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# ARHIMEDE
# MATHEMATICAL JOURNAL

*Articles*
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ARHIMEDE ASSOCIATION

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**Solutions**

# *Articles*

Arhimede Mathematical Journal aims to publish interesting and attractive papers with elegant mathematical exposition. Articles should include examples, applications and illustrations, whenever possible. Manuscripts submitted should not be currently submitted to or accepted for publication in another journal.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

# *Numerical sequences and polynomials*

**Arkady M. Alt**

**Abstract**

In this paper, motivated by some problems of mathematical Olympiad caliber, we present links between numerical sequences and polynomials.

## 1 Basic Results

Hereafter, we present a basic general problem and solve it. Applying its solution, we will solve some other problems appeared elsewhere.

**Problem 1 (Basic Problem).** *For a given sequence $\{a_n\}_{n\ge 0}$ and for all $n \ge 0$, let $A_n(x)$ be the sequence of polynomials of degree at most $n$ such that*

$$A_n(k) = a_k; \quad \text{for } 0 \le k \le n.$$

*Find the value of $A(n+1)$.*

*Solution.* For $n = 0$ we have that $A_0(x) = \alpha_0$ is the constant polynomial and $A_0(1) = a_0$. So, $A_0(x) = a_0$ and $A_0(1) = a_0$.

For $n = 1$, let $A_1(x) = \alpha_0 + \alpha_1 x$. Since $A_1(0) = a_0$, then $A_1(x) = a_0 + \alpha_1 x$. On the other hand, $A_1(1) = a_1 \iff a_0 + \alpha_1 = a_1$ and $\alpha_1 = a_1 - a_0$. Thus, $A_1(x) = a_0 + (a_1 - a_0)x$, from which it follows that $A_1(2) = a_0 + 2(a_1 - a_0) = 2a_1 - a_0$.

For $n = 2$, let $A_2(x) = \alpha_0 + \alpha_1 x + \alpha_2 x(x-1)$. Since $A_2(x) = A_1(x)$ for $x = 0, 1$, then $A_1(x) = \alpha_0 + \alpha_1 x$ for all $x$ and, therefore, $A_2(x) = A_1(x) + \alpha_2 x(x-1)$. Using that $A_2(2) = a_2$ and $A_1(2) = 2a_1 - a_0$ we obtain

$$2!\,\alpha_2 + 2a_1 - a_0 = a_2 \qquad \text{and} \qquad \alpha_2 = \frac{a_2 - 2a_1 + a_0}{2!}.$$

Thus,

$$\begin{aligned} A_2(x) &= A_1(x) + \frac{a_2 - 2a_1 + a_0}{2!}\, x(x-1) \\ &= a_0 + \frac{a_1 - a_0}{1!}\, x + \frac{a_2 - 2a_1 + a_0}{2!}\, x(x-1) \end{aligned}$$

and

$$A_2(3) = a_0 - 3a_1 + 3a_2.$$

To emphasize that $A_n(n+1)$ is not, generally speaking, a term of the sequence $\{a_n\}_{n \ge 0}$, we set $b_n = A_n(n+1)$ for all $n \ge 0$. Thus, our main goal is to express $b_n$ through the terms of the sequence $\{a_n\}_{n \ge 0}$ and to find the polynomial $A_n(x)$. For example, we already have $b_0 = a_0$, $b_1 = 2a_1 - a_0$, $b_2 = a_0 - 3a_1 + 3a_2$ and, by the way, the polynomials $A_0(x)$, $A_1(x)$ and $A_2(x)$.

Now, assume that we already have the polynomials $A_0(x), A_1(x), \ldots, A_n(x)$ that satisfy the condition of the statement. We will find

$$A_{n+1}(x) = P(x) + \alpha_{n+1} \binom{x}{n+1},$$

where $\deg(P(x)) \le n$ and

$$\binom{x}{n+1} = \frac{x(x-1)(x-2)\cdots(x-n)}{(n+1)!},$$

as is well-known.

Since $A_{n+1}(x) = A_n(x) = P(x)$ for $x = 0, 1, 2, \ldots, n$, then $P(x) = A_n(x)$ for all $x$ and, therefore,

$$A_{n+1}(x) = A_n(x) + \alpha_{n+1} \binom{x}{n+1},$$

where coefficient $\alpha_{n+1}$ is determined by using $A_{n+1}(n+1) = a_{n+1}$. We have

$$a_{n+1} = A_{n+1}(n+1) = A_n(n+1) + \alpha_{n+1}\binom{n+1}{n+1}$$

or

$$a_{n+1} = b_n + \alpha_{n+1} \iff \alpha_{n+1} = a_{n+1} - b_n.$$

Thus,

$$A_{n+1}(x) = A_n(x) + (a_{n+1} - b_n)\binom{x}{n+1}.$$

Applying the $(n+1)$-times iterated difference operator $\Delta^{n+1}$ to the polynomial $A_{n+1}(x) = A_n(x) + \alpha_{n+1}\binom{x}{n+1}$, we obtain

$$\Delta^{n+1}(A_{n+1}(x)) = \Delta^{n+1}(A_n(x)) + \Delta^{n+1}\left(\alpha_{n+1}\binom{x}{n+1}\right)$$

or

$$\Delta^{n+1}(A_{n+1}(x)) = 0 + \alpha_{n+1} \implies \Delta^{n+1}(A_{n+1}(0)) = \alpha_{n+1},$$

from which, on account that $A_{n+1}(x)$ is a constant polynomial, $\Delta^{n+1}(a_0) = \alpha_{n+1}$ follows. Thus, $A_{n+1}(x) = A_n(x) + \Delta^{n+1}(a_0)\binom{x}{n+1}$ and, therefore, $A_{n+1}(n+1) = A_n(n+1) + \Delta^{n+1}(a_0)\binom{n+1}{n+1}$ or $a_{n+1} = b_n + \Delta^{n+1}(a_0)$, from which it follows that $b_n = a_{n+1} - \Delta^{n+1}(a_0)$.

Since for all $n \in \mathbb{N}$ we ahve

$$\Delta^{n+1}(a_0) = \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} a_{n+1-k},$$

then we have

$$\begin{aligned} A_n(n+1) = b_n &= a_{n+1} - \Delta^{n+1}(a_0) \\ &= a_{n+1} - \sum_{k=0}^{n+1} (-1)^k \binom{n+1}{k} a_{n+1-k} \\ &= \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} a_{n+1-k} \end{aligned}$$

and

$$A_n(x) = a_0 + \sum_{k=1}^{n} \Delta^k(a_0)\binom{x}{k} = \sum_{k=0}^{n} \Delta^k(a_0)\binom{x}{k}. \qquad \square$$

**Remark 1.** *As it is well-known, for any function $f(x)$ the difference operator $\Delta$ is defined by $\Delta f(x) = f(x+1) - f(x)$ and the $k$-times iterated difference operator $\Delta^k$ is defined recursively by $\Delta^0 f(x) = f(x)$ and $\Delta^k f(x) = \Delta(\Delta^{k-1} f(x))$ for $k \in \mathbb{N}$. Since $\Delta^0(c) = 0$, $\Delta\binom{x}{1} = \Delta(x) = 1$ and $\Delta\binom{x}{n} = \binom{x+1}{n} - \binom{x}{n} = \binom{x}{n-1}$, then*

$$\Delta^k\left(\binom{x}{n}\right) = \Delta^k\left(\binom{x}{n-k}\right) = \begin{cases} 0 & \text{if } k > n, \\ 1 & \text{if } k = n. \end{cases}$$

A natural generalization of Problem 1 is the following:

**Problem 2.** *For a given sequence $a_0, a_1, \ldots, a_n, \ldots$, let $A_{m;n}(x)$, $n \geq 0$, be a polynomial of degree at most $n$ such that $A_{m;n}(k) = a_{m+k}$ for $0 \leq k \leq n$. Find the value of $A_{m;n}(n+1)$.*

The answer is obvious and we have

$$\begin{aligned} A_{m;n}(n+1) &= a_{m+n+1} - \Delta^{n+1}(a_n) \\ &= a_{m+n+1} - \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k} a_{m+n+1-k} \\ &= \sum_{k=1}^{n+1}(-1)^{k-1}\binom{n+1}{k} a_{m+n+1-k} \end{aligned}$$

and

$$A_{m;n}(x) = a_m + \sum_{k=1}^{n} \Delta^k(a_m)\binom{x}{k} = \sum_{k=0}^{n} \Delta^k(a_m)\binom{x}{k}.$$

# 2 Applications

In what follows, some applications of the above results are given. We begin with the following.

**Problem 3 (IMO Short List 1981).** *Let $P(x)$ be a polynomial of degree $n$ such that*

$$P(k) = 1 \Big/ \binom{n+1}{k}; \quad \text{for } 0 \le k \le n.$$

*Find $P(n+1)$.*

*Solution.* Using the correlation

$$A_n(n+1) = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} a_{n+1-k}$$

for $a_k = 1/\binom{n+1}{k} = 1/\binom{n+1}{n+1-k} = a_{n+1-k}$, we obtain

$$P(n+1) = A_n(n+1) = \sum_{k=1}^{n+1} (-1)^{k-1} \left[ \binom{n+1}{k} \Big/ \binom{n+1}{n+1-k} \right]$$
$$= \sum_{k=1}^{n+1} (-1)^{k-1} = \frac{(-1)^n + 1}{2}.$$ □

The next application appeared in [1] and it is stated as follows:

**Problem 4.** *Let $A(x)$ be a polynomial with integer coefficients such that for $1 \le k \le n+1$ holds:*

$$A(k) = 5^k.$$

*Find the value of $A(n+2)$.*

*Solution.* We will solve a more general problem replacing 5 with any $a \neq 1$. Let $a_k = a^{k+1}$ $(0 \le k \le n)$, $A(x) = A_n(x)$ and $A(n+2) = A_n(n+1)$. Since

$$\Delta^k(a_0) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} a_{k-i} = \sum_{i=0}^{k} \binom{k}{i} a^{k-i+1}$$
$$= a \sum_{i=0}^{k} (-1)^i \binom{k}{i} a^{k-i} = a(a-1)^k,$$

then

$$A_n(x) = a \sum_{k=0}^{n} (a-1)^k \binom{x}{k}$$

and

$$A(n+2) = A_n(n+1) = a^{n+2} - \sum_{k=0}^{n+1} (-1)^{k-1} \binom{n+1}{k} a^{n+2-k}$$
$$= a(a^{n+1} - (a-1)^{n+1}).$$

Finally, setting $a = 5$ we get $A(n+2) = 5(5^{n+1} - 4^{n+1})$. □

**Remark 2.** *Note that, if $a_n = \alpha a^n + \beta b^n$, then*

$$A_n(x) = \sum_{k=0}^{n} \left[ \alpha a(a-1)^k + \beta b(b-1)^k \right]$$

*and $A(n+1) = \alpha a(a^{n+1} - (a-1)^{n+1}) + \beta b(b^{n+1} - (b-1)^{n+1})$.*

Finally, we close this paper by giving an application involving Fibonacci numbers.

**Problem 5.** *Let $\{f_n\}_{n \geq 0}$ be the Fibonacci sequence defined by $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = f_n + f_{n-1}$ for all $n \geq 1$. Let $F_{m;n}(x)$ be a polynomial of degree at most $n$ such that $F_{m;n}(k) = f_{m+k}$ for $0 \leq k \leq n$. Determine $F_{m;n}(n+1)$.*

*Solution.* First note that $\Delta(f_n) = f_{n+1} - f_n = f_{n-1}$. Then, $\Delta^k(f_n) = f_{n-k}$, $k \leq n$. But what happens if $k > n$? To get the answer to this question we need to extend the definition of Fibonacci sequence to negative values of $n$. We may define $f_{-n} = (-1)^{n+1} f_n$, as it is well-known. On account of the preceding we have

$$F_{m;n}(n+1) = f_{m+n+1} - \Delta^{n+1}(f_m) = f_{m+n+1} - f_{m-n-1}$$

and

$$F_{m;n}(x) = f_m + \sum_{k=1}^{n} \binom{x}{k} \Delta^k(f_m) = \sum_{k=0}^{n} \binom{x}{k} \Delta^k(f_m) = \sum_{k=0}^{n} \binom{x}{k} f_{m-k}.$$

In particular, if $m = 1$ we get

$$F_{1;n}(n+1) = f_{n+2} - f_{-n} = f_{n+2} - (-1)^{n+1} f_n = f_{n+2} + (-1)^n f_n$$

and

$$F_{1,n}(x) = \sum_{k=0}^{n} \binom{x}{k} f_{1-k} = 1 + \sum_{k=1}^{n} \binom{x}{k} f_{1-k} = 1 + \sum_{k=1}^{n} \binom{x}{k} (-1)^k f_{k-1}.$$

□

# References

[1] "Problem 64". *Mathproblems* 4.1 (2014), p. 244. ISSN: 2217-446X.

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# A Generalization of an AMJ Inequality Problem

Henry Ricardo

Abstract

In this article, we note a similar pattern in two inequalities and prove a generalization that includes both. As a bonus, an additional generalization is provided.

## 1 Introduction

Shortly after solving Problem 1 in this Journal [1], I encountered a similar problem, Problem 2, shortlisted for the 2002 Junior Balkan Mathematical Olympiad [2]:

Problem 1. Let $a, b, c, d$ be positive numbers such that $abcd = 16$. Show that

$$\frac{a^5+b^5+c^5+d^5}{4} \geq a^2\sqrt{b+c}+b^2\sqrt{c+d}+c^2\sqrt{d+a}+d^2\sqrt{a+b}.$$

Problem 2. If $a, b, c$ are positive real numbers such that $abc = 2$, then

$$a^3+b^3+c^3 \geq a\sqrt{b+c}+b\sqrt{c+a}+c\sqrt{a+b}.$$

## 2 Analysis

What can we say about these inequalities? First of all, these inequalities are not *homogeneous*—that is, all terms are not of the same degree. The left-hand side of Problem 1 has degree 5, whereas the right-hand side is roughly (inaccurately) of degree 5/2. However, in each case, if we square each term in the right-hand sum individually, we will have a homogeneous inequality—of degree 5 in Problem 1 and of degree 3 in Problem 2. There does not seem to be any way to "homogenize" these inequalities, and squaring both sides creates very messy algebraic expressions. It seems as if we have to think of some inequalities that deal with sums of products and terms that are squared.

Now there are many inequalities that involve sums of products, but arguably, the two simplest such inequalities are the Cauchy-Schwarz and Chebyshev inequalities (the latter a generalization of the rearrangement inequality):

*Cauchy-Schwarz*: For any real numbers $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$, it holds that

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n \le \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\,\sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}.$$

*Chebyshev*: If the real numbers $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ are ordered the same way—that is, both sequences are increasing or both decreasing—, then

$$n\sum_{i=1}^{n} x_iy_i \ge \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right).$$

(The inequality is reversed if the two sequences are ordered differently).

Finally, we note that the dominant sides of Problem 1 and Problem 2 and the given conditions on the product of the variables suggest power means:

*Power Mean Inequality*: If $x_1, x_2, \dots, x_n$ are positive real numbers and $r \ge s$ for positive real numbers $r$ and $s$, then

$$\left(\frac{x_1^r + x_2^r + \cdots + x_n^r}{n}\right)^{1/r} \ge \left(\frac{x_1^s + x_2^s + \cdots + x_n^s}{n}\right)^{1/s} \ge \sqrt[n]{x_1x_2\cdots x_n}.$$

This inequality is valid for any real numbers $r \ge s$, but the problems we are trying to solve involve positive values only.

# 3 Main Result

As a generalization of the inequalities given by Problem 1 and Problem 2, I propose the following:

Theorem 1. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \cdots a_n = C$ and $k$ is a nonnegative integer, then

$$a_1^{2k+1} + a_2^{2k+1} + \cdots + a_n^{2k+1} \geq M \left( a_1^k \sqrt{a_2 + a_3} + a_2^k \sqrt{a_3 + a_4} + \cdots + a_n^k \sqrt{a_1 + a_2} \right),$$

where $M = \sqrt{C^{(2k+1)/n}/2}$.

Proof. We apply Cauchy's inequality to get

$$\sum_{\text{cyclic}} a_j^k \sqrt{a_{j+1} + a_{j+2}} \leq \sqrt{2(a_1^{2k} + a_2^{2k} + \cdots + a_n^{2k})(a_1 + a_2 + \cdots + a_n)}. \tag{1}$$

Then, Chebyshev's inequality gives us

$$\sqrt{2(a_1^{2k} + a_2^{2k} + \cdots + a_n^{2k})(a_1 + a_2 + \cdots + a_n)} \leq \sqrt{2n(a_1^{2k+1} + a_2^{2k+1} + \cdots + a_n^{2k+1})}. \tag{2}$$

Finally, the power mean inequality yields

$$a_1^{2k+1} + a_2^{2k+1} + \cdots + a_n^{2k+1} \geq n \left( \sqrt[n]{a_1 a_2 \cdots a_n} \right)^{2k+1} = n \cdot C^{(2k+1)/n} = m. \tag{3}$$

Now we combine inequalities (1), (2) and (3) to conclude that

$$
\begin{aligned}
& a_1^{2k+1}+a_2^{2k+1}+\cdots+a_n^{2k+1} \\
&= \sqrt{(a_1^{2k+1}+\cdots+a_n^{2k+1})^2} \\
&\overset{(3)}{\geq} \sqrt{m(a_1^{2k+1}+a_2^{2k+1}+\cdots+a_n^{2k+1})} \\
&= \sqrt{\frac{m}{2n}\,2n(a_1^{2k+1}+a_2^{2k+1}+\cdots+a_n^{2k+1})} \\
&= \underbrace{\sqrt{\frac{m}{2n}}}_{M}\sqrt{2n(a_1^{2k+1}+a_2^{2k+1}+\cdots+a_n^{2k+1})} \\
&\overset{(2)}{\geq} M\sqrt{2(a_1^{2k}+a_2^{2k}+\cdots+a_n^{2k})(a_1+a_2+\cdots+a_n)} \\
&\overset{(1)}{\geq} M\sum_{\text{cyclic}} a_j^k\sqrt{a_{j+1}+a_{j+2}}\,. \qquad \square
\end{aligned}
$$

Note that

$$
M=\sqrt{\frac{m}{2n}}=\sqrt{\frac{n(\sqrt[n]{a_1a_2\cdots a_n})^{2k+1}}{2n}}=\sqrt{\frac{C^{(2k+1)/n}}{2}}.
$$

Thus, in Problem 1, we have $k=2$, $n=4$, $C=16$, giving us $M=\sqrt{16^{5/4}/2}=\sqrt{16}=4$. For Problem 2, we have $k=1$, $n=3$, $C=2$, so that $M=\sqrt{2^{3/3}/2}=1$.

There is an alternative solution for Problem 2 given in [2], but this method does not generalize easily.

# 4 A Bonus

Although the result described in the previous section generalizes the inequalities from Problem 1 and Problem 2, we can do more. The following inequality extends our main result to the case of even exponents in the dominant member of the inequality:

Theorem 2. If $k$ is a positive integer and $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 a_2 \cdots a_n = C$, then

$$a_1^{2k} + a_2^{2k} + \cdots + a_n^{2k} \geq M\left(\sqrt{a_1^{2k-1}(a_2 + a_3)} + \sqrt{a_2^{2k-1}(a_3 + a_4)} + \cdots + \sqrt{a_n^{2k-1}(a_1 + a_2)}\right),$$

where $M = \sqrt{C^{2k/n}/2}$.

The proof is essentially the same as the proof of Theorem 1.

# References

[1] Berindeanu, M. "Problem MH-67". *Arhimede Math. J.* 6.1 (2019), p. 42. ISSN: 2462-537X.

[2] Bin, X., Boreico, I., Can, V. Q. B., and Lascu, M. *Olympiad Inequalities*. Zalau, Romania: GIL, 2015.

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# Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.
2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to José Luis Díaz-Barrero, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

April 30, 2020

# Elementary Problems

**E–71.** Proposed by Marc Felipe i Alsina and José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let $F_n$ be the $n$-th Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Is the number $F_{370370367}$ odd or even?

**E–72.** Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Find all real solutions of the equation

$$11^x + 60^x = 61^x.$$

**E–73.** Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let $x, y, z$ be positive real numbers. Prove that

$$\sqrt[6]{\frac{xy}{z^2}} + \sqrt[6]{\frac{yz}{x^2}} + \sqrt[6]{\frac{zx}{y^2}} \geq 3.$$

**E–74.** Proposed by Marc Felipe i Alsina and José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let $ABC$ be a triangle. With the usual notations, show that

$$\frac{a^2 + \sin^2 A}{a + \sin A} + \frac{b^2 + \sin^2 B}{b + \sin B} + \frac{c^2 + \sin^2 C}{c + \sin C} = \frac{(a+b+c)^2 + (\sin A + \sin B + \sin C)^2}{a + b + c + \sin A + \sin B + \sin C}.$$

**E–75.** Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let $ABC$ and $ABD$ be two congruent triangles. Vertices $C$ and $D$ lie on opposite sides of $AB$. If $CD$ meets $AB$ in $X$, then prove that $XC = XD$.

**E–76.** Proposed by Marc Felipe i Alsina and José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let $n$ be a positive integer. Compute the following sum:

$$\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \binom{n}{3}^2 + \cdots + (-1)^n \binom{n}{n}^2.$$

# Easy–Medium Problems

**EM–71.** Proposed by Oriol Baeza Guasch, CFIS, BarcelonaTech, Terrassa, Spain. Let $ABC$ be a triangle with orthocenter $H$. Let $M$ be the midpoint of arc $BC$, and let $P$ be the reflection of $H$ over side $BC$. If $MP$ is perpendicular to $HB$, find the value of

$$\frac{\angle ABC}{\angle BCA}.$$

**EM–72.** Proposed by Mihaela Berindeanu, Bucharest, România. Show that the inequality

$$\frac{(x^2+2)^2}{3\sqrt[3]{4y^4z^4+4}}+\frac{(y^2+2)^2}{3\sqrt[3]{4z^4x^4+4}}+\frac{(z^2+2)^2}{3\sqrt[3]{4x^4y^4+4}} \geq 3$$

holds for all non negative reals $x, y, z$.

**EM–73.** Proposed by Andrés Sáez Schwedt, Universidad de León, Spain. Let $E$ be a point on side $CD$ of a square $ABCD$, and let $H$ be the orthocenter of triangle $ABE$. The lines $BE$ and $AE$ meet the circumcircles of triangles $AEH$ and $BEH$ again at points $P$ and $Q$, respectively. The lines $AP$ and $BQ$ intersect at $F$. Prove that the circle of diameter $AB$ is tangent to the incircle of triangle $ABF$.

**EM–74.** Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let $A(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1x + a_0$ be a polynomial of degree $n \geq 1$ with nonnegative coefficients. Prove that

$$\sum_{k=1}^{n} \frac{A(k)}{A(k)+A(n-k)} \geq \frac{n+1}{2}.$$

**EM–75.** Proposed by Oriol Baeza Guasch, CFIS BarcelonaTech, Terrassa, Spain. Let $ABC$ be a triangle with incenter $I$ and circumcenter $O$. Denote by $D$, $E$ and $F$ the contact points of the incircle with the sides $BC$, $CA$ and $AB$, respectively. Let $Q$ be the second intersection of the circumcircles of $ABC$ and $BDF$. Finally, let $N$ be the midpoint of arc $BC$ containing $A$. Prove that lines $NQ$, $IO$ and $AB$ are concurrent.

**EM–76.** Proposed by Mihaela Berindeanu, Bucharest, România.
Let $P$ be an interior point in $\triangle ABC$ so that $AP \cap BC = \{E\}$, $BP \cap AC = \{F\}$, $CP \cap AB = \{G\}$ and $\angle(BGC) = \angle(AEC) = \angle(BFA)$. Show that, if $\overrightarrow{AP} + \overrightarrow{BP} + \overrightarrow{CP} = \overrightarrow{0}$, then $ABC$ is an equilateral triangle.

# Medium–Hard Problems

**MH–71.** Proposed by Andrés Sáez Schwedt, Universidad de León, Spain. $ABCD$ is a convex quadrilateral satisfying $\angle CAD = \angle DCA$ and $\angle DCB = \angle CBA$. Let $r$ be the external bisector of the angle $\angle DAB$ and let $s$ be the reflection of the line $AC$ with respect to $BC$. Prove that the lines $r$, $s$ and $BD$ are parallel or concurrent.

**MH–72.** Proposed by Pedro H. O. Pantoja, Natal/RN, Brazil. Find the solution of the equation $\sin^8 x - \cos^8 x = 7\sqrt{3}/16$.

**MH–73.** Proposed by Oriol Baeza Guasch, CFIS BarcelonaTech, Terrassa, Spain. Let $ABC$ be a triangle ($AB \neq AC$) with incenter $I$ and circumcenter $O$. Let $E$ be the contact point of the incircle with side $CA$. Denote by $N$ the midpoint of arc $BC$ containing $A$, and denote by $M$ the midpoint of arc $AC$ not containing $B$. Finally, suppose line $IO$ intersects side $AB$ at point $P$. Show that lines $NP$ and $ME$ meet at the circumcircle of $ABC$.

**MH–74.** Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Solve in real numbers the following system of equations:

$$\left.\begin{aligned} a\sqrt{b} + b\sqrt{c} + c\sqrt{a} &= 3\sqrt{abc}, \\ \frac{a^2c}{b^3+abc} + \frac{b^2a}{c^3+abc} + \frac{c^2b}{a^3+abc} &= \frac{3}{2}. \end{aligned}\right\}$$

**MH–75.** Proposed by Mihály Bencze, Braşov, Roamnia. Let $n \geq 1$ be an integer number. Prove that

$$\sum_{k=1}^{2n} \left\lfloor \sqrt{16n^2+k} \right\rfloor = 7n^2 + \sum_{k=1}^{n} \left\lfloor \sqrt{n^2+k} \right\rfloor,$$

where $\lfloor x \rfloor$ represents the integer part of $x$.

**MH–76.** Proposed by Mihaela Berindeanu, Bucharest, România.

Calculate

$$\lim_{n\to\infty} \int_0^1 \frac{(4x^3+1)^{n+1}}{(5x^4+1)^n}\,dx.$$

# Advanced Problems

**A–71.** Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Evaluate

$$\int_0^{\pi/2} \frac{\cos^4 x + 2\sin^2 x}{2+\sin^4 x + \cos^4 x}\,dx.$$

**A–72.** Proposed by Ander Lamaison Vidarte, Berlin, Germany. For any matrix $A \in M_{5\times 2}(\mathbb{R})$ prove that $A \cdot A^t$ has at most $16$ strictly negative entries.

**A–73.** Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let $a$, $b$ be real numbers such that $0 < a < b$. Prove that for any $x_1; x_2; \ldots; x_n \in [a; b]$ there exists a number $\xi \in (a, b)$ such that

$$\arctan^5(x_1) + \arctan^5(x_2) + \cdots + \arctan^5(x_n) = n\arctan^5(\xi).$$

**A–74.** Proposed by Marc Felipe i Alsina, BarcelonaTech, Barcelona, Spain. Let $f : [0; 1] \to \mathbb{R}$ be a twice differentiable function such that $f''(x) < 0$ for all $x \in [0; 1]$. For all $n \geq 2$, prove that

$$\int_{1/n}^{1} f(x)\,dx \geq \frac{n-1}{n^2} f(1) + \left(\frac{n-1}{n}\right)^2 \int_0^1 f(x)\,dx.$$

**A–75.** Proposed by Nicolae Papacu, Slobozia, Romania. Let $A$, $B \in GL(n; \mathbb{R})$ be two matrices distinct from the identity and let $p \geq 1$ be an integer number. If $B^{3p} = I$ and $AB = B^{2p}A$, prove that:

1. $B^3 = I$,
2. $AB = BA$ if and only if $3 \mid (p-2)$.

**A–76.** Proposed by Mihály Bencze, Braşov, Romania, and José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let $a$, $b$, $c$ be three positive reals numbers such that $abc = 1$. Prove that

$$\frac{(\ a)}{1+a+ab} + \frac{(\ b)}{1+b+bc} + \frac{(\ c)}{1+c+ca} \quad 1,$$

where is the gamma function.

# Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to José Luis Díaz-Barrero, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

jose.luis.diaz@upc.edu

# Lifting the exponent

Ismael Morales López

## 1 Introduction

Many properties of the integers, like being a perfect power or being a sum of two squares, can be reformulated in terms of their prime factors. Sometimes, the information that is contained in the factorization of an integer is not enough to reformulate other notions related to integers, such as being a solution of a diophantine equation, but this data does definitely provide insightful restrictions on the possibilities of such solutions. In many cases, these restrictions are enough to completely solve the equation.

It is obvious that, if the greatest powers of $p$ which appear in the factorizations of $a$ and $b$ are $p^m$ and $p^n$, respectively, then $p^{m+n}$ is the greatest power of $p$ that appears when factoring $ab$. This can be summarized in the following lemma.

Lemma 1. For every prime number $p$ and every $a, b$ integers we have that $\nu_p(ab) = \nu_p(a) + \nu_p(b)$.

It is important to state the following algebraic lemma. It appears constantly when dealing with inequalities that involve a diference of perfect powers of the same exponent.

Lemma 2. Let $x > y$ be positive integers. For every positive integer $N$ we have that $x^N - y^N \geq N(x - y)$, with equality if and only if $N = 1$.

Proof. Observe that

$$\frac{x^N - y^N}{x - y} = \sum_{k=0}^{N-1} x^k y^{N-1-k} \geq \sum_{k=0}^{N-1} 1 = N.$$

If $N \geq 2$, $xy^{N-2} \geq x \geq 2$, and thus

$$\frac{x^N - y^N}{x - y} = \sum_{k=0}^{N-1} x^k y^{N-1-k} \geq N + 1.$$

So the equality holds if and only if we are in the case $N = 1$. □

Whenever we introduce quantities of arithmetic interest in number theory, it is also convenient to have some control over their size. A very simple bound of this type is the following.

Lemma 3 (Standard bound). Let $n$ be a positive integer. Let $p$ be a prime number. Then, we have that $\nu_p(n) \leq \log_p(n)$, with equality if and only if $n$ is a power of $p$.

Proof. Factor $n = p^k m$, where $p$ does not divide $m$. Then, it is clear that $\log_p(n) = \log_p(p^k) + \log_p(m) = k + \log_p(m) \geq k = \nu_p(n)$, with equality if and only if $\log_p(m) = 0$, which occurs exactly when $m = 1$, that is, when $n = p^k$ is a power of $p$. □

## 2 Statement of the lemma

Theorem 1 (LTE lemma, v1). Let $x$ and $y$ be two integers, let $n$ be a positive integer and $p$ an odd prime such that $p \mid x - y$ but $p$ does not divide $x$ (and hence neither $y$). Then, it is true that

$$\nu_p(x^n - y^n) = \nu_p(x - y) + \nu_p(n).$$

Notice that we can modify a little bit the result to have a theorem about the sum of powers of odd exponent, just replacing $y$ by $-y$.

Theorem 2 (LTE lemma, v2). Let $x$ and $y$ be two integers, let $n$ be an odd positive integer and $p$ an odd prime such that $p \mid x + y$ but $p$ does not divide $x$ (and hence neither $y$). Then, it is true that

$$\nu_p(x^n + y^n) = \nu_p(x + y) + \nu_p(n).$$

It is also convenient to have a version of this result for the case $p = 2$. It turns out that the equation varies a little bit because there also appears an extra factor $x + y$, but it is analogous. The proof of this version will be skipped because we are more interested in the applications of the lemma, but there are still some clever and valuable tricks used in the proof, which can be found elsewhere. However, the case where the involved prime $p$ is odd, i. e. Theorem 1, will be proved later after analysing some easy examples.

Theorem 3 (LTE Lemma, v3). Let $x$ and $y$ be two odd integers and let $n$ be an even positive integer. Then,

$$\nu_2(x^n - y^n) = \nu_2(x^2 - y^2) + \nu_2(n) - 1.$$

# 3 Examples

These three versions 1, 2 and 3 of the *Lifting the exponent* lemma will be simultaneously called LTE lemma. It should be clear from the context which is the one we are using.

Example 1. $7^3$ divides $9^{49} - 2^{49}$.

Proof. We are going to prove more than that. It is clear that $7 \mid 9 - 2$ and $7$ does not divide $9$. Hence, we can apply Theorem 1 to deduce that

$$\nu_7(9^{49} - 2^{49}) = \nu_7(9 - 2) + \nu_7(49) = 3,$$

so $7^3 \mid 9^{49} - 2^{49}$ but $7^4$ does not divide $9^{49} - 2^{49}$. □

Example 2. $11^5$ divides $10^{11^4} + 1$.

Proof. We are going to check that the greatest power of 11 which divides $10^{11^4} + 1$ is $11^5$. Observe that $11 \mid 10 + 1$, $11^4$ is odd and $11$ does not divide 10, so, by Theorem 2,

$$\nu_{11}(10^{11^4} + 1) = 5,$$

hence $11^5 \mid 10^{11^4} + 1$ but $11^6$ does not divide $10^{11^4} + 1$. □

We recall the following definitions before introducing the third example.

Let $a$ be coprime with $m$. We denote by $\operatorname{ord}_m(a)$ the minimum positive integer $n$ for which $a^n \equiv 1 \mod m$. It is a well known fact that $\operatorname{ord}_m a \mid \varphi(m)$. We say that $a$ is a primitive root module $m$ if $\operatorname{ord}_m a = \varphi(m)$.

Example 3. $2$ is a primitive root module $3^k$ for every positive integer $k$.

Proof. It is easy to compute $\varphi(3^k) = 2 \cdot 3^{k-1}$. Suppose that $2^n \equiv 1 \mod 3^k$, or equivalently,

$$\nu_3(2^n - 1) \geq k.$$

It is clear that $n$ must be even since otherwise $3$ cannot divide $2^n - 1$. We write $n = 2u$.

$$\nu_3(2^n - 1) = \nu_3(4^u - 1) = \nu_3(4 - 1) + \nu_3(u) = 1 + \nu_3(u),$$

which is greater or equal to $k$ if and only if $3^{k-1} \mid u$. We have just proved that $2 \cdot 3^{k-1}$ is the minimum $n$ with its above defining property. □

# 4 Proof of the lemma

The proof will consist on the combination of two lemmas. If we were trying to prove Theorem 1 by induction on $\omega_p(n)$, i. e., on the number of primes $p$ that appear on the factorization of $n$, then the following two lemmas would naturally arise. The first can be understood as the base case of the induction, and the second one is precisely the step of induction. In this way, it will be clear how to combine both lemmas in order to prove our first version of the LTE.

Lemma 4. Let $x$ and $y$ be two integers, let $n$ be a positive integer and $p$ a prime such that $p \mid x - y$ but $p$ does not divide $x$ (and hence neither $y$). Aditionally, assume that $(n, p) = 1$. Then, it is true that

$$\nu_p(x^n - y^n) = \nu_p(x - y).$$

*Proof.* First, we factor the expression

$$x^n - y^n = (x - y)\left(\sum_{k=0}^{n-1} x^k y^{n-1-k}\right).$$

We will prove that $p$ cannot divide $\sum_{k=0}^{n-1} x^k y^{n-1-k}$, and it will follow directly that $\nu_p(x^n - y^n) = \nu_p(x - y)$. Since $x \equiv y \mod p$, we have that $\sum_{k=0}^{n-1} x^k y^{n-1-k} \equiv nx^{n-1} \mod p$. The last congruence cannot be zero because neither $n$ nor $x^{n-1}$ are 0 modulo $p$. □

It will be in the following lemma where the fact that $p$ is prime is crucial.

**Lemma 5.** *Let $x$ and $y$ be two integers and $p$ an odd prime such that $p \mid x - y$ but $p$ does not divide $x$ (and hence neither $y$). Then, it is true that*

$$\nu_p(x^p - y^p) = \nu_p(x - y) + 1.$$

*Proof.* The idea is similar to the proof of the previous lemma, but the details will be a bit more subtle. Consider again the following factorization:

$$x^p - y^p = (x - y)\left[\sum_{k=0}^{p-1} x^k y^{p-1-k}\right].$$

In this case, we have to check that $\nu_p\left(\sum_{k=0}^{p-1} x^k y^{p-1-k}\right) = 1$. Notice that $\sum_{k=0}^{p-1} x^k y^{p-1-k} \equiv px^{p-1} \equiv 0 \mod p$. Now we have to check that $p^2$ does not divide $\sum_{k=0}^{p-1} x^k y^{p-1-k}$. Let $x = y + tp$ with $t \in \mathbb{Z}$. With the aid of the Binomial theorem combined with the fact that a prime $q$ always divides the binomial coefficients $\binom{q}{m}$ for $1 \le m \le q - 1$, it is deduced that, for every $1 \le k \le p - 1$, the following holds:

$$\begin{aligned} x^k &\equiv (y + tp)^k \equiv y^k + ky^{k-1}tp + (\text{terms which are divisible by } p^2) \\ &\equiv y^k + ky^{k-1}tp \mod p^2. \end{aligned}$$

This yields that $x^k y^{p-1-k} \equiv y^{p-1} + ky^{p-2}tp \pmod{p^2}$ for all $0 \le k \le p-1$, and adding these equations up we get

$$\sum_{k=0}^{p-1} x^k y^{p-1-k} \equiv \sum_{k=0}^{p-1}\left(y^{p-1} + ky^{p-2}tp\right) \equiv py^{p-1} + \left(\frac{(p-1)p}{2}\right) y^{p-2}tp$$
$$\equiv py^{p-1} + \left(\frac{p-1}{2}\right) y^{p-2}tp^2 \equiv py^{p-1} \not\equiv 0 \pmod{p^2},$$

where we have used that $\frac{p-1}{2} \in \mathbb{Z}$ ($p$ is odd) and that $p$ does not divide $y$. □

We already have all the ingredients to prove Theorem 1.

Proof of Theorem 1. Let $n = p^m t$, where $p$ is prime and $t$ is not divisible by $p$, so $\nu_p(n) = m$. Observe that, for every $k \ge 1$, the numbers $x^k$ and $y^k$ are not divisible by $p$ and their difference is a multiple of $x - y$, hence divisible by $p$. Consequently, we can apply Lemma 4 to prove that

$$\nu_p(x^n - y^n) = \nu_p\left(\left(x^{p^m}\right)^t - \left(y^{p^m}\right)^t\right) = \nu_p(x^{p^m} - y^{p^m}),$$

because $p$ does not divide $t$. On the other side, applying Lemma 5 to the pair $x^{p^{j-1}}$ and $y^{p^{j-1}}$ for each $1 \le j \le m$ we prove that

$$\nu_p\left(x^{p^j} - y^{p^j}\right) = \nu_p\left(x^{p^{j-1}} - y^{p^{j-1}}\right) + 1.$$

Adding these equations up we obtain

$$\nu_p(x^{p^m} - y^{p^m}) = \nu_p(x - y) + m.$$

In conclusion,

$$\nu_p(x^n - y^n) = \nu_p(x - y) + \nu_p(n),$$

and the proof is complete. □

# 5 Proposed problems

The application of LTE lemma may not be strictly necessary to solve the following problems, but it certainly provides the most elegant and natural solution for most of them.

**Problem 1.** Show that the only positive integer value of $a$ for which $4(a^n+1)$ is a perfect cube for all positive integers $n$ is 1.

*Solution.* It is clear that $a=1$ verifies that $4(a^n+1)=8$ for each positive integer $n$ and, hence, it is a perfect cube. Now assume that $a$ verifies such condition; our aim is to prove that $a=1$. This value is the unique one for which the expression $4(a^n+1)$ is not strictly increasing and this is the fact that we have to exploit. We will assume that $a>1$ to reach a contradiction. In the language of valuations, the condition given in the statement is equivalent to the fact that, for every prime $p$, the integer $\nu_p(a^n+1)$ is a multiple of 3 for every $n \geq 1$. The first observation is that $a$ is odd because otherwise $\nu_2(4(a+1))=2$. This implies that $a^3+1$ has the odd factor $a^2-a+1>1$, and so, $p \mid a^3+1$ for some odd prime $p$. By the LTE lemma, $\nu_p(a^{3p}+1)=\nu_p(a^3+1)+1$, which is not divisible by 3 simply because $\nu_p(a^3+1)$ is. We conclude the proof with the previous contradiction. □

**Problem 2.** For some positive integer $n>1$, the number $3^n-2^n$ is a perfect power of a prime. Prove that $n$ is a prime number.

*Solution.* Let $3^n-2^n=p^k$ for some positive integer $k \geq 0$ (in fact, $k \geq 1$ because $n>1$) and let $q$ be a prime divisor of $n$. Clearly, $p$ is not 2 or 3, so $p \geq 5$.

Let $t=n/q$. Since the odd prime factors of $3^q-2^q$ are also factors of $(3^q)^t-(2^q)^t=p^k$, we have that $3^q-2^q$ is also a perfect power of $p$. It is important to remind that a perfect power of a prime $p$ is completely determined by its $p$-valuation. We can apply the LTE lemma to obtain that

$$3^{qt}-2^{qt}=p^{\nu_p(3^n-2^n)}=p^{\nu_p(3^q-2^q)}p^{\nu_p(t)}=(3^q-2^q)p^{\nu_p(t)} \leq (3^q-2^q)t,$$

and hence we must be in the equality case of Lemma 2, which implies that $t=1$. Thus, $n=q$ is prime. □

**Problem 3.** Let $n$ be a square-free positive integer. Show that there do not exist coprime positive integers $x, y$ such that

$$(x+y)^3 \mid x^n+y^n.$$

Solution. Let us assume the opposite. Let $x, y$ be positive integers verifying the divisibility condition of the statement. We will first prove that $n$ cannot be even. If it were the case, then $x+y$ is odd because 4 cannot divide $x^n + y^n$ when $x$ and $y$ are coprime. We pick a prime $p$ dividing $x+y$. It must also divide $x^n + y^n \equiv 2x^n \mod p$. Since $p$ is odd, then $p$ divides $x$ but then $p$ would be a common factor of $x$ and $y$, which are assumed to be coprime. This contradiction proves that $n$ is odd. At this moment, we analyse two different cases.

- There exists an odd prime $p$ dividing $x+y$. And by the LTE lemma, $\nu_p(x^n + y^n) = \nu_p(x+y) + \nu_p(n) \leq \nu_p(x+y) + 1 < 3\nu_p(x+y) = \nu_p((x+y)^3)$. In particular, $(x+y)^3$ cannot divide $x^n + y^n$, which contradicts our initial assumption.
- The number $x+y$ is a power of 2, so $x$ and $y$ are odd. Then,

$$\frac{x^n + y^n}{x+y} = \sum_{k=0}^{n-1} x^k y^{n-1-k}(-1)^k$$

  is the sum of an odd number of odd numbers, so it is odd and it is deduced that $\nu_2(x^n + y^n) = \nu_2(x+y) < 3\nu_2(x+y)$. The contradiction follows as in the previous case. □

Problem 4. Find all positive integers $x, y$ such that $p^x - y^p = 1$, where $p$ is a prime.

Solution. Notice that the equation can be rewritten in such a way that the sum of powers $1 + y^p$ appears on the right side, being equal to a perfect power of a prime $p$. This is the right observation towards an application of the LTE lemma. Let us distinguish the parity of the prime $p$.

If $p = 2$, then we have $2^x = 1 + y^2$. It is clear that $x$ cannot be greater or equal to 2 because otherwise 4 would divide $1 + y^2$, which is impossible since a perfect square leaves a remainder of 0 or 1 modulo 4. Then, $x = 1$, which yields the trivial solution $x = y = 1$ and $p = 2$.

The case $p$ odd is more interesting. Notice that $1 + y \mid 1 + y^p = p^x$ and, thus, $1 + y$ must be a power of $p$. In particular, $p$ divides

$p \mid 1+y$, so we are in the right conditions for applying the LTE lemma. We have $\nu_p(1+y^p) = \nu_p(1+y)+1$, which implies that

$$1+y^p = p^{\nu_p(1+y^p)} = p^{\nu_p(1+y)}p = (1+y)p,$$

because $x = p^{\nu_p(x)}$ if $x$ is a perfect power of $p$. We use the Newton binomial expansion:

$$\begin{aligned} p(y-1)+2p-1 = y^p &= (1+(y-1))^p \\ &\geq (y-1)^p + p(y-1)^{p-1} + \frac{p(p-1)}{2}(y-1)^{p-2}+1 \\ &\geq (y-1)^2(1+p) + \frac{p(p-1)}{2}(y-1)+1 . \end{aligned}$$

It is easy to see that $\frac{p(p-1)}{2}(y-1) \geq p(y-1)$ and, hence, $(y-1)^2(1+p)+1 \leq 2p-1$, which is false if $y \geq 3$. Therefore, $y = 2$. And the last inequality can be written as $2+p \leq 2p-1$, so $p = 3$. By the initial equation, $3^x = 1+2^3 = 9$, and thus $x = 2$.

It can be easily checked that the obtained tuples $(x;y;p)$, $(1;1;2)$ and $(2;2;3)$, are indeed solutions. By the previous reasoning, there are no more. □

Problem 5. Let $x$, $y$ be two positive rational numbers such that $x^n - y^n$ is a positive integer for infinitely many positive integers $n$. Show that $x$ and $y$ are both positive integers.

Solution. We will proceed by contradiction. The first observation is that, if the sum of two rational numbers is an integer, then they have the same denominator when considered in their standard irreducible form. For some positive integer $n$, $x^n$ and $y^n$ have integer difference, hence $x^n$ and $y^n$ (or, equivalently, $x$ and $y$) have the same denominator. We can now write $x = a/c$ and $y = b/c$, with $(a;c) = (b;c) = 1$, $a > b$ and $c > 0$. In this context, the condition of the statement can be reformulated as follows: there exists an increasing sequence of positive integers $\{n_i\}_{i \geq 1}$ for which $c^{n_i}$ divides $a^{n_i} - b^{n_i}$. Since $x$ and $y$ are not integers, $c > 1$ and there exists a prime $p$ dividing $c$.

The idea is that $\nu_p(c^n) = n\nu_p(c)$ is linear in $n$ but $\nu_p(a^n - b^n)$ will be proved to have at most logarithmic asymptotical order. In particular, $\nu_p(c^n) > \nu_p(a^n - b^n)$ for a sufficiently large $n$. This is a contradiction because the previous inequality cannot hold for those $n$ such that $c^n$ divides $a^n - b^n$. Thus, it suffices to prove the previous assertion about the asymptotic behaviour of $\nu_p(a^n - b^n)$ and this work will be done with the aid of the LTE lemma. Consider the sequence $m_k = \gcd(n_1; \dots; n_k)$ of positive integers. It is clearly non-increasing and, consequently, it is eventually constant. We call that last constant $d$. It is easily proved that $d$ divides $m_i$ for all $i \geq 1$.

On the other hand, $a^{n_i} \equiv b^{n_i} \mod p$ for every $i \geq 1$. Recall that $d = \gcd(n_1; \dots; n_N)$ for all sufficiently large $N$, so $p$ also divides $a^d - b^d$ by Bézout's identity. Precisely, for those $N$ for which $d = \gcd(n_1; \dots; n_N)$ there exist $k_1; \dots; k_N$ such that $\sum_{i=1}^{N} k_i n_i = d$, which can be used to prove that $a^d \equiv \prod_{i=1}^{N} a^{k_i n_i} \equiv \prod_{i=1}^{N} b^{k_i n_i} \equiv b^d \mod p$.

At this point, we have all the ingredients. By Theorem 1 and the standard bound given in Lemma 3,

$$\nu_p(a^{n_i} - b^{n_i}) = \nu_p(a^d - b^d) + \nu_p\left(\frac{n_i}{d}\right) \leq \nu_p(a^d - b^d) + \log_p(n_i),$$

as expected. The proof is complete. □

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# On Basel Problem

Marc Felipe i Alsina and José Luis Díaz-Barrero

## 1 Introduction

The Basel problem is a mathematical analysis problem with relevance in number theory. It was posed by Pietro Mengoli in 1650, solved by Leonhard Euler in 1734, and read in 1735 at the St. Petersburg Academy of Sciences. Since the solution of the problem had resisted the attacks of the main mathematicians since 1650, Euler's solution gave him immediate fame when he was only twenty-eight years old. Euler generalized the problem considerably, and his ideas were used years later by Bernhard Riemann in his 1859 document entitled "On the number of primes less than a given magnitude" [3], in which he de ned the zeta function and proved its basic properties. Since then, the result is called the Basel problem, Euler's hometown, as well as the Bernoulli family's that attacked it without success.

## 2 The Basel problem

The Basel problem asks for the exact summation of the reciprocals of the squares of the positive integers. It is stated as follows:

Problem 1. Compute

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} +$$

Solution (Euler). We have to compute

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots = \lim_{n\to\infty}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\right).$$

On account of Cradan-Viète's formulae, we know that, if a polynomial

$$A(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

has nonzero zeros $x_1, x_2, \ldots, x_n$, then the sum of their reciprocals satisfies

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\cdots+\frac{1}{x_n} = -a_1.$$

Now, we consider the function

$$\begin{aligned}\sin(x) &= x - \frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots \\ &= x\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\frac{x^8}{9!}-\cdots\right) = xf(x)\end{aligned}$$

Since the roots of $\sin(x) = 0$ are $0, \pm\pi, \pm 2\pi, \pm 3\pi \ldots$, then the zeros of $f(x)$ are $\pm\pi, \pm 2\pi, \pm 3\pi \ldots$ We make the substitution $y = x^2$ in $f(x)$ and we get the function

$$g(y) = 1 - \frac{y}{3!}+\frac{y^2}{5!}-\frac{y^3}{7!}+\frac{y^4}{9!}-\cdots,$$

whose zeros are $\pi^2, (2\pi)^2, (3\pi)^2 \ldots$ Now, we count the sum of their reciprocals in two ways. From the expression of $g(y)$,

$$\sum_{n=1}^{\infty}\frac{1}{y_n} = -\left(-\frac{1}{3!}\right) = \frac{1}{6}.$$

On the other hand,

$$\sum_{n=1}^{\infty}\frac{1}{y_n} = \frac{1}{\pi^2}+\frac{1}{2^2\pi^2}+\frac{1}{3^2\pi^2}+\cdots = \frac{1}{\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right).$$

Equating the preceding expressions yields

$$\frac{1}{\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\right) = \frac{1}{6},$$

from which it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}. \qquad \square$$

Another classical interesting problem connected with the previous one is the following

Problem 2. Consider the set $\{1, 2, \ldots, n\}$, with $n \to \infty$. What is the probability that two positive integers from this set chosen at random are relatively prime?

Solution. First, we observe that the two numbers, say $R$ and $S$, are coprime if and only if they do not have any common prime factor $p$. For the case $p = 2$ we have to find the probability that $R$ and $S$ do not have the factor 2 in common. Instead of computing this probability, we will compute the probability that $R$ and $S$ do have 2 as a common factor, that is, both are even. Since the probability that $R$ is even is $1/2$ and the probability that $S$ is even is also $1/2$, then the probability that $R$ and $S$ independently chosen are both even is $1/2^2$. Something similar occurs for primes 3, 5 and, more generally, for any prime $p$. Thus, we conclude that $1 - \frac{1}{p^2}$ is the probability that numbers $R$ and $S$ do not have the common factor $p$. By the Chinese remainder theorem, for different primes $p$ the events "$p$ divides $R$ and $S$" are independent. Then, $R$ and $S$ are coprime with probability

$$P = \left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{5^2}\right)\cdots = \prod_p \left(1 - \frac{1}{p^2}\right),$$

where the product is taken over all primes $p$.

Let $P_d$ be the probability that $(R, S) = d$. Since the probability that $d \mid R$ is $1/d$ and the probability that $d \mid S$ is also $1/d$, then the probability that $d$ divides both $R$ and $S$ is $1/d^2$. Since $d = (R, S)$, then $(R/d, S/d) = 1$, and the probability of this event has previously been called $P$. Thus, the probability that $(R, S) = d$ is $P/d^2$ and, therefore, we may conclude that

$$\sum_{d=1}^{\infty} \frac{P}{d^2} = P \sum_{d=1}^{\infty} \frac{1}{d^2} = 1,$$

from which it follows that

$$P = \left(\sum_{d=1}^{\infty} \frac{1}{d^2}\right)^{-1} = \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\right)^{-1}.$$

Inverting both terms of the last expression and using the preceding yields

$$\prod_p \left[\frac{1}{1-\frac{1}{p^2}}\right] = \sum_{d=1}^{\infty} \frac{1}{d^2},$$

and on account that

$$\frac{1}{1-\frac{1}{p^2}} = \sum_{n=0}^{\infty} \left(\frac{1}{p^2}\right)^n = 1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots,$$

then we have to see that

$$\prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots\right) = \sum_{d=1}^{\infty} \frac{1}{d^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

really holds. Indeed, multiplying up the terms in the LHS and invoking the Fundamental Theorem of Arithmetics we get all the terms in the RHS and viceversa.

Finally, we obtain that

$$P = \frac{6}{\pi^2}$$

on account of Problem 1. □

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# On fast calculation of square roots and logarithms

S. Camosso

Abstract

In this brief article we review basic algorithms for a fast calculation of square roots and logarithms. The necessity to have "good" approximations of these mathematical entities arises from the ancient Chinese civilization.

## 1 Introduction

Ch'in Chiu–shao was a Chinese mathematician who proposed different ways to approximate roots. The following inequalities are a remarkable example:

$$a+\frac{r}{2a+1}<\sqrt{a^2+r}<a+\frac{r}{2a}.$$

The second is used in order to approximate the square root as $\sqrt{a^2+r} \approx a+\frac{r}{2a}$. We will prove this formula in the next section. This is not the only example presented by the Chinese mathematician, and other formulas can be found in [2].

Now, square roots are not the only mathematical object that needs to be approximated, we also have logarithms. In [1], Euler describes a method for the determination of logarithms by hand. The method uses different properties of logarithms and the formula

$$\log\sqrt{xy}=\frac{\log x+\log y}{2}.$$

Let us consider an example: the determination of $\log_3 42$. The first step is to change the base from 3 to 10: $\frac{\log 42}{\log 3}$. The second step consists of observing that $\log 42 = \log 4.2 \cdot 10 = 1 + \log 4.2$. We have to determine $\log 4.2$. We know that $\sqrt{10} \approx 3.162278$ and that

$$\log \sqrt{1 \cdot 10} = \frac{0 + 1}{2} = 0.5.$$

Now $\sqrt{10} < 4.2 \leq 10$, so $0.5 < \log 4.2 < 1$. We repeat the calculations with $\sqrt{31.62278} \approx 5.623413$ and

$$\log \sqrt{3.162278 \cdot 10} = \frac{0.5 + 1}{2} = 0.75.$$

Now $3.162278 < 4.2 < 5.623413$, so $0.5 < \log 4.2 < 0.75$. We repeat the calculations with $\sqrt{3.162278 \cdot 5.623413} \approx 4.216965$ and

$$\log \sqrt{3.162278 \cdot 5.623413} = \frac{0.5 + 0.75}{2} = 0.625.$$

One can proceed in this way with more precision. We stop here, so $\log 4.2 \approx 0.625$. The last step consists in the division by $\log 3 \approx 0.477$:

$$\log_3 42 \approx 3.41.$$

# 2 A fast square root

Let $n$ be a positive integer, and let $m$ be the nearest positive integer to $n$ such that $m \leq n$ and $m$ is a perfect square. We define the positive quantity $\varepsilon = n - m$ and we remember the Taylor expansion of $\sqrt{1 + x}$ when $x$ is in a neighborhood of $0$:

$$\sqrt{1 + x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots + \binom{\frac{1}{2}}{n} x^n + o(x^n).$$

The aim is to find an approximation of $\sqrt{n}$ that is fast to compute manually.

**Proposition 1.** Let $n$ be a positive integer and $m$ be the nearest positive integer to $n$ such that $m \le n$ and $\sqrt{m} = m'$. Then,

$$\sqrt{n} = m' + \frac{\epsilon}{2m'} + o\left(\frac{\epsilon}{m}\right),$$

where $\epsilon = n - m$.

*Proof.* We observe that $\sqrt{n} = \sqrt{m+\epsilon} = \sqrt{m}\sqrt{1+\frac{\epsilon}{m}}$. We call $m' = \sqrt{m}$ and, using the Taylor expansion, we find that

$$\sqrt{1+\frac{\epsilon}{m}} = 1 + \frac{\epsilon}{2m} + o\left(\frac{\epsilon}{m}\right). \tag{1}$$

We only consider the first two terms of the Taylor expansion (1), so

$$\sqrt{n} = \sqrt{m}\left(1 + \frac{\epsilon}{m}\right) \approx m' + m'\frac{\epsilon}{2m}.$$

Now, realizing that $m' = \sqrt{m}$, we are done. □

**Example 1.** We want a fast approximation of $\sqrt{5}$. So $n = 5$, $m = 4$ and $m' = 2$. We have that

$$\sqrt{5} \approx 2 + \frac{5-4}{2 \cdot 2} = 2 + \frac{1}{4} = 2.25.$$

This is a good approximation of simple computation.

# 3 A fast logarithm

For the logarithm we must remember the following Taylor expansion:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + (-1)^{n+1}\frac{x^n}{n} + o(x^n). \tag{2}$$

**Proposition 2.** Let $n$ be a positive integer and $m$ be the nearest positive integer to $n$ such that $m \le n$ and $\ln m = m'$. Then,

$$\ln n = m' + \frac{\epsilon}{m} + o\left(\frac{\epsilon}{m}\right),$$

where $\epsilon = n - m$.

*Proof.* We observe that $\ln n = \ln(m + \varepsilon) = \ln m + \log\left(1 + \frac{\varepsilon}{m}\right)$. We call $m' = \ln m$ and, using the Taylor expansion, we find that

$$\ln\left(1 + \frac{\varepsilon}{m}\right) = \frac{\varepsilon}{m} + o\left(\frac{\varepsilon}{m}\right).$$

We only consider the first term of the Taylor expansion (2), so

$$\ln n = \ln m + \ln\left(1 + \frac{\varepsilon}{m}\right) \approx \ln m + \frac{\varepsilon}{m}.$$

Now, realizing that $m' = \ln m$, we are done. □

*Example 2.* We want a fast approximation of $\ln 3$. So $n = 3$, $m = e$ and $m' = 1$. We have that

$$\ln 3 \approx 1 + \frac{3 - e}{e} = \frac{3}{e} \approx 1.104.$$

This is a good approximation of simple computation.

# 4 A generalization

We can generalize the first proposition by considering the Taylor expansion in two variables $x$ and $y$:

$$\sqrt{1 + x + y} \approx 1 + \frac{1}{2}x + \frac{1}{2}y - \frac{1}{8}x^2 - \frac{1}{8}y^2 - \frac{1}{4}xy + \dots \quad (3)$$

*Proposition 3.* Let $n$ be a positive integer and $m$, $p$, $t$ be three positive integers such that $m, p, t \leq n$, $n = m + p + t$ and $\sqrt{m} = m'$. Then,

$$\sqrt{n} = m' + \frac{\delta}{2m'} + o\left(\frac{\delta}{m}\right),$$

where $\delta = p + t$.

*Proof.* The proof is similar to the first proposition where we apply the Taylor expansion (3) to

$$\sqrt{n} = \sqrt{m}\sqrt{1 + \frac{p}{m} + \frac{t}{m}} \approx m' + \frac{p}{2m'} + \frac{t}{2m'}.$$ □

Figure 1: Representation of $z = \sqrt{1+x+y}$ and the plane $z = 1 + \frac{1}{2}x + \frac{1}{2}y$.

Example 3. We want to approximate $\sqrt{6}$, so $n = 6$, $m = 4$, $p = t = 1$ and using the formula we have that

$$\sqrt{6} \approx 2 + \frac{1}{4} + \frac{1}{4} \approx 2.5.$$

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# Contests

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to nd interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to José Luis Díaz-Barrero, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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# Problems and solutions from the 60th edition of the International Mathematical Olympiad (IMO)

Óscar Rivero Salgado

## 1 Introduction

The 60th edition of the International Mathematical Olympiad took place in July 2019 in Bath (United Kingdom). The number of countries attending the competition with at least one contestant was 114, a new record for the IMO. The number of contestants was also impressive: there were 643 this year. The competition was developed in two consecutive days, and contestants had to solve three problems each day in a maximum time of four hours and a half. Each problem was graded with an integer mark between 0 and 7 points, so the maximum possible score was 42 points, achieved this year by six students. According to the usual standards, at most half of the students can get a medal, and then these are awarded in the proportion 1:2:3 for gold, silver and bronze, respectively. Under these circumstances, the number of points needed to get a bronze distinction were 17; for a silver medal, 24 points were necessary, and only those who reached 31 points got a gold medal. By countries, both the People's Republic of China and the United States of America got 227 points out of 252, while the Republic of Korea was just one point below.

Spain made an outstanding performance, with a new record of points (110) and relative position. They finished in the 42nd position and, for the first time, five of the students got a bronze medal, while the remaining one got an honorable mention. The team was composed by Pau Cantos (22 points), Leonardo Costa (21), Albert López (19), Oriol Baeza (17), Pablo Soto (17), and Juan Brieva (14). The delegation was completed by María Gaspar, as the chief of the delegation, and by Óscar Rivero, as the deputy leader.

We present now the four problems which were solved by at least one Spaniard contestant (problems 1, 2, 4 and 5), and include the solutions given to them by our team (in some case slightly modified by the deputy leader). In all the cases, the solutions follow the ideas presented by the contestants, but we have done some little modifications to ease the exposition.

## 2 Problems and solutions

We now present some problems that appeared in the paper, as well as some solutions and comments to them.

**Problem 1.** Let $\mathbb{Z}$ be the set of integers. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers $a$ and $b$,

$$f(2a) + 2f(b) = f(f(a+b)).$$

*Comment.* Five of the students gave complete solutions to this question. All of them require an algebraic manipulation before getting that any solution to the functional equation is of the form $f(x) = Cx + d$, where $C$ and $d$ are integer constants; after that, one has to guess which are the possible values for the constants. There are two main approaches for the first (and harder) part: either by different substitutions (as sketched in the first solution we present) or by passing to a Cauchy equation, whose solutions are well-known (as presented in the second solution, which assumes certain knowledge of functional equations by the reader of this note).

**Solution 1 by Juan Brieva Ramírez.** Setting $a = 0$ and $b = 0$, we obtain

$$3f(0) = f(f(0)). \tag{1}$$

When we evaluate the given expression at $(a; 0)$ and $(0; a)$ and subtract both equations, one gets

$$f(2a) + f(0) = 2f(a). \tag{2}$$

Using now (2), from the original equation we have

$$2f(a) + 2f(b) = f(f(a+b)) + f(0). \tag{3}$$

If we do $b \mapsto -a$ in (3), and then use (1), we obtain

$$f(a) + f(-a) = 2f(0). \tag{4}$$

We finally substitute $(-a; a+1)$ in the given equation and, using (4), one gets

$$f(a+1) - f(a) = C,$$

where $C = \frac{f(f(1)) - 4f(0)}{2}$. Since the difference between two consecutive integers is a constant, we have that all possible functions are of the form $f(x) = Cx + d$.

From the condition (1),

$$3d = f(d) = Cd + d,$$

and then, either $d = 0$ or $C = 2$. In the latter case, an easy check shows that all the solutions of the form $f(x) = 2x + d$ are valid and satisfy the given condition. If $d = 0$, one easyly gets that either $C = 2$ (and this solution is already included in the previous case) or $C = 0$ (and this solution clearly satisfies the given condition).

We conclude that the possible solutions are either $f(x) \equiv 0$, or $f(x) = 2x + d$, where $d \in \mathbb{Z}$.

**Solution 2 by Oriol Baeza Guasch.** As in the previous solution, a straightforward substitution gives

$$2f(a) + f(0) = f(f(a)),$$

and setting $a = x + y$,

$$2f(x+y) + f(0) = f(f(x+y)) = 2f(x) + 2f(y) - f(0). \quad (5)$$

Therefore,

$$f(x+y) - f(0) = (f(x) - f(0)) + (f(y) - f(0)). \quad (6)$$

Defining $g(x) = f(x) - f(0)$, equation (6) gives

$$g(x+y) = g(x) + g(y). \quad (7)$$

But (7) is a Cauchy equation, whose unique solutions are given by $g(x) = Cx$, where $C \in \mathbb{Z}$. Then, all possible solutions have the form $f(x) = Cx + f(0)$. Proceeding as before by substituting in the original equation, we get that either $f(0) = 0$ or $C = 2$, and by a careful analysis of each case we get that either $f(x) \equiv 0$, or $f(x) = 2x + f(0)$, with $f(0) \in \mathbb{Z}$.

**Problem 2.** In triangle $ABC$, point $A_1$ lies on side $BC$ and point $B_1$ lies on side $AC$. Let $P$ and $Q$ be points on segments $AA_1$ and $BB_1$, respectively, such that $PQ$ is parallel to $AB$. Let $P_1$ be a point on line $PB_1$ such that $B_1$ lies strictly between $P$ and $P_1$, and $\angle PP_1C = \angle BAC$. Similarly, let $Q_1$ be a point on line $QA_1$ such that $A_1$ lies strictly between $Q$ and $Q_1$, and $\angle CQ_1Q = \angle CBA$. Prove that points $P$, $Q$, $P_1$, and $Q_1$ are concyclic.

*Solution by Pablo Soto Martín.* Let $X$ stand for the cut point of the circumcircle of $\triangle A_1CQ_1$ with $AA_1$. Similarly, let $Y$ denote the intersection of the circumcircle of $\triangle B_1CP_1$ with $BB_1$.

We begin by claiming that both $X$ and $Y$ belong to the circumcircle of $\triangle ABC$. This is true because

$$\angle A_1XC = \angle A_1Q_1C = \angle ABC$$

and

$$\angle B_1YC = \angle B_1P_1C = \angle BAC.$$

We now prove that the following three quadrilaterals are cyclic: $PQXQ_1$, $PQYP_1$, and $PQXY$. In particular, these three facts together imply that $PQP_1Q_1XY$ is cyclic, and we are done.

Begin with $PQXQ_1$. In this case,

$$\angle QPX = \angle BAX = \angle BCX = \angle A_1Q_1X = \angle QQ_1X. \qquad (8)$$

For $PQYP_1$, the proof is symmetric, following the same angle chasing procedure as in (8). Finally, for $PQXY$ we just observe that

$$\angle PQY = \angle ABY = \angle AXY = \angle PXY,$$

and hence the conclusion follows.

**Problem 4.** Find all pairs $(k; n)$ of positive integers such that

$$k! = (2^n - 1)(2^n - 2)(2^n - 4)\cdots(2^n - 2^{n-1}).$$

*Notation.* As a general piece of notation for the two solutions we present, for a given prime $p$, let $v_p(n)$ stand for the greatest integer such that $p^{v_p(n)}$ divides $n$ but $p^{v_p(n)+1}$ does not. All the approaches to the problem are based on comparing the growth of both sides: one option is by using two different primes (as shown in the second solution), and the other one is using one prime and the usual absolute value (as shown in the first solution, which is maybe the most natural approach to the problem, although in the last part it requires some nasty computations). The second solution uses as a well-known fact the *lifting the exponent lemma*.

*Solution 1 by Pau Cantos Coll.* We begin by observing that

$$v_2(k!) = \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{k}{4} \right\rfloor + \left\lfloor \frac{k}{8} \right\rfloor + \cdots < \frac{k}{2} + \frac{k}{4} + \frac{k}{8} + \cdots = k.$$

Since

$$v_2((2^n - 1)(2^n - 2)(2^n - 4)\cdots(2^n - 2^{n-1})) = \frac{n(n-1)}{2},$$

we obtain that

$$\frac{n(n-1)}{2} < k. \qquad (9)$$

On the other hand, we have

$$2^{n^2} > (2^n - 1)(2^n - 2)(2^n - 4)\cdots(2^n - 2^{n-1}) = k!. \qquad (10)$$

Putting (9) and (10) together, one gets

$$2^{n^2} > \left(\frac{n(n-1)}{2}\right)!. \qquad (11)$$

For $n = 7$, this is false, since

$$21! = 2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 > 2^{18} \cdot 2^9 \cdot 2^8 \cdot 2^6 \cdot 2^3 \cdot 2^3 \cdot 2^4 \cdot 2^4 > 2^{49}.$$

We now show by induction that, indeed, the opposite inequality as that of (11) holds for $n \geq 7$. For $n = 7$ it has already been checked. Then, using the induction hypothesis, we obtain

$$\left(\frac{(n+1)n}{2}\right)! > \left(\frac{n(n-1)}{2}\right)^n \left(\frac{n(n-1)}{2}\right)! > 8^n \cdot 2^{n^2}$$
$$= 2^{n^2+3n} > 2^{n^2+2n+1} = 2^{(n+1)^2},$$

and we have proved that there are no solutions for $n \geq 7$.

Small cases have to be checked by hand.

When $n = 6$, the right hand side is $63 \cdot 62 \cdot 60 \cdot 56 \cdot 48 \cdot 32$. Since $31$ is a prime dividing $62$, $k \geq 31$ and then the left hand side would be clearly greater than the right hand side.
When $n = 5$, the right hand side is $31 \cdot 30 \cdot 28 \cdot 24 \cdot 16$. Since $31$ is prime, $k \geq 31$ and then, as before, the left hand side would be greater than the right hand side.
When $n = 4$, the right hand side is $15 \cdot 14 \cdot 12 \cdot 8$. Then, $k \geq 7$; if $k = 7$, the $2$-adic valuation of both sides does not agree. If $k \geq 14$, the left hand side is bigger than the right hand side.
When $n = 3$, the right hand side is $7 \cdot 6 \cdot 4 = 168$, which is not a factorial of an integer number.
When $n = 2$, the right hand side is $6 = 3!$, so $(3; 2)$ is a solution.
When $n = 1$, we obtain the solution $(1; 1)$.

Therefore, the unique possible solutions are $(3; 2)$ and $(1; 1)$.

Solution 2 by Albert López Bruch. We compare the $5$-adic and $7$-adic valuations of both sides to reach a contradiction. We begin with the prime 7. In this case, for determining

$$R_7(n) := v_7((2^n - 1)(2^{n-1} - 1) \cdots (2^2 - 1)(2 - 1)),$$

we just recall that, by virtue of the *lifting the exponent lemma*, $v_7(2^\ell - 1) = 0$ when $\ell$ is not a multiple of $3$, and

$$v_7(2^\ell - 1) = v_7(\ell) + 1 \quad \text{for all } \ell \in 3\mathbb{N}.$$

Then,

$$R_7(n) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{3 \cdot 7} \right\rfloor + \left\lfloor \frac{n}{3 \cdot 7^2} \right\rfloor + \cdots \geq \frac{n-2}{3}.$$

Similarly, for

$$R_5(n) := v_5((2^n - 1)(2^{n-1} - 1) \cdots (2^2 - 1)(2 - 1)),$$

we have that $v_5(2^\ell - 1) = 0$ if $\ell$ is not a multiple of $4$, and

$$v_5(2^\ell - 1) = v_5(\ell) + 1 \quad \text{for all } \ell \in 4\mathbb{N}.$$

We have that

$$R_5(n) = \left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{4 \cdot 5} \right\rfloor + \left\lfloor \frac{n}{4 \cdot 5^2} \right\rfloor + \cdots < \frac{5n}{16}.$$

Then, note that

$$\frac{5n}{16} > R_5(n) = v_5(k!) \geq v_7(k!) = R_7(n) \geq \frac{n-2}{3}. \tag{12}$$

Hence, $n < 32$. This means that, indeed, since only the first two terms can contribute, $R_5(n) < \frac{n}{4} + \frac{n}{20} = \frac{3n}{10}$ and, therefore,

$$\frac{3n}{10} \geq \frac{n-2}{3}. \tag{13}$$

Then, $n \leq 20$. We distinguish now several cases.

- If $n$ is congruent to 0 modulo 3, then equation (12) can be refined to $5n/16 \geq n/3$, which never holds.
- If $n$ is congruent to 1 modulo 3, and moreover $n < 20$, equation (13) becomes
$$\frac{n}{4} \geq \frac{n-1}{3}.$$
When $n = 1$ we get the solution $(1; 1)$. When $n = 4$, we proceed as in the previous solution to see that there are no possible solutions. All the cases corresponding to $n > 4$ are automatically excluded.

When $n = 20$, $R_5(20) = R_7(20) = 6$. Hence, $v_5(k!) = 6$, so $25 \leq k \leq 29$. At the same time, $v_7(k!) = 6$, so $42 \leq k \leq 48$, and this is a contradicition.

If $n < 20$ and $n$ is congruent to 2 modulo 3, equation (13) is replaced by

$$\frac{n}{4} \quad \frac{n-2}{3},$$

so $n \leq 8$. If $n = 8$, $R_5(8) = R_7(8) = 2$. In this case, $v_5(k!) = 2$, so $10 \leq k \leq 14$, but at the same time $v_7(k!) = 2$, and this forces $14 \leq k \leq 20$. Hence, the unique option is $k = 14$, but one can easily check that the pair $(14; 8)$ does not work.

The case $n = 5$ is discarded as before, and when $n = 2$ we get the solution $(3; 2)$.

Hence, the only possible solutions are $(3; 2)$ and $(1; 1)$.

**Problem 5.** The Bank of Bath issues coins with an $H$ on one side and a $T$ on the other. Harry has $n$ of these coins arranged in a line from left to right. He repeatedly performs the following operation: if there are exactly $k > 0$ coins showing $H$, then he turns over the $k$-th coin from the left; otherwise, all coins show $T$ and he stops. For example, if $n = 3$ the process starting with the configuration $THT$ would be $THT \to HHT \to HTT \to TTT$, which stops after three operations.

(a) Show that, for each initial configuration, Harry stops after a finite number of operations.

(b) For each initial configuration $C$, let $L(C)$ be the number of operations before Harry stops. For example, $L(THT) = 3$ and $L(TTT) = 0$. Determine the average value of $L(C)$ over all $2^n$ possible initial configurations $C$.

Comment. Before properly presenting the solutions given by the contestants, we point out that there are several straightforward ways to solve just part (a) without simultaneously solving part (b), as it will be the case with the two solutions we discuss here, due to Leonardo Costa and Albert López. For the sake of completeness, we indicate a strategy for part (a), following the script of Pablo Soto.

Assume that the process does not nish; since any con guration completely determines the subsequent movements, this would mean that we get a cycle. Suppose that we are in a position of the cycle where the number of appearances of the $H$ is minimum, and say that at that moment we have $k$ times that letter, with $k$ $1$. Therefore, at position $k$ we have the letter $T$ (otherwise, we would move to a situation with fewer than $k$ $H$ 's). This means that, in the following movement, we continue towards the right of the sequence, until we nd a letter $H$. If there were no more $H$ 's beyond the initial position, that initial letter would be necessarily an $H$, by the conditions of the problem. When we nd the rst $H$, we switch its state, so this becomes a $T$, and we begin the movement towards the left, until one nds a new $T$. Observe in particular that at position $k$ there is the letter $H$, and we do pass through it, since all the intermediate positions until the moment were $H$, by construction. When this happens, it becomes a $T$. Hence, since the state of the intermediate positions has not changed (we have gone through them exactly twice), we get a con guration with strictly fewer $H$ 's than the initial one. This is a contradiction, that comes from having supposed that the process never nishes.

Solution 1 by Albert López Bruch. Let $c_n$ denote the average number of operations for sequences of length $n$. We proceed by induction to solve both parts of the problem. For $n = 1$ it is easy, and one gets $c_1 = 1/2$.

Consider a sequence of length $n + 1$. If it nishes with a $T$, this does not affect the sequence of operations at all, since we never reach that position (it would require $n + 1$ appearances of the $H$, including the last one), and hence the number of operations is the same as those needed for the word of length $n$ obtained by deleting the last coin. Moreover, in this case the process nishes, since the induction hypothesis guarantees this.

We now take a word of length $n + 1$ nishing with an $H$. Say that the word is of the form $a_1 a_2 \ldots a_n H$. We establish a bijection between words of length $n + 1$ nishing with an $H$ and words of length $n$ by sending

$$a_1 a_2 \ldots a_n H \mapsto a_n \ldots a_2 a_1,$$

where $a_i \in \{H, T\}$ and $\overline{a_i}$ is $H$ when $a_i$ is $T$ and viceversa. This bijection commutes with the operation described in the statement of the problem; moreover, it is indeed a bijection, with the inverse given by

$$a_1 a_2 \ldots a_n \mapsto a_n \ldots a_2 a_1 H .$$

To see that it commutes with the operation, say that after applying it to the initial word we get $a_1 a_2 \ldots a_k \ldots a_n H$; if we apply now the bijection we get $a_n \ldots a_k \ldots a_2 a_1$. But this is precisely the result of applying the operation to $a_n \ldots a_2 a_1$, since this word has $n-(k-1) = n-k+1$ times the letter $H$, and precisely its $(n-k+1)$-th element is $a_k$, that now gets reversed. Hence, the process arrives to a situation of $H \ldots HH$. Moreover, this shows that, after the same number of operations needed in the case of length $n$ to get all $T$'s, we get a solution of all $H$'s. From here, and after $n+1$ operations, we can finish.

Therefore, we get

$$c_{n+1} = \frac{c_n}{2} + \frac{c_n + n + 1}{2} = c_n + \frac{n+1}{2},$$

and by the induction hypothesis one easily concludes. The answer is just

$$c_n = \frac{n(n+1)}{4}.$$

Solution 2 by Leonardo Costa Lesage. We keep the same notations as before, where $c_n$ stands for the expected number of movements. In this case, we proceed again by induction, but we need to consider two base cases, those being $c_1 = 1/2$ and $c_2 = 3/2$.

There are three kinds of sequences of length $n+2$: (a) those finishing with a $T$; (b) those beginning with an $H$; and (c) those beginning with a $T$ and finishing with an $H$. The proportion of the first two kinds is $1/2$, but the intersection is non-empty: those sequences beginning with an $H$ and finishing with a $T$ (proportion $1/4$) are being counted twice!

Let us determine the number of operations in all three cases, showing that the process always finishes. In the first one, since we never reach the last position (it would require $n+2$ appearances of the $H$), it is the same as the sequence of length $n+1$ removing the last coin. This gives an average number of $c_{n+1}$. In the second case, the rst coin shifts everything by one position, and therefore we can neglect it until the end, where we get an extra move. The average is $c_{n+1}+1$. We now consider its intersection, given by the sequences beginning with a $H$ and nishing with a $T$. Taking into account the previous considerations, we take the $n$ central coins, and since until the end the coins placed in the extremes do not affect the operations, we reach a con guration of the form $HTT:::TT$, and therefore the average number of movements is now $c_n+1$.

In the third and last case, everything goes as in the second one, and after the movements corresponding to the $n$ central positions, we get $TT:::TH$. From here, it takes $2n+3$ moves to nish. Observe that we have established that in all the cases the process terminates by reducing to shorter sequences where the induction hypothesis applies. Therefore,

$$c_{n+2} = \frac{c_{n+1}}{2} + \frac{c_{n+1}+1}{2} \quad \frac{c_n+1}{4} + \frac{c_n+2n+3}{4} = c_{n+1} + \frac{n+2}{2}.$$

We already know that $c_1 = 1/2$ and $c_2 = 3/2$, and hence we can prove by a direct induction argument that

$$c_n = \frac{n(n+1)}{4}.$$

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# Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

Please, send submittals to José Luis Díaz-Barrero, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

jose.luis.diaz@upc.edu

## Elementary Problems

E–65. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. In a deck of cards there are six diamond cards, ve club cards, four heart cards and three spade cards. We choose some of them, in such a way that we pick at least one card from each suit. In how many ways can we make such a choice?

Solution by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom. Given any set of $n$ distinct elements, the number of ways to choose a subset with at least one element is $2^n \quad 1$ (there are $2^n$ possible subsets, and we ignore the empty set). Thus, we can choose diamonds, clubs, hearts and spades in $2^6 \quad 1$, $2^5 \quad 1$, $2^4 \quad 1$ and $2^3 \quad 1$ ways, respectively. Therefore, by the rule of product, the total number $N$ of all possible ways to make such choices is

$$N = (2^6 \quad 1)(2^5 \quad 1)(2^4 \quad 1)(2^3 \quad 1) = 205065 .$$

Also solved by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and the proposer.

**E–66.** Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Find the number of all positive divisors of $46189^{12}$ which are not cubes of a positive integer.

Solution by Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain. The factorization is $46189 = 4199 \quad 11 = 323 \quad 13 \quad 11 = 19 \quad 17 \quad 13 \quad 11$. So $46189^{12} = 19^{12} \quad 17^{12} \quad 13^{12} \quad 11^{12}$.

The divisors of this number are of the form $19^a \quad 17^b \quad 13^c \quad 11^d$, where $a$, $b$, $c$ and $d$ are between $0$ and $12$. With these exponents, each divisor can be visualized as a vector of the form $(a; b; c; d)$ where each vector is associated with the divisor $19^a \quad 17^b \quad 13^c \quad 11^d$, with $a$, $b$, $c$ and $d$ in the same range mentioned before.

As there are $13$ options for each exponent, there are $13^4$ divisors. In order to count how many of them are not cubes, we are going to count how many are cubes, and then subtract this number from the number of total divisors.

In order for a number to be a cube, it is necessary that the primes in its factorization have exponents that are multiples of $3$. In this case, as the exponents move between $0$ and $12$, the only options for them to be a multiple of $3$ are $0$, $3$, $6$, $9$ and $12$. Therefore, there are $5$ options for each exponent, and we have to choose $4$ exponents, so there are $5^4$ divisors of the number $46189^{12}$ that are cubes and $13^4 \quad 5^4 = 27936$ that are not cubes.

Also solved by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom; Henry Ricardo, Westchester Area Math Circle, NY, USA, and the proposer.

**E–67.** Proposed by José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain. Find all polynomials $A(x)$ of degree $4$ such that $A(1) = A(\quad 2) = A(3) = A(\quad 4) = 3$ and $A(0) = 6$ .

Solution 1 by Sarah B. Seales, Northern Arizona University, Arizona, USA. We are asked to nd all polynomials $A(x)$ of degree

4 such that $A(1) = A(-2) = A(3) = A(-4) = 3$ and $A(0) = 6$. We will construct such a polynomial and show that it is the only solution.

The polynomial $A(x) = a(x-1)(x+2)(x-3)(x+4)+3$ satisfies $A(1) = A(-2) = A(3) = A(-4) = 3$, and so we use the condition $A(0) = 6$ to find $a$. We have

$$A(0) = a(-1)(2)(-3)(4) + 3 = 6,$$

which simplifies to $a = \frac{1}{8}$, giving us

$$A(x) = \frac{1}{8}x^4 + \frac{1}{4}x^3 - \frac{13}{8}x^2 - \frac{7}{4}x + 6.$$

Now, let us assume for contradiction that there is another polynomial $B(x)$ such that $B(1) = B(-2) = B(3) = B(-4) = 3$ and $B(0) = 6$. By the Identity Theorem for polynomials of degree four, if $B(x) = A(x)$ for five or more values, then $B(x)$ is the same polynomial as $A(x)$. Since

$$B(1) = A(1) = 3,$$

$$B(-2) = A(-2) = 3,$$

$$B(3) = A(3) = 3,$$

$$B(-4) = A(-4) = 3,$$

$$B(0) = A(0) = 6,$$

we see that $B(x)$ is the same polynomial as $A(x)$. Therefore, the only solution is

$$A(x) = \frac{1}{8}x^4 + \frac{1}{4}x^3 - \frac{13}{8}x^2 - \frac{7}{4}x + 6.$$

Solution 2 by Henry Ricardo, Westchester Area Math Circle, NY, USA. The polynomial is given by

$$A(x) = \frac{1}{8}x^4 + \frac{1}{4}x^3 - \frac{13}{8}x^2 - \frac{7}{4}x + 6,$$

and it is unique since an $n$-th degree polynomial is determined by its value at $n+1$ points.

To prove this, we use the *Lagrange interpolating polynomial*, the unique polynomial $A(x)$ of degree at most $n-1$ that passes through the $n$ points $(x_i; y_i)$, where $y_i = A(x_i)$ for $i = 1; 2; \dots; n$. In our problem, we have

$$A(x) = \sum_{j=1}^{5} A_j(x), \text{ where } A_j(x) = y_j \prod_{k=1;\, k\neq j}^{5} \frac{x - x_k}{x_j - x_k},$$

with $x_k = (-1)^k(k-1)$ for $k = 1; 2; \dots; 5$, $y_1 = 6$, and $y_j = 3$ for $j = 2; 3; 4; 5$.

After a bit of tedious algebra, this yields

$$A(x) = \frac{1}{8}x^4 + \frac{1}{4}x^3 - \frac{13}{8}x^2 - \frac{7}{4}x + 6.$$

**Solution 3 by Henry Ricardo, Westchester Area Math Circle, NY, USA.** The polynomial is given by

$$A(x) = \frac{1}{8}x^4 + \frac{1}{4}x^3 - \frac{13}{8}x^2 - \frac{7}{4}x + 6,$$

and it is unique since an $n$-th degree polynomial is determined by its value at $n+1$ points.

To prove this, let $A(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. The given values of $A(x)$ at the points $x = 0; 1; -2; 3$, and $-4$ yield $a_0 = 6$ and the system of linear equations

$$\left.\begin{aligned} a_4 + a_3 + a_2 + a_1 &= -3 \\ 16a_4 - 8a_3 + 4a_2 - 2a_1 &= -3 \\ 81a_4 + 27a_3 + 9a_2 + 3a_1 &= -3 \\ 256a_4 - 64a_3 + 16a_2 - 4a_1 &= -3 \end{aligned}\right\},$$

which we can write in the form

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 16 & -8 & 4 & -2 \\ 81 & 27 & 9 & 3 \\ 256 & -64 & 16 & -4 \end{pmatrix} \begin{pmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ -3 \\ -3 \end{pmatrix}.$$

The inverse of the matrix of coefficients is easily (but tediously) found to be

$$\begin{pmatrix} \frac{1}{30} & \frac{1}{60} & \frac{1}{210} & \frac{1}{280} \\ \frac{1}{10} & 0 & \frac{1}{42} & \frac{1}{140} \\ \frac{1}{3} & \frac{13}{60} & \frac{1}{105} & \frac{1}{56} \\ \frac{4}{5} & \frac{1}{5} & \frac{4}{105} & \frac{3}{140} \end{pmatrix},$$

and so we have

$$\begin{pmatrix} a_4 \\ a_3 \\ a_2 \\ a_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{30} & \frac{1}{60} & \frac{1}{210} & \frac{1}{280} \\ \frac{1}{10} & 0 & \frac{1}{42} & \frac{1}{140} \\ \frac{1}{3} & \frac{13}{60} & \frac{1}{105} & \frac{1}{56} \\ \frac{4}{5} & \frac{1}{5} & \frac{4}{105} & \frac{3}{140} \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{1}{8} \\ \frac{1}{4} \\ \frac{13}{8} \\ \frac{7}{4} \end{pmatrix}.$$

Also solved by Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain; José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain; Irene Fernández Fernández, IES El Carmen, Murcia, Spain; Rovsen Pirguliyev, Sumgait City, Azerbaijan; Henry Ricardo, Westchester Area Math Circle, NY, USA (one more solution), and the proposer.

**E–68.** Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Find all positive integers $a$ and $b$ such that $ab = 10648$ and $\operatorname{lcm}\{a, b\} = 968$.

*Solution 1 by Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain.* We know that $ab = \operatorname{lcm}(a, b) \cdot \gcd(a, b)$. Therefore, as $10648 = 968 \cdot 11$, we have that $\gcd(a, b) = 11$. Also, note that $968 = 2^3 \cdot 11^2$, so $10648 = 2^3 \cdot 11^3$. Since the greatest common divisor of these two numbers is $11$, we can write $a = 11a'$ and $b = 11b'$, where $a'$ and $b'$ are coprimes. The other $11$ in the prime factorization of $10648$ will be in the prime factorization of $a'$ or in the one of $b'$. And, as the least common multiple is $11^2 \cdot 2^3$, we have that either $a'$ or $b'$ must be $2^3$. So the only two options are:

$$\begin{cases} a' = 11 \cdot 2^3,\ b' = 1; \\ a' = 2^3,\ b' = 11, \end{cases}$$

which implies that either $a = 11^2 \cdot 2^3$ and $b = 11$, or $a = 11 \cdot 2^3$ and $b = 11^2$ (and *vice versa*, that is, interchanging $a$ and $b$).

**Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.** The greatest common divisor of $a$ and $b$ is

$$\frac{ab}{\operatorname{lcm}(a, b)} = \frac{10648}{968} = 11 .$$

Consequently, $a = 11a'$ and $b = 11b'$, where $a'$ and $b'$ are relatively prime natural numbers and, hence, using the assumption that $ab = 10648$, we obtain $a'b' = 88$.

From this,

$$(a', b') = (1, 88), (8, 11), (11, 8), (88, 1)$$

and

$$(a, b) = (11, 968), (88, 121), (121, 88), (968, 11).$$

Also solved by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom; Irene Fernández Fernández, IES El Carmen, Murcia, Spain; Rovsen Pirguliyev, Sumgait City, Azerbaijan; Henry Ricardo, Westchester Area Math Circle, NY, USA, and the proposer.

**E–69.** Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let $ABC$ be a triangle such that $3\angle A = 2\angle B$. Bisector of $BC$ meets $CA$ at point $X$. If $AB = BX$, then find the measure of the angles of triangle $ABC$.

**Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.** In isosceles triangle $ABX$, the base angle $BXA$ is also $\angle A$, and its vertical angle $ABX$ is $180^\circ - 2\angle A$.

Since $X$ lies on the perpendicular bisector of $BC$, then $BX = XC$, making $\triangle BXC$ isosceles and $\angle XBC = \angle C$.

We have in figure 1 that

$$\angle B = \angle ABX + \angle XBC = (180^\circ - 2\angle A) + \angle C,$$

that is,

$$2A + B - C = 180^\circ . \tag{1}$$

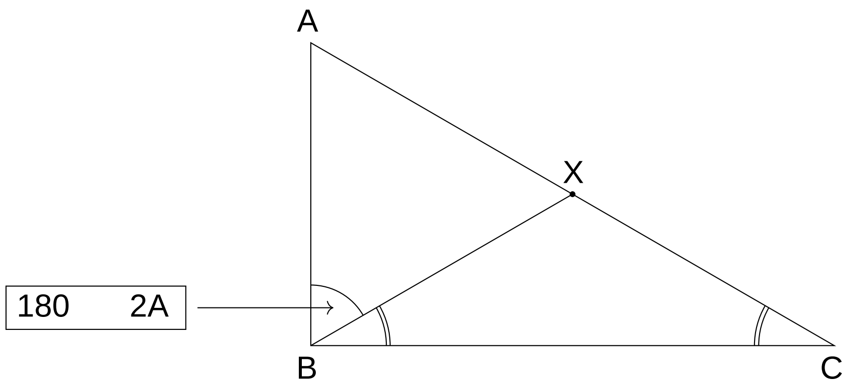


Figure 1: Scheme for the solution of Problem E–69.

We also have

$$3A \quad 2B = 0 \,, \qquad (2)$$
$$A + B + C = 180 \,. \qquad (3)$$

Solving (1), (2) and (3) simultaneously, we get

$$\backslash A = 60 \qquad \backslash B = 90 \qquad \backslash C = 30 \,.$$

Also solved by Sarah B. Seales, Northern Arizona University, Arizona, USA, and the proposer.

E–70. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. A 2019 side square is divided, by drawing parallel lines to the sides, into 4076361 equal squares. What is the total number of squares that appear in this gure?

Solution 1 by Irene Fernández Fernández, IES El Carmen, Murcia, Spain. To solve the problem, we use a square divided into 25 equal squares as an example (see gure 2). There are 25 one-unit-sided squares. We have that, if we want to nd out the number of two-unit-sided different squares that can come out of it, we must calculate $[5 \quad (2 \quad 1)]^2$, as there are 4 different two-unit columns and rows (delimited by the red, green, orange and light blue lines) and when distributing the columns and rows in groups of two there is one column that cannot be paired. The same happens with the three and four-unit-sided squares (the columns and rows that cannot be paired are $(3-1)$ and $(4-1)$ respectively). Hence, the total number of different squares that can be formed in a five-unit-sided square which is divided into $25$ equal squares is

$$[5-(1-1)]^2+[5-(2-1)]^2+[5-(3-1)]^2+[5-(4-1)]^2+[5-(5-1)]^2.$$

Figure 2: Scheme for Solution 1 of Problem E–70.

To find out the number of $k$-unit-sided different squares that can be formed in an $n$-unit-sided square, taking into account that the number of columns and rows that cannot be paired in each case are $(k-1)$, we must calculate

$$[n-(1-1)]^2+[n-(2-1)]^2+[n-(3-1)]^2+\dots+[n-(k-1)]^2.$$

Therefore, the general procedure to find out all the squares that can be formed in an $n$-unit-sided square is

$$\begin{aligned}[n-(1-1)]^2+[n-(2-1)]^2+[n-(3-1)]^2+\dots\\ +[n-(k-1)]^2+\dots+[n-(n-1)]^2.\end{aligned}$$

Hence, if we apply that reasoning to the 2019-unit-sided square (as $\sqrt{4076361}=2019$), we have that

$$\begin{aligned}&[2019-(1-1)]^2+[2019-(2-1)]^2+\dots+[2019-(2019-1)]^2\\ &=2019^2+2018^2+2017^2+\dots+2^2+1^2.\end{aligned}$$

Therefore, applying the sum-of-squares formula $n(n+1)(2n+1)/6$, the total number of squares is

$$\frac{2019\cdot(2019+1)\cdot(2\cdot 2019+1)}{6}=\frac{2019\cdot 2020\cdot 4039}{6}=2745429470.$$

Solution 2 by Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain. Note that $4076361 = 2019^2$, so we have a side square of dimension $2019\times 2019$. To count how many squares can be formed with all these $2019^2$ little squares, we are going to fix the square in position $(i, j)$, for $1\le i, j\le 2019$, and we are going to count how many squares can be obtained that have the square in position $(i, j)$ as the inferior left corner.

For example, we can begin with $i = j = 1$, that is, the lower left corner of the side square. When we fix this square, we can form 2019 squares: one of side 1, one of side 2, and all the way up to one of side 2019 (which would be the whole side square).

Now, let us consider the positions $(1, 2)$, $(2, 2)$ and $(2, 1)$. We can form squares of dimension $1, 2, \ldots, 2018$, but not one of dimension 2019, because we would include the square in the position $(1, 2 + 2018) = (1, 2020)$, which is not in the side square.

In general, for the position $(i, j)$ with $\max\{i, j\} = k$ we can form $2019-(k-1) = 2020-k$ squares. Since there are $2k-1$ positions $(i, j)$ with $\max\{i, j\} = k$, we have that, summing up for $k = 1, \ldots, 2019$, the number of squares we can build is

$$\begin{aligned}&1\cdot 2019+3\cdot 2018+5\cdot 2017+\cdots+(2\cdot 2019-1)\cdot 1\\&=\sum_{k=1}^{2019}(2k-1)(2020-k)=2745429470.\end{aligned}$$

Solution 3 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. The answer is 2745429470. In problem E 1572 [The American Mathematical Monthly, 71 (1964) 92–93] I showed that the number of squares on an $m\times n$ chess board with $m\ge n$ is

$$\frac{n(n+1)(3m+1-n)}{6},$$

which reduces to

$$\frac{n(n+1)(2n+1)}{6}$$

for $m = n$. The last expression, in the case $n = 2019$, yields the solution to the proposed problem.

Also solved by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom; Henry Ricardo, Westchester Area Math Circle, NY, USA, and the proposer.

# Easy–Medium Problems

**EM–65.** Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let $n$ be a positive integer. Show that

$$(n^7 - n)(7^{14n+7} + 11^{2n+1})$$

is a multiple of $4242$.

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Since 7 is a prime number, $n^7$ is congruent to $n$ with respect to the modulus $7$, that is, $n^7 - n$ is divisible by 7.

On the other hand, applying the identity

$$A^{2n+1} + B^{2n+1} = (A+B)\left[A^{2n} - A^{2n-1}B + \cdots - AB^{2n-1} + B^{2n}\right]$$

with $A = 7^7$ and $B = 11$, it follows that $7^{14n+7} + 11^{2n+1}$ is divisible by $7^7 + 11 = 823554$ and, hence, it is also divisible by 606 (since $606 \mid 823554$).

We conclude that the product $(n^7 - n)(7^{14n+7} + 11^{2n+1})$ is divisible by $7 \cdot 606 = 4242$, as desired.

Solution 2 by Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain. We need to prove that the product

$$(n^7 - n)(7^{14n+7} + 11^{2n+1}) = n(n^3+1)(n^3-1)[(7^7)^{2n+1} + 11^{2n+1}]$$

is divisible by $4242 = 101 \cdot 2 \cdot 3 \cdot 7$. So we need to check that the product is divisible by $2, 3, 7$ and $101$.

Divisible by $2$:

If $n \equiv 0 \pmod 2$, then the product is too.
If $n \equiv 1 \pmod 2$, then $n^3+1$ and $n^3 - 1$ are divisible by two, so the product is too.

Divisible by $3$:

If $n \equiv 0 \pmod 3$, then the product is too.

If $n \quad 1 \pmod 3$, then $n^3 \quad 1 \pmod 3$, hence $n^3 \quad 1 \quad 0 \pmod 3$, so then the product is too.
If $n \quad 2 \pmod 3$, then $n^3 \quad 2^3 = 8 \quad 2 \pmod 3$, hence $n^3 + 1 \quad 0 \pmod 3$ and, therefore, the product is too.

Last, when taking modulo $7$, we have the following cases:

$$n \quad 0 \pmod 7 \ = ) \quad n^3 \quad 0 \pmod 7 .$$
$$n \quad 1 \pmod 7 \ = ) \quad n^3 \quad 1^3 = 1 \pmod 7 .$$
$$n \quad 2 \pmod 7 \ = ) \quad n^3 \quad 2^3 = 8 \quad 1 \pmod 7 .$$
$$n \quad 3 \pmod 7 \ = ) \quad n^3 \quad 3^3 = 27 \quad 1 \pmod 7 .$$
$$n \quad 4 \pmod 7 \ = ) \quad n^3 \quad 4^3 \quad 1 \pmod 7 .$$
$$n \quad 5 \pmod 7 \ = ) \quad n^3 \quad 5^3 \quad 1 \pmod 7 .$$
$$n \quad 6 \quad 1 \pmod 7 \ = ) \quad n^3 \quad ( \quad 1)^3 = \quad 1 \pmod 7 .$$

There are two cases: if $n$ is divisible by $7$, which implies that the product is divisible too; if $n^3$ has remainder $1$ or $1$ when divided by $7$, which added to $1$ or $1$ is $0$. That is, $n^3 + 1$ or $n^3 \quad 1$ will be divisible by $7$.

We have already seen that the product will by divisible by $42$. All that is left is to check that it is divisible by $101$. In order to do that, we are not going to get any information from the rst $3$ factors, so let us study directly the last one: $(7^7)^{2n+1} + 11^{2n+1}$. It is easier if we write it this way:

$$7^7 = 7^{3} \quad 7^3 \quad 7 = 343^{2} \quad 7 \quad 40^2 \quad 7 = 1600 \quad 7 = (101 \quad 15 + 85) \quad 7$$
$$85 \quad 7 = 595 \quad 90 \quad 11 \pmod{101} .$$

Therefore,

$$(7^7)^{2n+1} + 11^{2n+1} \quad ( \quad 11)^{2n+1} + 11^{2n+1} = 0 \pmod{101} ,$$

were the last equality is obtained because the exponents are odd.

Solution 3 by the proposer. Since $4242 = 6 \quad 7 \quad 101$, then it will suf ce to prove the following congruences:

$$(n^7 \quad n) \text{€} 7^{14n+7} + 11^{2n+1} \text{Š} \quad 0 \pmod 6 ,$$
$$(n^7 \quad n) \text{€} 7^{14n+7} + 11^{2n+1} \text{Š} \quad 0 \pmod 7 ,$$
$$(n^7 \quad n) \text{€} 7^{14n+7} + 11^{2n+1} \text{Š} \quad 0 \pmod{101} .$$

Since $n^7 - n = (n-1)n(n+1)(n^2+n+1)(n^2-n+1)$, then we have that $n^7 - n \equiv 0 \pmod 3$. On the other hand, $7^{14n+7} + 11^{2n+1}$ is even and, therefore, $(n^7 - n)(7^{14n+7} + 11^{2n+1}) \equiv 0 \pmod 6$. Likewise, applying Fermat's Little Theorem we have that $n^7 - n \equiv 0 \pmod 7$. Finally, we have to prove that $7^{14n+7} + 11^{2n+1} \equiv 0 \pmod{101}$. Indeed, we have

$$
\begin{aligned}
7^1 &\equiv 7 \pmod{101},\\
7^2 &\equiv 49 \pmod{101},\\
7^3 &\equiv 40 \pmod{101},\\
7^4 &\equiv 78 \pmod{101},\\
7^5 &\equiv 41 \pmod{101},\\
7^6 &\equiv 85 \pmod{101},\\
7^7 &\equiv 90 \equiv -11 \pmod{101}.
\end{aligned}
$$

Therefore, $7^{14n} \equiv (7^7)^{2n} \equiv (-11)^{2n} \equiv 11^{2n} \pmod{101}$. Multiplying up both by $7^7 \equiv -11 \pmod{101}$ yields $7^{14n}7^7 \equiv 11^{2n}(-11) \pmod{101}$, from which we get $7^{14n+7} \equiv -11^{2n+1} \pmod{101}$, and the claim follows.

Also solved by Rovsen Pirguliyev, Sumgait City, Azerbaijan.

**EM–66.** Proposed by Mihaela Berindeanu, Bucharest, România.
Let $a, b, c$ be reals larger than or equal to one. Prove that

$$\sum_{cyclic} \frac{a^{2019}bc + 2019}{bc} \geq 3\left(1 + \frac{6057}{a+b+c}\right).$$

Solution by the proposer. Note that, since $a; b; c \geq 1$, we have that $(a^{2019} - 1)(bc - 1) \geq 0$, which means tht $a^{2019}bc - bc - a^{2019} + 1 \geq 0$ and, therefore, $a^{2019}bc + 1 \geq a^{2019} + bc$.

Using the AM-GM inequality, we have that

$$
\begin{aligned}
a^{2019} + 2018 \geq 2019a &\implies a^{2019}bc + 1 \geq 2019a - 2018 + bc\\
&\implies a^{2019}bc + 2019 \geq bc + 2019a\\
&\implies \frac{a^{2019}bc + 2019}{bc} \geq \frac{bc + 2019a}{bc}
\end{aligned}
$$

$$\Longrightarrow \quad \frac{a^{2019}bc + 2019}{bc} \geq 1 + \frac{2019a}{bc}. \qquad (1)$$

Analogously,

$$\frac{b^{2019}ac + 2019}{ac} \geq 1 + \frac{2019b}{ac} \qquad (2)$$

and

$$\frac{c^{2019}ab + 2019}{ab} \geq 1 + \frac{2019c}{ab}. \qquad (3)$$

Adding (1), (2) and (3) up yields

$$\sum_{cyclic} \frac{a^{2019}bc + 2019}{bc} \geq 3 + 2019\left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right). \qquad (4)$$

Observe that

$$2019\left(\frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right) \geq 2019 \cdot 3\sqrt[3]{\frac{abc}{a^2b^2c^2}} = \frac{2019 \cdot 3}{\sqrt[3]{abc}} \geq \frac{2019 \cdot 3}{\frac{a+b+c}{3}} = \frac{2019 \cdot 9}{a+b+c}.$$

Substituting this in (4), we have

$$\sum_{cyclic} \frac{a^{2019}bc + 2019}{bc} \geq 3 + \frac{2019 \cdot 9}{a+b+c}$$

$$\Longleftrightarrow \quad \sum_{cyclic} \frac{a^{2019}bc + 2019}{bc} \geq 3\left(1 + \frac{6057}{a+b+c}\right).$$

Equality is reached for $a = b = c = 1$.

**EM–67.** Proposed by Oriol Baeza Guasch, Institut de Terrassa, Terrassa, Spain. Let $ABC$ be a triangle with incenter $I$. Let $A'$ be the intersection of ray $AI$ with side $BC$. Define $B'$ and $C'$ similarly. Then, prove that

$$AI \cdot BI \cdot CI \geq 8\, A'I \cdot B'I \cdot C'I.$$

When does equality occur?

Solution 1 by Andrés Sáez Schwedt, Universidad de León, Spain. Let $a, b, c$ be the lengths of the sides $BC$, $CA$, $AB$, respectively. Since $\frac{A'B}{A'C} = \frac{AB}{AC} = \frac{c}{b}$ (by the bisector's theorem) and $A'B + A'C = BC = a$, we deduce that $A'B = \frac{ac}{b+c}$ and $A'C = \frac{ab}{b+c}$.

Looking at triangle $AA'B$ with bisector $BI$, another application of the bisector's theorem yields

$$\frac{IA}{IA'} = \frac{BA}{BA'} = \frac{c}{\frac{ac}{b+c}} = \frac{b+c}{a}.$$

Similar calculations lead immediately to

$$\frac{AI \cdot BI \cdot CI}{A'I \cdot B'I \cdot C'I} = \frac{(b+c)(c+a)(a+b)}{abc},$$

and one has to prove that this last quantity is at least $8$.

But we know from the AM-GM inequality that

$$b+c \geq 2\sqrt{bc}, \quad c+a \geq 2\sqrt{ca}, \quad a+b \geq 2\sqrt{ab}.$$

Multiplication of the three inequalities above gives the desired result.

Solution 2 by the proposer. By Van Aubel's theorem, we can compute the ratio

$$\frac{AI}{A'I} = \frac{AC'}{C'B} + \frac{AB'}{B'C}.$$

And these other ratios can be found easily by the angle bisector theorem:

$$\frac{AC'}{C'B} = \frac{b}{a}, \qquad \frac{AB'}{B'C} = \frac{c}{a}.$$

Therefore,

$$\frac{AI}{A'I} = \frac{b+c}{a}.$$

If we multiply the three ratios, we obtain

$$\frac{AI}{A'I}\frac{BI}{B'I}\frac{CI}{C'I} = \frac{(b+c)(c+a)(a+b)}{abc},$$

Figure 3: Construction for Solution 2 of Problem EM–67.

so the statement is equivalent to proving that

$$\mathrm{LHS} = \frac{(b+c)(c+a)(a+b)}{abc} \quad 8.$$

Manipulating,

$$\mathrm{LHS} = \frac{2abc + a(b^2+c^2) + b(c^2+a^2) + c(a^2+b^2)}{abc} \quad 8.$$

Now, using the arithmetic-geometric mean inequality, we can obtain

$$\mathrm{LHS} \quad \frac{2abc + a(2bc) + b(2ca) + c(2ab)}{abc} = \frac{8abc}{abc} = 8\,.$$

This way, we have proved the statement, and because we used AM-GM, the case of equality happens when $a = b = c$, i.e., when the triangle is equilateral.

Comment. Scott H. Brown, Auburn University, Montgomery AL, USA, commented that a similar problem and its solution can be found in the *American Mathematical Monthly* in Vol. 60, 1953, No. 6, p. 421, E 1043. A solution of the problem can be found in the *Mathematics Magazine*, Vol. 36, No. 4, 1963, p. 244; and other solutions of the problem can be found in the *American Mathematical Monthly* by Oppenheim (Vol. 68, No. 3, 1961, pp. 226–230) and Carlitz (Vol 71, No. 8, 1964, pp. 881–885).

Also solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Scott H. Brown, Auburn University, Montgomery AL, USA, and José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.

**EM–68.** Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. [Correction] Without the aid of a computer, show that

$$\left(\arctan^2\frac{1}{9}+\arctan^2\frac{4}{5}+\frac{\pi^2}{16}\right)^2 = 2\left(\arctan^4\frac{1}{9}+\arctan^4\frac{4}{5}+\frac{\pi^4}{256}\right).$$

Solution by Rovsen Pirguliyev, Sumgait City, Azerbaijan. Setting $a=\arctan\frac{1}{9}$ and $b=\arctan\frac{4}{5}$, then we have

$$a+b=\arctan\frac{1}{9}+\arctan\frac{4}{5}=\frac{\pi}{4}$$

because $\tan(a+b)=\dfrac{\tan a+\tan b}{1-\tan a\tan b}=1$. Putting these values into the well-known Candido's identity,

$$(a^2+b^2+(a+b)^2)^2=2(a^4+b^4+(a+b)^4),$$

we get

$$\left(\arctan^2\frac{1}{9}+\arctan^2\frac{4}{5}+\frac{\pi^2}{16}\right)^2 = 2\left(\arctan^4\frac{1}{9}+\arctan^4\frac{4}{5}+\frac{\pi^4}{256}\right).$$

and we are done.

Also solved by the proposer.

**EM–69.** Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let $a$, $b$, $c$ and $A$, $B$, $C$ be the lengths of the sides and the measure in radians of the angles of an acute triangle $ABC$. Prove that

$$\frac{a}{\tan B\tan C}+\frac{b}{\tan A\tan C}+\frac{c}{\tan A\tan B}\geq\frac{2}{3}s,$$

where $s$ is the semiperimeter of triangle $ABC$.

Solution by the proposers. WLOG, we may assume that $a \geq b \geq c$, and then $A \geq B \geq C$, as is well-known. On the other hand, the function $f : [0, \pi/2) \to \mathbb{R}$ defined by $f(x) = \tan x$ is increasing. Therefore, $\tan A \geq \tan B \geq \tan C$. Applying Chebyshev's inequality to the sequences $a \geq b \geq c$ and $\tan A \geq \tan B \geq \tan C$ yields

$$3(a \tan A + b \tan B + c \tan C) \geq (a+b+c)(\tan A + \tan B + \tan C).$$

Since $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, as is well-known, then the above inequality becomes

$$3(a \tan A + b \tan B + c \tan C) \geq (a+b+c)(\tan A \tan B \tan C),$$

from which it follows that

$$\frac{a}{\tan B \tan C} + \frac{b}{\tan A \tan C} + \frac{c}{\tan A \tan B} \geq \frac{a+b+c}{3} = \frac{2}{3}s.$$

Equality holds when $\triangle ABC$ is equilateral, and we are done.

**EM–70.** Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Without the aid of a computer, show that $x^4 + x^3 + x^2 + x + 1$ divides

$$A(x) = x^{99} + 3x^{98} + x^{97} + 2x^{54} + x^{52} + x^{27} + x^6 + 2x + 3,$$

and find the remainder of the division of $A(x)$ by $x^2 - 1$.

Solution by the proposers. Polynomial $B(x) = x^4 + x^3 + x^2 + x + 1$ divides $A(x)$ if and only if the zeroes of $B(x)$ are zeroes of $A(x)$. Since $x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$, then the roots $\omega_i$ $(1 \leq i \leq 4)$ of $B(x) = 0$ are fifth roots of unity and they satisfy the equation $\omega^5 = 1$. We have that

$$\begin{aligned} A(\omega) &= \omega^{99} + 3\omega^{98} + \omega^{97} + 2\omega^{54} + \omega^{52} + \omega^{27} + \omega^6 + 2\omega + 3 \\ &= \omega^4 + 3\omega^3 + \omega^2 + 2\omega^4 + \omega^2 + \omega^2 + \omega + 2\omega + 3 \\ &= 3(\omega^4 + \omega^3 + \omega^2 + \omega + 1) = 0, \end{aligned}$$

and so $B(x)$ divides $A(x)$.

Let $m; n$ be distinct integer numbers. Then, we have

$$A(x) = (x - m)(x - n)C(x) + ax + b.$$

Since $A(m) = am + b$ is the remainder obtained when dividing $A(x)$ by $x - m$ and $A(n) = an + b$ is the remainder obtained when dividing $A(x)$ by $x - n$, then the remainder obtained when dividing $A(x)$ by $(x - m)(x - n)$ is given by

$$ax + b = \frac{A(m) - A(n)}{m - n}x + \frac{mA(n) - nA(m)}{m - n}.$$

In the particular case $m = -1$ and $n = 1$ we get $A(-1) = 5$ and $A(1) = 15$, from which it follows that $ax + b = 5x + 10$.

# Medium–Hard Problems

**MH–65.** Proposed by Oriol Baeza Guasch, Institut de Terrassa, Terrassa, Spain. Let $ABC$ be a triangle such that $\angle CAB = 2\angle ABC$. Denote the incenter by $I$ and the circumcircle by $\Gamma$. From a point $P$ on $\Gamma$, drop perpendiculars to lines $AC$, $AI$, $AB$, and denote by $X$, $Y$, $Z$ the respective feet. Let $M$ be the intersection of lines $XY$ and $PZ$. Show that lines $CM$, $AY$ and the perpendicular bisector of $BC$ are concurrent.

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Since $AI$ (extended) and the perpendicular bisector of side $BC$ meet at $L$, the midpoint of the arc $BC$ of $\Gamma$ not containing $A$, it suffices to prove that the line $CM$ passes through $L$.

The angles in $\triangle ABC$ add up to $180^\circ$ and $\angle CAB = 2\angle ABC$. Then, we have

$$\angle BCA = 180^\circ - 3\angle ABC. \tag{1}$$

Let $\{Q\} = XZ \cap BC$. Since $XZ$ is the Simson line of the point $P$ with respect to $\triangle ABC$, we have $PQ \perp BC$. Since $\angle PXA$ is a right angle, the circle on $PA$ as diameter passes through $X$. Similarly, it passes through $Y$ and $Z$. Thus, the five points $P$, $A$, $X$, $Y$, $Z$ all lie on a circle, and on chord $YZ$ we have

$$\angle MXQ = \angle YXZ = \angle YAZ = \frac{1}{2}\angle CAB = \angle ABC. \tag{2}$$

Moreover, cyclic quadrilateral $PZAX$ implies that the exterior angle $CAZ$ is equal to the interior and opposite angle $P$. Thus,

$$\angle XPZ = \angle CAZ = \angle CAB = 2\angle ABC. \tag{3}$$

Now, the right angles at $X$ and $Q$ make $PQCX$ a cyclic quadrilateral, implying that

$$\angle XPQ = 180^\circ - \angle QCX,$$

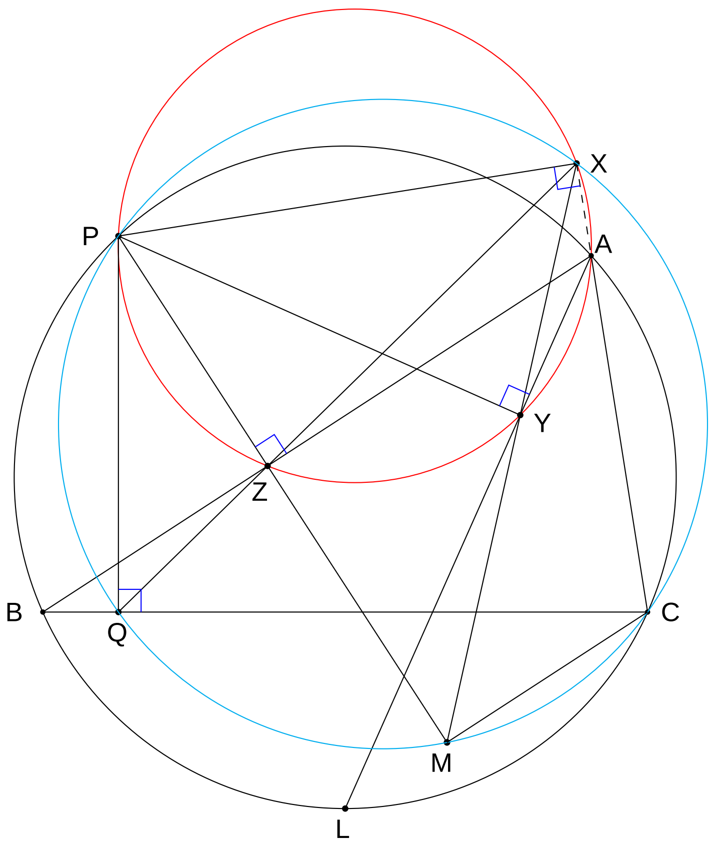


Figure 4: Scheme for Solution 1 of Problem MH–65.

that is,

$$\angle XPZ + \angle ZPQ = 180^\circ - \angle BCA .$$

Substituting from (1) and (3) for $\angle BCA$ and $\angle XPZ$, we obtain

$$\angle ZPQ = \angle ABC ,$$

that is,

$$\angle MPQ = \angle ABC . \qquad (4)$$

By (2) and (4),

$$\angle MXQ = \angle MPQ \quad (= \angle ABC ),$$

so $QM$ subtends equal angles at $X$ and $P$, making $PQMX$ cyclic. Thus, points $P$, $Q$, $M$, $C$, $X$ are concyclic, and on chord $QM$ we have

$$\angle MCB = \angle MCQ = \angle MPQ = \angle ABC = \frac{1}{2}\angle CAB = \angle LCB,$$

where the third equality follows by (4). Thus,

$$\angle MCB = \angle LCB$$

and points $L$, $C$, $M$ are collinear, which is equivalent to what was to be proved.

Solution 2 by the proposer. First, it should be noted that line $AY$ is the same as line $AI$. On the other hand, since the perpendicular bisector of side $BC$ divides arc $\overset{\frown}{BC}$ in two equal halves, the angles that look at these arcs must be the same. Therefore, the intersection between the perpendicular bisector and the circumcircle, say point $A'$, belongs to the angle bisector of $\angle CAB$.

Figure 5: Construction for Solution 2 of Problem MH–65.

Then, it will be enough to prove that $A'$ also belongs to line $CM$.

Let us assume, for the sake of contradiction, that it does not. That means that $CM$ cuts at a point $A''$. Let us now take a look at triangle $ACA'$. Note that the circumcircle of $ABC$, , is also the circumcircle of this triangle.

Next, if we drop a perpendicular from $P$ to $CA'$, by Simson's line theorem, we will have that $X$, $Y$ and the feet of this perpendicular (let us denote it $M'$) are collinear. But we also know, by problem construction, that $X$, $Y$ and $M$ are collinear, so $X$, $Y$, $M$ and $M'$ belong to the same line.

On the other hand, since $\angle CAB = 2\angle ABC$, $\angle CA'A = \angle A'AB$, that means $CA' \parallel AB$, and that implies that $PM \perp CA'$. That also tells us that the lines through $P$, $Y$ and $M$ and through $P$, $Y$ and $M'$ are the same. In other words, $P$, $Y$, $M$ and $M'$ are collinear.

However, that means that $M$ and $M'$ are collinear in two different lines, which is impossible unless $M = M'$. Thus, since $C$, $M$, $M'$, $A'$ and $A''$ are collinear, $A'$ and $A''$ are the same point, and we have come to a contradiction. Therefore, the three lines, $CM$, $AY$ and the perpendicular bisector of $BC$ concur.

**MH–66.** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $a, b, c, d$ be the roots of $x^4+6x^3+7x^2+6x+1 = 0$. Find the value of

$$\frac{7-2a}{1-a}+\frac{7-2b}{1-b}+\frac{7-2c}{1-c}+\frac{7-2d}{1-d}.$$

*Solution 1 by Andrés Sáez Schwedt, Universidad de León, Spain.* Note that $x^4+6x^3+7x^2+6x+1 = (x^2+x+1)(x^2+5x+1)$. Without loss of generality, we may assume that $a, b$ are the roots of $x^2+x+1$, and $c, d$ are those of $x^2+5x+1$, with the known relations

$$a+b = -1, \quad ab = 1, \quad c+d = -5, \quad cd = 1.$$

Now, we transform the proposed sum into an expression in terms of $\{a+b; ab; c+d; cd\}$, as follows:

$$\frac{7-2a}{1-a}+\frac{7-2b}{1-b}=\frac{(7-2a)(1-b)+(7-2b)(1-a)}{(1-a)(1-b)}$$
$$=\frac{14-9(a+b)+4ab}{1-(a+b)+ab}=\frac{14+9+4}{1+1+1}=9,$$

and, similarly,

$$\frac{7-2c}{1-c}+\frac{7-2d}{1-d}=\frac{14-9(c+d)+4cd}{1-(c+d)+cd}=\frac{14+45+4}{1+5+1}=9.$$

Therefore, the desired sum is equal to $9+9=18$.

Solution 2 by Sarah B. Seales, Northern Arizona University, Arizona, USA. Let $a, b, c, d$ be the roots of $x^4+6x^3+7x^2+6x+1$. We are asked to find the value of

$$\frac{7-2a}{1-a}+\frac{7-2b}{1-b}+\frac{7-2c}{1-c}+\frac{7-2d}{1-d}.$$

The terms $\frac{7-2x}{1-x}$ simplify to $2-\frac{5}{x-1}$, so the expression becomes

$$2-\frac{5}{a-1}+2-\frac{5}{b-1}+2-\frac{5}{c-1}+2-\frac{5}{c-1}$$

or

$$8-5\left(\frac{1}{a-1}+\frac{1}{b-1}+\frac{1}{c-1}+\frac{1}{d-1}\right).$$

We wish to find the sum $\frac{1}{a-1}+\frac{1}{b-1}+\frac{1}{c-1}+\frac{1}{d-1}$.

The polynomial

$$(1+x)^4+6(1+x)^3+7(1+x)^2+6(1+x)+1$$
$$=x^4+10x^3+31x^2+42x+21$$

has roots $a-1, b-1, c-1, d-1$.

Reversing the coefficients, we obtain the polynomial

$$21x^4+42x^3+31x^2+10x+1,$$

which has roots

$$\frac{1}{a-1},\quad \frac{1}{b-1},\quad \frac{1}{c-1},\quad \frac{1}{d-1}.$$

By Vieta's formula for the sum of the roots of a fourth degree polynomial, we get

$$\frac{1}{a-1}+\frac{1}{b-1}+\frac{1}{c-1}+\frac{1}{d-1}=-\frac{42}{21}=-2.$$

Plugging this in gives us

$$8-5\left(\frac{1}{a-1}+\frac{1}{b-1}+\frac{1}{c-1}+\frac{1}{d-1}\right)=8+10=18\,.$$

Also solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain (two solutions), and the proposer.

**MH–67.** Proposed by Mihaela Berindeanu, Bucharest, România. Let $a, b, c, d$ be positive numbers such that $abcd = 16$. Show that

$$a^2\sqrt{b+c}+b^2\sqrt{c+d}+c^2\sqrt{d+a}+d^2\sqrt{a+b}\le\frac{a^5+b^5+c^5+d^5}{4}.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY, USA. Appying the Cauchy-Schwarz inequality, we get

$$\sum_{\text{cyclic}} a^2\sqrt{b+c}\le\sqrt{2(a^4+b^4+c^4+d^4)(a+b+c+d)}.\qquad(1)$$

Using Chebychev's inequality, we get

$$\sqrt{2(a^4+b^4+c^4+d^4)(a+b+c+d)}\le\sqrt{8(a^5+b^5+c^5+d^5)}.\qquad(2)$$

Now, the power mean inequality yields

$$a^5+b^5+c^5+d^5\ge4\left(\sqrt[4]{abcd}\right)^5=128\,.\qquad(3)$$

Combining (1), (2) and (3), we conclude that

$$\begin{aligned} a^5+b^5+c^5+d^5 &= \sqrt{(a^5+b^5+c^5+d^5)^2} \\ &\geq \sqrt{128(a^5+b^5+c^5+d^5)} \\ &= 4\sqrt{8(a^5+b^5+c^5+d^5)} \\ &\geq 4\sum_{cyclic} a^2\sqrt{b+c}. \end{aligned}$$

Equality holds if and only if $a=b=c=d=2$.

Solution 2 by Sarah B. Seales, Northern Arizona University, Arizona, USA. The function $f(x)=\sqrt{x}$ is concave on the set of positive real numbers. Placing the variables inside the radicals, the left hand side is

$$\sqrt{a^4b+a^4c}+\sqrt{b^4c+b^4d}+\sqrt{c^4d+c^4a}+\sqrt{d^4a+d^4b} \leq 4\sqrt{\frac{a^4b+a^4c+b^4c+b^4d+c^4d+c^4a+d^4a+d^4b}{4}}$$

by Jensen's inequality. Without loss of generality, we may assume that $a \geq b \geq c \geq d$. By the rearrangement inequality,

$$a^4b+a^4c+b^4c+b^4d+c^4d+c^4a+d^4a+d^4b \leq 2(a^5+b^5+c^5+d^5),$$

so it suffices to show that

$$2\sqrt{2(a^5+b^5+c^5+d^5)} \leq \frac{a^5+b^5+c^5+d^5}{4}.$$

Squaring both sides and simplifying gives the equivalent inequality

$$128(a^5+b^5+c^5+d^5) \leq (a^5+b^5+c^5+d^5)^2,$$

and since all variables are positive, we reduce this to

$$128 \leq a^5+b^5+c^5+d^5.$$

Using the AM-GM inequality and $abcd=16$, we have $a^5+b^5+c^5+d^5 \geq 4(abcd)^{\frac{5}{4}}=128$. Since all of our steps are reversible, the inequality we wished to prove follows.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain; Andrés Sáez Schwedt, Universidad de León, Spain, and the proposer.

**MH–68.** Proposed by Pedro Henrique O. Pantoja, Natal/RN, Brazil. Are there $m, n, k, s \in \mathbb{N}$, $s > 2$, such that

$$\frac{(127m - 1)(127n + 1)}{2^k} = 486\underbrace{11\ldots11}_{s}19\underbrace{44\ldots44}_{s+2}10?$$

*Solution by the proposer.* Let $k = 4$, so $2^k = 8 \cdot 2$. Note that $486\underbrace{11\ldots11}_{s}0 \cdot 8 = 3\underbrace{88\ldots88}_{s+3}0$ and $19\underbrace{44\ldots44}_{s+2}1 \cdot 2 = 3\underbrace{88\ldots88}_{s+3}2$. Therefore, $(3\underbrace{88\ldots88}_{s+3}1 + 1)(3\underbrace{88\ldots88}_{s+3}1 - 1) = 3\underbrace{88\ldots88}_{s+3}1^2 - 1$. Making $m = n$, we have $(127m)^2 - 1$, so it suffices to prove that there exists $s > 0$ such that $3\underbrace{88\ldots88}_{s+3}1$ is divisible by $127$.

*Lemma.* There exists a number of the form

$$3\underbrace{8\ldots8}_{q}1$$

with more than two figures equal to $8$ which is a multiple of $381$.

*Proof.* Consider the numbers $38881$, $388881$, $3888881\ldots$ By the pigeonhole principle, two of these numbers leave the same residue when divided by $381$. Subtracting these two numbers, we obtain a number of the form $38\underbrace{8\ldots8}_{t}5000\ldots$, which is divisible by $381$. Since $381$ is coprime with $10$, the number $38\underbrace{8\ldots8}_{t}5 + 381 = 3\underbrace{8\ldots8}_{t+2}1$ is divisible by $381$. □

As $381 = 3 \cdot 127$, the result follows.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain, and José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.

**MH–69.** Proposed by Mihály Bencze, Braşov, Romania. Find all real solutions of the equation

$$3^x + 2 \cdot 5^x + 7^{-x} = \frac{95}{102}|x|^3.$$

*Solution by the proposer.* By inspection, we find that $x = -1$ is a solution. Since the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3^x + 2 \cdot 5^x + 7^x$ is increasing on $\mathbb{R}$ and the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = \frac{95}{102}|x|^3$ is decreasing on $(-\infty; 0]$ and increasing in $(0; +\infty)$, then the only solution in $(-\infty; 0]$ is $x = -1$. Now, we will prove that this is the unique solution of the given equation.

Indeed, for all $x > 0$ there exists a positive integer $n$, $n = \lfloor x \rfloor$, such that $n \le x < n + 1$. It follows that

$$f(x) = 3^x + 2 \cdot 5^x + 7^x \ge 3^n + 2 \cdot 5^n + 7^n.$$

On the other hand, by using mathematical induction, we may prove that

$$g(x) = \frac{95}{102}|x|^3 = \frac{95}{102}x^3 < (n+1)^3 < 3^n + 2 \cdot 5^n + 7^n.$$

So, $g(x) < f(x)$ for all $x > 0$ and we conclude that $x = -1$ is the only solution of the equation.

*Also solved by* José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain, and José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.

**MH–70.** *Proposed by Mihaela Berindeanu, Bucharest, România.* Let $ABC$ be a triangle with incircle $\omega$ and incenter $I$. The tangent point between $\omega$ and $BC$ is $D$, and between $AC$ and $\omega$ is $E$. The tangent to $\omega$ from a point $X \in AE$ cuts $AB$ in $Y$. Let $Z \in BC$ be a point so that $BZ = AB$. Show that if $Y, I, Z$ are collinear points, then $X, I, D$ are also collinear points.

*Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.* As usual, let $a$, $b$, $c$ denote the sides of $\triangle ABC$ and $s$ its semiperimeter.

We shall prove the stronger statement that $Y, I, Z$ are collinear *if and only if* $X, I, D$ are collinear, by means of the following lemma (see figure 6).

**Lemma 1.** Let $X$ and $Y$ be the intersections of $CA$ and $AB$ with a tangent to the incircle of $\triangle ABC$. Then, the following homographic and symmetrical relationship holds:

$$s \cdot AX \cdot AY = bc(AX + AY) - bc(s - a). \qquad (1)$$

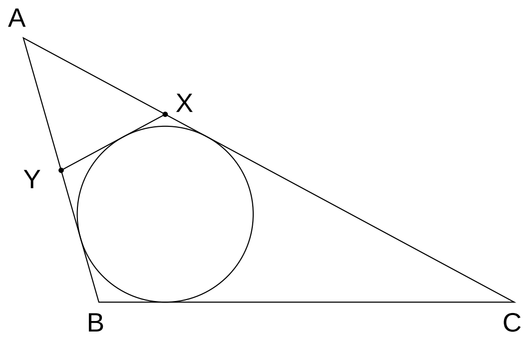


Figure 6: Construction for Lemma 1.

*Proof.* By the Briggs's formulae, applied to $\triangle ABC$,

$$\sin^2 \frac{A}{2} = \frac{(s-b)(s-c)}{bc},$$

and applied to $\triangle AXY$, whose semiperimeter is $s - a$,

$$\sin^2 \frac{A}{2} = \frac{((s-a) - AX)((s-a) - AY)}{AY \cdot AY}.$$

Hence,

$$\frac{(s-b)(s-c)}{bc} = \frac{(s-a-AX)(s-a-AY)}{AX \cdot AY},$$

which, when simplified, becomes equivalent to (1). □

Suppose first that $Y$, $I$, $Z$ are collinear. Figure 7 shows , the incircle of $\triangle ABC$, touching the side $AB$ at $F$. Let $AI$ (extended) meet $BC$ at $L$.

Since the tangents $BF$ and $BD$ are equal, subtracting $BF = BD$ from each side of $AB = BZ$, we have

$$AB - BF = BZ - BD,$$

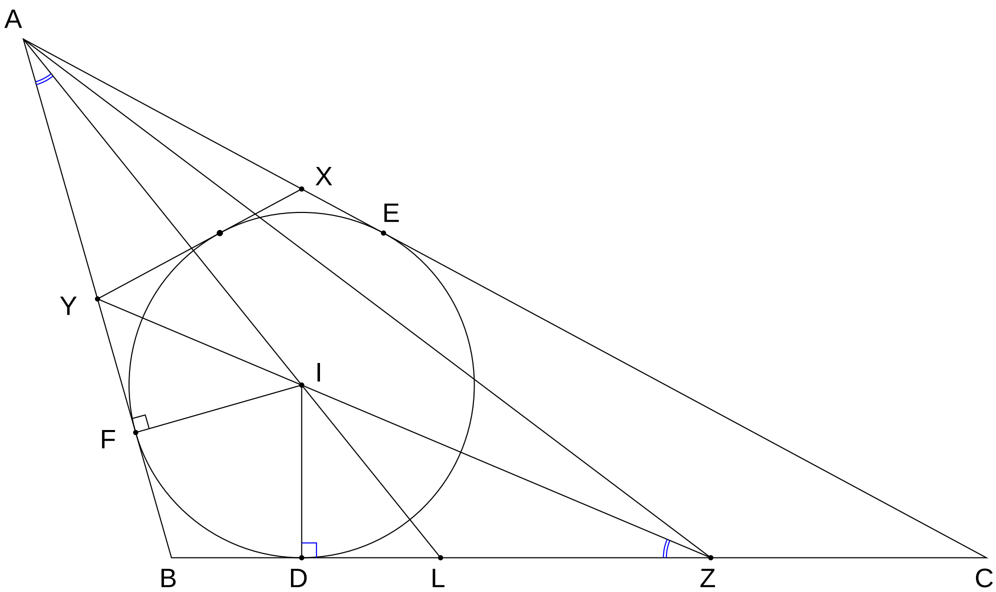


Figure 7: Scheme for Solution 1 of Problem MH–70.

so that

$$AF = DZ,$$

implying right-angled triangles $AFI$ and $ZDI$ are congruent (their corresponding arms are equal) with $\angle DZI = \angle FAI$. That is,

$$\angle BZY = \angle BAL.$$

Therefore, triangles $ABL$ and $ZBY$ are congruent (A-S-A) with $AL = YZ$.

By the law of sines applied to $\triangle ABY$, where $\angle BZY = \angle BAL = \frac{1}{2}\angle A$,

$$\frac{AL}{\sin B} = \frac{YZ}{\sin B} = \frac{BY}{\sin(\angle BZY)} = \frac{BY}{\sin\frac{A}{2}}$$

and

$$BY = \frac{AL\sin\frac{A}{2}}{\sin B}.$$

Since the length of the angle bisector $AL$ may be expressed as

$\dfrac{2bc\cos\frac{A}{2}}{b+c}$, we have

$$BY = \frac{bc\sin A}{(b+c)\sin B},$$

or, equivalently (by the law of sines),

$$BY = \frac{ca}{b+c},$$

and thus

$$AY = AB - BY = c - \frac{ca}{b+c} = \frac{2c(s-a)}{b+c}.$$

By hypothesis, $XY$ is tangent to . Then, substituting this expression for $AY$ into (1) and solving for $AX$, we get

$$AX = \frac{2b(b-c)(s-a)}{a^2+b^2-c^2}.$$

Hence,

$$CX = CA - AX = b - \frac{2b(b-c)(s-a)}{a^2+b^2-c^2} = \frac{2ab(s-c)}{a^2+b^2-c^2},$$

and because the length of the tangent from vertex $C$ of $\triangle ABC$ to the incircle is $s-c$, we have

$$CX = \frac{2ab}{a^2+b^2-c^2}\, CD.$$

Remembering that $c^2 = a^2+b^2-2ab\cos C$, it follows that $CD = CX\cos C$, showing that $\angle XDC$ is a right angle, that is, $XD \perp BC$. Since $ID \perp BC$, then, we have $X$, $I$, $D$ collinear. This completes the proof of this implication.

Let us now prove the converse. Suppose $X$, $I$, $D$ are collinear (see figure 8). Then, triangle $XDC$ is right-angled at $D$, so

$$CX = \frac{CD}{\cos C} = \frac{s-c}{\frac{a^2+b^2-c^2}{2ab}} = \frac{ab(a+b-c)}{a^2+b^2-c^2}$$

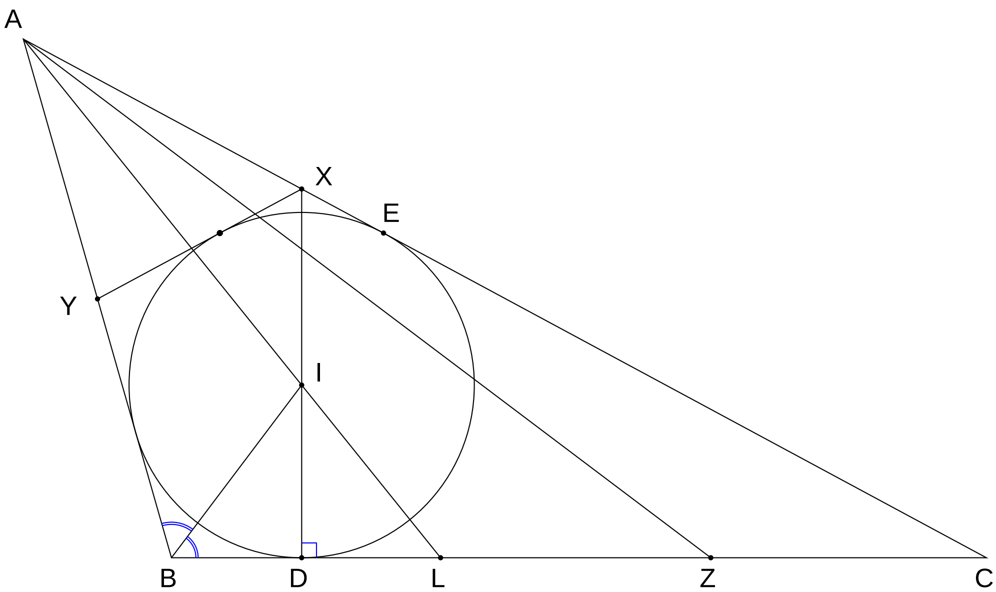


Figure 8: Construction for the converse implication of Problem MH–70.

and

$$AX = CA - CX = b - \frac{ab(a+b-c)}{a^2+b^2-c^2} = \frac{2b(b-c)(s-a)}{a^2+b^2-c^2}.$$

By hipothesis, $XY$ is tangent to . Then, substituting this expression for $AX$ into (1) and solving for $AY$, we get

$$AY = \frac{2c(s-a)}{b+c}.$$

Hence,

$$BY = AB - AY = c - \frac{2c(s-a)}{b+c} = \frac{ca}{b+c} = BL \qquad (2)$$

by the angle bisector theorem applied to $\triangle ABC$ at $A$, whence

$$LZ = BZ - BL = AB - BY = AY. \qquad (3)$$

Moreover, by the angle bisector theorem applied to $\triangle ABL$ at $B$,

$$\frac{LI}{IA} = \frac{BL}{AB}. \tag{4}$$

Therefore, by (2), (3) and (4), the product of the ratios into which $Y$, $I$, and $Z$ divide the sides of $\triangle ABL$ is

$$\frac{AY}{YB} \cdot \frac{BZ}{ZL} \cdot \frac{LI}{IA} = \frac{AY}{BL} \cdot \frac{AB}{AY} \cdot \frac{BL}{AB} = 1,$$

implying $Y$, $I$, and $Z$ are collinear by the theorem of Menelaus.

As a bonus, we now state and prove the following exercise:

Let $ABC$ be a triangle with incenter $I$ and incircle $\omega$ touching the side $BC$ at $D$. Let $DI$ (extended) meet side $CA$ at $X$. Let $Z$ denote the point on side $BC$ such that $AB = BZ$ and let $ZI$ (extended) intersects $AB$ at $Y$. Then, $XY$ is tangent to $\omega$.

Proof. Since $X$, $I$, and $D$ are collinear, as are $Y$, $I$, and $Z$, we have $AX = \dfrac{2b(b-c)(s-a)}{a^2+b^2-c^2}$ and $AY = \dfrac{2c(s-a)}{b+c}$ (see the results above).

By the law of cosines applied to $\triangle AXY$, we have

$$XY^2 = AX^2 + AY^2 - 2 \cdot AX \cdot AY \cdot \cos A,$$

which, on substitution, yields

$$XY^2 = \frac{4a^4b^2(s-a)^2}{(b+c)^2(a^2+b^2-c^2)^2}$$

since $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$. It follows that

$$XY = \frac{2a^2b(s-a)}{(b+c)(a^2+b^2-c^2)},$$

and so, by letting $s$ be the semiperimeter of $\triangle AXY$, we have

$$\begin{aligned} s &= \frac{1}{2}(AX + AY + XY) \\ &= \frac{1}{2}\left(\frac{2b(b-c)(s-a)}{a^2+b^2-c^2} + \frac{2c(s-a)}{b+c} + \frac{2a^2b(s-a)}{(b+c)(a^2+b^2-c^2)}\right) \\ &= (s-a)\frac{b(b^2-c^2)+c(a^2+b^2-c^2)+a^2b}{(b+c)(a^2+b^2-c^2)} \\ &= (s-a)\frac{b(a^2+b^2-c^2)+c(a^2+b^2-c^2)}{(b+c)(a^2+b^2-c^2)} \\ &= s-a, \end{aligned}$$

which is the length of the tangent from vertex $A$ of $\triangle AXY$ to , implying that is the excircle of $\triangle AXY$ beyond the side $XY$. Thus, $XY$ is tangent to , as claimed. □

Solution 2 by Andrés Sáez Schwedt, Universidad de León, Spain. We begin by drawing a picture, which includes two new points $V$ and $X'$: $V$ belongs to $AC$, with $VY \parallel BC$, and $X' = DI \cap AC$ (see figure 9). We claim that $X' = X$, and this will solve the problem.

Figure 9: Construction for Solution 2 of Problem MH–70.

Since $BA = BZ$, it follows that the points $A$, $Z$ are symmetric with respect to the line $BI$, therefore one has $\angle BZI = \angle BAI$. From this and the fact that $VY \parallel BC$, we obtain the equalities

$$\angle VAI = \angle BAI = \angle BZI = \angle VYI,$$

implying that the quadrilateral $AYIV$ is cyclic. Moreover, $IV = IY$ because $I$ lies on the bisector of $\angle VAY$. This means that the line $IX'$ is a symmetry axis with $V$, $Y$ being reflections of each other. Note that this symmetry axis passes through the center of . Finally, the fact that the line $X'V$ is tangent to implies that its reflection $X'Y$ is also tangent, therefore $X' = X$, as we claimed, and we are finished.

Also solved by the proposer.

# Advanced Problems

**A–65.** Proposed by Marc Felipe i Alsina, BarcelonaTech, Barcelona, Spain. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a function satisfying

$$|f(a+b) - f(a)| \leq \frac{b}{a}$$

for all positive reals $a$ and $b$. Show that $|f(x) - f(1)| \leq |\ln x|$ for all $x \in \mathbb{R}^+$.

*Solution by Moti Levy, Rehovot, Israel.* We begin by assuming that $x > 1$: We divide the interval $(1; x)$ into $n$ subintervals, each of length $h$ so that $h = \frac{x-1}{n}$. Then,

$$\begin{aligned} |f(1) - f(x)| &= \left|\sum_{k=0}^{n-1} f(1+(k+1)h) - f(1+kh)\right| \\ &\leq \sum_{k=0}^{n-1} |f(1+(k+1)h) - f(1+kh)| \\ &= \sum_{k=0}^{n-1} |f((1+kh)+h) - f(1+kh)|. \end{aligned}$$

The inequality $|f(a+b) - f(a)| \leq \frac{b}{a}$ implies that

$$|f((1+kh)+h) - f(1+kh)| \leq \frac{h}{1+kh},$$

and

$$|f(x) - f(1)| \leq \sum_{k=0}^{n-1} \frac{h}{1+kh}.$$

The limit $\sum_{k=0}^{n-1} \frac{h}{1+kh}$ exists and is equal to $\ln x$, therefore

$$|f(x) - f(1)| \leq \lim_{h \to 0} \sum_{k=0}^{n-1} \frac{h}{1+kh} = \int_1^x \frac{1}{x}\,dx = \ln x.$$

Now assume that $x < 1$. We divide the interval $(x; 1)$ into $n$ subintervals, each of length $h$ so that $h = \frac{1-x}{n}$. Then,

$$
\begin{aligned}
|f(x) - f(1)| &= \left| \sum_{k=0}^{n-1} f(1-(k+1)h) - f(1-kh) \right| \\
&\le \sum_{k=0}^{n-1} |f(1-(k+1)h) - f(1-kh)| \\
&= \sum_{k=0}^{n-1} |f((1-kh)-h) - f(1-kh)|.
\end{aligned}
$$

The inequality $|f(a+b) - f(a)| \le \frac{b}{a}$ implies that

$$|f(1-kh) - f((1-kh-h))| \le \frac{h}{1-kh},$$

and

$$|f(x) - f(1)| \le \sum_{k=0}^{n-1} \frac{h}{1-kh}.$$

The limit $\sum_{k=0}^{n-1} \frac{h}{1-kh}$ exists and is equal to $-\ln x$, therefore

$$|f(x) - f(1)| \le \lim_{h \to 0} \left( \sum_{k=0}^{n-1} \frac{h}{1-kh} \right) = \int_x^1 \frac{1}{x} dx = -\ln x.$$

We conclude that $|f(x) - f(1)| \le |\ln x|$ for all $x \in \mathbb{R}^+$.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain, and the proposer.

**A–66.** Proposed by Mihaela Berindeanu, Bucharest, România. Let $A, B \in M_2(\mathbb{Z})$ be two square matrices satisfying the following properties: $AB = BA$, $\det(A + B) = 3$ and $\det(A^3 + B^3) = 9$. Prove that $\det(A^2 + B^2) \in \{-5; 1\}$.

Solution 1 by Moti Levy, Rehovot, Israel. The fact that $AB = BA$ implies that $A$ and $B$ are simultaneously triangularizable, that is, there exists an invertible matrix $P$ such that

$$A = P\begin{pmatrix}\lambda_1 & r\\ 0 & \lambda_2\end{pmatrix}P^{-1},$$
$$B = P\begin{pmatrix}\mu_1 & s\\ 0 & \mu_2\end{pmatrix}P^{-1}.$$

Thus, we have

$$A^n + B^n = P\begin{pmatrix}\lambda_1^n+\mu_1^n & *\\ 0 & \lambda_2^n+\mu_2^n\end{pmatrix}P^{-1}.$$

It follows that

$$\det(A+B) = (\lambda_1+\mu_1)(\lambda_2+\mu_2) = 3,$$
$$\det(A^3+B^3) = \left(\lambda_1^3+\mu_1^3\right)\left(\lambda_2^3+\mu_2^3\right) = 9.$$

Our goal is to find all possible values of

$$\det(A^2+B^2) = \left(\lambda_1^2+\mu_1^2\right)\left(\lambda_2^2+\mu_2^2\right).$$

Let

$$\begin{cases}\lambda_1+\mu_1 = a,\\ \lambda_2+\mu_2 = \dfrac{3}{a},\\ \lambda_1^3+\mu_1^3 = b,\\ \lambda_2^3+\mu_2^3 = \dfrac{9}{b}.\end{cases} \tag{1}$$

One of the solutions of the system (1) is given by

$$\begin{cases}\lambda_1 = \dfrac{1}{2}a - \dfrac{1}{6}\sqrt{3}\sqrt{\dfrac{1}{a}(4b-a^3)},\\ \lambda_2 = \dfrac{3}{a} + \dfrac{1}{2a}\left(\sqrt{\dfrac{1}{b}(-3b+4a^3)} - 3\right),\\ \mu_1 = \dfrac{1}{2}a + \dfrac{1}{6}\sqrt{3}\sqrt{\dfrac{1}{a}(4b-a^3)},\\ \mu_2 = -\dfrac{1}{2a}\left(\sqrt{\dfrac{1}{b}(-3b+4a^3)} - 3\right),\end{cases} \tag{2}$$

which yields

$$\det(A^2+B^2) = \left(\ _1^2 + \ _1^2\right)\left(\ _2^2 + \ _2^2\right) = 2\left(\frac{b}{a^3} + \frac{2}{3}\,\frac{a^3}{b}\right) + \frac{7}{3}. \quad (3)$$

It is worth noting that the other three solutions of (1) give the same value of $\det(A^2+B^2)$.

Since $\mathrm{Tr}(A+B) \in \mathbb{Z}$, then $a + \frac{3}{a} \in \mathbb{Z}$. Since $\mathrm{Tr}(A^3+B^3) \in \mathbb{Z}$, then $b + \frac{9}{b} \in \mathbb{Z}$. Since $\det(A^2+B^2) \in \mathbb{Z}$, then $2\left(\frac{b}{a^3} + \frac{2}{3}\frac{a^3}{b}\right) + \frac{7}{3} \in \mathbb{Z}$. Therefore, $a \in \{1, -1, 3, -3\}$ and $b \in \{1, -1, 3, -3, 9, -9\}$.

Exhaustive check of the 24 pairs of $a$ and $b$ results in $\det(A^2+B^2) \in \{-5, 1\}$.

Solution 2 by the proposer. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(x) = \det(A + xB)$. Observe that, since $A, B \in M_2$, $f$ is a quadratic function with integer coefficients. Let

$$f(x) = ax^2 + bx + c,$$

with $a, b, c \in \mathbb{Z}$. We know that $\det(A+B) = 3$, so $f(1) = 3$, which means that

$$a + b + c = 3. \quad (4)$$

We also know that $AB = BA$, so

$$A^3 + B^3 = (A+B)(A+\varepsilon B)\left(A + \varepsilon^2 B\right),$$

with $\varepsilon^2 + \varepsilon + 1 = 0$. In turn, this means that

$$\begin{aligned} \det\left(A^3 + B^3\right) &= \det(A+B)\det(A+\varepsilon B)\det\left(A+\varepsilon^2 B\right) \\ &= f(1) f(\varepsilon) f\left(\varepsilon^2\right). \end{aligned} \quad (5)$$

Finally, we also have that $\det(A^3+B^3) = 9$, so

$$9 = 3\left(a\varepsilon^2 + b\varepsilon + c\right)\left(a\varepsilon^4 + b\varepsilon^2 + c\right).$$

Using in calculus $\varepsilon^3 = 1$, we obtain that

$$a^2 + b^2 + c^2 - ab - ac - bc = 3. \quad (6)$$

From (4) we have that $c = 3 - a - b$, so (6) becomes

$$
\begin{aligned}
a^2 + b^2 + (3 - a - b)^2 - ab - (a+b)(3-a-b) = 3 \\
\Leftrightarrow \quad 3a^2 + 3b^2 - 9a - 9b + 3ab + 6 = 0 \\
\Leftrightarrow \quad a^2 + b^2 - 3a - 3b + ab + 2 = 0 \\
\Leftrightarrow \quad a^2 + a(b-3) + b^2 - 3b + 2 = 0 .
\end{aligned}
$$

Since $a \in \mathbb{R}$, we have that $\Delta \geq 0$, and so,

$$
\begin{aligned}
(b-3)^2 - 4b^2 + 12b - 8 \geq 0 \\
\Leftrightarrow \quad -3b^2 + 6b + 1 \geq 0 \\
\Leftrightarrow \quad -3b^2 + 6b - 3 + 4 \geq 0.
\end{aligned}
$$

Thus, we conclude that $4 \geq 3(b-1)^2$, which means that $(b-1)^2 \in \{0; 1\}$, that is, $b \in \{0; 1; 2\}$. Now we consider several cases:

For $b = 0$ we have that $a \in \{1; 2\}$ and $c = 3 - a - b$, so the solutions $(a; b; c)$ are $(1; 0; 2)$ and $(2; 0; 1)$.
For $b = 1$ we have $a \in \{0; 2\}$ and $c = 3 - a - b$, so the solutions $(a; b; c)$ are $(0; 1; 2)$ and $(2; 1; 0)$.
For $b = 2$ we have $a \in \{0; 1\}$ and $c = 3 - a - b$, so the solutions $(a; b; c)$ are $(0; 2; 1)$ and $(1; 2; 0)$.

For the six solutions, we compute the value of

$$\det\left(A^2 + B^2\right) = \det[(A + Bi)(A - iB)] = f(i) f(-i).$$

The results are summarised in the following table:

| Sol. triples $(a; b; c)$ | $f(x)$ | $\det(A^2 + B^2)$ |
|---|---|---|
| $(1; 2; 0)$ | $f(x) = -x^2 + 2x$ | 5 |
| $(1; 0; 2)$ | $f(x) = -x^2 + 2$ | 1 |
| $(2; 1; 0)$ | $f(x) = 2x^2 - x$ | 5 |
| $(2; 0; 1)$ | $f(x) = 2x^2 + 1$ | 1 |
| $(0; 1; 2)$ | $f(x) = -x + 2$ | −5 |
| $(0; 2; 1)$ | $f(x) = 2x + 1$ | −5 |

In conclusion, we have that

$$\det\left(A^2 + B^2\right) \in \{-5; 1\}.$$

**A–67.** Proposed by Rica Zamfir, Bucharest, România. Consider a continuous function $f : [a,b] \to \mathbb{R}$ satisfying that $\int_a^b f(x)\,dx = 0$. Show that there exists a real number $c \in (a,b)$ such that

$$\int_a^c f(x)\,dx = \left(\frac{a+b}{2} - c\right) f(c).$$

*Solution 1 by Henry Ricardo, Westchester Math Circle, NY.*
Define $g(x) = \left(\frac{a+b}{2} - x\right)\int_a^x f(t)\,dt$. Since $g(a) = 0 = g(b)$, by Rolle's theorem there exists $c \in (a,b)$ such that

$$0 = g'(c) = \left(\frac{a+b}{2} - c\right) f(c) - \int_a^c f(t)\,dt,$$

or

$$\int_a^c f(t)\,dt = \left(\frac{a+b}{2} - c\right) f(c).$$

*Solution 2 by the proposer.* Consider the function $g : [a,b] \to \mathbb{R}$ defined by

$$g(x) = (x-a)\int_a^x f(t)\,dt + (b-x)\int_x^b f(t)\,dt.$$

Since $g$ verifies Rolle's Theorem, then there exists a $c \in (a,b)$ such that $g'(c) = 0$. We have

$$\begin{aligned} g'(x) &= \int_a^x f(t)\,dt + (x-a)f(x) - \int_x^b f(t)\,dt - (b-x)f(x) \\ &= \int_a^x f(t)\,dt - \int_x^b f(t)\,dt - (a+b-2x)\,f(c). \end{aligned}$$

Then, from $g'(c) = 0$, we obtain

$$\int_a^c f(t)\,dt - \int_c^b f(t)\,dt - (a+b-2c)f(c) = 0.$$

On account that $\int_a^b f(x)\,dx = 0$, we have $\int_a^c f(t)\,dt = -\int_c^b f(t)\,dt$, and the preceding becomes

$$2\int_a^c f(t)\,dt = (a+b-2c)f(c),$$

from which, after division by 2, the statement follows.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.

**A–68.** Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Without the aid of a computer, find the value of

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{k^2 + n^2}{n^3}\right).$$

Solution 1 by Moti Levy, Rehovot, Israel. Let

$$L_N = \prod_{k=1}^{N} \left(1 + \frac{k^2 + N^2}{N^3}\right).$$

Then,

$$\ln L_N = \sum_{k=1}^{N} \ln\left(1 + \frac{k^2 + N^2}{N^3}\right) = \sum_{k=1}^{N} \frac{k^2 + N^2}{N^3} + O\left(\frac{1}{N}\right).$$

Observe that

$$\sum_{k=1}^{N} \frac{k^2 + N^2}{N^3} = \sum_{k=1}^{N} \left(1 + \left(\frac{k}{N}\right)^2\right)\frac{1}{N} \to \int_0^1 \left(1 + x^2\right) dx = \frac{4}{3}$$

as $N \to \infty$. We also have

$$\ln L_N \to \ln \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{k^2 + n^2}{n^3}\right)$$

as $N \to \infty$. Therefore,

$$\ln \lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{k^2 + n^2}{n^3}\right) = \frac{4}{3},$$

or

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{k^2 + n^2}{n^3}\right) = e^{\frac{4}{3}}.$$

Solution 2 by Henry Ricardo, Westchester Math Circle, NY.
Let $P_n = \prod_{k=1}^{n} \left(1 + \frac{k^2+n^2}{n^3}\right)$. Note that $(k^2 + n^2)/n^3 \le 2/n < 1$ for

$n > 2$. Then,

$$\begin{aligned}\ln P_n &= \sum_{k=1}^{n} \ln\left(1 + \frac{k^2+n^2}{n^3}\right)\\ &= \sum_{k=1}^{n}\left(\frac{k^2+n^2}{n^3} + O\left(\frac{1}{n^2}\right)\right)\\ &= \frac{1}{n^3}\sum_{k=1}^{n} k^2 + 1 + O(1/n)\\ &= \frac{n(n+1)(2n+1)}{6n^3} + 1 + O(1/n) \to \frac{4}{3}\end{aligned}$$

as $n \to \infty$. Thus, $\lim_{n\to\infty} P_n = e^{4/3}$.

**Solution 3 by the proposer.** To compute the limit we need the following result.

**Lemma 2.** *If $f : [0,1] \to \mathbb{R}^+$ is an integrable function, then*

$$\lim_{n\to\infty} \prod_{k=1}^{n}\left[1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right] = e^{\int_0^1 f(x)\,dx}.$$

*Proof.* Since $f$ is integrable, then it is bounded and there exists $M > 0$ such that $|f(x)| \le M$ for all $x \in [0,1]$. Putting $x_k = \frac{1}{n} f(k/n)$ for $1 \le k \le n$ in the well-known inequality

$$x - \frac{x^2}{2} \le \ln(1+x) \le x, \quad 0 \le x \le 1,$$

we get

$$\frac{1}{n} f\left(\frac{k}{n}\right) - \frac{1}{2n^2} f\left(\frac{k}{n}\right)^2 \le \ln\left(1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right) \le \frac{1}{n} f\left(\frac{k}{n}\right).$$

Adding up the preceding expressions yields

$$\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) - \sum_{k=1}^{n} \frac{1}{2n^2} f\left(\frac{k}{n}\right)^2 \le \sum_{k=1}^{n} \ln\left(1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right) \le \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right).$$

Taking limits when $n$ tends to $+\infty$ we obtain

$$\int_0^1 f(x)\,dx \le \lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right) \le \int_0^1 f(x)\,dx$$

because

$$0 \le \sum_{k=1}^{n} \frac{1}{2n^2} f\left(\frac{k}{n}\right)^2 \le \frac{M^2}{2n^2}$$

and this tends to zero when $n$ goes to $+\infty$. Thus,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left(1+\frac{1}{n} f\left(\frac{k}{n}\right)\right) = \int_0^1 f(x)\,dx$$

and

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1+\frac{1}{n} f\left(\frac{k}{n}\right)\right) = e^{\int_0^1 f(x)\,dx}. \qquad \square$$

Applying the above lemma to the function $f:[0,1] \to \mathbb{R}^+$ defined by $f(x) = x^2+1$, we have

$$f\left(\frac{k}{n}\right) = \frac{k^2}{n^2} + 1 = \frac{k^2+n^2}{n^2}$$

and

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1+\frac{k^2+n^2}{n^3}\right) = e^{\int_0^1 (x^2+1)\,dx} = e^{4/3}.$$

***A–69.*** *Proposed by Pedro Henrique O. Pantoja, Natal/RN, Brazil.*

(i) Find all matrices $A \in M_{2\times 2}(\mathbb{Z})$ with trace equal to zero such that

$$(A-3I)(A^2+A+3I) = \begin{pmatrix} 2 & 1 \\ 80 & 20 \end{pmatrix}.$$

(ii) Let $A$ be a matrix satisfying (i). Compute $A^{2020}$.

**Solution 1 by Moti Levy, Rehovot, Israel.** (i) Every matrix $A$ has at least one eigenvalue, hence the restriction that $\mathrm{Tr}(A) = 0$ implies $A = P\begin{pmatrix} \lambda & 0 \\ c & -\lambda \end{pmatrix}P^{-1}$, where $\lambda$, $-\lambda$ are the eigenvalues of $A$. Thus, we have

$$\left(P\begin{pmatrix} \lambda & 0 \\ c & -\lambda \end{pmatrix}P^{-1} - 3I\right)\left(P\begin{pmatrix} \lambda & 0 \\ c & -\lambda \end{pmatrix}^2 P^{-1} + P\begin{pmatrix} \lambda & 0 \\ c & -\lambda \end{pmatrix}^2 P^{-1} + 3I\right)$$
$$= \begin{pmatrix} 2 & 1 \\ 80 & 20 \end{pmatrix}$$

or (after some simplification)

$$P\begin{pmatrix} \lambda^3 - 2\lambda^2 & 0 \\ 0 & -\lambda^3 - 2\lambda^2 \end{pmatrix} P^{-1} = \begin{pmatrix} 7 & 1 \\ -80 & -11 \end{pmatrix}. \qquad (1)$$

The eigenvalues of $\begin{pmatrix} 7 & 1 \\ -80 & -11 \end{pmatrix}$ are $-1$ and $-3$. Hence, $\lambda^3 - 2\lambda^2 = -1$, which implies that $\lambda = 1$.

Let $A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$. Then, $\det A = -a^2 - bc = -\lambda^2$. It follows that $a^2 + bc = 1$. Thus, the eigenvectors of $A$ are $\begin{bmatrix} \frac{1}{c}\left(a + \sqrt{a^2 + bc}\right) \\ 1 \end{bmatrix} = \begin{pmatrix} \frac{1}{c}(a+1) \\ 1 \end{pmatrix}$ and $\begin{bmatrix} \frac{1}{c}\left(a - \sqrt{a^2 + bc}\right) \\ 1 \end{bmatrix} = \begin{pmatrix} \frac{1}{c}(a-1) \\ 1 \end{pmatrix}$. Hence,

$$P = \begin{pmatrix} \frac{1}{c}(a+1) & \frac{1}{c}(a-1) \\ 1 & 1 \end{pmatrix}$$

and

$$P^{-1} = \begin{pmatrix} \frac{1}{2}c & \frac{1}{2} - \frac{1}{2}a \\ -\frac{1}{2}c & \frac{1}{2}a + \frac{1}{2} \end{pmatrix}.$$

Now, from (1) we have

$$P\begin{pmatrix} \lambda^3 - 2\lambda^2 & 0 \\ 0 & -\lambda^3 - 2\lambda^2 \end{pmatrix} P^{-1}$$

$$= \begin{pmatrix} \frac{1}{c}(a+1) & \frac{1}{c}(a-1) \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} \frac{1}{2}c & \frac{1}{2} - \frac{1}{2}a \\ -\frac{1}{2}c & \frac{1}{2}a + \frac{1}{2} \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ -80 & -11 \end{pmatrix},$$

that is,

$$\begin{pmatrix} a - 2 & -\frac{1}{c}(a^2 - 1) \\ c & -a - 2 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ -80 & -11 \end{pmatrix}.$$

It follows that $a = 9$ and $c = -80$. The condition $a^2 + bc = 1$ implies that $b = \frac{1 - a^2}{c} = 1$.

We conclude that the matrix $\begin{pmatrix} 9 & 1 \\ -80 & -9 \end{pmatrix}$ is the only matrix in $M_2(\mathbb{Z})$ which satisfies the equation.

(ii) $A^{2020} = P\begin{pmatrix} \lambda^{2020} & 0 \\ 0 & (-\lambda)^{2020} \end{pmatrix} P^{-1} = \lambda^{2020} P \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} P^{-1} = I.$

**Solution 2 by the proposer.** (i) Simplifying the equality of the statement, it suffices to verify that

$$A^3 - 2A^2 = \begin{pmatrix} 7 & 1 \\ -80 & -11 \end{pmatrix}.$$

Let

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}.$$

A simple calculation shows that we want to verify that

$$\begin{pmatrix} a^3+abc & a^2b+b^2c \\ a^2c+bc^2 & -a^3-abc \end{pmatrix} - \begin{pmatrix} 2a^2+2bc & 0 \\ 0 & 2a^2+2bc \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ -80 & -11 \end{pmatrix}.$$

From the first matrix equality we obtain that $a^2(a-2) = 7 - bc(a-2)$. This implies that $a-2$ divides $7$, that is, either $a=3$ and $bc=-2$, or $a=9$ and $bc=-80$, or $a=1$ and $bc=-8$, or $a=-5$ and $bc=-26$. From the second matrix equality we obtain $a^2b+(bc)b=1$. If $a=3$ and $bc=-2$, then $7b=1$, which is impossible. If $a=9$, then $b=1$ and $c=-80$. From the third matrix equality we obtain $c^2+81c=-80$, so $c=-1$ or $c=-80$. Finally, from the fourth matrix equality we conclude that $a=9$, $b=1$ and $c=-80$ is a valid answer. If $a=1$ and $bc=-8$, then $-7b=1$, impossible. If $a=-5$ and $bc=-26$, then $b=-1$, which means that $c=26$. From the third matrix equality we obtain $a^2c+bc^2=-26\neq-80$. Therefore, $a=9$, $b=1$ and $c=-80$. Note that

$$\begin{pmatrix} 9 & 1 \\ -80 & -9 \end{pmatrix}^2 = I.$$

This means that

$$A^3 - 2A^2 = A - 2I = \begin{pmatrix} 9 & 1 \\ -80 & -9 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ -80 & -11 \end{pmatrix},$$

as we wanted to see. So the answer is

$$A = \begin{pmatrix} 9 & 1 \\ -80 & -9 \end{pmatrix}.$$

(ii) $A^{2020} = (A^2)^{1010} = I^{1010} = I$.

***A–70.*** *Proposed by Mihály Bencze, Braşov, România.* Let $G$ be the set of positive reals. We define the composition of any two $x, y \in G$ by

$$a^x + a^{x \circ y} + a^y = 2 + a^{x+y},$$

where $a > 1$. Prove that if $x_1, x_2, \ldots, x_n$ are elements of $G$, then

$$\frac{n(n-1)}{2} + \sum_{1 \le i<j \le n} \sqrt{a^{x_i \circ x_j} - 1} \le \frac{n-1}{2} \sum_{k=1}^{n} a^{x_k}.$$

**Solution 1 by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.** For all $x, y \in G$ and $a > 1$ we have that

$$x \circ y = \log_a(1 + (a^x - 1)(a^y - 1)).$$

It can be easily proven that $(G, \circ)$ is an abelian group. Now we have, on account of the AM-GM inequality, that

$$a^{x_i \circ x_j} - 1 = (a^{x_i} - 1)(a^{x_j} - 1) \le \left(\frac{a^{x_i} - 1 + a^{x_j} - 1}{2}\right)^2$$

or

$$2\sqrt{a^{x_i \circ x_j} - 1} \le a^{x_i} + a^{x_j} - 2.$$

Adding up the above expressions yields

$$2 \sum_{1 \le i<j \le n} \sqrt{a^{x_i \circ x_j} - 1} \le \sum_{1 \le i<j \le n} (a^{x_i} + a^{x_j} - 2) = (n-1) \sum_{k=1}^{n} a^{x_k} - 2\binom{n}{2}$$

or

$$2\binom{n}{2} + 2 \sum_{1 \le i<j \le n} \sqrt{a^{x_i \circ x_j} - 1} \le (n-1) \sum_{k=1}^{n} a^{x_k},$$

from which the statement follows.

**Solution 2 by Moti Levy, Rehovot, Israel.** By definition of the composition,

$$a^{x_i \circ x_j} - 1 = (a^{x_j} - 1)(a^{x_i} - 1).$$

By the power mean inequality,

$$\sum_{1 \le i<j \le n} \sqrt{(a^{x_j} - 1)(a^{x_i} - 1)} \le \sqrt{\frac{n(n-1)}{2}} \sqrt{\sum_{1 \le i<j \le n} (a^{x_j} - 1)(a^{x_i} - 1)}.$$

Therefore, it is enough to prove that

$$\sqrt{\frac{n(n-1)}{2}}\sqrt{\sum_{1\le i<j\le n}(a^{x_j}-1)(a^{x_i}-1)}\le\frac{n-1}{2}\sum_{k=1}^{n}(a^{x_k}-1),$$

or that

$$\sum_{1\le i<j\le n}(a^{x_j}-1)(a^{x_i}-1)\le\frac{1}{2}\frac{n-1}{n}\left(\sum_{k=1}^{n}(a^{x_k}-1)\right)^2. \qquad (1)$$

Clearly,

$$\sum_{1\le i<j\le n}(a^{x_j}-1)(a^{x_i}-1)=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a^{x_j}-1)(a^{x_i}-1)-\frac{1}{2}\sum_{k=1}^{n}(a^{x_k}-1)^2. \qquad (2)$$

We substitute (2) into (1) and get the equivalent inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a^{x_j}-1)(a^{x_i}-1)\le\sum_{k=1}^{n}(a^{x_k}-1)^2+\frac{n-1}{n}\left(\sum_{k=1}^{n}(a^{x_k}-1)\right)^2.$$

Again, by the power mean inequality we get

$$\frac{1}{n}\left(\sum_{k=1}^{n}(a^{x_k}-1)\right)^2\le\sum_{k=1}^{n}(a^{x_k}-1)^2,$$

hence it is enough to show that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a^{x_j}-1)(a^{x_i}-1)$$
$$\le\frac{1}{n}\left(\sum_{k=1}^{n}(a^{x_k}-1)\right)^2+\frac{n-1}{n}\left(\sum_{k=1}^{n}(a^{x_k}-1)\right)^2=\left(\sum_{k=1}^{n}(a^{x_k}-1)\right)^2,$$

which is obvious since

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a^{x_j}-1)(a^{x_i}-1)=\left(\sum_{k=1}^{n}(a^{x_k}-1)\right)^2.$$

**Also solved by** *the proposer.*

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