by Moti Levy, Rehovot, Israel. One may verify that

$$\begin{aligned} &\cos(t)\sin\left(e^{\cos t}\cos(\sin(t)-t)\right)\cosh\left(e^{\cos t}\sin(\sin(t)-t)\right) \qquad (1)\\ &-\sin(t)\cos\left(e^{\cos t}\cos(\sin(t)-t)\right)\sinh\left(e^{\cos t}\sin(\sin(t)-t)\right)\\ &=\operatorname{Re}\left(e^{it}\sin\left(e^{-it}e^{e^{it}}\right)\right). \end{aligned}$$

It follows from (1) that

$$A - B = \operatorname{Re}\int_0^{2\pi} e^{it}\sin\left(e^{-it}e^{e^{it}}\right)dt. \qquad (2)$$

The Taylor series of $\sin(x)$ is

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!},$$

hence

$$e^{it}\sin\left(e^{-it}e^{e^{it}}\right) = \sum_{n=0}^{\infty}(-1)^n e^{it}\frac{e^{-i(2n+1)t}e^{(2n+1)e^{it}}}{(2n+1)!}.$$

Integrating term by term and changing the order of summation and integration, we obtain,

$$A - B = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\operatorname{Re}\left(\int_0^{2\pi} e^{it}e^{-i(2n+1)t}e^{(2n+1)e^{it}}dt\right)$$

Let $z = e^{it}$ then

$$\operatorname{Re}\int_0^{2\pi} e^{it}e^{-i(2n+1)t}e^{(2n+1)e^{it}}dt = \operatorname{Re}\left(-i\oint_\Gamma \frac{e^{(2n+1)z}}{z^{2n+1}}dz\right),$$

where Γ is the unit circle $\{z : |z| = 1\}$.
Thus

$$A - B = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\operatorname{Re}\left(-i\oint_\Gamma \frac{e^{(2n+1)z}}{z^{2n+1}}dz\right). \tag{3}$$

The Taylor series of e^x is $\sum_{m=0}^{\infty} \frac{x^m}{m!}$, hence

$$e^{(2n+1)z} = \sum_{m=0}^{\infty} \frac{(2n+1)^m}{m!}\frac{z^m}{z^{2n+1}}$$

Again, integrating term by term and changing the order of summation and integration, we obtain,

$$\begin{aligned}\operatorname{Re}\left(-i\oint_\Gamma \frac{e^{(2n+1)z}}{z^{2n+1}}dz\right) &= \operatorname{Re}\left(\sum_{m=0}^{2n-1}(-i)\oint_\Gamma \frac{(2n+1)^m}{m!}\frac{z^m}{z^{2n+1}}dz\right)\\ &+ \operatorname{Re}\left(-i\oint_\Gamma \frac{(2n+1)^{2n}}{(2n)!}\frac{1}{z}dz\right)\\ &+ \operatorname{Re}\left(-i\oint_\Gamma \sum_{m=2n+1}^{\infty} \frac{(2n+1)^m}{m!}\frac{z^m}{z^{2n+1}}dz\right)\end{aligned}$$

By the Cauchy's integral theorem

$$\oint_\Gamma z^k dz = \begin{cases} 2\pi i & if \quad z = -1 \\ 0 & otherwise \end{cases}, \ k \in \mathbb{Z}.$$

Then

$$\operatorname{Re}\left(-i\oint_\Gamma \frac{e^{(2n+1)z}}{z^{2n+1}}dz\right) = 2\pi\frac{(2n+1)^{2n}}{(2n)!}. \tag{4}$$

Plugging (4) into (3), we get the required result

$$A - B = 2\pi\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)^{2n}}{(2n+1)!(2n)!} \cong 2.74804.$$

Remark: A nice reference on this type of integrals is

[1] R. P. Boas, Jr. and Lowell Schoenfeld, "*Indefinite Integration by Residues*", SIAM Review, Vol. 8., No. 2, April, 1966.

Solution 2 by Michel Bataille, Rouen, France. Let

$$C(t) = e^{\cos(t)}\cos(\sin(t)-t),\ S(t) = e^{\cos(t)}\sin(\sin(t)-t)$$

and $Z(t) = C(t) + iS(t)$ so that $C(t), S(t) \in \mathbb{R}$ and $Z(t) = e^{(e^{it})}e^{-it}$. Using $\cosh(S(t)) = \cos(iS(t))$, $\sinh(S(t)) = -i\sin(iS(t))$, we readily obtain that

$$A = \frac{1}{2}\int_0^{2\pi}\cos(t)[\sin(Z(t)) + \sin(\overline{Z(t)})]\,dt,$$

$$B = \frac{-i}{2}\int_0^{2\pi}\sin(t)[\sin(Z(t)) - \sin(\overline{Z(t)})]\,dt$$

and therefore

$$A - B = \frac{1}{2}\int_0^{2\pi}\left(e^{it}\sin(Z(t)) + e^{-it}\sin(\overline{Z(t)})\right)dt = \Re(I).$$

where $I = \int_0^{2\pi} e^{it}\sin(Z(t))\,dt$.
Now, we have

$$I = \frac{1}{i}\int_0^{2\pi}\sin\left(\frac{e^{(e^{it})}}{e^{it}}\right)d(e^{it}) = \frac{1}{i}\int_\gamma \sin\left(\frac{e^z}{z}\right)dz = \frac{1}{i}\cdot 2\pi i\sigma = 2\pi\sigma$$

where γ is the unit circle described positively and σ is the residue at 0 of the function $z \mapsto \sin\left(\frac{e^z}{z}\right)$.
Since

$$\sin\left(\frac{e^z}{z}\right) = \sum_{n=0}^{\infty}(-1)^n\frac{e^{(2n+1)z}}{z^{2n+1}(2n+1)!} \text{ and } e^{(2n+1)z} = \sum_{k=0}^{\infty}\frac{(2n+1)^k z^k}{k!},$$

the coefficient σ of $\frac{1}{z}$ in the Laurent expansion of $\sin\left(\frac{e^z}{z}\right)$ is $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\cdot\frac{(2n+1)^{2n}}{(2n)!}$, a real number. Thus

$$A - B = 2\pi\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^{2n}}{(2n+1)!(2n)!}.$$

Also solved by *the proposer.*

A–108. *Proposed by Michel Bataille, Rouen, France.* Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{1}{k\binom{n+k-1}{n}}.$$

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Observe that

$$\begin{aligned}\frac{1}{\binom{n+k-1}{n}} &= \frac{n!(k-1)!}{(n+k-1)!} = n\frac{(n-1)!(k-1)!}{(n+k-1)!} = n\frac{\Gamma(n)\Gamma(k)}{\Gamma(n+k)}\\ &= nB(n,k) = n\int_0^1 (1-x)^{n-1}x^{k-1}\,dx,\end{aligned}$$

where $\Gamma(x)$ is the gamma function and $B(x,y)$ is the beta function. Thus,

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{1}{k\binom{n+k-1}{n}} &= n\int_0^1 (1-x)^{n-1}\sum_{k=1}^{\infty}\frac{x^{k-1}}{k}\,dx\\ &= -n\int_0^1\frac{(1-x)^{n-1}\ln(1-x)}{x}\,dx\\ &= -n\int_0^1\frac{x^{n-1}\ln x}{1-x}\,dx\\ &= -n\sum_{j=0}^{\infty}\int_0^1 x^{n+j-1}\ln x\,dx\\ &= n\sum_{j=0}^{\infty}\frac{1}{(n+j)^2} = n\sum_{j=n}^{\infty}\frac{1}{j^2}.\end{aligned}$$

Finally, by the Stolz-Cesaro theorem,

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{1}{k\binom{n+k-1}{n}} = \lim_{n\to\infty} n\sum_{j=n}^{\infty}\frac{1}{j^2} = \lim_{n\to\infty}\frac{-1/n^2}{1/(n+1)-1/n} = 1.$$

Solution 2 by G. C. Greubel, Newport News, VA, USA. The series can be evaluated in the following way.

$$\begin{aligned}S_n &= \sum_{k=1}^{\infty}\frac{1}{k\binom{n+k-1}{n}} = \sum_{k=1}^{\infty}\frac{n!\,(k-1)!}{k\,(n+k-1)!}\\ &= \sum_{k=0}^{\infty}\frac{n!\,k!}{(k+1)\,(n+k)!} = \sum_{k=0}^{\infty}\frac{n+k+1}{k+1}B(n+1,k+1),\end{aligned}$$

where $B(x,y)$ is the Beta function defined by

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}\,(1-t)^{y-1}\,dt.$$

Now, by using the integral form of the Beta function, then

$$\begin{aligned}
S_n &= \sum_{k=0}^{\infty}\left(1+\frac{n}{k+1}\right)\int_0^1 t^n\,(1-t)^k\,dt \\
&= \int_0^1\left(\frac{t^n}{1-(1-t)} + n\,t^n\sum_{k=0}^{\infty}\frac{(1-t)^k}{k+1}\right)dt \\
&= \int_0^1\left(t^{n-1} + n\,t^n\,\frac{\ln(t)}{t-1}\right)dt \\
&= \int_0^1 t^{n-1}\,dt + n\int_0^1 \frac{t^n\,\ln(t)}{t-1}\,dt \\
&= \frac{1}{n} + n\,\psi'(n+1) \\
&= n\,\psi'(n),
\end{aligned}$$

where $\psi'(x)$ is the trigamma function. The asymptotic expansion of the trigamma function is

$$\psi'(x) \approx \frac{1}{x}\left(1+\frac{1}{2\,x}+\frac{1}{6\,x^2}+\mathcal{O}\left(\frac{1}{x^4}\right)\right)$$

and leads the series in question to the form

$$S_n \approx 1+\frac{1}{2\,n}+\frac{1}{6\,n^2}+\mathcal{O}\left(\frac{1}{n^4}\right)$$

and when the limit is taken yields the result

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{1}{k\binom{n+k-1}{n}} = 1.$$

Solution 3 by Moti Levy, Rehovot, Israel. This problem is solved here using hypergeometric function. A very good reference to the application of hypergeometric functions to the evaluation of binomial series is the wonderful book of Graham, Knut and Patashnik, "Concrete Mathematics".

The first step is to express the binomial series as a hypergeometric function.

The second step is application of some classical hypergeometric theorems and identities.

The following lemma explains how to express the binomial sum as a hypergeometric function:

Lemma : Let $(\beta_k)_{k\geq 0}$ be a sequence which satisfies the following conditions:

$$\beta_0 = 1,$$
$$\frac{\beta_{k+1}}{\beta_k} = \frac{1}{k+1}\frac{(k+a)(k+b)(k+c)}{(k+d)(k+e)}z.$$

Then

$$\sum_{k=0}^{\infty}\beta_k = {}_3F_2\left[\begin{matrix} a & b & c \\ & d & e \end{matrix}\middle| z\right], \tag{1}$$

where ${}_3F_2\left[\begin{matrix} a & b & c \\ & d & e \end{matrix}\middle| z\right]$ is a hypergeometric function.

Let

$$\beta_k := \frac{1}{(k+1)\binom{n+k}{n}},$$

then

$$\frac{\beta_{k+1}}{\beta_k} = \frac{1}{k+1}\frac{(k+1)^3}{(k+n+1)(k+2)}.$$

Hence by the lemma,

$$\sum_{k=1}^{\infty}\frac{1}{k\binom{n+k-1}{n}} = {}_3F_2\left[\begin{matrix} 1 & 1 & 1 \\ & n+1 & 2 \end{matrix}\middle| 1\right].$$

Now we apply two known hypergeometric identities:

$$ {}_3F_2\left[\begin{matrix} a & b & c \\ & d & e \end{matrix}\middle| 1\right] = \frac{\Gamma(d)\Gamma(e)\Gamma(s)}{\Gamma(a)\Gamma(b+s)\Gamma(c+s)}\,{}_3F_2\left[\begin{matrix} d-a & e-a & s \\ & b+s & c+s \end{matrix}\middle| 1\right], \tag{2}$$

$$s := d+e-a-b-c;\quad \mathrm{Re}(a) > 0,\quad \mathrm{Re}(s) > 0.$$

$$ {}_3F_2\left[\begin{matrix} a & a & 1 \\ & a+1 & a+1 \end{matrix}\middle| 1\right] = a^2 \sum_{k=0}^{\infty} \frac{1}{(a+k)^2}. \tag{3} $$

Application of (2) gives

$$ \begin{aligned} {}_3F_2\left[\begin{matrix} 1 & 1 & 1 \\ & n+1 & 2 \end{matrix}\middle| 1\right] &= \frac{\Gamma(n+1)\Gamma(2)\Gamma(n)}{\Gamma(1)\Gamma(n+1)\Gamma(n+1)} {}_3F_2\left[\begin{matrix} n & n & 1 \\ & n+1 & n+1 \end{matrix}\middle| 1\right] \\ &= \frac{1}{n} {}_3F_2\left[\begin{matrix} n & n & 1 \\ & n+1 & n+1 \end{matrix}\middle| 1\right]. \end{aligned} $$

Application of (3) gives

$$ {}_3F_2\left[\begin{matrix} n & n & 1 \\ & n+1 & n+1 \end{matrix}\middle| 1\right] = n^2 \sum_{k=0}^{\infty} \frac{1}{(n+k)^2}. $$

Thus

$$ \sum_{k=1}^{\infty} \frac{1}{k\binom{n+k-1}{n}} = n \sum_{k=0}^{\infty} \frac{1}{(n+k)^2} = \frac{1}{n} \sum_{k=0}^{\infty} \frac{1}{\left(1+\frac{k}{n}\right)^2} $$

Now

$$ \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{\infty} \frac{1}{\left(1+\frac{k}{n}\right)^2} = \int_0^{\infty} \frac{1}{(1+x)^2} = 1. $$

Solution 4 by the proposer. First, it is readily checked that

$$ \sum_{k=1}^{\infty} \frac{1}{k\binom{n+k-1}{n}} = n! \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{1}{k(k+1)\cdots(k+n-1)}. $$

Then, we prove that for $n \in \mathbb{N}$, the following equality holds:

$$ \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{1}{k(k+1)\cdots(k+n-1)} = \frac{R_n}{(n-1)!} \tag{1} $$

where $R_n = \sum\limits_{k=n}^{\infty} \frac{1}{k^2}$.

We use induction on n. The result is obvious when $n = 1$. Assume that (1) holds for some positive integer n. Then we calculate the sum

$$ S_n = \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{1}{k(k+1)\cdots(k+n)} $$

as follows

$$S_n = \sum_{k=1}^{\infty} \frac{1}{k} \cdot \frac{1}{n}\left(\frac{1}{k(k+1)\cdots(k+n-1)} - \frac{1}{(k+1)\cdots(k+n)}\right)$$
$$= \frac{1}{n} \cdot \frac{R_n}{(n-1)!} - \frac{1}{n} \cdot \frac{1}{n\cdot(n!)} = \frac{1}{n!}\left(R_n - \frac{1}{n^2}\right) = \frac{R_{n+1}}{n!}.$$

(We have used the well-known telescopic series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)\cdots(k+n)}$$
$$= \sum_{k=1}^{\infty} \frac{1}{n}\left(\frac{1}{k(k+1)\cdots(k+n-1)} - \frac{1}{(k+1)\cdots(k+n)}\right)$$
$$= \frac{1}{n\cdot(n!)}.$$

This completes the induction step and the proof.
From (1), we now deduce that

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{k\binom{n+k-1}{n}} = \lim_{n\to\infty} nR_n.$$

But we have $R_n = \frac{1}{n^2} + \sum\limits_{k=n+1}^{\infty} \frac{1}{k^2}$ and we know that for $\alpha > 1$, $\sum\limits_{k=n+1}^{\infty} \frac{1}{k^\alpha} \sim \frac{1}{(\alpha-1)n^{\alpha-1}}$ as $n \to \infty$. It follows that $\lim\limits_{n\to\infty} nR_n = 1$ and we conclude

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} \frac{1}{k\binom{n+k-1}{n}} = 1.$$

A–109. *Proposed by Florică Anastase, "Alexandru Odobescu" high school, Lehliu-Gară, Călăraşi, Romania.* Let $\{\omega_n\}_{n\geq 1}$ be the sequence defined by $\omega_n = (2n+1)\left(\sum\limits_{k=0}^{2n} \tan\left(x + \frac{k\pi}{2n+1}\right)\right)^{-1}$. Find

$$\Omega = \lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2n+1}}\left(\frac{\cot x}{\cot\frac{\pi}{2n+1}}\right)^{\omega_n}\right).$$

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. We start by establishing a closed form expression for ω_n. Using the complex exponential representation for the sine function,

$$\begin{aligned}\prod_{k=0}^{2n} \sin\left(x+\frac{k\pi}{2n+1}\right) &= \prod_{k=0}^{2n} \frac{e^{i\left(x+\frac{k\pi}{2n+1}\right)} - e^{-i\left(x+\frac{k\pi}{2n+1}\right)}}{2i} \\ &= \frac{1}{(2i)^{2n+1}} \prod_{k=0}^{2n} e^{i\frac{k\pi}{2n+1}-ix}\left(e^{2ix} - e^{-i\frac{2k\pi}{2n+1}}\right) \\ &= \frac{1}{(2i)^{2n+1}} e^{in\pi - i(2n+1)x} \prod_{k=0}^{2n}\left(e^{2ix} - e^{-i\frac{2k\pi}{2n+1}}\right).\end{aligned}$$

Now, substitute $z = e^{2ix}$ into the equation

$$z^{2n+1} - 1 = \prod_{k=0}^{2n}\left(z - e^{-i\frac{2k\pi}{2n+1}}\right)$$

to obtain

$$\prod_{k=0}^{2n}\left(e^{2ix} - e^{-i\frac{2k\pi}{2n+1}}\right) = e^{2i(2n+1)x} - 1.$$

Thus,

$$\begin{aligned}\prod_{k=0}^{2n} \sin\left(x+\frac{k\pi}{2n+1}\right) &= \frac{1}{(2i)^{2n+1}} e^{in\pi}\left(e^{i(2n+1)x} - e^{-i(2n+1)x}\right) \\ &= \frac{1}{2^{2n}} \sin(2n+1)x.\end{aligned}$$

Taking the natural logarithm on both sides of this last expression, differentiating and then replacing x by $x + \frac{\pi}{2}$ yields

$$\sum_{k=0}^{2n} \tan\left(x+\frac{k\pi}{2n+1}\right) = (2n+1)\tan((2n+1)x+\pi) = (2n+1)\tan(2n+1)x.$$

Thus,

$$\omega_n = (2n+1)\left(\sum_{k=0}^{2n} \tan\left(x+\frac{k\pi}{2n+1}\right)\right)^{-1} = \cot(2n+1)x.$$

Next, let

$$y = \lim_{x \to \frac{\pi}{2n+1}} \left(\frac{\cot x}{\cot \frac{\pi}{2n+1}} \right)^{\omega_n}.$$

Then,

$$\begin{aligned} \ln y &= \lim_{x \to \frac{\pi}{2n+1}} \frac{\ln\left(\frac{\cot x}{\cot \frac{\pi}{2n+1}}\right)}{\tan(2n+1)x} = \lim_{x \to \frac{\pi}{2n+1}} \frac{-\tan x \csc^2 x}{(2n+1)\sec^2(2n+1)x} \\ &= -\frac{1}{2n+1} \csc \frac{\pi}{2n+1} \sec \frac{\pi}{2n+1}, \end{aligned}$$

and

$$\lim_{x \to \frac{\pi}{2n+1}} \left(\frac{\cot x}{\cot \frac{\pi}{2n+1}} \right)^{\omega_n} = \exp\left(-\frac{1}{2n+1} \csc \frac{\pi}{2n+1} \sec \frac{\pi}{2n+1} \right).$$

Finally, with

$$\lim_{n \to \infty} \left(-\frac{1}{2n+1} \csc \frac{\pi}{2n+1} \sec \frac{\pi}{2n+1} \right) = -\frac{1}{\pi},$$

it follows that

$$\Omega = \lim_{n \to \infty} \left(\lim_{x \to \frac{\pi}{2n+1}} \left(\frac{\cot x}{\cot \frac{\pi}{2n+1}} \right)^{\omega_n} \right) = e^{-1/\pi}.$$

Solution 2 by Moti Levy, Rehovot, Israel. We rephrase the problem as follows:

Let $\{\nu_n\}_{n \geq 1}$ be the sequence defined by

$$\nu_n = \frac{1}{2n+1} \sum_{k=0}^{2n} \tan\left(\left(\frac{(k+1)\pi}{2n+1} + \varepsilon \right) \right)$$

Find

$$\Omega = \lim_{n \to \infty} \left(\lim_{\varepsilon \to 0} \left(\frac{\tan\left(\frac{\pi}{2n+1}\right)}{\tan\left(\frac{\pi}{2n+1} + \varepsilon\right)} \right)^{\frac{1}{\nu_n}} \right).$$

Now we find asymptotic expansion of ν_n in terms of ε :

$$\frac{d\nu_n}{d\varepsilon}\bigg|_{\varepsilon=0} = \frac{1}{2n+1}\sum_{k=0}^{2n}\left(1+\tan^2\left(\left(\frac{(k+1)\pi}{2n+1}\right)\right)\right)$$

The following identity is well known (see outline of proof below):

$$\sum_{k=0}^{2n}\tan^2\left(\left(\frac{(k+1)\pi}{2n+1}\right)\right) = 2n(2n+1),$$

which implies

$$\frac{1}{2n+1}\sum_{k=0}^{2n}\left(1+\tan^2\left(\left(\frac{(k+1)\pi}{2n+1}\right)\right)\right) = 2n+1.$$

Hence,

$$\nu_n = (2n+1)\varepsilon + O(\varepsilon^2). \tag{1}$$

Now we find asymptotic expansion of $\frac{\tan\left(\frac{\pi}{2n+1}\right)}{\tan\left(\frac{\pi}{2n+1}+\varepsilon\right)}$ in terms of ε :

$$\begin{aligned}
\frac{\tan\left(\frac{\pi}{2n+1}\right)}{\tan\left(\frac{\pi}{2n+1}+\varepsilon\right)} &= \frac{\tan\left(\frac{\pi}{2n+1}\right)\left(1-\tan\left(\frac{\pi}{2n+1}\right)\tan(\varepsilon)\right)}{\tan\left(\frac{\pi}{2n+1}\right)+\tan(\varepsilon)} \\
&= \left(1-\varepsilon\tan\left(\frac{\pi}{2n+1}\right)\right)\left(1-\frac{\varepsilon}{\tan\left(\frac{\pi}{2n+1}\right)}\right)+O(\varepsilon^2) \\
&= 1-\varepsilon\left(\tan\left(\frac{\pi}{2n+1}\right)+\cot\left(\frac{\pi}{2n+1}\right)\right)+O(\varepsilon^2) \\
&= 1-\frac{2\varepsilon}{\sin\left(\frac{2\pi}{2n+1}\right)}+O(\varepsilon^2) \qquad (2)
\end{aligned}$$

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\left(\frac{\tan\left(\frac{\pi}{2n+1}\right)}{\tan\left(\frac{\pi}{2n+1}+\varepsilon\right)}\right)^{\frac{1}{\nu_n}} \\
&= \lim_{\varepsilon\to 0}\left(1-\frac{2\varepsilon}{\sin\left(\frac{2\pi}{2n+1}\right)+O(\varepsilon^2)}\right)^{\frac{1}{(2n+1)\varepsilon+O(\varepsilon^2)}} \\
&= \lim_{\varepsilon\to 0}\left(1-\frac{2\varepsilon}{\sin\left(\frac{2\pi}{2n+1}\right)}\right)^{\frac{1}{(2n+1)\varepsilon}} = \exp\left(-\frac{2}{(2n+1)\sin\left(\frac{2\pi}{2n+1}\right)}\right).
\end{aligned}$$

$$\Omega=\lim_{n\to\infty}\exp\left(-\frac{2}{(2n+1)\sin\left(\frac{2\pi}{2n+1}\right)}\right)=e^{-\frac{1}{\pi}}\cong 0.727\,38.$$

Finally, we give a proof of the identity (taken from StackExchange):

$$\sum_{k=1}^{2n+1}\tan^2\left(\frac{k\pi}{2n+1}\right)=2\sum_{k=1}^{n}\tan^2\left(\frac{k\pi}{2n+1}\right).$$

By Euler's formula,

$$(-1)^k=\cos(k\pi)+i\sin(k\pi).$$

$$(-1)^k=\left(\cos\left(\frac{k\pi}{2n+1}\right)+i\sin\left(\frac{k\pi}{2n+1}\pi\right)\right)^{2n+1}$$

The imaginary part of $(-1)^k$ is zero, hence

$$\begin{aligned}0&=\operatorname{Im}\left(\cos\left(\frac{k\pi}{2n+1}\right)+i\sin\left(\frac{k\pi}{2n+1}\pi\right)\right)^{2n+1}\\&=\sum_{r=0}^{n}\binom{2n+1}{2r+1}\left(\cos\left(\frac{k\pi}{2n+1}\right)\right)^{2n-2r}\left(i\sin\left(\frac{k\pi}{2n+1}\pi\right)\right)^{2r+1}.\end{aligned}$$

Dividing by $\left(\cos\left(\frac{k\pi}{2n+1}\right)\right)^{2n+1}$,

$$\sum_{r=0}^{n}\binom{2n+1}{2r+1}\left(i\tan\left(\frac{k\pi}{2n+1}\right)\right)^{2r+1}=0.$$

Dividing by $\tan\left(\frac{k\pi}{2n+1}\right)$,

$$\sum_{r=0}^{n}\binom{2n+1}{2r+1}\left(-\tan^2\left(\frac{k\pi}{2n+1}\right)\right)^{r}=0.$$

Setting $x=\tan^2\left(\frac{k\pi}{2n+1}\right)$, the left hand side is a polynomial of degree n in x,

$$P_n(x):=\sum_{r=0}^{n}\binom{2n+1}{2r+1}(-x)^r,$$

whose n roots are $\left(\tan^2\left(\frac{k\pi}{2n+1}\right)\right)_{1\le k\le n}$.

By Vieta's formulas, the sum of the n roots is equal to minus the coefficient of x^{n-1} divided by the coefficient of x^n,

$$\sum_{k=1}^{n}\tan^2\left(\frac{k\pi}{2n+1}\right)=\frac{-(-1)^{n-1}\binom{2n+1}{2(n-1)+1}}{(-1)^n}=n(2n+1).$$

Solution 3 by Michel Bataille, Rouen, France. Recall that if $a \in [0, \frac{\pi}{2})$, then $\tan(a+h) - \tan(a) \sim (1+\tan^2(a))h$ as $h \to 0$ (this directly follows from $\tan'(a) = 1 + \tan^2(a)$).
Let $h = x - \frac{\pi}{2n+1}$. Then $h \to 0$ as $x \to \frac{\pi}{2n+1}$ and the sum

$$T = \sum_{k=0}^{2n} \tan\left(x + \frac{k\pi}{2n+1}\right) = \sum_{k=0}^{2n} \tan\left(h + \frac{k\pi}{2n+1}\right)$$

satisfies $T = \sum_{k=0}^{2n}\left(1 + \tan^2 \frac{k\pi}{2n+1}\right)h + o(h)$. Since $\sum\limits_{k=0}^{2n} \tan^2 \frac{k\pi}{2n+1} = 2n(2n+1)$ (see proof at the end), we obtain $T = (2n+1)^2 h + o(h)$ and $\omega_n \sim \frac{1}{(2n+1)h}$ as $h \to 0$.
Now, consider

$$\ln\left(\frac{\cot x}{\cot \frac{\pi}{2n+1}}\right) = \ln\left(\frac{\tan \frac{\pi}{2n+1}}{\tan x}\right) = \ln\left(1 - \left(1 - \frac{\tan \frac{\pi}{2n+1}}{\tan x}\right)\right).$$

We have

$$1 - \frac{\tan \frac{\pi}{2n+1}}{\tan x} = \frac{\tan\left(h + \frac{\pi}{2n+1}\right) - \tan \frac{\pi}{2n+1}}{\tan\left(h + \frac{\pi}{2n+1}\right)} \sim \frac{1 + \tan^2 \frac{\pi}{2n+1}}{\tan \frac{\pi}{2n+1}} \cdot h$$

hence

$$\ln\left(\frac{\cot x}{\cot \frac{\pi}{2n+1}}\right) \sim -\frac{1 + \tan^2 \frac{\pi}{2n+1}}{\tan \frac{\pi}{2n+1}} \cdot h$$

as $h \to 0$. Thus,

$$\lim_{x \to \frac{\pi}{2n+1}} \omega_n \ln\left(\frac{\cot x}{\cot \frac{\pi}{2n+1}}\right) = -\frac{1 + \tan^2 \frac{\pi}{2n+1}}{\tan \frac{\pi}{2n+1}} \cdot \frac{1}{2n+1}.$$

As $n \to \infty$, the latter tends to $-\frac{1}{\pi}$ (since $\tan \frac{\pi}{2n+1} \sim \frac{\pi}{2n+1}$) and we readily deduce that $\Omega = e^{-\frac{1}{\pi}}$.

Proof of $\sum\limits_{k=0}^{2n} \tan^2 \frac{k\pi}{2n+1} = 2n(2n+1)$.
Consider the polynomial $A(x) = (1+x)^{2n+1} - (1-x)^{2n+1}$. Using the binomial theorem, we easily obtain $A(x) = 2xP(x^2)$ where the polynomial P is defined by

$$P(x) = \sum_{k=0}^{n} \binom{2n+1}{2k+1} x^k.$$

Now, the roots of $A(x)$ are the complex numbers x_k such that $\frac{1+x_k}{1-x_k} = \exp\left(\frac{2k\pi i}{2n+1}\right)$ ($k = 0, 1, \ldots, 2n$). A simple calculation gives $x_k = i\tan\frac{k\pi}{2n+1}$ and (since $\tan\frac{(n+j)\pi}{2n+1} = -\tan\frac{(n+1-j)\pi}{2n+1}$ for $j = 1, \ldots, n$) it readily follows that

$$A(x) = 2x\prod_{k=1}^{n}\left(x - i\tan\frac{k\pi}{2n+1}\right)\left(x + i\tan\frac{k\pi}{2n+1}\right)$$

and

$$P(x) = \prod_{k=1}^{n}\left(x + \tan^2\frac{k\pi}{2n+1}\right).$$

We deduce that

$$\sum_{k=1}^{n}\left(-\tan^2\frac{k\pi}{2n+1}\right) = -\binom{2n+1}{2n-1} = -n(2n+1)$$

and the result follows.

Also solved by *the proposer.*

A–110. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* The plane is partitioned into n regions by three families of parallel lines in general position. What is the least number of lines to ensure that $n \geq 2023$?

Solution 1 by Moti Levy, Rehovot, Israel. Generally, a collection of n lines in the plane are said to be in *general position* if no two are parallel and no three are concurrent. So in this problem, we assume that there are no three lines which intersect at a common point. We can divide the lines into three subsets of lines. The lines in each subsets are parallel.

Let μ_1, μ_2, μ_3 be the number of lines in each subset.

$$n = \mu_1 + \mu_2 + \mu_3. \tag{1}$$

Now we prove by mathematical induction that the number of regions is

$$R_n(\mu_1, \mu_2, \mu_3) = 1 + \binom{n+1}{2} - \binom{\mu_1}{2} - \binom{\mu_2}{2} - \binom{\mu_3}{2}. \tag{2}$$

Suppose (2) is true. We add (without loss of generality) one line to the first subset, that is the number of lines of the first subset becomes $\mu_1 + 1$ and the total number of lines becomes $n + 1$.

One can observe that the additional line intersects the lines of the second and third subsets, thus creating additional $1 + \mu_2 + \mu_3$ regions.

Therefore,

$$R_{n+1}(\mu_1 + 1, \mu_2, \mu_3) = R_n + 1 + \mu_2 + \mu_3$$

$$= 1 + \binom{n+1}{2} - \binom{\mu_1}{2} - \binom{\mu_2}{2} - \binom{\mu_3}{2} + 1 + \mu_2 + \mu_3$$

$$= 1 + \binom{n+1}{2} - \binom{\mu_1}{2} - \binom{\mu_2}{2} - \binom{\mu_3}{2} + 1 + n - \mu_1$$

$$= 1 + \binom{n+2}{2} - \binom{\mu_1+1}{2} - \binom{\mu_2}{2} - \binom{\mu_3}{2}.$$

One can check that

$$R_{76}(26, 25, 25) = 1 + \binom{76+1}{2} - \binom{26}{2} - \binom{25}{2} - \binom{25}{2} = 2002,$$

$$R_{77}(26, 26, 25) = 1 + \binom{77+1}{2} - \binom{26}{2} - \binom{26}{2} - \binom{25}{2} = 2054.$$

We conclude that the least number of lines to ensure the number of regions exceeds 2023 is 77.

Solution 2 by the proposer. Suppose that there are x, y and z lines in the three families. Assume that no point is common to three distinct lines. The $x + y$ lines of the first two families partition the plane into $(x + 1)(y + 1)$ regions. Let α be one of the lines of the third family. It is cut into $x + y + 1$ parts by the lines in the first two families, so the number of regions is increased by $x + y + 1$. Since this happens z times, the number of regions that the plane is partitioned into by the three families of lines is

$$n = (x+1)(y+1) + z(x+y+1) = (x+y+z) + (xy+yz+zx) + 1.$$

Let $u = x + y + z$ and $v = xy + yz + zx$. Then $v \leq x^2 + y^2 + z^2$, as can be easily proven, so that $u^2 = x^2 + y^2 + z^2 + 2v \geq 3v$. Therefore,

$$n \leq u + \frac{u^2}{3} + 1.$$

This is a value less than 2003 when $u = 76$. However, when $(x, y, z) = (26, 26, 25)$, then $u = 77$, $v = 1976$ and $n = 2044$. Therefore, we need at least 77 lines, so a conveniently chosen set of 77 lines will suffice.

A–111. *Proposed by Félix Moreno Peñarrubia, Charles University, Prague, Czech Republic.* Consider the function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ defined as:

- $f(n, 1) = n$ for all positive integers n.
- For $m \geq 2$, $f(n, m)$ is the smallest multiple of m which is greater than or equal to $f(n, m - 1)$.

We call a positive integer n *special* if all elements of the infinite sequence $\{f(n, 1), f(n, 2), \ldots\}$ are distinct. Prove that there are at least 2022 special integers less than $5\,000\,000$.

Solution 1 by Ander Lamaison Vidarte, Brno, Czech Republic. We claim that for all positive integers k, there is a special number such that $f(n, k) = k^2$, and so $f(n, m) = km$ for all $m \geq k$. To prove this, we consider the sequence with $a_m = km$ for all $m \geq k$ and with a_m being the greatest multiple of $m - 1$ strictly smaller than $f(n, m)$ for $2 \leq m \leq k$. We can show that, for $n = a_1$, this sequence satisfies the recurrence in the statement ($f(n, m)$ is divisible by m and $1 \leq f(n, m) - f(n, m - 1) \leq m - 1$), so n is a special number. Note that the generated numbers are different, since they create different sequences, and that the number n generated from k is less than or equal to $f(n, k) = k^2$. Therefore, for $k = 1, 2, \ldots, 2022$, all generated special numbers are less than $2022^2 < 5\,000\,000$.

Solution 2 by the proposer. We will prove the following two claims:

1. For each special number n, there exists exactly one m such that $f(n,m) = m^2$.
2. For each positive integer m, there exists exactly one special number n so that $f(n,m) = m^2$.

(Note: in fact, it is true that the m-th special number satisfies $f(n,m) = m^2$, and this can be used to compute the m-th special number efficiently). It is clear that these two facts taken together imply what we want to prove, since therefore there are 2022 distinct special numbers less than or equal to $1^2, 2^2, \ldots, 2022^2$ respectively, and $2022^2 < 5\,000\,000$.

Let's prove the first claim. Consider the integer-valued function $g(n,m) = \frac{f(n,m)}{m}$. We have the following facts:

1. $g(n,m)$ is decreasing in m.
   Proof: $(m+1)g(n,m)$ is a multiple of $m+1$ which is greater than $f(n,m)$ and therefore $f(n,m+1) \leq (m+1)g(n,m) \implies g(n,m+1) \leq g(n,m)$.
2. If $g(n,m) > m$ and $g(n,m+1) < m+1$, then $f(n,m) = f(n,m+1)$.
   Proof: $f(n,m+1) = (m+1)g(n,m+1) \leq (m+1)m \leq g(n,m)m = f(n,m)$, and $f(n,m+1) \geq f(n,m)$ by definition.
3. If $g(n,m) \leq m$, then $g(n,m+1) = g(n,m)$.
   Proof: Note that $(m+1)(g(n,m)-1) = f(n,m) + g(n,m) - m - 1 < f(n,m)$, so $f(n,m+1) = (m+1)g(n,m)$.

By the first and second facts, if n is a special number there must exist a m_0 such that $f(n,m_0) = m_0^2$. By the third fact, this m_0 has to be unique since for $m > m_0$, we have $f(n,m) = m \cdot m_0$. So the first claim is proven.

Let's prove the second claim. For each m, define the function $h_m(x)$ as:

- For $x \geq m$, $h_m(x) = x \cdot m$.
- For $1 \leq x < m$, $h_m(x) = h_m(x+1) - x$ if $h_m(x+1)$ is a multiple of x, $h_m(x) = h_m(x+1) - \text{Res}(h_m(x+1), x)$ otherwise, where $\text{Res}(a,b)$ is the residue of dividing a by b.

Note that $h_m(x)$ must be a positive integer for all positive integers x (even if we subtracted x at each x from $m-1$ to 1, it wouldn't reach 0), and $h_m(x+1)$ is the smallest multiple of $x+1$ greater than or equal to $h_m(x)$, and that $h_m(x+1) > h_m(x)$ for all x. Therefore, we must have $h_m(x) = f(h_m(1), x)$, and $h_m(1)$ is the special number we are looking for. To see that this special number is unique, note that if we impose $f(n,m) = mk$ for any given m, k and we require n to be a special number, then the m values that $f(n,m), f(n,m-1), \ldots, f(n,1)$ should have (if a solution exists) are uniquely determined by a recursive formula like the one for the function $h_m(x)$. So the second claim is proven and we are done.

A–112. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b be positive real numbers. Prove that

$$\int_0^1 t^a (1-t)^{b-1} \ln(t)\, dt \geq \frac{a\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)} \ln\left(\frac{a}{a+b}\right),$$

where $\Gamma(x)$ is the Euler Gamma Function.

Solution 1 by G. C. Greubel, Newport News, VA, USA. Starting with the Beta function,

$$\int_0^1 t^{x-1}\,(1-t)^{y-1}\, dt = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

and taking a derivative with respect to x leads to

$$\int_0^1 t^{x-1}\,(1-t)^{y-1}\, \ln(t)\, dt = B(x,y)\,(\psi(x) - \psi(x+y)).$$

Now using a form of expansion for the digamma function, namely,

$$\psi(x+1) = \ln(x) + \frac{1}{2\,x} - \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)\,x^{2j}},$$

where B_n are the Bernoulli numbers, shows that

$$\psi(x+1) \geq \ln(x)$$

and

$$\int_0^1 t^{x-1}\,(1-t)^{y-1}\,\ln(t)\,dt \geq B(x,y)\,\ln\!\left(\frac{x-1}{x+y-1}\right).$$

Setting $x = a+1$ and $y = b$ yields

$$\int_0^1 t^{a}\,(1-t)^{b-1}\,\ln(t)\,dt \geq B(a+1,b)\,\ln\!\left(\frac{a}{a+b}\right).$$

This is valid for $a, b > 0$.

Additional Notes Similarly, if a derivative with respect to y is taken then

$$\begin{aligned}\int_0^1 t^{x-1}\,(1-t)^{y-1}\,\ln(1-t)\,dt &= B(x,y)\,(\psi(y)-\psi(x+y))\\ &\geq B(x,y)\,\ln\!\left(\frac{y-1}{x+y-1}\right).\end{aligned}$$

Setting $x = a+1$ and $y = b+1$ then

$$\int_0^1 t^{a}\,(1-t)^{b}\,\ln(1-t)\,dt \geq B(a+1,b+1)\,\ln\!\left(\frac{b}{a+b+1}\right)$$

which is valid for $b > 0$.

If a second derivative is applied, with respect to x, then

$$\begin{aligned}I_2 &= \int_0^1 t^{x-1}\,(1-t)^{y-1}\,\ln^2(t)\,dt\\ &= B(x,y)\left(\psi'(x)-\psi'(x+y)+(\psi(x)-\psi(x+y))^2\right).\end{aligned}$$

Using $\psi'(x) \geq \dfrac{1}{x}$ then

$$I_2 \geq B(x,y)\left(\frac{y}{x\,(x+y)}+\ln^2\!\left(\frac{x}{x+y}\right)\right).$$

Setting $x = a+1$ and $y = b+1$ leads to

$$\int_0^1 t^{a}\,(1-t)^{b}\,\ln^2(t)\,dt$$

$$\geq B(a+1,b+1)\left(\frac{b+1}{(a+1)\,(a+b+2)}+\ln^2\!\left(\frac{a+1}{a+b+2}\right)\right).$$

Solution 2 by Moti Levy, Rehovot, Israel.

$$I(a,b) := \int_0^1 t^a (1-t)^{b-1} \ln(t) dt$$

The Beta function $B(x,y)$ is defined as

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

$$\begin{aligned} I(a,b) &= \int_0^1 \frac{\partial t^a}{\partial a}(1-t)^{b-1} dt = \frac{\partial}{\partial a}\int_0^1 t^a (1-t)^{b-1} dt \\ &= \frac{\partial}{\partial a} B(b, a+1) = \frac{\partial}{\partial a}\frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)} \\ &= \frac{a}{a+b}\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}(\psi(a+1) - \psi(a+b+1)), \end{aligned} \tag{1}$$

where $\psi(x)$ is the Digamma function defined as

$$\psi(x) := \frac{d}{dx}\ln(\Gamma(x)) = \frac{\Gamma'(x)}{\Gamma(x)}.$$

It follows from (1), that the original inequality is proved once we show that $\psi(a+1) - \psi(a+b+1) \geq \ln\left(\frac{a}{a+b}\right)$, or that

$$\psi(a+1) - \ln(a) \geq \psi(a+b+1) - \ln(a+b). \tag{2}$$

To this end,we show that the function $\psi(x+1) - \ln(x)$ is monotone decreasing for $x > 0$.

$$\frac{d(\psi(x+1) - \ln(x))}{dx} = \psi'(x+1) - \frac{1}{x}. \tag{3}$$

Now we use a well known inequality (see "Polygamma function" entry in Wikipedia)

$$\psi'(y) \leq \frac{1}{y} + \frac{1}{y^2} \tag{4}$$

to show that

$$\psi'(x+1) \leq \frac{1}{x+1} + \frac{1}{(x+1)^2} = \frac{x+2}{(x+1)^2} \leq \frac{1}{x}, \quad for \quad x > 0. \tag{5}$$

It follows from (5) that $\frac{d(\psi(x+1)-\ln(x))}{dx} \leq 0$ for all $x > 0$, hence the function $\psi(x+1) - \ln(x)$ is monotone decreasing for $x > 0$.

Solution 3 by Michel Bataille, Rouen, France. Let $B(x,y)$ denote the Beta Function, defined by $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$ for $x, y > 0$.
We know that for $x, y > 0$, $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ and $\Gamma(x+1) = x\Gamma(x)$.
It follows that

$$\int_0^1 t^a(1-t)^{b-1}\,dt = B(a+1,b) = \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+1+b)} = \frac{a\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)}.$$

In consequence, the requested inequality will be obtained if we prove that

$$\int_0^1 t^a(1-t)^{b-1}\left[\ln(t) - \ln\left(\frac{a}{a+b}\right)\right]dt \geq 0. \qquad (1)$$

From the classical inequality $\ln(x) \leq x - 1$, we deduce that for $t > 0$ we have

$$\ln\left(\frac{a}{a+b}\right) - \ln(t) = \ln\left(\frac{\frac{a}{a+b}}{t}\right) \leq \frac{\frac{a}{a+b}}{t} - 1$$

so that $t\Big(\ln(t) - \ln\big(\frac{a}{a+b}\big)\Big) \geq t - \frac{a}{a+b}$.
We first deduce that

$$t^a(1-t)^{b-1}\left[\ln(t) - \ln\left(\frac{a}{a+b}\right)\right] \geq t^a(1-t)^{b-1} - \frac{a}{a+b}\cdot t^{a-1}(1-t)^{b-1}$$

and then, if I denotes the integral on the left of (1),

$$I \geq B(a+1,b) - \frac{a}{a+b}\cdot B(a,b) = 0,$$

as desired.

Solution 4 by the proposer. To proof the statement we will need to apply Jensen's inequality for integrals. Namely, if $f : [a,b] \to \mathbb{R}$ is a convex function and $h : [a,b] \to \mathbb{R}_+^*$ and $u : [a,b] \to \mathbb{R}_+$ are continuous functions, then

$$f\left(\frac{\int_a^b h(x)u(x)\,dx}{\int_a^b h(x)\,dx}\right) \leq \frac{\int_a^b h(x)f(u(x))\,dx}{\int_a^b h(x)\,dx}$$

Let X be a Beta random variable which probability density function is

$$\begin{aligned} h_X(t) &= \frac{1}{B(a,b)} t^{a-1}(1-t)^{b-1} I\{0 < t < 1\} \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} t^{a-1}(1-t)^{b-1} I\{0 < t < 1\} \end{aligned}$$

with expectation given by $E(X) = \dfrac{a}{a+b}$.
Putting $u(t) = t$ and $f(t) = t\ln(t)$ (that is convex in $(0,1)$) into Jensen's inequality, we have

$$\begin{aligned} \left(\frac{a}{a+b}\right) \ln\left(\frac{a}{a+b}\right) &\leq \frac{1}{B(a,b)} \int_0^1 t^{a-1}(1-t)^{b-1} t\, \ln(t)\, dt \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \int_0^1 t^{a-1}(1-t)^{b-1} t \ln(t)\, dt \end{aligned}$$

from which the statement follows.

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