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Some variations of Walker's inequality

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Abstract

In this note, we revisit some inequalities involving the elements of a triangle. At first glance, there is nothing in common between them, but actually all of them are equivalent to Walker's inequality. It provides a very convenient and useful quadratic $R; r$ minorant for the square of the semiperimeter s of an acute triangle. Although the problems listed below are borrowed from different sources, all the solutions featured are due to the author of this note.

1 Main results

In what follows, we present some problems involving the circumradius R, the inradius r, and the semiperimeter s of and acute triangle ABC. We begin with an inequality used by Skopets [3].

Problem 1. *Prove that in an acute triangle with side lengths a, b and c, circumradius R and inradius r we have that*

$$a^2 + b^2 + c^2 \quad 4(R + r)^2. \tag{1}$$

Solution. On account of the sine law, $a^2 + b^2 + c^2 \quad 4(R + r)^2$ is equivalent to

$$4R^2 \ \sin^2 A + \sin^2 B + \sin^2 C \qquad 4R^2 \ \ 1 + \frac{r}{R}^{\ 2}$$

or

$$\sin^2 A + \sin^2 B + \sin^2 C \quad (\cos A + \cos B + \cos C),^2$$

which is equivalent to

$$\sum_{\text{cyc}} 1 \quad \cos^2 A \qquad \sum_{\text{cyc}} \cos^2 A + 2 \sum_{\text{cyc}} \cos B \cos C$$

or

$$\sum_{\text{cyc}} (\cos A + \cos B)^2 \quad 3.$$

To prove the above inequality, we use Cauchy's inequality and we have

$$(\cos A + \cos B)^2 \quad (a \cos B + b \cos A)\left(\frac{\cos B}{a} + \frac{\cos A}{b}\right)$$
$$= c\left(\frac{\cos B}{a} + \frac{\cos A}{b}\right).$$

Then,

$$\sum_{\text{cyc}} (\cos A + \cos B)^2 \quad \sum_{\text{cyc}} \left(\frac{c \cos B}{a} + \frac{c \cos A}{b}\right)$$
$$= \frac{c \cos B}{a} + \frac{c \cos A}{b} + \frac{a \cos C}{b} + \frac{a \cos B}{c} + \frac{b \cos A}{c} + \frac{b \cos C}{a}$$
$$= \sum_{\text{cyc}} \frac{c \cos B + b \cos C}{a} = \sum_{\text{cyc}} \frac{a}{a} = 3. \qquad \square$$

Remark 1. *During the above solution, we obtained the inequality*

$$\sum_{\text{cyc}} (\cos A + \cos B)^2 \quad 3 \tag{2}$$

which is a trigonometric equivalent of the inequality claimed in Problem 1 [1]. It holds in any acute triangle ABC.

Remark 2. *Since $a^2 + b^2 + c^2 = 2(s^2 \quad 4Rr \quad r^2)$, we have that $a^2 + b^2 + c^2 \quad 4(R + r)^2 \quad (\;) \quad 2(s^2 \quad 4Rr \quad r^2) \quad 4(R + r)^2$ or, equivalently,*

$$s^2 \quad 2R^2 + 8Rr + 3r^2. \tag{W}$$

*This is known as **Walker's Inequality** for an acute triangle.*

Problem 2. *Prove that the following inequality holds in any triangle:*

$$\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)^2+\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2+\left(\sin\frac{C}{2}+\sin\frac{A}{2}\right)^2\le 3. \tag{3}$$

Solution. Since dealing with the sine of a half angle is not very convenient, we will use the cosine theorem to obtain a metrical representation for $\sin\frac{A}{2}; \sin\frac{B}{2}; \sin\frac{C}{2}$ and also a metrical equivalent of the inequality claimed in the statement. Namely,

$$c^2 = a^2+b^2-2ab\cos C = (a-b)^2+4ab\sin^2\frac{C}{2},$$

from which it follows that

$$\sin^2\frac{C}{2}=\frac{(s-a)(s-b)}{ab}.$$

Likewise, we get

$$\sin^2\frac{A}{2}=\frac{(s-b)(s-c)}{bc}$$

and

$$\sin^2\frac{B}{2}=\frac{(s-a)(s-c)}{ac}.$$

Then, by Cauchy's inequality, we have

$$\begin{aligned}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)^2 &= \frac{s-c}{c}\left(\sqrt{s-b}\,\frac{1}{\sqrt{b}}+\sqrt{s-a}\,\frac{1}{\sqrt{a}}\right)^2\\ &\le \frac{s-c}{c}\left(\frac{1}{a}+\frac{1}{b}\right)(s-a+s-b)\\ &= \frac{(a+b-c)(a+b)c}{2abc}\\ &= \frac{(a^2+b^2)c-c^2(a+b)+2abc}{2abc}\\ &= 1+\frac{(a^2+b^2)c-c^2(a+b)}{2abc}\end{aligned}$$

and, therefore,

$$\begin{aligned}\sum_{cyc}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)^2 &\le 3+\sum_{cyc}\frac{(a^2+b^2)c-c^2(a+b)}{2abc}\\ &= 3+\frac{1}{2abc}\left(\sum_{cyc}\left(a^2+b^2\right)c-\sum_{cyc}c^2(a+b)\right)\\ &= 3.\end{aligned}$$ □

Remark 3. *We have*

$$\sum_{cyc}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)^2 \le 3 \iff \left(\sum_{cyc}\sin\frac{A}{2}\right)^2 \le \sum_{cyc}\left(1-\sin^2\frac{A}{2}\right),$$

which is equivalent to the inequality

$$\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2 \le \cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2} \tag{4}$$

presented by Andreescu and Dospinescu [2].

Remark 4. *We have that*

$$\sum_{cyc}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)^2 \le 3 \iff 2\sum_{cyc}\sin^2\frac{A}{2}+2\sum_{cyc}\sin\frac{A}{2}\sin\frac{B}{2} \le 3$$

and

$$\begin{aligned}\sum_{cyc}\sin\frac{A}{2}\sin\frac{B}{2} &\le \sum_{cyc}\left(1-2\sin^2\frac{A}{2}\right)\\ \iff 2\sum_{cyc}\sin\frac{A}{2}\sin\frac{B}{2} &\le \sum_{cyc}\cos A,\end{aligned}$$

from which it follows that

$$\sum_{cyc}\sin\frac{A}{2}\sin\frac{B}{2} \le \frac{1}{2}\left(1+\frac{r}{R}\right). \tag{5}$$

Remark 5. *Let* $\alpha = \frac{\pi - A}{2}$, $\beta = \frac{\pi - B}{2}$ *and* $\gamma = \frac{\pi - C}{2}$. *Since* $\alpha; \beta; \gamma \in (0; \pi/2) > 0$ *and* $\alpha + \beta + \gamma = \pi$ *and, therefore, there exists an acute triangle with angles* $\alpha; \beta; \gamma$, *which has side lengths* $a; b; c$, *circumradius* R *and inradius* r *for some values of* $a; b; c; R; r$. *Then, inequality* (4) *becomes*

$$\left(\sum_{\text{cyc}} \cos\left(\frac{\pi}{2} - \frac{A}{2}\right)\right)^2 \le 2\sum_{\text{cyc}} \sin^2\left(\frac{\pi}{2} - \frac{A}{2}\right)$$

$$\iff \left(\sum_{\text{cyc}} \cos\alpha\right)^2 \le 2\sum_{\text{cyc}} \sin^2\alpha$$

$$\iff \left(1 + \frac{r}{R}\right)^2 \le 2\sum_{\text{cyc}} \frac{a^2}{4R^2} \iff (1).$$

Problem 3. *Let* $x; y; z > 0$ *be any real numbers such that* $x^2 + y^2 + z^2 + 2xyz = 1$. *Prove that*

$$x^2 + y^2 + z^2 + xy + yz + zx \le 3/2. \tag{6}$$

Solution. We have that

$$x^2 + y^2 + z^2 + xy + yz + zx \le 3/2 \iff \sum_{\text{cyc}} (x + y)^2 \le 3.$$

Since any positive solution $(x; y; z)$ of the equation $x^2 + y^2 + z^2 + 2xyz = 1$ can be represented as

$$(x; y; z) = \left(\sqrt{\frac{bc}{(a+b)(c+a)}}; \sqrt{\frac{ca}{(b+c)(a+b)}}; \sqrt{\frac{ab}{(c+a)(b+c)}}\right)$$

for all $a, b, c > 0$, then, using Cauchy's inequality, we obtain

$$\sum_{cyc}(x+y)^2 = \sum_{cyc}\left(\sqrt{\frac{bc}{(a+b)(c+a)}}+\sqrt{\frac{ca}{(b+c)(a+b)}}\right)^2$$
$$= \sum_{cyc}\frac{c}{a+b}\left(\sqrt{\frac{b}{c+a}}+\sqrt{\frac{a}{b+c}}\right)^2$$
$$= \sum_{cyc}\frac{c}{a+b}\left(\sqrt{b}\,\frac{1}{\sqrt{c+a}}+\sqrt{a}\,\frac{1}{\sqrt{b+c}}\right)^2$$
$$\le \sum_{cyc}\frac{c}{a+b}(b+a)\left(\frac{1}{c+a}+\frac{1}{b+c}\right)$$
$$= \sum_{cyc}\left(\frac{c}{c+a}+\frac{c}{b+c}\right) = 3. \qquad \square$$

Remark 6. Taking into account that

$$\{(x, y, z) \mid x, y, z > 0 \text{ and } x^2+y^2+z^2+2xyz = 1\}$$
$$= \{(\cos\alpha, \cos\beta, \cos\gamma) \mid x, y, z > 0;\ \alpha, \beta, \gamma \in (0, \pi/2);\ \alpha+\beta+\gamma = \pi\},$$

we can rewrite inequality (6) as

$$\sum_{cyc}\cos^2\alpha + \sum_{cyc}\cos\alpha\cos\beta \le 3/2$$
$$\iff 2\sum_{cyc}\cos^2\alpha + 2\sum_{cyc}\cos\alpha\cos\beta \le 3 \iff (2).$$

(Here, numbers α, β and γ are interpreted as angles of an acute triangle.)

Finally, up to notations and interpretations, here is the chain of equivalent inequalities:

$$(W) \iff (1) \iff (2) \iff (3) \iff (4) \iff (5) \iff (6).$$

We conclude that a proof of any of them is at the same time a proof for all others.

References

[1] Andreescu, T. and Dospinescu, G. “Problem 12”. *Problems from the Book*. XYZ Press, 2008, p. 22.

[2] Andreescu, T. and Dospinescu, G. “Problem 9”. *Problems from the Book*. XYZ Press, 2008, p. 21.

[3] Skopets, Z. A. “Problem M1178”. *Kvant* (1989), p. 153.

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An Elementary Proof of Weighted Power Means Inequality

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Abstract

By making use of a simple idea [1, 2], we provide a new proof for the weighted power means inequality.

1 Introduction

The weighted power mean of order p of $x_1, x_2, \ldots, x_n$ is defined by

$$M_p(x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^{n} w_i x_i^p \right)^{\frac{1}{p}},$$

where $\sum_{i=1}^{n} w_i = 1$. Moreover, it can be shown that

$$M_0(x_1, x_2, \ldots, x_n) = \lim_{p \to 0} M_p(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i^{w_i}.$$

By setting all $w_i = \frac{1}{n}$, the weighted power mean reduces to the well-known harmonic mean (HM), geometric mean (GM), arithmetic mean (AM) and quadratic mean (QM) for $p = -1, 0, 1$ and 2, respectively. In this short paper, we use the simple idea proposed by Razminia [1, 2] to prove the weighted power means inequality

$$M_p(x_1, x_2, \ldots, x_n) \geq M_q(x_1, x_2, \ldots, x_n) \tag{1}$$

for positive real x_i's and all real p and q, where $p > q$. Equality occurs if and only if all x_i's are equal. Consequently, the special cases QM-AM-GM-HM mean inequalities can be concluded.

2 Main results

To prove that (1) holds, we show that

$$h(x_1; x_2; :::; x_n) = M_p(x_1; x_2; :::; x_n) \quad M_q(x_1; x_2; :::; x_n)$$

is always non-negative and its minimum value is zero, when all x_i's are equal. According to Weierstrass's extreme value theorem, the continuous function h has a global minimum on $I = [\min f x_i g; \max f x_i g]^n$. We prove that this minimum value is zero and it occurs at a point with equal elements. Let us assume that the minimum occurs at the point $(\ _1; \ _2; :::; \ _n)$ with two unequal elements, without loss of generality, say $\ _2 > \ _1$ (contradiction approach). Then, the following inequality must hold:

$$\begin{aligned} & h(\ _1; \ _2; :::; \ _n) \\ & h \left((w_1^0 \ _1^q + w_2^0 \ _2^q)^{\frac{1}{q}}; (w_1^0 \ _1^q + w_2^0 \ _2^q)^{\frac{1}{q}}; \ _3; :::; \ _n \right), \end{aligned} \qquad (2)$$

where $w_i^0 = \frac{w_i}{w_1 + w_2}$ for $i = 1; 2$. It should be noted that the point $(w_1^0 \ _1^q + w_2^0 \ _2^q)^{\frac{1}{q}}; (w_1^0 \ _1^q + w_2^0 \ _2^q)^{\frac{1}{q}}; \ _3; \ _4; :::; \ _n$ belongs to the cell I, because $\min f x_i g \quad _1 < (w_1^0 \ _1^q + w_2^0 \ _2^q)^{\frac{1}{q}} < \ _2 \quad \max f x_i g$.

The right hand side of (2) is

$$\begin{aligned} & \left((w_1 + w_2)(w_1^0 \ _1^q + w_2^0 \ _2^q)^{\frac{p}{q}} + \sum_{i=3}^{n} w_i \ _i^p \right)^{\frac{1}{p}} \\ & \left((w_1 + w_2)(w_1^0 \ _1^q + w_2^0 \ _2^q) + \sum_{i=3}^{n} w_i \ _i^q \right)^{\frac{1}{q}}, \end{aligned} \qquad (3)$$

or (after simplifying the second term in (3))

$$\left((w_1 + w_2)(w_1^0 \ _1^q + w_2^0 \ _2^q)^{\frac{p}{q}} + \sum_{i=3}^{n} w_i \ _i^p \right)^{\frac{1}{p}} \quad M_q(\ _1; \ _2; :::; \ _n). \qquad (4)$$

Replacing the right hand side of (2) by (4), (2) becomes (after cancelling out the similar terms on both sides)

$$\left(\sum_{i=1}^{n} w_i \; {}_i^{p}\right)^{\frac{1}{p}} \quad (w_1 + w_2)(w_1^0 \; {}_1^q + w_2^0 \; {}_2^q)^{\frac{p}{q}} + \left(\sum_{i=3}^{n} w_i \; {}_i^{p}\right)^{\frac{1}{p}}. \qquad (5)$$

After a few manipulations, (5) becomes

$$(w_1^0 \; {}_1^p + w_2^0 \; {}_2^p)^{\frac{1}{p}} \quad (w_1^0 \; {}_1^q + w_2^0 \; {}_2^q)^{\frac{1}{q}}, \qquad (6)$$

which is obviously wrong, according to Sándor [3, Lemma 2.1]. This lemma says that $(w_1^0 \; {}_1^p + w_2^0 \; {}_2^p)^{\frac{1}{p}}$ is a strictly increasing function of p (Sándor's lemma is only for the case $w_1^0 = w_2^0 = \frac{1}{2}$, but it can be easily extended to the general case for all w_1^0 and w_2^0, where $w_1^0 + w_2^0 = 1$). Therefore, the assumption that minimum can occur at some points with unequal elements is wrong and the minimum value of $h(x_1; x_2; :::; x_n)$ occurs at a point with equal elements

$$h(\; _1; \; _2; :::; \; _n) = M_p(\; _1; \; _2; :::; \; _n) \quad M_q(\; _1; \; _2; :::; \; _n) = 0 .$$

References

[1] Razminia, K. “A short proof of symmetric inequalities”. The College Mathematics Journal 46 (2015), pp. 364–366.

[2] Razminia, K. “Hölder's inequality revisited (to appear)”. Mathematical Gazettel (2019).

[3] Sándor, J. “A note on certain inequalities for bivariate means”. Notes on Number Theory and Discrete Mathematics 20 (2014), pp. 11–13.

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Re nements of Gerretsen's inequality

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Abstract

Gerretsen's inequality states that, in a triangle, the sum of the squares of the sides is at most a linear combination of the squares of R and r, where R is the radius of the circumscribed circle and r is the radius of the inscribed circle. The purpose of this paper is to nd a chain of inequalities which represent re nements of Gerretsen's inequality. In the chain of the inequalities we include rational functions of R and r and the square root of a fourth degree polynomial in R and r.

1 Introduction

Let ABC be a triangle. We shall denote $a = BC$, $b = AC$, $c = AB$, $s = \frac{a+b+c}{2}$ the semiperimeter, R the radius of the circumscribed circle and r the radius of the inscribed circle, as usual.

J. C. Gerretsen [6] proved that the inequality $a^2+b^2+c^2 \quad 8R^2+4r^2$ holds in any triangle ABC. Radulescu, Dr agan, and Maftei [9] studied the following problem: Find the best constants $\;;\;;\; 2\; R$ with $\quad 0$ such that the following inequality holds:

$$a^2+b^2+c^2 \quad R^2 + \quad Rr + \quad r^2.$$

L. Panaitopol [8] proved that, if $\; = 0$, then $\; = 8$ and $\; = 4$ are the best constants for which the above inequality holds. R adulescu, Dr agan, and Maftei [9] proved that the constants $\; = 8$, $\; = 0$

and = 4 are the best constants for which the above inequality holds. In other words, it was proved that the constants in the inequality of Gerretesen are the best constants.

This means that, if , and are real numbers with 0 and with the property that the inequality $a^2 + b^2 + c^2 \quad R^2 + \quad Rr + \quad r^2$ is true in every triangle ABC, then we have that the inequality $8R^2 + 4r^2 \quad R^2 + \quad Rr + \quad r^2$ is true in every triangle ABC.

W. J. Blundon [5] proposed the inequality

$$js^2 \quad (2R^2 + 10Rr \quad r^2)j \quad 2(R \quad 2r)^p \overline{R(R \quad 2r)}$$

and later gave a proof [4]. In the following, we shall refer to the preceding inequality as Blundon's inequality. It is important to note that Makovski [7] gave another proof of Blundon's inequality.

Radulescu, Dr agan, and Maftei [9] proved the inequality

$$a^2 + b^2 + c^2 \quad \frac{36(8R^4 + \ tr^4)}{36R^2 + (\ t \quad 16)r^2}; \quad \text{for each } \ t\ 2\ [\quad 2; 6]. \tag{1}$$

If we take $t = 6$ in (1), we obtain the inequality

$$a^2 + b^2 + c^2 \quad \frac{36(4R^4 + 3r^4)}{18R^2 \quad 5r^2}. \tag{2}$$

The inequality

$$a^2 + b^2 + c^2 \quad \frac{8(27R^4 + 2r^4 + 8Rr^3)}{27R^2 \quad 8r^2} \tag{3}$$

was proved by A. Bager [1]. It represents an improvement on the inequality

$$a^2 + b^2 + c^2 \quad \frac{(4R + r)(4R^2 \quad 3Rr + 2r^2)}{2R \quad r}. \tag{4}$$

One can easily see that (2) represents an improvement of (3). Also, inequality (4) represents an improvement on the inequality

$$a^2 + b^2 + c^2 \quad \frac{72R^4}{9R^2 \quad 4r^2}$$

proposed by I. V. Maftei in the *Arhimede International Symposium of Pure and Applied Mathematics 2008* and solved by D. Băiţan [2].

Consider the function $f : [-2; +\infty) \to \mathbb{R}$ defined by

$$f(t) = \frac{36(8R^4 + tr^4)}{36R^2 + (t - 16)r^2}. \qquad (5)$$

Radulescu, Drăgan, and Maftei [9] proved that f is decreasing. As a consequence, it results that

$$f(6) \le f(1) \le f(0) \le f(-1) \le f(-2).$$

The above chain of inequalities may be written as follows:

$$\begin{aligned} a^2 + b^2 + c^2 &\le \frac{36(4R^4 + 3r^4)}{18R^2 - 5r^2} \le \frac{8(27R^4 + 2r^4 + 8Rr^3)}{27R^2 - 8r^2} \\ &\le \frac{(4R + r)(4R^2 - 3Rr + 2r^2)}{2R - r} \le \frac{72R^4}{9R^2 - 4r^2} \\ &\le 8R^2 + 4r^2, \end{aligned}$$

which proves that (2) is the best from this chain. Note that the above chain of inequalities contains inequalities (2), (3) and (4).

2 Main results

In the following we will determine the constants $\alpha; \beta; \gamma; \delta$ such that the inequality

$$a^2 + b^2 + c^2 \le \frac{\alpha R^4 + \beta r^4}{\gamma R^2 + \delta r^2} \le 8R^2 + 4r^2 \qquad (6)$$

holds in any triangle ABC and that the inequality

$$a^2 + b^2 + c^2 \le \frac{\alpha R^4 + \beta r^4}{\gamma R^2 + \delta r^2}$$

is the best of this type. We denote $u = \frac{\alpha}{\gamma}$, $v = \frac{\delta}{\gamma}$ and $t = \frac{\beta}{\gamma}$. Inequality (6) may be written as

$$a^2 + b^2 + c^2 \le \frac{uR^4 + tr^4}{R^2 + vr^2} \le 8R^2 + 4r^2. \qquad (7)$$

If we consider the case of an isosceles degenerate triangle with sides $a = 0$, $b = c = 1$, we have that $R = \frac{1}{2}$, $r = 0$. By replacing this in (7), we obtain $u = 8$. If we consider the case of an equilateral triangle, from (7) we obtain that $v = \frac{t-16}{36}$. Thus, (7) may be written equivalently as

$$a^2 + b^2 + c^2 \leq \frac{36(8R^4 + tr^4)}{36R^2 + (t-16)r^2} \leq 8R^2 + 4r^2. \qquad (8)$$

The right side of (8), after we substitute $x = \frac{R}{r}$, may be written as

$$\frac{72x^4 + 9t}{32x^2 + t - 16} \leq 2x^2 + 1$$

and, after some computations, we obtain

$$\frac{(2t+4)(-x^2+4)}{36x^2 + t - 16} \leq 0 \text{ for each } x \geq 2. \qquad (9)$$

If we take $x \to \infty$ in (9), we obtain $t \geq -2$.

Baiţan [2] proved that the function f defined in (5) is decreasing and $a^2 + b^2 + c^2 \leq f(t) \leq 8R^2 + 4r^2$ for every $t \in [-2; 6]$. We propose to find the greatest $t_0 \in (6; \infty)$ for which the above inequality holds in any triangle. Our result is stated in the following theorem.

Theorem 1. *In any triangle ABC, for each $t \in [-2; t_0]$, the following inequality holds:*

$$a^2 + b^2 + c^2 \leq \frac{36(8R^4 + tr^4)}{36R^2 + (t-16)r^2} \leq 8R^2 + 4r^2, \qquad (10)$$

where x_0 is the unique positive root of the equation

$$18x^4 - 82x^3 - 4x^2 - 3x - 2 = 0,$$

$x_0 = 4.61269707$, F is the function $F : (2; +\infty) \to \mathbb{R}$ defined by

$$F(x) = \frac{72x^4 + 72x^3 - 216x^2 - 60x + 40}{12x + 5} - \frac{(72x^3 + 144x^2 - 36x - 72)\sqrt{x^2 - 2x}}{12x + 5}$$

and $t_0 = F(x_0)$.

Proof. From the equality

$$a^2 + b^2 + c^2 = 2\left(s^2 - r^2 - 4Rr\right),$$

the left side of (10) may be written as

$$s^2 - r^2 + 4Rr + \frac{18(8R^4 + tr^4)}{36R^2 - (16 - t)r^2}. \tag{11}$$

From Blundon's theorem, we have

$$s^2 \le 2R^2 + 10Rr - r^2 + 2\sqrt{R(R - 2r)^3}.$$

We denote $x = \frac{R}{r}$. It turns out that, since it is known that the right side of Blundon's inequality is the best of the type $s^2 \le f(R; r)$ where $f(R; r)$ is an homogeneous function, in order to find the best real number t for which (11) holds it will suffice to find the best real number for which the following inequality is true:

$$2R^2 + 10Rr - r^2 + 2\sqrt{R(R - 2r)^3} \ge r^2 + 4Rr + \frac{18(8R^4 + tr^4)}{36R^2 - (16 - t)r^2}. \tag{12}$$

Inequality (12) may be written as

$$x^2 + 3x - 1 + \sqrt{x(x - 2)^3} \ge \frac{9(8x^4 + t)}{36x^2 + t - 16}. \tag{13}$$

After some computations, we obtain

$$t \ge \frac{72x^4 - (36x^2 - 16)(x^2 + 3x - 1) - (36x^2 - 16)\sqrt{x(x - 2)^3}}{x^2 + 3x - 10 + \sqrt{x(x - 2)^3}}. \tag{14}$$

If we take $x = 2$, the inequality (13) becomes equality in the case of the equilateral triangle. Now we consider $x \neq 2$. After dividing by $x - 2$, inequality (13) becomes

$$t \ge \frac{36x^3 - 36x^2 - 20x + 8 - (36x^2 - 16)\sqrt{x(x - 2)}}{x + 5 + \sqrt{x(x - 2)}} \tag{15}$$

for each $x > 2$.

We observe that the right hand side of inequality (15) is obtained after using the conjugate and rationalize just the function F from the statement. From (15), it follows that the best real number is the minimum of the function on $(2; +\infty)$.

After we calculate F', we obtain

$$F'(x) = \frac{\sqrt{x^2-2x}\,(648x^2-360x-108)-648x^3+1008x^2+108x-72}{\left(x+5+\sqrt{x^2-2x}\right)^2\sqrt{x^2-2x}}. \tag{16}$$

The equation $F'(x) = 0$ may be written equivalently as

$$\left(x^2-2x\right)\left(18x^2-10x-3\right)^2 = \left(18x^3-28x^2-3x+2\right)^2$$

or

$$18x^4-82x^3-4x^2-3x-2 = 0. \tag{17}$$

From Descartes's theorem, it follows that equation (17) has at most one positive root. We consider the polynomial

$$g(x) = 18x^4-82x^3-4x^2-3x-2.$$

Since $g(4)\,g(5) < 0$, it follows that the equation $g(x) = 0$ has a unique root in $(4; 5)$. But since $F'(4) < 0$ and $F'(5) > 0$, it follows that x_0 is a minimum point of F on $(2; +\infty)$. It results that the minimum of F on $(2; +\infty)$ is $t_0 = f(x_0)$. In conclusion,

$$a^2+b^2+c^2 \geq \frac{36(8R^4+tr^4)}{36R^2+(t-16)r^2} \quad \text{for each } t \in [2; t_0].$$

We consider the function $h : [2; t_0] \to \mathbb{R}$ defined by

$$h(t) = \frac{36(t+8x^4)}{t+36x^2-16}.$$

We have that

$$h'(t) = \frac{288(x^2-4)\left(x^2-\frac{1}{2}\right)}{(t+36x^2-16)^2} \quad \text{for each } t \in [2; t_0].$$

It results that h is a decreasing function on $[\ \ 2; t_0]$, so we have that

$$h(t_0) \quad h(6) \quad h(t) \quad h(\ \ 2).$$

Finally, we get the best chain of inequalities

$$a^2 + b^2 + c^2 \quad \frac{36(8R^4 + \ t_0 r^4)}{36R^2 + (\ t_0 \quad 16) r^2} \quad \frac{36(4R^4 + 3\ r^4)}{18R^2 \quad 5r^2}$$
$$\frac{36(8R^4 + \ tr^4)}{36R^2 + (\ t \quad 16) r^2} \quad 8R^2 + 4r^2$$

for each $t\ 2\ [\ \ 2; 6]$. □

In the following we give an irrational refinement of Gerretsen's theorem. For this, we find the best real numbers $\ ,\ ,\ $ such that the inequality

$$a^2 + b^2 + c^2 \quad \sqrt{\ R^4 + \ R^2 r^2 + \ r^4} \quad 8R^2 + 4r^2 \tag{18}$$

is true in any triangle ABC and the inequality

$$a^2 + b^2 + c^2 \quad \sqrt{\ R^4 + \ R^2 r^2 + \ r^4}$$

is the best of this type.

In the case of the isosceles degenerate triangle with $a = b = 1$, $c = 0$, $R = \frac{1}{2}$, $r = 0$, from (18) we obtain

$$\ = 64. \tag{19}$$

In the case of equilateral triangle, from (18) we have

$$\frac{1}{12}\sqrt{1024 + 4\ \ + \ } = 3,$$

from where we obtain that

$$4\ \ + \ = 272. \tag{20}$$

From (18), (19) and (20) it follows that

$$a^2 + b^2 + c^2 \quad \sqrt{64R^4 + \ R^2 r^2 + (272 \quad 4\ ) r^4} \quad 8R^2 + 4r^2. \tag{21}$$

We denote $x = \frac{R}{r}$. From inequality (21) it follows that

$$64x^4 + \ x^2 + 272 \quad 4 \qquad 64x^4 + 64x^2 + 16$$

or, in an equivalent form,

$$+ \frac{272 \quad 4}{x^2} \quad 64 \text{ for each } x \quad 2.$$

If we take $x \, ! \, 1$, we obtain $\quad 64$.

In the following we will nd the best real number $\quad 64$ for which the inequality $a^2 + b^2 + c^2 \quad \sqrt{64R^4 + \ R^2r^2 + (272 \quad 4 \ )r^4}$ is the best of this type.

Theorem 2. In any triangle ABC, the inequality

$$a^2 + b^2 + c^2 \quad \sqrt{64R^4 + \ R^2r^2 + (272 \quad 4 \ )r^4} \quad 8R^2 + 4r^2 \qquad (22)$$

holds for each $\quad 2 \ [\ _0; 64]$, where $\ _0 = F(x_0)$, x_0 is a unique real positive root of the equation

$$4x^5 \quad 5x^4 \quad 82x^3 \quad 164x^2 \quad 14x \quad 4 = 0 \, ; \ x_0 \, 2 \, (4; 5),$$

and $F : (2; +1 \ ) \, ! \ R$ is de ned by

$$F(x) = \frac{2\sqrt{x(x \quad 2)}(x^2 + 3x \quad 1) + x(x \quad 2)^2 \quad 3x^3 + 7x + 8}{x + 2}.$$

Proof. From the identity

$$a^2 + b^2 + c^2 = 2 \ s^2 \quad r^2 \quad 4Rr \ ,$$

the left side of (22) becomes

$$2 \ s^2 \quad r^2 \quad 4Rr \qquad \sqrt{64R^4 + \ R^2r^2 + (272 \quad 4 \ )r^4}. \qquad (23)$$

From Blundon's theorem we have

$$s^2 \quad 2R^2 + 10Rr \quad r^2 + 2\sqrt{R(R \quad 2r)^3}.$$

It follows that, in order to nd the best real number 64 for which (23) is true, it suf ces to nd the best real number 64 for which the following inequality is true:

$$2 \quad 2R^2 + 6Rr \quad 2r^2 + 2\sqrt{R(R \quad 2r)^3}$$
$$\sqrt{64R^4 + \quad R^2 r^2 + (272 \quad 4 \quad) r^4}. \qquad (24)$$

Inequality (24) may be written in an equivalent form as

$$x^2 + 3x \quad 1 + \sqrt{x(x \quad 2)^3} \quad \sqrt{4x^4 + \frac{}{16} x^2 + \frac{68}{4}}. \qquad (25)$$

After some calculation, inequality (25) may be written as

$$\frac{2\sqrt{x(x \quad 2)}(x^2 + 3x \quad 1) + x(x \quad 2)^2 \quad 3x^3 + 7x + 8}{x + 2} \qquad (26)$$

for each $x > 2$. If we consider the function stated in (26), it follows that $F(x)$ for each $x \quad 2$. Hence, the best real number we may nd is the maximum of the function F on $(2; +1 \;)$, $_0 = \max_{x\,2\,(2;+1\;)} F(x)$. We have

$$F^0(x) = \frac{4x^4 + 12x^3 \quad 2x^2 \quad 42x + 4 \quad (4x^3 + 16x^2 + 16x \quad 14)\sqrt{x^2 \quad 2x}}{(x + 2)\sqrt{x^2 \quad 2x}}.$$

The equation $F^0(x) = 0$ is equivalent to

$$(2x^4 + 6x^3 \quad x^2 \quad 21x + 2)^2 = (2x^3 + 8x^2 + 8x \quad 7)^2 (x^2 \quad 2x),$$

or

$$4x^5 \quad 5x^4 \quad 82x^3 \quad 164x^2 \quad 14x \quad 4 = 0. \qquad (27)$$

Since we have just one change of sign of the coef cients of equation (27), from Descartes's theorem it follows that (27) has at most one root.

We consider the polynomial

$$g(x) = 4x^5 \quad 5x^4 \quad 82x^3 \quad 164x^2 \quad 14x \quad 4.$$

Since $g(5) \quad g(6) < 0$, it follows that $x_0 \; 2 \; (5; 6)$ is the only positive root of equation (27). But since $F^0(5) > 0$ and $F^0(6) < 0$, it follows that x_0 is a maximum point of F. □

Remark 1. We observe that = 35 is the best natural number for which inequality (22) is true. So in any triangle ABC, the following inequality holds:

$$a^2 + b^2 + c^2 \quad \sqrt{64R^4 + 35R^2r^2 + 132r^4} \quad 8R^2 + 4r^2.$$

Theorem 3. In any triangle ABC the following chain of inequalities holds:

$$\begin{aligned} a^2 + b^2 + c^2 & \quad \sqrt{64R^4 + \quad {}_0 R^2r^2 + (272 \quad 4 \; {}_0)r^4} \\ & \quad \sqrt{64R^4 + 35R^2r^2 + 132r^4} \\ & \quad \frac{8(27R^4 + 2r^4 + 8Rr^3)}{27R^2 \quad 8r^2} \\ & \quad \frac{(4R + r)(4R^2 \quad 3Rr + 2r^2)}{2R \quad r} \\ & \quad \frac{36(4R^4 + r^4)}{18R^2 \quad 7r^2} \quad \frac{36(8R^4 + r^4)}{36R^2 \quad 15r^2} \\ & \quad \frac{72R^4}{9R^2 \quad 4r^2} \quad 8R^2 + 4r^2. \end{aligned}$$

Here, ${}_0$ 2 (34:49; 34:58) is the positive real number from Theorem 2.

Proof. Inequality

$$\frac{(4R + r)(4R^2 \quad 3Rr + 2r^2)}{2R \quad r} \quad \frac{36(4R^4 + r^4)}{18R^2 \quad 7r^2}$$

is equivalent to $(x \quad 2)(22x^2 \quad 48x + 11) \quad 0$ for each $x \quad 2$, which is true since the roots of $22x^2 \quad 48x + 11 = 0$ are lower than 2.

It remains to prove that

$$\sqrt{64R^4 + \quad {}_0 R^2r^2 + (272 \quad 4 \; {}_0)r^4} \quad \sqrt{64R^4 + 35R^2r^2 + 132r^2} \quad \frac{8(27R^4 + 2r^4 + 8Rr^3}{27R^2 \quad 8r^2}; \tag{28}$$

the rest of the inequalities follow by Theorem 2 and from the introduction.

To prove the left side of inequality (28), let $u = \frac{R^2}{r^2}$ and consider the function $h : [\lambda_0; 64] \to \mathbb{R}$, $h(\lambda) = r^2\sqrt{64u^2 + \lambda u + 272 - 4\lambda}$. Note that

$$h'(\lambda) = \frac{r^2(u-4)}{2\sqrt{64u^2 + \lambda u + 272 - 4\lambda}} \geq 0,$$

hence h is an increasing function. We have $h(\lambda_0) \leq h(\lambda)$ for each $\lambda \in [\lambda_0; 64]$. Therefore, $h(\lambda_0) \leq h(35)$.

The left side of inequality (28) may be written as

$$\sqrt{64x^4 + 35x^2 + 132} \geq \frac{8(27x^4 + 8x + 2)}{27x^2 - 8}$$

for each $x \geq 2$ or, after performing some calculations,

$$2133x^6 + 27648 - 78292x^4 + 58880x^2 + 2048x - 81192 \geq 0$$

for each $x \geq 2$, or

$$(x-2)(2133x^5 + 31914x^4 - 14464x^3 - 28928x^2 + 1024x + 4096) \geq 0$$

for each $x \geq 2$, inequality which is true since

$$\begin{aligned} &2133x^5 + 31914x^4 - 14464x^3 - 28928x^2 + 1024x + 4096 \\ &= 2133x^2(x^3 - 8) + 7232x^3(x-2) + 2966x^2(x^2-4) + 21716x^4 \\ &\quad + 1024x + 4096 \geq 0 \end{aligned}$$

for each $x \geq 2$. □

In the following, we use the following notation:

$$U = \sqrt{64R^4 + 35R^2r^2 + 132r^4}, \qquad V = \frac{36(4R^4 + 3r^4)}{18R^2 - 5r^2},$$

$$S = \sqrt{64R^4 + \lambda_0 R^2r^2 + (272 - 4\lambda_0)r^4}, \qquad T = \frac{36(8R^4 + t_0 r^4)}{36R^2 + (t_0 - 16)r^2},$$

$$W = \frac{8(27R^4 + 2r^4 + 8Rr^3)}{27R^2 - 8r^2}, \qquad \lambda_0 = \sqrt{\frac{1561 + 8\sqrt{39544}}{90}},$$

$$A = 73728 \quad 4608t_0 \quad 1296 \quad {}_0,$$
$$B = \quad 64t_0^2 + 4352\, t_0 \quad 73984 + 1152 \quad {}_0 \quad 72 \quad {}_0 t_0,$$
$$C = \quad 256t_0^2 \quad 2176t_0 + 17408 \quad {}_0 t_0^2 + 32 \quad {}_0 t_0 \quad 256 \quad {}_0,$$
$${}_0 = \sqrt{\frac{B + \sqrt{B^2 \quad 4AC}}{2A}},$$

where t_0 and ${}_0$ were de ned in Theorem 1 and Theorem 2.

Theorem 4. In any triangle ABC the following chains of inequalities hold:

i)
$$a^2 + b^2 + c^2 \quad S \quad U \quad V \quad W \quad \text{if } \frac{R}{r} \quad {}_0,$$
$$a^2 + b^2 + c^2 \quad T \quad V < U \quad W \quad \text{if } 2 \quad \frac{R}{r} < {}_0.$$

ii)
$$a^2 + b^2 + c^2 \quad S \quad T \quad V \quad W \quad \text{if } \frac{R}{r} \quad {}_0,$$
$$a^2 + b^2 + c^2 \quad T < S \quad U \quad W \quad \text{if } 2 \quad \frac{R}{r} < {}_0.$$

Proof. i) First we shall prove that $V \quad W$ for every triangle ABC . This is equivalent to

$$\frac{36(4x^4 + 3)}{18x^2 \quad 5} \quad \frac{8(27x^4 + 8\, x + 2)}{27x^2 \quad 8},$$

or $4(x \quad 2)(18x^3 + 324x^2 \quad 9x \quad 98) \quad 0$, which is true for each $x \quad 2$.

In the following we search the values of $\frac{R}{r}$ for which $U \quad V$, or

$$\sqrt{64R^4 + 35R^2r^2 + 132r^4} \quad \frac{36(4R^4 + 3r^4)}{18R^2 \quad 5r^2}. \qquad (29)$$

If we write $\frac{R^2}{r^2} = y$, inequality (29) may be rewritten as

$$\sqrt{64y^2 + 35y + 132} \quad \frac{36(4y^2 + 3)}{18y \quad 5}, \quad \text{for each } y \quad 4,$$

or $(y \quad 4)(180y^2 \quad 6244y \quad 2091) \quad 0$ for each $y \quad 4$, or $y \quad {}_0^2$. It results that $U \quad V$ if $\frac{R}{r} \quad {}_0$ and $U > V$ if $2 \quad \frac{R}{r} < {}_0$. The inequality $S \quad U$ was proved in Theorem 3. Furthermore, the inequality $T \quad V$ follows from the monotony of function $f : R\, ! \quad R$, de ned by $f(t) = \dfrac{36(8R^4 + tr^4)}{36R^2 + (t \quad 16)r}$, since $f(t_0) \quad f(6)$.

ii) We search the values of $\frac{R}{r}$ for which $S \quad T$ or

$$p\,\overline{64y^2 + \quad {}_0 y + 272 \quad 4 \quad {}_0} \qquad \frac{36(8y^2 + t_0)}{36y + t_0 \quad 16}$$

or, after squaring and some computations, $(y \quad 4)(Ay^2 + By + C)$ 0 for each $y \quad 4$. Obviously, $A > \quad 0$, $B < \quad 0$ and $C < \quad 0$, so we have $y \quad {}^2_0$. We conclude that $S \quad T$ if $\frac{R}{r} > \quad {}_0$ and $S > T$ if $2 \quad \frac{R}{r} < \quad {}_0$. □

References

[1] Bager, A. “A family of goniometric inequalities”. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 338-352 (1971), pp. 5–25.

[2] Baiţan, D. “On some re nements of Gerretsen's inequality”. Arhimede 2 (2011), pp. 70–73.

[3] Bil cev, S. J. and Velikova, E. A. “Matem. i Matem. Obraz.” So a BAN (1988), pp. 626–628.

[4] Blundon, W. “Inequalities associated with the triangle”. Canad. Math. Bull. 8 (1966), pp. 615–626.

[5] Blundon, W. “Problem 1935”. Amer. Math. Monthly 73 (1966), p. 1122.

[6] Gerretsen, J. C. “Ongelijkheden in de Driehoek”. Nieuw Tijdschr, Wisk 41 (1953), pp. 1–7.

[7] Makovski, A. “Solution of problem E1935”. Amer. Math. Monthly 75 (1968), p. 404.

[8] Panaitopol, L. “A geometric inequality”. Gazeta Matematica Seria B 4 (1982), pp. 113–114.

[9] Radulescu, S., Dr agan, M., and Maftei, I. V. “Some consequences of W. J. Blundon's inequality”. Gazeta Matematica Seria B 1 (2011), pp. 3–9.

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About some trigonometric sums

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Abstract

In this short note, we will calculate some sums involving trigonometric functions, such as sine and cosine. Using these sums, some problems of the mathematical Olympiads will be very immediate to solve.

1 Main results

First, we will calculate some sums involving cosines:

Proposition 1. Let $t \in \mathbb{Z}$. The following identitites hold:

i) $$\sum_{k=1}^{n} \cos kx = \frac{\sin \frac{(2n+1)x}{2}}{2\sin \frac{x}{2}} - \frac{1}{2}, \text{ for } x \neq 2t\pi;$$

ii) $$\sum_{k=1}^{n} \cos(2k-1)x = \frac{\sin 2nx}{2\sin x}, \text{ for } x \neq t\pi;$$

iii) $$\sum_{k=1}^{n} k\cos kx = \frac{n\sin \frac{(2n-1)x}{2}}{2\sin \frac{x}{2}} - \frac{1-\cos nx}{4\sin^2 \frac{x}{2}}, \text{ for } x \neq 2t\pi;$$

iv) $$\sum_{k=1}^{n} (-1)^k \cos kx = -\frac{1}{2} + \frac{(-1)^n \cos \frac{(2n+1)x}{2}}{2\cos \frac{x}{2}}, \text{ for } x \neq (2t+1)\pi;$$

v) $$\sum_{k=1}^{n} \cos^2 kx = \frac{n}{2} + \frac{\cos(n+1)x \sin nx}{2\sin x}, \text{ for } x \neq t\pi;$$

vi) $\sum_{k=1}^{n} \cos^3 kx = \dfrac{3\cos\frac{(n+1)\ x}{2}\sin\frac{nx}{2}}{4\sin\frac{x}{2}} + \dfrac{\cos\frac{3(n+1)\ x}{2}\sin\frac{3nx}{2}}{4\sin\frac{3x}{2}}$, for $x \neq 2t\ ;\ \dfrac{2t}{3}$;

vii) $\sum_{k=1}^{n} \cos^4 kx = \dfrac{3n}{8}\quad \dfrac{\cos(n+1)\ x\ \sin nx}{2\sin x} + \dfrac{\cos 2(n+1)\ x\ \sin 2nx}{8\sin 2x}$, for $x \neq t\ ;\ \dfrac{t}{2}$.

Proof. i) If we multiply the sum by $2\sin\frac{x}{2}$ we get

$$\sum_{k=1}^{n} 2\sin\frac{x}{2}\cos kx = \sum_{k=1}^{n}\ \sin\ \ k+\frac{1}{2}\ \ x\quad \sin\ \ k\quad \frac{1}{2}\ \ x$$
$$= \sin\ \ n+\frac{1}{2}\ \ x\quad \sin\frac{x}{2}.$$

The result follows by dividing by $2\sin\frac{x}{2}$.

ii) First we notice equality

$$\sum_{k=1}^{n} [\cos(2k\quad 1)x + \cos(2\ k)x] = \sum_{k=1}^{2n} \cos kx\ .$$

By the previous identity, we get

$$\sum_{k=1}^{2n} \cos kx = \frac{\sin\frac{(4n+1)\ x}{2}}{2\sin\frac{x}{2}}\quad \frac{1}{2} = \frac{2\sin\frac{(4n+1)\ x}{2}\cos\frac{x}{2}}{2\sin x}\quad \frac{1}{2}$$
$$= \frac{\sin(2n+1)\ x + \sin 2\ nx}{2\sin x}\quad \frac{1}{2},$$

and

$$\sum_{k=1}^{n} \cos 2kx = \frac{\sin(2n+1)\ x}{2\sin x}\quad \frac{1}{2}.$$

The result follows easily.

iii) Differentiating the identity of Proposition 2 item i) in relation to x, we obtain the result.

iv) Observe that $(\quad 1)^k \cos kx = \cos k \quad \cos kx$, therefore using item i) we have

$$\sum_{k=1}^{n} (\quad 1)^k \cos kx = \frac{1}{2}\sum_{k=1}^{n} [\cos(x + \quad)k + \cos(x \quad)k]$$
$$= \frac{1}{2}\left[\frac{\sin\frac{(2n+1)(x+ \quad)}{2}}{2\sin\frac{(x+ \quad)}{2}} + \frac{\sin\frac{(2n+1)(x \quad)}{2}}{2\sin\frac{(x \quad)}{2}} \quad 1\right]. \qquad (\quad)$$

Simplifying, we get

$$\sin\frac{(x + \quad)}{2} = \cos\frac{x}{2}$$

and

$$\sin\frac{(2n + 1)(x + \quad)}{2} = \cos \quad nx + \frac{x}{2} + n \quad = (\quad 1)^n \cos\frac{(2n + 1) x}{2}.$$

Likewise,

$$\sin\frac{(x \quad)}{2} = \quad \cos\frac{x}{2}$$

and

$$\sin\frac{(2n + 1)(x \quad)}{2} = \quad \cos \quad nx + \frac{x}{2} \quad n$$
$$= (\quad 1)^{n+1} \cos\frac{(2n + 1) x}{2}.$$

Replacing these in ( ), the result follows immediately.

v) Using $\cos^2 x = (1 + \cos 2 \ x)=2$ we have

$$\sum_{k=1}^{n} \cos^2 kx = \frac{n}{2} + \frac{1}{2}\sum_{k=1}^{n} \cos 2kx = \frac{n}{2} + \frac{1}{2}\quad\frac{\sin(2n + 1) x}{2\sin x} \quad \frac{1}{2}$$
$$= \frac{n}{2} \quad \frac{\sin(2n + 1) x \quad \sin x}{4\sin x} = \frac{n}{2} + \frac{\cos(n + 1) x \sin nx}{2\sin x}.$$

vi) By the identity $4\cos^3 x = \cos 3x + 3\cos x$ we have

$$4\sum_{k=1}^{n}\cos^3 kx = \sum_{k=1}^{n}\cos 3kx + 3\sum_{k=1}^{n}\cos kx$$
$$= \frac{\sin\frac{(2n+1)3x}{2}}{2\sin\frac{3x}{2}} - \frac{1}{2} + 3\left(\frac{\sin\frac{(2n+1)x}{2}}{2\sin\frac{x}{2}} - \frac{1}{2}\right)$$
$$= \frac{3\cos\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin\frac{x}{2}} + \frac{\cos\frac{3(n+1)x}{2}\sin\frac{3nx}{2}}{\sin\frac{3x}{2}}.$$

vii) Just use the identity $8\cos^4 x = \cos 4x + 8\cos^2 x - 1$. □

The next proposition is analogous to the previous one. Just make the appropriate changes to the sine function.

Proposition 2. Let $t \in \mathbb{Z}$. We have that

i) $\displaystyle\sum_{k=1}^{n}\sin kx = \frac{\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}}{\sin\frac{x}{2}}$, for $x \neq 2t\pi$;

ii) $\displaystyle\sum_{k=1}^{n}\sin(2k-1)x = \frac{\sin^2 nx}{\sin x}$, for $x \neq t\pi$;

iii) $\displaystyle\sum_{k=1}^{n}(-1)^{k+1}\sin(2k-1) = (-1)^{n+1}\frac{\sin 2nx}{2\cos x}$, for $x \neq \frac{\pi}{2} + t\pi$;

iv) $\displaystyle\sum_{k=1}^{n}k\sin kx = \frac{\sin\frac{(2n+1)x}{2}\cos\frac{x}{2} - (2n+1)\cos\frac{(2n+1)x}{2}\sin\frac{x}{2}}{4\sin^2\frac{x}{2}}$, for $x \neq 2t\pi$;

v) $\displaystyle\sum_{k=1}^{n}\sin^2 kx = \frac{n}{2} - \frac{\cos(n+1)x\sin nx}{2\sin x}$, for $x \neq t\pi$;

vi) $\displaystyle\sum_{k=1}^{n}\sin^3 kx = \frac{3\sin\frac{(n+1)x}{2}\sin\frac{nx}{2}}{4\sin\frac{x}{2}} - \frac{\sin\frac{3(n+1)x}{2}\sin\frac{3nx}{2}}{4\sin\frac{3x}{2}}$, for $x \neq 2t\pi; \frac{2t\pi}{3}$;

vii) $\displaystyle\sum_{k=1}^{n}\sin^4 kx = \frac{3n}{8} + \frac{\cos(n+1)x\sin nx}{2\sin x} + \frac{\cos 2(n+1)x\sin 2nx}{8\sin 2x}$, for $x \neq t\pi; \frac{t\pi}{2}$.

Proof. Left as an exercise to the reader. □

2 Aplications

At this point we will see how these propositions can help to solve some problems. The vast majority of these problems can be solved using complex numbers, formulas of sum-to-product transformations, etc. The rst example was proposed in the International Mathematical Olympiad in 1963.

Example 1 (IMO-63). Prove that $\cos\frac{}{7}\quad \cos\frac{2}{7} + \cos\frac{3}{7} = \frac{1}{2}$.

Solution 1. By Proposition 1, i), for $x = \ = 7$ and $n = 3$ we get

$$\cos\frac{}{7} + \cos\frac{2}{7}\quad \cos\frac{3}{7} = \quad \frac{1}{2}\quad \frac{\cos\frac{7}{14}}{2\cos\frac{}{14}} = \quad \frac{1}{2},$$

because $\cos\frac{}{2} = 0$. The result follows by multiplying the equation by 1. □

Solution 2. By Proposition 1, iv), for $x = 2 \ = 7$ and $n = 3$ we have

$$\cos\frac{2}{7} + \cos\frac{4}{7} + \cos\frac{6}{7} = \quad \frac{1}{2}.$$

But $\cos\frac{}{7} = \quad \cos\frac{6}{7}$ and $\cos\frac{3}{7} = \quad \cos\frac{4}{7}$, so

$$\cos\frac{2}{7}\quad \cos\frac{2}{7}\quad \cos\frac{}{7} = \quad \frac{1}{2}.$$

The result follows by multiplying the equation by 1. □

Example 2. Prove that

$$\cos\frac{}{11} + \cos\frac{3}{11} + \cos\frac{5}{11} + \cos\frac{7}{11} + \cos\frac{9}{11} = \frac{1}{2}.$$

Solution. By Proposition 1, ii), for $x = \ = 11$ and $n = 5$ we have

$$\cos\frac{}{11} + \cos\frac{3}{11} + \cos\frac{5}{11} + \cos\frac{7}{11} + \cos\frac{9}{11} = \frac{\sin\frac{10}{11}}{2\sin\frac{}{11}} = \frac{1}{2},$$

since $\frac{10}{11} + \frac{}{11} =$, and so, $\sin \frac{10}{11} = \sin \frac{}{11}$. □

The following example was proposed in the Mexican selection test for IMC 2016.

Example 3 (Mexico TST 2016). Determine the value of the sum

$$\sin^2 4 + \sin^2 8 + \sin^2 12 + \quad + \sin^2 176 .$$

Solution. By Proposition 2, v), for $x = 4$ and $n = 44$ we have

$$\sum_{k=1}^{44} \sin^2 4k = \frac{44}{2} + \frac{\cos 180 \ \sin 176}{2 \sin 4} = \frac{44}{2} + \frac{1}{2} = \frac{45}{2}.$$ □

The next problem was proposed in the American Invitational Mathematics Examination in 1999.

Example 4 (AIME-99). Given that

$$\sum_{k=1}^{35} \sin 5k = \tan \frac{m}{n},$$

where angles are measured in degrees, and m and n are relatively prime positive integers that satisfy $\frac{m}{n} < 90$, nd $m + n$.

Solution. By Proposition 2, i), for $x = 5$ and $n = 35$ we get

$$\sum_{k=1}^{35} \sin 5k = \frac{\sin 90 \sin \frac{175}{2}}{\sin \frac{5}{2}} = \frac{\sin \frac{175}{2}}{\cos \frac{175}{2}} = \tan \frac{175}{2},$$

because $\frac{175}{2} + \frac{5}{2} = 90$. The answer is $m + n = 177$. □

By Proposition 1, we can obtain the exact value of the trigonometric ratios of 72 degrees.

Example 5. Find $\sin 72$ and $\cos 72$.

Solution. By Proposition 1, i), for $x = \frac{2}{5}$ and $n = 2$ we get

$$\cos \frac{2}{5} + \cos \frac{4}{5} = \frac{1}{2},$$

so

$$\cos\frac{2}{5} + 2\cos^2\frac{2}{5} \quad 1 = \quad \frac{1}{2}.$$

Consider $y = \cos\frac{2}{5} > 0$. We have that

$$2y^2 + y = \frac{1}{2} \implies 4y^2 + 2y \quad 1 = 0,$$

so $y = \frac{1 \quad \sqrt{5}}{4}$. Therefore, using the fact that $\sin^2 \quad + \cos^2 \quad = 1$, we get

$$\cos\frac{2}{5} = \frac{\sqrt{5} \quad 1}{4},$$
$$\sin\frac{2}{5} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}. \qquad \square$$

Example 6. Compute the following sums:

i) $\cos^2 10 + \cos^2 50 + \cos^2 70$;
ii) $\sin^2 10 + \sin^2 50 + \sin^2 70$.

Solution. i) Observe that

$$\cos^2 10 + \cos^2 50 + \cos^2 70 = \sin^2 80 + \sin^2 40 + \sin^2 20 .$$

By Proposition 2, v), for $x = 20$ and $n = 4$ we have

$$\begin{aligned}\sin^2 20 + \sin^2 40 + \sin^2 60 + \sin^2 80 &= 2 \quad \frac{\cos 100 \quad \sin 80}{2\sin 20}\\ &= 2 \quad \frac{\cos 100 \quad \sin 80}{2\sin 160}\\ &= 2 \quad \frac{\cos 80 \quad \sin 80}{4\sin 80 \quad \sin 80} = \frac{9}{4}.\end{aligned}$$

Now, $\sin^2 60 = 3=4$ and, therefore,

$$\sin^2 80 + \sin^2 40 + \sin^2 20 = \frac{3}{2}.$$

ii) From i), $\sin^2 10 + \sin^2 50 + \sin^2 70 = \frac{3}{2}$. $\square$

The next problem was proposed in the United States of America Mathematical Olympiad in 1996.

Example 7 (USAMO-96). Prove that the average of the numbers $n \sin n$ $(n = 2; 4; 6; \ldots; 180)$ is $\cot 1$.

Solution. By Proposition 2, iv), for $x = 2$ and $n = 90$ we get

$$S = 2 \quad \frac{\sin 181 \quad \cos 1 \qquad 181 \cos 181 \quad \sin 1}{4 \sin^2 1}$$
$$= \frac{1}{2} \quad \frac{181 \cos 1 \quad \sin 1 \qquad \sin 1 \quad \cos 1}{\sin^2 1}$$
$$= \frac{1}{2} \frac{180 \sin 1 \quad \cos 1}{\sin^2 1} = 90 \cot 1 \quad . \qquad \square$$

The following example was proposed in the Iberoamerican Olympiad for University Students in 2015.

Example 8 (OIMU-2015). Prove that

$$\cos \frac{2}{7} \cos \frac{4}{7} + \cos \frac{4}{7} \cos \frac{8}{7} + \cos \frac{8}{7} \cos \frac{2}{7} = \quad \frac{1}{2}.$$

Solution. Due to equalities $\cos \frac{8}{7} = \quad \cos \frac{}{7}$, and $\cos \frac{3}{7} = \quad \cos \frac{4}{7}$, and by Example 2, we have already proved that

$$\cos \frac{2}{7} + \cos \frac{4}{7} + \cos \frac{8}{7} = \quad \frac{1}{2}.$$

Now, by Proposition 1, v), with $n = 3$ and $x = 2 \quad = 7$ we have

$$\cos^2 \frac{2}{7} + \cos^2 \frac{4}{7} + \cos^2 \frac{8}{7} = \frac{5}{4}.$$

Using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$, we nally get

$$\cos \frac{2}{7} \cos \frac{4}{7} + \cos \frac{4}{7} \cos \frac{8}{7} + \cos \frac{8}{7} \cos \frac{2}{7} = \quad \frac{1}{2}. \qquad \square$$

Example 9. Prove that

$$\sin \frac{2}{7} + \sin \frac{4}{7} + \sin \frac{8}{7} = \frac{\sqrt{7}}{2}.$$

Solution. By Proposition 2, v), for $x = \; = 7$ and $n = 7$ we have

$$\sin^2 \frac{}{7} + \sin^2 \frac{2}{7} + \sin^2 \frac{3}{7} = \frac{7}{4}$$

and

$$\sin \frac{}{7} \quad \sin \frac{2}{7} + \sin \frac{3}{7} \quad = 2 \sin \frac{}{7} \sin \frac{5}{14} \cos \frac{}{14}$$
$$= 2 \sin \frac{}{7} \cos \frac{2}{14} \sin \frac{6}{14}$$
$$= \sin \frac{2}{7} \sin \frac{3}{7}.$$

Using the identity $(a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$ for $a = \frac{}{7}$, $b = \frac{2}{7}$ and $c = \frac{3}{7}$, we nd that $a^2 + b^2 + c^2 = 7 =4$ and $ab + bc + ca = 0$. Therefore,

$$0 < \sin \frac{2}{7} + \sin \frac{3}{7} \quad \sin \frac{}{7} = \frac{\sqrt{7}}{2}.$$ □

Example 10. Prove that

$$\sin \frac{}{7} \sin \frac{2}{7} \sin \frac{3}{7} = \frac{\sqrt{7}}{8}.$$

Solution. To simplify the notation, let $a = \cos \frac{}{7}$, $b = \cos \frac{2}{7}$ and $c = \cos \frac{3}{7}$. So it is enough to prove that

$$\sin^2 \frac{}{7} \sin^2 \frac{2}{7} \sin^2 \frac{3}{7} = \frac{7}{64}$$
$$\Rightarrow \quad 1 \quad \cos^2 \frac{}{7} \quad 1 \quad \cos^2 \frac{2}{7} \quad 1 \quad \cos^2 \frac{3}{7} = \frac{7}{64}.$$

We know from Example 9 that $a^2 + b^2 + c^2 = \frac{5}{4}$ and, by Proposition 1, vii), with $n = 3$, $x = \frac{2}{7}$ and $a^4 + b^4 + c^4 = \frac{13}{16}$, then

$$a^2b^2 + b^2c^2 + c^2a^2 = \frac{1}{2}[(a^2 + b^2 + c^2)^2 \quad (a^4 + b^4 + c^4)].$$

This implies that $a^2b^2 + b^2c^2 + c^2a^2 = \frac{3}{8}$. Now, $abc = \frac{1}{8}$. Indeed,

$$\cos\frac{}{7}\cos\frac{2}{7}\cos\frac{3}{7} = \frac{\sin\frac{2}{7}\cos\frac{2}{7}\cos\frac{4}{7}}{2\sin\frac{2}{7}} = \frac{\sin\frac{4}{7}\cos\frac{4}{7}}{4\sin\frac{}{7}}$$

$$= \frac{\sin\frac{8}{7}}{8\sin\frac{}{7}} = \frac{\sin\ \ +\frac{}{7}}{8\sin\frac{}{7}} = \frac{1}{8}.$$

Finally, using the identity

$$(1\quad a^2)(1\quad b^2)(1\quad c^2) = 1\quad (a^2+b^2+c^2)+(\ a^2b^2+b^2c^2+c^2a^2)\quad a^2b^2c^2,$$

we conclude that

$$(1\quad a^2)(1\quad b^2)(1\quad c^2) = 1\quad \frac{5}{4} + \frac{3}{8}\quad \frac{1}{64} = \frac{7}{64}.$$

This ends the proof. □

Example 11. Prove that

$$\cos\frac{2}{21} + \cos\frac{8}{21} + \cos\frac{10}{21} = \frac{1+\sqrt{21}}{4}.$$

Solution. Denote $S = \cos\frac{2}{21} + \cos\frac{8}{21} + \cos\frac{10}{21}$. Then,

$$S = \cos\ \frac{2}{21}\quad \cos\ \frac{13}{21}\ + \cos\ \frac{10}{21}$$

$$= \cos\ \frac{}{3}\ \frac{3}{7}\quad \cos\ \frac{}{3} + \frac{2}{7}\ + \cos\ \frac{}{3} + \frac{}{7}\ .$$

Using the well-known formulas $\cos(a\quad b) = \cos a\cos b\quad \sin a\sin b$, we obtain, after due simpli cation,

$$S = \cos\frac{}{3}\ \cos\frac{}{7}\quad \cos\frac{2}{7} + \cos\frac{3}{7}\ + \sin\frac{}{3}\ \sin\frac{3}{7} + \sin\frac{2}{7}\quad \sin\frac{}{7}$$

$$= \frac{1}{2}\ \frac{1}{2} + \frac{\sqrt{3}}{2}\ \frac{\sqrt{7}}{2} = \frac{1+\sqrt{21}}{4},$$

where we used the results of Examples 1 and 9, respectively. □

3 Independent study problems

This last section is devoted to stating some problems for the reader. All these can be solved with the techniques covered in this note.

Problem 1. Compute the sums

$$\sum_{k=1}^{n-1} \sin\frac{k\pi}{n} \quad \text{and} \quad \sum_{k=1}^{n-1} \sin^2\frac{k\pi}{n}.$$

Problem 2. Prove that $\cos 2^\circ + \cos 4^\circ + \cos 6^\circ + \cdots + \cos 178^\circ = 0$.

Problem 3 (IMO-1962). Solve the equation

$$\cos^2 x + \cos^2 2x + \cos^2 3x = 1.$$

Problem 4 (AIME-1997). Let

$$x = \frac{\sum_{n=1}^{44} \cos n^\circ}{\sum_{n=1}^{44} \sin n^\circ}.$$

What is the greatest integer that does not exceed $100x$?

Problem 5. Compute the following sums:

i) $\cos^4 4^\circ + \cos^4 8^\circ + \cos^4 12^\circ + \cdots + \cos^4 176^\circ$;
ii) $\sin^4 4^\circ + \sin^4 8^\circ + \sin^4 12^\circ + \cdots + \sin^4 176^\circ$.

Problem 6. Prove that

$$\cos^3\frac{2\pi}{7} + \cos^3\frac{4\pi}{7} + \cos^3\frac{8\pi}{7} = -\frac{1}{2}.$$

Problem 7 (USSR). Show that

$$\cos\frac{2\pi}{2n+1} + \cos\frac{4\pi}{2n+1} + \cos\frac{6\pi}{2n+1} + \cdots + \cos\frac{2n\pi}{2n+1} = -\frac{1}{2}.$$

Problem 8. Prove that

$$\cot\frac{\pi}{14} - 4\sin\frac{\pi}{7} = \sqrt{7}.$$

Problem 9. For every positive integer n, prove that

$$\sum_{k=0}^{n-1} (-1)^k \cos^n \frac{k\pi}{n} = \frac{n}{2^{n-1}}.$$

Problem 10 (REOIM [2]). Evaluate

$$\lim_{n\to\infty} \frac{\cos\frac{\pi}{2n} + \cos\frac{2\pi}{2n} + \cdots + \cos\frac{n\pi}{2n}}{\frac{\pi}{2n} + \frac{2\pi}{2n} + \cdots + \frac{n\pi}{2n}}.$$

Problem 11. Prove that

$$\cos\frac{2\pi}{13} + \cos\frac{6\pi}{13} + \cos\frac{8\pi}{13} = \frac{\sqrt{13}-1}{4}.$$

References

[1] Andreescu, T. and Andrica, D. *Complex Numbers from A to Z*. Birkhaüser, 2005.

[2] Bellot, F. "Problemas Propuestos: Problema 2 suggested by F. Bellot (Absolutorial Aufgabe, Baviera 1874)". *Revista Escolar de la Olimpiada Iberoamericana de Matemática (REOIM)* 1 (2002).

[3] Gradshteyn, I. S. and Ryzhik, I. M. *Table of Integrals, Series, and Products*. 8th. Academic Press, Inc., 1980.

[4] Parvardi, A. H. *Trigonometry*. Problems-art of problem solving, 2011.

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Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.

2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to José Luis Díaz-Barrero, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to
jose.luis.diaz@upc.edu

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

October 31, 2019

Elementary Problems

E–65. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. In a deck of cards there are six diamond cards, five club cards, four heart cards and three spade cards. We choose some of them, in such a way that we pick at least one card from each suit. In how many ways can we make such a choice?

E–66. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Find the number of all positive divisors of 46189^{12} which are not cubes of a positive integer.

E–67. Proposed by José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain. Find all polynomials $A(x)$ of degree 4 such that $A(1) = A(-2) = A(3) = A(-4) = 3$ and $A(0) = 6$.

E–68. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Find all positive integers a and b such that $ab = 10648$ and $\operatorname{lcm}\{a; b\} = 968$.

E–69. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let ABC be a triangle such that $3\angle A = 2\angle B$. Bisector of BC meets CA at point X. If $AB = BX$, then find the measure of the angles of triangle ABC.

E–70. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. A 2019 side square is divided, by drawing parallel lines to the sides, into 4076361 equal squares. What is the total number of squares that appear in this figure?

Easy–Medium Problems

EM–65. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let n be a positive integer. Show that

$$(n^7 - n)(7^{14n+7} + 11^{2n+1})$$

is a multiple of 4242.

EM–66. Proposed by Mihaela Berindeanu, Bucharest, România. Let a, b, c be reals larger than or equal to one. Prove that

$$\sum_{\text{cyclic}} \frac{a^{2019}bc + 2019}{bc} \geq 3\left(1 + \frac{6057}{a+b+c}\right).$$

EM–67. Proposed by Oriol Baeza Guasch, Institut de Terrassa, Terrassa, Spain. Let ABC be a triangle with incenter I. Let A' be the intersection of ray AI with side BC. Define B' and C' similarly. Then, prove that

$$AI \cdot BI \cdot CI \geq 8\, A'I \cdot B'I \cdot C'I.$$

When does equality occur?

EM–68. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Without the aid of a computer, show that

$$\left[\left(\arctan\frac{1}{9}\right)^2 + \left(\arctan\frac{4}{5}\right)^2 + \frac{\pi^2}{16}\right]^2 = 2\left[\left(\arctan\frac{1}{9}\right)^4 + \left(\arctan\frac{4}{5}\right)^4 + \frac{\pi^4}{256}\right].$$

EM–69. Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let a, b, c and A, B, C be the lengths of the sides and the measure in radians of the angles of an acute triangle ABC. Prove that

$$\frac{a}{\tan B \cdot \tan C} + \frac{b}{\tan A \cdot \tan C} + \frac{c}{\tan A \cdot \tan B} \geq \frac{2}{3}s,$$

where s is the semiperimeter of triangle ABC.

EM–70. Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Without the aid of a computer, show that $x^4 + x^3 + x^2 + x + 1$ divides

$$A(x) = x^{99} + 3x^{98} + x^{97} + 2x^{54} + x^{52} + x^{27} + x^6 + 2x + 3,$$

and nd the remainder of the division of $A(x)$ by $x^2 \quad 1$.

Medium–Hard Problems

MH–65. Proposed by Oriol Baeza Guasch, Institut de Terrassa, Terrassa, Spain. Let ABC be a triangle such that $\angle CAB = 2\angle ABC$. Denote the incenter by I and the circumcircle by Ω. From a point P on Ω, drop perpendiculars to lines AC, AI, AB, and denote by X, Y, Z the respective feet. Let M be the intersection of lines XY and PZ. Show that lines CM, AY and the perpendicular bisector of BC are concurrent.

MH–66. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let a, b, c, d be the roots of $x^4+6x^3+7x^2+6x+1=0$. Find the value of

$$\frac{7-2a}{1-a}+\frac{7-2b}{1-b}+\frac{7-2c}{1-c}+\frac{7-2d}{1-d}.$$

MH–67. Proposed by Mihaela Berindeanu, Bucharest, România. Let a, b, c, d be positive numbers such that $abcd = 16$. Show that

$$a^2\sqrt{b+c}+b^2\sqrt{c+d}+c^2\sqrt{d+a}+d^2\sqrt{a+b} \le \frac{a^5+b^5+c^5+d^5}{4}.$$

MH–68. Proposed by Pedro Henrique O. Pantoja, Natal/RN, Brazil. Are there $m, n, k, s \in \mathbb{N}$, $s > 2$, such that

$$\frac{(127m-1)(127n+1)}{2^k} = 486\underbrace{11\ldots11}_{s}\cdot 19\underbrace{44\ldots44}_{s+2}10?$$

MH–69. Proposed by Mihály Bencze, Braşov, Romania. Find all real solutions of the equation

$$3^x + 2\cdot 5^x + 7^x = \frac{95}{102}|x|^3.$$

MH–70. Proposed by Mihaela Berindeanu, Bucharest, România. Let ABC be a triangle with incircle ω and incenter I. The tangent point between and BC is D , and between AC and is E . The tangent to from a point X 2 AE cuts AB in Y . Let Z 2 BC be a point so that BZ = AB . Show that if Y , I , Z are collinear points, then X , I , D are also collinear points.

Advanced Problems

A–65. Proposed by Marc Felipe Alsina, BarcelonaTech, Barcelona, Spain. Let $f : \mathbb{R}^+ \to \mathbb{R}$ be a function satisfying

$$|f(a+b) - f(a)| \leq \frac{b}{a}$$

for all positive reals a and b. Show that $|f(x) - f(1)| \leq |\ln x|$ for all $x \in \mathbb{R}^+$.

A–66. Proposed by Mihaela Berindeanu, Bucharest, România. Let $A, B \in M_2(\mathbb{Z})$ be two square matrices satisfying the following properties: $AB = BA$, $\det(A+B) = 3$ and $\det(A^3+B^3) = 9$. Prove that $\det(A^2+B^2) \in \{-5, 1\}$.

A–67. Proposed by Rica Zamfir, Bucharest, România. Consider a continuous function $f : [a,b] \to \mathbb{R}$ satisfying that $\int_a^b f(x)\,dx = 0$. Show that there exists a real number $c \in (a,b)$ such that

$$\int_a^c f(x)\,dx = \left(\frac{a+b}{2} - c\right) f(c).$$

A–68. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Without the aid of a computer, find the value of

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{k^2+n^2}{n^3}\right).$$

A–69. Proposed by Pedro Henrique O. Pantoja, Natal/RN, Brazil.

(i) Find all matrices $A \in M_{2\times 2}(\mathbb{Z})$ with trace equal to zero such that

$$(A - 3I)(A^2 + A + 3I) = \begin{pmatrix} 2 & 1 \\ 80 & 20 \end{pmatrix}.$$

(ii) Let A be a matrix satisfying (i). Compute A^{2020}.

A–70. Proposed by Mihály Bencze, Braşov, România. Let G be the set of positive reals. We define the composition of any two $x, y \in G$ by

$$a^x + a^{x\circ y} + a^y = 2 + a^{x+y},$$

where $a > 1$. Prove that if $x_1, x_2, \ldots, x_n$ are elements of G, then

$$\frac{n(n-1)}{2} + \sum_{1 \le i<j \le n} \sqrt{a^{x_i\circ x_j} - 1} \le \frac{n-1}{2} \sum_{k=1}^{n} a^{x_k}.$$

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to José Luis Díaz-Barrero, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Chebyshev's inequality and some of its applications

Marc Felipe Alsina and José Luis Díaz-Barrero

1 Introduction

A classical inequality that can be used to get new ones is the well-known Chebyshev's inequality. It states that the product of the arithmetic means of two monotonic sequences is always at most the arithmetic mean of their term-wise product, or always greater than or equal to it, depending on whether they have the same monotonicity or not. In this note, we present two proofs of this inequality. In the first one, Chebyshev's inequality is obtained from an identity of Korkin [2], and in the second one, the inequality is obtained as a consequence of the rearrangement inequality. Finally, we give some applications of this inequality.

2 Chebyshev's inequality

Hereafter, we present the two results that will be used to prove Chebyshev's inequality. We start by stating and proving Korkin's identity.

Theorem 1 (Korkin's inequality). Given any two sequences of real numbers $a_1; a_2; \ldots; a_n$ and $b_1; b_2; \ldots; b_n$, the following holds:

$$\frac{1}{n}\sum_{k=1}^{n} a_k b_k = \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) + \frac{1}{n^2}\sum_{1 \le i<j \le n} (a_i - a_j)(b_i - b_j).$$

Proof. To prove the preceding identity, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i - a_j)(b_i - b_j)\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}(a_i b_i + a_j b_j - a_i b_j - a_j b_i)\\
&= n\sum_{i=1}^{n} a_i b_i + n\sum_{j=1}^{n} a_j b_j - \left(\sum_{j=1}^{n} b_j\right)\left(\sum_{i=1}^{n} a_i\right) - \left(\sum_{i=1}^{n} b_i\right)\left(\sum_{j=1}^{n} a_j\right)\\
&= 2n\sum_{k=1}^{n} a_k b_k - 2n^2\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right),
\end{aligned}$$

from which, by dividing both sides by $2n^2$, we obtain

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n} a_k b_k &= \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) + \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}(a_i - a_j)(b_i - b_j)\\
&= \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) + \frac{1}{n^2}\sum_{1\le i<j\le n}(a_i - a_j)(b_i - b_j),
\end{aligned}$$

and we are done. □

The second result is the rearrangement inequality. It is stated as follows.

Theorem 2 (Rearrangement inequality). Let $a_1; a_2; \dots; a_n$ and $b_1; b_2; \dots; b_n$ be sequences of positive real numbers satisfying $a_1 \ge a_2 \ge \cdots \ge a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$, and let $c_1; c_2; \dots; c_n$ be a permutation of $b_1; b_2; \dots; b_n$. The sum $S = a_1c_1 + a_2c_2 + \cdots + a_nc_n$ attains its maximum when $c_1 \ge c_2 \ge \cdots \ge c_n$, and it attains its minimum when $c_1 \le c_2 \le \cdots \le c_n$.

Proof. Let $a_i < a_j$. Consider the sums

$$\begin{aligned}
S_1 &= a_1c_1 + \cdots + a_ic_i + \cdots + a_jc_j + \cdots + a_nc_n,\\
S_2 &= a_1c_1 + \cdots + a_ic_j + \cdots + a_jc_i + \cdots + a_nc_n.
\end{aligned}$$

We have obtained S_2 from S_1 by switching the positions of c_i and c_j. Then,

$$S_1 - S_2 = a_ic_i + a_jc_j - a_ic_j - a_jc_i = (a_j - a_i)(c_j - c_i).$$

Therefore,

$$c_j < c_i \Rightarrow S_1 < S_2 \quad \text{and} \quad c_j > c_i \Rightarrow S_1 > S_2.$$

If S_1 is maximal, then it cannot happen that $a_i < a_j$ and $c_j < c_i$, so we will have that $S_1 = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$. If S_1 is minimal, then it cannot happen that $a_i < a_j$ and $c_j > c_i$, so we will have that $S_1 = a_1 b_n + a_2 b_n + \cdots + a_n b_n$. This completes the proof. □

Now, we state and proof the main result of this note.

Theorem 3 (Chebyshev). If $a_1 \geq a_2 \geq \cdots \geq a_n$, $b_1 \geq b_2 \geq \cdots \geq b_n$, then

$$\frac{1}{n}\sum_{k=1}^{n} a_k b_{n+1-k} \leq \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) \leq \frac{1}{n}\sum_{k=1}^{n} a_k b_k.$$

Proof 1. Both parts of Chebyshev's inequality immediately follow from the identity of Korkin. Indeed, we have that

$$\frac{1}{n}\sum_{k=1}^{n} a_k b_k = \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) + \frac{1}{n^2}\sum_{1 \leq i<j \leq n} (a_i - a_j)(b_i - b_j)$$
$$\geq \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right)$$

because the term

$$\frac{1}{n^2}\sum_{1 \leq i<j \leq n} (a_i - a_j)(b_i - b_j)$$

is non-negative. Likewise,

$$\frac{1}{n}\sum_{k=1}^{n} a_k b_{n+1-k}$$
$$= \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right) + \frac{1}{n^2}\sum_{1 \leq i<j \leq n} (a_i - a_j)(b_{n+1-i} - b_{n+1-j})$$
$$\leq \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)\left(\frac{1}{n}\sum_{k=1}^{n} b_k\right)$$

because the term

$$\frac{1}{n^2}\sum_{1\le i<j\le n}(a_i-a_j)(b_{n+1-i}-b_{n+1-j})$$

is non-positive. □

The second proof is an immediate consequence of the rearrangement inequality.

Proof 2. We have that

$$
\begin{aligned}
a_1b_n+\cdots+a_nb_1 &= a_1b_1+a_2b_2+\cdots+a_nb_n \le a_1b_1+\cdots+a_nb_n,\\
a_1b_n+\cdots+a_nb_1 &\le a_1b_2+a_2b_3+\cdots+a_nb_1 \le a_1b_1+\cdots+a_nb_n,\\
a_1b_n+\cdots+a_nb_1 &\le a_1b_3+a_2b_4+\cdots+a_nb_2 \le a_1b_1+\cdots+a_nb_n,\\
&\;\;\vdots \qquad\qquad\qquad\qquad\qquad \vdots\\
a_1b_n+\cdots+a_nb_1 &\le a_1b_n+a_2b_1+\cdots+a_nb_{n-1} \le a_1b_1+\cdots+a_nb_n.
\end{aligned}
$$

Adding up the preceding inequalities, we have

$$n(a_1b_n+\cdots+a_nb_1)\le(a_1+\cdots+a_n)(b_1+\cdots+b_n)\le n(a_1b_1+\cdots+a_nb_n).$$

The statement follows immediately from the above inequality, and the proof is complete. □

Remark 1. Notice the following.

a. Equality holds if and only if $a_1=a_2=\cdots=a_n$ or $b_1=b_2=\cdots=b_n$.

b. If $a_1,a_2,\ldots,a_n$ and $b_1,b_2,\ldots,b_n$ are monotonic non-increasing sequences (instead of monotonic non-decreasing), then the inequality also holds. If both sequences have different monotonicity, then the inequality is reversed.

c. A generalization is the weighted form of Chebyshev's inequality. It is presented here without proof: under the same hypotheses, the following holds:

$$\frac{p_1 a_1b_n+p_2 a_2b_{n-1}+\cdots+p_n a_nb_1}{p_1+p_2+\cdots+p_n}\le\frac{p_1a_1+p_2a_2+\cdots+p_na_n}{p_1+p_2+\cdots+p_n}\cdot\frac{p_1b_1+p_2b_2+\cdots+p_nb_n}{p_1+p_2+\cdots+p_n}\le\frac{p_1 a_1b_1+p_2 a_2b_2+\cdots+p_n a_nb_n}{p_1+p_2+\cdots+p_n},$$

where $p_1; p_2; :::; p_n > 0$. When all the weights are equal ($p_1 = p_2 = ::: = p_n$), we recover the original Chebyshev inequality.

Other extensions of Chebyshev's inequality can be found in a paper of Bencze [1].

3 Applications

In what follows, we give some applications. We begin using Chebyshev's inequality to prove the well-known inequality of Nesbitt [3].

Problem 1. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \quad \frac{3}{2}.$$

Solution. WLOG we may assume that $a \quad b \quad c$. Then, we have $a+b \quad a+c \quad b+c$ and

$$\frac{1}{b+c} \quad \frac{1}{c+a} \quad \frac{1}{a+b}.$$

On account of Chebyshev's inequality, we have

$$3 \quad \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \qquad (a+b+c) \quad \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \quad .$$

Now, it will suf ce to prove that

$$(a+b+c) \quad \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \qquad \frac{9}{2}.$$

The above can be written in an equivalent form as

$$2(a+b+c) \quad \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \qquad 9,$$

or

$$[(a+b)+(b+c)+c+a)] \quad \frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b} \qquad 9,$$

which is an immediate consequence of the AM-HM inequality. Equality holds when $a = b = c$, and we are done. □

Another example is the following.

Problem 2. Let $x_1, x_2, \ldots, x_n$ be positive real numbers. For any integer $k \geq 2$, show that

$$(x_1^k + x_2^k + \cdots + x_n^k)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n(x_1^{k-1} + x_2^{k-1} + \cdots + x_n^{k-1}).$$

Solution. First, we write the given inequality as

$$\frac{x_1^k + x_2^k + \cdots + x_n^k}{x_1^{k-1} + x_2^{k-1} + \cdots + x_n^{k-1}} \geq n\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)^{-1}$$

and, inserting the term $\frac{x_1 + x_2 + \cdots + x_n}{n}$, we get

$$\frac{x_1^k + x_2^k + \cdots + x_n^k}{x_1^{k-1} + x_2^{k-1} + \cdots + x_n^{k-1}} \geq \frac{x_1 + x_2 + \cdots + x_n}{n} \geq n\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right)^{-1}.$$

The RHS inequality is a consequence of the AM-HM inequality. To prove the LHS inequality, we observe that the sequences $x_1^k, x_2^k, \ldots, x_n^k$ and $x_1^{k-1}, x_2^{k-1}, \ldots + x_n^{k-1}$ are sorted in the same way. Then, on account of Chebyshev's inequality, we get

$$n(x_1^k + x_2^k + \cdots + x_n^k) \geq (x_1 + x_2 + \cdots + x_n)(x_1^{k-1} + x_2^{k-1} + \cdots + x_n^{k-1}),$$

from which the inequality to be proven follows. Equality holds when $x_1 = x_2 = \cdots = x_n$. □

We close this section by applying the preceding result for solving a nonlinear system of equations in the reals.

Problem 3. Find all positive solutions of the following system of equations:

$$\begin{cases} ab + bc + ca = abc, \\ a^3 + b^3 + c^3 = 3(a^2 + b^2 + c^2). \end{cases}$$

Solution. By inspection we see that $a = b = c = 3$ is a solution. To see that it is unique, on account of Problem 2, we have that, if a, b, c are positive reals, then

$$(a^3 + b^3 + c^3)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \quad 3(a^2 + b^2 + c^2).$$

Since $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$, then the preceding inequality becomes

$$a^3 + b^3 + c^3 \quad 3(a^2 + b^2 + c^2).$$

If a, b and c also satisfy the second equality, we have a case of equality. Equality only holds when $a = b = c$, for which, in order to satisfy the rst equality as well, they must be equal to 3 and, therefore, the only solution of the given system is

$$(a; b; c) = (3 ; 3; 3).$$ □

References

[1] Bencze, M. "A generalization of Chebyshev's inequality". *Arhimede math. j.* 1 (2014), pp. 46–51.

[2] Korkine, A. N. "Sur un théorème de M. Tchebychef". *C. R. Acdad. Sci. Paris* 96 (1883), pp. 326–327.

[3] Nesbitt, A. M. "Problem 15114". *Educational Times* 3 (1903).

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Contests

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to nd interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to José Luis Díaz-Barrero, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Problems and solutions from the 6th edition of the BarcelonaTech Mathcontest

Óscar Rivero Salgado

1 Problems and solutions

Hereafter, we present the four problems that appeared in the paper given to the contestants of Mathcontest 2019, as well as their official solutions.

Problem 1. Let $p \geq 3$ be a prime number. If $a < b$ are positive integers, then find the value of $\gcd(p^{2^a}+1, p^{2^b}+1)$.

Solution. Using induction on b, we have

$$p^{2^b}-1=(p-1)(p+1)(p^2+1)(p^{2^2}+1)\cdots(p^{2^{b-1}}+1),$$

and since $a < b$ then $p^{2^a}+1$ is a divisor of $p^{2^b}-1$. Thus, the gcd of $p^{2^a}+1$ and $p^{2^b}+1$ divides both $p^{2^b}-1$ and $p^{2^b}+1$. Therefore, it divides their difference

$$(p^{2^b}+1)-(p^{2^b}-1)=2,$$

and it is exactly 2.

Problem 2. Let D, E, F denote the feet of the altitudes of an acute triangle ABC, and let $(X_1; X_2)$, $(Y_1; Y_2)$, $(Z_1; Z_2)$ denote the feet of perpendiculars from D, E, F, respectively, upon the other two sides of the triangle. Prove that the six points X_1, X_2, Y_1, Y_2, Z_1, Z_2 lie on a circle.

Solution. Let H be the orthocenter of $\triangle ABC$, and let Z_2 and Y_1 lie on BC, X_2 and Z_1 on CA, and Y_2 and X_1 on AB, as shown in the figure below.

Figure 1: Scheme for solving Problem 2.

From the fact that $\triangle BDX_1 \sim \triangle BFZ_2$, we get

$$\frac{BX_1}{BZ_2} = \frac{BD}{BF};$$

since $\triangle BEY_2 \sim \triangle BHF$, we have

$$\frac{BY_2}{BF} = \frac{BE}{BH} \iff \frac{BY_2}{BE} = \frac{BF}{BH}.$$

Likewise, from the fact that $\triangle BEY_1 \sim \triangle BHD$, we obtain

$$\frac{BY_1}{BD} = \frac{BE}{BH} \iff \frac{BY_1}{BE} = \frac{BD}{BH}.$$

Dividing up the last two equalities yields

$$\frac{BY_1}{BY_2} = \frac{BD}{BF} = \frac{BX_1}{BZ_2} \iff BY_1 \cdot BZ_2 = BX_1 \cdot BY_2.$$

Hence, Y_2, X_1, Z_2, Y_1 lie on a circle Γ_1; similarly, Z_2, Y_1, X_2, Z_1 lie on a circle Γ_2, and X_2, Z_1, Y_2, X_1 lie on a circle Γ_3. If Γ_1 and Γ_2, for example, coincide, the statement is proved. If they are distinct their radical axis is the line BC. Thus, if our statement is not true, we have three distinct circles Γ_1, Γ_2 and Γ_3, whose radical axes, taking the circles in pairs, form the sides of triangle ABC, which contradicts the theorem that such radical axes are either concurrent or parallel, as it is well-known.

Problem 3. Let OX and OY be two segments in the plane that form an acute angle α. Let $P = X_1Y_1X_2Y_2 \cdots X_nY_n$ $(n \geq 2)$ be a closed polygonal line with $X_i \in OX$ and $Y_i \in OY$ for all $1 \leq i \leq n$. The sides of P are colored red or blue in such a way that X_iY_i is red and Y_iX_{i+1} is blue for all $1 \leq i \leq n$ (subscripts are taken modulo n). If R and B are the total length of segments colored red and blue, respectively, then prove that

$$\frac{R}{B} \geq \sin\frac{\alpha}{2}.$$

Solution. Let $R = \sum_{i=1}^{n} X_iY_i$ and $B = \sum_{i=1}^{n} Y_iX_{i+1}$. For all $1 \leq i \leq n$; we have

$$Y_iX_{i+1} \leq OY_i + OX_{i+1}$$

on account of triangular inequality. Adding up these expressions yields

$$B = \sum_{i=1}^{n} Y_iX_{i+1} \leq \sum_{i=1}^{n} (OY_i + OX_{i+1})$$

Using the cosine law and the AM-GM-QM inequalities, we have

$$\begin{aligned} X_iY_i^2 &= OX_i^2 + OY_i^2 - 2\,OX_i \cdot OY_i \cos\alpha \\ &\geq OX_i^2 + OY_i^2 - (OX_i^2 + OY_i^2)\cos\alpha \\ &= (OX_i^2 + OY_i^2)(1 - \cos\alpha) \\ &\geq \frac{(OX_i + OY_i)^2}{2}(1 - \cos\alpha), \end{aligned}$$

from which if follows that

$$X_iY_i \qquad (OX_i + OY_i)\sqrt{\frac{2}{1 - \cos\theta}} = \frac{1}{\sin\theta/2}.$$

Adding up the above expressions, we get

$$B \qquad \sum_{i=1}^{n}(OY_i + OX_{i+1}) \qquad \frac{1}{\sin\theta/2}\sum_{i=1}^{n} X_iY_i, = \frac{R}{\sin\theta/2}$$

from which the statement follows.

Problem 4. Consider the binomial coefficients $\binom{m}{n}$, where $1 \leq m \leq 1919$ and $0 \leq n \leq m$ (that is, the numbers corresponding to the rows of Tartaglia's triangle from 1 to 1919). Determine how many of these numbers are odd.

Solution. We first prove that the cardinality of the set of odd numbers in a row of Tartaglia's triangle is given by

$$2^{\#\text{ ones in the binary expansion of } n}$$

(for instance, the number of odd number between $\binom{5}{i}$, where $0 \leq i \leq 5$, is 2^2). To see this, recall first that $(a+b)^2 \equiv a^2 + b^2 \pmod 2$. Next, consider the binary decomposition of n, as $n = 2^{a_1} + \dots + 2^{a_k}$. Therefore,

$$\sum_{i=0}^{n}\binom{n}{i}x^i = (1+x)^n = (1+x)^{2^{a_1}+\dots+2^{a_k}}$$
$$\equiv (1+x^{2^{a_1}})\cdots(1+x^{2^{a_k}}) \pmod 2.$$

Expanding the latter product we get 2^k summands, each of the form x raised to a sum of powers of 2; since each of them is different, we get that in $\sum\binom{n}{i}x^i$ there are exactly 2^k non-zero terms, as claimed.

Now, we just have to do the corresponding sum between the rows 1 and 1919; for simplicity, let us first count the number of odd terms in the first 2047 rows (the closest power of two to 1919). There are $\binom{11}{i}$ numbers between 1 and 2047 with exactly i ones in its decomposition, for $1 \le i \le 11$. This is because these numbers can be seen as words of length 11 where each letter is either 0 or 1. Hence, we have that the desired value is given by

$$\sum_{i=1}^{11} \binom{11}{i} 2^i.$$

Observe that

$$(1+x)^{11} = \sum_{i=0}^{11} \binom{11}{i} x^i,$$

and hence

$$\sum_{i=1}^{11} \binom{11}{i} 2^i = 3^{11} - 1.$$

Now, we just have to discount the number of odd numbers in the rows $2047 - i$, with $0 \le i \le 127$. The number of ones in the binary decomposition of $2047 - i$ is 11 minus the number of ones in the binary decomposition of i. But between 0 and 127 we have $\binom{7}{i}$ numbers with i ones, $0 \le i \le 7$, and hence we have to subtract

$$\sum_{i=0}^{7} \binom{7}{i} 2^{11-i} = 2^{11} \sum_{i=0}^{7} \binom{7}{i} \left(\frac{1}{2}\right)^i = 2^{11} \sum_{i=0}^{7} \binom{7}{i} \left(\frac{1}{2}\right)^i$$
$$= 2^{11} \cdot (3/2)^7 = 2^4 \cdot 3^7.$$

Then, the number of odd numbers in the first 1919 rows is

$$3^{11} - 2^4 \cdot 3^7 - 1.$$

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Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

Please, send submittals to José Luis Díaz-Barrero, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Elementary Problems

E–59. Proposed by Oriol Baeza Guasch, Institut de Terrassa, Terrassa, Spain. Let ABC be a triangle with circumcentre O, and let P be the point of intersection of ray AO with side BC. Denote by D and E the feet of the perpendiculars from P to sides AB and AC, respectively. Show that $BD = CE$ if and only if $AB = AC$.

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Figure 1 shows the circumcircle of $\triangle ABC$ and diameter AA' through A. As shown, $\angle ABA'$ is a right angle, since it is inscribed in a semicircle. Thus,

$$A'B \perp AB.$$

Since PD and $A'B$ are both perpendicular to the side AB, they are parallel. By Thales's theorem, then,

$$\frac{BD}{PA'} = \frac{AB}{AA'}.$$

Analogously,

$$\frac{CE}{PA'} = \frac{CA}{AA'}:$$

The quotient of these equations is

$$\frac{BD}{CE} = \frac{AB}{CA},$$

which implies the required equivalence.

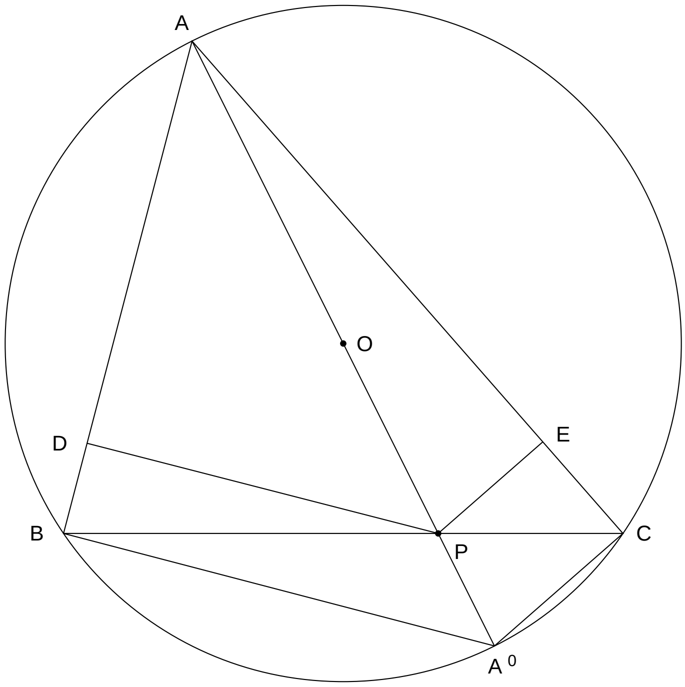

Figure 1: Construction for Solution 1 of Problem E–59.

Solution 2 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. Let X, Y be the midpoints of AB and AC, respectively. Since the circumcenter is be the intersection all the perpendicular bisectors, OX and OY are perpendicular to AB and AC, respectively, just as DP and EP are perpendicular to AB and AC, respectively, as well.

Therefore, on the one hand, triangles $\triangle AOX$ and $\triangle APD$ are similar, and, on the other, $\triangle AOY$ and $\triangle APE$ are similar as well. Using these similarity relations, we get $\frac{AO}{AX} = \frac{AP}{AD}$ and $\frac{AO}{AY} = \frac{AP}{AE}$. This leads to $\frac{AP}{AO} = \frac{AD}{AX} = \frac{AE}{AY}$. As $AX = \frac{AB}{2}$ and $AY = \frac{AC}{2}$, we will get $\frac{AD}{AB} = \frac{AE}{AC}$. Finally, since $AD = AB - BD$ and $AE = AC - CE$, we get

$$\frac{AB - BD}{AB} = \frac{AC - CE}{AC} \Longleftrightarrow 1 - \frac{BD}{AB} = 1 - \frac{CE}{AC} \Longleftrightarrow \frac{BD}{AB} = \frac{CE}{AC} \Longleftrightarrow \frac{BD}{CE} = \frac{AB}{AC}.$$

We wanted to prove $BD = CE$ if and only if $AB = AC$ or, what is equivalent, $\frac{BD}{CE} = 1$ if and only if $\frac{AB}{AC} = 1$, which we have just shown since $\frac{BD}{CE} = \frac{AB}{AC}$.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain; Aleix Torres i Camps, CFIS, BarcelonaTech, Barcelona, Spain, and the proposer

E–60. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Find all real numbers a, b such that

$$(a^2 + \sqrt{1 + a^4})(b^2 + \sqrt{1 + b^4}) = 1 .$$

Solution 1 by Aleix Torres i Camps, CFIS, BarcelonaTech, Barcelona, Spain. Using that $a^4 \ge 0$ and $b^4 \ge 0$, and then that $a^2 \ge 0$ and $b^2 \ge 0$, we get

$$1 \ge (a^2 + \sqrt{1+0})(b^2 + \sqrt{1+0}) = (a^2+1)(b^2+1) \ge (0+1)(0+1) = 1 .$$

As the expression is, at the same time, at least one and at most one, the only possible solution occurs when a^2, a^4, b^2 and b^4 are all of them 0. Therefore, the solution is unique and it is $(0;0)$.

Solution 2 by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom. First, note that $(0;0)$ is a solution for the equation in the statement. Now let us show that this is the only solution. Indeed, observe that $a^2+\sqrt{1+a^4}\geq 1$, with equality if and only if $a=0$, because $a^2; a^4 \geq 0$ and $\sqrt{1+x}$ is increasing in x, and likewise for $b^2+\sqrt{1+b^4}$. The product of two numbers greater or equal than 1 only equals 1 if both numbers equal 1.

Solution 3 by Henry Ricardo, Westchester Area Math Circle, NY, USA. Assume that a and b are real numbers satisfying the given equation. Since $a^4; b^4 \geq 0$, we have

$$\begin{aligned} 1 = (a^2+\sqrt{1+a^4})(b^2+\sqrt{1+b^4}) &\geq (a^2+1)(b^2+1) \\ &= a^2b^2+a^2+b^2+1, \end{aligned}$$

or

$$a^2b^2+a^2+b^2 \leq 0,$$

which is possible only for $a=b=0$. Thus, $(a;b)=(0;0)$ is the only real solution.

Also solved by José Gibergans Báguena, BarcelonaTech, Barcelona, Spain; Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain, and the proposer.

E–61. Proposed by Mihaela Berindeanu, Bucharest, România. If $\lfloor a \rfloor$ is the integer part of the real number a, solve in $\mathbb{R}$ the equation

$$\left\lfloor \frac{x^2-x}{x^2-x+1} \right\rfloor + \frac{2x^2+1}{3x} = 0.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. Note that $x \neq 0$. If $x<0$ or $x \geq 1$, then $0 \leq x^2-x < x^2-x+1$, so $\left\lfloor \frac{x^2-x}{x^2-x+1} \right\rfloor = 0$ and the proposed equation has no real solutions. If $0<x<1$, then $x^2<x$, so $\left\lfloor \frac{x^2-x}{x^2-x+1} \right\rfloor = -1$ and the equation is equivalent to $2x^2+1=3x$, with solutions $x=1$ and $x=1/2$. Since x is supposed to be in $(0;1)$, the unique solution of the given equation is $x=1/2$.

Solution 2 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. Let $f(x) = \frac{x^2 - x}{x^2 - x + 1}$. We will show that, for all x, $-\frac{1}{3} \le f(x) < 1$.

As $g(x) = x^2 - x = x(x-1)$ is a parabola having a minimum of $-\frac{1}{4}$, $x^2 - x + 1 \ge 1 - \frac{1}{4} > 0$. Therefore, if $f(x) \ge 1$ for some x, by multiplying both sides by $x^2 - x + 1$ (which, as we have seen, is always positive) we would have $x^2 - x \ge x^2 - x + 1$, which would imply $0 \ge 1$, getting a contradiction.

In the same way, if $f(x) < -\frac{1}{3}$ for some x, by multiplying both sides by $x^2 - x + 1$, which is always positive, we would have $x^2 - x < \frac{-x^2 + x - 1}{3}$, which would imply $\frac{4}{3}(x^2 - x) < -\frac{1}{3}$. As $x^2 - x \ge -\frac{1}{4}$, this would mean that $-\frac{1}{3} > \frac{4}{3}(x^2 - x) \ge -\frac{1}{4}\frac{4}{3} = -\frac{1}{3}$, getting a contradiction again.

Finally, if $-\frac{1}{3} \le f(x) < 1$, this can only mean that $\lfloor f(x) \rfloor$ must be either -1 or 0. If it is 0, letting $h(x) = -\frac{2x^2+1}{3x}$, we would need $h(x) = 0$, which would mean $2x^2 + 1 = 0$, which is not possible since $2x^2 + 1$ is always positive. Hence, the only option is having $\lfloor f(x) \rfloor = -1$, which would mean $h(x) = -1$. For this case, we would have $2x^2 + 1 = 3x$. Solving the equation $2x^2 - 3x + 1$ we obtain two solutions, $x = 1$ and $x = \frac{1}{2}$. Substituting in $f(x)$, we have $f(\frac{1}{2}) = -\frac{1}{3}$, satisfying $\lfloor f(\frac{1}{2}) \rfloor = -1$. But for $x = 1$, $f(1) = 0$, which does not satisfy $\lfloor f(1) \rfloor = -1$. Thus, the only solution of the equation happens at $x = \frac{1}{2}$.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain; Aleix Torres i Camps, CFIS, BarcelonaTech, Barcelona, Spain, and the proposer.

E–62. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. On a 2019×2019 board symmetrical with respect to the main diagonal, the numbers from 1 to 2019 are placed so that in each row and column each number appears once and only once. Show that all the numbers appear on the main diagonal.

Solution 1 by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom. First observe that any given number must appear 2019 times, as it must appear exactly once in each of the 2019 rows. Now suppose that some number does not appear in the main diagonal. Since the numbers are placed symmetrically with respect to the main diagonal, that means that the number appears an even number of times (exactly twice the number of times it appears below the main diagonal). But this contradicts the fact that it must appear 2019 times. Therefore, the statement must be true.

Solution 2 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. For each number $k \in \{1, \ldots, 2019\}$, let l_k, u_k, d_k be the number of times that the number k appears strictly below the main diagonal, strictly above it, and on the diagonal itself, respectively. By symmetry, for each value k, the two relations $l_k = u_k$ and $l_k + u_k + d_k = 2019$ will hold. Plugging the first relation into the second, we will get $2019 = 2l_k + d_k$, which implies that d_k is odd, which will imply d_k cannot be 0. As d_k must be a non-negative integer, this will mean $d_k \geq 1$, meaning that each number k will appear on the diagonal.

Remark. Since the diagonal part of the board has 2019 entries, this will mean $2019 = d_1 + \ldots + d_{2019}$. As $d_k \geq 1$ for each k, this will imply d_k is exactly 1 for every k, meaning that each number appears exactly once in the main diagonal. Note also that the same property we proved in this problem will also hold changing 2019 for any odd natural.

Also solved by Aleix Torres i Camps, CFIS, BarcelonaTech, Barcelona, Spain, and the proposer.

E–63.

Proposed by Mihaela Berindeanu, Bucharest, România. Find all pairs (p, q) of prime numbers for which

$$p\left(p^2 + p + 3\right) = q(q + 4).$$

Solution 1 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. We will start by showing that $p < q$. Indeed, if

$p \geq q$, then

$$0 = p(p^2 + p + 3) - q(q + 4) \geq p^3 + p^2 + 3p - p^2 - 4p$$
$$= p^3 - p = p(p^2 - 1) > 0.$$

The last equality holds since $p \geq 2$, taking us to a contradiction.

In particular, as $p \neq q$, this means that $\gcd(p; q) = 1$, and as p is a factor of the LHS, it has to be a factor of the RHS, and as it cannot divide q, it must divide $q + 4$. Thus, let $k = \frac{q+4}{p}$, which is an integer. We can write q as $q = kp - 4$. Plugging it into the main relation we get $p(p^2 + p + 3) = (kp - 4)kp$, which means $p^2 + p + 3 = k^2 p - 4k$. Reducing this equality modulo p, we see that $4k + 3$ must be divisible by p, or, as $k = \frac{q+4}{p}$, this means $\frac{4\frac{q+4}{p}+3}{p}$ is an integer. In other words, p^2 must divide $4q + 16 + 3p$.

We will try to prove the following inequality. If $p > 7$ (that is, if $p \geq 11$, since p is prime), then $p^{\frac{3}{2}} < q < (p + 1)^{\frac{3}{2}}$.

Let us begin by proving the first inequality by contradiction. If $p^{\frac{3}{2}} \geq q$, then $p^3 + p^2 + 3p = q(q + 4) \leq p^3 + 4p^{\frac{3}{2}}$, which means that $p^2 + 3p \leq 4p^{\frac{3}{2}}$. Dividing by p we get $0 \geq p - 4\sqrt{p} + 3 = (\sqrt{p} - 1)(\sqrt{p} - 3)$ which is false since $p \geq 11$.

We claim that, if $0 \leq y < x$, then $\sqrt{x - y} > \sqrt{x} - \sqrt{y}$. Indeed, if that was not true, $\sqrt{x - y} + \sqrt{y} \leq \sqrt{x}$, and squaring both sides, which are positive, we would get $x \geq x - y + y + 2\sqrt{x - y}\sqrt{y} = x + 2\sqrt{x - y}\sqrt{y}$, which is false.

With this, we are ready to prove that $q < (p + 1)^{\frac{3}{2}}$. Assume the contrary. We can observe that we can rewrite our equality in this form: $\sqrt{p^3 + p^2 + 3p + 4} - 2 = q$. Let us set $x = p^3 + p^2 + 3p$, $y = 4$, (as $p > 1$, $x > 1 + 1 + 3 > 4 = y$). We would have $(p + 1)^{3/2} \leq q = \sqrt{x + y} - \sqrt{y} < \sqrt{x} = \sqrt{p^3 + p^2 + p}$. Squaring both sides and expanding $(p + 1)^3$, this would be equivalent to $p^3 + 3p^2 + 3p + 1 < p^3 + p^2 + 3p$, and would mean that $1 + 2p^2 < 0$, which is not possible.

We will consider the special cases in which $p \leq 7$. We will point out that $p = 2$ cannot be part of a solution. In that case, $p^3 + p^2 +$

$3p + 4 = 22$, which is not a square. As $p^3 + p^2 + 3p + 4 = (q + 2)^2$, $p = 2$ cannot be part of a solution. In a similar way, if $p = 7$, $p^3 + p^2 + 3p + 4 = 417$. As it is not a square, it cannot be a part of a solution either.

For $p = 3$, we see that $p^3 + p^2 + 3p + 4 = 49$, which is a square, which leads to $q = 5$, which is a prime. Doing the same for $p = 5$, we would get $p^3 + p^2 + 3p + 4 = 169$, which is a square, from where we deduce $q = 11$, which is a prime.

Now, assuming $p \geq 11$, p is odd. As p^2 divides $4q + 16 + 3p$, and both p^2 and $4q + 16 + 3p$ are odd, this would mean $r = \frac{4q+16+3p}{p^2}$ is an odd integer. As $p^{\frac{3}{2}} < q < (p + 1)^{\frac{3}{2}}$, we have, on the one hand,

$$r > \frac{4p^{3/2} + 16 + 3p}{p^2} = \frac{4\sqrt{p^3}}{p^2} + \frac{16}{p^2} + \frac{3}{p} = \frac{4}{\sqrt{p}} + \frac{16}{p^2} + \frac{3}{p},$$

and on the other,

$$r < \frac{4(p + 1)^{3/2} + 16 + 3p}{p^2} = \frac{4\sqrt{p^3 + 3p^2 + 3p + 1}}{p^2} + \frac{16}{p^2} + \frac{3}{p}$$

$$= 4\sqrt{\frac{1}{p} + \frac{3}{p^2} + \frac{3}{p^3} + \frac{1}{p^4}} + \frac{16}{p^2} + \frac{3}{p}.$$

The functions $f(x) = \frac{4}{\sqrt{x}} + \frac{16}{x^2} + \frac{3}{x}$ and $g(x) = 4\sqrt{\frac{1}{x} + \frac{3}{x^2} + \frac{3}{x^3} + \frac{1}{x^4}} + \frac{16}{x^2} + \frac{3}{x}$ are clearly decreasing in the real positive integers, and they satisfy $f(x) < r < g(x)$. For $x = 7$, we have $r < f(11) \approx 1.77915 < 3$. This means that, for $p \geq 11$, $r < 3$, and as r is a positive odd integer, r will have to be exactly 1, or, in other words, $p^2 = 4q + 16 + 3p$.

However, if $p \leq 19$, then $r > f(p) \geq f(19) \approx 1.11988 > 1$ and, similarly, if $p \geq 29$, then $r < g(p) \leq g(29) \approx 0.904004 < 1$. This means that p cannot be a part of a solution if $11 \leq p \leq 19$ or $p \geq 29$. The only prime number left between 19 and 29 is $p = 23$. For $p = 23$, $p^3 + p^2 + 3p + 4 = 12769 = 113^2$, which implies $q = 111$. But q is not a prime since $q = 3 \cdot 37$ and, therefore, the only pairs $(p; q)$ of solutions are $(3; 5)$ and $(5; 11)$.

Solution 2 by the proposer. We consider the following cases:

1. Case $p = q$. We have that

$$p = q \Rightarrow p\left(p^2 + p + 3\right) = p(p + 4)$$
$$\Rightarrow p^2 + p + 3 = p + 4 \Rightarrow p^2 = 1 \Rightarrow p = 1.$$

Since 1 is not a prime number, the case $p = q$ is excluded.

2. Case $p \neq q$. We have that

$$\left.\begin{array}{l} q(q+4) \mid p \\ p \text{ prime} \end{array}\right\} \Rightarrow q + 4 \mid p.$$

This means that there exists $k \in \mathbb{N}$, $k \geq 1$, such that $q + 4 = kp$. We then have that

$$p\left(p^2 + p + 3\right) = q(q+4) \Rightarrow p\left(p^2 + p + 3\right) = kp(kp - 4)$$
$$\Rightarrow p^2 + p + 3 = k^2 p - 4k$$
$$\Rightarrow p^2 + p\left(1 - k^2\right) + 4k + 3 = 0.$$

Since $p \in \mathbb{N}$, $\Delta = (1 - k^2)^2 - 16k - 12$ must be a perfect square. Observe that

$$\Delta < \left(1 - k^2\right)^2. \qquad (1)$$

We now further consider several cases:

If $\Delta > (2 - k^2)^2$, then $(1 - k^2)^2 > \Delta > (2 - k^2)^2$. In this case, Δ cannot be a perfect square because its value is between two perfect consecutive squares. Therefore, this case is excluded.

If $\Delta \leq (2 - k^2)^2$, then we must have $k^4 - 2k^2 + 1 - 16k - 12 \leq 4 - 4k^2 + k^4$, which is equivalent to $2k^2 - 16k - 15 \leq 0$, which implies that $k \leq 8$. By (1) and the fact that $k \in \mathbb{N}$, we conclude that $k > 1$, so the only possibilities are $k \in \{2; 3; 4; 5; 6; 7; 8\}$. Studying each case, we have that

k		Remarks for p values
$k = 2$	$p^2 - 3p + 11 = 0$	$p \notin \mathbb{R}$
$k = 3$	$p^2 - 8p + 15 = 0$	$p \in \{3, 5\}$
$k = 4$	$p^2 - 15p + 19 = 0$	$p \notin \mathbb{N}$
$k = 5$	$p^2 - 24p + 23 = 0$	$p \in \{1, 23\}$
$k = 6$	$p^2 - 35p + 27 = 0$	$p \notin \mathbb{N}$
$k = 7$	$p^2 - 48p + 31 = 0$	$p \notin \mathbb{N}$
$k = 8$	$p^2 - 63p + 35 = 0$	$p \notin \mathbb{N}$

The only values admitted are $k \in \{3, 5\}$, which means that $p \in \{3, 5, 23\}$. We then have that

$$\left.\begin{array}{llll} p = 3 & \Longrightarrow & q = 5 & \text{Accepted,} \\ p = 5 & \Longrightarrow & q = 11 & \text{Accepted,} \\ p = 23 & \Longrightarrow & q = 111 & \text{Excluded } (3 \mid 111), \end{array}\right\}$$

which means that the pairs (p, q) are $(3, 5)$ and $(5, 11)$.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.

E–64. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let a, b, c be three positive real numbers such that $ab + bc + ca = 2abc$. Prove that

$$\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \leq 2.$$

Solution 1 by Sarah B. Seales, USA. The condition that $ab + bc + ca = 2abc$ can be rewritten as $\frac{1}{c} + \frac{1}{a} + \frac{1}{b} = 2$, which makes the given inequality

$$\frac{1}{c} + \frac{1}{a} + \frac{1}{b} \geq \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}}.$$

Since a, b, c are positive real numbers, we are free to square both sides knowing that this is a reversible step. After some algebra, the inequality becomes

$$\left(\frac{1}{c}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{a}\right)^2 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq 2\left(\frac{1}{b\sqrt{ac}} + \frac{1}{c\sqrt{ba}} + \frac{1}{a\sqrt{bc}}\right).$$

We notice that

$$\left(\frac{1}{c}-\frac{1}{\sqrt{ba}}\right)^2=\left(\frac{1}{c}\right)^2-2\frac{1}{c\sqrt{ba}}+\frac{1}{ab}.$$

This gives us

$$\left(\frac{1}{c}-\frac{1}{\sqrt{ba}}\right)^2+\left(\frac{1}{b}-\frac{1}{\sqrt{ca}}\right)^2+\left(\frac{1}{a}-\frac{1}{\sqrt{bc}}\right)^2\geq 0,$$

which is true by a trivial inequality.

Solution 2 by Irene Fernández Fernández, IES El Carmen, Murcia, Spain. If we use the arithmetic-geometric mean inequality we find that

$$\frac{\frac{1}{a}+\frac{1}{b}}{2}\geq\sqrt{\frac{1}{a}\cdot\frac{1}{b}},\quad \frac{\frac{1}{b}+\frac{1}{c}}{2}\geq\sqrt{\frac{1}{b}\cdot\frac{1}{c}},\quad \frac{\frac{1}{c}+\frac{1}{a}}{2}\geq\sqrt{\frac{1}{c}\cdot\frac{1}{a}}.$$

Therefore,

$$\frac{\frac{1}{a}+\frac{1}{b}}{2}+\frac{\frac{1}{b}+\frac{1}{c}}{2}+\frac{\frac{1}{c}+\frac{1}{a}}{2}\geq\sqrt{\frac{1}{a}\cdot\frac{1}{b}}+\sqrt{\frac{1}{b}\cdot\frac{1}{c}}+\sqrt{\frac{1}{c}\cdot\frac{1}{a}}$$

or, equivalently,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}.$$

If we rationalize,

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq\frac{\sqrt{ab}}{ab}+\frac{\sqrt{bc}}{bc}+\frac{\sqrt{ca}}{ca}.$$

Then, we multiply by the least common multiple (l.c.m.) to obtain

$$ab+bc+ca\geq c\sqrt{ab}+a\sqrt{bc}+b\sqrt{ca}.$$

As $ab+bc+ca=2abc$, if we replace $ab+bc+ca$ for $2abc$, the inequality does not change. Hence,

$$2abc\geq c\sqrt{ab}+a\sqrt{bc}+b\sqrt{ca}.$$

Dividing by abc yields

$$2 \quad \frac{\sqrt{ab}}{ab} + \frac{\sqrt{bc}}{bc} + \frac{\sqrt{ca}}{ca}.$$

Therefore,

$$2 \quad \frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}}.$$

Solution 3 by Aleix Torres i Camps, CFIS, BarcelonaTech, Barcelona, Spain. Since a, b, c are positive real numbers, consider the following change of variables:

$$x^2 = ab \qquad y^2 = bc \qquad z^2 = ca.$$

Therefore, the inequality can be expressed as

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \quad 2,$$

which is equivalent to

$$xy + yz + zx \quad 2xyz.$$

From the equality condition, we know that $2xyz = x^2 + y^2 + z^2$, so we just have to prove that

$$xy + yz + zx \quad x^2 + y^2 + z^2.$$

Applying GM AM to the pairs $(x^2; y^2)$, $(y^2; z^2)$ and $(z^2; x^2)$, and adding them up, we get the inequality we wanted. Since we have applied geometric arithmetic mean, we can assure that the equality case holds when $x^2 = y^2 = z^2$, which implies $a = b = c = \frac{3}{2}$.

Solution 4 by Scott H. Brown, Auburn University at Montgomery, Montgomery, Alabama, USA. Rewrite the given inequality and simplify to obtain

$$c\sqrt{ab} + a\sqrt{bc} + b\sqrt{ac} \quad 2abc.$$

Substitute the given value for 2abc into the above inequality and multiply each term on both sides by 2 to obtain

$$2c\sqrt{ab} + 2a\sqrt{bc} + 2b\sqrt{ac} \leq 2ab + 2bc + 2ca.$$

The RHS of the last inequality can be arranged as follows

$$2c\sqrt{ab} + 2a\sqrt{bc} + 2b\sqrt{ac} \leq (bc + ac) + (ab + ac) + (ab + bc).$$

From the preceding inequality we have three inequalities, which hold true according to [1]. Namely,

$$2c\sqrt{ab} \leq (bc + ac) \Leftrightarrow 2\sqrt{ab} \leq a + b$$
$$2a\sqrt{bc} \leq (ab + ac) \Leftrightarrow 2\sqrt{bc} \leq b + c$$
$$2b\sqrt{ac} \leq (ab + bc) \Leftrightarrow 2\sqrt{ac} \leq a + c$$

Therefore the last inequality holds true, which implies that the inequality claimed in the statement holds true. Equality holds when $a = b = c$.

Reference [1]: Geometric Inequalities, O. Bottema et al, 1969.

Solution 5 by the proposer. We will use Cauchy's inequality. So, we have to choose two vectors $\vec{u}; \vec{v}$ that satisfy $(\vec{u} \cdot \vec{v})^2 \leq \|\vec{u}\|^2 \|\vec{v}\|^2$. If $\vec{u} = \left(\frac{1}{\sqrt{a}}; \frac{1}{\sqrt{b}}; \frac{1}{\sqrt{c}}\right)$ and $\vec{v} = \left(\frac{1}{\sqrt{b}}; \frac{1}{\sqrt{c}}; \frac{1}{\sqrt{a}}\right)$, then we have

$$\left(\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}}\right)^2 \leq \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)\left(\frac{1}{b} + \frac{1}{c} + \frac{1}{a}\right).$$

Taking square roots we obtain

$$\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 2$$

on account that from $ab + bc + ca = 2abc$ we get $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 2$. Equality holds when $a = b = c = 3/2$, and we are done.

Also solved by Arkady Alt, San Jose, California, USA; Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain; Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain, and Henry Ricardo, Westchester Area Math Circle, NY, USA.

Easy–Medium Problems

EM–59. Proposed by Pedro Henrique O. Pantoja, University of Campina Grande, Brazil. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2y + y^2z + z^2x) = xf(x) + f(f(y) + z) + f(zx)$$

for all real numbers x, y, z.

Solution 1 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. Let us substitute $x = 0$, $y = 0$, $z = -f(0) + t$, with $t \in \mathbb{R}$, into the equation. We conclude that

$$f(0) = 0 \cdot f(0) + f(f(0) - f(0) + t) + f(0),$$

which implies $f(t) = 0$ for every $t \in \mathbb{R}$. Therefore, only the function $f(x) \equiv 0$ satisfies the functional equation.

Solution 2 by the proposer. The answer is $f(x) = 0$. Indeed, by substituting $x = 0$ into the original equation, we obtain $f(y^2z) = f(f(y) + z) + f(0)$. Now let us substitute $z = -f(y)$, and we get $2f(0) = f(-y^2f(y))$. By substituting $y = 0$, we obtain $2f(0) = f(0) \Rightarrow f(0) = 0$, thus $f(f(0)) = 0$. Now, by substituting $y = z = 0$, we get $f(0) = xf(x) + f(f(0)) + f(0) \Rightarrow xf(x) = 0$, therefore if $x \neq 0$ we have $f(x) = 0$. Altogether, we have shown that $f(x) = 0$ for every x is the only possible solution, and it clearly is a solution.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain, and José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.

EM–60. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

Solution 1 by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom. Since each of the six problems was solved by at least 120 students, the total number of correct solutions must be at least 720. Suppose that the statement is false, that is, there is no pair of students who, together, solved all the problems. We now divide the proof into several cases:

- If one student solved all six problems we reach a contradiction, as this student together with any other gives the desired pair.
- If there is a student who solved five of the problems we reach another contradiction, as the sixth problem, by assumption, was solved by at least 120 people. That is, the first student together with any of these other students would give the desired pair.
- If some student has solved four problems we reach yet another contradiction. In this case, observe that, as each problem is solved by at least 120 students and there are a total of 200, at least 40 students must have solved each pair of problems (by the pigeonhole principle). Any of the 40 students who solved the two problems the first student did not solve, together with this student, gives the desired pair.
- We are left with the case in which each student solves at most three problems. But then, the overall number of correct solutions is at most 600, which contradicts the first observation.

Remark. The same proof strategy works even if each problem is solved by at least 101 students.

Solution 2 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. Let a_i, with $i \in \{1, \ldots, 200\}$ be the number of correct problems solved by the i-th student, and b_j, with $j \in \{1, \ldots, 6\}$ be the number of students that solved the j-th problem. By hypothesis, $b_j \geq 120$ for all $j \in \{1, \ldots, 6\}$.

Now, by double counting, we have

$$\sum_{i=1}^{200} a_i = \sum_{j=1}^{6} b_j \geq 6 \cdot 120 = 720 .$$

This means that there is some i such that $a_i \geq 4$; otherwise, if

a_i 3 fr all i, $\sum_{i=1}^{200} a_i$ 200 3 = 600, which would lead us to a contradiction.

Now, we can see that, for each pair of problems $(P_r; P_s)$, at most 80 students did not solve P_r, and at most 80 did not solve P_s. This means that at most 160 students failed at solving at least one of those problems, or, in other words, at least 40 solved both.

Now let S_i be one of the students that managed to solve four problems or more. Let P_a, P_b, P_c, P_d be the four problems that they managed to solve, and let P_e, P_f be the other two. By what we have just seen, at least 40 students will have solved P_e and P_f. Let S_j be one of the at least 39 students, different that S_i, that have solved both P_e and P_f. So, every problem will have been solved by S_i, S_j or both, as we wanted to prove.

Solution 3 by the proposer. We argue by contradiction. Let us assume that for every two students, there is some problem that neither of them solved. This prompts us to count the pairs of students with their unsolved problem. Let us consider the incidence matrix of this con guration. We have six rows, each representing a problem, and 200 columns, each representing a student. In light of the above remark, we make an entry of the matrix 1 if the student corresponding to the column did not solve the problem corresponding to the row, and make the entry 0 otherwise. The setup is illustrated below.

$$\begin{array}{l} \text{Problem } 1 \\ \text{Problem } 2 \\ \text{Problem } 3 \\ \text{Problem } 4 \\ \text{Problem } 5 \\ \text{Problem } 6 \end{array} \begin{pmatrix} 0 & 1 & 0 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 1 \end{pmatrix}$$

Let P denote the set of pairs of 1's that belong in the same row. Let us count the cardinality of P from two different perspectives.

Counting by columns: We assumed that for every two students, there was a problem that neither of them solved. Thus, for every two columns, there is at least one pair of 1's among these two columns that belong in the same row. So we can find an element of P in every pair of columns. Since there are $\binom{200}{2}$ pairs of columns, we have

$$|P| \geq \binom{200}{2} = 19900 .$$

Counting by rows: We are told that each problem was solved by at least 120 students. This means that there are at most 80 ones in each row. So each row contains at most $\binom{80}{2}$ pairs of 1's. Since there are six rows, we have

$$|P| \leq 6\binom{80}{2} = 18960 .$$

Combining the above two inequalities, we get $19900 \leq |P| \leq 18960$, which is clearly false. Therefore, our initial assumption must be false. So there must be two students such that every problem was solved by at least one of these two students.

Also solved by José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.

EM–61. Proposed by Mihaela Berindeanu, Bucharest, România. Let x, y, z be real numbers. Show that

$$\sum_{\text{cyclic}} \frac{1}{(1+2^{y+1-x})(1+2^{z+1-x})} \geq \frac{1}{3}.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. By changing variables as $u = 2^x$, $v = 2^y$, $w = 2^z$, the inequality reads as

$$\sum_{\text{cyclic}} \frac{u^2}{(u+2v)(u+2w)} \geq \frac{1}{3}.$$

Now, by the Cauchy-Schwarz inequality, we have

$$\sum_{\text{cyclic}} \frac{u^2}{(u+2v)(u+2w)} \geq \frac{(u+v+w)^2}{\sum_{\text{cyc}}(u^2+2uw+2uv+4vw)} \geq \frac{(u+v+w)^2}{3(u+v+w)^2} = \frac{1}{3}.$$

Solution 2 by Sarah B. Seales, USA. The inequality rewrites as

$$\sum_{\text{cyclic}} \frac{2^{2x}}{(2^x + 2^{y+1})(2^x + 2^{z+1})} \geq \frac{1}{3}.$$

Let $a = 2^x$, $b = 2^y$, $c = 2^z$. Since $2^m > 0$ for all m, we may now say that $a; b; c$ are positive real numbers. Using the substitution, this gives us

$$\sum_{\text{cyclic}} \frac{a^2}{(a+2b)(a+2c)} \geq \frac{1}{3}.$$

Since

$$\begin{aligned}(a+2b)(a+2c) &= a^2 + 2(ca + ba + bc) \\ &\leq a^2 + b^2 + c^2 + 2(ca + ba + bc) = (a+b+c)^2,\end{aligned}$$

we have

$$\sum_{\text{cyclic}} \frac{a^2}{(a+2b)(a+2c)} \geq \sum_{\text{cyclic}} \frac{a^2}{(a+b+c)^2}.$$

We must show that

$$\sum_{\text{cyclic}} \frac{a^2}{(a+b+c)^2} \geq \frac{1}{3},$$

but this is just

$$\frac{a^2 + b^2 + c^2}{(a+b+c)^2} \geq \frac{1}{3}.$$

Rewriting, we get

$$3(a^2 + b^2 + c^2) \geq a^2 + b^2 + c^2 + 2(ab + bc + ca),$$

which is equivalent to

$$a^2 + b^2 + c^2 \geq ab + bc + ca,$$

a known true inequality by the rearrangement inequality. Equality happens when $a = b = c$.

Solution 3 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. We have that

$$\sum_{\text{cyclic}} \frac{1}{(1+2^{y+1-x})}\frac{1}{(1+2^{z+1-x})} \geq \sum_{\text{cyclic}} \frac{1}{\left(\frac{1+2^{y+1-x}+1+2^{z+1-x}}{2}\right)^2}$$

$$= \sum_{\text{cyclic}} \frac{1}{(1+2^{y-x}+2^{z-x})^2} = \sum_{\text{cyclic}} \frac{1}{\left(1+\frac{2^y}{2^x}+\frac{2^z}{2^x}\right)^2}$$

$$= \sum_{\text{cyclic}} \frac{1}{\left(\frac{2^x+2^y+2^z}{2^x}\right)^2} = \frac{1}{(2^x+2^y+2^z)^2}\sum_{\text{cyclic}} 2^{2x} = \frac{2^{2x}+2^{2y}+2^{2z}}{(2^x+2^y+2^z)^2}$$

$$= 3\,\frac{\frac{(2^x)^2+(2^y)^2+(2^z)^2}{3}}{(2^x+2^y+2^z)^2} \geq 3\,\frac{\left(\frac{2^x+2^y+2^z}{3}\right)^2}{(2^x+2^y+2^z)^2} = \frac{3}{9} = \frac{1}{3},$$

where the used inequalities are, in order, the AM-GM inequality (applied in the denominator) and the QM-AM inequality.

Also solved by Arkady Alt, San Jose, California, USA, and the proposer.

EM–62. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let $a_1 \leq a_2 \leq \cdots \leq a_n$ be $n \geq 1$ real numbers lying in the interval $(0; \pi/2)$. Show that

$$a_1 \leq \arctan\left(\frac{\sin a_1 + \sin a_2 + \cdots + \sin a_n}{\cos a_1 + \cos a_2 + \cdots + \cos a_n}\right) \leq a_n.$$

Solution by Henry Ricardo, Westchester Area Math Circle, NY, USA. On the interval $(0; \pi/2)$, the arctangent and sine functions are increasing, while the cosine is decreasing. In particular, $\sin a_1 = \min_{1\leq k\leq n}\{\sin a_k\}$, $\sin a_n = \max_{1\leq k\leq n}\{\sin a_k\}$, $\cos a_n = \min_{1\leq k\leq n}\{\cos a_k\}$, and $\cos a_1 = \max_{1\leq k\leq n}\{\cos a_k\}$. Thus, we can write

$$\begin{aligned} a_1 = \arctan(\tan a_1) &= \arctan\left(\frac{n\sin a_1}{n\cos a_1}\right) \\ &\leq \arctan\left(\frac{\sin a_1 + \sin a_2 + \cdots + \sin a_n}{\cos a_1 + \cos a_2 + \cdots + \cos a_n}\right) \\ &\leq \arctan\left(\frac{n\sin a_n}{n\cos a_n}\right) = \arctan(\tan a_n) = a_n. \end{aligned}$$

Also solved by Arkady Alt, San Jose, California, USA; José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain; Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain; Pirkuliyev Rovsen, Sumgayit, Azerbaijan, and the proposer

EM–63. Proposed by Oriol Baeza Guasch, Institut de Terrassa, Terrassa, Spain. Let ABC be a triangle with usual notation. Show that

$$\left(\frac{1 \quad \cos}{\sin}\right)\left(\frac{1 \quad \cos}{\sin}\right) = 1 \quad \frac{2a}{a+b+c}.$$

Solution 1 by Pirkuliyev Rovsen, Sumgayit, Azerbaijan. On account of the well-known relations

$$\frac{1 \quad \cos}{\sin} = \tan\frac{}{2} = s\frac{(p \quad a)(p \quad c)}{p(p \quad b)}$$

and

$$\frac{1 \quad \cos}{\sin} = \tan\frac{}{2} = s\frac{(p \quad a)(p \quad b)}{p(p \quad c)},$$

where $p = (a+b+c)=2$, we have

$$\tan\frac{}{2} \quad \tan\frac{}{2} = s\frac{(p \quad a)(p \quad c)}{p(p \quad b)} \quad s\frac{(p \quad a)(p \quad b)}{p(p \quad c)}$$

$$= \frac{p \quad a}{p} = 1 \quad \frac{a}{p} = 1 \quad \frac{2a}{a+b+c}.$$

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. We will prove the desired equality in the equivalent form

$$\tan\frac{}{2}\tan\frac{}{2} = \frac{s \quad a}{s}, \qquad (1)$$

where $s = \frac{a+b+c}{2}$, applying the identities

$$1 \quad \cos x = 2\sin^2\frac{x}{2}, \qquad \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$$

with $x = \ ; \ .$

By Briggs's formulae,

$$\tan \frac{}{2} = \sqrt{\frac{(s\quad c)(s\quad a)}{s(s\quad b)}}, \qquad \tan \frac{}{2} = \sqrt{\frac{(s\quad a)(s\quad b)}{s(s\quad c)}}.$$

Multiplying these two equations, we get (1) and we are done.

Remark. One can also arrive at the solution in a more routine manner. Let r be the inradius of the given triangle and use Heron's formula to write $r^2 s = (s\quad a)(s\quad b)(s\quad c)$. This yields (see gure 2)

$$\tan \frac{}{2} \tan \frac{}{2} = \frac{r}{s\quad b}\ \frac{r}{s\quad c} = \frac{r^2}{(s\quad b)(s\quad c)} = \frac{s\quad a}{s}.$$

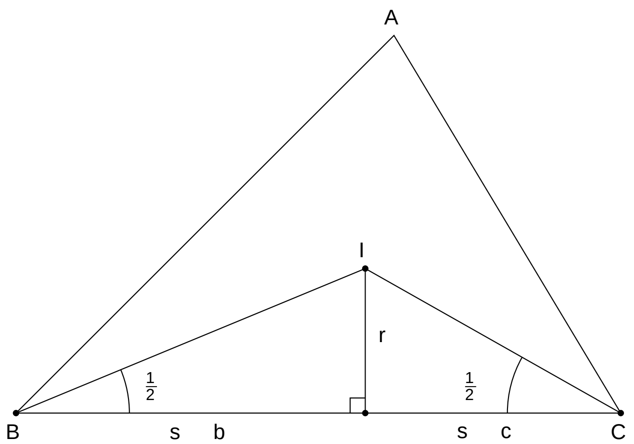

Figure 2: Construction for Solution 2 of Problem EM–63.

Solution 3 by the proposer. To begin, it will be helpful to transform the expression a little.

First of all, it can be easily shown that

$$\frac{1\quad \cos}{\sin} = \frac{\sqrt{1\quad \cos}\sqrt{1\quad \cos}}{\sqrt{1\quad \cos^2}} = \frac{\sqrt{1\quad \cos}}{\sqrt{1+\cos}} = \tan(\quad = 2).$$

On the other hand,

$$1 \quad \frac{2a}{a+b+c} = \frac{a+b+c \quad 2a}{a+b+c} = \frac{p \quad a}{p},$$

where p denotes the semiperimeter of the triangle.

With this new information, we have that the statement is equivalent to

$$\tan(\; = 2)\tan(\; = 2) = \frac{p \quad a}{p}.$$

Given that we have the half angles, it seems logical to construct a triangle and look at the incenter, say I. And the fact that p and $p \quad a$ also appear suggests to construct the ex-incenter too.

Figure 3: Construction for Solution 3 of Problem EM–63.

Now, we can see that $\backslash IBP \; = \; = 2$, so we can express $\tan(\; = 2) = \frac{IP}{BP}$. We can also see that $\backslash ICP \; = \; = 2$, but this angle also appears on another place that will lead to the solution quicker, that is, $\backslash QJC = 90 \quad \backslash QCJ = 90 \quad \frac{1}{2}(180 \quad) = \; = 2$.

This happens to be more helpful because, now, $\tan(\; = 2) = \frac{QC}{QJ}$, and it is well known that $BP = QC$ (because $BP = p \quad b$ and $QC = CC^0 = AC^0 \quad AC = p \quad b$, where p is the semiperimeter).

Therefore,

$$\tan(\; = 2)\tan(\; = 2) = \frac{IP}{BP}\frac{QC}{QJ} = \frac{IP}{QJ} = \frac{r}{R_a},$$

where r is the inradius and R_a is the A-exradius. But this is easy to compute, just drop the perpendicular from I to AB, say IF, and we have two similar triangles: $\triangle AIF$ and $\triangle AJB'$. Then,

$$\frac{IF}{B'J} = \frac{r}{R_a} = \frac{AF}{AB'} = \frac{p-a}{p},$$

which is exactly what we wanted to show.

Also solved by Arkady Alt, San Jose, California, USA; José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain; José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain, and Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain.

EM–64. Proposed by Mihaela Berindeanu, Bucharest, România.
If $z \in \mathbb{C}$ and $|z^2-1| = |4z-i|$, then show that $|z| \le \sqrt{9+\sqrt{85}}$.

Solution 1 by Arkady Alt, San Jose, California, USA. Let $z = x+iy$. Then,

$$|z^2-1| = |4z-i| \iff |z^2-1|^2 = |4z-i|^2,$$

from which it follows that $(x^2-y^2-1)^2+4x^2y^2 = 16x^2+16y^2-8y+1$ or, equivalently,

$$\left(x^2+y^2\right)^2 - 18\left(x^2+y^2\right)+4y^2+8y+1=0$$
$$\iff \left(x^2+y^2-9\right)^2+4(y+1)^2 = 85.$$

This implies $(x^2+y^2-9)^2 \le 85$. Hence, $x^2+y^2 \le \sqrt{85}+9$ or, equivalently, $|z| \le \sqrt{9+\sqrt{85}}$, and we are done.

Solution 2 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. We know that, for any complex number x, its modulus $|x|$ satisfies $|x|^2 = x\bar{x}$, where $\bar{x}$ is its complex conjugate. The complex conjugate also satisfies $\overline{x^r} = \bar{x}^r$ for any complex r. Squaring both sides of the equation, we get $|z^2-1|^2 = |4z-i|^2$.

Now, we get the following:

$$
\begin{aligned}
& |z^2 - 1|^2 = |4z - i|^2 \\
\iff\ & (z^2 - 1)\overline{(z^2 - 1)} = (4z - i)\overline{(4z - i)} \\
\iff\ & z^2\bar{z}^2 - z^2 - \bar{z}^2 + 1 = 16z\bar{z} - 4zi + 4\bar{z}i + 1 \\
\iff\ & |z|^4 - 16|z|^2 = z^2 - 4zi + \bar{z}^2 + 4\bar{z}i \\
\iff\ & |z|^4 - 16|z|^2 = (z - 2i)^2 + (\bar{z} + 2i)^2 + 8 \\
\iff\ & |z|^4 - 16|z|^2 - (z - 2i)^2 - (\overline{z - 2i})^2 = 8 \\
\iff\ & |z|^4 - 16|z|^2 - 2\mathrm{Re}\left((z - 2i)^2\right) = 8 .
\end{aligned}
$$

We will now try to find the maximum value that $\mathrm{Re}((z-2i)^2)$ can achieve, subject to $|z| = r$ for a certain positive real r. We can write z in the more convenient form $z = a + bi$, which will mean $r^2 = a^2 + b^2$. Now, $(z - 2i)^2 = (a + (b-2)i)^2 = a^2 - (b-2)^2 + 2a(b-2)i$, which means $\mathrm{Re}((z-2i)^2) = a^2 - (b-2)^2 = r^2 - b^2 - (b-2)^2$.

We want to find the maximum of that expression taking in account that r is fixed. Let $f(x) = -x^2 - (x-2)^2$. This function is infinitely differentiable. Let us find a maximum:

$$f'(x) = 0 \iff -2x - 2(x-2) = 0 \iff -4x + 4 \iff x = 1 .$$

As its second derivative is $f''(x) = -4 < 0$ everywhere, it is indeed a maximum.

With this, we can see that $r^2 - b^2 - (b-2)^2 \leq r^2 + f(1) = |z|^2 - 2$. Therefore,

$$
\begin{aligned}
0 &= |z|^4 - 16|z|^2 - 2\mathrm{Re}\left((z-2i)^2\right) - 8 \\
&\geq |z|^4 - 16|z|^2 - 2|z|^2 + 4 - 8 = |z|^4 - 18|z|^2 - 4,
\end{aligned}
$$

Let $g(x) = x^4 - 18x^2 - 4$. As $g(x)$ is a polynomial, we can take its derivative $g'(x) = 4x^3 - 36x = 4x(x-3)(x+3)$. Now it is trivial to see that this function is strictly increasing for $x \geq \sqrt{9 + \sqrt{85}}$. As $\sqrt{9 + \sqrt{85}} > \sqrt{9} = 3$, all three factors of $g'(x)$ are strictly positive, so the function is increasing. Now we see that

$$g\left(\sqrt{9+\sqrt{85}}\right) = \left(9+\sqrt{85}\right)^2 - 18\left(9+\sqrt{85}\right) - 4$$
$$= 81 + 18\sqrt{85} + 85 - 162 - 18\sqrt{85} - 4 = 0\,.$$

Hence, for $x > \sqrt{9+\sqrt{85}}$, $g(x) > 0$. Therefore, if $|z| > \sqrt{9+\sqrt{85}}$, the necessary condition $0 \geq g(|z|)$ cannot be satisfied, thus completing the proof.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain; José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain; Pirkuliyev Rovsen, Sumgayit, Azerbaijan; Sarah B. Seales, USA, and the proposer.

Medium–Hard Problems

MH–59. Proposed by Arkady Alt, San Jose, California, USA. Prove that $2nF_{n+1} - (n+1)F_n$ is divisible by 5 for any integer $n \geq 1$. Here, F_n is the n-th Fibonacci number defined by $F_0 = 0$, $F_1 = 1$ and, for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. When $n = 1$, the result follows trivially. On account of the definition of F_n, we have

$$\begin{aligned} 2(n+1)F_{n+2} - (n+2)F_{n+1} &= 2(n+1)(F_{n+1} + F_n) - (n+2)F_{n+1} \\ &= nF_{n+1} + 2(n+1)F_n. \end{aligned}$$

Adding and subtracting $4nF_{n+1}$ to the right hand side enables us to write

$$2(n+1)F_{n+2} - (n+2)F_n = 5nF_{n+1} - 2(2nF_{n+1} - (n+1)F_n).$$

Hence the desired result follows by induction.

Solution 2 by Henry Ricardo, Westchester Area Math Circle, NY, USA. In what follows, we use the well known fact that

$$5 \mid n \implies 5 \mid F_n.$$

For any positive integer n, let $P(n) = 2nF_{n+1} - (n+1)F_n$. Applying the usual Fibonacci recursion formula to the *larger* of the two Fibonacci numbers in each line and collecting terms, we can write

$$\begin{aligned} P(n) &= 2nF_{n+1} - (n+1)F_n && (0) \\ &= (n-1)F_n + 2nF_{n-1} && (1) \\ &= (3n-1)F_{n-1} + (n-1)F_{n-2} && (2) \\ &= (4n-2)F_{n-2} + (3n-1)F_{n-3} && (3) \\ &= (7n-3)F_{n-3} + (4n-2)F_{n-4}. && (4) \end{aligned}$$

Now assume that $n \equiv r \pmod 5$, $r \in \{0; 1; 2; 3; 4\}$. In line (r), we have a linear combination

$$a_r F_{n-r+1} + b_r F_{n-r},$$

and $n \quad r \pmod 5$ implies that $a_r \quad 0 \pmod 5$ and $5 \text{ j } F_{n \quad r}$. Thus, $5 \text{ j } P(n)$ for all $n \quad 1$.

Solution 3 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. Let us calculate the 25 rst elements of the Fibonacci sequence modulo 5:

$$0\,1\,1\,2\,3 \quad 0\,3\,3\,1\,4 \quad 0\,4\,4\,3\,2 \quad 0\,2\,2\,4\,1 \quad 0\,1\,1\,2\,3 \quad :::$$

As two consecutive values of the Fibonacci determine all the values that come after it, this will also be true if we take the sequence modulo n, for any natural n, which will mean, in this case, that the sequence $F_n \bmod 5$ is cyclic with cycle length 20. In the same way, the sequence $n \bmod 5$ is cyclic with cycle length 5. Therefore, $F_{n+1} \bmod 5$ will by cyclic with cycle length 20, and $n+1 \bmod 5$ and $2n \bmod 5$ will both be cyclic with cycle length 5.

Finally, as the sequence $G_n = (2nF_{n+1} \quad (n+1)F_n) \bmod 5$ combines elements that are cyclic modulo either 5 or 20, it will have cycle length divisible by $\operatorname{lcm}(5;20) = 20$. We can calculate by hand $G_0; : : : G_{19}$ to see that they are all identically 0, which will mean $G_n = 0$ for all n, which means $2nF_{n+1} \quad (n+1)F_n \quad 0 \pmod 5)$ for all n, as we wanted to show.

Solution 4 by the proposer. For any nonnegative integer, let

$$g_n = \frac{2nF_{n+1} \quad (n+1)F_n}{5}.$$

Then, $g_0 = 0$, $g_1 = 0$, and

$$\begin{aligned} g_{n+1} &= \frac{2(n+1)F_{n+2} \quad (n+2)F_{n+1}}{5} \\ &= \frac{2(n+1)(F_n + F_{n+1}) \quad (n+2)F_{n+1}}{5} \\ &= \frac{nF_{n+1} + 2(n+1)F_n}{5}. \end{aligned}$$

Thus,

$$
\begin{aligned}
g_{n+2} &= \frac{(n+1)F_{n+2} + 2(n+2)F_{n+1}}{5} \\
&= \frac{(n+1)(F_n + F_{n+1}) + 2(n+2)F_{n+1}}{5} \\
&= \frac{(n+1)F_n + (3n+5)F_{n+1}}{5}
\end{aligned}
$$

From the above, we get

$$
\begin{aligned}
& g_{n+2} - g_{n+1} - g_n \\
&= \frac{(n+1)F_n + (3n+5)F_{n+1}}{5} - \frac{nF_{n+1} + 2(n+1)F_n}{5} \\
&\quad - \frac{2nF_{n+1} - (n+1)F_n}{5} = F_{n+1}.
\end{aligned}
$$

Since $g_{n+1} = g_n + g_{n-1} + F_n$ for all $n \geq 1$ and $g_0 = 0$, $g_1 = 0$, then by induction it follows that $g_n \in \mathbb{N} \cup \{0\}$, and we are done.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain, and José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.

MH–60. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Prove that in any acute triangle ABC with the usual notations the following inequality holds:

$$
\frac{\sin A \sin B}{\cos C} + \frac{\sin B \sin C}{\cos A} + \frac{\sin C \sin A}{\cos B} \geq \frac{9}{4}.
$$

Solution 1 by José Gibergans-Báguena, BarcelonaTech, Barcelona. Applying Cauchy's inequality to the vectors

$$
\vec{u} = \left(\sqrt{\frac{\cos A}{\sin B \sin C}}; \sqrt{\frac{\cos B}{\sin A \sin C}}; \sqrt{\frac{\cos C}{\sin A \sin B}} \right)
$$

and

$$
\vec{v} = \left(\sqrt{\frac{\sin B \sin C}{\cos A}}; \sqrt{\frac{\sin A \sin C}{\cos B}}; \sqrt{\frac{\sin A \sin B}{\cos C}} \right)
$$

yields

$$9 \le \left(\sum_{cyclic} \frac{\sin A \cdot \sin B}{\cos C}\right)\left(\sum_{cyclic} \frac{\cos C}{\sin A \cdot \sin B}\right).$$

Now we claim that

$$\frac{\cos A}{\sin B \cdot \sin C} + \frac{\cos B}{\sin A \cdot \sin C} + \frac{\cos C}{\sin A \cdot \sin B} = 2\,.$$

Indeed, multiplying both sides of the given identity by $\sin A \cdot \sin B \cdot \sin C$, we get

$$\sin A \cos A + \sin B \cos B + \sin C \cos C = 2 \sin A \sin B \sin C$$

or

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C\,.$$

Taking into account trigonometric formulae for sums and product, we have

$$\begin{aligned} \sin 2A + \sin 2B &= 2\sin(A + B)\cos(A - B) \\ &= 2\sin(\pi - C)\cos(A - B) \\ &= 2\sin C \cos(A - B)\,. \end{aligned}$$

Then,

$$\begin{aligned} \sin 2A + \sin 2B + \sin 2C &= 2\sin C\cos(A - B) + 2\sin C\cos C \\ &= 2\sin C\,[\cos(A - B) + \cos C] \\ &= 2\sin C\,[\cos(A - B) - \cos(A + B)] \\ &= 4\sin A \sin B \sin C\,, \end{aligned}$$

and the claim is proven.

Finally, we have

$$9 \le \left(\sum_{cyclic} \frac{\sin A \sin B}{\cos C}\right)\left(\sum_{cyclic} \frac{\cos C}{\sin A \sin B}\right) = 2\left(\sum_{cyclic} \frac{\sin A \sin B}{\cos C}\right),$$

from which

$$\frac{\sin A \cdot \sin B}{\cos C} + \frac{\sin B \cdot \sin C}{\cos A} + \frac{\sin C \cdot \sin A}{\cos B} \ge \frac{9}{4}$$

follows. Equality holds when $\triangle ABC$ is equilateral, and we are done.

Solution 2 by Henry Ricardo, Westchester Area Math Circle, NY, USA. We will prove that the cyclic sum is greater than or equal to $9=2$, an improved lower bound. First we note that $\cos C = \cos(\quad (A + B)) = \quad \cos(A + B) = \quad \cos A \cos B + \sin A \sin B$, so that

$$\sin A \sin B = \cos C = 1 + (\cos \quad A \cos B) = \cos C$$
$$= 1 + \tan \quad C = (\tan A + \tan B).$$

Now we use Nesbitt's inequality to see that

$$\sum_{cyclic} \frac{\sin A \sin B}{\cos C} = \sum_{cyclic} 1 + \frac{\tan C}{\tan A + \tan B}$$
$$= 3 + \sum_{cyclic} \frac{\tan C}{\tan A + \tan B} \quad 3 + \frac{3}{2} = \frac{9}{2}.$$

Equality holds if and only if $A = B = C = \quad = 3$.

Solution 3 by Arkady Alt, San Jose, California, USA. We will prove the following re nement of the claimed inequality. That is,

$$\sum \frac{\sin A \sin B}{\cos C} \quad \frac{9}{2}.$$

Indeed, let $\quad = \quad 2A; \quad = \quad 2B; \quad = \quad 2C$. Then,

$$\sum \frac{\sin A \sin B}{\cos C} = \frac{\sum \sin A \sin B \cos A \cos B}{\cos A \cos B \cos C} = \frac{\sum \sin 2A \sin 2B}{4 \cos A \cos B \cos C}$$
$$= \frac{\sum \sin(\quad 2A) \sin(\quad 2B)}{4 \sin \frac{\quad 2A}{2} \sin \frac{\quad 2B}{2} \sin \frac{\quad 2C}{2}}$$
$$= \frac{\sum \sin \quad \sin}{4 \sin \frac{}{2} \sin \frac{}{2} \sin \frac{}{2}}.$$

Since $\quad + \quad + \quad = \quad$ and $A; B; C < \frac{}{2}$ () $\quad ; \; ; \quad > \quad 0$, then $; \; ;$ can be considered as angles of some triangle $A_1 B_1 C_1$. Let

a, b, c, R, r and s be, respectively, the sidelengths, circumradius, inradius and semiperimeter of $\triangle A_1B_1C_1$. Then,

$$\frac{\sum \sin\alpha \sin\beta}{4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}} = \frac{(ab+bc+ca)/4R^2}{r/R} = \frac{ab+bc+ca}{4Rr}.$$

Denoting $x = s-a$, $y = s-b$, $z = s-c$, $p = xy+yz+zx$, $q = xyz$ and assuming $s = 1$ (due to homogeneity), we obtain $x, y, z > 0$, $x+y+z = 1$, $a = 1-x$, $b = 1-y$, $c = 1-z$, $ab+bc+ca = 1+p$, $r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \sqrt{q}$ and $R = \frac{abc}{4rs} = \frac{p-q}{4\sqrt{q}}$.

Taking into account that $3p = xy+yz+zx \le (x+y+z)^2 = 1$ and $9q \ge 4p-1$ (Schur's inequality $\sum x(x-y)(x-z) \ge 0$ in p, q notation) and normalizing by $x+y+z = 1$, we obtain

$$\frac{ab+bc+ca}{4Rr} = \frac{1+p}{p-q} \ge \frac{1+p}{p-\frac{4p-1}{9}} = \frac{9(p+1)}{5p+1} \ge \frac{9\left(\frac{1}{3}+1\right)}{5\cdot\frac{1}{3}+1} = \frac{9}{2}$$

(because $\frac{9(p+1)}{5p+1} = \frac{36}{5(5p+1)} + \frac{9}{5}$ decreases by $p > 0$). Thus, $\sum \frac{\sin A \sin B}{\cos C} \ge \frac{9}{2}$.

Remark. Another way to prove the inequality is the following:

$$\frac{ab+bc+ca}{4Rr} \ge \frac{9}{2} \iff ab+bc+ca \ge 18Rr.$$

Noting that $ab+bc+ca = s^2+4Rr+r^2$ and using the known inequalities $s^2 \ge 16Rr-5r^2$ (Gerretsen's inequality) and $R \ge 2r$ (Euler's inequality), we obtain

$$ab+bc+ca-18Rr = s^2+4Rr+r^2-18Rr = s^2+r^2-14Rr \ge 16Rr-5r^2+r^2-14Rr = 2r(R-2r) \ge 0.$$

Also solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain, and the proposer.

MH–61. Proposed by Mihály Bencze, Braşov, Romania. [Correction] Suppose that $ABCDA_1B_1C_1D_1$ is a rectangle parallelepiped with sides a, b, c and diagonal d. Prove that

$$\frac{a+b+c}{d} + \frac{d^3}{abc} \geq 4\sqrt{3}.$$

Solution by Pirkuliyev Rovsen, Sumgayit, Azerbaijan, and the proposer (same solution). We claim that if x, y, z are positive reals such that $x^2 + y^2 + z^2 = 1$, then

$$x + y + x + \frac{1}{xyz} \geq 4\sqrt{3}.$$

Indeed, applying AM-GM inequality, we have

$$x+y+x+\frac{1}{xyz} = (x+y+z)(x^2+y^2+z^2) + \frac{1}{xyz} \geq 9xyz + \frac{1}{xyz} = 9t + \frac{1}{t}$$

with $t = xyz$. Since $1 = x^2 + y^2 + z^2 \geq 3\sqrt[3]{x^2y^2z^2} = 3\sqrt[3]{t^2}$, we have $t \leq \frac{1}{3\sqrt{3}} < \frac{1}{\sqrt{3}}$. Thus,

$$9t + \frac{1}{t} \geq 4\sqrt{3} \iff 9t^2 - 4\sqrt{3}t + 1 \geq 0$$
$$\iff \left(t - \frac{1}{3\sqrt{3}}\right)\left(t - \frac{1}{\sqrt{3}}\right) \geq 0,$$

and the claim is proved. Finally, putting in the above inequality $x = a/d$, $y = b/d$ and $z = c/d$ yields

$$\frac{a+b+c}{d} + \frac{d^3}{abc} \geq 4\sqrt{3}.$$

Equality holds when $a = b = c$, that is, when the parallelepiped is a cube.

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.

MH–62. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Find all integer solutions of the equation

$$(x+1)(y-1) = x^2y^2.$$

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. For $x = 0$, $y = 1$; for $x = -1$, $y = 0$. For $x \notin \{-1; 0\}$ and $y \notin \{0; 1\}$, suppose that there exist integers x, y satisfying the given equation,

$$(x+1)(y-1) = x^2y^2. \tag{1}$$

This implies that both $x+1$ and $y-1$ divide x^2y^2. Since the consecutive integers x and $x+1$ are relatively primes, as are $y-1$ and y, it easily follows that $(x+1; x^2) = 1$ and $(y-1; y^2) = 1$ as well, which, by the fundamental theorem of arithmetic, implies that

$$x+1 \mid y^2 \quad \text{and} \quad y-1 \mid x^2.$$

We then have

$$y^2 = u(x+1) \quad \text{and} \quad x^2 = v(y-1),$$

where u, v are integers. From these and (1) we obtain $uv = 1$, implying that either $u = v = 1$ or $u = v = -1$.

For $u = v = 1$ we obtain $y^2 = x+1$, $x^2 = y-1$, and eliminating x we have $y^3 + y^2 - y - 2 = 0$, which has no integer solution.

For $u = v = -1$, elimination of x from the equations $y^2 = -x-1$, $x^2 = -y+1$ gives $y^4 + 2y^2 + y = 0$, which has no solution in nonzero integers.

Thus, the two solutions listed above are the only integral solutions of the given equation.

Solution 2 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. Let us consider the change of variable $z = -y$. As z is an integer if and only if y is also an integer, we can rewrite our problem as finding the integer solutions of $(x+1)(-z-1) = x^2z^2$ or, equivalently, $(x+1)(z+1) = -x^2z^2$.

If we force x to be 0, this implies $z = \ 1$. Analogously, if $x = \ 1$, then $z = 0$. If $x = 1$, then $2(z + 1) = \ z^2$, which implies $0 = z^2 + 2z + 2 = (z + 1)^2 + 1 \ 0$, which does not have an integer solution.

As the equation is symmetric in the two variables, this means that the only solutions with at least one of the variables having absolute value at most 1 are $(x; z) = (0; \ 1)$ and $(x; z) = (\ 1; 0)$. Therefore, for the rest of the solutions, $jxj \ 2$ and $jzj \ 2$. We can rewrite the equation as $\frac{1}{x} + \frac{1}{x^2} \ \frac{1}{z} + \frac{1}{z^2} = \ 1$. Taking the absolute value, we see that $\frac{1}{x} + \frac{1}{x^2} \ \frac{1}{z} + \frac{1}{z^2} = 1$. The rest is now trivial:

$$1 = \frac{1}{x} + \frac{1}{x^2} \ \frac{1}{z} + \frac{1}{z^2} \qquad \frac{1}{x} + \frac{1}{x^2} \qquad \frac{1}{z} + \frac{1}{z^2}$$

$$\frac{1}{2} + \frac{1}{4} \quad \frac{1}{2} + \frac{1}{4} = \frac{3}{4}^{2} < 1,$$

which is not possible. Therefore, reverting the change $z = \ y$, the only two integer solutions $(x; y)$ of the equation are $(0; 1)$ and $(\ 1; 0)$.

Solution 3 by the proposer. Since $(x; x + 1) = 1$, then $(x^2; x + 1) = 1$. Likewise, $(y; y \ 1) = 1$ implies $(y^2; y \ 1) = 1$. Therefore, the equation holds when $x + 1 = \ y^2$ and $y \ 1 = \ x^2$, where the signs in both equations are the same. If both signs are positive, then the first equation becomes $x = y^2 \ 1 = (y + 1)(y \ 1)$ and the second equation, $y \ 1 = x^2$, from which we obtain

$$x = x^2(x^2 + 2).$$

If $x = 0$, then $y = 1$ and $(x; y) = (0; 1)$ is a solution. If $x > 0$, then dividing by x we get $1 = x(x^2 + 2)$, which is impossible because $x^2 + 2 > 1$, and there are no solutions.

If both signs are negative, then multiplying by $\ 1$ we obtain $y + 1 = x^2$ and $x \ 1 = y^2$, from which we get that the only solution is $(x; y) = (\ 1; 0)$.

Also solved by José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.

MH–63. Proposed by Pedro Henrique O. Pantoja, University of Campina Grande, Brazil. Let $0 \le a, b, c < \frac{\pi}{2}$. Prove that

$$8\tan\frac{a}{2}\tan\frac{b}{2}\tan\frac{c}{2} \le \tan\frac{(a+b)}{2}\tan\frac{(b+c)}{2}\tan\frac{(c+a)}{2} < \frac{(1+\tan^2\frac{a}{2})(1+\tan^2\frac{b}{2})(1+\tan^2\frac{c}{2})}{(1-\tan^2\frac{a}{2})(1-\tan^2\frac{b}{2})(1-\tan^2\frac{c}{2})}.$$

Solution 1 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. We will start by proving that $2\sqrt{\tan\frac{a}{2}\tan\frac{b}{2}} \le \tan\frac{a+b}{2}$. We can see that, using AM-GM inequality,

$$2\sqrt{\tan\frac{a}{2}\tan\frac{b}{2}} \le 2\frac{\tan\frac{a}{2}+\tan\frac{b}{2}}{2} = \tan\frac{a}{2}+\tan\frac{b}{2}.$$

Now, we can use the facts that $\tan(x+y) = \frac{\tan x+\tan y}{1-\tan x\tan y}$, that $\frac{a}{2}, \frac{b}{2} \in \left[0, \frac{\pi}{4}\right)$, and that, if $x \in \left[0, \frac{\pi}{4}\right)$, then $0 \le \tan x < 1$, to see that

$$\tan\frac{a+b}{2} = \frac{\tan\frac{a}{2}+\tan\frac{b}{2}}{1-\tan\frac{a}{2}\tan\frac{b}{2}} \ge \tan\frac{a}{2}+\tan\frac{b}{2}.$$

As all the tangents here are positive, we can now prove the first inequality:

$$\begin{aligned}
&8\tan\frac{a}{2}\tan\frac{b}{2}\tan\frac{c}{2} \\
&= \left(2\sqrt{\tan\frac{a}{2}\tan\frac{b}{2}}\right)\left(2\sqrt{\tan\frac{b}{2}\tan\frac{c}{2}}\right)\left(2\sqrt{\tan\frac{c}{2}\tan\frac{a}{2}}\right) \\
&\le \tan\frac{a+b}{2}\tan\frac{b+c}{2}\tan\frac{c+a}{2}.
\end{aligned}$$

We will prove now the second inequality. To start, we can easily see that

$$\frac{1+\tan^2 x}{1-\tan^2 x} = \frac{1+\frac{\sin^2 x}{\cos^2 x}}{1-\frac{\sin^2 x}{\cos^2 x}} = \frac{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}}{\frac{\cos^2 x-\sin^2 x}{\cos^2 x}} = \frac{1}{\cos 2x}.$$

Let us try to prove that, if $a, b \in \left(0, \frac{\pi}{2}\right)$, then $\tan\left(\frac{a+b}{2}\right) < \sqrt{\frac{1}{\cos a}\frac{1}{\cos b}}$, or, equivalently, as $\tan x$ and $\cos x$ are non-negative if $x \in \left(0, \frac{\pi}{2}\right)$, that $\tan^2\left(\frac{a+b}{2}\right) < \frac{1}{\cos a}\frac{1}{\cos b}$. We can check that

$$\tan^2\frac{a+b}{2} = \frac{\sin^2\frac{a+b}{2}}{\cos\frac{a+b}{2}} < \frac{1}{\cos^2\frac{a+b}{2}},$$

where we used that, as $0 \le \frac{a+b}{2} < \frac{\pi}{2}$, $0 \le \sin\frac{a+b}{2} < 1$.

Now, let $g(x)$ be a twice differentiable function, and let $h(x) = \log(g(x))$ be its natural logarithm. We can see that $\frac{d^2}{dx^2}h(x) = \frac{d}{dx}\frac{g'(x)}{g(x)} = \frac{g(x)g''(x) - (g'(x))^2}{g^2(x)}$, which means that, if I is an open interval of $\mathbb{R}$ such that, for all $x \in I$, $g(x)g''(x) - (g'(x))^2 > 0$, then $h(x)$ is convex in I. Analogously, if $g(x)g''(x) - (g'(x))^2 < 0$, it will be concave in I.

We have to prove now that $\frac{1}{\cos^2\frac{a+b}{2}} \le \frac{1}{\cos a}\frac{1}{\cos b}$ or, equivalently, that $2\log\left(\cos\frac{a+b}{2}\right) \ge \log(\cos(a)) + \log(\cos(b))$.

Taking $g(x) = \cos x$, we get that $g(x)g''(x) - (g'(x))^2 = -\cos^2 x - (-\sin x)^2 = -1 < 0$, meaning that $\log(\cos(x))$ is concave. Now, using Jensen's inequality, it is immediate to see that the inequality $2\log\cos\left(\frac{a+b}{2}\right) \ge \log(\cos(a)) + \log(\cos(b))$ holds.

With this, we can now prove the second inequality of the problem:

$$\begin{aligned}
&\tan\left(\frac{a+b}{2}\right)\tan\left(\frac{b+c}{2}\right)\tan\left(\frac{c+a}{2}\right) \\
&< \sqrt{\frac{1}{\cos a}\frac{1}{\cos b}}\sqrt{\frac{1}{\cos b}\frac{1}{\cos c}}\sqrt{\frac{1}{\cos c}\frac{1}{\cos a}} = \frac{1}{\cos a}\frac{1}{\cos b}\frac{1}{\cos c} \\
&= \frac{1+\tan^2\frac{a}{2}}{1-\tan^2\frac{a}{2}}\,\frac{1+\tan^2\frac{b}{2}}{1-\tan^2\frac{b}{2}}\,\frac{1+\tan^2\frac{c}{2}}{1-\tan^2\frac{c}{2}},
\end{aligned}$$

thus completing the proof.

Solution 2 by the proposer. First notice that

$$\frac{\sin a + \sin b}{\cos a + \cos b} = \frac{2\sin((a+b)/2)\cos((a-b)/2)}{2\cos((a+b)/2)\cos((a-b)/2)} = \tan(a+b)/2.$$

Furthermore,

$$\prod_{cyc}\tan\frac{a+b}{2}=\prod_{cyc}\frac{\sin a+\sin b}{\cos a+\cos b}<\prod_{cyc}\frac{2}{\cos a+\cos b}$$
$$<\prod_{cyc}\frac{2}{2\sqrt{\cos a\cos b}}=\frac{1}{\cos a\cos b\cos c}$$
$$=\frac{(1+\tan^2\frac{a}{2})(1+\tan^2\frac{b}{2})(1+\tan^2\frac{c}{2})}{(1-\tan^2\frac{a}{2})(1-\tan^2\frac{b}{2})(1-\tan^2\frac{c}{2})}.$$

Now, (changing $(a+b)/2$ by $a+b$, etc.) the inequality we want to prove is equivalent to

$$\tan(a+b)\tan(b+c)\tan(a+c)\geq 8\tan a\tan b\tan c$$
$$\Leftrightarrow\quad \tan^2(a+b)\tan^2(b+c)\tan^2(a+c)\geq 64\tan^2 a\tan^2 b\tan^2 c.$$

Let us prove that $\tan(a+b)\geq\tan a+\tan b$ for $0<a,b,c<\frac{\pi}{4}$. Indeed,

$$\tan(a+b)=\frac{\sin(a+b)}{\cos(a+b)}=\frac{\sin a\cos b+\sin b\cos a}{\cos a\cos b-\sin a\sin b}$$
$$\geq\frac{\sin a\cos b+\sin b\cos a}{\cos a\cos b}=\tan a+\tan b.$$

By the AM$-$GM inequality,

$$\tan(a+b)\geq 2\sqrt{\tan a\tan b}\quad\Leftrightarrow\quad\tan(a+b)^2\geq 4\tan a\tan b.$$

Similarly, we obtain that $\tan(b+c)^2\geq 4\tan b\tan c$ and $\tan(c+a)^2\geq 4\tan c\tan a$. Multiplying these three expressions yields the result. Equality equality occurs, for example, when $a=b=0$ or $b=c=0$ or $a=c=0$.

MH–64. Proposed by Marc Felipe Alsina, BarcelonaTech, Barcelona, Spain. Let ABC be an isosceles triangle with $\angle ABC>90^\circ$. Let D be the projection of C onto the line AB. Let Γ_1 be the circle centered at A that passes through C and let Γ_2 be the circle centered at D that passes through A. Let Γ_1 and Γ_2 intersect at X and Y. Prove that X and Y belong to the perpendicular to AB through B.

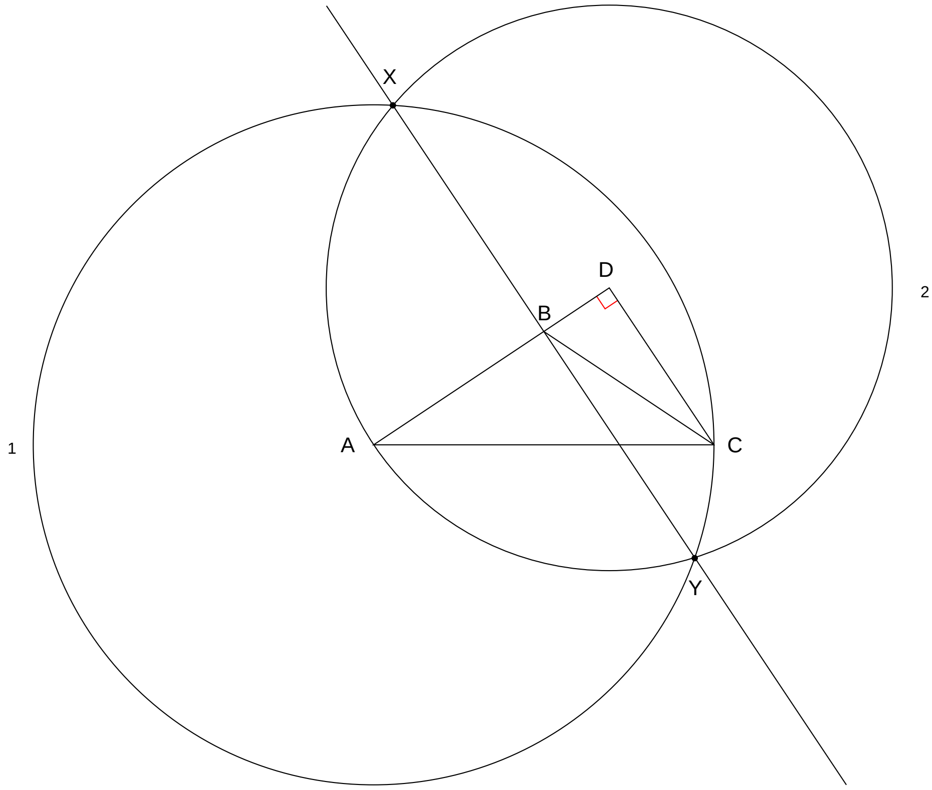

Figure 4: Construction for Solution 1 of Problem MH–64.

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Since the two circles $_1$ and $_2$ intersect at X and Y, their radical axis is simply the line XY. Since the radical axis of two non-concentric circles is perpendicular to the line of centers of the two circles, XY is perpendicular to AD. That is,

$$XY \ ? \ AB.$$

It remains only to prove that XY passes through B. To do this, it suf ces to show that B has equal power for both circles $_1$ and $_2$.

The power of point B with respect $_1$ is $\overline{BA}^2 \quad \overline{AC}^2$ and with respect to $_2$ is $\overline{BD}^2 \quad \overline{DA}^2$. We nd, by the Pythagorean theorem, that

$$\overline{BA}^2 \quad \overline{AC}^2 = \overline{BA}^2 \quad \overline{AD}^2 + \overline{CD}^2 \tag{1}$$

and

$$\overline{BD}^2 \quad \overline{DA}^2 = \quad \overline{BC}^2 \quad \overline{DC}^2 \quad \overline{DA}^2. \tag{2}$$

Substracting (2) from (1) gives

$$\overline{BA}^2 \quad \overline{AC}^2 \qquad \overline{BD}^2 \quad \overline{DA}^2 = \overline{BA}^2 \quad \overline{BC}^2. \tag{3}$$

Since 4 ABC is isosceles with \ $ABC >$ 90 , we must have

$$\overline{BA} = \overline{BC}. \tag{4}$$

By (3) and (4), then,

$$\overline{BA}^2 \quad \overline{AC}^2 \qquad \overline{BD}^2 \quad \overline{DA}^2 = 0$$

and

$$\overline{BA}^2 \quad \overline{AC}^2 = \overline{BD}^2 \quad \overline{DA}^2,$$

which is what we set out to prove.

Solution 2 by the proposer. Draw the circle centered at B that passes through A and C, and call it ! . Let E be the symmetric point of C with respect to D. Name the lines XY and CE as r and s, respectively. Since r is the radical axis of $_1$ and $_2$, it is perpendicular to the line through their centers, so XY is perpendicular to AB. We now need to show that B belongs to r.

Consider inversion with xed circle $_1$. Since $_2$ passes through the center of $_1$, it is converted into a line, which passes through xed points X and Y, so it must become the line r. Similarly, ! passes through A as well, so it also gets converted into a line, passing through xed points C and E, so it is converted into s. Since s passes through the center of $_2$, they are orthogonal, so their inversions must also be so. This means r and ! are orthogonal, and the only way this can happen is if r passes through the center B. So B belongs to r, as we wanted to prove.

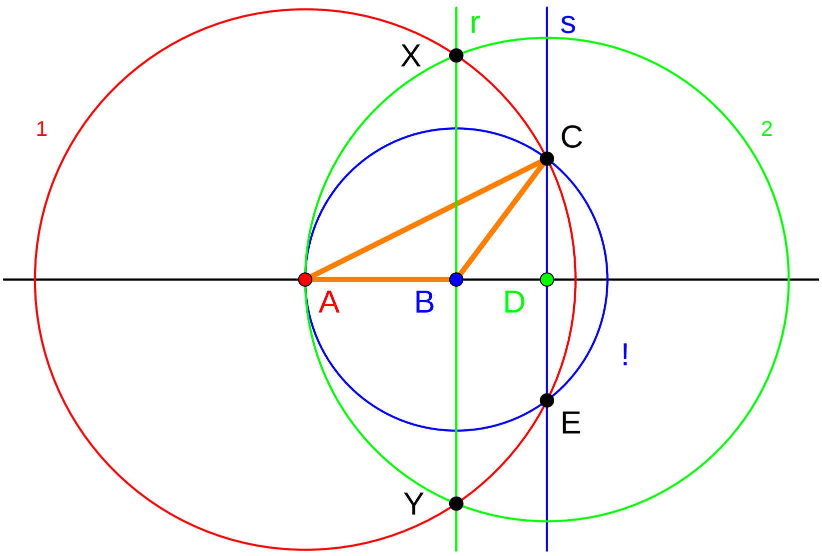

Figure 5: Construction for Solution 2 of Problem MH–64.

Advanced Problems

A–59. Proposed by Víctor Martín Chabrera, FME, BarcelonaTech, Spain. Compute

$$\sum_{s=2}^{\infty} (\zeta(s) - 1),$$

where $\zeta(s)$ is the Riemann zeta function.

Solution 1 by Sarah B. Seales, USA. The Riemann zeta function is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

when the real part $\Re(s)$ is greater than 1. Since $s \geq 2$, this suffices. The term of -1 in the summation makes it handy to reindex. The problem rewrites to

$$\sum_{s=2}^{\infty} \sum_{n=2}^{\infty} \frac{1}{n^s},$$

and this rearranges to

$$\sum_{n=2}^{\infty} \sum_{s=2}^{\infty} \frac{1}{n^s}.$$

Wishing to get the sum to telescope, we zero in on

$$\sum_{s=2}^{\infty} \frac{1}{n^s} = \sum_{s=2}^{\infty} \left(\frac{1}{n}\right)^s.$$

We consider the related sum

$$1 + \frac{1}{n} + \left(\frac{1}{n}\right)^2 + \left(\frac{1}{n}\right)^3 + \cdots$$

which evaluates to $\dfrac{1}{1-\frac{1}{n}}$ since it is a convergent geometric series for each natural number $n \geq 2$. This implies that

$$\sum_{s=2}^{\infty}\frac{1}{n^s}=\frac{1}{1-\frac{1}{n}}-\left(1-\frac{1}{n}\right),$$

but in a form useful for telescoping it is $\frac{1}{n-1}-\frac{1}{n}$. The problem becomes

$$\sum_{s=2}^{\infty}(\zeta(s)-1)=\sum_{n=2}^{\infty}\sum_{s=2}^{\infty}\frac{1}{n^s}=\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right).$$

Notice that the finite sum

$$\sum_{n=2}^{p}\left(\frac{1}{n-1}-\frac{1}{n}\right)=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots+\left(\frac{1}{p-1}-\frac{1}{p}\right)$$

telescopes to $1-\frac{1}{p}$ and the infinite sum evaluates to

$$\sum_{s=2}^{\infty}(\zeta(s)-1)=\lim_{p\to\infty}\left(1-\frac{1}{p}\right)=1.$$

Solution 2 by Pedro Henrique O. Pantoja, Natal/RN, Brazil. Note that

$$\sum_{s=2}^{\infty}(\zeta(s)-1)=\sum_{s=2}^{\infty}\sum_{n=2}^{\infty}\frac{1}{n^s}\underset{!}{=}\sum_{n=2}^{\infty}\sum_{s=2}^{\infty}\frac{1}{n^s}=\sum_{n=2}^{\infty}\left(\frac{1}{n^2}+\frac{1}{n^3}+\frac{1}{n^4}+\cdots\right)$$
$$=\sum_{n=2}^{\infty}\frac{\frac{1}{n^2}}{1-\frac{1}{n}}=\sum_{n=2}^{\infty}\frac{1}{n(n-1)},$$

where in the last expression we use the sum of an infinite geometric progression of ratio equal to $|1/n|<1$. Therefore,

$$\sum_{s=2}^{\infty}(\zeta(s)-1)=\sum_{n=2}^{\infty}\frac{1}{n(n-1)}.$$

Let us calculate this sum as follows:

$$\sum_{n=2}^{\infty}\frac{1}{n(n-1)}=\lim_{n\to\infty}\sum_{k=2}^{n}\frac{1}{k(k-1)}=\lim_{n\to\infty}\sum_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k}\right)$$
$$=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1.$$

Also solved by *the proposer.*

A–60. *Proposed by Mihály Bencze, Braşov, Romania.* Let $A \in M_2(\mathbb{C})$ with $\mathrm{Tr}(A) = \sqrt{2}/2$. Show that

$$\det\left(A^2 + \frac{3\sqrt{2}}{2}A + 3I_2\right) - \det\left(A^2 - \frac{\sqrt{2}}{2}A\right) = 15.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. The characteristic polynomial of A is

$$p_A(x) = \det(A - xI_2) = x^2 - (\sqrt{2}/2)x + d,$$

where $d = \det A$. The Cayley-Hamilton theorem gives us $A^2 - (\sqrt{2}/2)A + dI_2 = 0$, or $A^2 - (\sqrt{2}/2)A = -dI_2$ and $A^2 = (\sqrt{2}/2)A - dI_2$.

This implies that

$$A^2 + \frac{3\sqrt{2}}{2}A + 3I_2 = \left(\frac{\sqrt{2}}{2}A - dI_2\right) + \frac{3\sqrt{2}}{2}A + 3I_2 = 2\sqrt{2}A - (d-3)I_2.$$

It follows that

$$\begin{aligned}
D &= \det\left(A^2 + \frac{3\sqrt{2}}{2}A + 3I_2\right) - \det\left(A^2 - \frac{\sqrt{2}}{2}A\right)\\
&= \det(2\sqrt{2}A - (d-3)I_2) - d^2\\
&= (2\sqrt{2})^2 \det\left(A - \frac{d-3}{2\sqrt{2}}I_2\right) - d^2\\
&= 8p_A\left(\frac{d-3}{2\sqrt{2}}\right) - d^2\\
&= 8\left(\left(\frac{d-3}{2\sqrt{2}}\right)^2 - \frac{\sqrt{2}}{2}\left(\frac{d-3}{2\sqrt{2}}\right) + d\right) - d^2 = 15.
\end{aligned}$$

Solution 2 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. We will use the following properties:

The sum of the eigenvalues of a matrix is its trace.

> The product of the eigenvalues of a matrix is its determinant.
> If $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of a certain matrix $B \in M_n(\mathbb{C})$, then, for each $z \in \mathbb{C}$, the eigenvalues of $B - zI_n$ are $\lambda_1 - z, \ldots, \lambda_n - z$.
> If $P, Q \in M_n(\mathbb{C})$, then $\det(PQ) = \det(P)\det(Q)$.

Now, let $\lambda_1, \lambda_2 \in \mathbb{C}$ be the eigenvalues of A. By the first property above, $\lambda_1 + \lambda_2 = \frac{\sqrt{2}}{2}$. We can factor $A^2 + \frac{3\sqrt{2}}{2}A + 3$ as $(A - \alpha_1 I_2)(A - \alpha_2 I_2)$, where α_1, α_2 are the complex roots of the polynomial $P(x) = x^2 + \frac{3\sqrt{2}}{2}x + 3$.

Using the fourth property, we have

$$\det\left(A^2 + \frac{3\sqrt{2}}{2}A + 3I_2\right) = \det(A - \alpha_1 I_2)\det(A - \alpha_2 I_2).$$

Now, combining the second and third properties we can see that $\det(A - zI) = (\lambda_1 - z)(\lambda_2 - z)$. With this, we have

$$\begin{aligned} &\det(A - \alpha_1 I_2)\det(A - \alpha_2 I_2) \\ &= (\lambda_1 - \alpha_1)(\lambda_1 - \alpha_2)(\lambda_2 - \alpha_1)(\lambda_2 - \alpha_2) = P(\lambda_1)P(\lambda_2). \end{aligned}$$

Applying a similar argument for the second determinant, we get

$$\begin{aligned} \det\left(A^2 - \frac{\sqrt{2}}{2}A\right) &= \det(A)\det\left(A - \frac{\sqrt{2}}{2}I_2\right) \\ &= \lambda_1\lambda_2\left(\lambda_1 - \frac{\sqrt{2}}{2}\right)\left(\lambda_2 - \frac{\sqrt{2}}{2}\right). \end{aligned}$$

Taking into account that $\lambda_1 + \lambda_2 = \frac{\sqrt{2}}{2}$, we see that we can convert that last expression into

$$\lambda_1\lambda_2\left(\lambda_1 - \frac{\sqrt{2}}{2}\right)\left(\lambda_2 - \frac{\sqrt{2}}{2}\right) = \lambda_1\lambda_2(-\lambda_2)(-\lambda_1) = \lambda_1^2\lambda_2^2.$$

Therefore, we have the following:

$$\det\left(A^2 + \frac{3\sqrt{2}}{2}A\right)\det\left(A^2 - \frac{\sqrt{2}}{2}A\right) = P(\lambda_1)P(\lambda_2)\lambda_1^2\lambda_2^2.$$

Letting $\alpha_1 = x$ we have $\alpha_2 = \frac{\sqrt{2}}{2} - \alpha_1 = \frac{\sqrt{2}}{2} - x$.

Now,

$$\begin{aligned}
P(\alpha_1)P(\alpha_2) - \alpha_1^2\alpha_2^2 &= P(x)P\left(\frac{\sqrt{2}}{2} - x\right) - x^2\left(\frac{\sqrt{2}}{2} - x\right)^2\\
&= \left(x^2 + \frac{3\sqrt{2}}{2}x + 3\right)\left(\left(\frac{\sqrt{2}}{2} - x\right)^2 + \frac{3\sqrt{2}}{2}\left(\frac{\sqrt{2}}{2} - x\right) + 3\right) - x^2\left(\frac{\sqrt{2}}{2} - x\right)^2\\
&= \left(x^2 + \frac{3\sqrt{2}}{2}x + 3\right)\left(\frac{1}{2} - \sqrt{2}x + x^2 + \frac{3}{2} - \frac{3\sqrt{2}}{2}x + 3\right) - x^2\left(\frac{1}{2} - \sqrt{2}x + x^2\right)\\
&= \left(x^2 + \frac{3\sqrt{2}}{2}x + 3\right)\left(x^2 - \frac{5\sqrt{2}}{2}x + 5\right) - \left(x^4 + \sqrt{2}x^3 - \frac{x^2}{2}\right).
\end{aligned}$$

Expanding the first product, we get

$$\begin{aligned}
&\left(x^2 + \frac{3\sqrt{2}}{2}x + 3\right)\left(x^2 - \frac{5\sqrt{2}}{2}x + 5\right) - \left(x^4 + \sqrt{2}x^3 - \frac{x^2}{2}\right)\\
&= x^4 - \sqrt{2}x^3 + \frac{x^2}{2} + 15 - \left(x^4 + \sqrt{2}x^3 - \frac{x^2}{2}\right) = 15,
\end{aligned}$$

as we wanted to see.

Also solved by *José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain; José Gibergans Báguena, BarcelonaTech, Barcelona, Spain, and the proposer*

A–61. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* If the coefficients of the power series $\sum_{n=0}^{\infty} a_n z^n$ are given by the recurrence $a_0 = a_1 = 1$ and for all $n \geq 2$, $14a_n + 3a_{n-1} - a_{n-2} = 0$, then find the radius of convergence of the series and the function to which it converges in its disc of convergence.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. Let $A(z)$ denote the ordinarity power generating function of sequence $(a_n)_{n\geq 0}$, that is, $A(z) = \sum_{n=0}^{\infty} a_n z^n$. The recurrence relation may be written as $a_{n+2} = -\frac{3}{14}a_{n+1} + \frac{1}{14}a_n$. By multiplying by z^n and summing up, we obtain

$$\sum_{n=0}^{\infty} a_{n+2}z^n = -\frac{3}{14}\sum_{n=0}^{\infty} a_{n+1}z^n + \frac{1}{14}\sum_{n=0}^{\infty} a_n z^n,$$

$$\frac{A(z) - 1 - z}{z^2} = -\frac{3}{14}\left(\frac{A(z) - 1}{z}\right) + \frac{1}{14}A(z),$$

from where $A(z) = \dfrac{14 + 17z}{14 + 3z - z^2}$. Therefore, the radius of convergence of the series, say R, is the absolute value of the root of the denominator nearest to zero. Since $z^2 - 3z - 14 = 0$ for $z = \frac{1}{2}\left(3 - \sqrt{65}\right)$ and $z = \frac{1}{2}\left(3 + \sqrt{65}\right)$, then $R = \frac{1}{2}\left(\sqrt{65} - 3\right)$.

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. Using the given recurrence relation, we write

$$\begin{aligned} F(z) &= \sum_{n=0}^{\infty} a_n z^n = 1 + z + \sum_{n=2}^{\infty}\left(-\frac{3}{14}a_{n-1} + \frac{1}{14}a_{n-2}\right)z^n \\ &= 1 + z - \frac{3}{14}\sum_{n=2}^{\infty} a_{n-1}z^n + \frac{1}{14}\sum_{n=2}^{\infty} a_{n-2}z^n \\ &= 1 + z - \frac{3z}{14}\sum_{n=2}^{\infty} a_{n-1}z^{n-1} + \frac{z^2}{14}\sum_{n=2} a_{n-2}z^{n-2} \\ &= 1 + z - \frac{3z}{14}(F(z) - 1) + \frac{z^2}{14}F(z). \end{aligned}$$

Solving for $F(z)$, we find that

$$F(z) = \frac{17z + 14}{14 + 3z - z^2}.$$

The denominator of $F(z)$ has zeros at $(3 \pm \sqrt{65})/2$, which indicates that the disc of convergence (centered at the origin) extends to the zero closer to the origin —that is, to $(3 - \sqrt{65})/2$. Thus, the radius of convergence for the power series is $|(3 - \sqrt{65})/2| \approx 2.5311$.

Solution 3 by the proposer. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Using the recursion, we have

$$\begin{aligned}
&14f(z) + 3zf(z) - z^2 f(z)\\
&= 14\sum_{n=0}^{\infty} a_n z^n + 3\sum_{n=0}^{\infty} a_n z^{n+1} - \sum_{n=0}^{\infty} a_n z^{n+2}\\
&= 14a_0 + 14za_1 + 14\sum_{n=2}^{\infty} a_n z^n + 3a_0 z + 3\sum_{n=2}^{\infty} a_{n-1} z^n - \sum_{n=2}^{\infty} a_{n-2} z^n\\
&= 14a_0 + 14za_1 + 3a_0 z + \sum_{n=2}^{\infty} (14a_n + 3a_{n-1} - a_{n-2}) z^n\\
&= 14 + 17z.
\end{aligned}$$

Thus,

$$f(z) = \frac{14 + 17z}{14 + 3z - z^2}.$$

Its poles are the roots of $14 + 3z - z^2 = 0$. That is,

$$z \in \left\{ \frac{1}{2}\left(3 - \sqrt{65}\right), \frac{1}{2}\left(3 + \sqrt{65}\right) \right\}.$$

The smallest absolute value of such a pole is $\frac{1}{2}\left(\sqrt{65} - 3\right)$, so this is the radius of convergence of the series.

Remark. *The series has positive radius of convergence because the coefficients, being solutions of a recurrence, grow at most exponentially. Therefore, it is the Taylor series for the function at $z = 0$, and converges in the largest disc centered at $z = 0$ over which the function is holomorphic.*

Also solved by *José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.*

A–62. *Proposed by Nicolae Papacu, Slobozia, Romania.* Assume that $A(+,\cdot)$ is a ring such that $1 + 1 + 1$ is invertible. If $x, y \in A$ verify that $x + y = 1$ and $x^3 = x$, then prove that the elements $1 - xy$ and $1 - yx$ are invertible.

Solution by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. We can write y as $y = 1 - x$. With this, we can see that $1 - xy$ and $1 - yx$ are equal: $1 - xy = 1 - x(1 - x) = 1 - x + x^2$ and $1 - yx = 1 - (1 - x)x = 1 - x + x^2$, since in a ring the sum operation is commutative and associative and the product operation of the ring is distributive with respect to the sum.

Now, we want to see an element z of the ring which is an inverse of $1 - x + x^2$. We will try to write z as a polynomial in x. The property $x = x^3$, though, means that any factor of the form x^n, with $n \geq 3$, can be written as $x^n = x^{n-3}x^3 = x^{n-3}x = x^{n-2}$, which means we think of z as just a polynomial of degree 2 in x. Therefore, let us write $z = a + bx + cx^2$, for some $a, b, c \in A$. Now, $z(1 - x + x^2) = 1$ implies $1 = (a + bx + cx^2)(1 - x + x^2) = a + (b - a)x + (a - b + c)x^2 + (b - c)x^3 + cx^4$.

Applying now the property we showed before ($x^n = x^{n-2}$ if $n \geq 3$), we can rewrite the above expression as $a + (b - a)x + (a - b + c)x^2 + (b - c)x + cx^2 = a + (2b - a - c)x + (a - b + 2c)x^2$.

Now we will want to choose $a, b, c \in A$ such that $a = 1$, $2b - a - c = 0$ and $a - b + 2c = 0$. By plugging the first equation into the other two, we get $2b - c = 1$ and $2c - b = -1$. Adding twice the first equation to the second, we get $3b = 1$. Now, as $3 = 1 + 1 + 1$ is invertible in A, let $r \in A$ be its inverse. We will have $b = r$. Now, in a similar way, adding twice the second equation to the first, we will get $3c = -1$. Left-multiplying by r, we will get $(r \cdot 3)c = r \cdot (-1)$. As $r \cdot (-1) = -r$ (this is easy to prove, since, using the properties of rings, $0 = r \cdot 0 = r \cdot (-1 + 1) = r \cdot (-1) + r \cdot 1 = r \cdot (-1) + r$, which implies that $r \cdot (-1) = -r$ by adding $-r$ to each side), and as $r \cdot 3 = 1$ by definition of r, we get $c = -r$. This way we have found an inverse of $1 - x + x^2$, which will be $z = 1 + rx - rx^2$.

Also solved by *José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain; José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain, and the proposer.*

A–63. *Proposed by Óscar Rivero Salgado, BarcelonaTeach, Barcelona, Spain.* Let $k \geq 1$ be a fixed positive integer, and let $n \geq 0$

be a non-negative integer. Show that

$$\sum_{j=0}^{n}\binom{kn}{kj}=\frac{2^{kn}}{k}+\frac{2^{kn+1}}{k}\sum_{j=1}^{\lfloor n/2\rfloor}(-1)^{jn}\cos^{kn}\left(\frac{j\pi}{n}\right).$$

Solution by the proposer. For any $0 \le a \le k-1$, observe that

$$(1+\omega_k^a)^{kn}=\sum_{j=0}^{kn}\binom{kn}{j}\omega_k^{aj},$$

and summing over the k possible values of a, one gets that

$$\sum_{a=0}^{n-1}(1+\omega_k^a)^{kn}=k\sum_{j=0}^{n}\binom{kn}{kj}.$$

We have to distinguish the following cases:

- For $a=0$, $(1+\omega_k^a)^{kn}=2^{kn}$.
- If k is even and $a=k/2$, then $1+\omega_k^a=0$.
- For $a<k/2$,

$$\begin{aligned}(1+\omega_k^a)^{kn}&=\left(\sqrt{2+2\cos(2\pi a/n)}\right)^{kn}\left(\frac{1+\omega_k^a}{\sqrt{2+2\cos(2\pi a/n)}}\right)^{kn}\\&=\left(\sqrt{2+2\cos(2\pi a/n)}\right)^{kn}\omega_{2k}^{kan}\\&=2^{kn}(-1)^{an}\cos^{kn}\left(\frac{\pi a}{n}\right).\end{aligned}$$

One may argue that one gets the same result for a and for $n-a$.

Therefore, we conclude that

$$\sum_{j=0}^{n}\binom{kn}{kj}=\frac{2^{kn}}{k}+\frac{2^{kn+1}}{k}\sum_{j=1}^{\lfloor n/2\rfloor}(-1)^{jn}\cos^{kn}\left(\frac{j\pi}{n}\right).$$

A–64. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Assume that polynomial $A(z)$ with leading coefficient one and degree n has distinct zeros $\alpha_1, \alpha_2, \dots, \alpha_n$. Prove that

$$|\Delta(A)| \le 2^{n(n-1)} \left(\prod_{k=1}^{n} \max\{1, |\alpha_k|\} \right)^{2n-2},$$

where $\Delta(A)$ is the discriminant of $A(z)$.

Solution 1 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. If $B(x) = b_n x^n + \dots + b_0$ is a polynomial of degree n with complex roots $r_1, \dots, r_n$, its determinant is defined as $\Delta(B) = (-1)^{n(n-1)/2} b_n^{2n-2} \prod_{i \ne j}(r_i - r_j)$. In this case, if $A(x) = a_n x^n + \dots + a_0$, as $a_n = 1$, $|\Delta(A)| = \prod_{i \ne j} |\alpha_i - \alpha_j|$. Therefore, we want to prove that

$$\prod_{i \ne j} |\alpha_i - \alpha_j| \le 2^{n(n-1)} \left(\prod_{k=1}^{n} \max(1, |\alpha_k|) \right)^{2(n-1)}$$

or, equivalently, dividing by $2^{n(n-1)}$, and distributing each of the $n(n-1)$ factors of 2 into the $n(n-1)$ factors of the LHS,

$$\prod_{i \ne j} \frac{|\alpha_i - \alpha_j|}{2} \le \left(\prod_{k=1}^{n} \max(1, |\alpha_k|) \right)^{2(n-1)}.$$

Let us define now $\beta_k = \max(1, |\alpha_k|)$. By definition, $\beta_k \in \mathbb{R}$ and $\beta_k \ge 1$ for all k. On the one hand, the LHS satisfies

$$\prod_{i \ne j} \frac{|\alpha_i - \alpha_j|}{2} \le \prod_{i \ne j} \frac{|\alpha_i| + |\alpha_j|}{2} \le \prod_{i \ne j} \frac{\beta_i + \beta_j}{2}.$$

On the other, the RHS can be rewritten as

$$\left(\prod_{k=1}^{n} \max(1, |\alpha_k|) \right)^{2} = \left(\prod_{k=1}^{n} \beta_k \right)^{2(n-1)},$$

so it would be enough to prove that

$$\prod_{i \ne j} \frac{\beta_i + \beta_j}{2} \le \left(\prod_{k=1}^{n} \beta_k \right)^{2(n-1)}.$$

Taking logarithms, since $\log x$ is an increasing function, this would be equivalent to

$$\sum_{i\neq j} \log \frac{\alpha_i + \alpha_j}{2} \le 2(n-1)\sum_{k=1}^{n} \log \alpha_k .$$

Since $\log x$ is increasing and $\log x \ge 0$ if $x \ge 1$, if $x; y \ge 1$, and assuming WLOG that $x \ge y$,

$$\log \frac{x+y}{2} \le \log \frac{x+x}{2} = \log x \le \log x + \log y$$

Applying this property, since $\alpha_k \ge 1\ \forall k$, we conclude that

$$\begin{aligned}\sum_{i\neq j} \log \frac{\alpha_i + \alpha_j}{2} &\le \sum_{i\neq j} \log \alpha_i + \log \alpha_j \\ &= \sum_{k=1}^{n} (n-1)\log \alpha_k + \sum_{k=1}^{n} (n-1)\log \alpha_k \\ &= 2(n-1)\sum_{k=1}^{n} \log \alpha_k ,\end{aligned}$$

as we wanted to show.

Solution 2 by the proposer. Arranging the zeros of $A(z)$ so that

$$|\alpha_1| \ge |\alpha_2| \ge \dots \ge |\alpha_n|,$$

we have

$$\begin{aligned}|\Delta(A)|^{1/2} = {} & |\alpha_1 - \alpha_n|\,|\alpha_2 - \alpha_n| \dots |\alpha_{n-1} - \alpha_n| \\ & \cdot |\alpha_1 - \alpha_{n-1}|\,|\alpha_2 - \alpha_{n-1}| \dots |\alpha_{n-2} - \alpha_{n-1}| \\ & \vdots \\ & \cdot |\alpha_1 - \alpha_3|\,|\alpha_2 - \alpha_3| \\ & \cdot |\alpha_1 - \alpha_2| .\end{aligned}$$

It follows by the triangle inequality that

$$\begin{aligned}
|\lambda_1 - \lambda_2| &\le |\lambda_1| + |\lambda_2| \le 2|\lambda_1|,\\
|\lambda_1 - \lambda_3||\lambda_2 - \lambda_3| &\le 2^2|\lambda_1||\lambda_2|,\\
&\vdots\\
|\lambda_1 - \lambda_n||\lambda_2 - \lambda_n|\cdots|\lambda_{n-1} - \lambda_n| &\le 2^{n-1}|\lambda_1||\lambda_2|\cdots|\lambda_{n-1}|,
\end{aligned}$$

and, after multiplication, we get

$$\begin{aligned}
|\Delta(A)|^{1/2} &\le 2^{n(n-1)/2}|\lambda_1|^{n-1}|\lambda_2|^{n-2}\cdots|\lambda_{n-1}|\\
&\le 2^{n(n-1)/2}\prod_{k=1}^{n}\max\{1,|\lambda_k|\}^{n-1}.
\end{aligned}$$

Finally, squaring both terms of the preceding inequality, the statement follows.

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