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`jose.luis.diaz@upc.edu`

A Short Proof of Jensen's Inequality

Kambiz Razminia

Abstract

In this short note, using the extreme value theorem another proof of Jensen's inequality is presented.

1 Main results

By adopting the idea proposed by Razminia ([1],[3],[2]) as how to introduce a new function and then apply the extreme value theorem (the Weierstrass Theorem), a simple and short proof is given for the Jensen's inequality.

Theorem. *The discrete finite form of the Jensen's inequality claims that for $\lambda_i \in \mathbb{R}_{\geq 0}$, where $\sum_{i=1}^{n} \lambda_i = 1$, any real-valued continuous function f that is convex on the real closed interval I satisfies*

$$0 \leq h(x_1, x_2, \cdots, x_n) = \sum_{i=1}^{n} \lambda_i f(x_i) - f\left(\sum_{i=1}^{n} \lambda_i x_i\right),$$

where $x_i \in I$. Moreover, equality holds if and only if all x_i's are equal or f is affine.

Proof. According to the extreme value theorem (the Weierstrass Theorem), the continuous function h has a global minimum, say at the argument $A = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, on the closed cell $J = I^n$. Without loss of generality, let us assume $\alpha_1 = \min_{1 \leq i \leq n}(\alpha_i)$

and $\alpha_2 = \max_{1\leq i\leq n}(\alpha_i)$. Hence, for any point in J such as $B = (\alpha_1', \alpha_2', \alpha_3, \cdots, \alpha_n)$, where $\alpha_1' = \alpha_2' = \frac{\lambda_1}{\lambda_1+\lambda_2}\alpha_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}\alpha_2$ (obviously, $\alpha_1 \leq \alpha_1' \leq \alpha_2$, so B belongs to J), we have

$$h(A) = \sum_{i=1}^{n} \lambda_i f(\alpha_i) - f\Big(\sum_{i=1}^{n} \lambda_i \alpha_i\Big) \leq h(B)$$

$$= \lambda_1 f(\alpha_1') + \lambda_2 f(\alpha_2') + \sum_{i=3}^{n} \lambda_i f(\alpha_i) - f\Big(\sum_{i=1}^{n} \lambda_i \alpha_i\Big), \qquad (1)$$

in which we have used the fact that $\lambda_1\alpha_1' + \lambda_2\alpha_2' = (\lambda_1+\lambda_2)\alpha_1' = (\lambda_1+\lambda_2)\big(\frac{\lambda_1}{\lambda_1+\lambda_2}\alpha_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}\alpha_2\big) = \lambda_1\alpha_1 + \lambda_2\alpha_2$ on the right-hand side of (1). By canceling out the similar terms on both sides of (1), we have

$$\lambda_1 f(\alpha_1) + \lambda_2 f(\alpha_2) \leq \lambda_1 f(\alpha_1') + \lambda_2 f(\alpha_2') = (\lambda_1+\lambda_2) f(\alpha_1'),$$

or, equivalently

$$\frac{\lambda_1}{\lambda_1+\lambda_2} f(\alpha_1) + \frac{\lambda_2}{\lambda_1+\lambda_2} f(\alpha_2) \leq f\Big(\frac{\lambda_1}{\lambda_1+\lambda_2}\alpha_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}\alpha_2\Big). \qquad (2)$$

Because of the convexity of f, (2) holds if and only if $\alpha_1 = \alpha_2$ (consequently, all α_i's are equal) or f is an affine function. In either case, the value of h at A is zero. Indeed,

- If the α_i are equal for $1 \leq i \leq n$, then

$$h(\alpha_1, \alpha_2, \cdots, \alpha_n) = \sum_{i=1}^{n} \lambda_i f(\alpha_i) - f(\sum_{i=1}^{n} \lambda_i \alpha_i)$$

$$= f(\alpha_1)\sum_{1}^{n} \lambda_i - f(\alpha_1 \sum_{1}^{n} \lambda_i) = f(\alpha_1) - f(\alpha_1) = 0.$$

- If f is an affine function, then

$$h(\alpha_1, \alpha_2, \cdots, \alpha_n) = \sum_{i=1}^{n} \lambda_i f(\alpha_i) - f(\sum_{i=1}^{n} \lambda_i \alpha_i) = 0.$$

Therefore, h is always non-negative. It is zero (global minimum) if and only if all α_i's are equal or f is affine. It completes the proof.

Additional Note. The proof can be divided into two parts. In the first part, we proved that h has a global minimum if and only if all α_i's are equal or f is affine without using the property $\sum_{i=1}^{n}\lambda_i = 1$. In the second part, by using $\sum_{i=1}^{n}\lambda_i = 1$, we concluded that the global minimum is zero. Let us assume $\sum_{i=1}^{n}\lambda_i = \lambda$, in which λ is not necessarily one. If f is affine, the global minimum is zero for any λ. Otherwise, the global minimum is

$$h(\alpha_1, \alpha_2, \cdots, \alpha_n) = \sum_{i=1}^{n} \lambda_i f(\alpha_i) - f\left(\sum_{i=1}^{n} \lambda_i \alpha_i\right)$$

$$= f(\alpha_1)\sum_{1}^{n}\lambda_i - f\left(\alpha_1 \sum_{1}^{n}\lambda_i\right) = \lambda f(\alpha_1) - f(\lambda\alpha_1),$$

which is equal to zero when $\lambda = 1$.

References

[1] Razminia, K. "A short proof of symmetric inequalities". *The College Mathematics Journal* 46 (2015), pp. 364–366.

[2] Razminia, K. "An elementary proof of weighted power means inequality". *Arhimede math. j.* 6.1 (2019), pp. 9–11.

[3] Razminia, K. "Hölder's inequality revisited". *The Mathematical Gazette* 103 (2019), pp. 512–514.

Kambiz Razminia
Persian Gulf University, Bushehr, Iran
kambiz.razminia@gmail.com

Note on Ionescu–Weitzenböck's inequality

Bătineţu-Giurgiu, D. M., Stanciu, N., and Díaz-Barrero, J. L.

Abstract. This note presents refinements of Ionescu-Weitzenböck's inequality and some applications of it.

1 Introduction

Ion Ionescu, the founder of Romanian Magazine *Gazeta Matematică*, published in 1897 problem 273 [4]. He stated that there is no triangle for which the inequality $4\sqrt{3}S > a^2 + b^2 + c^2$ can be satisfied. The solution appeared in [5]. Twenty two years later, in the year 1919, Roland Weitzenböck published in *Mathematische Zeitschrift*, Vol.5, No.1-2, pp.137-146 the article *Uber eine Ungleichung in der Dreiecksgeometrie*, where he stated and proved that in any triangle, with usual notations, holds the inequality $a^2 + b^2 + c^2 \geq 4\sqrt{3}\,S$. In the year 2013 [1] a proof was given based on [5] and consequently we think that the Weitzenb "ock result should be called the Ionescu-Weitzenb "ock inequality.

In this work, refinements of Ionescu-Weitzenböck's inequality and alternative solutions to problems previously published are presented.

2 Refinements

Two refinements are presented below of Ionescu-Weitzenböck's inequality. Be begin with

Theorem 1. *Let m_a, m_b, m_c be the medians of a triangle ABC with area S. Then,*

$$a^2 + b^2 + c^2 \geq \frac{2\sqrt{3}}{3}\,(am_a + bm_b + cm_c) \geq 4\sqrt{3}\,S.$$

Proof. First, we claim that $a^2 + b^2 + c^2 \geq 2\sqrt{3}\,am_a$ (cyclic). Indeed, squaring both terms in the preceding inequality, we get

$$(a^2 + b^2 + c^2)^2 \geq 12a^2m_a^2 = 3a^2\,4m_a^2) = 3a^2(2b^2 + 2c^2 - a^2) \quad \text{or}$$

$$a^4 + b^4 + c^4 + 2(a^2b^2 + b^2c^2 + c^2a^2) - (6a^2b^2 + 6a^2c^2 - 3a^4) \geq 0$$

from which

$$4a^4 + b^4 + c^4 - 4a^2b^2 - 4c^2a^2 + 2b^2c^2 = (b^2 + c^2 - 2a^2)^2 \geq 0$$

follows, and the claim is proven. Adding up the last inequalities, yields $3(a^2 + b^2 + c^2) \geq 2\sqrt{3}\,(am_a + bm_b + cm_c)$ from which LHS inequality immediately follows. To prove RHS inequality, we have

$$\frac{2\sqrt{3}}{3}\,(am_a + bm_b + cm_c) \geq \frac{2\sqrt{3}}{3}\,(ah_a + bh_b + ch_c) = 4\sqrt{3}\,S,$$

where h_a, h_b, h_c are the heights of the triangle.

Theorem 2. *Let h_a, h_b, h_c and w_a, w_b, w_c be the heights and the interior angle bisectors of a triangle ABC with area S. Then,*

$$a^2 + b^2 + c^2 \geq \frac{4S\sqrt{3}}{3}\left(\frac{w_a}{h_a} + \frac{w_b}{h_b} + \frac{w_c}{h_c}\right) \geq 4\sqrt{3}\,S.$$

Proof. WLOG we may assume that $a \leq b \leq c$ and we have $w_a \geq w_b \geq w_c$ and $h_a \geq h_b \geq h_c$, as it is well-known. By applying Chebyshev's inequality, we obtain

$$\sum_{cyc}\frac{w_a}{h_a} \leq \frac{1}{3}\left(\sum_{cyc} w_a\right)\left(\sum_{cyc}\frac{1}{h_a}\right).$$

On the other hand, we have

$$\sum_{cyc}\frac{1}{h_a}=\sum_{cyc}\frac{a}{ah_a}=\frac{1}{2S}(a+b+c)=\frac{s}{S},$$

where $s=(a+b+c)/2$ is the semiperimenter of triangle ABC, and

$$w_a=\frac{2bc}{b+c}cos\frac{A}{2}=\frac{2bc}{b+c}\sqrt{s(s-a)bc}=\frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)}$$

$$\leq\frac{2\sqrt{bc}}{2\sqrt{bc}}\sqrt{s(s-a)}=\sqrt{s(s-a)}\text{ (cyclic)}$$

on account that $b+c\geq 2\sqrt{bc}$. Using the inequality $x^2+y^2+z^2\geq (x+y+z)^2/3$ and the preceding, we get

$$\sum_{cyc}s(s-a)\geq\frac{1}{3}\sum_{cyc}\Big(\sqrt{s(s-a)}\Big)^2$$

from which

$$\sum_{cyc}wa=\sum_{cyc}\sqrt{s(s-a)}\leq s\sqrt{3}$$

follows. Combining the preceding, we obtain

$$\sum_{cyc}\frac{w_a}{h_a}\leq\frac{1}{3}\left(\sum_{cyc}w_a\right)\left(\sum_{cyc}\frac{1}{h_a}\right)\leq\frac{s\sqrt{3}}{3}\cdot\frac{s}{S}$$

$$=\frac{(a+b+c)^2}{4\sqrt{3}S}\leq\frac{3(a^2+b^2+c^2)}{4\sqrt{3}S}=\frac{\sqrt{3}}{4S}(a^2+b^2+c^2).$$

Since $w_a\geq ha$ (cyclic), then

$$3\leq\sum_{cyc}\frac{w_a}{h_a}\leq\frac{3(a^2+b^2+c^2)}{4\sqrt{3}S}$$

and

$$a^2+b^2+c^2\geq\frac{4S\sqrt{3}}{3}\left(\frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c}\right)\geq 4\sqrt{3}\,S.$$

3 Applications

In the following alternative solutions to some problems involving Weitzenböck's inequality that were published in magazines with problem columns around the world are presented. We begin with

Problem 1 ([2]). *Let $m \in \mathbb{R}^+$. With the usual notations for triangles, calling S the area of the triangle ABC, show that it holds:*

$$\frac{a^{m+2}}{(b \cdot R + c \cdot r)^m} + \frac{b^{m+2}}{(b \cdot R + b \cdot r)^m} + \frac{c^{m+2}}{(a \cdot R + b \cdot r)^m} \geq \frac{4\sqrt{3}}{(R+r)^m} S.$$

Solution. Let us denote

$$W = \frac{a^{m+2}}{(b \cdot R + c \cdot r)^m} + \frac{b^{m+2}}{(b \cdot R + b \cdot r)^m} + \frac{c^{m+2}}{(a \cdot R + b \cdot r)^m}.$$

Then, we have

$$\begin{aligned} W &= \sum_{cyc} \frac{a^{m+2}}{(b \cdot R + b \cdot r)^m} = \sum_{cyc} \frac{a^{2(m+1)}}{(ab \cdot R + ac \cdot r)^m} \\ &\geq 2^m \sum_{cyc} \frac{(a^2)^{m+1}}{(a^2+b^2)R + (a^2+c^2)r)^m}. \end{aligned}$$

Applying Radon's inequality, we obtain

$$W \geq 2^m \cdot \left(\sum_{cyc} (a^2)^{m+1}\right) \left(\sum_{cyc} (a^2+b^2)R + \sum_{cyc})(a^2+c^2)r\right)^{-m}$$

$$= \frac{1}{(R+r)^m} \sum_{cyc} a^2 \geq \frac{4\sqrt{3}}{(R+r)^m} S$$

on account of Ionescu–Weitzenböck's inequality. Equality holds when triangle ABC is equilateral.

Problem 2 ([3]). *Show that, for any triangle ABC, it holds*

$$(ab + bc + ca)^2 + 2(a^2 + b^2 + c^2)^2 \geq 16\sqrt{3}s^3 r,$$

where a, b, c are the sides, s is the semi-perimeter and r the radius of the circle inscribed in the triangle.

Solution. We begin observing that for any positive reals m, n, p it holds:

$$\frac{mn}{p}+\frac{np}{m}+\frac{pm}{n}\geq m+n+p.$$

Indeed, putting $w_1=\sqrt{mn/p}, w_2=\sqrt{np/m}, w_3=\sqrt{mp/n}$ in the well-known inequality $w_1^2+w_2^2+w_3^2\geq w_1w_2+w_2w_3+w_3w_1$, we get the inequality claimed. Equality occurs when $m=n=p$. Now, for $x, y, z>0$, put $m=x^2+y^2+z^2, n=xy+yz+zx, p=xy+yz+zx$ in the preceding to obtain

$$\frac{(x^2+y^2+z^2)(xy+yz+zx)}{xy+yz+zx}+\frac{(xy+yz+zx)(xy+yz+zx)}{x^2+y^2+z^2}$$

$$+\frac{(x^2+y^2+z^2)(xy+yz+zx)}{xy+yz+zx}\geq x^2+y^2+z^2+2(xy+yz+zx)$$

from which, we get

$$(x^2+y^2+z^2)^2+2\,(xy+yz+zx)^2\geq (x^2+y^2+z^2)(x+y+z)^2.$$

Finally, letting $a=x, b=y, c=z$ in the last inequality and using Ionescu–Weitzenböck's result, we get

$$(ab+bc+ca)^2+2(a^2+b^2+c^2)^2\geq 16\sqrt{3}s^3r,$$

on account that $S=sr$. Equality holds when triangle ABC is equilateral.

Problem 3. *Let x, y, z, t be positive reals. Then for any triangle ABC, with usual notations, prove that*

$$\sum_{cyc}\frac{(xm_a+ym_b)^{m+1}}{(za^2+tw_a)^m}\geq\frac{3^{m+3/2}(x+y)^{m+1}}{(4z+3t)^m}\,S,$$

where $m\geq 0$.

Solution. From $w_a=\dfrac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)}\leq\sqrt{s(s-a)}$, we obtain $w_a^2\leq s(s-a)$ (cyclic). Adding up, the preceding yields

$$w_a^2+w_b^2+w_c^2\leq s^2=\frac{(a+b+c)^2}{4}\leq\frac{3}{4}\,(a^2+b^2+c^2).$$

On the other hand, we have $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$, as it is well-known. Applying Radon's inequality, and taking into account the preceding, we have

$$\sum_{cyc} \frac{(xm_a^2 + ym_b^2)^{m+1}}{(za^2 + tw_a^2)^m} \geq \left(\sum_{cyc}(xm_a^2 + ym_b^2)\right)^{m+1} \left(\sum_{cyc}(za^2 + tw_a^2)\right)^{-m}$$
$$= \left((x+y)\sum_{cyc} m_a^2\right)^{m+1} \left(z\sum_{cyc} a^2 + t\sum_{cyc} w_a^2\right)^{-m}$$
$$\geq \left(3(x+y)/4\sum_{cyc} m_a^2\right)^{m+1} \left(z\sum_{cyc} a^2 + 3t/4\sum_{cyc} a^2\right)^{-m}$$
$$= \frac{3^{m+1}(x+y)^{m+1}}{4(4z+3t)^m}\sum_{cyc} a^2 \geq \frac{3^{m+1}(x+y)^{m+1}}{4(4z+3t)^m} 4\sqrt{3}S$$

on account of Ionescu–Weitzenböck's inequality. Finally, we get

$$\sum_{cyc} \frac{(xm_a + ym_b)^{m+1}}{(za^2 + tw_a)^m} \geq \frac{3^{m+3/2}(x+y)^{m+1}}{(4z+3t)^m} S,$$

and we are done.

References

[1] Bătineţu-Giurgiu, D. M. and Stanciu, N. "Inegalităţi de tip Ionescu-Wietzenbök". *Gazeta Matematică* CXIX.1 (2013), pp. 1–10.

[2] Bătineţu-Giurgiu, D. M. and Stanciu, N. "Problema 242". *REOIM* 49 (2013).

[3] Bătineţu-Giurgiu, D. M. and Stanciu, N. "Problema 277". *REOIM* 55 (2016).

[4] Ionescu, I. "Problem 273". *Gazeta Matematică* III.2 (1897), p. 52.

[5] Ionescu, I. "Solution Problem 273". *Gazeta Matematică* III.12 (1898), pp. 281–283.

Dumitru Bătineţu-Giurgiu
“Matei Basarab”, National College
Bucharest
Romania
dmb@yahoo.com

Neculai Stanciu
“George Emil Palade” School
Buzău
Romania
stanciuneculai@yahoo.com

José Luis Díaz-Barrero
Civil and Environmental Engineering
BarcelonaTech
Barcelona, Spain
jose.luis.diaz@upc.edu

Asymptotics of differences of inverse integer powers

Anastasios Kotronis

Abstract

We study sums in the form $\sum_{1\le k\le n}(-1)^k\binom{n}{k}k^{-m}$, where m is a positive integer and an asymptotic expansion in terms of Gamma's nested derivatives is given.

1 Introduction and known results

In this paper, we define

$$S_{n,m}=\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{-m},$$

where m is an integer. The case $m=0$ is trivial and the case $m<0$ was discussed in [2]. Here, the author gave a closed form

$$S_{n,m}=(-1)^n\left\{{-m \atop n}\right\}n!,$$

where $\left\{{-m \atop n}\right\}$ is the Stirling number of the second kind. For the case $m>0$ (that we are interested in), Flajolet and Sedgewick gave an asymptotic expansion to $S_{n,m}$ in [1] using techniques of complex analysis that we are going to improve using real analysis.

The main result of this paper is stated as

$$\begin{aligned}S_{n,m} &= \sum_{k=0}^{m}(-1)^{k+1}\frac{\binom{m}{k}\Gamma^{(k)}(1)}{m!}\ln^{m-k} n\\ &+ \sum_{k=0}^{m-1}(-1)^{k+1}\frac{\binom{m-1}{k}\Gamma^{(k)}(2)}{2(m-1)!}\frac{\ln^{m-k-1} n}{n}+\mathcal{O}\big(n^{-2}\ln^{m-1} n\big).\end{aligned}$$

for integers $m>0$. It will be proven in the next section.

2 Main results

Starting by the relation

$$\int_0^{+\infty} t^{a-1}e^{-nt}\,dt = \Gamma(a)n^{-a}, \qquad a>0, \tag{1}$$

and differentiating m times with respect to a we get for $m \geq 0$,

$$\int_0^{+\infty} t^{a-1}\ln^m t e^{-nt}\,dt = n^{-a}\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}\Gamma^{(r)}(a)\ln^{m-r} n. \tag{2}$$

Now the following Lemma holds.

Lemma 1. *Let m be a non negative integer and let a, c be real numbers such that $a > -1$, and $0 < c < 1$. Then, when $n \to +\infty$, it holds:*

$$\int_0^{c} t^{a}\ln^m t e^{-nt}\,dt = n^{-a-1}\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}\Gamma^{(r)}(a+1)\ln^{m-r} n+\mathcal{O}\big(n^{-1}e^{-nc}\big).$$

Proof. Indeed, for $a \geq 0$ and putting $x = nt$, we have

$$\left|\int_c^{+\infty} t^{a}\ln^m t e^{-nt}\,dt\right| = \frac{1}{n^{a+1}}\left|\int_{nc}^{+\infty} x^{a}\ln^m \frac{x}{n}e^{-x}\,dx\right|.$$

Making the change of variables $x = (1+u)nc$ and taking into account that $1+u \leq e^u$, we get

$$\begin{aligned}
\frac{1}{n^{a+1}}\left|\int_{nc}^{+\infty} x^{a}\ln^m \frac{x}{n}e^{-x}\,dx\right| &= c^{a+1}\left|\int_0^{+\infty}(1+u)^a\ln^m((1+u)c)e^{-(1+u)nc}\,du\right| \\
&\leq c^{a+1}\int_0^{+\infty}(1+u)^a\Big|\ln((1+u)c)\Big|^m e^{-(1+u)nc}\,du \\
&\leq c^{a+1}e^{-nc}\int_0^{+\infty}(\max\{-\ln c, u\})^m e^{-u(nc-a)}\,du \\
&= c^{a+1}e^{-nc}\left((-\ln c)^m\int_0^{-\ln c} e^{-u(nc-a)}\,du + \int_{-\ln c}^{+\infty} u^m e^{-u(nc-a)}\,du\right)
\end{aligned}$$

$$\leq c^{a+1}e^{-nc}\Big((-\ln c)^m \int_0^{+\infty} e^{-u(nc-a)}\,du + \int_0^{+\infty} u^m e^{-u(nc-a)}\,du\Big)$$
$$= c^{a+1}e^{-nc}\big(\mathcal{O}(n^{-1}) + \mathcal{O}(n^{-m-1})\big) = \mathcal{O}\big(n^{-1}e^{-nc}\big).$$

For $-1 < a < 0$ the steps are almost the same taking into account that $(1+u)^a$ is bounded. From (2) the result is immediate.

Now for a small real number c with $0 < c < 1$ and m a positive integer we can write

$$\begin{aligned} S_{n,m} &= \frac{1}{\Gamma(m)} \sum_{k=1}^{n} (-1)^k \binom{n}{k} \int_0^{+\infty} t^{m-1} e^{-kt}\,dt \\ &= \frac{1}{\Gamma(m)} \int_0^{+\infty} t^{m-1} \sum_{k=1}^{n} (-1)^k \binom{n}{k} e^{-kt}\,dt \\ &= \frac{1}{\Gamma(m)} \int_0^{+\infty} t^{m-1} \Big(\sum_{k=0}^{n} (-1)^k \binom{n}{k} e^{-kt} - 1\Big)\,dt \\ &= \frac{1}{\Gamma(m)} \int_0^{+\infty} t^{m-1}\big((1-e^{-t})^n - 1\big)\,dt. \end{aligned} \tag{3}$$

Hence we expressed $S_{n,m}$ as an improper integral which is easier to study than the sum form. Integrating by parts, and making $\ln(1-e^{-t}) = -x$, we get

$$\begin{aligned} S_{n,m} &= \frac{t^m\big((1-e^{-t})^n - 1\big)}{\Gamma(m+1)}\Bigg|_0^{+\infty} - \frac{n}{\Gamma(m+1)} \int_0^{+\infty} t^m e^{-t}(1-e^{-t})^{n-1}\,dt \\ &= -\frac{n}{\Gamma(m+1)} \int_0^{+\infty} \frac{t^m e^{-t}}{1-e^{-t}} e^{-n\ln\frac{1}{1-e^{-t}}}\,dt \\ &= -\frac{n}{\Gamma(m+1)} \int_0^{+\infty} \Big(\ln\frac{1}{1-e^{-x}}\Big)^m e^{-nx}\,dx. \end{aligned} \tag{4}$$

Note that

$$\ln\frac{1}{1-e^{-x}} = -\ln\big(1-e^{-x}\big) = -\ln\Big(x - \frac{x^2}{2} + \mathcal{O}(x^2)\Big)$$
$$= -\ln x - \ln\Big(1-\frac{x}{2}\Big) + \mathcal{O}(x^2) = -\ln x - \ln\Big(1-\frac{x}{2}\Big) + \mathcal{O}(x^2)$$
$$= -\ln x + \frac{x}{2} + \mathcal{O}(x^2).$$

Next, we substitute this expansion in (4) and compute the integral in two parts to get

$$S_{n,m} = -\frac{n}{\Gamma(m+1)}\Big(\int_0^c \Big(-\ln x + \frac{x}{2} + \mathcal{O}(x^2)\Big)^m e^{-nx}\,dx$$
$$+ \int_c^{+\infty}\Big(\ln\frac{1}{1-e^{-x}}\Big)^m e^{-nx}\,dx\Big)$$
$$= (-1)^{m+1}\frac{n}{\Gamma(m+1)}\int_0^c \Big(\ln^m x - \frac{m}{2}x\ln^{m-1}x + \mathcal{O}\big(x^2\ln^{m-1}x\big)\Big)e^{-nx}\,dx$$
$$+\mathcal{O}(e^{-nc})$$
$$= (-1)^{m+1}\frac{n}{\Gamma(m+1)}\Big(\int_0^c \ln^m x e^{-nx}\,dx - \frac{m}{2}\int_0^c x\ln^{m-1}xe^{-nx}\,dx$$
$$+\mathcal{O}\Big(\int_0^c x^2\ln^{m-1}xe^{-nx}\,dx\Big)\Big)+\mathcal{O}(e^{-nc}) = \sum_{k=0}^{m}(-1)^{k+1}\frac{\binom{m}{k}\Gamma^{(k)}(1)}{\Gamma(m+1)}\ln^{m-k}n$$
$$+\sum_{k=0}^{m-1}(-1)^{k+1}\frac{\binom{m-1}{k}\Gamma^{(k)}(2)}{2\Gamma(m)}\cdot\frac{\ln^{m-k-1}n}{n} + \mathcal{O}\big(n^{-2}\ln^{m-1}n\big),$$

where the last three equalities are true because $(-\ln(1-e^{-x}))^m$ is bounded on $[c,+\infty)$ and on account of the Lemma.
In particular, for $m = 1, 2$, with the values $\Gamma(1) = \Gamma(2) = 1$, $\Gamma^{(1)}(1) = -\gamma$, $\Gamma^{(2)}(1) = \gamma^2 + \frac{\pi^2}{6}$, $\Gamma^{(1)}(2) = 1-\gamma$ we get

$$S_{n,1} = -\ln n - \gamma - \frac{1}{2n} + \mathcal{O}(n^{-2}),$$
$$S_{n,2} = -\ln^2 n - \gamma\ \ln n - \frac{\gamma^2}{2} - \frac{\pi^2}{12} + \frac{1-\gamma}{2n} + \mathcal{O}(n^{-2}\ln n).$$

Remark: The above is an ad hoc approach, with elementary methods, of *example 1* in [1], p.107,
(`http://algo.inria.fr/flajolet/Publications/FlSe95.pdf`),
which appears as an application of a more general method for estimating asymptotically high order differences of a fixed numerical sequence f_k:

$$D_n[f] = \sum_{k=0}^{n}(-1)^k\binom{n}{k}f_k.$$

One can note that the above method can give more accurate estimates if the integrand in (4) is expanded in more terms.

References

[1] Flajolet, P. and Sedgewick, R. “Mellin Transforms and asymptotics: Finite differences and Rice’s integrals”. *Theoretical Computer Science,* 144 (1995), pp. 101–124.

[2] Spivey, M. Z. “Combinatorial Sums and Finite Differences”. *Discrete Mathematics* 307.24 (2007), pp. 3130–3146.

Anastasios Kotronis
Athens, Greece
akotronis@gmail.com

Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.

2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

April 30, 2023

Elementary Problems

E–107. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Suppose the polynomial with complex coefficients $A(z) = z^5 + a_4z^4 + a_3z^3 + a_2z^2 + a_1z + a_0$ has distinct zeros and that any three of them are noncollinear. Show that we can always choose four of these zeros as vertices of a convex quadrilateral.

E–108. *Proposed by Rovsen Pirkuliyev, Sumgait City, Azerbaijan.* Solve in $\mathbb{R}$ the following equation

$$2022\sin^{2022}x+\cos^4 x-3\sin 2x\cos x-13\cos^2 x+11\sin x+2022 = 0.$$

E–109. *Proposed by Michel Bataille, Rouen, France.* Let $f(x) = \dfrac{\cos(x/2)}{1-\sin(x/2)}$ and let α, β, γ be the angles of a triangle. Prove that

$$f(\alpha) + f(\beta) + f(\gamma) = f(\alpha)\cdot f(\beta)\cdot f(\gamma).$$

E–110. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Homer and Flanders have an appointment at a certain time, and each will arrive at the meeting place with a delay between 0 and 1 hour, with all pairs of delays equally likely. The first to arrive will wait for 20 minutes and will leave if the other has not yet arrived. What is the probability that they will meet?

E–111. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Prove that if n is a nonnegative integer, then prove that 2023 divides

$$68^{2n+1} + 197\cdot 51^{2n+1}.$$

E–112. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* In the acute triangle ABC, F is the foot of the altitude from A and D, E are the projections of F on AB, respectively AC. The tangents to the circumcircle $\triangle ADE$ in D and E intersect in X. If $AX \cap BC = \{Y\}$, prove that triangle ABY and ACY have the same area.

Easy–Medium Problems

EM–107. *Proposed by Mihály Bencze, Braşov, Romania.* Prove that

$$\sum_{0\le i<j\le n}(-1)^{i+j-1}(i+j)\binom{n}{i}\binom{n}{j}=n\binom{2n-1}{n}.$$

EM–108. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.* Triangles are erected on the altitudes AA', BB', CC' of a triangle ABC, whose other sides are parallel to the other altitudes. Let S_a, S_b, S_c denote the areas of these triangles and let S be the area of $\triangle ABC$. Prove that

$$\frac{4}{S}=2\sum_{\text{cyclic}}\frac{1}{\sqrt{S_aS_b}}-\sum_{\text{cyclic}}\frac{1}{S_a}.$$

EM–109. *Proposed by Michel Bataille, Rouen, France.* Let p be an odd prime and let r be an integer with $0\le r<p$. Prove that

$$\sum_{k=r}^{p-1}(-1)^{k-r}\binom{k}{r}\equiv 2^r \pmod{p}.$$

EM–110. *Proposed by Goran Conar, Varaždin, Croatia.* If $\alpha_1,\alpha_2,\dots,\alpha_n$ are the angles of a convex polygon with $n\ge 3$ vertices, then prove that it holds

$$\sin\Big(\frac{\alpha_1}{\pi}\Big)\sin\Big(\frac{\alpha_2}{\pi}\Big)\dots\sin\Big(\frac{\alpha_n}{\pi}\Big)+\cos\Big(\frac{\alpha_1}{\pi}\Big)\cos\Big(\frac{\alpha_2}{\pi}\Big)\dots\cos\Big(\frac{\alpha_n}{\pi}\Big)<1.$$

EM–111. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* How many are the different pairs of integers x and y from 1 to 2023, for which $(x^2+y^2)/49$ is an integer? (The pairs (x,y) and (y,x) are considered equal).

EM–112. *Proposed by Toma–Ioan Dumitrescu, Bucharest, Romania.* Find all surjective functions $f:(0,+\infty)\to(0,+\infty)$ such that for every positive reals x,y, satisfy

$$f\Big(\frac{xf(x)}{f(y)}\Big)=f^4(y)f\Big(\frac{f(y)}{x}\Big).$$

Medium–Hard Problems

MH–107. *Proposed by Michel Bataille, Rouen, France.* Let a, b be real numbers such that the polynomial $x^4 - ax^3 + (b-a)x^2 + bx - 1$ has three positive real roots. Prove that

$$ab \geq 4 + |a - b|.$$

MH–108. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all positive integers n for which the equation

$$4\left(a_1^4 + a_2^4 + \ldots + a_n^4\right) = 7\left(a_1^3 + a_2^3 + \ldots + a_n^3\right)$$

has solution in the set of positive integers and determine them.

MH–109. *Proposed by Ruben Carpenter, Barcelona, Spain.* Given a triangle $\triangle ABC$, let A_1, B_1 and C_1 be the feet of the internal angle bisectors from A, B and C, respectively. Further, let the circumscribed circle ω of $\triangle A_1B_1C_1$ intersect AB and AC at $C_2 \neq C_1$ and $B_2 \neq B_1$, respectively. Assume that circle ω is tangent to BC. Show that either $|AB| = |AC|$ or $|AB_1| = |AC_2|$.

MH–110. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let x_{ij}, $(1 \leq i \leq m, 1 \leq j \leq n)$ be nonnegative real numbers. Find the minimum value of

$$\prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m} \frac{\sqrt[3]{x_{ij}}}{1 + 2\sqrt[3]{x_{ij}}}\right) + \prod_{i=1}^{m}\left(1 - \prod_{j=1}^{n} \frac{1 + \sqrt[3]{x_{ij}}}{1 + 2\sqrt[3]{x_{ij}}}\right).$$

MH–111. *Proposed by Félix Moreno Peñarrubia, Charles University, Prague, Czech Republic.* Let n be a positive integer, and let $\mathbb{Q}_n = \{\frac{a}{b} \in \mathbb{Q} : 1 \leq a, b \leq n\}$. Let S be a subset of $\mathbb{Q}_n$ with $|S| = n$. Prove that there exist two distinct (not necessarily nonempty) subsets $A, B \subseteq S$ such that $\prod_{x \in A} x = \prod_{y \in B} y$.

MH–112. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* For an integer $n \geq 3$ we consider a circle containing n vertices. To each vertex we assign a positive integer, and these integers do not necessarily have to be distinct. Such an assignment of integers is called stable if the product of any three adjacent integers is n. For how many values of n with $3 \leq n \leq 2023$ does there exist a stable assignment?

Advanced Problems

A–107. *Proposed by Joseph Santmyer, US Federal Government (retired), Las Crues, New Mexico, USA.* Let us denote by A and B the following integrals

$$\begin{aligned} A &= \int_0^{2\pi} \cos(t)\sin[e^{\cos(t)}\cos(\sin(t)-t)]\cosh[e^{\cos(t)}\sin(\sin(t)-t)]dt, \\ B &= \int_0^{2\pi} \sin(t)\cos[e^{\cos(t)}\cos(\sin(t)-t)]\sinh[e^{\cos(t)}\sin(\sin(t)-t)]dt. \end{aligned}$$

Prove that

$$2\pi\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)^{2n}}{(2n+1)!(2n)!} = A - B.$$

(Correction of Problem A-103)

A–108. *Proposed by Michel Bataille, Rouen, France.* Evaluate

$$\lim_{n\to\infty}\sum_{k=1}^{\infty}\frac{1}{k\binom{n+k-1}{n}}.$$

A–109. *Proposed by Florică Anastase, "Alexandru Odobescu" high school, Lehliu-Gară, Călăraşi, Romania.* Let $\{\omega_n\}_{n\geq 1}$ be the sequence defined by $\omega_n = (2n+1)\left(\sum_{k=0}^{2n}\tan\left(x+\frac{k\pi}{2n+1}\right)\right)^{-1}$. Find

$$\Omega = \lim_{n\to\infty}\left(\lim_{x\to\frac{\pi}{2n+1}}\left(\frac{\cot x}{\cot\frac{\pi}{2n+1}}\right)^{\omega_n}\right).$$

A–110. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* The plane is partitioned into n regions by three families of parallel lines in general position. What is the least number of lines to ensure that $n \geq 2023$?

A–111. *Proposed by Félix Moreno Peñarrubia, Charles University, Prague, Czech Republic.* Consider the function $f : \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+$ defined as:

- $f(n,1) = n$ for all positive integers n.
- For $m \geq 2$, $f(n,m)$ is the smallest multiple of m which is greater than or equal to $f(n,m-1)$.

We call a positive integer n *special* if all elements of the infinite sequence $\{f(n,1), f(n,2), \ldots\}$ are distinct. Prove that there are at least 2022 special integers less than $5\,000\,000$.

A–112. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b be positive real numbers. Prove that

$$\int_0^1 t^a (1-t)^{b-1} \ln(t)\, dt \geq \frac{a\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)} \ln\left(\frac{a}{a+b}\right),$$

where $\Gamma(x)$ is the Euler Gamma Function.

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

Some Applications of Chebyshev's integral inequality

Florică Anastase and José Luis Díaz-Barrero

Abstract

In this paper, some integral inequalities derived using classical Chebyshev's inequality are presented.

1 Introduction

In [7] was published a discrete version of Chebyshev's inequality and some of its applications. In this note, following a similar procedure, we state the integral counterpart of Korkine's identity [3], and using it we derive the integral version of Chebyshev's inequality. Finally, the weighted version of it and some of its applications similar the ones appeared in ([4], [6], [5]) are also given.

2 Basic results

The following result, known in the literature as Chebyshev's inequality, is the natural continuous form of its discrete version. It is stated in the following

Lemma 1. *Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions both increasing or both decreasing. Then, it holds*

$$(b-a)\int_a^b f(x)g(x)\,dx \geq \int_a^b f(x)\,dx \int_a^b g(x)\,dx \tag{1}$$

The integral version of Korkine's identity in where integrals appears instead of finite sums is known as Andréief's identity [3]. It states that

$$\begin{aligned}(b-a)&\int_a^b f(x)g(x)\,dx - \int_a^b f(x)\,dx \int_a^b g(x)\,dx\\ &= \frac{1}{2}\int_a^b\int_a^b (f(x)-f(y)(g(x)-g(y))\,dxdy\end{aligned}$$

Note that if f and g have the same monotony, then

$$\frac{1}{2}\int_a^b\int_a^b (f(x)-f(y)(g(x)-g(y))\,dxdy \geq 0$$

and (1) immediately follows. A particular case is the following

Corollary 1. *Let $f, g : [0, 1] \to \mathbb{R}$ be integrable functions both increasing or both decreasing. Then, it holds*

$$\int_0^1 f(x)g(x)\,dx \geq \int_0^1 f(x)\,dx \int_0^1 g(x)\,dx.$$

Remark. A generalized version of the inequality given in Lemma 1 is the following claim. *It holds that*

$$\begin{aligned}(b-a)^{n-1}&\int_a^b f_1(x)f_2(x)\dots f_n(x)dx\\ &\geq \int_a^b f_1(x)dx \int_a^b f_2(x)dx \dots \int_a^b f_n(x)dx,\end{aligned}$$

where $f_1, f_2, \dots, f_n$ are nonnegative integrable functions on $[a, b]$ which are all either monotonic increasing or monotonic decreasing.

Finally, a weighted version of (1) is presented in the next

Theorem 1. *Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions both increasing or both decreasing and let $p : [a, b] \to (0, \infty)$ be an integrable function. Then*

$$\int_a^b p(x)\,dx \int_a^b p(x)f(x)g(x)\,dx \geq \int_a^b p(x)f(x)\,dx \int_a^b p(x)g(x)\,dx.$$

If one of the functions f or g is nonincreasing or nondecreasing the preceding inequality reverses.

Proof. Suppose that f, g are similarly monotonous, then for all $x, y \in [a, b]$ we have:

$$[f(x) - f(y)][g(x) - g(y)] \geq 0.$$

Now, interchanging variable x to y, clearly holds

1. $\displaystyle\int_a^b p(x)\,dx \int_a^b p(y)f(y)g(y)\,dy = \int_a^b p(y)\,dy \int_a^b p(x)f(x)g(x)\,dx$
2. $\displaystyle\int_a^b p(x)f(x)\,dx \int_a^b p(y)f(y)\,dy = \int_a^b p(y)f(y)\,dy \int_a^b p(x)f(x)\,dx$

Hence, it follows that

$$\begin{aligned}
&\int_a^b p(x)\,dx \int_a^b p(x)f(x)g(x)\,dx - \int_a^b p(x)f(x)\,dx \int_a^b p(x)g(x)\,dx \\
&= \frac{1}{2}\left[2\int_a^b p(x)\,dx \int_a^b p(x)f(x)g(x)\,dx - 2\int_a^b p(x)f(x)\,dx \int_a^b p(x)g(x)\,dx\right] \\
&= \frac{1}{2}\left[\int_a^b p(x)\,dx \int_a^b p(y)f(y)g(y)\,dy + \int_a^b p(y)\,dy \int_a^b p(x)f(x)g(x)\,dx\right] \\
&\quad - \frac{1}{2}\left[\int_a^b p(x)f(x)\,dx \int_a^b p(y)g(y)\,dy + \int_a^b p(y)f(y)\,dy \int_a^b p(x)g(x)\,dx\right] \\
&= \frac{1}{2}\int_a^b \int_a^b [p(x)p(y)f(y)g(y) + p(y)p(x)f(x)g(x)]\,dy\,dx \\
&\quad - \frac{1}{2}\int_a^b \int_a^b [p(x)p(y)f(x)g(y) + p(y)p(x)f(y)g(x)]\,dy\,dx \\
&= \frac{1}{2}\int_a^b \int_a^b p(x)p(y)[f(x) - f(y)][g(x) - g(y)]\,dy\,dx \geq 0.
\end{aligned}$$

Therefore,

$$\left(\int_a^b p(x)\,dx\right)\left(\int_a^b p(x)f(x)g(x)\,dx\right) \geq \left(\int_a^b p(x)f(x)\,dx\right)\left(\int_a^b p(x)g(x)\,dx\right).$$

From the above inequality immediately follows:

Theorem 2. *Let $f, u : [a, b] \to \mathbb{R}$ be two function that is continuous and increasing on $[a, b]$. If p is continuous and non negative on $[a, b]$, then*

$$\int_a^b p(x)\,dx \int_a^b p(x)u(x)f(u(x))\,dx \geq \int_a^b p(x)u(x)\,dx \int_a^b p(x)f(u(x))\,dx$$

holds.

3 Applications

Hereafter, using the previous results, some integral inequalities are derived. We begin with

Problem 1. *Let $f : [a, b] \to \mathbb{R}$ be a continuous and increasing function. Prove that*

$$\int_a^b xf(x)\,dx \geq \frac{a+b}{2} \int_a^b f(x)\,dx.$$

Solution. Equality holds when $a = b$ or f is a constant function, as can be easily checked. Let $a < b$ and f be a continuous and increasing function on $[a, b]$. Let $g(x) = x$ for all $x \in [a, b]$. Since f and g have same monotony on $[a, b]$, using Lemma 1, we get

$$\left(\int_a^b f(x)\,dx\right)\left(\int_a^b x\,dx\right) \leq (b-a) \int_a^b xf(x)\,dx$$

$$\frac{b^2-a^2}{2} \int_a^b f(x)\,dx \leq (b-a) \int_a^b xf(x)\,dx$$

Therefore,

$$\int_a^b xf(x)\,dx \geq \frac{a+b}{2} \int_a^b f(x)\,dx.$$

Problem 2. *If $f : [a, b] \to \mathbb{R}$ is a continuous and derivable function such that $xf'(x) \leq f(x)$ for all $x \in [a, b]$, then prove that it holds*

$$\int_a^b f(x)\,dx \leq \frac{b+a}{2} \int_a^b \frac{f(x)}{x}\,dx.$$

Solution. Since $\left(\frac{f(x)}{x}\right)' = \frac{xf'(x) - f(x)}{x^2} \leq 0$ then $x \to \frac{f(x)}{x}$ is a decreasing function. Using Lemma 1, we get

$$\int_a^b f(x)\,dx = \int_a^b \frac{x\,f(x)}{x}\,dx \leq \frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b x\,dx\right)$$

$$= \frac{b^2 - a^2}{2(b-a)}\int_a^b \frac{f(x)}{x}\,dx = \frac{b+a}{2}\int_a^b \frac{f(x)}{x}\,dx.$$

Problem 3. *If $f, g : [0,1] \to (0, +\infty)$ are continuous functions and f, and f/g are both increasing functions, then prove that*

$$\int_0^1 \left(\int\limits_0^x f(t)\,dt \Big/ \int\limits_0^x g(t)\,dt\right) dx \leq 2\int_0^1 \frac{f(t)}{g(t)}\,dt.$$

Solution. On account of (1), we have

$$\int_0^x f(t)\,dt \int_0^x \frac{g(t)}{f(t)}\,dt \leq x\int_0^x f(t)\,\frac{g(t)}{f(t)}\,dt = x\int_0^x g(t)\,dt$$

Using Cauchy's inequality integral form, we have

$$\int_0^x \frac{g(t)}{f(t)}\,dt \int_0^x \frac{t^2 f(t)}{g(t)}\,dt \geq \frac{x^4}{4}.$$

Hence, for all $x \in [0,1]$, it holds

$$\frac{\int\limits_0^x f(x)\,dt}{\int\limits_0^x g(t)\,dt} \leq \frac{4}{x^3}\int_0^x \frac{t^2 f(t)}{g(t)}\,dt,$$

and we have

$$\int_0^1 \left(\frac{\int\limits_0^x f(t)\,dt}{\int\limits_0^x g(t)\,dt}\right) dx \leq 4\int_0^1 \left(\frac{1}{x^3}\int_0^x \frac{t^2 f(t)}{g(t)}\,dt\right) dx$$

$$= 2\int_0^1 \frac{f(x)}{g(x)}(1 - x^2)\,dx \leq 2\int_0^1 \frac{f(t)}{g(t)}\,dt.$$

The last equality was established using integration by parts. Indeed, we have

$$\int_0^1 \left(\frac{1}{x^3}\int_0^x \frac{t^2 f(t)}{g(t)}\,dt\right)dx = -\frac{1}{2}\int_0^1 \left(\frac{1}{x^2}\right)'\left(\int_0^x \frac{t^2 f(t)}{g(t)}\,dt\right)dx =$$
$$= -\frac{1}{2}\cdot\frac{1}{x^2}\int_0^x \frac{t^2 f(t)}{g(t)}dt\Big|_0^1 + \frac{1}{2}\int_0^1 \frac{1}{x^2}\cdot\frac{x^2 f(x)}{g(x)}\,dx$$
$$= -\frac{1}{2}\int_0^1 \frac{x^2 f(x)}{g(x)}\,dx + \frac{1}{2}\lim_{x\to 0}\frac{1}{x^2}\int_0^x \frac{x^2 f(x)}{g(x)}dx + \frac{1}{2}\int_0^1 \frac{f(x)}{g(x)}\,dx$$
$$= \frac{1}{2}\int_0^1 (1-x^2)\frac{f(x)}{g(x)}\,dx.$$

Problem 4. *Let $h : [0,1] \to (0,\infty)$ be a decreasing function. Prove that*

$$\int\limits_0^1 x\,h^2(x)\,dx \Big/ \int\limits_0^1 x\,h(x)\,dx \le \int\limits_0^1 h^2(x)\,dx \Big/ \int\limits_0^1 h(x)\,dx\,.$$

Solution. Putting $f(x) = x, g(x) = 1/h(x), p(x) = h^2(x)$ in Theorem 1, it follows

$$\int_0^1 h^2(x)\,dx \int_0^1 h^2(x)\,x\,\frac{1}{h(x)}dx$$
$$\ge \int_0^1 xh^2(x)\,dx \Big/ \int_0^1 h^2(x)\,\frac{1}{h(x)}\,dx$$
$$\int_0^1 h^2(x)\,dx \int_0^1 xh(x)\,dx \ge \int_0^1 xh^2(x)\,dx \int_0^1 h(x)\,dx,$$

and the statement follows.

Problem 5. *Let $f : [a,b] \to \mathbb{R}$ be an increasing function with second derivative on $[a,b]$. If $f(a) = 0$ and $M = \max\limits_{x\in[a,b]} |f(x)|$, then prove that for all $p \ge 1$, it holds*

$$(b-a)^{p-1}\int_a^b |f'(x)|^p\,dx \ge M^p.$$

Solution. We have

$$\int_a^b |f'(x)|^p \, dx = \int_a^b |f'(x)|\, |f'(x)|^{p-1} \, dx$$

$$\geq \frac{1}{b-a} \int_a^b |f'(x)| \, dx \int_a^b |f'(x)|^{p-1} \, dx$$

$$= \frac{f(b)}{b-a} \int_a^b |f'(x)|^{p-1} \, dx = \frac{f(b)}{b-a} \int_a^b |f'(x)|\, |f'(x)|^{p-2} \, dx$$

$$\geq \frac{f^2(b)}{(b-a)^2} \int_a^b |f'(x)|^{p-2} \, dx \geq \ldots \geq \frac{f^{p-1}(b)}{(b-a)^{p-1}} \int_a^b |f'(x)|^{p-(p-1)} \, dx$$

$$= \frac{f^p(b)}{(b-a)^{p-1}} = \frac{M^p}{(b-a)^{p-1}}$$

from which

$$(b-a)^{p-1} \int_a^b |f'(x)|^p \, dx \geq M^p$$

follows.

Problem 6. *Let $f : [a, b] \to [a, b]$ with $0 < a < b$, be a second derivable function such that $a \leq f'(x) \leq b$. If f and f' are increasing functions and f' is a convex function on $[a, b]$, then prove*

1. $f(f'(b)) - f(f'(a)) \geq (f'(b) - f'(a)) \cdot f'\left(\dfrac{f(b) - f(a)}{b-a}\right)$
2. $\tan\left(\tan^2 \dfrac{1}{2}\right) - \tan^2 \dfrac{1}{2} \geq \tan^2 \dfrac{1}{2} \cdot \tan^2\left(2 \tan \dfrac{1}{2} - 1\right)$

Solution. 1. Since $f'(x) \geq 0$, for all $x \in [a, b]$, then $f, f', f \circ f$ are increasing functions on $[a, b]$. On account that f' is convex function on $[a, b]$, and using Theorem 1, we get

$$f(f'(b)) - f(f'(a)) = \int_a^b (f \circ f')'(x) \, dx = \int_a^b f'(f'(x)) \cdot f''(x) \, dx$$

$$\geq \frac{1}{b-a} \int_a^b f'(f'(x)) \, dx \cdot \int_a^b f''(x) \, dx = \frac{f'(b) - f'(a)}{b-a} \cdot \int_a^b f'(f'(x)) \, dx$$

$$\geq (f'(b) - f'(a)) \cdot f'\left(\frac{1}{b-a} \int_a^b f'(x) \, dx\right) = (f'(b) - f'(a)) \cdot f'\left(\frac{f(b) - f(a)}{b-a}\right)$$

2. We apply the previous result. Indeed, let $f : [0, \frac{1}{2}] \to [0, \frac{1}{2}]$ defined by $f(x) = \tan x - x > 0$. Then, we have $f'(x) = \dfrac{1}{\cos^2 x} - 1 > 0$, $f''(x) = \dfrac{2}{\cos^2 x} \cdot \tan x > 0$ and $f'''(x) = \dfrac{2 + 4\sin^2 x}{\cos^4 x} > 0$. From the above result, it follows

$$f\left(f'\left(\frac{1}{2}\right)\right) - f(f'(0)) \geq f'\left(\frac{1}{2} - f'(0)\right) f'\left(\frac{f(1) - f(0)}{1 - 0}\right)$$

and

$$\tan\left(\tan^2 \frac{1}{2}\right) - \tan^2 \frac{1}{2} \geq \tan^2 \frac{1}{2} \cdot \tan^2\left(2\tan \frac{1}{2} - 1\right).$$

Problem 7. *If $0 < a < b < 1$, then prove that*

$$\int_0^{\frac{\pi}{4}} \frac{a + b\sin x}{b + a\sin x} dx \int_0^{\frac{\pi}{4}} \frac{1}{b + a\sin x} dx \geq \frac{\pi}{4}\left(\frac{1}{b} - \frac{\sqrt{2}}{2b + a\sqrt{2}}\right).$$

Solution. Consider the functions

$$f, g : \left[0, \frac{\pi}{4}\right] \to \mathbb{R}, f(x) = \frac{a + b\sin x}{b + a\sin x}, g(x) = \frac{1}{b + a\sin x}$$

$$f'(x) = \frac{(b^2 - a^2)\cos x}{(b + a\sin x)^2} > 0 \text{ and } g'(x) = -\frac{a\cos x}{(b + a\sin x)^2} < 0.$$

So, $f-$is increasing function and $g-$is decreasing function. Applying previous results, we get

$$\int_0^{\frac{\pi}{4}} \frac{a + b\sin x}{b + a\sin x} dx \int_0^{\frac{\pi}{4}} \frac{1}{b + a\sin x} dx \geq \frac{\pi}{4} \int_0^{\frac{\pi}{4}} \frac{a + b\sin x}{(b + a\sin x)^2} dx.$$

$$\begin{aligned} I &= \int_0^{\frac{\pi}{4}} \frac{a + b\sin x}{(b + a\sin x)^2} dx = \frac{b}{a} \int_0^{\frac{\pi}{4}} \frac{\frac{a^2}{b} - b + (b + a\sin x)}{(b + a\sin x)^2} dx \\ &= \frac{a^2 - b^2}{a} \int_0^{\frac{\pi}{4}} \frac{dx}{(b + a\sin x)^2} + \frac{b}{a} \int_0^{\frac{\pi}{4}} \frac{dx}{b + a\sin x} dx \quad (2) \end{aligned}$$

Putting

$$t = \frac{\cos x}{b + a\sin x},\ dt = -\frac{b}{a}\left(\frac{1}{b + a\sin x} + \frac{a^2 - b^2}{b(b + a\sin x)^2}\right) dx$$

and then t can also be expressed as

$$t = -\frac{b}{a}\int_0^{\frac{\pi}{4}} \frac{dx}{b + a\sin x} - \frac{a^2 - b^2}{b}\int_0^{\frac{\pi}{4}} \frac{dx}{(b + a\sin x)^2} \quad (3)$$

From (2) and (3), we get:

$$I = \frac{-\cos x}{b + a\sin x}\Big|_0^{\frac{\pi}{4}} = \frac{1}{b} - \frac{\sqrt{2}}{2b + a\sqrt{2}},$$

and therefore,

$$\int_0^{\frac{\pi}{4}} \frac{a + b\sin x}{b + a\sin x}\, dx \int_0^{\frac{\pi}{4}} \frac{1}{b + a\sin x}\, dx \geq \frac{\pi}{4}\left(\frac{1}{b} - \frac{\sqrt{2}}{2b + a\sqrt{2}}\right).$$

Problem 8. *Let a, b, c, d be positive reals such that $bc = ad$. Then prove that it holds*

$$\int_0^{\frac{a}{b}} \frac{\log(a + bx)}{\sqrt{c^2 + d^2x^2}}\, dx \geq \frac{1}{d}\log(1 + \sqrt{2})\log(a\sqrt{2}).$$

Solution. We have

$$\int_0^{\frac{a}{b}} \frac{\log(a + bx)}{\sqrt{c^2 + d^2x^2}} dx = \int_0^{\frac{a}{b}} \frac{\sqrt{c^2 + d^2x^2}}{c^2 + d^2x^2}\log(a + bx)\, dx.$$

Next, we apply Weighted Chebyshev's Integral Inequality. Indeed, let $p(x) = \frac{1}{c^2 + d^2x^2}$, (continuous function) and $f(x) = \sqrt{c^2 + d^2x^2}$ and $g(x) = \log(a + bx)$ (increasing functions), then

$$\int_0^{\frac{a}{b}} \frac{1}{c^2 + d^2x^2}\, dx \int_0^{\frac{a}{b}} \frac{\log(a + bx)}{\sqrt{c^2 + d^2x^2}}\, dx$$

$$\geq \int_0^{\frac{a}{b}} \frac{\sqrt{c^2 + d^2x^2}}{c^2 + d^2x^2}\, dx \int_0^{\frac{a}{b}} \frac{\log(a + bx)}{c^2 + d^2x^2}\, dx$$

Now, we compute

$$I_1 = \int_0^{\frac{a}{b}} \frac{1}{c^2+d^2x^2} dx = \frac{1}{d^2}\int_0^{\frac{a}{b}} \frac{1}{(\frac{c}{d})^2+x^2}\, dx = \frac{\pi}{4cd},$$

$$I_2 = \int_0^{\frac{a}{b}} \frac{\sqrt{c^2+d^2x^2}}{c^2+d^2x^2}\, dx = \int_0^{\frac{a}{b}} \frac{dx}{\sqrt{c^2+d^2x^2}} = \frac{1}{d}\log\left(1+\sqrt{2}\right),$$

$$I_3 = \int_0^{\frac{a}{b}} \frac{log(a+bx)}{c^2+d^2x^2}\, dx = \int_0^{\frac{a}{b}} \frac{\log a + log(1+\frac{bx}{a})}{d^2((\frac{c}{d})^2+x^2)}\, dx$$

$$= \frac{\log a}{d^2}\int_0^{\frac{a}{b}} \frac{dx}{(\frac{c}{d})^2+x^2} + \frac{1}{c^2}\int_0^{\frac{a}{b}} \frac{\log\left(1+\frac{bx}{a}\right)}{1+(\frac{dx}{c})^2}\, dx$$

$$= \frac{\pi\log a}{4cd} + \frac{1}{c^2}\int_0^{\frac{a}{b}} \frac{\log\left(1+\frac{bx}{a}\right)}{1+(\frac{bx}{a})^2}\, dx = \frac{\pi\log a}{4cd} + \frac{a}{bc^2}\int_0^{1} \frac{\log(1+y)}{1+y^2}\, dy$$

$$= \frac{\pi\log a}{4cd} + \frac{a}{bc^2}\int_0^{\frac{\pi}{4}} \log(1+\tan z)\, dz = \frac{\pi}{4cd}\log\left(a\sqrt{2}\right),$$

and thus, we have

$$\int_0^{\frac{a}{b}} \frac{\log(a+bx)}{\sqrt{c^2+d^2x^2}}\, dx \geq \frac{1}{d}\log\left(1+\sqrt{2}\right)\,\log\left(a\sqrt{2}\right).$$

Putting $a=c=\pi$ and $b=d=1$ in [1] it was obtained

$$\int_0^{\pi} \frac{\log(\pi+x)}{\sqrt{\pi^2+x^2}}\, dx \geq \log\left(1+\sqrt{2}\right)\,\log\left(\pi\sqrt{2}\right)$$

Putting $a=c=e$ and $b=d=1$ we have obtained in [2] the inequality

$$\int_0^{e} \frac{\log(e+x)}{\sqrt{e^2+x^2}}\, dx \geq \log\left(1+\sqrt{2}\right)\cdot\log\left(e\sqrt{2}\right)$$

Next, we apply Cauchy's Integral Inequality inequality (CBS) to solve the following

Problem 9. *Let* a, b *be positive reals. If* $f : [0, \frac{a}{b}] \to [0, \infty)$ *is a continuous and convex function such that* $f(0) = 0, f(\frac{a}{b}) = 1$*, then it holds*

$$\int_0^{\frac{a}{b}} \sqrt{c^2+d^2x^2}\,(f'(x))^2\, dx \geq \frac{d}{\log\left(1+\sqrt{2}\right)}.$$

Solution. Applying CBS Integral Inequality, it follows that

$$1 = f\Big(\frac{a}{b}\Big) - f(0) = \int_0^{\frac{a}{b}} f'(x)\,dx = \int_0^{\frac{a}{b}} \frac{\sqrt[4]{c^2+d^2x^2}}{\sqrt[4]{c^2+d^2x^2}}\, f'(x)\,dx$$

$$\leq \Big(\int_0^{\frac{a}{b}} \frac{dx}{\sqrt{c^2+d^2x^2}}\Big)^{\frac{1}{2}} \Big(\int_0^{\frac{a}{b}} \sqrt{c^2+d^2x^2}\,(f'(x))^2 dx\Big)^{\frac{1}{2}}$$

$$= \Big(\frac{\log(1+\sqrt{2})}{d}\Big)^{\frac{1}{2}} \Big(\int_0^{\frac{a}{b}} \sqrt{c^2+d^2x^2}\,(f'(x))^2 dx\Big)^{\frac{1}{2}}.$$

Therefore,

$$\int_0^{\frac{a}{b}} \sqrt{c^2+d^2x^2}\,(f'(x))^2 dx \geq \frac{d}{\log(1+\sqrt{2})}.$$

Problem 10. *Let a, b, c, d be positive reals such that $bc = ad$. If $f : [0, \frac{a}{b}] \to [0; \infty)$ is a continuous and convex function such that $f(0) = 0, f(\frac{a}{b}) = 1$, then prove that*

$$\int_0^{\frac{a}{b}} \sqrt{c^2+d^2x^2}\, \log(a+bx)\,(f'(x))^2\,dx \geq \frac{\log(a\sqrt{2})}{\log(1+\sqrt{2})}.$$

Solution. We have

$$\int_0^{\frac{a}{b}} \sqrt{c^2+d^2x^2}\, \log(a+bx)\,(f'(x))^2\,dx$$

$$= \int_0^{\frac{a}{b}} \frac{1}{\sqrt{c^2+d^2x^2}}\, \log(a+bx)\,(c^2+d^2x^2)(f'(x))^2 dx.$$

Next, we apply Weighted Chebyshev's Integral Inequality. Indeed, let $p(x) = \frac{1}{\sqrt{c^2+d^2x^2}}$, $u(x) = \log(a+bx)$ $v(x) = (c^2 + d^2x^2)(f'(x))^2$, with u, v increasing, then taking into account the previous results, it follows that

$$\Big(\int_0^{\frac{a}{b}} \frac{dx}{\sqrt{c^2+d^2x^2}}\Big)\Big(\frac{1}{\sqrt{c^2+d^2x^2}}\, \log(a+bx)\,(c^2+d^2x^2)(f'(x))^2\,dx\Big)$$

$$\geq \Big(\int_0^{\frac{a}{b}} \frac{\log(a+bx)}{\sqrt{c^2+d^2x^2}}\,dx\Big)\Big(\int_0^{\frac{a}{b}} \frac{c^2+d^2x^2}{\sqrt{c^2+d^2x^2}} \cdot (f'(x))^2\,dx\Big)$$

$$= \Big(\int_0^{\frac{a}{b}} \frac{\log(a+bx)}{\sqrt{c^2+d^2x^2}}\, dx \Big) \Big(\int_0^{\frac{a}{b}} \sqrt{c^2+d^2x^2}\, (f'(x))^2\, dx \Big)$$

$$\geq \frac{a}{bc} \log(1+\sqrt{2})\, (a\sqrt{2}) \Big(\int_0^{\frac{a}{b}} \sqrt{c^2+d^2x^2}\, (f'(x))^2 dx \Big)$$

$$\geq \frac{a}{bc} \log(1+\sqrt{2}) \log(a\sqrt{2})\, \frac{d}{\log(1+\sqrt{2})} = \log(a\sqrt{2}).$$

Therefore,

$$\int_0^{\frac{a}{b}} \sqrt{c^2+d^2x^2}\, \log(a+bx)\, (f'(x))^2 dx \geq \frac{\log(a\sqrt{2})}{\log(1+\sqrt{2})}.$$

References

[1] Anastase, F. “Problem PP.34852”. *Octogon Mathematical Magazine* 28.1 (2020), p. 357.

[2] Anastase, F. “Problem 5698”. *School Science and Mathematics* 122.6 (2022), p. 1.

[3] Andréief, K. A. “Notes sur une relation entre intégrales définies des produits des functions”. *Mém. Soc. Sci. Bourdeaux* 3.2 (1883), pp. 1–14.

[4] Bencze, M. and Sitaru, D. *699 Olympic Mathematical Challenges*. Iasi, Romania: Studis, 2017.

[5] Bencze, M. and Sitaru, D. *Olympic Mathematical Energy*. Iasi, Romania: Studis, 2018.

[6] Bencze, M. and Sitaru, D. *Quantum Mathematical Power*. Iasi, Romania: Studis, 2018.

[7] Felipe Alsina, M. and Díaz-Barrero, J. L. “Chebyshev’s inequality and some of its applications”. *Arhimede math. j.* 6.1 (2019), pp. 47–53.

Florică Anastase
“Alexandru Odobescu” high school
Lehliu-Gară
Călăraşi, Romania
anastaseflorica@gmail.com

José Luis Díaz-Barrero
Civil and Environmental Engineering
BarcelonaTech
Barcelona, Spain
jose.luis.diaz@upc.edu

Some rational symmetrical and homogenous inequalities

Mihály Bencze and Marius Drăgan

Abstract

In [2] is presented a new method to prove and refine symmetrical and homogeneous algebraic inequalities. In this work, using it a collection of applications is given.

1 Introduction

In this paper we will denote $s = x + y + z$, $p = xyz$, $\alpha = (x + y + z)\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right) \geq 9$, $t = \frac{(x+y+z)^3}{xyz} \geq 27$. In [2] was published the following

Lemma 1. *Let* $t = s^3/p$, $t_3 = 4\alpha - 9$ *and* $t_4 = \alpha^3/(4\alpha - 9)$. *Then* $t_3 \leq t_1 \leq t \leq t_2 \leq t_4$, *where*

$$t_{1,2} = \frac{\alpha^2 + 18\alpha - 27 \pm \sqrt{(\alpha-1)(\alpha-9)^3}}{8}.$$

Using this result, below a collection of symmetric and homogeneous inequalities are proven and some of them refined.

2 The results

Hereafter, all the sums considered are cyclic. Next, we present the main results of this work.

Problem 1. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^3+2abc}{b+c}+\frac{b^3+2abc}{c+a}+\frac{c^3+2abc}{a+b}\geq\frac{1}{2}\,(a+b+c)^2.$$

Solution. We have

$$\sum\frac{a^3+2abc}{b+c}-\frac{1}{2}\left(\sum a\right)^2=s^2f(t),\ \forall\, t\in[t_1,t_2],$$

where

$$f(t)=\frac{t-4\alpha+7+\frac{2\alpha^2}{t}}{\alpha-1}-\frac{1}{2},\ f'(t)=\frac{t^2-2\alpha^2}{(\alpha-1)t^2},\ t\in[t_1,t_2].$$

Since $\alpha\sqrt{2}\leq 4\alpha-9$, $\forall\,\alpha\geq 9$, hence f is increasing on $[t_1,t_2]$. So $f(t)\geq f(t_1)$ and since $f(t_1)\geq 0$ (We have check it using the Computer Algebra System Wolphram Alpha) it follows that

$$\sum\frac{a^3+3abc}{a+c}\geq\frac{1}{2}(a+b+c)+s^2f(t_1)\geq\frac{1}{2}(a+b+c)^2.$$

Problem 2. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^2+2bc}{b+c}+\frac{b^2+2ca}{c+a}+\frac{c^2+2ab}{a+b}\geq\frac{3}{2}\,(a+b+c).$$

Solution. We have $\sum\frac{a^2+2bc}{b+c}-\frac{3}{2}\sum a=sf(t)$, $t\in[t_1,t_2]$ where $f(t)=\frac{1}{\alpha-1}\left(t-3\alpha+6+\frac{2\alpha^2}{t}\right)-\frac{3}{2}$, $f'(t)=\frac{t^2-2\alpha^2}{(\alpha-1)t^2}$, $t\in[t_1,t_2]$.
We have $\alpha\geq 9>\frac{9}{4-\sqrt{2}}$ or $\alpha\sqrt{2}\leq 4\alpha-9$.

So f is increasing on $[t_1, t_2]$ and $f(t_1) \leq f(t)$. So

$$\frac{\sum a^2 + 2bc}{b+c} \geq \frac{3}{2}\sum a + (a+b+c)f(t_1) \geq \frac{3}{2}\sum a$$

Since $f(t_1) \geq 0$, $\forall\, \alpha \geq 9$, the conclusion follows.

If $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 10$ then $\sum \frac{a^3+2bc}{b+c} \geq \frac{91}{60}(a+b+c)$.

Problem 3. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^3+3abc}{b+c} + \frac{b^3+3abc}{c+a} + \frac{c^3+3abc}{a+b} \geq 2(ab+bc+ca)$$

(V. Cârtoaje)

Solution. We have $\sum \frac{a^3+3abc}{b+c} - 2\sum ab = \frac{p}{s(\alpha-1)}f(t)$, $t \in [t_1, t_2]$ where $f(t) = t^2 - (4\alpha - 8)t + 3\alpha$, $\forall\, t \in [t_1, t_2]$ with roots $t'_{1,2} = 2\alpha - 4 \pm \sqrt{4\alpha^2 - 19\alpha + 16}\,(t'_1 < t'_2)$. We have (Wolfram Alpha) $t'_1 < t_3 < t'_2 < t_4$, $\forall\, \alpha \geq 9$. Also $t_r = 2\alpha - 4 < t_3$ or $t'_1 < t_v < t_3 < t'_2 < t_4$.

So f is increasing on $[t_3, t_4]$. So on $[t_1, t_2] \subset [t_3, t_4]$. It follows that

$$f(t) \geq f(t_1) = \frac{1}{32}\big(\alpha^4 - 12\alpha^3 + 14\alpha^2 + 132\alpha - 135 - \\ - (\alpha^2 + 2\alpha + 5)\sqrt{(\alpha-1)(\alpha-9)^3}\big) = g(\alpha) \geq 0,\ \alpha \geq 9$$

We obtain the refinement

$$\sum \frac{a^3+3abc}{b+c} \geq 2(ab+bc+ca) + \frac{abc}{(\alpha-1)(a+b+c)}\,g(\alpha)$$

If $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 10$ we obtain

$$\sum \frac{a^3+3abc}{b+c} \geq 2(ab+bc+ca) + \frac{35}{48}\,\frac{abc}{a+b+c}.$$

Problem 4. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^2(b+c)^2}{b^2+c^2}+\frac{b^2(c+a)^2}{c^2+a^2}+\frac{c^2(a+b)^2}{a^2+b^2}\geq 2(ab+bc+ca).$$

Solution. We have $\sum\frac{a^2(b+c)^2}{b^2+c^2}-2\sum ab=\frac{p}{s}f(t)$, $t\in[t_1,t_2]$ with

$$f(t)=\frac{(\alpha^2-2\alpha+5)t^2-(4\alpha^3-4\alpha^2+8\alpha)t+4\alpha^4}{-2t^2+(\alpha^2+4\alpha-1)t-2\alpha^3}-2\alpha,\ t\in[t_1,t_2]\ \text{ and}$$

$$f'(t)=\frac{(\alpha^2-6\alpha+5)\left[(\alpha^2-1)t^2-4\alpha^3t+4\alpha^4\right]}{\left[-2t^2+(\alpha^2+4\alpha-1)t-2\alpha^3\right]^2},\ \forall\, t\in[t_1,t_2]$$

with the roots $t_1'=\dfrac{2\alpha^2}{\alpha-1}$, $t_2'=\dfrac{2\alpha^2}{\alpha+1}$, $t_2'<t_1'$.

We have $t_1'\leq t_3\Longleftrightarrow\alpha>\dfrac{13+\sqrt{97}}{4}\simeq 5,7,$ so if $\alpha\geq 9$.

So since $t_3\leq t_1\leq t\Rightarrow t_2'\leq t_1'\leq t$ or f is increasing on $[t_1,t_2]$, or $f(t)\geq f(t_1)\geq 0$, $(f(t_1)\underset{\text{(WA)}}{\geq}0,\ \forall\,\alpha\geq 9)$.

We obtain the refinement

$$\sum\frac{a^2(b+c)^2}{b^2+c^2}\geq\sum ab+\frac{abc}{a+b+c}f(t_1)$$

If $\sum a\sum\dfrac{1}{a}=10$, then $\sum\dfrac{a^2(b+c)^2}{b^2+c^2}\geq 2\sum ab+\dfrac{abc}{2\sum a}$.

Problem 5. *Let a, b, c be positive real numbers. Prove that*

$$\frac{2a^2+5bc}{(b+c)^2}+\frac{2b^2+5ca}{(c+a)^2}+\frac{2c^2+5ab}{(a+b)^2}\geq\frac{21}{2}$$

Solution. We have $\sum\dfrac{2a^2+5bc}{(b+c)^2}-\dfrac{21}{4}=f(t)$, $t\in[t_3,t_4]$ with

$$f(t)=\frac{1}{(\alpha-1)^2}\left[21+2t^2-8\alpha t+17t+4\alpha^2-13\alpha+\frac{5\alpha^3}{t}\right]-\frac{21}{4}\quad\text{and}$$

$$f(t)=\frac{1}{t^2(\alpha-1)^2}\left[4t^3+(17-8\alpha)t^2-5\alpha^3\right],\ t\in[t_3,t_4]$$

We have

$$4t^3-(8\alpha-17)t^2-5\alpha^3=t^2(4t-8\alpha+17)-5\alpha^3\geq$$
$$\geq t_1^2(44-8\alpha+17)-5\alpha^3\geq 0 \quad \text{(WA)}.$$

It follows that f is increasing on $[t_1,t_2]$ and $f(t)\geq f(t_1)>0$ (WA).

We obtain $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)=10$. Then $\sum\frac{2a^2+5bc}{(a+c)^2}\geq\frac{43}{8}$.

Problem 6. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^2+2bc}{(b+c)^2}+\frac{b^2+2ca}{(c+a)^2}+\frac{c^2+2ab}{(a+b)^2}\geq\frac{9}{4}.$$

*(*V. Cârtoaje*)*

Solution. We have for all $t\in[t_3,t_4]$,

$$\sum\frac{a^2+2bc}{(b+c)^2}-\frac{9}{4}=$$
$$=\frac{1}{(\alpha-1)^2}\left(9+t^2-4\alpha t+8t+2\alpha^2-6\alpha+\frac{2\alpha^3}{t}\right)-\frac{9}{4}=f(t),$$

with $f'(t)=\dfrac{2t^3+(8-4\alpha)t-2\alpha^3}{t^2(\alpha-1)^2}$, $\forall\, t\in[t_3,t_4]$.

The function $g(t)=2t^3+(8-4\alpha)t-2\alpha^3$ with $g'(t)\geq 6(4\alpha-9)^2-4\alpha+8>0$, $\forall\,\alpha>19/8$. So g is increasing or $g(t)\geq g(4\alpha-3)>0$, $\forall\,\alpha>3{,}077$. So f is increasing on $[t_3,t_4]$. We obtain the refinement

$$\sum\frac{a^2+2bc}{(b+c)^2}\geq\frac{9}{4}+\frac{(\alpha-9)^2(4\alpha-7)}{4(\alpha-1)^2(4\alpha-9)},$$

where $\alpha=\sum a\sum\frac{1}{a}$.

Problem 7. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^2+2bc}{b+c}+\frac{b^2+2ca}{c+a}+\frac{c^2+2ab}{a+b}\geq\frac{3}{2}(a+b+c).$$

*(*V. Cârtoaje*)*

Solution. We have $\sum \dfrac{a^2+2bc}{b+c} - \dfrac{3}{2}\sum a = sf(t),\ t \in [t_1, t_2]$ with $f(t) = \dfrac{1}{(\alpha-1)t}\left[t^2+(6-3\alpha)t+2\alpha^2\right] - \dfrac{3}{2},\ t \in [t_1, t_2]$ and $f'(t) = \dfrac{t^2-2\alpha^2}{(\alpha-1)t^2}$ with the roots $t'_{1,2} = \pm\alpha\sqrt{2}$ $(t'_1 < t'_2)$. Since $t'_2 < t_3 < t_1 \le t \le t_2$ it follows that f is increasing on $[t_1, t_2]$ or $f(t) \ge f(t_1) \ge 0$ (WA). We obtain

$$\sum \frac{a^2+2bc}{b+c} \ge \frac{3}{2}\sum a + \sum a g(\alpha) \text{ where } \alpha = \sum a \sum \frac{1}{a} \text{ and}$$

$$g(\alpha) = (\alpha-1)(\alpha^2+18\alpha-27-\sqrt{(\alpha-1)(\alpha-9)^3}\big(\alpha^4-14\alpha^3+40\alpha^2+$$
$$+54\alpha+27-(\alpha^2+3)\sqrt{(\alpha-1)(\alpha-9)^3}\big) \ge 0,\ \forall\,\alpha \ge 9$$

Also if $\sum a \sum \dfrac{1}{a} = 10$, then $\sum \dfrac{a^2+2bc}{b+c} \ge \dfrac{3}{2}\sum a + \dfrac{\sum a}{60}$.

Problem 8. *Let a, b, c be positive real numbers. Prove that*

$$\sum \frac{ab+4bc+ca}{b^2+c^2} \ge 4.$$

Solution. We have $\sum \dfrac{ab+4bc+ca}{b^2+c^2} - 4 = f(t),\ t \in [t_3, t_4]$ where

$$f(t) = \frac{(\alpha+3)t^2-(4\alpha^2+17\alpha-9)t+8\alpha^3}{-2t^2+(\alpha^2+4\alpha-1)t-2\alpha^3} - 4,\ \forall\, t \in [t_3, t_4] \quad \text{and}$$

$$f'(t) = \frac{(\alpha-5)\left[(\alpha^2+4\alpha-3)t^2-4\alpha^3 t+2\alpha^3\right]}{\left[2t^2-(\alpha^2+4\alpha-1)t+2\alpha^3\right]},\ t \in [t_3, t_4]$$

Since

$$(\alpha^2+4\alpha-3)t^2-4\alpha^3 t+2\alpha^3 = t\left[(\alpha^2+4\alpha-3)t-4\alpha^3\right]+2\alpha^3 >$$
$$> t_3^2(\alpha^2+4\alpha-3)-4\alpha^3 t_3+2\alpha^3 > 0, \quad \forall\,\alpha > 6,238$$

So f is increasing on $[t_3, t_4]$ or $f(t) \ge f(t_3) > 0$, $\forall\,\alpha \ge 9$.

We obtain the refinement

$$\sum \frac{ab+4bc+ca}{b^2+c^2} \ge 4+\frac{44\alpha^2-362\alpha+774}{2\alpha^3-25\alpha^2+104\alpha-153} \ge 4,\ \alpha = \sum a \sum \frac{1}{a}.$$

Problem 9. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{1}{(a+b)^2}\geq\frac{9}{4(ab+bc+ca)}.$$

*(*Iran *1996)*

Solution. We have $\sum\dfrac{1}{(b+c)^2}-\dfrac{9}{4\sum ab}=\dfrac{s}{p}f(t)$, $t\in[t_3,t_4]$ with

$$f(t)=\frac{t-2\alpha+4+\alpha^2/t}{(\alpha-1)^2}-\frac{9}{4\alpha},\quad\forall\, t\in[t_3,t_4]\quad\text{and}$$

$$f'(t)=\frac{(t-\alpha)(t+\alpha)}{(\alpha-1)^2t^2}\quad\text{with positive root}\quad t_1'=\alpha$$

Since $t_1'<t_3$ it follows that f is increasing on $[t_3,t_4]$.

So $f(t)>f(t_3)=\dfrac{(\alpha-9)^2}{4(\alpha-1)\alpha(4\alpha-9)}\geq 0$. We obtain the refinement

$$\sum\frac{1}{(b+c)^2}\geq\frac{9}{4\sum ab}+\frac{(a+b+c)}{abc}\,\frac{(\alpha-9)^2}{4(\alpha-1)^2\alpha(\alpha-9)}$$

where $\alpha=\sum a\sum\dfrac{1}{a}$.

Problem 10. *Let a, b, c be positive real numbers. Prove that*

$$\frac{ab-2bc+ca}{b^2-bc+c^2}+\frac{bc-2ca+ab}{c^2-ca+a^2}++\frac{ca-2ab+bc}{a^2-ab+b^2}\geq 0.$$

*(*V. Cârtoaje*)*

Solution. We have $\sum\dfrac{ab-2bc+ca}{b^2-bc+c^2}=f(t)$, $\forall\, t\in[t_3,t_4]$ with

$$f(t)=\frac{(\alpha-3)t^2-(5\alpha^2-19\alpha+36)t+4\alpha^3}{-3t^2+(\alpha^2+10\alpha-8)t-3\alpha^3},\ \forall\ \in[t_3,t_4]\quad\text{with}$$

$$f'(t)=\frac{(\alpha-7)(\alpha^2-\alpha+12)t^2-6\alpha^3t+11\alpha^4-20\alpha^3}{(-3t^2+(\alpha^2+10\alpha-8)t-3\alpha^3)^2},\ \forall\, t\in[t_3,t_4]$$

Since $(\alpha^2-\alpha+12)t^2-6\alpha^3t+11\alpha^4-20\alpha^3=0$, $\Delta=-\alpha^3(\alpha-4)^2(2\alpha-15)<0$.

It follows $f'(t)>0$, $\forall\, t\in[t_3,t_4]$ so f is increasing on $[t_3,t_4]$ and $f(t)\geq\dfrac{(\alpha-9)^2}{\alpha^3-17\alpha^2+94\alpha-171}=f(t_3)$ with refinement

$$\sum\frac{ab-2bc+ca}{b^2-bc+c^2}\geq\frac{(\alpha-9)^2}{\alpha^3-17\alpha^2+94\alpha-171},\quad \alpha=\sum a\sum\frac{1}{a}$$

Also since $[t_1,t_2]\subset[t_3,t_4]$ it follows that f is increasing on $[t_1,t_2]$ or
$f(t)\geq f(t_1)$. So

$$\sum\frac{ab-2bc+ca}{b^2-bc+c^2}\geq\frac{1}{\alpha}\left[\frac{(\alpha-3)t^2-(5\alpha^2-19\alpha+36)t+4\alpha^3}{-3t^2+(\alpha^2+10\alpha-8)t-3\alpha^3}\right]$$

Also if $\sum a\sum\frac{1}{a}=10$, we have $\sum\dfrac{ab-2bc+ca}{b^2-bc+c^2}\geq\dfrac{1}{3}$.

Problem 11. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^2}{(2a+b)(2a+c)}+\frac{b^2}{(2b+c)(2b+a)}+\frac{c^2}{(2c+a)(2c+b)}\leq\frac{1}{3}$$

(Tigran Sloyan)

Solution. We have $\sum\dfrac{a^2}{(2a+b)(2a+c)}-\dfrac{1}{3}=f(t)$, $t\in[t_3,t_4]$ with

$$f(t)=\frac{t^2+(27+2\alpha^2-9\alpha)t+4\alpha^3}{4t^2+(8\alpha^2-18\alpha+27)t+4\alpha^3}-\frac{1}{3},\quad t\in[t_3,t_4]\quad\text{and}$$
$$f'(t)=\frac{3(3t+2\alpha^2)\left[(2\alpha-9)t-4\alpha^3+6\alpha^2\right]}{\left[4t^2+(8\alpha^2-18\alpha+27)t+4\alpha^3\right]},\quad t\in[t_3,t_4]$$

We have $(2\alpha-9)t-4\alpha^3+6\alpha^2\leq(2\alpha-9)t_4-4\alpha^3+6\alpha^2<0$, $\forall\,\alpha>\dfrac{9}{4}$.

So f is decreasing on $[t_3,t_4]$ or $f(t)\leq f(t_3)\leq 0$.

We obtain the refinement

$$\sum \frac{a}{(2a+b)(2b+c)} \leq \frac{1}{3} - \frac{(\alpha-9)(34\alpha-63)}{3(36\alpha^2-80\alpha^2-18\alpha+81)} \leq 0$$

Since $[t_1, t_2] \subset [t_3, t_4]$ and f is decreasing on $[t_1, t_2]$ we obtain $f(t) \leq f(t_3) \leq 0$. So $\sum \frac{a}{(2a+b)(2a+c)} \leq \frac{1}{3}+f(t_1) \leq \frac{1}{3}+f(t_3) \leq \frac{1}{3}$.

We obtain $f(t) \leq f(t_1)$, so $\sum \frac{a}{(2a+b)(2a+c)} \leq \frac{1}{3} + f(t_1) \leq \frac{1}{3} + f(t_3) \leq \frac{1}{3}$.

If $\sum a \sum \frac{1}{a} = 10$, then $\sum \frac{a}{(2a+b)(2a+c)} \leq \frac{1}{3} - \frac{1}{240} = \frac{79}{240}$.

Problem 12. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{5(a^2+b^2)-ab}+\frac{1}{5(b^2+c^2)-bc}+\frac{1}{5(c^2+a^2)-ca} \geq \frac{1}{a^2+b^2+c^2}.$$

(V. Cârtoaje)

Solution. We have $\sum \frac{1}{5(a^2+b^2)-ab} - \frac{1}{\sum a^2} = \frac{s}{p} f(t)$, $t \in [t_3, t_4]$ with

$$f(t) = \frac{25t^2-(105\alpha+54)t+135\alpha^2}{-275t^2+(125\alpha^2+630\alpha-216)t-275\alpha^3} - \frac{1}{t-2\alpha} \quad \text{and}$$

$$f'(t) = \frac{5(5\alpha-27)(6\alpha-5t)(80\alpha^2-36\alpha-(25\alpha+30)t)}{\left[-275t^2+(125\alpha^2+630\alpha-216)t-27+\alpha^3\right]^2} + \frac{1}{(t-\alpha)^2},$$
$$\forall\, t \in [t_3, t_4]$$

We have $5t \geq 5(4\alpha-9) = 20\alpha-45 \geq 6\alpha$, true since $\alpha \geq 9$.

Also $(25\alpha+30)t \geq (25\alpha+30)t_3 \geq 80\alpha^2-36\alpha$ if $\alpha \geq 5,78$, $t \in [t_3, t_4]$.

We obtain $80\alpha^2 - 36\alpha - (25\alpha + 30)t < 0$. So $f'(t) \geq 0$, $\forall\, t \in [t_3, t_4]$ or $f(t) \geq f(t_3) \geq 0$, $\forall\, t \in [t_3, t_4]$. We obtain the refinement

$$\sum \frac{1}{5(a^2+b^2)-c^2} \geq$$
$$\geq \frac{1}{\sum a^2} + \frac{\sum a}{abc} \cdot \frac{(\alpha-9)(5\alpha^2-127\alpha+252)}{(2\alpha-9)(225\alpha^2-3005\alpha^2+13266\alpha-20331)},$$

where $\alpha = \sum a \sum \frac{1}{a}$.

Remark. Since $[t_1, t_2] \subset [t_3, t_4]$ it follows that f is increasing on $[t_1, t_2]$. So $f(t) \geq f(t_1)$.

If $\alpha = 10$, $t_1 = \dfrac{125}{4}$. So $f(t) \geq f\left(\dfrac{125}{4}\right) = \dfrac{472481}{3021265}$.

If $\sum a \sum \dfrac{1}{a} = 10$, $\quad \sum \dfrac{1}{5(a^2+b^2)-ab} \geq \dfrac{1}{\sum a^2} + \sum a \dfrac{472481}{3021265}$.

Problem 13. *Let a, b, c be positive real numbers. Prove that*

$$\frac{4a^2-b^2-c^2}{a(b+c)} + \frac{4b^2-c^2-a^2}{b(c+a)} + \frac{4c^2-a^2-b^2}{c(a+b)} \leq 3.$$

Solution. We have $\sum \dfrac{4a^2-b^2-c^2}{a(b+c)} - 3 = f(t)$, $t \in [t_3, t_4]$ where $f(t) = \dfrac{15 + 4t - \alpha^2 - 8\alpha + \frac{2\alpha^3}{t}}{\alpha - 1} - 3$, $\quad f'(t) = \dfrac{2(2t^2-\alpha^3)}{(\alpha-1)t^2}$, $t \in [t_3, t_4]$.

We distinguish the cases:

Case 1. $t_1' < t_3 < t_2' < t_4$, if $\alpha > 26{,}86$.

In this case $f(t) \leq f(t_4) = -\dfrac{3\alpha(\alpha-9)}{(\alpha-1)(4\alpha-9)}$.

Case 2. $\alpha \in (26,86,\ +\infty)$

We have $$f(t) \le \max\{f(t_3), f(t_4)\} =$$
$$= \max\left\{\frac{-3(\alpha-9)\alpha}{(\alpha-1)(4\alpha-9)}, \frac{-(\alpha-9)(2\alpha^2-11\alpha+18)}{(\alpha-1)(4\alpha-9)}\right\} = \frac{-3(\alpha-9)\alpha}{(\alpha-1)(4\alpha-9)}.$$

We obtain the refinement

$$\sum \frac{4a^2-b^2-c^2}{a(b+c)} \le 3 - \frac{3\alpha(\alpha-9)}{(\alpha-1)(4\alpha-9)}, \text{ where } \alpha = \sum a \sum \frac{1}{a}.$$

Problem 14. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \ge \frac{3(a^3+b^3+c^3)}{2(a^2+b^2+c^2)}.$$

Solution. We have $\sum \frac{a^2}{b+c} - \frac{3\sum a^3}{2\sum a^2} = sf(t),\ t \in [t_3, t_4]$ with

$$f(t) = \frac{t-3\alpha-4}{\alpha-1} - \frac{3(t-3\alpha+3)}{3(t-2\alpha)},\ f'(t) = \frac{2t^2-8\alpha t+9\alpha^2+12\alpha-9}{2(\alpha-1)(2\alpha-t)^2}$$

with roots $t'_{1,2} = \dfrac{4\alpha \pm \sqrt{6(\alpha^2-4\alpha+3)}}{2}$ $(t'_1 < t_2)$.

We have $t'_1 < t'_2 < t_3 < t_4$, $\forall \alpha > 8,32$ so f is increasing on $[t_3, t_4]$ or $f(t) \ge f(t_3) = \dfrac{(\alpha-8)(\alpha-9)}{2(\alpha-1)(2\alpha-9)}$. We obtain the refinement

$$\sum \frac{a^2}{b+c} \ge \frac{3\sum a^3}{2\sum a^2} + (a+b+c)\frac{(\alpha-8)(\alpha-9)}{2(\alpha-1)(\alpha-9)} \text{ where } \alpha = \sum a \sum \frac{1}{a}.$$

Since $[t_1, t_2] \subset [t_3, t_4]$ it follows that $f(t) \ge f(t_1)$. So

$$\sum \frac{a^2}{b+c} \ge \frac{3\sum a^3}{2\sum a^2} + (a+b+c)f(t_1) \quad \text{or}$$
$$\sum \frac{a^2}{b+c} \ge \frac{3\sum a^3}{2\sum a^2} + \frac{\sum a}{60} \quad \text{if } \alpha = 10.$$

Problem 15. *Let a, b, c be positive real numbers. Prove that*

$$\frac{ab-bc+ca}{b^2+c^2}+\frac{bc-ca+ab}{c^2+a^2}+\frac{ca-ab+bc}{a^2+b^2}\geq\frac{3}{2}.$$

Solution. We have $\sum\dfrac{ab-bc+ca}{b^2+c^2}-\dfrac{3}{2}=f(t)$ where

$$f(t)=\frac{(\alpha-2)t^2-(4\alpha^2-8\alpha+6)t+3\alpha^3}{-2t^2+(\alpha^2+4\alpha-1)t-2\alpha^3}-\frac{3}{2},\quad t\in[t_3,t_4]\quad\text{and}$$

$$f'(t)=\frac{(\alpha-5)\left[(\alpha^2-\alpha+2)t^2-4\alpha^3t+5\alpha^4-3\alpha^3\right]}{\left[2t^2-(\alpha^2+4\alpha-1)t+2\alpha^3\right]^2},\quad t\in[t_3,t_4]$$

with $\Delta=4(6-\alpha)\alpha^3(\alpha-1)^2<0$. So $f'(t)>0$, $\forall\, t\in[t_3,t_4]$ or $f(t)\geq f(t_3)=\dfrac{3(\alpha-9)^2}{2(2\alpha^3-25\alpha^2+104\alpha-153)}$, $\forall\,\alpha\geq 9$, so

$$\sum\frac{ab-bc+ca}{b^2+c^2}\geq\frac{3}{2}+\frac{3(\alpha-9)^2}{2(2\alpha^3-25\alpha^2+104\alpha-153)}\geq 0$$

Problem 16. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a(b+c)}{b^2+bc+c^2}+\frac{b(c+a)}{c^2+ac+a^2}+\frac{c(a+b)}{a^2+ab+b^2}\geq 2.$$

Solution. We have $\sum\dfrac{a(b+c)}{b^2+c^2+bc}-2=f(t)$ where

$$f(t)=\frac{(\alpha-1)t^2-(3\alpha^2-3\alpha)t+2\alpha^3}{-t^2+\alpha^2t-\alpha^3}-2,\ t\in[t_3,t_4]\quad\text{and}$$

$$f'(t)=\frac{\alpha(\alpha-3)\left[(\alpha-1)t^2-2\alpha^2t+\alpha^3\right]}{(t^2-\alpha^2t+\alpha^3)^2},\ t\in[t_3,t_4]\ \text{with roots}$$

$$t'_{1,2}=\frac{\alpha^3-3\alpha^2\pm\sqrt{\alpha^5-6\alpha^4+9\alpha^3}}{\alpha^2-4\alpha+3},\ (t'_1\leq t'_2)$$

We have $t'_2<t_3$ if $\alpha>3$. So $t'_1<t'_2<t_3\leq t_4$.

So f is increasing on $[t_3,t_4]$ or $f(t)\geq f(t_3)$. So

$$\sum\frac{a(b+c)}{b^2+bc+c^2}\geq 2+\frac{(\alpha-9)^2}{3\alpha^3-25\alpha^2+72\alpha-81}\geq 2,\ \text{where}\ \alpha=\sum a\sum\frac{1}{a}.$$

Problem 17. *Let a, b, c be positive real numbers. Prove that*

$$\frac{b+c}{2a^2+bc}+\frac{c+a}{2b^2+ca}+\frac{a+b}{2c^2+ab}\le\frac{6}{a+b+c}.$$

(V. Cârtoaje, M.S. 2006)

Solution. We have $\sum\frac{b+c}{2a^2+bc}-\frac{6}{\sum a}=\frac{1}{s}f(t)$ where

$$f(t)=\frac{(2\alpha-6)t^2-2\alpha^2t}{2t^2+(27-18\alpha)t+4\alpha^3}-6,\ t\in[t_3,t_4]\quad\text{and}$$

$$f'(t)=-\frac{2(16\alpha^2-81\alpha+81)t^2+(24\alpha^3-8\alpha^4)t+4\alpha^5}{(2t^2+(27-18\alpha)t+4\alpha^3)^2},\ t\in[t_3,t_4]$$

with roots $t'_{1,2}=\dfrac{2\left(2\alpha^4-6\alpha^3\pm\sqrt{(\alpha-1)\alpha^5(2\alpha-9)^2}\right)}{16\alpha^2-81\alpha+81}$ $(t'_1\le t'_2)$.

We have $t'_2>t_4,\ \forall\,\alpha>4,9$. It follows that $t'_1<t_3\le t_4<t'_2$ and since $16\alpha^2-81\alpha+81>0$ if $\alpha>3,69$ we obtain that f is increasing on $[t_3,t_4]$ or $f(t)\ge f(t_3)=\dfrac{18\alpha(\alpha-9)}{4\alpha^3-40\alpha^2+126\alpha-81}$.
We obtain the refinement

$$\sum\frac{b+c}{2a^2+bc}\ge\frac{6}{a+b+c}+\frac{18\alpha(\alpha-9)}{(4\alpha^3-40\alpha^2+126\alpha-81)\sum a},$$

where $\alpha=\sum a\sum\frac{1}{a}$.

Problem 18. *Let a, b, c be positive real numbers. Prove that*

$$\frac{(b+c)^2}{a^2+bc}+\frac{(c+a)^2}{b^2+ca}+\frac{(a+b)^2}{c^2+ab}\ge 6.$$

(Peter Scolze, Darij Grinberg)

Solution. We have $\sum\frac{(b+c)^2}{a^2+bc}-6=f(t)$ where

$$f(t)=\frac{t^2(\alpha+1)+(12-3\alpha^2-5\alpha)t+2\alpha^3}{t^2+(8-6\alpha)t+\alpha^3}-6,\ t\in[t_3,t_4]\quad\text{and}$$

$$f'(t)=\frac{(\alpha-1)\left[(3\alpha-4)t^2-2\alpha^3t+3\alpha^4-4\alpha^3\right]}{\left(t^2+(8-6\alpha)t+\alpha^3\right)^2},\ t\in[t_3,t_4]$$

with roots $t'_{1,2} = \dfrac{\alpha^3 \pm \sqrt{(\alpha-4)^2(\alpha-1)\alpha^3}}{3\alpha-4}$ $(t'_1 < t'_2)$. We have $t_4 < t'_2$, $\forall\, \alpha > 4,36982$. We obtain $t'_1 < t_3 < t_4 < t'_2$, $\forall\, \alpha \geq 3$.

So f is decreasing on $[t_3, t_4]$ or $f(t) \geq f(t_4)$, so

$$\sum \frac{(b+c)^2}{a^2+bc} \geq 6+\frac{\alpha(\alpha-3)(\alpha-5)(\alpha-9)}{\alpha^3-8\alpha^2+14\alpha+9} \geq 6, \text{ where } \alpha = \sum a \sum \frac{1}{a}.$$

Problem 19. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab} \geq \frac{2}{ab+bc+ca}.$$

(V. Cârtoaje)

Solution. We have $\sum \dfrac{1}{a^2+2bc} - \dfrac{2}{\sum ab} = \dfrac{s}{p} f(t)$ where

$$f(t) = \frac{2\alpha t - 3\alpha^2}{4t^2+(27-18\alpha)t+2\alpha^3} - \frac{2}{\alpha},\ t \in [t_3, t_4]$$

$$f'(t) = \frac{\alpha(-8t^2+24\alpha t+4\alpha^3-54\alpha^2+81\alpha)}{(4t^2+(27-18\alpha)t+2\alpha^3)^2},\ t \in [t_3, t_4]$$

with roots $t'_{1,2} = \dfrac{1}{4}\Big(6\alpha \pm \sqrt{2\alpha(2\alpha-9)^2}\Big)$ $(t'_1 \leq t'_2)$.

We have $t'_2 < t_3$ if $\alpha \in (9; 14,783)$ and $t_3 < t'_2 < t_4$ if $\alpha > 14,783$.

Also $t'_1 < t_3$ if $\alpha > 2,73$. We obtain:

Case 1. $\alpha \in [9; 14,783)$.

We have $t'_1 < t'_2 < t_3 < t_4$. In this case f decreasing on $[t_3, t_4]$ or

$$f(t) \geq f(t_4) = \frac{14\alpha^2-36\alpha-81}{\alpha(4\alpha^3-40\alpha^2+126\alpha-81)} > 0 \quad \text{if } \alpha > 4,013$$

Case 2. $\alpha > 14,783$.

We get $t_1' < t_3 < t_2' < t_4$. So $f(t) \geq \min\{f(t_3), f(t_4)\} = f(t_4) > 0$.

We obtain the refinement

$$\sum \frac{1}{a^2+2bc} - \frac{2}{\sum ab} \geq \frac{14\alpha^2 - 36\alpha - 81}{\alpha(4\alpha^3 - 40\alpha^2 + 126\alpha - 81)},$$

where $\alpha = \sum a \sum \frac{1}{a}$.

Problem 20. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ac+c^2} \geq \frac{9}{(a+b+c)^2}.$$

(V. Cârtoaje)

Solution. We have

$$\sum \frac{1}{a^2+ab+b^2} - \frac{9}{(a+b+c)^2} = \frac{s}{p}\,\frac{f(t)}{t(-t^2+\alpha^2 t - \alpha^3)} \quad \text{where}$$

$$f(t) = t^3 + (9-3\alpha)t^2 - 6\alpha^2 t + 9\alpha^3,\ t \in [t_3, t_4] \quad \text{and}$$
$$f'(t) = 3t^2 + (18-6\alpha)t - 6\alpha^2,\ t \in [t_3, t_4]$$

with roots $t_{1,2}' = \alpha - 3 \pm \sqrt{2\alpha^3 + \alpha^2 - 6\alpha + 9}$ $(t_1' \leq t_2')$. We have $t_2' \geq t_3$, $\forall\, \alpha > -246$, $t_1' \leq t_3$ and $t_2' \leq t_4$ if $\alpha > 34,8$ and $t_4 \leq t_2'$ if $\alpha \in [9,\ 34,8)$. We distinguish the cases:

Case 1. $\alpha > 34,83$ or $t_1' \leq t_3 \leq t_2' \leq t_4$.

So $f(t) \geq f(t_2) > 0$ since (WA) $f(t_2) > 0$, $\forall\, \alpha > 0$.

Case 2. $\alpha \in [9,\ 34,83...)$.

In this case $t_1' \leq t_3 \leq t_4 \leq t_2'$ so f is decreasing on $[t_3, t_4]$ or $f(t) \geq f(t_4)$. But $f(t_4) \geq 0$ (WA), $\forall\, \alpha \geq 0$. So $f(t) \geq 0$, $\forall\, \alpha \geq 9$, $t \in [t_3, t_4]$.

It remain to prove that $t^2 - \alpha^2 t + \alpha^3 < 0$, $\forall\, t \in [t_3, t_4]$ or $(t - t_3')(t - t_4') < 0$ where $t_{3,4}' = \dfrac{\alpha^2 \pm \sqrt{\alpha^4 - 4\alpha^3}}{2}$ $(t_3' < t_4')$.

We have $t_1' < t_3 \leq t_4 < t_2'$, $\forall\, \alpha > 9$ so $t^2 - \alpha^2 t + \alpha^3 < 0$ $t \in [t_3, t_4]$ from which it follows the statement.

Problem 21. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^3+3abc}{(b+c)^2}+\frac{b^3+3abc}{(c+a)^2}+\frac{c^3+3abc}{(a+b)^2}\geq a+b+c.$$

Solution. We have $\sum\frac{a^3+3abc}{(b+c)^2}-\sum a=\frac{s}{(\alpha-1)t}f(t)$, where

$f(t)=(18+4\alpha^2-15\alpha)t+t^3+(10-5\alpha)t^2+3\alpha^2,\ t\in[t_3,t_4]$ and with $f'(t)=3t^2+(20-10\alpha)t+4\alpha^2-15\alpha+18,$

with roots $t'_{1,2}=\frac{1}{3}\left(5\alpha-10\pm\sqrt{13\alpha^2-55\alpha+46}\right)$ $(t_1<t'_2)$.

But $t'_2<t_3$ if $\alpha\geq 3,08$. So $t'_1<t'_2<t_3<t_4$, so f is increasing on $[t_3,t_4]\subset[t_1,t_2]$ or $f(t)>f(t_1)$. But $f(t_1)>0$, $\forall\,\alpha>0$, from which it follows the statement.

Problem 22. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^2(b+c)}{b^2+c^2}+\frac{b^2(c+a)}{c^2+a^2}+\frac{c^2(a+b)}{a^2+b^2}\geq a+b+c.$$

*(*Darij Grinberg/*MS 2004)*

Solution. We have $\sum\frac{a^2(b+c)}{b^2+c^2}-\sum a=s\,f(t)$ where

$$f(t)=\frac{(\alpha+1)t^2+(3-6\alpha^2+3\alpha)t+8\alpha^3-10\alpha^2}{-2t^2+(\alpha^2+4\alpha-1)t-2\alpha^3},\ t\in[t_3,t_4]\quad\text{with}$$

$$f'(t)=\frac{(\alpha-5)\left[(\alpha^2-2\alpha-1)t^2-(4\alpha^3-8\alpha^2)t+4\alpha^4-8\alpha^3+2\alpha^2\right]}{\left[2t^2-(\alpha^2+4\alpha-1)t+2\alpha^3\right]^2},$$

$t\in[t_3,t_4]$

with roots $t'_{1,2}=\frac{2\alpha^3-4\alpha^2\pm\sqrt{2}(\alpha^2-\alpha)}{\alpha^2-2\alpha-1}$ $(t'_1<t'_2)$.

Since $t'_2<t_3$, $\forall\,\alpha>5,72...$, it follows that $t'_1<t'_2<t_3<t_4$, so f is increasing on $[t_3,t_4]$, so on $[t_1,t_2]\subset[t_3,t_4]$ or $f(t)>f(t_1)\geq 0$.

We obtain the refinement

$$\sum \frac{a^2(b+c)}{b^2+c^2} - \sum a^2 \geq (a+b+c)g(\alpha) \geq 0 \text{ where } \alpha = \sum a \sum \frac{1}{a} \text{ and}$$

$$g(\alpha) = \frac{\alpha^4 - 14\alpha^3 + 40\alpha^2 + 54\alpha - 81 - (\alpha^2-3)\sqrt{(\alpha-1)(\alpha-9)^3}}{2(\alpha-5)\left(\alpha^2+18\alpha-27-\sqrt{(\alpha-1)(\alpha-9)^3}\right)}$$

If $\sum a \sum \frac{1}{a} = 10$, then $\sum \frac{a^2(b+c)}{b^2+c^2} - \sum a^2 \geq \frac{42}{625}(a+b+c)$.

Problem 23. *Let a, b, c be positive numbers satisfying the identity $a^2+b^2+c^2=3$. Then*

$$\frac{a^2+b^2}{a+b} + \frac{b^2+c^2}{b+c} + \frac{c^2+a^2}{c+a} \geq 3$$

Solution. We have $\frac{1}{3}\left(\sum \frac{a^2+b^2}{a+b} - \sqrt{3(a^2+b^2+c^2)}\right) = f(t)$ where

$$f(t) = \left(2\alpha - 4 - \frac{2\alpha^2}{t}\right)\frac{1}{\alpha-1} - \sqrt{3\left(1-\frac{2\alpha}{t}\right)}, \ t \in [t_3, t_4], \text{ with}$$

$$f'(t) = \frac{\alpha}{t^2}\left(\frac{2\alpha}{\alpha-1} - \sqrt{\frac{3t}{t-2\alpha}}\right) \text{ with roots } t_1' = \frac{8\alpha^3}{\alpha^2+6\alpha-3} < t_3,$$
$\forall \alpha > 3,55$.

It follows that f is increasing on $[t_3, t_4]$ or $f(t) \geq f(t_3) \geq 0$. So

$$\frac{1}{a+b+c}\left(\sum \frac{a^2+b^2}{a+b} - \sqrt{3(a^2+b^2+c^2)}\right) \geq$$
$$\geq \frac{3}{2} - \frac{29\alpha-45}{2(\alpha-1)(4\alpha-9)} - \sqrt{\frac{3(4\alpha-18)}{2(4\alpha-9)}} \geq 0.$$

Problem 24. *Let a, b, c be positive numbers satisfying the identity $(a+b+c)^3 = 32abc$. Find the* min *and* max *of* $E = \frac{a^4+b^4+c^4}{(a+b+c)^4}$.

*(*Tan Num Dung, *Vietnam 2004)*

Solution. We have $t_1 \le t \le t_2$ and $t = 32$, so $t_1 \le 32 \le t_2$ or $10 \le \alpha \le 5\sqrt{5}-1$. We have $E = f(\alpha)$ where $f(\alpha) = \left(1 - \dfrac{\alpha}{16}\right)^2 + \dfrac{1}{8} - \dfrac{\alpha^2}{512}$, $\alpha \in [10, 5\sqrt{5}-1]$. So

$$\min_{\alpha\in[10,5\sqrt{5}-1]} f(\alpha) = \frac{383-165\sqrt{5}}{256} \quad \text{and} \quad \max_{\alpha\in[10,5\sqrt{5}-1]} = \frac{9}{128}.$$

Problem 25. *Let a, b, c be positive numbers such that $abc = 1$. Then*

$$a^3 + b^3 + c^3 + 15 \ge 6\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

*(*Michael Rosemberg*, MS 2006)*

Solution. After deconditionated we obtain $f(t) \ge 0$ where $f(t) = t - \dfrac{6\alpha}{t^{\frac{1}{3}}} - 3\alpha + 18$, $f'(t) = 1 + \dfrac{2\alpha}{t^{\frac{4}{3}}}$, $t \in [t_1, t_2]$ which is increasing or $f(t) \ge f(t_1)$. But $f(t_1) \ge 0$, $\forall\, \alpha \ge 9$ from which it follows the statement.

Problem 26. *Let a, b, c be positive numbers. Then*

$$\begin{aligned} 0 &\le \frac{16\alpha^2 - 183v + 513}{\alpha^4 - 17\alpha^3 + 94\alpha^2 - 171\alpha} \le \\ &\le \frac{abc}{a+b+c}\left(\sum \frac{1}{a^2 - ab + b^2} - \frac{3}{ab+bc+ca}\right) \le \\ &\le \frac{\alpha^4 - 23\alpha^3 + 200\alpha^2 + 68\alpha - 1080}{\alpha^4 - 17\alpha^3 + 94\alpha^2 - 171\alpha} \quad \text{where } \alpha = \sum a \sum \frac{1}{a}. \end{aligned}$$

Solution. We have $\sum \dfrac{1}{a^2 - ab + b^2} - \dfrac{3}{\sum ab} = \dfrac{s}{p} f(t)$ where

$$f(t) = \frac{t^2 - (5\alpha + 2)t + 7\alpha^2}{-3t^2 + (\alpha^2 + 10\alpha - 8)t - 3\alpha^3} - \frac{3}{\alpha},\ t \in [t_3, t_4] \quad \text{and}$$

$$f'(t) = \frac{(\alpha - 7)\left[(\alpha + 2)t^2 - 6\alpha^2 t + 8\alpha^3 - 3\alpha^2\right]}{(-3t^2 + (\alpha^2 + 10\alpha - 8)t - 3\alpha^3)^2},\ t \in [t_3, t_4]$$

with roots $t_1' = 2\alpha$, $t_2' = \dfrac{4(\alpha^2-\alpha)}{\alpha+2}$. We have $2\alpha < \dfrac{4(\alpha^2-\alpha)}{\alpha+2} < t_3$ if $\alpha > 6$.

So $t_1' < t_2' < t_3 < t_4$. So f is increasing on $[t_3, t_4]$ or $f(t_3) \leq f(t) \leq f(t_4)$, from which it follows the statement.

Problem 27. *Let a, b, c be positive numbers and $\alpha = \sum a \sum \frac{1}{a}$. Then*

$$2 < \frac{2\alpha^3 - 13\alpha^2 + 33\alpha - 54}{\alpha^3 - 8\alpha^2 + 14\alpha + 9} \leq$$
$$\leq \frac{a(b+c)}{a^2+bc} + \frac{b(b+c)}{b^2+ca} + \frac{c(c+a)}{a^2+ab} \leq \frac{\alpha - 3\sqrt{\alpha} + 6}{2\left(\sqrt{\alpha}-2\right)}.$$

Solution. We have $\sum \dfrac{a(b+c)}{a^2+bc} = f(t)$ with

$$f(t) = \frac{(\alpha^2 + 7\alpha - 12)t - 2t^2 - 2\alpha^3}{t^2 + (8-6\alpha)t + \alpha^3}$$
$$f'(t) = \frac{(\alpha^3 - t^2)(\alpha^2 - 5\alpha + 4)}{\left[t^2 + (8-6\alpha)t + \alpha^3\right]^2}, \quad t \in [t_3, t_4]$$

It results that $\min\{f(t_3), f(t_4)\} \leq f(t) \leq f\left(\sqrt{\alpha^3}\right)$.

But $\dfrac{2\alpha^3 - 13\alpha^2 + 33\alpha - 54}{\alpha^3 - 8\alpha^2 + 14\alpha + 9} > 2$, $\forall\, \alpha > 4{,}13$, from which it follows the statement.

Problem 28. *Let a, b, c be positive numbers. Then*

$$\frac{1}{a^2+bc} + \frac{1}{b^2+ca} + \frac{1}{c^2+ab} - \frac{3}{ab+bc+ca} \geq \frac{7\alpha^2 - 24\alpha - 7}{\alpha^3 - 8\alpha^2 + 14\alpha + 9},$$

where $\alpha = \sum a \sum \frac{1}{a}$.

Solution. We have $\sum \dfrac{1}{a^2+bc} - \dfrac{3}{\sum ab} = \dfrac{s}{\alpha p} f(t)$ where

$$f(t) = \frac{-3t^2+(\alpha^2+16\alpha-24)t-4\alpha}{t^2+(8-6\alpha)t+\alpha^3} \quad \text{and}$$

$$f'(t) = \frac{(2-\alpha)t^2+2\alpha^2 t+\alpha^4-8\alpha^3+8\alpha^2}{\left[t^2+(8-6\alpha)t+\alpha^3\right]^2},\ t\in[t_3,t_4]$$

with roots $t'_{1,2} = \dfrac{\alpha^2 \pm \sqrt{(\alpha-4)^2\alpha^2(\alpha-1)}}{\alpha-2}$ $(t'_1 < t'_2)$. We have $t_3 < t'_2 < t_4$ if $\alpha > 14,01$ and $t_4 < t'_2$ if $\alpha \in [9,\ 14,01]$, $t'_1 < t_3$, $\forall\, \alpha > 4,13$.

We distinguish

Case 1. $\alpha > 14,01$. We have

$$f(t) \geq \min\left\{f(t_3), f(t_4)\right\} = \begin{cases} \dfrac{7\alpha^2-24\alpha-7}{\alpha^3-8\alpha^2+14\alpha+9}, & \alpha > 9,58... \\ \dfrac{\alpha^3-9\alpha^2+48\alpha-108}{\alpha^3-8\alpha^2+14\alpha+9}, & \alpha\in[9;9,58...) \end{cases} =$$

$$= \frac{7\alpha^2-24\alpha-7}{\alpha^3-8\alpha^2+14\alpha+9} \ \text{ if } \alpha > 14,012$$

Case 2. $\alpha \in [9,\ 14,01...)$. So $f(t) \geq f(t_3) = \dfrac{7\alpha^2-24\alpha-7}{\alpha^3-8\alpha^2+14\alpha+9}$ from which it follows the statement.

Problem 29. *Let a, b, c be positive numbers. Then*

$$\frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab} \leq \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}.$$

*(*V. Cârtoaje*)*

Solution. We have $\sum \dfrac{a}{a^2+bc} - \sum \dfrac{1}{a+b} = f(t)$ we have

$$f(t) = \frac{1}{s}\frac{g(t)}{h(t)},\quad g(t) = -t^3+(t^3+(4\alpha-7)t^2+(3\alpha^2-6\alpha)t-\alpha^4,$$

$$h(t) = (\alpha-1)(t^2+(8-6\alpha)t+\alpha^3),\ t\in[t_3,t_4].$$

Since $\Delta_h = (8-6\alpha)^2 - 4\alpha^3 = -4(\alpha-1)(\alpha+1)^2 < 0$, we have $h(t) > 0$, $\forall\, t \in [t_3, t_4]$. Also $g'(t) = -3t^2 + (8\alpha - 14)t + 3\alpha^2 - 6\alpha$ with roots $t'_{1,2} = \dfrac{1}{3}\left(4\alpha - 7 \pm \sqrt{25\alpha^2 - 74\alpha + 49}\right)$ $(t'_1 < t'_2)$. We have $t'_2 < t_3$, $\forall\, \alpha > 4,12$.

So g is decreasing on $[t_3, t_4]$ or $g(t) < g(t_3) = -(\alpha-9)(\alpha^3 - 3\alpha^2 - 8\alpha + 18) < 0$.

Also $h'(t) = (\alpha-1)(2t + 8 - 6\alpha) > (\alpha-1)(2\alpha - 10) > 0$ so h is increasing on $[t_3, t_4]$. So $h(t) = \dfrac{1}{s}\dfrac{g(t)}{h(t)} \le \dfrac{1}{s}\dfrac{g(t_3)}{h(t)} \le \dfrac{1}{s}\dfrac{g(t_3)}{h(t_4)}$. So

$$\sum \frac{a}{a^3 + bc} \le$$
$$\le \sum \frac{1}{b+c} - \frac{1}{\sum a} \frac{(\alpha-9)(\alpha^3 - 3\alpha^2 - 8\alpha + 18)(4\alpha - 9)^2}{\alpha^5(\alpha-1)(\alpha^3 - 8\alpha^2 + 14\alpha + 9)} \le 0.$$

Problem 30. *Let a, b, c be positive numbers. Then*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{9(a+b+c)}{ab+bc+ca} \ge 8\left(\frac{a}{a^2+bc} + \frac{b}{b^2+ca} + \frac{c^2}{c^2+ab}\right).$$

(Ngugen Viet Hung / *9471 MR)*

Solution. From Cârtoaje above inequality [1], we have $\sum \dfrac{a^2}{a^2+bc} \le \sum \dfrac{1}{a+b} - A$, where

$$A = \frac{1}{\sum a} \frac{(\alpha-9)(\alpha^3 - 3\alpha^2 - 8\alpha + 18)(4\alpha - 9)^2}{\alpha^3(\alpha-1)(\alpha^3 - 8\alpha^2 + 14\alpha + 9)} \ge 0.$$

We compute

$$\sum \frac{1}{a} + \frac{9\sum a}{\sum ab} - 8\sum \frac{1}{a+b} = \frac{1}{s} f(t) \quad \text{where}$$

$$f(t) = \alpha + \frac{9t}{\alpha} - \frac{8(t+\alpha)}{\alpha - 1}, \quad \text{with} \quad f'(t) = \frac{\alpha - 9}{(\alpha^2 - \alpha)^2} \ge 0,\ \forall\, t \in [t_3, t_4]$$

So f is increasing on $[t_3, t_4]$ or $f(t) \geq f(t_3) = B = \dfrac{(\alpha-9)(\alpha^2+4\alpha-9)}{\alpha(\alpha-1)}$

or $\sum \dfrac{1}{a} + \dfrac{9\sum a}{\sum ab} - 8\sum \dfrac{1}{a+b} \geq B$ or

$$\sum \frac{1}{a} + \frac{9\sum a}{\sum ab} \geq 8\sum \frac{1}{a+b} + B \geq 8\Big(\sum \frac{a}{a^2+bc} + A\Big) + B =$$
$$= 8\sum \frac{a}{a^2+bc} + 8A + B \geq 8\sum \frac{a}{a^2+bc}$$

So
$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{9\sum a}{\sum ab} \geq 8\sum \frac{a}{a^2+bc} +$$
$$+\frac{1}{\sum a}\,\frac{\alpha-9}{\alpha(\alpha-1)}\left[\alpha^2+4\alpha-9+\frac{(\alpha^3-3\alpha^2-8\alpha+18)(4\alpha-9)^2}{\alpha^2(\alpha^3-8\alpha^2+14\alpha+9)}\right],$$

where $\alpha = \sum a \sum \dfrac{1}{a}$.

References

[1] Cârtoaje, V. *Algebraic inequalities, Old and New Method*. GIL Publishing House, 2006.

[2] Drăgan, M. "A method to solve and refine some algebraic inequalities". *Recreaţii Matematice* XXIII.1 (2021), pp. 14–18.

Mihály Bencze
Str. Hărmanului 6, 505600 Săcele-Négyfalu, Jud. Braşov, Romania
benczemihaly@yahoo.com

Marius Drăgan
61311 bd. Timişoara Nr. 35, Bl. 0D6, Sc. E, et. 7, Ap. 176, Sect. 6, Bucureşti, Romania
marius.dragan2005@yahoo.com

Contests

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to find interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

Problems and solutions from the 63rd edition of the International Mathematical Olympiad (IMO)

Óscar Rivero Salgado

1 Problems and solutions

Below, we present now the four problems which were solved at least by some Spaniard contestant (problems 1, 2, 4 and 5), and include the solutions given to them by our team (in some case slightly modified by the deputy leader). In all the cases, the solutions follow the ideas presented by the contestants, but we have done some little modifications to ease the exposition.

Problem 1. The Bank of Oslo issue two types of coin: aluminium (denoted A) and bronze (denoted B). Marianne has n aluminium coins and n bronze coins, arranged in a row in some arbitrary initial order. A *chain* is any subsequence of consecutive coins of the same type. Given a fixed positive integer $k \leq 2n$, Marianne repeatedly performs the following operations: she identifies the longest chain containing the k-th coin from the left, and moves all coins in that chain to the left end of the row. For example, if $n = 4$ and $k = 4$, the process starting from the ordering $AABBBABA$ would be

$$AAB\underline{B}BABA \to BBB\underline{A}AABA \to AAA\underline{B}BBBA \to BBB\underline{B}AAAA \to \cdots .$$

Find all pairs (n, k) with $1 \leq k \leq 2n$ such that for every initial ordering, at some moment during the process, the leftmost n coins will all be of the same type.

Solution by Javier Badesa. We distinguish four different cases depending on the value of k.

(i) Case $k \leq n-1$. In this case, the arrangement described in the statement is not possible. For instance, put k coins of type A at the beginning of the line, then a coin of type B and finally distribute the remaining ones arbitrarily.

(ii) Case $k > 2n - \lfloor n/2 \rfloor$. Again, this is not possible by considering first a block of $\lfloor n/2 \rfloor$ coins of type A; then a block of $\lfloor n/2 \rfloor$ coins of type B; then a block of $n - \lfloor n/2 \rfloor$ coins of type A; and finally a block of $n - \lfloor n/2 \rfloor$ coins of type B. Observe that in this case the blocks never get merged.

(iii) Case $k = n$. Let m be the minimum number of chains one can get after performing repeatedly the operation of the statement. Since the number of chains is a decreasing quantity, this number m is well defined. If $m = 2$ then we are done, so assume that this is not the case. Number the chains as $C_1, C_2, \ldots, C_m$, and let C_i be the chain containing the position k. Then, by the given conditions, it must be $i > 1$ and $i < m$, so it is an intermediate chain. When we remove it, the chains which are adjacent to it are merged and the number of chains decreases, contradicting the minimality of m.

(iv) Case $k \in [n+1, 2n - \lfloor n/2 \rfloor]$. Use the same notations as in case (iii), and note again that the minimum number of chains, m, must be strictly greater than 2. If m is odd there are two options: either the chain we move is at the end and gets merged with the chain at the beginning, which is of the same color and thus reduces the value of m; or the chain we move is in the middle and then the two chains adjacent to it get merged, reducing again the value of m. The case in which m is even (and then $m \geq 4$) is slightly different: if the chain we move is (in some moment) in the middle, again the two chains adjacent to it get merged and the value of m decreases; however, if it is at the end, the movement of the different chains is cyclic and all the chains arrive to the last position in some moment. By the pigeonhole principle, there exists a chain whose length is at most $\lfloor 2n/4 \rfloor = \lfloor n/2 \rfloor$, and hence the chain we move cannot be in this last block.

Hence, all the possible values are given by $k \in [n, 2n - \lfloor n/2 \rfloor]$.

Problem 2. Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying

$$xf(y) + yf(x) \leq 2.$$

Solution by Roger Lidón. As a piece of notation, fixed a positive real number x, write $y(x)$ for the unique element such that

$$xf(y(x)) + y(x)f(x) \leq 2.$$

We say that x is self-conjugate if $y(x) = x$.

We begin by observing that when x is self-conjugate, then $xf(x) \leq 1$, which is clear from the definition of $y(x)$. Conversely, we claim that if x is not self-conjugate, then $xf(x) > 1$. If this were not the case, $xf(x) + xf(x) \leq 2$, thus contradicting the characterization of $y(x)$ as the unique element satisfying that condition.

We now show that f is strictly decreasing. We proceed by contradiction, assuming that there exists a pair (x, y) such that $x > y$ and $f(x) \geq f(y)$. Then,

$$yf(y(x)) + f(y)y(x) < xf(y(x)) + f(x)y(x) \leq 2.$$

This means that $y(y(x)) = y$, but $y(y(x)) = x$, which is a contradiction.

This is enough to show that all x are self-conjugate. Indeed, if there were a pair of non self-conjugate numbers (x, y), this would mean that

$$2 \geq xf(y) + yf(x) > xf(x) + yf(y) > 1 + 1 = 2,$$

where the second inequality following by noting that $(x-y)(f(y)-f(x)) \geq 0$ and the third one is immediate by the claim that $xf(x) > 1$ for x non self-conjugate.

Consider now two arbitrary real numbers x, y, with $x \neq y$. Then,

$$2 < xf(y) + yf(x) < \frac{x}{y} + yf(x),$$

which may be rewritten as

$$f(x)y^2 - 2y + x > 0.$$

This is a quadratic equation in y; in particular, this means that

$$y \notin \left[\frac{1-\sqrt{1-xf(x)}}{f(x)}, \frac{1+\sqrt{1-xf(x)}}{f(x)}\right].$$

Since $0 < xf(x) \leq 1$, the interval is strictly contained in the set of positive real numbers. However, we know that the inequality is true for all $y \neq x$, and hence the unique possibility is that the interval degenerates to one single point. This happens if and only if $xf(x) = 1$, and in that case the unique point of the interval is precisely x.

This leaves as the unique possible candidate $f(x) = \frac{1}{x}$, and it is immediate to check that this function satisfies the condition, because

$$\frac{x}{y} + \frac{y}{x} = \frac{x^2 + y^2}{xy} \geq 2,$$

and equality holds if and only if $x = y$. This is because $(x-y)^2 \geq 0$ and $(x-y)^2 = 0$ if and only if $x = y$.

Problem 4. Let $ABCDE$ be a convex pentagon such that $BC = DE$. Assume that there is a point T inside $ABCDE$ with $TB = TD$, $TC = TE$ and $\angle ABT = \angle TEA$. Let line AB intersect lines CD and CT at points P and Q, respectively. Assume that the points P, B, A, Q occur on their line on that order. Let line AE intersect lines CD and DT at points R and S, respectively. Assume that the points R, E, A, S occur on their line in that order. Prove that the points P, S, Q, R lie on a circle.

Solution by Álvaro Gamboa. We use directed angles along this solution. Let $X = DT \cap AB$ and $Y = CT \cap AE$. We begin by observing that triangles TDE and TBC are congruent, since $TB = TD$, $TC = TE$ and $BC = DE$, according to the statement.

We now show that points Q, S, X, Y are concyclic. Indeed,

$$\begin{aligned}\angle XQY &= \angle BQC = 180^\circ - \angle TBA - \angle CBT - \angle TCB = \\ &= 180^\circ - \angle AET - \angle EDT - \angle TED = \angle DSE = \angle XSY,\end{aligned}$$

where we have used that $\angle ABT = \angle TEA$ and the similarity of triangles TDE and TBC.

We claim that the lines XY and CD are parallel. First of all, we observe that triangles TBX and TEY are similar, since $\angle YET = \angle AET = \angle TBA = \angle TBX$, and

$$\angle ETY = 180^\circ - \angle CTD - \angle DTE = 180^\circ - \angle CTD - \angle BTC = \angle XTB.$$

Then, TEY is similar to TBX, so we have

$$\frac{TY}{TX} = \frac{TE}{TB} = \frac{TC}{TD},$$

since according to the statement $TB = TD$ and $TC = TE$. From this equality, together with Thales' theorem, the parallelism of XY and CD follows.

Using power of a point with respect to Q, S, X, Y, it holds that $\frac{AS}{AQ} = \frac{AX}{AY}$. Since XY and CD are parallel, $\frac{AX}{AY} = \frac{AP}{AR}$. Hence, we have

$$AP \cdot AQ = AR \cdot AS.$$

This shows that $PQRS$ is cyclic, and we are done.

Problem 5. Find all triples (a, b, p) of positive integers with p prime and

$$a^p = b! + p.$$

Notation. As a general piece of notation, for a given prime p, let $v_p(n)$ stand for the greatest integer such that $p^{v_p(n)}$ divides n but $p^{v_p(n)+1}$ does not. The solution uses as a well-known fact the *lifting the exponent lemma*.

Solution by Martín Padrón. We will show that the unique solutions are $(a, b, p) = (2, 2, 2)$ and $(a, b, p) = (3, 4, 3)$, which clearly work. We divide the problem in four different cases.

(i) Case $b < p$. Here, $p \nmid b! + p$ and $q \nmid b! + p$, for any prime $q \leq b$. Since $b! + p$ has at least one prime divisor q, we must have $q > b$, and since $q \mid a$, we must have $a^p \geq q^p$.
We now claim that $q^p > 2p$ for all $q > 1$, $p > 2$ and $q^p > 2b!$ for p, $q > b$. These inequalities are immediately verified by induction. Adding up both equalities and dividing by 2, we get that $q^p > p + b!$ for p, $q > b$.
This means that, for $p > 2$,
$$a^p \geq q^p > p + b!,$$
which is a contradiction. When $p = 2$ (and $b = 1$), $a^2 = 1 + 2$ does not have a solution in the set of positive integers.

(ii) Case $b = p$. In this case, $p \mid p! + p$ implies that $p \mid a$, and therefore $a^p \geq p^p$. Using again a straightforward induction argument, we have that $n^n > n! + n$ for all $n > 2$, which is again a contradiction.
In the case $p = 2$, we get $a^2 = 2 + 2 = 4$, and therefore $a = 2$.

(iii) Case $b \geq 2p$. Proceeding as before, $p \mid b! + p$, which implies that $p \mid a$. From here, $p^2 \mid p^p \mid a^p = b! + p$, but since $p^2 \mid b!$ and $p^2 \nmid p$, we reach a contradiction.

(iv) Case $p < b < 2p$. We may rewrite the given equation as
$$a^p = p\Big(\frac{b!}{p} + 1\Big),$$
from where $a = pa'$. Observe that a' cannot have prime factors smaller or equal than b and different from p. Hence, if $a' > 1$, $a' > p$ and in particular $a^p \geq p^{2p}$.
We now claim that $n^{2n} > (2n-1)!$ for all $n > 1$. Indeed, using the inequality between the arithmetic mean and the geometric mean,
$$(2n-1)! < \Big(\frac{\sum_{i=1}^{2n-1} i}{2n-1}\Big)^{2n-1} = n^{2n-1} < n^{2n}.$$
Then, $a^p \geq p^{2p} > (2p-1) \geq b!$, which is a contradiction.

Hence, it must hold that $a = p$ and $\frac{b!}{p} = p^{p-1} - 1$. Assume first that $p > 2$. In this case,

$$v_2(p^{p-1} - 1) = v_2(p-1) + v_2(p+1) + v_2(p-1) - 1.$$

However,

$$v_2(b!/p) = v_2(b!) \geq v_2(p+1)+v_2(p-1)+v_2((p-1)/2)+v_2(2) > v_2(p^{p-1}-1),$$

which is a contradiction if $\frac{p-1}{2} > 2$. Then, if $p > 5$, the equality cannot be satisfied. We can check the cases $p = 2, 3, 5$ and see that the unique solution is $(a, b, p) = (3, 4, 3)$, and we are done.

Óscar Rivero Salgado
University of Warwick
UK
riverosalgado@gmail.com

Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

Elementary Problems

E–101. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Determine two positive integers which sum is 3948 and their lest common multiple is 82861.

Solution 1 by the proposer. We begin proving that for any positive integers a, b it holds

$$(a, b) = (a + b, [a, b]).$$

First, we observe that if $(c, d) = 1$, then $(c + d, cd) = 1$. Indeed, suppose that $p \mid c + d$ and $p \mid cd$ for some prime p. Then WLOG $p \mid c$, and so $p \mid (c + d) - c = d$. This means $p \mid (c, d)$, a contradiction.

On the other hand, we know that if $c > 0$, then $(ac, bc) = c(a, b)$. Now, we suppose that $g = (a, b)$, then $a = gn$ and $b = gm$ for some positive integers m and n with $(m,n)=1$. On account of the preceding, we have

$$\begin{aligned}(a+b,[a,b]) &= \left(a+b,\frac{ab}{g}\right)=\frac{1}{g}\left(g(a+b),ab\right)\\ &= g(m+n,gmn)=g(m+n,mn)=g=(a,b),\end{aligned}$$

and the prove is complete.

Finally, to find a, b, we have $a+b=3948$ and

$$ab=(a,b)\,[a,b]=(a+b,[a,b])\,[a,b]=47\cdot 82861=3894467.$$

Then, a and b are the solutions of the quadratic equation $t^2-3948t+3894467=0$. That is, 1927 and 2021, the years of my father's birth and passing away.

Solution 2 by Henry Ricardo, Westchester Area Math Circle, NY,USA. We write these two numbers as ad and bd, where a, b, d are positive integers with $(a,b)=1$. It is easy to show that $(ab, a+b)=1$. Since

$$\frac{a+b}{ab}=\frac{ad+bd}{abd}=\frac{3948}{82861}=\frac{47\cdot 84}{47\cdot 1763}=\frac{84}{1763}, \quad (*)$$

and the first and last fractions in $(*)$ are in lowest terms, we must have $a+b=84$ and $ab=1763$. Thus a and b are the roots of the quadratic equation $x^2-84x-1763=0$—that is, $(a,b)=(43,41)$. It follows from $(*)$ that $d=47$, from which we derive $(ad,bd)=(43\cdot 47, 41\cdot 47)=(2021,1927)$. This is the only solution.

Solution 3 by Michel Bataille, Rouen, France. We easily find that $3948=2^2\cdot 3\cdot 7\cdot 47$ and $82861=41\cdot 43\cdot 47$.
Let a, b be two positive integers such that $a+b=3948$ and $\operatorname{lcm}(a,b)=82861$, and let $\gcd(a,b)=d$. Let $a=da_1, b=db_1$ where a_1, b_1 are coprime. We have $d(a_1+b_1)=3948$ and $d^2a_1b_1=ab=d\cdot 82861$ so that $82861=da_1b_1$. We see that d divides 3948 and 82861, hence divides their gcd, which is 47. Thus $d=1$ or $d=47$ (since 47 is a prime). However, $d=1$

cannot occur since otherwise $a_1b_1 = 82861$ and $a_1 + b_1 = 3948$; since $(a_1+b_1)^2 - 4a_1b_1 = (a_1-b_1)^2$, the integer $1974^2 - 82861 = 3813815$ would be a perfect square, which is untrue (3813815 is divisible by 5, not by 25). Thus, we must have $d = 47$ and $a_1b_1 = 41\cdot 43, a_1+b_1 = 2^2\cdot 3\cdot 7 = 84$, that is, $\{a_1, b_1\} = \{41, 43\}$ and $\{a, b\} = \{41\cdot 47, 43\cdot 47\} = \{1927, 2021\}$ follows.
Conversely, it is readily checked that the sum and least common multiple of 1927 and 2021 are 3948 and 82861, respectively.
In conclusion, the problem has a unique solution, namely 1927, 2021.

Solution 4 by Morgan Orr (student) and Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. The prime factorization of 3948 is

$$3948 = 2^2 \cdot 3 \cdot 7 \cdot 47,$$

and the prime factorization of 82861 is

$$82861 = 41 \cdot 43 \cdot 47.$$

Because $2^2 \cdot 3 \cdot 7 = 84 = 41 + 43$,

$$41 \cdot 47 + 43 \cdot 47 = 84 \cdot 47 = 3948,$$

and

$$\mathrm{lcm}(41 \cdot 47, 43 \cdot 47) = 41 \cdot 43 \cdot 47 = 82861.$$

The two positive integers we seek are $41\cdot 47 = 1927$ and $43\cdot 47 = 2021$.

Also solved by *Miguel Amengual Covas, Cala Figuera, Mallorca, Spain (two solutions); Rovsen Pirkuliyev, Sumgait City, Azerbaijan; Brittany Turner, Jessica Hiauw and Ryan Dipaola both at SUNY Brockport, Brockport, NY; Scott H. Brown Auburn University Montgomery, Montgomery AL; Daniel Văcaru, Pitești, Romania; and Hyunbin Yoo, South Korea.*

E–102. *Proposed by Michel Bataille, Rouen, France.* Let ABC be a triangle with sides a, b, c and opposite angles

$$A = \frac{6\pi}{11},\ B = \frac{3\pi}{11},\ C = \frac{2\pi}{11}.$$

Prove that $a^2(a-b) + b^2(b-c) + c^2(c-a) = bc^2$.

Solution 1 by the proposer. Let R denote the circumradius of ΔABC. Then, we have

$$a = 2R\sin\frac{6\pi}{11}, b = 2R\sin\frac{3\pi}{11}, c = 2R\sin\frac{2\pi}{11}$$

so that $a = 4R\sin\frac{3\pi}{11}\cos\frac{3\pi}{11} = 2b\cos\frac{3\pi}{11}$, that is,

$$\cos\frac{3\pi}{11} = \frac{a}{2b}. \tag{1}$$

Similarly,

$$b = 2R\sin\frac{8\pi}{11} = 4R\sin\frac{4\pi}{11}\cos\frac{4\pi}{11} = 4c\cos\frac{2\pi}{11}\cos\frac{4\pi}{11}$$

so that

$$b = 2c\left(\cos\frac{6\pi}{11} + \cos\frac{2\pi}{11}\right). \tag{2}$$

From (2) and the Law of Cosines we obtain

$$\frac{b}{c} = \frac{b^2 + c^2 - a^2}{bc} + \frac{a^2 + b^2 - c^2}{ab}$$

which gives

$$a^3 + c^3 = b^2c + c^2a + ca^2. \tag{3}$$

Similarly, (1) leads to $\frac{a}{2b} = \frac{c^2+a^2-b^2}{2ca}$, or

$$b^3 + ca^2 = bc^2 + a^2b. \tag{4}$$

Now, using (3) and (4), we obtain

$$a^2(a-b) + b^2(b-c) + c^2(c-a) = (a^3 + c^3 - b^2c - c^2a) + (b^3 - a^2b)$$

$$= ca^2 + (bc^2 - ca^2) = bc^2,$$

as required.

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. To solve the problem we will use the following lemmas. Let ABC be a triangle with sides a, b, c opposite to A, B, C.

LEMMA 1. *The condition $\angle A = 2\cdot\angle B$ is equivalent to $a^2 = b(b + c)$.*

[1].

LEMMA 2. *The condition* $\angle A = 3 \cdot \angle C$ *implies* $(a^2 - c^2)(a - c) = b^2c$. This fact that we shall establish at the end of this solution. The left side of the equality to be proved may be written in the form

$$\left(a^2 - c^2\right)(a - c) - \left(a^2 - b^2\right)(b - c).$$

Since $A : B : C = 6 : 3 : 2$, substituting b^2c for $(a^2 - c^2)(a - c)$ from LEMMA 2 and bc for $a^2 - b^2$ from LEMMA 1, this expression is equal to

$$b^2c - bc(b - c) = bc^2,$$

as desired.

We now complete the argument by proving: LEMMA 2. Let D be a point on side BC such that AD trisects $\angle BAC$. Similar triangles ABD and ABC yield

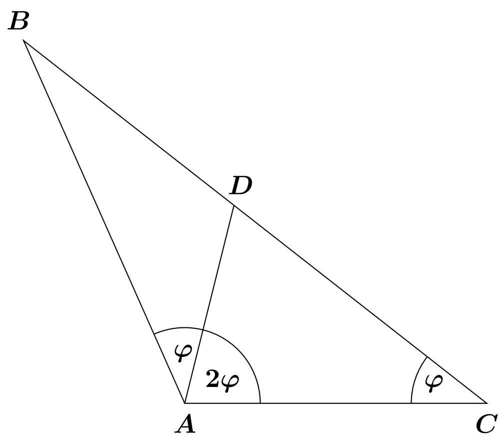
$$\frac{AB}{BC} = \frac{AD}{CA} = \frac{BD}{AB},$$

that is,

$$\frac{c}{a} = \frac{AD}{b} = \frac{BD}{c},$$

and hence

$$AD = \frac{bc}{a}, \qquad BD = \frac{c^2}{a}, \qquad DC = BC - BD = a - \frac{c^2}{a} = \frac{a^2 - c^2}{a}.$$



By LEMMA 1, applied to $\triangle ADC$, where $\angle DAC = 2 \cdot \angle ACD$,

$$CD^2 = AD(AD + CA)$$

which, on substitution, yields

$$\left(\frac{a^2 - c^2}{a}\right)^2 = \frac{bc}{a}\left(\frac{bc}{a} + b\right),$$

which, when simplified, becomes

$$\left(a^2 - c^2\right)(a - c) = b^2 c.$$

Reference

[1] Amengual Covas, M. "Triangles with one angle twice another" *Arhimede math. j.* (8.2) (2021).

Also solved by *Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA*

E–103. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Find the smallest integer value of $n \geq 1$, such that

$$2022 = {a_1}^3 + {a_2}^3 + \cdots + {a_n}^3$$

for integers $a_1, a_2, \ldots, a_n$.

Solution 1 by the proposer. For $k = 3m$, $k^3 = 27m^3 \equiv 0 \pmod 9$, for integer m.

For $k = 3m + 1$, $k^3 = 27m^3 + 27m^2 + 9m + 1 \equiv 1 \pmod 9$.

For $k = 3m - 1$, $k^3 = 27m^3 - 27m^2 + 9m - 1 \equiv -1 \pmod 9$.

Hence for all integers k, $k^3 \equiv 0, 1, -1 \pmod 9$.

$2022 \equiv 6 \equiv -3 \pmod 9$, so $n \geq 3$.

A direct check show that $2022 = 38^3 + (-37)^3 + (-13)^3$ is a solution, so the smallest integer value is $n = 3$.

Solution 2 by Hyunbin Yoo, South Korea. Since $\pm 12^3 = \pm 1728$ and $\pm 13^3 = \pm 2197$, there is no integral solution to $a_1^3 = 2022$. While calculating $(m+1)^3 - m^3$ it can be found that $38^3 - 37^3 - 13^3 = 2022$. Now we will show that there is no integral solution to $a_1^3 + a_2^3 = 2022$. We will divide the problem into two cases, $x^3 + y^3 = 2022$ and $x^3 - y^3 = 2022$ where x and y are both positive integers.

- If $x^3 + y^3 = 2022$, then 2022 can be factored as $2 \cdot 3 \cdot 337$. We distinguish the following cases: (1) Both are even, then $(2k)^3 + (2k')^3 = 8k^3 + 8k'^3 = 8(k^3 + k'^3)$ however 2022 is not a multiple of 8 so it is impossible. (2) One odd and one even, then the sum of their cubes is odd so that is also impossible. (3) Both are odd, then Factoring x^3+y^3 gives $(x+y)(x^2-xy+y^2)$. $x + y$ is even while $x^2 - xy + y^2$ is odd. Thus $x + y$ is one of $2, 6, 674, 2022$. The first two cases can easily be ruled plugging in all possible values. To determine whether the latter two are valid, note that $x^2 - xy + y^2 = (x + y)^2 - 3xy$. If we let $x = k + d$ and $y = k - d$, $x^2 - xy + y^2 = k^2 + 3d^2$. Since it is clear that $x^2 - xy + y^2 < k^2 = \left(\frac{x+y}{2}\right)^2$ in both, there are no solutions.
- If $x^3 - y^3 = 2022$, then both are odd on account of the preceding and $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$. $x^2 + xy + y^2 = (x-y)^2 + 3xy$ so $x - y < (x-y)^2 + 3xy$. Since $x - y$ is even it is one of $2, 6, 674, 2022$. The latter two are ruled out due to the inequality above. If $x - y = 2$, $(x - y)^2 + 3xy = 1011$. Here $3xy = 1007$ so it is invalid. If $x - y = 6$, $(x-y)^2 + 3xy = 737$. $3xy = 701$ so it is also invalid.

Thus there are no integer solutions to $a_1^3 + a_2^3 = 2022$ and the minimum n is 3.

E–104. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $a_1, a_2, \ldots, a_{20}$ be distinct integer numbers. Show that among them, there are two a_i and a_j such that $a_i - a_j$ is a multiple of 19.

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY, USA. There are 19 possible remainders when an integer is divided by 19. If we take the remainders as "pigeonholes" and the 20 numbers as "pigeons", then by the Pigeonhole Principle, there will be at least two pigeons sharing the same hole—that is, two numbers with the same remainder when divided by 19. The difference of these two numbers is thus divisible by 19.

Solution 2 by Michel Bataille, Rouen, France. We consider 19 boxes numbered $0, 1, \ldots, 18$ and we put the integer a_k in box number r if $a_k \equiv r \pmod{19}$. Since we have 20 such a_k, two of these integers must be in the same box, say $a_i \equiv r \pmod{19}$ and $a_j \equiv r \pmod{19}$. Then $a_i - a_j \equiv 0 \pmod{19}$, meaning that $a_i - aj$ is a multiple of 19.

Solution 3 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. For $j = 2, 3, \ldots, 20$, calculate

$$a_1 - a_j \pmod{19}.$$

If one these values is zero, we are done. Otherwise, among these 19 differences modulo 19 there are only 18 possible values, so the Pigeonhole principle guarantees there are at least two differences, say $a_1 - a_i$ and $a_1 - a_j$, which are congruent modulo 19. Thus,

$$a_i - a_j \equiv 0 \pmod{19};$$

that is, $a_i - a_j$ is a multiple of 19.

Solution 4 by Hyunbin Yoo, South Korea. All integers n can be expressed in the form $n = 19q + r$ where q and r are integers and $0 \leq r < 19$. So there are 19 different possible r. Furthermore, $a_k = 19q_k + r_k$ for all $1 \leq k \leq 20$. Then $a_i - a_j = (19q_i + r_i) - (19q_j + r_j) = 19(q_i - q_j) + (r_i - r_j)$. Since $19(q_i - q_j)$ is a multiple of 19, $a_i - a_j$ is a multiple of 19 if and only if $r_i - r_j$ is a multiple of 19. As there are 20 r_k but they can only take 19 different values, according to the pigeonhole principle, there is at least one pair (i, j) that satisfies $r_i = r_j$ and thus $r_i - r_j = 0$, which is a multiple of 19. Thus there is at least one (i, j) where $a_i - a_j$ is a multiple of 19, and the proposition is proved.

Also solved by *Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Rovsen Pirkuliyev, Sumgait City, Azerbaijan; Daniel Văcaru, Pitești, Romania; and the proposer.*

E–105. *Proposed by Toyesh Prakash Sharma, Agra College, India.* **[Correction]** If $1 < a, b, c < 6$ and $a + b + c = 6$ then show that

$$\frac{\ln a}{b+c} + \frac{\ln b}{c+a} + \frac{\ln c}{a+b} \geq \frac{3}{4}\ln 2.$$

Solution by Henry Ricardo, Westchester Area Math Circle, NY, USA. The condition $a + b + c = 6, 1 < a, b, c < 6$, allows us to rewrite the given inequality as $\sum_{cyclic} \ln a/(6 - a) \geq (3\ln 2)/4$. We note that if the function $f(x) = \ln x/(6 - x)$ were convex on $(1, 6)$, then Jensen's inequality would yield

$$\sum_{cyclic} \frac{\ln a}{6-a} \;\geq\; 3 \cdot \frac{\ln\left(\frac{a+b+c}{3}\right)}{6 - \left(\frac{a+b+c}{3}\right)} \;=\; \frac{3}{4}\ln 2.$$

However, since the function $f(x) = \ln x/(6 - x)$ has an inflection point at $x \approx 1.7$, we cannot use Jensen's inequality and instead apply Cîrtoaje's Right-Convex Function (RCF) theorem:

> Let $f(u)$ be a function defined on an interval $\mathbb{I} \subset \mathbb{R}$ and convex for $u \geq s, s \in \mathbb{I}$. If
>
> $$f(x_1)+f(x_2)+\cdots+f(x_n) \geq nf\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \qquad (1)$$
>
> for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ such that
>
> $$x_1 + x_2 + \cdots + x_n = ns \text{ and } x_2 = x_3 = \cdots = x_n \geq s,$$
>
> then (1) holds for all $x_1, x_2, \ldots, x_n \in \mathbb{I}$ such that $x_1 + x_2 + \cdots + x_n \geq ns$.

Taking $f(u) = \ln u/(6 - u)$ and $\mathbb{I} = (1, 6)$, an easy calculation shows that $f''(u) \geq 0$ for $u \geq s = 2$. We have seen that (1) holds for $a+b+c = 3\cdot 2 = 6$, and now we prove that $\sum_{cyclic} \ln a/(6-a) \geq (3\ln 2)/4$ when $b = c \geq 2$. Assuming that $b = c \geq 2$, we have

$a + b + c = a + 2b = 6$, and we must show that

$$\sum_{cyclic} \frac{\ln a}{6-a} \;=\; \frac{\ln(6-2b)}{2b} \;+\; \frac{2\ln b}{6-b} \;\geq\; \frac{3}{4}\ln 2.$$

Defining $g(x) = \ln(6-2x)/(2x) + 2\ln x/(6-x)$, an elementary calculation establishes that $g'(2) = 0$, $g''(2) = 3\ln 2/16 > 0$, so that g attains a minimum value of $g(2) = \ln 2/4 + 2\ln 2/4 = (3\ln 2)/4$. Thus the conditions of the RCF theorem are satisfied, and we conclude that the proposed inequality (1) holds for all $a, b, c \in (1, 6)$ such that $a+b+c = 6$. We have equality if and only if $a = b = c = 2$.

Also solved by *Rovsen Pirkuliyev, Sumgait City, Azerbaijan.*

E–106. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let ABC be a triangle with the circumcenter O and the orthocenter H. Points A_1, B_1, C_1 are the circumcenters of triangles HBC, HAC and HAB, respectively. Show that if $\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = \overrightarrow{0}$, then ABC is an equilateral triangle.

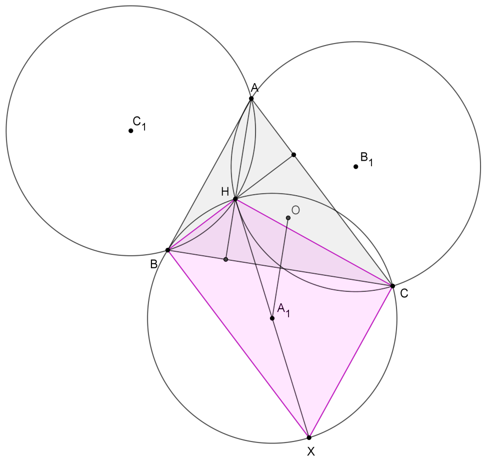
Solution 1 by Michel Bataille, Rouen, France. Let Γ be the circumcircle of ΔABC. We know that the reflection of H in BC is on Γ. It follows that the reflection of the circumcircle of HBC is Γ and therefore A_1 is the reflection of O in BC. In addition, if M is the midpoint of BC, then $2\overrightarrow{OM} = \overrightarrow{AH}$ so that $\overrightarrow{OA_1} = \overrightarrow{AH}$. Similarly, we have $\overrightarrow{OB_1} = \overrightarrow{BH}$ and $\overrightarrow{OC_1} = \overrightarrow{CH}$ so that

$$\begin{aligned}\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} &= \overrightarrow{AO} + \overrightarrow{BO} + \overrightarrow{CO} + \overrightarrow{AH} + \overrightarrow{BH} + \overrightarrow{CH} \\ &= 2(\overrightarrow{AO} + \overrightarrow{BO} + \overrightarrow{CO}) + 3\overrightarrow{OH} \\ &= 2\overrightarrow{HO} + 3\overrightarrow{OH} = \overrightarrow{OH}.\end{aligned}$$

As a result, if $\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = \overrightarrow{0}$, then $O = H$ and ΔABC is equilateral.

Solution 2 by the proposer. Let R, R_1, R_2, R_3 be the circumradius of circumcircles ABC, HBC, HAC and respectively HAB. Let $\alpha = \measuredangle A$. Let X be a point that forms together with A, B and C a parallelogram, so

$$\begin{cases} BH \perp AC \\ AC \parallel BX \end{cases} \Rightarrow BH \perp BX$$



Scheme for solving problem E-106

Analogously $CH \perp CX \Rightarrow BXCH$ is an inscribed quadrilateral with the diameter HX, so A_1 is the middle point of HX.

From the Sine Theorem, in $\triangle BHC$ $\begin{cases} \dfrac{BC}{\sin BHC} = 2R_1, \\ \dfrac{BC}{\sin(180^\circ - \alpha)} = 2R_1, \\ \dfrac{BC}{\sin \alpha} = 2R_1. \end{cases}$

In $\triangle ABC$, we have $\dfrac{BC}{\sin \alpha} = 2R$, so $R_1 = R$. Since $A_1B = A_1C = OB = OC = R$, then OBA_1C is a rhombus and $\overrightarrow{AA_1} = \overrightarrow{AO} + \overrightarrow{OA_1} = \overrightarrow{AO} + \overrightarrow{OB} + \overrightarrow{OC}$.

Analogously $\overrightarrow{BB_1} = \overrightarrow{BO} + \overrightarrow{OA} + \overrightarrow{OC}$ and $\overrightarrow{CC_1} = \overrightarrow{CO} + \overrightarrow{OA} + \overrightarrow{OB}$.

According to Sylvester's relationship, we have $\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OH}$. On account that $\overrightarrow{AA_1} + \overrightarrow{BB_1} + \overrightarrow{CC_1} = \overrightarrow{0}$, then $\overrightarrow{OH} = 0 \Rightarrow H \equiv O \Rightarrow ABC$ is an equilateral triangle.

Easy–Medium Problems

EM–101. *Proposed by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.* Let ℓ be an exterior common tangent to congruent external circles Ω and Ω' of radius r. The circle Ω_1 touches ℓ and also touches Ω and Ω' externally. The circle Ω_2 touches Ω_1, Ω and Ω' externally. The circle Ω_3 touches Ω_2, Ω and Ω' externally. The circle Ω_4 touches Ω_3, Ω and Ω' externally and also touches the line of centers of Ω and Ω'. Prove that the radius of Ω_2 is twice the length of the radius of Ω_4.

Solution 1 by the proposer. Let r_i denote the radii of Ω_i, $i = 1, 2, 3, 4$. Then

$$2r_1 + 2r_2 + 2r_3 + 2r_4 = r \tag{1}$$

and half the distance between the centers of Ω and Ω' is $2\sqrt{rr_1}$.

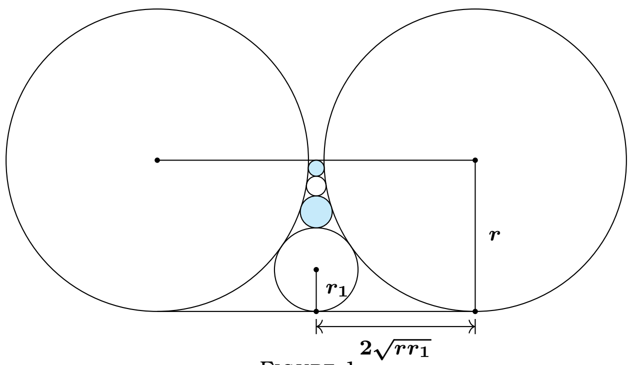


FIGURE 1

Applying three times the Pythagorean theorem, we obtain

$$\left(2\sqrt{rr_1}\right)^2 + r_4^2 = (r_4 + r)^2, \tag{2}$$

$$\left(2\sqrt{rr_1}\right)^2 + (2r_4 + r_3)^2 = (r_3 + r)^2, \tag{3}$$

and

$$\left(2\sqrt{rr_1}\right)^2 + (2r_4 + 2r_3 + r_2)^2 = (r_2 + r)^2. \tag{4}$$

Equation (2) simplifies to

$$2r_4 = 4r_1 - r. \tag{5}$$

In (3) we substitute $r - 2r_1 - 2r_2 - r_3$ for $2r_4 + r_3$ from (1) and solve for r_3, obtaining

$$r_3 = \frac{(r_1 + r_2)^2 - rr_2}{r - (r_1 + r_2)}. \tag{6}$$

In (4) we substitute $r - 2r_1 - r_2 - r_3$ for $2r_4 + 2r_3 + r_2$ from (1) and solve for r_2, obtaining

$$r_2 = \frac{r_1^2}{r - r_1}. \tag{7}$$

Substituting for r_2 into (6), we obtain

$$r_3 = \frac{r_1^3}{(r - r_1)(r - 2r_1.)}$$

Substitution in (1) yields

$$2r_1 + \frac{2r_1^2}{r - r_1} + \frac{2r_1^3}{(r - r_1)(r - 2r_1.)} + (4r_1 - r) = r,$$

equivalent to

$$(r_1 - r)\left(5r_1^2 - 5rr_1 + r^2\right) = 0,$$

implying

$$5r_1^2 - 5rr_1 + r^2 = 0,$$

which we rewrite as

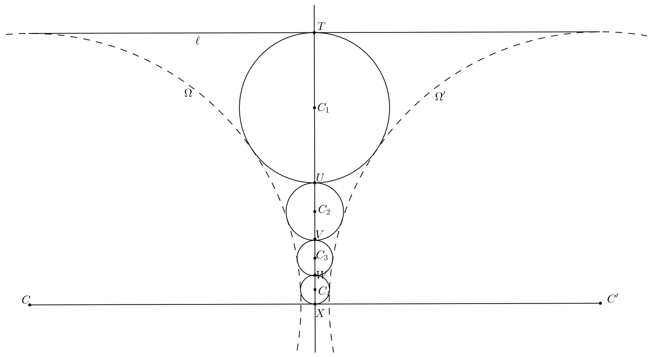
$$r_1^2 - (r - r_1)(4r_1 - r) = 0.$$

This, in turn, is equivalent to $(r - r_1)(r_2 - 2r_4) = 0$, implying $r_2 - 2r_4 = 0$, as desired.

We note, in passing, that

$$r_1 : r_2 : r_3 : r_4 : r \quad = \quad \frac{5 - \sqrt{5}}{10} : \frac{5 - 2\sqrt{5}}{5} : -15 + 7\sqrt{5} : \frac{5 - 2\sqrt{5}}{10} : 1$$

Solution 2 by Michel Bataille, Rouen, France. Let C, C', C_i be the centers of $\Omega, \Omega', \Omega_i$, respectively, and let r_i be the radius of Ω_i $(i = 1, 2, 3, 4)$. Let $d = \frac{CC'}{2}$. We denote by T, U, V, W, X the points of tangency as on the figure above and introduce two inversions: I in the circle with center X and radius $\sqrt{d^2 - r^2}$ and J in the circle with center T and radius d. Since $d^2 - r^2$ and d^2 are the power of X and T, respectively, with respect to Ω and Ω', these circles are invariant under I and J.



Scheme for solving problem EM-101

Since inversion preserves tangency, we have $\mathrm{I}(C_4) = \ell$, hence $\mathrm{I}(W) = T$, that is, $2r_4 \cdot r = d^2 - r^2$ and therefore

$$r_4 = \frac{d^2 - r^2}{2r}. \tag{1}$$

Similarly, $\mathrm{J}(C_1) = \ell'$, the reflection of ℓ in the line CC', so that $2r_1 \cdot 2r = d^2$ and $r_1 = \frac{d^2}{4r}$.
The circle $\mathrm{J}(C_2)$, which is tangent to Ω, Ω', ℓ', is the reflection of C_1 in CC'. It follows that $\frac{r_1}{r_2} = \frac{d^2}{TU \cdot TV} = \frac{d^2}{2r_1(2r_1 + 2r_2)}$. A short calculation yields

$$r_2 = \frac{d^4}{4r(4r^2 - d^2)}. \tag{2}$$

On the other hand, we have $\mathrm{I}(\Omega_1) = \Omega_3$ and $\mathrm{I}(\Omega_2) = \Omega_2$. The latter gives $XU \cdot XV = d^2 - r^2$, that is, $(2r_4 + 2r_3)(2r_4 + 2r_3 + 2r_2) = d^2 - r^2$. But since $2r_4 + 2r_3 + 2r_2 = r - 2r_1$, we have

$2r_4 + 2r_3 = \frac{2r(d^2-r^2)}{2r^2-d^2}$ so that

$$2r_2 \cdot \frac{2r(d^2 - r^2)}{2r^2 - d^2} + \left(\frac{2r(d^2 - r^2)}{2r^2 - d^2}\right)^2 = d^2 - r^2.$$

We easily deduce that

$$r_2 = \frac{d^4 - 8r^2d^2 + 8r^4}{4r(2r^2 - d^2)}. \tag{3}$$

Comparing (2) and (3) readily leads to the constraint $5d^4-20r^2d^2+16r^4 = 0$ that r, d must verify. Now, from (1) and (2), we derive

$$r_2 - 2r_4 = \frac{d^4}{4r(4r^2 - d^2)} - \frac{d^2 - r^2}{r} = \frac{5d^4 + 16r^4 - 20r^2d^2}{4r(4r^2 - d^2)} = 0$$

so that $r_2 = 2r_4$, as desired.

EM–102. *Proposed by Michel Bataille, Rouen, France.* Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) = f(y) \cdot f(xf(y))$$

for all real numbers x, y.

Solution 1 by Félix Moreno Peñarrubia, Charles University, Prague, Czech Republic. The constant functions $f(x) = 0$ and $f(x) = 1$ can be checked to be solutions. We will prove that those are the only ones. Assume we have a function f which is not constantly 0 or constantly 1. If there exists an y_0 such that $f(y_0) = 0$, then plugging $y = y_0$ in the equation gives $f(x + y_0) = 0$ for all x, so f is constantly 0, contradiction. So 0 is not in the image of f. Since f is not constantly 1, there must exist $1 \neq z \in \Im(f)$. Let $y = y_z$, where $f(y_z) = z$, and let $x = \frac{y_z}{z-1}$. Then we have $f\left(\frac{y_z z}{z-1}\right) = zf\left(\frac{y_z z}{z-1}\right)$. Since we determined $f\left(\frac{y_z z}{z-1}\right) \neq 0$, we must have $z = 1$, contradiction.

Solution 2 by the proposer. The constant functions $x \mapsto 0$ and $x \mapsto 1$ are obvious solutions. We show that there is no other solution. Let f be an arbitrary solution and let $E(x, y)$ denote the equality $f(x + y) = f(y) \cdot f(xf(y))$. Then, $E(0, 0)$ gives $f(0) = (f(0))^2$ so that either $f(0) = 0$ or $f(0) = 1$. If $f(0) = 0$, then $E(x, 0)$ yields $f(x) = 0$ for all $x \in \mathbb{R}$ and so f is the constant function $x \mapsto 0$. Now, consider the case when $f(0) = 1$ and assume that $f(a) \neq 1$ for some nonzero real number a. Then the equality $E\left(\frac{a}{f(a)-1}, a\right)$ rewrites as

$$f\left(\frac{a}{f(a)-1} + a\right) = f(a)f\left(\frac{af(a)}{f(a)-1}\right),$$

that is, $f(b)(1 - f(a)) = 0$ if we set

$$b = \frac{af(a)}{f(a)-1}.$$

Since $f(a) \neq 1$, it follows that $f(b) = 0$, but then $E(-b, b)$ gives

$$1 = f(0) = f(-b + b) = f(b)f(-bf(b)) = 0,$$

a clear contradiction. As a result, we must have $f(x) = 1$ for all $x \in \mathbb{R}$ and f is the constant function $x \mapsto 1$.

Also solved by *Rovsen Pirkuliyev, Sumgait City, Azerbaijan.*

EM–103. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a_0, a_1, a_2 be real numbers such that $3a_0 \geq a_1a_2$. If the polynomial with real coefficients $A(z) = a_0 + a_1z + a_2z^2 + z^3$ has all its zeros real and non-negative, then prove that

$$9a_0^2 + a_1^2a_2^2 \geq \frac{4}{3}a_1^3 + 6a_0a_1a_2.$$

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Let u, v, and w be the real, non-negative zeros of the polynomial $a_0+a_1z+a_2z^2+z^3$. By Viete's formulas,

$$a_0 = -uvw, \quad a_1 = uv + vw + wu, \quad \text{and} \quad a_2 = -(u + v + w).$$

Multiplying the desired inequality by 3 and substituting for a_0, a_1, a_2 in terms of u, v, and w yields

$$27u^2v^2w^2 + 3(uv + vw + wu)^2(u + v + w)^2$$

$$\geq 4(uv + vw + wu)^3 + 18uvw(u + v + w)(uv + vw + wu).$$

Clearing parentheses, this becomes

$$3(u^4v^2+u^2v^4+v^4w^2+v^2w^4+u^4w^2+u^2w^4)+6(u^4vw+uv^4w+uvw^4)$$

$$+2(u^3v^3 + v^3w^3 + w^3u^3) \geq 6u^2v^2w^2$$

$$+6(u^3v^2w + u^3vw^2 + u^2v^3w + uv^3w^2 + u^2vw^3 + uv^2w^3). \quad (1)$$

Now, by the arithmetic mean - geometric mean inequality,

$$2(u^3v^3 + v^3w^3 + w^3u^3) \geq 6\sqrt[3]{u^6v^6w^6} = 6u^2v^2w^2.$$

Next, again by the arithmetic mean - geometric mean inequality,

$$\frac{u^2v^4 + u^4w^2}{2} \geq u^3v^2w, \ \frac{u^4v^2 + u^2w^4}{2} \geq u^3vw^2, \ \frac{u^4v^2 + v^4w^2}{2} \geq u^2v^3w,$$

$$\frac{v^2w^4 + u^2v^4}{2} \geq uv^3w^2, \ \frac{u^4w^2 + v^2w^4}{2} \geq u^2vw^3, \ \frac{v^4w^2 + u^2w^4}{2} \geq uv^2w^3.$$

Adding these last six inequalities and multiplying by 3 yields

$$3(u^4v^2 + u^2v^4 + v^4w^2 + v^2w^4 + u^4w^2 + u^2w^4)$$

$$\geq 3(u^3v^2w + u^3vw^2 + u^2v^3w + uv^3w^2 + u^2vw^3 + uv^2w^3).$$

Lastly, apply the AM-GM inequality to obtain

$$2u^3 + v^3 \geq 3u^2v, \quad 2u^3 + w^3 \geq 3u^2w, \quad u^3 + 2v^3 \geq 3uv^2,$$

$$2v^3 + w^3 \geq 3v^2w, u^3 + 2w^3 \geq 3uw^2, \quad v^3 + 2w^3 \geq 3vw^2.$$

Adding these last six inequalities and multiplying by uvw yields

$$6(u^4vw + uv^4w + uvw^4)$$

$$\geq 3(u^3v^2w + u^3vw^2 + u^2v^3w + uv^3w^2 + u^2vw^3 + uv^2w^3).$$

The sum of previous inequalities is (1).

Note: The condition $3a_0 \geq a_1 a_2$ is not needed. Moreover, because

$$(u+v+w)(uv+vw+wu) \geq 3\sqrt[3]{uvw} \cdot 3\sqrt[3]{u^2v^2w^2} = 9uvw,$$

it follows that

$$-a_1a_2 \geq -9a_0 \quad \text{or} \quad 9a_0 \geq a_1a_2.$$

As a_0 is non-positive, $3a_0 \geq 9a_0$, so $3a_0 \geq a_1a_2$ is a direct consequence of the zeroes of the polynomial $a_0 + a_1z + a_2z^2 + z^3$ being real and non-negative.

Solution 2 by Michel Bataille, Rouen, France. We discard the first sentence of the statement of the problem. Let x_1, x_2, x_3 denote the zeros of $A(z)$, so that $a_0 = -x_1x_2x_3$, $a_1 = x_1x_2 + x_2x_3 + x_3x_1$, $a_2 = -(x_1+x_2+x_3)$ and let $L = 9a_0^2 + a_1^2a_2^2 - \frac{4}{3}a_1^3 - 6a_0a_1a_2$. If one among x_1, x_2, x_3 is 0, say $x_1 = 0$, then

$$L = a_1^2\left(a_2^2 - \frac{4}{3}a_1\right) = a_1^2\left((x_2+x_3)^2 - \frac{4}{3}x_2x_3\right)$$

and $L \geq 0$ holds since $(x_2+x_3)^2 \geq 4x_2x_3$.
Now suppose $x_1, x_2, x_3 > 0$. Observing that L is a homogeneous polynomial in x_1, x_2, x_3, we may even suppose $x_1 + x_2 + x_3 = 1$ and are reduced to proving $9a_0^2 + a_1^2 - \frac{4}{3}a_1^3 + 6a_0a_1 \geq 0$ or

$$9X^2 + 1 - \frac{4}{3}a_1 - 6X \geq 0 \tag{1}$$

if we set $X = \dfrac{x_1x_2x_3}{x_1x_2 + x_2x_3 + x_3x_1} = \dfrac{-a_0}{a_1}$.
The well known inequality $x_1^2 + x_2^2 + x_3^2 \geq x_1x_2 + x_2x_3 + x_3x_1$ gives $1 = (x_1+x_2+x_3)^2 \geq 3(x_1x_2 + x_2x_3 + x_3x_1)$ that is, $a_1 \leq \frac{1}{3}$, and therefore

$$9X^2 + 1 - \frac{4}{3}a_1 - 6X \geq 9X^2 - 6X + \frac{5}{9} = 9(X - \frac{5}{9})(X - \frac{1}{9}).$$

Now, let H be the harmonic mean of x_1, x_2, x_3. Then,

$$\frac{3}{H} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{1}{X}$$

so that $0 < X = \dfrac{H}{3} \leq \dfrac{1}{3} \cdot \dfrac{x_1+x_2+x_3}{3} = \dfrac{1}{9}$. Thus, $9(X - \frac{5}{9})(X - \frac{1}{9}) \geq 0$ and (1) follows, completing the proof.

Solution 3 by the proposer. Let us denote by a, b, c the zeros of $A(z)$. From Cardan-Viète formulas, we have $a_0 = -abc, a_1 = ab + bc + ca$ and $a_2 = -(a + b + c)$. Then,

$$9a_0^2 + a_1^2 a_2^2 \geq \frac{4}{3}a_1^3 + 6a_0a_1a_2 \Leftrightarrow 9a_0^2 + a_1^2a_2^2 - 6a_0a_1a_2 \geq \frac{4}{3}a_1^3,$$

or

$$(3a_0 - a_1a_2)^2 \geq \frac{4}{3}\,a_1^3 \Rightarrow 3a_0 - a_1a_2 \geq \frac{2}{\sqrt{3}}\,a_1^{3/2}$$

on account of the constrain. (Note that when $3a_0 - a_1a_2 = 0$ then a_1 must be zero and the statement trivially holds). Substituting the preceding expressions of a_0, a_1, a_2 in terms of a, b, c, we obtain that

$$3a_0 - a_1a_2 \geq \frac{2}{\sqrt{3}}\,a_1^{3/2}$$

is equivalent to

$$(a + b + c)(ab + bc + ca) \geq \frac{2}{\sqrt{3}}(ab + bc + ca)^{3/2} + 3abc. \qquad (1)$$

By the AM-GM inequality, we have

$$\frac{a + b + c}{3} \geq \sqrt[3]{abc},$$
$$\frac{ab + bc + ca}{3} \geq \sqrt[3]{a^2b^2c^2}.$$

Multiplying up the preceding inequalities yields

$$\frac{1}{3}(a + b + c)(ab + bc + ca) \geq 3abc. \qquad (2)$$

On the other hand, from $(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0$, we get $(a + b + c)^2 \geq 3\,(ab + bc + ca)$. After multiplying both terms by $(ab+bc+ca)^2$, we obtain $(a+b+c)^2(ab+bc+ca)^2 \geq 3\,(ab+bc+ca)^3$. Computing the square root of both terms, yields

$$(a + b + c)(ab + bc + ca) \geq \sqrt{3}\,(ab + bc + ca)^{3/2}$$

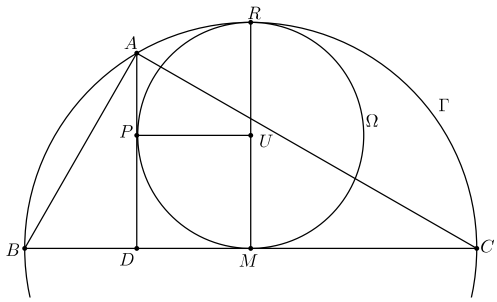
or equivalently,

$$\frac{2}{3}(a + b + c)(ab + bc + ca) \geq \frac{2}{\sqrt{3}}(ab + bc + ca)^{3/2}. \qquad (3)$$

Now, (1) immediately follows adding up (2) and (3). Note that equality holds when $a = b = c$.

EM–104. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let ABC be a right triangle with $\angle BAC = 90°$. If M is the middle point of BC, D is the foot of the altitude from A, Γ is the circumcircle of $\triangle ABC$ and exists the circle Ω tangent to BC at M, tangent to AD at P, and tangent to circle Γ at R, then calculate $\angle ABC$ and $\angle ACB$, respectively.

Solution 1 by Michel Bataille, Rouen, France. Let U be the center of Ω. Since M is the center of Γ, the perpendicular to BC at M, which contains U (since Ω is tangent to BC at M), intersects Γ at R. The point U is the midpoint of MR and P



Scheme for solving problem EM-104

is the projection of U onto AD (see figure). The quadrilateral $UPDM$ has three right angles, hence is a rectangle; it is even a square since $UP = UM$ (= the radius of Ω). It follows that $DM = MU = \frac{MR}{2} = \frac{MB}{2}$ and therefore D is the midpoint of BM. Since AD is the altitude and the median from its vertex A the triangle BAM is isosceles (with $AB = AM$), even equilateral since $MA = MB$ (= the radius of Γ). We can conclude: $\angle ABC = 60°$ and $\angle ACB = 30°$.

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. We first observe that Ω is the incircle of the mixtilinear triangle ADC.

Referring to Figure 1 and denoting the radius of Ω by r, our task is to locate the point D in such a way that

$$DM = r. \tag{1}$$

We have $BC = BD + DM + MC = BD + DM + \frac{1}{2}BC$, so

$$DM = \frac{1}{2}BC - BD. \tag{2}$$

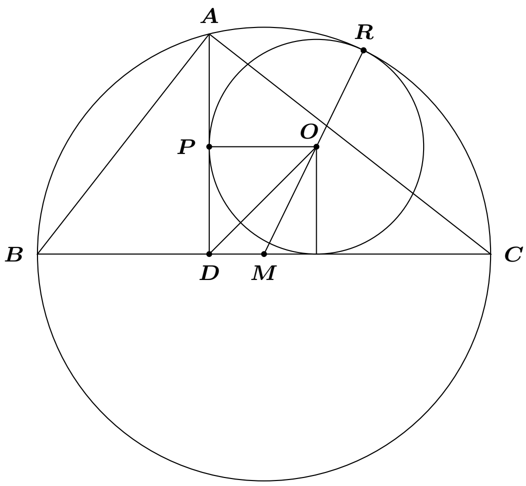


FIGURE 1

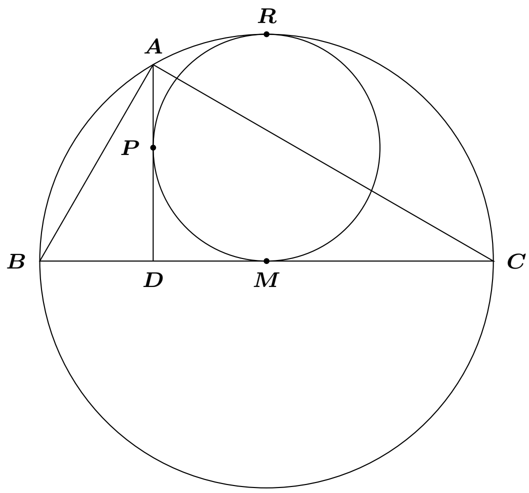


FIGURE 2

Let O be the center of Ω. By the law of cosines, applied to $\triangle ODM$, where $\angle ODM = 45^o$, $OD = OP \csc 45^o = r\sqrt{2}$ and $MO = MR - MO = \frac{1}{2}BC - r$,

$$\left(\frac{1}{2}BC - r\right)^2 = \left(\frac{1}{2}BC - BD\right)^2 + 2r^2 - 2\left(\frac{1}{2}BC - BD\right)^2 \cdot r\sqrt{2} \cdot \cos 45^o,$$

so

$$r = \sqrt{BC \cdot BD} - BD. \tag{3}$$

Sustituting for DM and r from (2) and (3) in (1), we get

$$BD = \frac{1}{4}BC.$$

By an standard mean proportion, then,

$$AB = \sqrt{BC \cdot BD} = \sqrt{(4 \cdot BD) \cdot BD} = 2 \cdot BD,$$

implying $\angle DAB = 30^o$ (Figure 2).

We conclude that $\angle ABC = 60^o$ and $\angle BCA = 30^\circ$.

Also solved by *by the proposer.*

EM–105. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* The union of nineteen planar surfaces, each of area equal to 1, has a total area equal to 10. Prove that the overlap of some two of these surfaces has an area greater than or equal to $1/19$.

Solution 1 by the proposer. Arguing by contradiction, let us assume that the area of the overlap of any two surfaces is less than $1/19$. In this case, if $S_1, S_2, \ldots, S_{19}$ denote the nineteen surfaces, then the area of $S_1 \cup S_2$ is greater than $1 + \dfrac{18}{19}$ because

$$[S_1 \cup S_2] = [S_1] + [S_2] - [S_1 \cap S_2] > 1 + 1 - \frac{1}{19} = 1 + \frac{18}{19},$$

the area of $S_1 \cup S_2 \cup S_3$ is greater than $1 + \dfrac{18}{19} + \dfrac{17}{19}$ because

$$[S_1 \cup S_2 \cup S_3] = [S_1] + [S_2] + [S_3] - [S_1 \cap S_2] - [S_2 \cap S_3] - [S_3 \cap S_1] + [S_1 \cap S_2 \cap S_3]$$

$$> 1 + 1 + 1 - \frac{1}{19} - \frac{1}{19} - \frac{1}{19} + [S_1 \cap S_2 \cap S_3] > 1 + \frac{18}{19} + \frac{17}{19},$$

and so on. Finally, the area of $S_1 \cup S_2 \cup \ldots \cup S_{19}$ is greater than

$$1 + \frac{18}{19} + \frac{17}{19} + \ldots + \frac{1}{19} = 10$$

a contradiction. Hence the conclusion.

Solution 2 by Hyunbin Yoo, South Korea. Since there are 19 surfaces, we can define $\frac{19 \cdot 18}{2} = 19 \cdot 9$ surfaces s_1 to $s_{19 \cdot 9}$ which correspond to the overlaps of two of the 19 surfaces. If two surfaces do not overlap that s_k is defined to be zero. WLOG, let $s_i \geq s_j$ if $i < j$. Then $1 \geq s_1 \geq s_2 \geq \cdots \geq s_{19 \cdot 9} \geq 0$. Since there are zero or more overlaps between different s_k themselves, $\sum_{k=1}^{19\cdot 9} s_k \geq 9$. Combining the two inequalities we get

$$(19\cdot 9)s_1 \geq \sum_{k=1}^{19\cdot 9} s_k \geq 9.$$

Divide by $19\cdot 9$ and we end up with $s_1 \geq \frac{1}{19}$. In conclusion, there is at least one overlap whose area is equal to or greater than $1/19$.

Solution 3 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. Let $S_1, S_2, \ldots, S_{19}$ denote the given planar surfaces and let $A(S)$ denote the area of a surface S. Then the *principle of inclusion-exclusion* yields

$$\begin{aligned} 10 \;=\; A(\bigcup_{i=1}^{19} S_i) \;&=\; \sum_{i=1}^{19} A(S_i) \;-\; \sum_{i<j} A(S_i \cap S_j) \;+\; R \\ &=\; 19 \;-\; \sum_{i<j} A(S_i \cap S_j) \;+\; R, \end{aligned}$$

where $R \geq 0$. That is, $10 > 19 \;-\; \sum_{i<j} A(S_i \cap S_j)$.

Now suppose that $A(S_i \cap S_j) < 1/19$ for all $i \neq j$. Then

$$\sum_{i<j} A(S_i \cap S_j) < \binom{19}{2} \cdot (1/19) = 9$$

and $19 \;-\; \sum_{i<j} A(S_i \cap S_j) > 19 - 9 = 10$. But this says that $10 > 10$, and this contradiction shows that the intersection of two of the surfaces must have an area greater than or equal to $1/19$.

Solution 4 by Michel Bataille, Rouen, France. Let $|A|$ denote the area of the surface A. We first show that if $A_1, A_2, \ldots, A_n$ ($n \geq 2$) are surfaces (with finite area), then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| \geq |A_1| + |A_2| + \cdots + |A_n| - \sum_{1\leq i<j\leq n} |A_i \cap A_j|. \quad (1)$$

The proof is by induction. For $n = 2$, equality holds. Let $n \geq 2$ be an integer, and assume that (1) holds for surfaces $A_1, \ldots, A_n$. Let A_{n+1} be a supplementary surface. Then, we have

$$|A_1 \cup A_2 \cup \cdots \cup A_{n+1}|$$

$$= |A_1 \cup A_2 \cup \cdots \cup A_n| + |A_{n+1}| - |A_{n+1} \cap (A_1 \cup A_2 \cup \cdots \cup A_n|$$
$$\geq |A_1| + |A_2| + \cdots + |A_n| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + |A_{n+1}| -$$
$$|(A_{n+1} \cap A_1) \cup (A_{n+1} \cap A_2) \cup \cdots (A_{n+1} \cap A_n)|$$
$$\geq |A_1| + |A_2| + \cdots + |A_{n+1}| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| - \sum_{k=1}^{n} |A_{n+1} \cap A_k|$$
$$= \sum_{k=1}^{n+1} |A_k| - \sum_{1 \leq i < j \leq n+1} |A_i \cap A_j|$$

(where the first inequality follows from (1) and seond one from the obvious $|X_1 \cup \cdots X_n| \leq |X_1| + \cdots + |X_n|$.
Returning to the problem, we now have $n = 19$, $|A_k| = 1$ for $k = 1, \ldots, 19$ and $|A_1 \cup A_2 \cup \cdots \cup A_{19}| = 10$. Should we have $|A_i \cap A_j| < 1/19$ for all pairs (i, j) such that $1 \leq i < j \leq 19$, we would have (from (1)):

$$10 > 19 - (1/19)\binom{19}{2} = 19 - 9 = 10,$$

a contradiction. Thus, we must have $|A_i \cap A_j| \geq 1/19$ for some pair (i, j).

EM–106. *Proposed by Nicolae Papacu, Slobozia, Romania.* Find all positive integers x, y, z such that $2^{x+1} + 45^y = z^2$.

Solution 1 by Michel Bataille, Rouen, France. The triple $(x, y, z) = (1, 1, 7)$ is an obvious solution. We show that there is no other solution.

Let (x, y, z) be a solution. Clearly, z is odd, hence $z^2 \equiv 1 \pmod{8}$.

Assume that $x > 1$. Then $2^{x+1} \equiv 0 \pmod{8}$ so that $45^y \equiv 1 \pmod{8}$. It follows that y is even (otherwise $45^y \equiv 45 \equiv 5 \pmod{8}$). Let $y = 2r$ where r is a positive integer. Since $2^{x+1} = (z - 45^r)(z + 45^r)$, we have $z - 45^r = 2^u$, $z + 45^r = 2^v$ for some integers u, v such that $0 \leq u < v$ and $u + v = x + 1 \geq 3$. We deduce that $2 \cdot 45^r = 2^v - 2^u = 2^u(2^{v-u} - 1)$ and it follows that $u = 1,\ v = x$ and

$$2^{x-1} = 45^r + 1. \tag{1}$$

Modulo 3, (1) gives $(-1)^{x-1} \equiv 1$, hence x is odd and so $x \geq 3$. But then (1) gives $0 \equiv 2 \pmod 4$, a contradiction. Thus, we must have $x = 1$ and y, z satisfy $3^{2y}5^y = (z-2)(z+2)$. Since 5 cannot divide both $z - 2$ and $z + 2$ (otherwise it would divide $4 = (z+2) - (z-2)$), 5^y divides either $z - 2$ or $z + 2$.

If $z+2 = q5^y$ for some positive integer $q \geq 1$, then $3^{2y} = q(q5^y - 4)$ so that $q = 3^a$, $q5^y - 4 = 3^b$ for some integers a, b satisfying $0 \leq a \leq b$ and $a + b = 2y$. From $3^a5^y - 3^b = 4$, we deduce that we must have $a = 0$. But this leads to $5^y - 9^y = 4$, which cannot occur (since $5^y - 9^y < 0$). Thus, the only possibility is $z - 2 = s5^y$ for some $s \geq 1$. In the same way as in the previous case, we now obtain $9^y - 5^y = 4$. But for $y \geq 2$, we have $9^y - 5^y = (5+4)^y - 5^y \geq 4y \cdot 5^{y-1} > 4$, hence $y = 1$. It follows that $x = y = 1$ and $z = 7$. This completes the proof.

Solution 2 by Daniel Văcaru, Pitești, Romania and the proposer (*Modified by the editor*). Notice that z is an odd number. For $x = 1$, we have $(z-2)(z+2) = 45^y$. Since $(z-2, z+2) = 1$ and $z - 2 < z + 2$, then we have

$$\left.\begin{aligned} z - 2 &= 1 \\ z + 2 &= 45^y \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} z - 2 &= 5^y \\ z + 2 &= 9^y \end{aligned}\right\}$$

The first system has no solutions because $z = 3$ and $5 = 45^y$ implies that y is not a positive integer. Subtracting the two equations in the second system, we obtain $9^y - 5^y = 4$. This equation has the solution $y = 1$. If $y \geq 2$, we have

$$9^y - 5^y = 5^y\left(\left(\frac{9}{5}\right)^y - 1\right) \geq 25\left(\left(\frac{9}{5}\right)^2 - 1\right) = 4.$$

Therefore for $x = 1$ we have $y = 1$ and $z = 7$ as a solution.

If $x \geq 2$, we have $2^{x+1} \equiv 0 \pmod 8$ and then $45^y \equiv 1 \pmod 8$. If y is and odd positive integer, $y = 2t + 1$, $t \geq 1$, then

$$45^y = 45^{2t+1} = 45^{2t} \cdot 45 \equiv 1 \pmod 8.$$

Since $45^{2t} \equiv 1 \pmod 8$ and $45 \equiv 5 \pmod 8$, then $45^y \equiv 5 \pmod 8$ in contradiction with $45^y \equiv 1 \pmod 8$. Hence y is even, say $y = 2t$, $t \geq 1$, and the equation becomes $5^{2t} + 2^{x+1} = z^2$ or $z^2 - 45^{2t} = 2^{x+1}$ or $(z - 45^t)(z + 45^t) = 2^{x+1}$ from which we obtain

$$\left.\begin{aligned} z - 45^t &= 2^v \\ z + 45^t &= 2^{x+1-v} \end{aligned}\right\}$$

with $0 \leq v < x + 1 - v$. Subtracting the two equations, we have $2 \cdot 45^t = 2^v(2^{x+1-2v} - 1)$. For $v = 1$, we get $45^t + 1 = 2^{x+1}$. Since $5^t + 1 \equiv 2 \pmod 4$, then $2^{x-1} \equiv 2 \pmod 4$ and $x - 1 = 1$ or $x = 2$. Then, we have $45^t = 1$ and $t = 1$ and $y = 0$. Contradiction because y is a positive integer. In conclusion the only solution of the equation is $1, 1, 7$.

Also solved by *José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.*

Medium–Hard Problems

MH–101. *Proposed by Michel Bataille, Rouen, France.* Find the extreme of

$$\frac{ab(a+b-1)}{(a^2+b^2+1)^3}$$

when a, b are real numbers satisfying $a^2+b^2=a+b$.

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Let $x=a+b$ and $y=a-b$. Then

$$\begin{aligned}\frac{ab(a+b-1)}{(a^2+b^2+1)^3} &= \frac{((a+b)^2-(a^2+b^2))(a+b-1)}{2(a+b+1)^3}\\ &= \frac{((a+b)^2-(a+b))(a+b-1)}{2(a+b+1)^3}\\ &= \frac{(x^2-x)(x-1)}{2(x+1)^3} = \frac{x(x-1)^2}{2(x+1)^3},\end{aligned}$$

and the condition $a^2+b^2=a+b$ becomes $x^2+y^2=2x$. Thus, determining the extrema of

$$\frac{ab(a+b-1)}{(a^2+b^2+1)^3}$$

when a, b are real numbers satisfying $a^2+b^2=a+b$ is equivalent to determining the extrema of

$$f(x)=\frac{x(x-1)^2}{2(x+1)^3}, \quad 0 \le x \le 2.$$

With

$$f'(x)=\frac{(x-1)(5x-1)}{2(x+1)^4},$$

the critical points of f are $x=\frac{1}{5}$ and $x=1$. Now,

$$f(0)=f(1)=0 \quad \text{and} \quad f\left(\frac{1}{5}\right)=f(2)=\frac{1}{27}.$$

Thus, the minimum value of f is 0, which occurs when $x = y = 0$ and when $x = 1$ and $y = \pm 1$; the maximum value of f is $\frac{1}{27}$, which occurs when $x = 2$ and $y = 0$ and when $x = \frac{1}{5}$ and $y = \pm\frac{3}{5}$.

Returning to the original problem, the minimum value of

$$\frac{ab(a+b-1)}{(a^2+b^2+1)^3}$$

when a, b are real numbers satisfying $a^2 + b^2 = a + b$ is 0, which occurs when $a = b = 0$, when $a = 1, b = 0$ and when $a = 0, b = 1$; the maximum value is $\frac{1}{27}$, which occurs when $a = b = 1$, when $a = \frac{2}{5}, b = -\frac{1}{5}$, and when $a = -\frac{1}{5}, b = \frac{2}{5}$.

Solution 2 by the proposer. Let

$$Q = \frac{ab(a+b-1)}{(a^2+b^2+1)^3}.$$

We show that under the constraint $a^2 + b^2 = a + b$ the minimum of Q is 0 and its maximum is $\frac{1}{27}$. Indeed, we have $Q = 0$ for $a = b = 0$ and $Q = \frac{1}{27}$ when $a = b = 1$, hence it remains to prove that $0 \le Q \le \frac{1}{27}$ holds for any a, b satisfying $a^2 + b^2 = a + b$.
Let a, b be such real numbers and let $u = \frac{2a}{1+a+b}$, $v = \frac{2b}{1+a+b}$, $w = \frac{1-a-b}{1+a+b}$. It is readily checked that $u + v + w = u^2 + v^2 + w^2 = 1$ so that $uv + vw + wu = 0$. Also $uvw = -4Q$ and we obtain that u, v, w are the roots of the polynomial

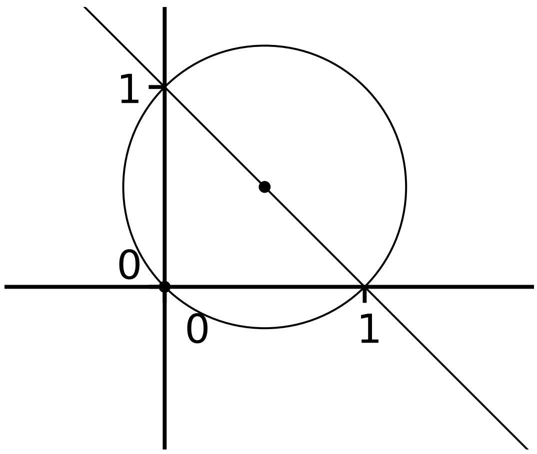
$$\begin{aligned} P(X) &= (X-u)(X-v)(X-w) = X^3 - X^2 + 4Q \\ &= \left(X - \frac{1}{3}\right)^3 - \frac{1}{3}\left(X - \frac{1}{3}\right) + 4Q - \frac{2}{27}. \end{aligned}$$

As a result, the polynomial $X^3 - \frac{X}{3} + 4Q - \frac{2}{27}$ has three real roots and the following condition must hold:

$$27\left(4Q - \frac{2}{27}\right)^2 + 4\left(-\frac{1}{3}\right)^3 \le 0.$$

The latter yields $\left|4Q - \frac{2}{27}\right| \le \frac{2}{27}$ and $0 \le Q \le \frac{1}{27}$ follows.

Note. The inequality $Q \geq 0$ can also be proved *via* the following geometrical argument. The condition $a^2 + b^2 = a + b$ means that the point with coordinates (a, b) is on the circle γ with equation $x^2 + y^2 - x - y = 0$, with centre $(1/2, 1/2)$ and radius $1/\sqrt{2}$.



Scheme for solving problem MH-101

The line ℓ with equation $x + y = 1$ divides γ into two half-circles: on the one which contains the origin we have $x + y - 1 \leq 0$ and either $x \geq 0, y \leq 0$ or $x \leq 0, y \geq 0$, hence $xy(x+y-1) \geq 0$, while on the other one we have $x, y, x+y-1 \geq 0$ and $xy(x+y-1) \geq 0$ again (see figure). It immediately follows that $Q \geq 0$.

Solution 3 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. The answer is $\frac{1}{27}$. Indeed, we rewrite the given equation $a^2 + b^2 = a + b$ in the form $a(a-1) = b(1-b)$ or, equivalently,

$$\frac{a}{b} = \frac{1-b}{a-1}.$$

Denoting by t the common value of these ratios, we have

$$\frac{a}{b} = \frac{1-b}{a-1} = t.$$

Solving for a and b simultaneously, we get

$$a = \frac{t(1+t)}{1+t^2} \qquad b = \frac{1+t}{1+t^2}.$$

When these are substituted into $\frac{ab(a+b-1)}{(a^2+b^2+1)^3}$, it becomes

$$\frac{(t(t+1))^2}{4(1+t+t^2)^3},$$

which we rewrite as

$$\frac{\left(\frac{1}{4}-\left(t+\frac{1}{2}\right)^2\right)^2}{4\left(\frac{3}{4}+\left(t+\frac{1}{2}\right)^2\right)^3}. \tag{1}$$

Maximizing (1) is equivalent to maximize the numerator and minimize the denominator. Therefore (1) is a maximum when $t+\frac{1}{2}=0$ i.e., $t=-\frac{1}{2}$. Thus (1) takes on its maximum value $\frac{1}{27}$, as claimed.

MH–102. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be three positive numbers. Prove that

$$\sqrt[3]{(a^a+b^a+c^a)(a^b+b^b+c^b)(a^c+b^c+c^c)} \geq a^{\frac{a+b+c}{3}}+b^{\frac{a+b+c}{3}}+c^{\frac{a+b+c}{3}}.$$

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. By Hölder's inequality,

$$\sqrt[3]{(a^a+b^a+c^a)(a^b+b^b+c^b)(a^c+b^c+c^c)}$$

$$\geq \sqrt[3]{a^a a^b a^c}+\sqrt[3]{b^a b^b b^c}+\sqrt[3]{c^a c^b c^c} = a^{\frac{a+b+c}{3}}+b^{\frac{a+b+c}{3}}+c^{\frac{a+b+c}{3}}.$$

Equality holds when $a=b=c$.

Solution 2 by Rovsen Pirkuliyev, Sumgayit City, Azerbaijan. To prove the inequality claimed, we consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \ln(a^x+b^x+c^x)$. We have $f'(x) = \dfrac{a^x \ln a + b^x \ln b + c^x \ln c}{a^x+b^x+c^x}$ and

$$f''(x) = \frac{\left(a^x \ln^2 a + b^x \ln^2 b + c^x \ln^2 c\right)(a^x+b^x+c^x)}{(a^x+b^x+c^x)^2}$$

$$-\frac{(a^x \ln a + b^x \ln b + c^x \ln c)^2}{(a^x+b^x+c^x)^2}.$$

We claim that $f''(x) \geq 0$. Indeed, we have to see that

$$\left(a^x \ln^2 a + b^x \ln^2 b + c^x \ln^2 c\right)(a^x+b^x+c^x) \geq (a^x \ln a + b^x \ln b + c^x \ln c)^2.$$

This inequality can be easily obtained applying CBS inequality to vectors $\vec{u} = \left(\sqrt{a^x}, \sqrt{b^x}, \sqrt{c^x}\right)$ and $\vec{v} = \left(\sqrt{a^x}\ln a, \sqrt{b^x}\ln b, \sqrt{c^x}\ln c\right)$. Therefore, f is convex and applying Jensen's inequality, we get

$$f\left(\frac{a+b+c}{3}\right) \leq \frac{1}{3}(f(a) + f(b) + f(c))$$

or

$$\ln\left(a^{\frac{a+b+c}{3}} + b^{\frac{a+b+c}{3}} + c^{\frac{a+b+c}{3}}\right)$$

$$\leq \frac{1}{3}\left[\ln(a^a + b^a + c^a) + \ln\left(a^b + b^b + c^b\right) + \ln(a^c + b^c + c^c)\right].$$

Now, taking into account the properties of the logarithm function, the statement follows. Equality holds when $a = b = c$, and we are done.

Solution 3 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. We apply Hölder's inequality in the form

$$(a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3) \geq \left(\sqrt[3]{a_1a_2a_3} + \sqrt[3]{b_1b_2b_3} + \sqrt[3]{c_1c_2c_3}\right)^3,$$

to obtain $\quad (a^a + b^a + c^a)(a^b + b^b + c^b)(a^c + b^c + c^c)$

$$\geq \left(\sqrt[3]{a^a a^b a^c} + \sqrt[3]{b^a b^b b^c} + \sqrt[3]{c^a c^b c^c}\right)^3 = \left(a^{\frac{a+b+c}{3}} + b^{\frac{a+b+c}{3}} + c^{\frac{a+b+c}{3}}\right)^3,$$

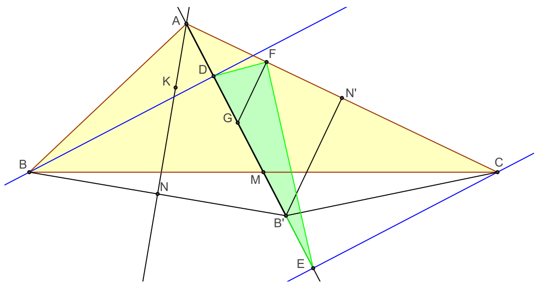
which is equivalent to the proposed inequality. Equality holds if and only if $a = b = c$.

Also solved by *Michel Bataille, Rouen, France; and the proposer.*

MH–103. *Proposed by Todor Zaharinov, Sofia, Bulgaria.* Let ABC be a scalene triangle with centroid G and symmedian point K, such that AK is the angle bisector of the angle BAG. D, E and F are the feet of perpendiculars from B to the line AG, from C to the line AG and from G to AC respectively. Prove that the area of triangle ABC is $[ABC] = 6[DEF]$.

Solution 1 by the proposer. It is well known that the isogonal conjugate of the centroid G is the symmedian point K, so $\angle BAK = \angle GAC$. It follows that

$$\angle BAK = \angle KAG = \angle GAC = \alpha;\ \alpha < 60^\circ$$



Scheme for solving problem MH-103 (Case 1)

Let N be the foot of the perpendicular from B to AK, B' be symmetric of B with respect to the line AK, and N' be symmetric of N with respect to the line AG. It is clear that $B' \in AG, N' \in AC$ and $\triangle ABN \cong \triangle AB'N \cong \triangle AB'N'$. Hence $AB' = AB = c, AN' = AN = c\cos\alpha$.

Let M be the midpoint of BC.

$$[BAM] = \frac{1}{2}BA.AM.\sin 2\alpha$$

$$[MAC] = \frac{1}{2}CA.AM.\sin\alpha$$

$$[BAM] = [MAC] = \frac{1}{2}[ABC]$$

$$\frac{1}{2}AM.c\sin 2\alpha = \frac{1}{2}AM.b\sin\alpha$$

$$2c\sin\alpha\cos\alpha = b\sin\alpha$$

$$\cos\alpha = \frac{b}{2c} \tag{1}$$

But $AC = b = 2c\cos\alpha = 2AN'$. Hence N' is the midpoint of AC and $\triangle CB'N' \cong \triangle AB'N'$; $CB' = AB' = c$ and $\angle B'CN' = \angle B'AN' = \alpha$.

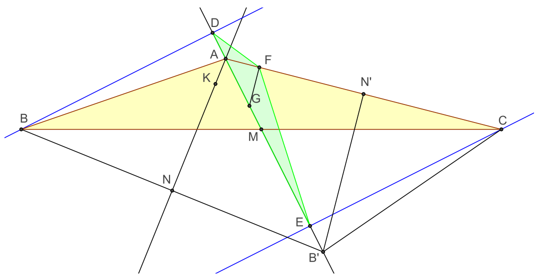
Case 1: D is between A and E.

$$\angle ECB' = \angle ECA - \alpha = 90^\circ - 2\alpha = \angle DBA$$

It follows that $\triangle ECB' \cong \triangle DBA$ and $B'E = AD$.

$$DE = AB' + B'E - AD = AB' = AB = c.$$

Case 2: A is between D and E.



Scheme for solving problem MH-103 (Case 2)

$$\angle EB'C = 180^\circ - 2\alpha = \angle DAB$$

It follows that $\triangle ECB' \cong \triangle DBA$ and $B'E = AD$.

$$DE = DA + AB' - B'E = AB' = AB = c.$$

Hence $DE = AB' = c$.

Points A, D, E, B' are collinear and $DE = AB'$, so $[DEF] = [AB'F]$.

$$\begin{aligned}[DEF] = [AB'F] &= \frac{1}{2}AB'.AF.\sin\alpha = \frac{1}{2}cAG\cos\alpha.\sin\alpha \\ &\overset{(1)}{=} \frac{1}{2}cAG\frac{b}{2c}\sin\alpha = \frac{1}{2}AG\frac{b}{2}\sin\alpha = \frac{1}{2}AG.AN'\sin\alpha \\ &= [AGN'] = \frac{1}{2}[AGC] = \frac{1}{6}[ABC]\end{aligned}$$

from which the desired result follows.

Solution 2 by Michel Bataille, Rouen, France. Let $a = BC, b = CA, c = AB$, M be the midpoint of BC and let $\theta = \angle BAK = \angle KAG$. Since AK and AG are symymmetrical in the angle bisector of $A = \angle BAC$, we have $\theta + \frac{\theta}{2} = \frac{A}{2}$, hence $\theta = \frac{A}{3}$.
From

$$\frac{a/2}{\sin(2A/3)} = \frac{c}{\sin(\angle AMB)} \quad \text{and} \quad \frac{a/2}{\sin(A/3)} = \frac{b}{\sin(\angle AMC)}$$

(by the law of sines) and $\sin(\angle AMC) = \sin(\angle AMB)$, we deduce that $c \sin(2A/3) = b \sin(A/3)$ from which $\cos(A/3) = \frac{b}{2c}$ follows. This said, we use the barycentric coordinates relatively to (A, B, C). Since $F = tA + (1-t)C$ for some real t and $3G = A + B + C$, we have

$$\overrightarrow{GF} = F - G = \left(t - \frac{1}{3}\right)A - \frac{1}{3}B + \left(\frac{2}{3} - t\right)C = \left(t - \frac{1}{3}\right)\overrightarrow{CA} - \frac{1}{3}\overrightarrow{CB}.$$

From the dot products $\overrightarrow{GF} \cdot \overrightarrow{CA} = 0$ and $\overrightarrow{CA} \cdot \overrightarrow{CB} = \frac{a^2+b^2-c^2}{2}$, we deduce that $0 = tb^2 - \frac{3b^2+a^2-c^2}{6}$, that is, $t = \frac{3b^2+a^2-c^2}{6b^2}$ and therefore

$$6b^2F = (3b^2 + a^2 - c^2)A + (3b^2 - a^2 + c^2)C.$$

The point D is on the median $y = z$, hence $D = uA + v(B + C)$ for u, v satisfying $u + 2v = 1$ and $\overrightarrow{BD} = u\overrightarrow{BA} + v\overrightarrow{BC}$. Since $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{BC} - \overrightarrow{BA}$ and $\overrightarrow{BD} \cdot \overrightarrow{AM} = 0$, we obtain $u(a^2 - b^2 - 3c^2) + 2v(b^2 - c^2) = 0$ so that

$$2(a^2 - 2b^2 - 2c^2)D = 2(c^2 - b^2)A + (a^2 - b^2 - 3c^2)(B + C).$$

Since M is also the midpoint of DE, we have $E = 2M - D = B + C - D$, hence

$$2(a^2 - 2b^2 - 2c^2)E = 2(b^2 - c^2)A + (a^2 - c^2 - 3b^2)(B + C).$$

It follows that $\frac{[DEF]}{[ABC]} = [24b^2(a^2 - 2b^2 - 2c^2)^2]^{-1} \cdot |\delta|$ where δ is the determinant

$$\begin{vmatrix} 3b^2 + a^2 - c^2 & 2(c^2 - b^2) & 2(b^2 - c^2) \\ 0 & a^2 - 3c^2 - b^2 & a^2 - 3b^2 - c^2 \\ 3b^2 - a^2 + c^2 & a^2 - 3c^2 - b^2 & a^2 - 3b^2 - c^2 \end{vmatrix}.$$

Subtracting row 2 from row 3 and then adding column 2 to column 3 shows that

$$\delta = \begin{vmatrix} 3b^2 + a^2 - c^2 & 2(c^2 - b^2) & 0 \\ 0 & a^2 - 3c^2 - b^2 & 2(a^2 - 2b^2 - 2c^2) \\ 3b^2 - a^2 + c^2 & 0 & 0 \end{vmatrix}$$

so that $\delta = 4(a^2 - 2b^2 - 2c^2)(c^2 - b^2)(3b^2 - a^2 + c^2)$.
Thus, to answer the problem, it suffices to prove that $(c^2-b^2)(3b^2-a^2+c^2) = b^2(a^2-2b^2-2c^2)$, that is, $b^4+c^2a^2 = c^4+4b^2c^2$. We are done since a short calculation shows that this directly follows from $\cos A = 4\cos^3\frac{A}{3} - 3\cos\frac{A}{3}$, $\cos A = \frac{b^2+c^2-a^2}{2bc}$ and $\cos(A/3) = \frac{b}{2c}$.

MH–104. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* On a 2023×2023 board symmetrical with respect to the main diagonal, the integer numbers from 1 to 2023 are placed so that in each row and column each number appears once and only once. Show that all the numbers appear on the main diagonal.

Solution 1 by the proposer. Suppose that one of the numbers in $\{1, 2, 3, \ldots, 2023\}$ does not appear in the main diagonal. WLOG we may assume that this number is 1. Since it must appear in all the rows we have 2023 one's that must be placed symmetrically above and below the main diagonal. But by symmetry there must be the same number of 1's above and below the main diagonal which is impossible if there are and odd number of them. Hence, the statement is true and we are done.

Solution 2 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Each integer from 1 to 2023 appears once and only once in each row and each column, so each integer from 1 to 2023 appears exactly 2023 times on the board. Let j be an arbitrary integer from 1 to 2023. Each time j appears on the board at a location off the main diagonal, that appearance is paired with another at the location symmetric with respect to the main diagonal. Because j appears 2023 times on the board, an odd number of times, it follows that at least one appearance must be along the main diagonal. As this is true for every integer from 1 to 2023 and there are only 2023 locations along the main diagonal of the board, each integer from 1 to 2023 must appear exactly once along the main diagonal.

Note: The general case of an $n \times n$ board symmetrical with respect to the main diagonal and with n an odd integer is Problem 1 from the 1967 edition of the Indiana College Mathematics Competition.

Also solved by *Félix Moreno Peñarrubia, Charles University, Prague, Czech Republic.*

MH–105. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* If $[1, 2, \ldots, 2n]$ represents the least common multiple of the integers $1, 2, \ldots, 2n$, then show that

$$\frac{\Big[1, 2, \ldots, 2n\Big]}{\binom{n}{0}^2 + \binom{n}{1}^2 + \ldots + \binom{n}{n}^2}$$

is an integer number.

Solution 1 by Michel Bataille, Rouen, France. It is well-known that

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$$

hence we have to prove that

$$\frac{[1, 2, \ldots, 2n]}{\binom{2n}{n}} \in \mathbb{N}. \tag{1}$$

Now, consider the integral $I_n = \int_0^1 x^{n-1}(1-x)^n \, dx$. On the one hand, we have

$$I_n = B(n, n+1) = \frac{\Gamma(n)\Gamma(n+1)}{\Gamma(2n+1)} = \frac{(n-1)!n!}{(2n)!} = \frac{1}{n\binom{2n}{n}}$$

and on the other hand

$$I_n = \int_0^1 x^{n-1}\left(\sum_{j=0}^{n}\binom{n}{j}(-1)^j x^j\right) dx = \sum_{j=0}^{n}(-1)^j\binom{n}{j}\int_0^1 x^{n-1+j} \, dx$$

so that

$$I_n = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \cdot \frac{1}{n+j}.$$

It follows that $[1, 2, \ldots, 2n] \cdot I_n \in \mathbb{N}$ and therefore $\frac{[1,2,\ldots,2n]}{n\binom{2n}{n}} \in \mathbb{N}$, which implies (1).

Solution 2 by the proposer. Let $L = [1, 2, 3, \ldots, 2n]$ and let $M = \binom{2n}{n}$. To prove that $M \mid L$, it suffices to show that $e_p(M) \leq e_p(L)$ for each prime p. Since L is the smallest positive integer that is divisible by each of $1, 2, 3, \ldots, 2n$, it follows that $e_p(L)$ is the largest integer m such that $p^m \leq 2n$, namely $m = \lfloor \log(2n)/\log(p) \rfloor$. By Legendre's formula, we have

$$e_p(M) = e_p(2n!) - 2e_p(n!) = \sum_{r=1}^{m} \left\{ \left\lfloor \frac{2n}{p^r} \right\rfloor - 2 \left\lfloor \frac{n}{p^r} \right\rfloor \right\}.$$

(Since $p^r > 2n$ for all $r > m$, we have included all of the non vanishing terms in Legendre's formula.) Note that $\lfloor 2x \rfloor - 2\lfloor x \rfloor$ is either 0 or 1 (depending on whether or not $x - \lfloor x \rfloor$ is less than 1/2). In view of this fact, each of the terms in the above sum is either 0 or 1, and thus $e_p(M) \leq m = e_p(L)$.

MH–106. *Proposed by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA.* If n is a multiple of 8, prove that the number of subsets of $\{1, 2, \ldots, n\}$ having cardinality divisible by 4 is $2^{n-2} + 2^{(n-2)/2}$.

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. Let $n = 8m$ for some positive integer m. The number of subsets of $\{1, 2, \ldots, n\}$ having cardinality divisible by 4 is then

$$\sum_{k=0}^{2m} \binom{8m}{4k}.$$

By the binomial theorem,

$$\sum_{k=0}^{8m} \binom{8m}{k} x^k = (1+x)^{8m}.$$

It follows that

$$\sum_{k=0}^{4m} \binom{8m}{2k} x^{2k} = \frac{(1+x)^{8m} + (1-x)^{8m}}{2},$$

and

$$\sum_{k=0}^{2m} \binom{8m}{4k} x^{4k} = \frac{(1+x)^{8m} + (1-x)^{8m} + (1+ix)^{8m} + (1-ix)^{8m}}{4}.$$

Thus,

$$\begin{aligned}
\sum_{k=0}^{2m} \binom{8m}{4k} &= \left.\frac{(1+x)^{8m} + (1-x)^{8m} + (1+ix)^{8m} + (1-ix)^{8m}}{4}\right|_{x=1} \\
&= \frac{2^{8m} + (1+i)^{8m} + (1-i)^{8m}}{4} \\
&= \frac{2^{8m} + (\sqrt{2})^{8m} + (\sqrt{2})^{8m}}{4} \\
&= 2^{8m-2} + 2^{4m-1} \\
&= 2^{n-2} + 2^{(n-2)/2}.
\end{aligned}$$

Solution 2 by Michel Bataille, Rouen, France. The subsets of $\{1, 2, \ldots, n\}$ having cardinality divisible by 4 are those of cardinalities $0, 4, 8, \ldots, 4 \cdot \frac{n}{4}$. Their total number is

$$S_n = \binom{n}{0} + \binom{n}{4} + \cdots + \binom{n}{n}.$$

Let $T_n = \binom{n}{2} + \binom{n}{6} + \cdots + \binom{n}{n-2}$ so that $S_n + T_n = \sum_{k=0}^{n/2} \binom{n}{2k} = 2^{n-1}$. On the other hand, from the binomial theorem, we have

$$(1+i)^n = \sum_{k=0}^{n} \binom{n}{k} i^k = \sum_{k=0}^{n/2} \binom{n}{2k} (-1)^k + i \sum_{k=1}^{n/2} \binom{n}{2k-1} (-1)^{k-1},$$

hence

$$S_n - T_n = \mathrm{Re}((1+i)^n) = \mathrm{Re}(2^{n/2} e^{in\pi/4}) = 2^{n/2} \cos \frac{n\pi}{4} = 2^{n/2},$$

where the last equality follows from the fact that n is a multiple of 8. From $S_n + T_n = 2^{n-1}$ and $S_n - T_n = 2^{n/2}$, we deduce that $S_n = \frac{1}{2}(2^{n-1} + 2^{n/2}) = 2^{n-2} + 2^{(n-2)/2}$, as desired.

Solution 3 by the proposer. As a consequence of the binomial theorem, we have the fact that for a set S of cardinality n *the number of subsets of even cardinality equals the number of subsets of odd cardinality.* Since the power set of S has cardinality 2^n. we can see that the number of subsets of S having *even* cardinality is of size $\frac{1}{2} \cdot 2^n = 2^{n-1}$. Furthermore, letting $N(a \bmod b)$ denote the number of subsets whose cardinality is $a \bmod b$, we have

$$N(0 \bmod 2) = N(0 \bmod 4) + N(2 \bmod 4) = A + B = 2^{n-1}. \quad (1)$$

Since $1+i = \sqrt{2}e^{i\pi/4}$ and $1-i = \sqrt{2}e^{-i\pi/4}$, we have $(1+i)^n = 2^{n/2}$ and $(1-i)^n = 2^{n/2}$. Looking at this another way, the binomial theorem yields, where all congruences are taken modulo 4,

$$(1+i)^n \;=\; \sum_{k=0}^{n}\binom{n}{k}i^k \;=\; \sum_{k\equiv 0} + \sum_{k\equiv 1} + \sum_{k\equiv 2} + \sum_{k\equiv 3}.$$

Referring to relation (1), we see that the real part of this last expansion is

$$2^{n/2} \;=\; \sum_{k\equiv 0}\binom{n}{k}i^k \;+\; \sum_{k\equiv 2}\binom{n}{k}i^k = A - B \qquad (2)$$

since the terms in $\sum_{k\equiv 0}\binom{n}{k}i^k$ involve $i^4 = 1$ and the terms in $\sum_{k\equiv 2}\binom{n}{k}i^k$ have $i^2 = -1$. Now (1) and (2) allow us to solve for A:

$$2A = 2^{n-1} + 2^{n/2}, \text{ or } \quad A = 2^{n-2} + 2^{(n-2)/2}.$$

Solution 4 by the proposer. As a consequence of the binomial theorem, we know that for a set S of cardinality n, *the number of subsets of even cardinality equals the number of subsets of odd cardinality.* Since the power set of S has cardinality 2^n, we see that the number of subsets of S having even cardinality is $\frac{1}{2} \cdot 2^n = 2^{n-1}$. Letting S_2 be the number of subsets whose size is even, S_4 the number with size divisible by 4, and T the number with size divisible by 2 but not 4, we have

$$S_2 \;=\; S_4 \;+\; T \;=\; 2^{n-1}. \quad (3)$$

Writing $1+i=\sqrt{2}e^{i\pi/4}$, where $i=\sqrt{-1}$, we have $(1+i)^n=2^{n/2}$ and

$$2^{n/2}=(1+i)^n=\sum_{k=0}^{n}\binom{n}{k}i^k$$

$$=\sum_{k\equiv 0}\binom{n}{k}i^k+\sum_{k\equiv 1}\binom{n}{k}i^k+\sum_{k\equiv 2}\binom{n}{k}i^k+\sum_{k\equiv 3}\binom{n}{k}i^k, \quad (4)$$

where the congruences are modulo 4. Since the terms in $\sum_{k\equiv 0}$ involve $i^4=1$ and the terms in $\sum_{k\equiv 2}$ involve $i^2=-1$, the real part of the right-hand side of (2) is

$$\sum_{k\equiv 0}(1) \ + \ \sum_{k\equiv 2}(-1) \ = \ S_4 \ - \ T \ = \ 2^{n/2}\,. \qquad (3)$$

Adding expressions (3) and (5), we find that $2S_4=2^{n-1}+2^{n/2}$, or $S_4=2^{n-2}+2^{(n-2)/2}$.

Also solved by *Hyunbin Yoo, South Korea; and Daniel Văcaru, Pitești, Romania.*

Advanced Problems

A–101. *Proposed by Michel Bataille, Rouen, France.* For each positive integer n, let A_n and G_n be the arithmetic mean and the geometric mean of the n integers $n^2+1, n^2+2, \ldots, n^2+n$. Find

$$\lim_{n\to\infty}(A_n - G_n).$$

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. For each positive integer n,

$$A_n = \frac{1}{n}\sum_{j=1}^{n}(n^2+j) = \frac{1}{n}\left(n^3 + \frac{n(n+1)}{2}\right) = n^2 + \frac{n}{2} + \frac{1}{2},$$

and

$$G_n = \sqrt[n]{\prod_{j=1}^{n}(n^2+j)} = n^2\sqrt[n]{\prod_{j=1}^{n}\left(1+\frac{j}{n^2}\right)}.$$

Next,

$$\begin{aligned}
\ln\prod_{j=1}^{n}\left(1+\frac{j}{n^2}\right) &= \sum_{j=1}^{n}\ln\left(1+\frac{j}{n^2}\right)\\
&= \sum_{j=1}^{n}\left(\frac{j}{n^2} - \frac{j^2}{2n^4} + \frac{j^3}{3n^6} + O\left(\frac{j^4}{n^8}\right)\right)\\
&= \frac{n(n+1)}{2n^2} - \frac{n(n+1)(2n+1)}{12n^4} + \frac{n^2(n+1)^2}{12n^6} + O\left(\frac{1}{n^3}\right)\\
&= \frac{1}{2} + \frac{1}{3n} - \frac{1}{6n^2} + O\left(\frac{1}{n^3}\right),
\end{aligned}$$

so

$$\begin{aligned}
G_n &= n^2\sqrt[n]{\exp\left(\frac{1}{2} + \frac{1}{3n} - \frac{1}{6n^2} + O\left(\frac{1}{n^3}\right)\right)}\\
&= n^2\exp\left(\frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{6n^3} + O\left(\frac{1}{n^4}\right)\right)\\
&= n^2\left(1 + \frac{1}{2n} + \frac{1}{3n^2} - \frac{1}{6n^3} + \frac{1}{8n^2} + \frac{1}{6n^3} + \frac{1}{48n^3} + O\left(\frac{1}{n^4}\right)\right)\\
&= n^2 + \frac{n}{2} + \frac{11}{24} + O\left(\frac{1}{n}\right).
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}(A_n - G_n) = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{11}{24} - O\left(\frac{1}{n}\right)\right) = \frac{1}{24}.$$

Solution 2 by Moti Levy, Rehovot, Israel. We have

$$A_n = \frac{1}{n}\sum_{k=1}^{n}(n^2+k) = n^2 + \frac{1}{2}n + \frac{1}{2}.$$

$$G_n = \sqrt[n]{\prod_{k=1}^{n}(n^2+k)} = \sqrt[n]{\frac{(n+n^2)!}{(n^2)!}}.$$

We use the following Stirling's series,

$$\ln(n!) = \left(n+\frac{1}{2}\right)\ln(n) - n + \frac{1}{2}\ln(2\pi) + O\left(\frac{1}{n}\right),$$

and apply it on $n+n^2$ and on n^2 :

$$\ln\left((n+n^2)!\right) = \left(n+n^2+\frac{1}{2}\right)\ln\left(n+n^2\right) - n - n^2 + O\left(\frac{1}{n}\right)$$

$$\ln\left((n^2)!\right) = \left(n^2+\frac{1}{2}\right)\ln\left(n^2\right) - n^2 + O\left(\frac{1}{n}\right)$$

$$\begin{aligned}\ln(G_n) &= \frac{(\ln((n+n^2)!) - \ln((n^2)!))}{n}\\ &= \left(1-n-\frac{1}{2n}\right)\ln(n) + \left(1+n+\frac{1}{2n}\right)\ln(1+n) - 1 + O\left(\frac{1}{n}\right)\end{aligned}$$

$$\begin{aligned}\lim(A_n - G_n) &= \lim_{n\to\infty}\left(n^2+\frac{1}{2}n+\frac{1}{2} - \frac{\left(1+\frac{1}{n}\right)^n}{e}n^2\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^{\frac{1}{2n}}\right)\\ &= \lim_{n\to\infty}\left(n^2+\frac{1}{2}n+\frac{1}{2} - \frac{\left(1+\frac{1}{n}\right)^n}{e}n^2\left(1+\frac{1}{n}\right)\left(1+\frac{1}{2n^2}\right)\right)\\ &= \lim_{n\to\infty}\left(-\frac{n}{2} + \left(1-\frac{\left(1+\frac{1}{n}\right)^n}{e}\right)\left(n^2+n+\frac{1}{2}\right)\right)\\ &\lim_{n\to\infty}\left(\left(1-\frac{\left(1+\frac{1}{n}\right)^n}{e}\right)n^2 - \frac{n}{2}\right) + \lim_{n\to\infty}\left(\left(1-\frac{\left(1+\frac{1}{n}\right)^n}{e}\right)n\right)\end{aligned}$$

By applying L'Hôpital's rule twice on each of the following limits, we obtain,

$$\lim_{n\to\infty}\left(\left(1-\frac{\left(1+\frac{1}{n}\right)^n}{e}\right)n^2-\frac{n}{2}\right)=-\frac{11}{24},$$
$$\lim_{n\to\infty}\left(\left(1-\frac{\left(1+\frac{1}{n}\right)^n}{e}\right)n\right)=\frac{1}{2}.$$

It follows that $\lim_{n\to\infty}(A_n-G_n)=\frac{1}{24}$.

Also solved by *the proposer.*

A–102. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In how many ways can the rational $2022/2021$ be written as the product of two rational numbers of the form $(n+1)/n$, where n is a positive integer?

Solution 1 by the proposer. We begin by proving a more general result. Let a be a positive integer, and denote by $\tau(a)$ and $\tau(a+1)$, the number of positive divisors of a and $a+1$ respectively. Suppose that
$$\frac{a+1}{a}=\frac{p+1}{p}\cdot\frac{q+1}{q},$$
where p and q are positive integers exceeding a. Then $(a+1)pq=a(p+1)(q+1)$, which reduces to $(p-a)(q-a)=a(a+1)$. It follows that $p=a+u$ and $q=a+v$, where $uv=a(a+1)$. Hence, every representation of $(a+1)/a$ corresponds to a factorization of $a(a+1)$.

On the other hand, observe that, if $uv=a(a+1)$, then

$$\frac{a+u+1}{a+u}\cdot\frac{a+v+1}{a+v}=\frac{a^2+a(u+v+2)+uv+(u+v)+1}{a^2+a(u+v)+uv}$$
$$=\frac{a^2+(a+1)(u+v)+a(a+1)+2a+1}{a^2+a(u+v)+a(a+1)}$$
$$=\frac{(a+1)^2+(a+1)(u+v)+a(a+1)}{a^2+a(u+v)+a(a+1)}$$

$$= \frac{(a+1)\,[(a+1)+(u+v)+a]}{a[a+(u+v)+a+1]} = \frac{a+1}{a}.$$

Hence, there is a one-one correspondence between representations and unordered pairs (u, v) of complementary factors of $a(a+1)$. (Since $a(a+1)$ is not square, u and v are always distinct.) Since a and $a+1$ are coprime, the number of factors of $a(a+1)$ is equal to $\tau(a)\tau(a+1)$, and so the number of representations is

$$N = \frac{\tau(a)\tau(a+1)}{2}.$$

Now consider the case that $a = 2021$. Since $2021 = 43 \cdot 47$, then $\tau(2021) = 4$ and from $2022 = 2 \cdot 3 \cdot 337$ we get $\tau(2022) = 8$. Hence, the desired number of representations is $N = 16$.

Solution 2 by Moti Levy, and Moshe Goldstein, Rehovot, Israel. We are asked to find positive numbers n, m such that

$$\frac{n+1}{n}\frac{m+1}{m} = \frac{2022}{2021}. \tag{1}$$

Simple manipulation of (1) leads to necessary condition on the positive numbers n, m:

$$2021(m+n+1) = mn. \tag{2}$$

Clearly, there must be a positive integer k such that

$$mn = 2021k. \tag{3}$$

From (2) and (3), we have the following equations:

$$m+n+1 = k \tag{4}$$

$$m+n+1 = \frac{mn}{2021} = k \tag{5}$$

Using (4) and (5), we express n in terms of m,

$$n = \frac{2021(m+1)}{m-2021}, \tag{6}$$

It follows from (6) that $m - 2021$ must divide $m(m+1)$,

$$(m - 2021) \mid m(m+1). \tag{7}$$

Setting $l = m - 2021$, (7) becomes

$$l \mid (l + 2021)(l + 2022). \tag{8}$$

It follows from (8) that l must divide $2021 * 2022$.

Since $2021 \cdot 2022 = 2 \cdot 3 \cdot 43 \cdot 47 \cdot 337$, there are 2^5 divisors.
In order to avoid duplicates we require that $m < n,$. That is, $m < \frac{2021(m+1)}{m-2021}$, which implies $m \leq 4042$ and $l \leq 2021$. The divisors of $202 \cdot 2022$ which are less or equal 2021 are

$l \in \{1, 2, 3, 6, 43, 47, 86, 94, 129, 141, 258, 282, 337, 674, 1011, 2021\}$.

One can check that the pairs (m, n) belonging to the set

$$\left\{ \begin{array}{c} \{2022, 4088483\}, \{2023, 2045252\}, \{2024, 1364175\}, \{2027, 683098\}, \\ \{2064, 97055\}, \{2068, 88967\}, \{2107, 49538\}, \{2115, 45494\}, \\ \{2150, 33699\}, \{2162, 31003\}, \{2279, 17860\}, \{2303, 16512\}, \\ \{2358, 14147\}, \{2695, 8084\}, \{3032, 6063\}, \{4042, 4043\} \end{array} \right\},$$

satisfy $\frac{n+1}{n}\frac{m+1}{m} = \frac{2022}{2021}$. We conclude that there are 16 ways such that the rational $\frac{2022}{2021}$ can be written as the product of two rational numbers of the form $\frac{n+1}{n}$.

Also solved by *Michel Bataille, Rouen, France; Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA*

A–103. *Proposed by Joseph Santmyer, US Federal Government (retired), Las Crues, New Mexico, USA.* (Correction) Let us denote by A and B the following integrals

$$\begin{aligned} A &= \int_0^{2\pi} \cos(t) \sin[e^{\cos(t)} \cos(\sin(t) - t)] \cosh[e^{\cos(t)} \sin(\sin(t) - t)] dt, \\ B &= \int_0^{2\pi} \sin(t) \cos[e^{\cos(t)} \cos(\sin(t) - t)] \sinh[e^{\cos(t)} \sin(\sin(t) - t)] dt. \end{aligned}$$

Prove that
$$2\pi \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)^{2n}}{(2n+1)!(2n)!} = A - B.$$

Editor's comment. Solutions to this problem will appear in the next issue.

A–104. *Proposed by Gabriel T. Prăjitură, SUNY Brockport, Brockport, NY, USA.* Let f be a real function with the intermediate value property. If there is p a point at which f is not continuous, prove that there is an interval (α, β) such that for every $y \in (\alpha, \beta)$, $f^{-1}\{y\}$ has infinite cardinality.

Solution by the proposer. Since f is not continuous at p, there is a sequence (x_n) such that
$$\lim_n x_n = p \qquad \text{and} \qquad \lim_n f(x_n) \neq f(p)$$

Therefore either
$$\limsup_n f(x_n) \neq f(p) \qquad \text{or} \qquad \liminf_n f(x_n) \neq f(p).$$

I will assume the first case, for the second the proof can be easily modified accordingly.

Let $r = \limsup_n f(x_n)$. Then there is a subsequence (x_{k_n}) such that $r = \lim_n f(x_{k_n})$. Furthermore I will assume that $r > f(p)$, the proof being similar for the case $r < f(p)$. Let
$$s = \frac{r - f(p)}{2}$$
There is m such that
$$f(x_{k_n}) \in (r - s, r + s)$$
for all $n \geq m$. Let $y \in (f(p), f(p) + s)$. Then $y \in (f(p), f(x_{k_{m+1}}))$. Then there is $z_1 \in (p, x_{k_{m+1}})$ or in $(x_{k_{m+1}}, p)$ such that $f(z_1) = y$. Since $\lim_n x_{k_n} = p$, there is $v > m + 1$ such that $x_{k_v} \in (p, z_1)$ or in (z_1, p). As above we get that there is $z_2 \in (p, x_{k_v})$ or in (x_{k_v}, p) (and thus $z_1 \neq z_2$) such that $f(z_2) = y$. Continuing in the same way we obtain a sequence $z_n \to p$ such that $f(z_n) = y$.

A–105. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $A_n \in \mathcal{M}_n(\mathbb{R})$ be the matrix with entries $a_{ij} = \dfrac{1}{i+j-1}$, $1 \le i,j \le n$. Compute $\det(A_n)$ and the sum of the entries of A_n^{-1}.

Solution by the proposer. Subtracting the last row of A_n from the preceding ones and factoring, we get

$$\det(A_n) = \frac{(n-1)!^2}{(2n-1)!} \det\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n-1} & \frac{1}{n} & \frac{1}{n+1} & \cdots & \frac{1}{2n-2} \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$= \frac{(n-1)!^4}{(2n-1)!(2n-2)!} \det(A_{n-1}) = \frac{[1!2!\cdots(n-1)!]^3}{n!(n+1)!\cdots(2n-1)!}$$

on account that $\det(A_1)$ is nonzero.

To compute the sum of the entries of A_n^{-1} we consider the linear system

$$A_n \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

Then the sum of the entries of A_n^{-1} is $x_1 + x_2 + \ldots + x_n$. Now, we consider the function

$$f(x) = \frac{x_1}{x+1} + \frac{x_2}{x+2} + \ldots + \frac{x_n}{x+n} = \frac{A(x)}{(x+1)(x+2)\ldots(x+n)},$$

where $A(x)$ is a polynomial of degree at most $n-1$. since $f(0) = f(1) = \ldots = f(n-1) = 1$, then the zeros of polynomial $B(x) = (x+1)(x+2)\ldots(x+n) - A(x)$ are $0, 1, \ldots, n-1$. Since

$$B(x) = (x+1)(x+2)\ldots(x+n) - A(x) = (x(x-1)\ldots(x-(n-1))),$$

then equating coefficients, yields

$$\frac{n(n+1)}{2} - (x_1 + x_2 + \ldots + x_n) = -\frac{(n-1)n}{2}$$

from which $x_1 + x_2 + \ldots + x_n = n^2$ follows.

Also solved by *Moti Levy, Rehovot, Israel.*

A–106. *Proposed by Vasile Mircea Popa, "Lucian Blaga" University of Sibiu, Romania.* Calculate the following integral:

$$\int_1^\infty \frac{\sqrt{x}\ln^2 x}{x^3 + x\sqrt{x} + 1} dx.$$

Solution 1 by Brian Bradie, Department of Mathematics, Christopher Newport University, Newport News, VA. With the change of variable $x \to \frac{1}{x}$,

$$\int_1^\infty \frac{\sqrt{x}\ln^2 x}{x^3 + x\sqrt{x} + 1}\, dx = \int_0^1 \frac{\sqrt{x}\ln^2 x}{x^3 + x\sqrt{x} + 1}\, dx.$$

Now, multiply the numerator and denominator of the integrand by $1 - x^{3/2}$ to obtain

$$\int_1^\infty \frac{\sqrt{x}\ln^2 x}{x^3 + x\sqrt{x} + 1}\, dx = \int_0^1 \frac{(\sqrt{x} - x^2)\ln^2 x}{1 - x^{9/2}}\, dx.$$

Next, using the geometric series

$$\frac{1}{1 - x^{9/2}} = \sum_{k=0}^\infty x^{9k/2}$$

yields

$$\begin{aligned}
\int_1^\infty \frac{\sqrt{x}\ln^2 x}{x^3 + x\sqrt{x} + 1}\, dx &= \sum_{k=0}^\infty \int_0^1 \left(x^{(9k+1)/2} - x^{(9k+4)/2}\right)\ln^2 x\, dx \\
&= \sum_{k=0}^\infty \left(\frac{16}{(9k+3)^3} - \frac{16}{(9k+6)^3}\right) \\
&= \frac{16}{729}\sum_{k=0}^\infty \left(\frac{1}{(k+\frac{1}{3})^3} - \frac{1}{(k+\frac{2}{3})^3}\right) \\
&= -\frac{8}{729}\left(\psi_2\left(\frac{1}{3}\right) - \psi_2\left(\frac{2}{3}\right)\right),
\end{aligned}$$

where $\psi_2(x)$ is the tetragamma function. By the reflection formula

$$\psi_2(1-z)-\psi_2(z)=\pi\frac{d^2}{dz^2}\cot(\pi z)=2\pi^3\csc^2(\pi z)\cot(\pi z),$$

it follows that

$$\psi_2\left(\frac{1}{3}\right)-\psi_2\left(\frac{2}{3}\right)=2\pi^3\csc^2\left(\frac{2\pi}{3}\right)\cot\left(\frac{2\pi}{3}\right)=-\frac{8\pi^3}{3\sqrt{3}}.$$

Finally,

$$\int_1^\infty \frac{\sqrt{x}\ln^2 x}{x^3+x\sqrt{x}+1}\,dx=\frac{64\pi^3}{2187\sqrt{3}}=\frac{64\pi^3\sqrt{3}}{6561}.$$

Solution 2 by Moti Levy, Rehovot, Israel. By change of variable $u=\frac{1}{\sqrt{x}}$,

$$\begin{aligned} I &= \int_0^1 \frac{8u^2}{u^6+u^3+1}\ln^2(u)du &(1)\\ &= \int_0^1 \frac{8u^2(u^3-1)}{u^9-1}\ln^2(u)du \\ &= \int_0^1 \frac{8u^2}{1-u^9}\ln^2(u)du-\int_0^1 \frac{8u^5}{1-u^9}\ln^2(u)du &(2)\end{aligned}$$

Integration by parts twice gives

$$\int_0^1 u^k\ln^2(u)du=\frac{2}{(1+k)^3}. \qquad (3)$$

$$\begin{aligned}\int_0^1 \frac{8u^2}{1-u^9}\ln^2(u)du &= 8\int_0^1\sum_{k=0}^\infty u^{9k+2}\ln^2(u)du &(4)\\ &= 8\sum_{k=0}^\infty\int_0^1 u^{9k+2}\ln^2(u)du=\frac{16}{729}\sum_{k=0}^\infty\frac{1}{\left(k+\frac{1}{3}\right)^3}\\ &= \frac{16}{729}\zeta\left(3,\frac{1}{3}\right), &(5)\end{aligned}$$

where $\zeta(s,a)$ is the Hurwitz zeta function.

$$\int_0^1 \frac{8u^5}{1-u^9}\ln^2(u)du=\frac{16}{729}\zeta\left(3,\frac{2}{3}\right). \qquad (6)$$

By (2), (5) and (6),

$$I = \frac{16}{729}\left(\zeta\left(3, \frac{1}{3}\right) - \zeta\left(3, \frac{2}{3}\right)\right)$$

The values of the Hurwitz zeta function at rational values have been studied in [1]. Professor V. S. Adamchik gives the following results

$$\zeta\left(2n+1, \frac{1}{3}\right) = \frac{1}{2}\left(3^{2n+1} - 1\right)\zeta(2n+1) + \frac{(2\pi)^{2n+1}}{2\sqrt{3}(2n+1)!},$$
$$\zeta\left(2n+1, \frac{2}{3}\right) = \frac{1}{2}\left(3^{2n+1} - 1\right)\zeta(2n+1) - \frac{(2\pi)^{2n+1}}{2\sqrt{3}(2n+1)!}$$

$$\zeta\left(3, \frac{1}{3}\right) = \frac{1}{2}\left(3^{3} - 1\right)\zeta(3) + \frac{(2\pi)^{3}}{2\sqrt{3}(3)!} = 13\zeta(3) + \frac{2}{9}\sqrt{3}\pi^3,$$
$$\zeta\left(3, \frac{2}{3}\right) = 13\zeta(3) - \frac{2}{9}\sqrt{3}\pi^3.$$

We conclude that

$$I = \frac{64}{6561}\sqrt{3}\pi^3 \cong 0.523\,87.$$

Reference

[1] Adamchik, V.S., "*On the Hurwitz Function for Rational Arguments*", Applied Mathematics and Computation, Vol. 187, Issue 1, (2007), 3-12.

Solution 3 by Michel Bataille, Rouen, France. Let I denote the integral to be evaluated. The change of variables $x = u^{2/3}$ gives

$$I = \int_1^\infty \frac{u^{1/3} \cdot (4/9) \cdot (\ln u)^2}{u^2 + u + 1} \cdot \frac{2}{3} u^{-1/3}\, du = \frac{8}{27}\int_1^\infty \frac{(\ln u)^2}{u^2+u+1}\, du.$$

Since we have

$$\int_1^\infty \frac{(\ln u)^2}{u^2+u+1}\, du = \int_0^1 \frac{(\ln t)^2}{t^2+t+1}\, dt$$

(by the change of variables $u = \frac{1}{t}$), we readily see that $I = \frac{4}{27}J$ where

$$J = \int_0^\infty \frac{(\ln u)^2}{u^2+u+1}\,du.$$

To calculate J, we set $f(z) = \frac{(\log z)^3}{z^2+z+1}$ where $\log z = \ln(|z|) + i\theta$ with $0 \le \theta < 2\pi$. The method detailed in **[1]** leads to

$$\int_0^\infty \frac{(\ln u)^3}{u^2+u+1}\,du + \int_\infty^0 \frac{(\ln u + 2\pi i)^3}{u^2+u+1}\,du = 2\pi i\big(\mathrm{Res}(f,w) + \mathrm{Res}(f,w^2)\big). \tag{1}$$

Here $\mathrm{Res}(f,w)$ (resp. $\mathrm{Res}(f,w^2)$) is the residue of f at $w = \exp(2\pi i/3)$ (resp. $w^2 = \exp(4\pi i/3)$).
Since $\mathrm{Res}(f,w) = \frac{(\log w)^3}{2w+1} = \frac{(2\pi i/3)^3}{i\sqrt{3}} = -\frac{8\pi^3}{27\sqrt{3}}$ and $\mathrm{Res}(f,w^2) = \frac{(4\pi i/3)^3}{-i\sqrt{3}} = \frac{64\pi^3}{27\sqrt{3}}$, (1) yields

$$(-6\pi i)J + (12\pi^2)K + (8\pi^3 i)L = 2\pi i \cdot \frac{56\pi^3}{27\sqrt{3}}$$

where $K = \int_0^\infty \frac{\ln u}{u^2+u+1}\,du$ and $L = \int_0^\infty \frac{1}{u^2+u+1}\,du$. It follows that $K = 0$ and because

$$L = \frac{4}{3}\int_0^\infty \frac{du}{\left(\frac{2u+1}{\sqrt{3}}\right)^2 + 1} = \frac{2}{\sqrt{3}}\Big(\frac{\pi}{2} - \frac{\pi}{6}\Big) = \frac{2\pi}{3\sqrt{3}}$$

we finally obtain $J = \frac{16\pi^3}{81\sqrt{3}}$ and

$$I = \frac{64\pi^3}{2187\sqrt{3}}.$$

Reference

[1] Cartan, H. *Elementary Theory of Analytic Functions of One or Several Complex Variables*, Dover, 1995, ch. III, p. 109.

Solution 4 by the proposer. Let us denote $I = \displaystyle\int_1^\infty \frac{\sqrt{x}\ln^2 x}{x^3 + x\sqrt{x} + 1}dx.$
In this integral we make the variable change: $x = 1/y$. We get

$$I = \int_1^\infty \frac{\sqrt{x}\ln^2 x}{x^3 + x\sqrt{x} + 1}dx = \int_0^1 \frac{\sqrt{y}\ln^2 y}{y^3 + y\sqrt{y} + 1}dy.$$

Let us denote $J = \int_0^1 \frac{\sqrt{y}\ln^2 y}{y^3 + y\sqrt{y} + 1} dy$ In this integral we make the change of variable $y = z^{\frac{2}{3}}$ and we obtain

$$J = \frac{8}{27}\int_0^1 \frac{\ln^2 z}{z^2 + z + 1} dz.$$

We have, successively

$$\begin{aligned}
J &= \frac{8}{27}\int_0^1 \frac{(1-z)\ln^2 z}{1-z^3} dz, \\
J &= \frac{8}{27}\left(\int_0^1 \frac{\ln^2 z}{1-z^3} dz - \int_0^1 \frac{z\ln^2 z}{1-z^3} dz\right), \\
J &= \frac{8}{27}\left(\int_0^1 \sum_{n=0}^{\infty} z^{3n}\ln^2 z dz - \int_0^1 \sum_{n=0}^{\infty} z^{3n+1}\ln^2 z dz\right), \\
J &= \frac{8}{27}\sum_{n=0}^{\infty}\left(\int_0^1 z^{3n}\ln^2 z dz - \int_0^1 z^{3n+1}\ln^2 z dz\right).
\end{aligned}$$

Using the following relationship

$$\int_0^1 x^a \ln^2 x dx = \frac{2}{(a+1)^3}, \text{ where } a \in \mathbb{R}, a \geq 0,$$

we obtain

$$J = \frac{8}{27}\sum_{n=0}^{\infty}\left[\frac{2}{(3n+1)^3} - \frac{2}{(3n+2)^3}\right]$$

or

$$J = \frac{8}{27}\sum_{n=0}^{\infty}\left[\frac{\frac{2}{27}}{\left(n+\frac{1}{3}\right)^3} - \frac{\frac{2}{27}}{\left(n+\frac{2}{3}\right)^3}\right].$$

Using the following relationship

$$\psi_2(x) = -\sum_{n=0}^{\infty} \frac{1}{(x+n)^3}$$

where $\psi_2(x)$ is the tetragamma function, we have

$$J = \frac{8}{27^2}\left[-\psi_2\left(\frac{1}{3}\right) + \psi_2\left(\frac{2}{3}\right)\right]$$

We use the reflection formula for the tetragamma function

$$\psi_2(p) - \psi_2(1-p) = f(p),$$

where $f(x) = -\pi \frac{d^2}{dx^2} ctg(\pi x)$. We have

$$\psi_2\left(\frac{2}{3}\right) - \psi_2\left(\frac{1}{3}\right) = \frac{8\pi^3\sqrt{3}}{9}.$$

We obtained the value of the integral required in the problem statement

$$I = J = \frac{64\pi^3}{2187\sqrt{3}} = \left(\frac{4\pi}{9\sqrt{3}}\right)^3,$$

and we are done.

Also solved by *Daniel Văcaru, Pitești, Romania.*

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