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About one multidimensional sum with Fibonacci numbers

Arkady M. Alt

Abstract

This note is motivated by the following problem: Let $(f_n)_{n\geq 0}$ be the Fibonacci sequence defined by $f_0 = 0$, $f_1 = 1$ and, for all $n \geq 1$, $f_{n+1} = f_n + f_{n-1}$. Determine

$$h_n = \sum_{i,j,k} f_i f_j f_k,$$

where the sum is over $i, j, k > 0$ with $i + j + k = n$. This problem appeared in *Mathematical Horizons* [1], and is also motivated by a problem of Díaz-Barrero [2].

1 Main results

We will consider the general problem of computing the sum

$$S_m(n) := \sum_{\substack{i_1, i_2, \ldots, i_m \geq 0 \\ i_1 + i_2 + \cdots + i_m = n}} f_{i_1} f_{i_2} \cdots f_{i_m}$$

for any non negative integers m and n. (Note that $S_0(0) = 0$ as the sum is over the empty set.) It is easy to see that, in particular,

$S_m(0) = 0$ and $S_1(n) = f_n$. We have

$$
\begin{aligned}
S_m(n) &= \sum_{\substack{i_1,i_2,\ldots,i_m \geq 0\\ i_1+i_2+\cdots+i_m = n}} f_{i_1} f_{i_2} \cdots f_{i_m}\\
&= \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0\\ i_1+i_2+\cdots+i_{m-1} \leq n}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{n-(i_1+i_2+\cdots+i_{m-1})}\\
&= \sum_{k=1}^{n-1} \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0\\ i_1+i_2+\cdots+i_{m-1} = k}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{n-k}
\end{aligned}
$$

and

$$
\begin{aligned}
S_m(n+1) &= \sum_{k=1}^{n} \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0\\ i_1+i_2+\cdots+i_{m-1} = k}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{n+1-k}\\
&= \sum_{k=1}^{n-1} \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0\\ i_1+i_2+\cdots+i_{m-1} = k}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{n+1-k}\\
&\quad + \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0\\ i_1+i_2+\cdots+i_{m-1} = n}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_1\\
&= \sum_{k=1}^{n-1} \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0\\ i_1+i_2+\cdots+i_{m-1} = k}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{n-k}\\
&\quad + \sum_{k=1}^{n-1} \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0\\ i_1+i_2+\cdots+i_{m-1} = k}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{n-1-k}\\
&\quad + \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0\\ i_1+i_2+\cdots+i_{m-1} = n}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}}\\
&= \sum_{k=1}^{n-1} \sum_{\substack{i_1,i_2,\ldots,i_{m-1} \geq 0\\ i_1+i_2+\cdots+i_{m-1} = k}} f_{i_1} f_{i_2} \cdots f_{i_{m-1}} f_{n-k}
\end{aligned}
$$

$$+\sum_{k=1}^{n-2}\sum_{\substack{i_1,i_2,\ldots,i_{m-1}\geq 0\\ i_1+i_2+\cdots+i_{m-1}=k}} f_{i_1}f_{i_2}\cdots f_{i_{m-1}}f_{n-1-k}$$
$$+\sum_{\substack{i_1,i_2,\ldots,i_{m-1}\geq 0\\ i_1+i_2+\cdots+i_{m-1}=n}} f_{i_1}f_{i_2}\cdots f_{i_{m-1}}.$$

Thus,

$$S_m(n+1)=S_m(n)+S_m(n-1)+S_{m-1}(n),\quad n,m\in\mathbb{N}. \tag{1}$$

Using (1) we will constructively find explicit formulas for $S_2(n)$, $S_3(n)$ and $S_4(n)$. Note that, since $f_0=0$, then

$$h_n = S_3(n)=\sum_{\substack{i,j,k\geq 0\\ i+j+k=n}} f_i f_j f_k=\sum_{k=0}^{n}\sum_{\substack{i,j\geq 0\\ i+j=k}} f_i f_j f_{n-k}.$$

We will also use the notation g_n for $S_2(n)$ and s_n for $S_4(n)$. Namely, for $m=1;2;3$, equation (1) becomes, respectively,

$$g_{n+1}=g_n+g_{n-1}+f_n,\quad n\in\mathbb{N}, \tag{2}$$

$$h_{n+1}=h_n+h_{n-1}+g_n,\quad n\in\mathbb{N} \tag{3}$$

and

$$s_{n+1}=s_n+s_{n-1}+h_n,\quad n\in\mathbb{N}. \tag{4}$$

Consider now the Fibonacci operator F defined by $F(a_n)=a_{n+1}-a_n-a_{n-1}$ for any sequence $(a_n)_{n\geq 0}$ of real numbers and, in particular, for any integer k consider two special applications of operator F. Namely,

$$\begin{aligned}
F(a_n f_{n+k}) &= a_{n+1}f_{n+1+k}-a_n f_{n+k}-a_{n-1}f_{n-1+k}\\
&= a_{n+1}(f_{n+1+k}-f_{n+k}-f_{n-1+k})\\
&\quad + a_{n+1}f_{n+k}+a_{n+1}f_{n-1+k}-a_n f_{n+k}-a_{n-1}f_{n-1+k}\\
&=(a_{n+1}-a_n)f_{n+k}+(a_{n+1}-a_{n-1})f_{n-1+k}\\
&=(a_{n+1}-a_n)f_{n+k}+(a_{n+1}-a_{n-1})(f_{n+1+k}-f_{n+k})\\
&=(a_{n+1}-a_{n-1})f_{n+1+k}-(a_n-a_{n-1})f_{n+k},
\end{aligned}$$

so

$$F(a_n f_{n+k}) = (a_{n+1} - a_{n-1}) f_{n+1+k} - (a_n - a_{n-1}) f_{n+k}, \qquad (5)$$

and

$$\begin{aligned} F(a_n f_{n+k+1}) &= a_{n+1} f_{n+k+2} - a_n f_{n+k+1} - a_{n-1} f_{n+k} \\ &= a_{n+1}(f_{n+k+2} - f_{n+k+1} - f_{n+k}) \\ &\quad + a_{n+1} f_{n+k+1} + a_{n+1} f_{n+k} - a_n f_{n+k+1} - a_{n-1} f_{n+k} \\ &= (a_{n+1} - a_n) f_{n+k+1} + (a_{n+1} - a_{n-1}) f_{n+k}, \end{aligned}$$

so

$$F(a_n f_{n+k+1}) = (a_{n+1} - a_n) f_{n+k+1} + (a_{n+1} - a_{n-1}) f_{n+k}. \qquad (6)$$

Note that $F(g_n) = f_n$, $F(h_n) = g_n$ and $F(s_n) = h_n$. Note also that

$$F(a_n) = 0 \Leftrightarrow a_n = (a_1 - a_0) f_n + a_0 f_{n+1},$$

as can be easily proven by induction.

Now we are ready to find g_n, h_n and, afterwards, s_n. Applying (5) and (6) to $a_n = n$ we obtain, for any integer k,

$$F(nf_{n+k}) = 2f_{n+k} - f_{n+k-1} \text{ and } F(nf_{n+k+1}) = f_{n+k+1} + 2f_{n+k}.$$

Since for $k = 0$ we have

$$F(nf_{n+1}) = f_{n+1} + 2f_n \quad \text{and} \quad F(nf_n) = 2f_{n+1} - f_n,$$

thus

$$F(nf_{n+1}) + 2F(nf_n) = f_{n+1} + 2f_n + 2(2f_{n+1} - f_n) = 5f_{n+1}$$

and, therefore,

$$f_{n+1} = F\left(\frac{2f_{n+1} - nf_n}{5}\right) \qquad (7)$$

using the fact F is linear.

We also have

$$2F(nf_{n+1}) - F(nf_n) = 2f_{n+1} + 4f_n - (2f_{n+1} - f_n) = 5f_n,$$

from which

$$f_n = F \quad \frac{2nf_{n+1} \quad nf_n}{5} \quad . \tag{8}$$

Hence, equation (2) is equivalent to

$$F(g_n) = F \quad \frac{2nf_{n+1} \quad nf_n}{5} \quad () \quad F \quad g_n \quad \frac{2nf_{n+1} \quad nf_n}{5} \quad = 0\,,$$

from which we conclude that

$$g_n = \frac{2nf_{n+1} \quad nf_n}{5} + c_1 f_{n+1} + c_2 f_n\,.$$

From the fact that $g_0 = 0$ we get $c_1 \quad 1 + c_2 \quad 0 = 0$ and $c_1 = 0$. Likewise, from $g_1 = 0$ we get $c_1 \quad 1 + c_2 \quad 1 + \frac{2\,1 \quad 1}{5} = 0$ and $c_2 = \quad \frac{1}{5}$. Substituting in the above expression, we obtain

$$g_n = S_2(n) = \frac{2nf_{n+1} \quad (n+1)\,f_n}{5}, \tag{9}$$

and now we can nd h_n. Indeed, applying (5) and (6) to $a_n = n^2$ we obtain

$$\begin{aligned} F(n^2 f_{n+k}) &= ((n+1)^2 \quad (n \quad 1)^2) F_{n+k+1} \quad (n^2 \quad (n \quad 1)^2) f_{n+k} \\ &= 4\,nf_{n+k+1} \quad (2n \quad 1) f_{n+k}\,, \end{aligned}$$

or

$$F(n^2 f_{n+k}) = 4\,nf_{n+k+1} \quad (2n \quad 1) f_{n+k}\,, \tag{10}$$

and

$$\begin{aligned} F(n^2 f_{n+k+1}) &= ((n+1)^2 \quad n^2) F_{n+k+1} \quad (n+1)^2 \quad (n \quad 1)^2) f_{n+k} \\ &= (2\,n+1)\,f_{n+k+1} + 4\,nf_{n+k}\,, \end{aligned}$$

or

$$F(n^2 f_{n+k+1}) = (2\,n+1)\,f_{n+k+1} + 4\,nf_{n+k}\,. \tag{11}$$

In particular, for $k = 0$ in (10) and (11) we obtain

$$F(n^2 f_n) = 4\,f_{n+1} \quad (2n \quad 1) f_n$$

and

$$F(n^2 f_{n+1}) = (2\,n+1)\,f_{n+1} + 4\,nf_n\,.$$

Hence,

$$F(2n^2 f_n) + F(n^2 f_{n+1}) = 8\ nf_{n+1} \quad (4n \quad 2) f_n + (2\ n+1)\ f_{n+1} + 4\ f_n$$
$$= 10\ nf_{n+1} + f_{n+1} + 2\ f_n$$

and

$$\begin{aligned} & 10nf_{n+1} \\ = \ & F(2n^2 f_n + n^2 f_{n+1}) \quad 2f_n \quad f_{n+1} \\ = \ & F(2n^2 f_n + n^2 f_{n+1}) \\ & 2F \ \frac{2nf_{n+1} \quad nf_n}{5} \quad F \ \frac{nf_{n+1} + 2\ nf_n}{5} \\ = \ & F \quad n^2 f_{n+1} + 2\ n^2 f_n \quad \frac{2(2nf_{n+1} \quad nf_n)}{5} \quad \frac{nf_{n+1} + 2\ nf_n}{5} \\ = \ & F((n^2 \quad n) f_{n+1} + 2\ n^2 f_n), \end{aligned}$$

from which

$$nf_{n+1} = F \ \frac{(n^2 \quad n) f_{n+1} + 2\ n^2 f_n}{10} \ . \qquad (12)$$

Likewise, from

$$\begin{aligned} & F(2n^2 f_{n+1}) \quad F(n^2 f_n) \\ = \ & (4\ n+2)\ f_{n+1} + 8\ nf_n \quad (4nf_{n+1} \quad (2n \quad 1) f_n) \\ = \ & 10\ nf_n \quad f_n + 2\ f_{n+1} \end{aligned}$$

we get

$$nf_n = F \ \frac{2n^2 f_{n+1} \quad (n^2 + n) f_n}{10} = F \ \frac{ng_n}{2} \ . \qquad (13)$$

Then, using (13), (12) and (9) we obtain

$$5g_n = 2\ nf_{n+1} \quad (n+1)\ f_n = F \ \frac{(5n^2 + 3\ n) f_n \quad 6nf_{n+1}}{10} \ .$$

That is, $g_n = F \ \frac{(5n^2 + 3\ n) f_n \quad 6nf_{n+1}}{10}$ and, therefore, (2) is equivalent to $F \quad h_n \quad \frac{(5n^2 + 3\ n) f_n \quad 6nf_{n+1}}{50} = 0$ and

$$h_n = \frac{(5n^2 + 3\ n) f_n \quad 6nf_{n+1}}{50} + c_1 f_{n+1} + c_2 f_n .$$

Since $h_0 = 0 = c_1$ and $h_1 = 0 = \frac{2}{50} + c_2$, we get $c_2 = \frac{1}{25}$, from which it follows that

$$h_n = S_3(n) = \frac{(5n^2 + 3n \quad 2)f_n \quad 6nf_{n+1}}{50}.$$

Before considering the computation of s_n we present another way to obtain g_n. Note that $F(F(g_n)) = F(f_n) = 0$ and $F(F(F(h_n))) = F(F(g_n)) = F(f_n) = 0$. Since the characteristic polynomials of $F(F(g_n))$ and $F(F(F(h_n)))$ are $(x^2 \quad x \quad 1)^2$ and $(x^2 \quad x \quad 1)^3$, respectively, then

$$g_n; h_n = P(n) \quad ^n + Q(n)\,\overline{}^n,$$

where and $\overline{}$ are the roots of $x^2 \quad x \quad 1 = 0$ and P and Q are polynomials of degree at most 1 for g_n and at most 2 in the case of h_n. Since ^{n} and $\overline{}^n$ can be represented as linear combinations of f_{n+1} and f_n, then we may also represent g_n and h_n in the form $P(n)f_{n+1} + Q(n)f_n$, namely

$$g_n = (an + b)f_{n+1} + (cn + d)f_n = anf_{n+1}\,(cn + d)f_n$$

because $g_0 = 0$ and

$$\begin{aligned} h_n &= (an^2 + bn + c)f_{n+1} + (pn^2 + qn + r)f_n \\ &= (an^2 + bn)f_{n+1} + (pn^2 + qn + r)f_n \end{aligned}$$

because $h_0 = 0$, where $g_0 = g_1 = 0$, $g_2 = 1$, $g_3 = 2$, $h_0 = h_1 = h_2 = 0$, $h_3 = 1$ and $h_4 = 3$. Then, we get $a = \frac{2}{5}$, $c = \frac{1}{5}$ and $d = \frac{1}{5}$ and, therefore,

$$g_n = \frac{2nf_{n+1} \quad (n+1)f_n}{5}.$$

The same expression may be obtained substituting $g_n = anf_{n+1} + (cn + d)f_n$ in $g_{n+1} \quad g_n \quad g_{n \quad 1} = f_n$.

Now we consider the computation of $s_n = S_4(n)$. Applying (5) and (6) to $a_n = n^3$ for $k = 0$ we obtain

$$\begin{aligned} F(n^3 f_n) &= ((n+1)^3 \quad (n \quad 1)^3)f_{n+1} \quad (n^3 \quad (n \quad 1)^3)f_n \\ &= (6n^2 + 2)f_{n+1} \quad (3n^2 \quad 3n + 1)f_n \end{aligned} \qquad (14)$$

and

$$F(n^3 f_{n+1}) = ((n+1)^3 - n^3) f_{n+1} - ((n+1)^3 - (n-1)^3) f_n$$
$$= (3n^2 + 3n + 1) f_{n+1} + (6n^2 + 2) f_n . \qquad (15)$$

Since $g_n = \frac{2nf_{n+1} - (n+1) f_n}{5}$, $f_n = F(g_n)$, $nf_n = F\left(\frac{ng_n}{2}\right)$ and $nf_{n+1} = F\left(\frac{(n^2 - n)F_{n+1} + 2n^2 f_n}{10}\right)$, then

$$\begin{aligned} & F(2n^3 f_{n+1}) - F(n^3 f_n) \\ &= (6n^2 + 6n + 2) f_{n+1} + (12n^2 + 4) f_n - ((6n^2 + 2) f_{n+1} - (3n^2 - 3n + 1) f_n) \\ &= 15n^2 f_n + 6nf_{n+1} - (3n - 5) f_n \\ &= 15n^2 f_n + 6F\left(\frac{(n^2 - n) f_{n+1} + 2n^2 f_n}{10}\right) - 3F\left(\frac{ng_n}{2}\right) + 5F(g_n) \end{aligned}$$

and

$$\begin{aligned} 15n^2 f_n &= F(2n^3 f_{n+1}) - F(n^3 f_n) - 6F\left(\frac{(n^2 - n) f_{n+1} + 2n^2 f_n}{10}\right) \\ &\quad + 3F\left(\frac{ng_n}{2}\right) - 5F(g_n) \\ &= F\left(\frac{(10 + 7n - 15n^2 - 10n^3) f_n + (20n^3 - 14n) f_{n+1}}{10}\right), \end{aligned}$$

from which we conclude that

$$n^2 f_n = F\left(\frac{(10 + 7n - 15n^2 - 10n^3) f_n + (20n^3 - 14n) f_{n+1}}{150}\right).$$

Since $S_3(n) = h_n = \frac{(5n^2 + 3n - 2) f_n - 6nf_{n+1}}{50}$,

$$nf_{n+1} = F\left(\frac{(n^2 - n) f_{n+1} + 2n^2 f_n}{10}\right)$$

and

$$nf_n = F\left(\frac{2n^2 f_{n+1} - (n^2 + n) f_n}{10}\right).$$

Then, from the fact that $F(S_4(n)) = \quad S_3(n)$ we have

$$F(s_n) = \frac{1}{10}n^2 f_n + \frac{3}{50}nf_n \quad \frac{1}{25}f_n \quad \frac{3}{25}nf_{n+1}$$
$$= F\left(\frac{(11 + 5\;n \quad 30n^2 \quad 5n^3)f_n + (10\;n^3 \quad 10n)f_{n+1}}{750}\right).$$

Hence,

$$s_n = \frac{(11+5\;n \quad 30n^2 \quad 5n^3)f_n + (10\;n^3 \quad 10n)f_{n+1}}{750} + c_1 f_{n+1} + c_2 f_n.$$

Since $s_0 = \;s_1 = 0$, then $c_1 = 0$ and $c_2 = \frac{19}{750}$ and, therefore,

$$s_n = \frac{(11 + 5\;n \quad 30n^2 \quad 5n^3)f_n + (10\;n^3 \quad 10n)f_{n+1}}{750} + \frac{19}{750}f_{n+1}.$$

Finally, rearranging terms we obtain

$$s_n = S_4(n) = \frac{(n \quad 1)(n + 1)(2\;nf_{n+1} \quad (n + 6)\;f_n)}{150}. \tag{16}$$

Remark. We claim that $S_m(n) = \;P_m(n)f_{n+1} + \;Q_m(n)f_n$, where P, Q are polynomials of degree at most $\;m$, because $F^m(S_m(n)) = 0$. Here, $F^m = F \quad F \quad ::: \quad F$ with characteristic polynomial $(x^2 \quad x \quad 1)^m$. Then, the polynomials P and Q can be determined by substitution of $\;S_m(n)$ in (1) assuming that

$$S_{m \quad 1}(n) = \;P_{m \quad 1}(n)f_{n+1} + \;Q_{m \quad 1}(n)f_n$$

and that we know polynomials $P_{m \quad 1}$ and $Q_{m \quad 1}$. Using the fact that $S_m(0) = \;S_m(1) = 0$ we can determine both polynomials. Indeed, since

$$\begin{aligned}
&P_m(n + 1)\;f_{n+2} + \;Q_m(n + 1)\;f_{n+1} \quad P_m(n)f_{n+1}\\
&Q_m(n)f_n \quad P_m(n \quad 1)f_n \quad Q_m(n \quad 1)f_{n \quad 1}\\
=&(\;P_m(n + 1) \quad P_m(n) + \;Q_m(n + 1) \quad Q_m(n \quad 1))\,f_{n+1}\\
+&(\;P_m(n + 1) \quad P_m(n \quad 1) \quad Q_m(n) + \;Q_m(n \quad 1))\,f_n,
\end{aligned}$$

then the fact that $F(S_m(n)) = \;P_{m \quad 1}(n)f_{n+1} + \;Q_{m \quad 1}(n)f_n$ implies

$$P_m(n + 1) \quad P_m(n) + \;Q_m(n + 1) \quad Q_m(n \quad 1) = \;P_{m \quad 1}(n),$$

$$P_m(n + 1) \quad P_m(n \quad 1) + \;Q_m(n) + \;Q_m(n \quad 1) = \;Q_{m \quad 1}(n).$$

References

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Two re nements of power Lessels-Pelling inequality

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Abstract

The Lessels-Pelling inequality states that, in a triangle, the sum of two angle bisectors and a median is at most $\sqrt{3}$ of its semiperimeter. The purpose of this paper is to present re- nements of the powered Lessels-Pelling inequality.

1 Introduction

Let ABC be a triangle. We denote by $a = BC$, $b = CA$ and $c = AB$ the lengths of the sides; $s = \frac{1}{2}(a+b+c)$ the semiperimeter; m_a the median from A; w_b and w_c the bisectors from B and C, respectively.

In 1974, Garfunkel [5] conjectured, on basis of computer check, the following inequality:

$$m_a + w_b + h_c \le s\sqrt{3},$$

where h_c represents the altitude from C. Gardner [4] proved it in 1976 by using a sequence of elementary transformations and a differentiable function. In 1977, Lessels and Pelling [6] gave the stronger inequality

$$m_a + w_b + w_c \le s\sqrt{3}$$

using a computer check. In 1980, Patuwo, Thomas, and Wang [9] gave a proof of this inequality and some chains of inequalities. A new proof due to C. T anasescu appeared in the book of Mitrinović, Pečarić, and Volonec [7]. In 1981, Panaitopol [8] gave an elementary proof of the Lessels-Pelling inequality. Recently, Dr agan [2, 3] gave a simple proof and some re nements of the Lessels-Pelling inequality, and Bencze and Dr agan [1] presented a weighted power version of the Lessels-Pelling inequality. Our goal in this paper is to give re nements of this weighted power inequality.

2 Main results

Hereafter we use the following notation:

$$x = \frac{a}{s}, \quad y = \frac{b}{s}, \quad z = \frac{c}{s}, \quad u = \sqrt{1 \quad y}, \quad v = \sqrt{1 \quad z},$$
$$s_1 = u + v, \quad p_1 = uv, \quad s_2 = \quad u + \quad v,$$
$$E = \frac{\sqrt{2(y^2 + z^2) \quad x^2}}{4}, \quad t = \frac{3 \quad ^2 \quad 2 \quad + 1}{2}, \qquad (1)$$

where , , are positive numbers such that ( )(b c) 0 and + + = 1 .

Lemma 1. With the above notation, the following inequalities hold:

(i) $E \quad \sqrt{1 \quad \frac{s_1^2}{2}}.$

(ii) $E \quad \sqrt{1 \quad \frac{2}{(1 \quad)^2} s_2^2}.$

Proof. (i) From (1) we have

$$E=\sqrt{\frac{2(2+u^4+v^4-2u^2-2v^2)-(u^2+v^2)^2}{4}}$$
$$=\sqrt{\frac{2[2+(u^2+v^2)^2-2p_1^2-2(u^2+v^2)]-(u^2+v^2)^2}{4}}$$
$$=\sqrt{\frac{4+2(s_1^2-2p_1)^2-4p_1^2-4(s_1^2-2p_1)-(s_1^2-2p_1)^2}{4}}$$
$$=\sqrt{\frac{4+s_1^4-4s_1^2+4p_1(2-s_1^2)}{4}}$$
$$\sqrt{\frac{4-4s_1^2+2s_1^2}{4}}=\sqrt{1-\frac{s_1^2}{2}}.$$

(ii) We have

$$s_2=\alpha u+\beta v\ge\frac{1}{2}(\alpha+\beta)(u+v)=\frac{(1-\gamma)s_1}{2}\qquad(2)$$

since, as $(\alpha-\beta)(b-c)\ge 0$, it results that $(\alpha-\beta)(u-v)\ge 0$. The statement follows from (i) and (2). □

Theorem 1. In any triangle ABC the following holds:

$$m_a+w_b+w_c\le m_a+\sqrt{s(s-b)}+\sqrt{s(s-c)}\le\sqrt{\frac{2s^2-s\left(\sqrt{s-b}+\sqrt{s-c}\right)^2}{2}}$$
$$+\frac{1}{2}\left(\sqrt{s(s-b)}+\sqrt{s(s-c)}\right)\le s\sqrt{t},\qquad(3)$$

where α,β,γ are positive numbers such that $(\alpha-\beta)(b-c)\ge 0$ and $\alpha+\beta+\gamma=1$.

Proof. Since $w_b\le\sqrt{s(s-b)}$ and $w_c\le\sqrt{s(s-c)}$, to prove (3) it will be sufficient to prove that

$$m_a + \sqrt{s(s-b)} + \sqrt{s(s-c)} \ge \sqrt{\frac{2S^2 - s\left(\sqrt{s-b} + \sqrt{s-c}\right)^2}{2}} + \frac{1}{2}\left(\sqrt{s(s-b)} + \sqrt{s(s-c)}\right) \ge s\sqrt{t}. \tag{4}$$

On account of (1), inequality (4) may be written as

$$E + S_2 = \sqrt{\frac{2(y^2+z^2) - x^2}{4}} + \sqrt{1-y} + \sqrt{1-z} \ge \sqrt{\frac{2 - S_1^2}{2}} + \frac{1}{2} S_1 \ge \sqrt{t}. \tag{5}$$

From Lemma 1(i) and (2) we have that

$$E + S_2 \ge \sqrt{\frac{2 - S_1^2}{2}} + \frac{1}{2} S_1.$$

To prove (5), it will suffice to show that

$$\sqrt{\frac{2 - S_1^2}{2}} + \frac{1}{2} S_1 \ge \sqrt{t}. \tag{6}$$

We note that

$$S_1 = \sqrt{1-y} + \sqrt{1-z} \le \sqrt{2(2-y-z)} = \sqrt{2x} = \sqrt{\frac{2a}{s}} < \sqrt{2}.$$

So, we have $S_1 < \sqrt{2}$.

We consider the function $f : (0; \sqrt{2}) \to \mathbb{R}$ defined by

$$f(S_1) = \sqrt{\frac{2 - S_1^2}{2}} + \frac{(1 - \;)S_1}{2},$$

for which the derivative is

$$f^{0}(S_1) = \frac{S_1}{\sqrt{4 \quad 2S_1^2}} + \frac{1}{2}.$$

We solve the equation $f^{0}(S_1) = 0$ with the root

$$S_0 = \frac{2(1 \quad)}{\sqrt{6 \;^2 \quad 4 \quad + 2}}.$$

Since $f^{0}(0) \quad 0$ and $f^{0}\sqrt{2} \quad 0$, it follows that S_0 is a maximum point for f. So we have

$$f(S_1) \quad f(S_0) = \sqrt{\frac{1}{2}\, 2 \quad \frac{4(1 \quad)^2}{6 \;^2 \quad 4 \quad + 2}} + \frac{(1 \quad)^2}{\sqrt{6 \;^2 \quad 4 \quad + 2}}$$

$$= \sqrt{\frac{4 \;^2}{6 \;^2 \quad 4 \quad + 2}} + \frac{(1 \quad)^2}{\sqrt{6 \;^2 \quad 4 \quad + 2}}$$

$$= \sqrt{\frac{3 \;^2 \quad 2 \quad + 1}{2}} = \sqrt{t},$$

which nishes the proof of (6). □

Remark 1. Putting $= \quad =$ in (3) we obtain a re nement of the Lessels-Pelling inequality.

Theorem 2. In every triangle ABC the following inequalities hold:

(i) $m_a \quad \frac{\sqrt{t}}{2} \quad \frac{m_a^2}{s} + \frac{}{2\sqrt{t}}\, s;$

(ii) $\frac{\sqrt{t}}{2}\frac{m_a^2}{s} + \frac{}{2\sqrt{t}}\, s \quad \frac{t + \;^2}{2\sqrt{t}}\, s \quad \frac{\sqrt{t}}{(1 \quad)^2} \quad \sqrt{s \quad b} + \sqrt{s \quad c} \;^2;$

(iii) $m_a \quad \sqrt{s^2 \quad \frac{2}{(1 \quad)^2} \quad \sqrt{s \quad b} + \sqrt{s \quad c} \;^2}$

$$\frac{t + \;^2}{2\sqrt{t}}\, s \quad \frac{\sqrt{t}}{(1 \quad)^2} \quad \sqrt{s \quad b} + \sqrt{s \quad c} \;^2,$$

where , , are positive numbers such that + + = 1 and ( )(b c) 0.

Proof. (i) From (1), the inequality in (i) may be written as

$$E \quad \frac{\sqrt{t}}{2}E^{2} + \frac{}{2\sqrt{t}} \quad \text{or, equivalently,} \quad \left(\sqrt{t}E \quad\right)^{2} \quad 0.$$

(ii) From Lemma 1(ii), we have

$$\frac{\sqrt{t}}{2}E^{2} + \frac{}{2\sqrt{t}} \quad \frac{\sqrt{t}}{2} \quad \frac{\sqrt{t}}{(1 \quad)^{2}}S_2^2 + \frac{}{2\sqrt{t}}$$
$$= \left(\frac{t + }{2\sqrt{t}}\right)^{2} \quad \frac{\sqrt{t}}{(1 \quad)^{2}}S_2^2. \qquad (7)$$

Taking in (7) the notation from (1) we obtain the equality from the statement.

(iii) The rst inequality is just the inequality in Lemma 1(ii). We will prove the second inequality, which is equivalent to

$$\sqrt{1 \quad \frac{2}{(1 \quad)^{2}}S_2^2} \quad \left(\frac{t + }{2\sqrt{t}}\right)^{2} \quad \frac{\sqrt{t}}{(1 \quad)^{2}}S_2^2,$$

or

$$\left[2\sqrt{t \quad \frac{2t}{(1 \quad)^{2}}S_2^2 y} \quad\right]^{2} \quad 0. \qquad \square$$

Theorem 3. In every triangle ABC the following chains of inequalities are true:

(i)
$$m_a + w_b + w_c$$
$$m_a + \sqrt{s(s \quad b)} + \sqrt{s(s \quad c)}$$
$$\frac{\sqrt{t}}{2}\frac{ma^{2}}{s} + \frac{^{2}}{2\sqrt{t}}s + \sqrt{s(s \quad b)} + \sqrt{s(s \quad c)}$$
$$\left(\frac{t + }{2\sqrt{t}}\right)^{2} s \quad \frac{\sqrt{t}}{(1 \quad)^{2}} \quad \left(\sqrt{s \quad b} + \sqrt{s \quad c}\right)^{2}$$
$$+ \sqrt{s(s \quad b)} + \sqrt{s(s \quad c)} \quad s\sqrt{t};$$

(ii)
$$\begin{aligned}
m_a + w_b + w_c &\le \sqrt{s^2 - \frac{2}{(1-\alpha)^2}\left(\beta\sqrt{s-b}+\gamma\sqrt{s-c}\right)^2} + w_b + w_c \\
&\le \frac{t+\alpha^2}{2\sqrt{t}}\,s - \frac{\sqrt{t}}{(1-\alpha)^2}\left(\beta\sqrt{s-b}+\gamma\sqrt{s-c}\right)^2 + w_b + w_c \\
&\le \frac{t+\alpha^2}{2\sqrt{t}}\,s - \frac{\sqrt{t}}{(1-\alpha)^2}\left(\beta\sqrt{s-b}+\gamma\sqrt{s-c}\right)^2 \\
&\quad + \sqrt{s(s-b)} + \sqrt{s(s-c)} \le s\sqrt{t},
\end{aligned}$$

where α, β, γ are positive numbers such that $\alpha+\beta+\gamma = 1$ and $(\beta-\gamma)(b-c) \ge 0$.

Proof. (i) The first inequality follows from the inequalities $w_b \le \sqrt{s(s-b)}$ and $w_c \le \sqrt{s(s-c)}$. The second inequality results from Theorem 2(i). The third follows from Theorem 2(ii). It remains to prove that

$$\frac{t+\alpha^2}{2\sqrt{t}}\,s - \frac{\sqrt{t}}{2(1-\alpha)^2}\left(\beta\sqrt{s-b}+\gamma\sqrt{s-c}\right)^2 + \sqrt{s(s-b)} + \sqrt{s(s-c)} \le s\sqrt{t}. \tag{8}$$

Using (1), inequality (8) may be written as

$$\frac{t+\alpha^2}{2\sqrt{t}} - \frac{\sqrt{t}}{(1-\alpha)^2}S_2^2 + S_2 \le \sqrt{t}$$

or, equivalently,

$$tS_2^2 - 2\sqrt{t}(1-\alpha)^2S_2 + (t-\alpha^2)(1-\alpha)^2 \ge 0,$$

or

$$tS_2^2 - 2\sqrt{t}(1-\alpha)^2S_2 + \frac{(1-\alpha)^4}{2} \ge 0,$$

or

$$2\sqrt{t}S_2 - 1 + \alpha^2 \ge 0.$$

(ii) The first inequality from this chain follows from Theorem 2(iii). The second also results from Theorem 2(iii). The third is true since $w_b \le \sqrt{s(s-b)}$ and $w_c \le \sqrt{s(s-c)}$, and the last inequality from the chain is the inequality from (i). □

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Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.

2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to José Luis Díaz-Barrero, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

April 30, 2019

Elementary Problems

E–59. Proposed by Oriol Baeza Guasch, Institut de Terrassa, Terrassa, Spain. Let ABC be a triangle with circumcentre O, and let P be the point of intersection of ray AO with side BC. Denote by D and E the feet of the perpendiculars from P to sides AB and AC, respectively. Show that $BD = CE$ if and only if $AB = AC$.

E–60. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Find all real numbers a, b such that

$$\left(a^2 + \sqrt{1 + a^4}\right)\left(b^2 + \sqrt{1 + b^4}\right) = 1 .$$

E–61. Proposed by Mihaela Berindeanu, Bucharest, România. If $\lfloor a \rfloor$ is the integer part of the real number a, solve in $\mathbb{R}$ the equation

$$\frac{x^2 - x}{x^2 - x + 1} + \frac{2x^2 + 1}{3x} = 0 .$$

E–62. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. On a 2019×2019 board symmetrical with respect to the main diagonal, the numbers from 1 to 2019 are placed so that in each row and column each number appears once and only once. Show that all the numbers appear on the main diagonal.

E–63. Proposed by Mihaela Berindeanu, Bucharest, România. Find all pairs $(p; q)$ of prime numbers for which

$$p\left(p^2 + p + 3\right) = q(q + 4) .$$

E–64. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let $a; b; c$ be three positive real numbers such that $ab + bc + ca = 2abc$. Prove that

$$\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \le 2.$$

Easy–Medium Problems

EM–59. Proposed by Pedro Henrique O. Pantoja, University of Campina Grande, Brazil. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2y + y^2z + z^2x) = xf(x) + f(f(y) + z) + f(zx)$$

for all real numbers x, y, z.

EM–60. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that every problem was solved by at least one of these two students.

EM–61. Proposed by Mihaela Berindeanu, Bucharest, România. Let x, y, z be real numbers. Show that

$$\sum_{\text{cyclic}} \frac{1}{(1+2^{y+1-x})(1+2^{z+1-x})} \geq \frac{1}{3}.$$

EM–62. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let $a_1 \leq a_2 \leq \cdots \leq a_n$ be $n \geq 1$ real numbers lying in the interval $(0, \pi/2)$. Show that

$$a_1 \leq \arctan\frac{\sin a_1 + \sin a_2 + \cdots + \sin a_n}{\cos a_1 + \cos a_2 + \cdots + \cos a_n} \leq a_n.$$

EM–63. Proposed by Oriol Baeza Guasch, Institut de Terrassa, Terrassa, Spain. Let ABC be a triangle with usual notation. Show that

$$\left(\frac{1-\cos\beta}{\sin\beta}\right)\left(\frac{1-\cos\gamma}{\sin\gamma}\right) = 1 - \frac{2a}{a+b+c}.$$

EM–64. Proposed by Mihaela Berindeanu, Bucharest, România. If $z \in \mathbb{C}$ and $|z^2 - 1| = |4z - i|$, then show that $|z| \leq \sqrt{9+\sqrt{85}}$.

Medium–Hard Problems

MH–59. Proposed by Arkady Alt, San Jose, California, USA. Prove that $2nF_{n+1} - (n+1)F_n$ is divisible by 5 for any integer $n \geq 1$. Here, F_n is the n-th Fibonacci number defined by $F_0 = 0$, $F_1 = 1$ and, for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

MH–60. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Prove that in any acute triangle ABC with the usual notations the following inequality holds:

$$\frac{\sin A \sin B}{\cos C} + \frac{\sin B \sin C}{\cos A} + \frac{\sin C \sin A}{\cos B} \geq \frac{9}{4}.$$

MH–61. Proposed by Mihály Bencze, Braşov, Romania. Suppose that $ABCDA_1B_1C_1D_1$ is a rectangle parallelepiped with sides $a; b; c$ and diagonal d. Prove that

$$\frac{a+b+c}{d} + \frac{1}{abc} \geq 4\sqrt{3}.$$

MH–62. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Find all integer solutions of the equation

$$(x+1)(y-1) = x^2y^2.$$

MH–63. Proposed by Pedro Henrique O. Pantoja, University of Campina Grande, Brazil. Let $0 \leq a; b; c < \frac{\pi}{2}$. Prove that

$$8 \tan\frac{a}{2} \tan\frac{b}{2} \tan\frac{c}{2} \leq \tan\frac{(a+b)}{2} \tan\frac{(b+c)}{2} \tan\frac{(c+a)}{2} < \frac{(1+\tan^2\frac{a}{2})(1+\tan^2\frac{b}{2})(1+\tan^2\frac{c}{2})}{(1-\tan^2\frac{a}{2})(1-\tan^2\frac{b}{2})(1-\tan^2\frac{c}{2})}.$$

MH–64. Proposed by Marc Felipe Alsina, BarcelonaTech, Barcelona, Spain. Let ABC be an isosceles triangle with $\angle ABC > 90^\circ$. Let D be the projection of C onto the line AB. Let Γ_1 be the circle centered at A that passes through C and let $_2$ be the circle centered at D that passes through A. Let $_1$ and $_2$ intersect at X and Y. Prove that X and Y belong to the perpendicular to AB through B.

Advanced Problems

A–59. Proposed by Víctor Martín Chabrera, FME, BarcelonaTech, Spain. Compute

$$\sum_{s=2}^{\infty} (\zeta(s) - 1),$$

where $\zeta(s)$ is the Riemann zeta function.

A–60. Proposed by Mihály Bencze, Braşov, Romania. Let $A \in M_2(\mathbb{C})$ with $\mathrm{Tr}(A) = \sqrt{2}/2$. Show that

$$\det\left(A^2 + \frac{3\sqrt{2}}{2}A + 3I_2\right) - \det\left(A^2 - \frac{\sqrt{2}}{2}A\right) = 15.$$

A–61. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. If the coefficients of the power series $\sum_{n=0}^{\infty} a_n z^n$ are given by the recurrence $a_0 = a_1 = 1$ and for all $n \ge 2$, $14a_n + 3a_{n-1} - a_{n-2} = 0$, then find the radius of convergence of the series and the function to which it converges in its disc of convergence.

A–62. Proposed by Nicolae Papacu, Slobozia, Romania. Assume that $A(+,\cdot)$ is a ring such that $1 + 1 + 1$ is invertible. If $x, y \in A$ verify that $x + y = 1$ and $x^3 = x$, then prove that the elements $1 - xy$ and $1 - yx$ are invertible.

A–63. Proposed by Óscar Rivero Salgado, BarcelonaTeach, Barcelona, Spain. Let $k \ge 1$ be a fixed positive integer, and let $n \ge 0$ be a non-negative integer. Show that

$$\sum_{j=0}^{n} \binom{kn}{kj} = \frac{2^{kn}}{k} + \frac{2^{kn+1}}{k} \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^{jn} \cos^{kn}\left(\frac{j\pi}{n}\right).$$

A–64. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Assume that polynomial $A(z)$ with leading coefficient one and degree n has distinct zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$. Prove that

$$|\Delta(A)| \le 2^{n(n-1)} \left(\prod_{k=1}^{n} \max\{1, |\alpha_k|\} \right)^{2n-2},$$

where $\Delta(A)$ is the discriminant of $A(z)$.

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to José Luis Díaz-Barrero, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Theorems and problems in classical geometry

José Luis Díaz-Barrero and Marc Felipe Alsina

1 Introduction

In classical elementary plane geometry, there are two kinds of lines that usually are most drawn, straight lines and circles. In this note, three well-known results involving the above mentioned elements are presented. These results may be applied to solve problems that frequently appear in mathematical competitions, similar to the ones given in this Mathlesson.

2 Theorems

We begin with a result that establishes a relationship between the lengths of the sides and the length of a cevian in a triangle due to the Scottish mathematician Matthew Stewart, who published it in 1746 [3].

Theorem 1 (Stewart). Let D be a point in the side BC of triangle ABC. Let $m = BD$, $n = DC$, and $p = AD$. Then,

$$b^2 m + c^2 n = a(p^2 + mn).$$

Proof. Since $\angle ADB + \angle ADC = \quad$, then $\cos\angle ADB + \cos\angle ADC = 0$. Taking into account the law of cosines, we have

$$\frac{m^2 + p^2 \quad c^2}{2mp} + \frac{n^2 + p^2 \quad b^2}{2np} = 0,$$

Figure 1: Scheme for Stewart's theorem.

or equivalently,

$$n(m^2 + p^2 - c^2) + m(n^2 + p^2 - b^2) = 0\ ,$$

from which it follows that

$$b^2 m + c^2 n = (m + n)(p^2 + mn) = a(p^2 + mn),$$

as claimed. □

Corollary 1 (Length of the median). In any triangle ABC the length of the median is given by the expression

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \quad \text{(cyclic)}.$$

Proof. Applying Stewart's theorem with $m = n = a=2$ and $p = m_a$; we have

$$\frac{b^2 a}{2} + \frac{c^2 a}{2} = a\left(m_a^2 + \frac{a^2}{4}\right),$$

from which we get $4m_a^2 = 2b^2 + 2c^2 - a^2$, and the statement follows. □

The preceding expressions are also known in the literature as Apollonius formulae .

Next we present a fascinating result in classical Euclidean geometry, the Butter y theorem.

Theorem 2 (Butter y). *Let M be the midpoint of a chord PQ of a circle, through which two other chords AB and CD are drawn. Chords AD and BC intersect PQ at points X and Y, respectively. Then, M is the midpoint of XY.*

Figure 2: Scheme for Butter y's theorem

Proof. We begin by dropping perpendiculars $x_1 = XX''$ and $y_1 = YY''$ from X and Y to AB, and $x_2 = XX'$ and $y_2 = YY'$ from X and Y to CD, as shown in Figure 2. Letting $a = PM = MQ$, $x = XM$, $y = MY$, and observing the pairs of similar triangles $\triangle MXX'' \sim \triangle MYY''$, $\triangle MXX' \sim \triangle MYY'$, $\triangle AXX'' \sim \triangle CYY'$ and $\triangle DXX' \sim \triangle BYY''$, yields

$$\frac{MX}{MY} = \frac{XX''}{YY''} \Longrightarrow \frac{x}{y} = \frac{x_1}{y_1},$$
$$\frac{MX}{MY} = \frac{XX'}{YY'} \Longrightarrow \frac{x}{y} = \frac{x_2}{y_2},$$
$$\frac{AX}{CY} = \frac{XX''}{YY'} \Longrightarrow \frac{AX}{CY} = \frac{x_1}{y_2},$$
$$\frac{DX}{BY} = \frac{XX'}{YY''} \Longrightarrow \frac{DX}{BY} = \frac{x_2}{y_1}.$$

From the preceding, and taking into account the power of points X and Y respect to the circle, we get

$$\frac{x^2}{y^2} = \frac{x_1 x_2}{y_1 y_2} = \frac{x_1}{y_2} \cdot \frac{x_2}{y_1} = \frac{AX \cdot XD}{CY \cdot YB} = \frac{PX \cdot XQ}{PY \cdot YQ}$$
$$= \frac{(a - x)(a + x)}{(a + y)(a - y)} = \frac{a^2 - x^2}{a^2 - y^2} = \frac{(a^2 - x^2) + x^2}{(a^2 - y^2) + y^2} = \frac{a^2}{a^2} = 1,$$

from which $x = y$ follows, as we wanted to prove. □

According to Coxeter and Greitzer [1], one of the proofs to the Butterfly theorem was submitted in 1815 by W. G. Horner of Horner's method fame [2]. However, two recent discoveries have made it clear that Wallace's proof came before Horner's proof by about ten years. The first of those discoveries is an 1803 publication of *The Gentlemen's Mathematical Companion* containing a generalization of the problem by Wallace. The second is a correspondence from Sir William Herschel to Wallace in 1805. In the letter, Herschel presents the problem of proving that the two line segments XM and YM in the theorem are equal in length. In turn, he is asking Wallace for a proof of the generalized butterfly theorem.

Finally, we present and give three proofs of a classical result of Wittaker appeared in 1820 [4].

Theorem 3 (Napoleon's theorem). *If equilateral triangles AGB, BHC and CKA are constructed on the sides of any triangle ABC, then the centers of the circumcircles of those equilateral triangles D, E, F themselves form an equilateral triangle.*

Proof 1. We begin with the following well-known result.

Lemma 1 (Fermat's point). *Lines KB, HA and GC in Figure 3 meet at point P. (This point is called Fermat's point.)*

Proof. Let P be the intersection of AH and KB. Triangles KCB and ACH are congruent because the second is a 60° rotation of the first about point C. So, $\angle CKP = \angle CAP$ and $\angle CHP = \angle CBP$. From the preceding, it immediately follows that quadrilaterals $KAPC$ and $HBPC$ are concyclic. Thus,

Figure 3: Scheme for Fermat's theorem.

$\angle APC = \angle APB = \angle BPC = 120$. Since $\angle APB$ and $\angle AGC$ add up to 180, then $APBG$ is also concyclic. Hence, $\angle APG = \angle ABG = 60$ (both have the common arc AG). Finally, since $\angle APG + \angle APC = 180$, then points G, P and C are collinear. □

Figure 4: Scheme for proof 1 of Napoleon's theorem.

Now we give the first proof of the theorem. Indeed, let P be the intersection point of the circumcircles of the equilateral triangles ABG, BCH and CAK. Then, by the inscribed angle theorem (capacious arc), $\angle APC = 180 \quad \angle AKC$ (cyclic). That is,

$$\angle APC = \angle BPC = \angle APB = 120 .$$

Since PA, PB and PC are, respectively, radical axis of two circles, then $PQ \perp EF$, $PR \perp DF$ and $PS \perp DE$, as is well-known. Now, considering the quadrilateral $FRPQ$, we have $\angle RFQ + \angle APC = 180$ or, equivalently, $\angle RFQ + 120 = 180$, from which it follows that $\angle RFQ = 60$. Likewise, we obtain that $\angle RDS = \angle QSD = 60$, and the proof is complete. □

Proof 2. In Figure 5 we have that $\angle ACK = \angle BCH = 60$. Adding to each $\angle ACB$, we have $\angle BCK = \angle ACH$. Since sides $AC = KC$ and $CH = CB$, then $\triangle BKC$ and $\triangle AHC$ are congruent and $AH = BK$.

Figure 5: Scheme for proof 2 of Napoleon's theorem.

Extend BD and BE to L and M, respectively. Then, since D and E are the centers of the equilateral triangles ABG and CBH, then $\angle ABL = \angle CBM = 30$. Moreover, $BD = \frac{2}{3} BL$ and $BE = \frac{2}{3} BM$ as is well-known. Triangles BCM and ABL are similar. Therefore,

$$\frac{AB}{BC} = \frac{BL}{BM} = \frac{\sqrt{3}/2\ BD}{\sqrt{3}/2\ BE} = \frac{BD}{BE}.$$

Since $\angle CBE + \angle ABD = \angle CBH$. then adding $\angle ABC$ to each, we get $\angle DBE = \angle ABH$ and triangles DBE and ABH are similar. Likewise, triangles AKB and ADF are similar too. Hence,

$$\frac{AB}{AH} = \frac{BD}{DE} \quad \text{and} \quad \frac{AB}{BK} = \frac{AD}{DF}.$$

Since we have already seen that $BK = AH$ and $AD = BD$, then we get $DE = DF$. A similar procedure leads us to obtain that $DF = FE$. So, $\triangle DEF$ is equilateral and the proof is complete. □

Remark 1. The relation $\frac{AB}{AH} = \frac{BD}{DE}$ can also be obtained from the right triangles BMH and ALB. Indeed, in $\triangle BMH$ we have

$$\cos 30^\circ = \frac{BM}{BH} = \frac{\sqrt{3}/2\, BE}{BH},$$

and in $\triangle ALB$ we have

$$\cos 30^\circ = \frac{BL}{AB} = \frac{\sqrt{3}/2\, BD}{AB},$$

from which it follows that $\frac{AB}{AH} = \frac{BD}{DE}$.

Finally, we give a third proof using complex numbers. We need the following lemma.

Lemma 2. Let z_1, z_2, z_3 be the coordinates of the vertices A_1, A_2, A_3 of a positively oriented triangle. Then, the following holds:

$$\triangle A_1A_2A_2 \text{ is equilateral} \iff z_1 + e^{i2\pi/3}z_2 + e^{i4\pi/3}z_3 = 0.$$

Proof. It is well-known that $\triangle A_1A_2A_2$ is equilateral and positively oriented if and only if A_3 is obtained from A_2 by rotation about A_1 through an angle of $\pi/3$. Namely,

$$z_3 - z_1 = e^{i\pi/3}(z_2 - z_1),$$

from which it follows that

$$z_3 = z_1 + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)(z_2 - z_1) = \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) z_1 + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_2 .$$

Therefore,

$$\begin{aligned} & z_1 + e^{i2\pi/3} z_2 + e^{i4\pi/3} z_3 \\ &= z_1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_2 \\ &+ \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left[\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) z_1 + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_2\right] \\ &= z_1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_2 - z_1 + \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) z_2 = 0 , \end{aligned}$$

and the proof is complete. □

Proof 3. Let z_A, z_B, z_C be the affixes of vertices A, B and C, respectively (see Figure 3). On account of Lemma 2, we have

$$z_A + \omega z_C + \omega^2 z_k = 0 , \; z_G + \omega z_B + \omega^2 z_A = 0 , \; z_B + \omega z_H + \omega^2 z_C = 0 ,$$

where $\omega = e^{i2\pi/3}$. The centers of gravity of triangles $\triangle GAB$, $\triangle HBC$ and $\triangle KAC$ are, respectively,

$$\begin{aligned} z_D &= \frac{1}{3}\left(z_A + z_B + z_G\right) , \\ z_F &= \frac{1}{3}\left(z_A + z_C + z_K\right) , \\ z_E &= \frac{1}{3}\left(z_B + z_C + z_H\right) . \end{aligned}$$

Finally, we have

$$\begin{aligned} & 3(z_D + \omega z_E + \omega^2 z_F) \\ &= (z_A + z_B + z_G) + \omega(z_B + z_C + z_H) + \omega^2(z_A + z_C + z_K) \\ &= (z_A + \omega z_c + \omega^2 z_K) + (z_B + \omega z_H + \omega^2 z_C) + (z_G + \omega z_B + \omega^2 z_A) \\ &= 0 , \end{aligned}$$

and $\triangle DEF$ is equilateral. □

3 Problems

Hereafter, some examples we used in training sessions for Mathematical Contests are presented. We begin with the following.

Problem 1. On a circle of radius R, four points A, B, C, D are chosen such that AC is a chord perpendicular to BD and both intersect at a point P. Show that

$$\frac{1}{AP^2}+\frac{1}{BP^2}+\frac{1}{CP^2}+\frac{1}{DP^2}=\frac{4}{R^2}.$$

Solution. First, we claim that $AP^2+BP^2+CP^2+DP^2=4R^2$. Indeed, since $4R^2=(2R)^2$ is the square of a diameter of the circle, then it suggests to draw a diameter, say AE in Figure 6. We have that $\triangle ABE$ and $\triangle ACE$ are right triangles on account of Thales's theorem. By Pythagoras, applied to $\triangle ABE$, we get

$$4R^2=AE^2=AB^2+BE^2.$$

Figure 6: Scheme for solving Problem 1.

From right triangle APB we have that $AB^2=AP^2+BP^2$. Thus,

$$4R^2=AE^2=AB^2+BE^2=(AP^2+BP^2)+BE^2$$

and it only remains to prove that $BE^2 = CP^2 + DP^2$. On the other hand, in the right triangle DPC we have $DP^2 + PC^2 = DC^2$. Finally, it remains to prove that $DC^2 = BE^2$ or, taking square roots, $DC = BE$. Looking at the gure we see that $\angle BPC = \angle ACE = 90$, from which it follows that $BP \parallel CE$, and $\angle DBC = \angle BCE$ by Thales's theorem (a transversal BC cuts two parallel). Now, by the inscribed angle theorem (spinning arc) we conclude that $DC = BE$ because both are subtended by the same angle , and the claim is proven.

Now, we apply Cauchy's inequality to the vectors

$$\vec{u} = \frac{AP}{R}; \frac{BP}{R}; \frac{CP}{R}; \frac{DP}{R}$$

and

$$\vec{v} = \frac{R}{AP}; \frac{R}{BP}; \frac{R}{CP}; \frac{R}{DP}$$

to obtain

$$16 \quad \frac{AP^2}{R^2} + \frac{BP^2}{R^2} + \frac{CP^2}{R^2} + \frac{DP^2}{R^2} \qquad \frac{R^2}{AP^2} + \frac{R^2}{BP^2} + \frac{R^2}{CP^2} + \frac{R^2}{DP^2}$$

$$= 4 \quad \frac{R^2}{AP^2} + \frac{R^2}{BP^2} + \frac{R^2}{CP^2} + \frac{R^2}{DP^2},$$

from which we get

$$4 \quad \frac{R^2}{AP^2} + \frac{R^2}{BP^2} + \frac{R^2}{CP^2} + \frac{R^2}{DP^2}$$

and the statement follows. Equality holds when P is at the center of the circle and the chords are perpendicular diameters. □

The next problem appeared in the Spanish Mathematical Olympiad in 2018.

Problem 2. Let AB be the diameter of a semicircle . Let D be any point on the tangent to at B and lying on the same side of AB as , and let C be the mid-point of BD. The segments AC and AD intersect for the second time at points K and L, respectively. If M and N are the projections onto KL of A and B, respectively, show that $ML = LK = KN$.

Figure 7: Scheme for solving Problem 2.

Solution. Let P be the second point of intersection of the line NB and the circle . Then, $\angle BPL = \angle BAL$ and $\angle BPK = \angle BAK$, so that triangles LPN and DAB are similar. Hence, PK is the median of $\triangle LPN$, because AC is the median of $\triangle DAB$. Thus $LK = KN$.

Let X be the mid-point of LK. Since $OL = OK$, then $OX \perp LK$. On the other hand, O is the mid-point of AB and $ABNM$ is a trapezoid with right angles at M and N, whence X is also the midpoint of MN. It follows that $ML = KN$, which completes the proof. □

Finally, we give an example which appeared in IMO 2017.

Problem 3. Let R and S be points on a circle such that RS is not a diameter. Let ℓ be the tangent line to at R. Point T is such that S is the midpoint of the line segment RT. Point J is chosen in the shorter arc RS of so that the circumcircle ω of triangle JST intersects ℓ at two distinct points. Let A be the common point of ω and ℓ that is closer to R. Line AJ meets again at K. Prove that line KT is tangent to ω.

Figure 8: Scheme for solving Problem 3.

Solution. On gure 8 we observe that points A, J, S, T line on ω, so quadrilateral $AJST$ is cyclic and we have

$$\angle ATS = 180 \quad \angle SJA = \angle SJK = \angle SRK = \angle TRK,$$

from which $RK \parallel TA$. Now construct the parallelogram $RATP$. Since RST is a diagonal, then ASP is also a diagonal and A, S, P are collinear. Thus, by semi-inscribed angles we have $\angle SKR = \angle SRA = \angle PTR$. Since

$$\angle PKS = 180 \quad \angle SKR = 180 \quad \angle SRA = 180 \quad \angle PTR = 180 \quad \angle PTS,$$

then it follows that quadrilateral $PKST$ is cyclic. Then, we have

$$\angle KTS = \angle KPS = \angle SAT.$$

The last equality holds on account that AP is a transversal. Since in circle ω we have that $\angle KTS = \angle SAT$, then $\angle KTS$ is semi-inscribed and KT is tangent to ω. □

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Contests

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to nd interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to José Luis Díaz-Barrero, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Problems and solutions from the 59th edition of the International Mathematical Olympiad (IMO)

Óscar Rivero Salgado

1 Introduction

The 59th edition of the International Mathematical Olympiad took place in June 2018 in Cluj-Napoca. More than 100 countries sent delegations to the competition, formed by a maximum of six members. As usual, it was developed in two consecutive days, where the participants had to solve three problems each day in a maximum time of four hours and a half. The Spanish team was formed by the six gold medallists in the Spanish Mathematical Olympiad: Alejandro Epelde Blanco, Rodrigo Marlasca Aparicio, Félix Moreno Peñarrubia, Santiago Vázquez Sáez, Martín Gómez Abejón and José Pérez Cano. The delegation was completed by María Gaspar, as the chief of the delegation, and by Óscar Rivero, as the deputy leader.

We present now the four problems which were solved at least by some Spaniard contestant (problems 1, 2, 4 and 5) and include all the correct solutions that were given to them. It is worth mentioning that it is the second time (the rst being in 2008) in which all the contestants are able to fully solve at least one problem.

Santiago and Alejandro got a bronze medal, while Félix, Martín, Rodrigo and José got an honorable mention.

The solutions follow the ideas presented by the contestants, but we have done some little modi cations to ease the exposition.

2 Problems and solutions

We now present the four problems that appeared in the paper, as well as some solutions and comments to them.

Problem 1. Let be the circumcircle of acute-angled triangle ABC. Points D and E lie on segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of BD and CE intersect the minor arcs AB and AC of at points F and G, respectively. Prove that the lines DE and FG are parallel (or are the same line).

Solution 1 by Alejandro Epelde. Let M be the midpoint of the minor arc BC in . Then, DE is perpendicular to AM and it is enough to see that FG is perpendicular to AM, too. Let $_B$ be the circle centered at F and of radius $FB = FD$. We de ne $_C$ in the same way. Let X and Y denote the intersection of with $_B$ and $_C$, respectively. Since $\angle XAF = \angle FAB$, it holds that $FX = FD$, and since angles $\angle FXA = \angle FDA$ are obtuse, we have that triangles XAF and FAD are equal. Then, $AX = AD$ and, in the same way, $AY = AE$. Let be the angle seen by arcs AX and AY in ($= \angle ABX$). Then, if , and stand for the angles of triangle ABC,

$$\angle ABF = + \angle XBF = + \frac{\angle AMB \qquad \angle AMX}{2} = \frac{+}{2}.$$

In the same way,

$$\angle ACG = \frac{+}{2}.$$

Therefore, the angle between AM and GF is

$$\angle ABF + \angle GAM = \frac{+ \quad +}{2} = \quad = 2,$$

as desired.

Solution 2 by Martín Gómez. It is enough to prove that FG is perpendicular to the angle bisector of $\angle BAC$, that we call . To see this, let I and J stand for the midpoints of BD and CE, respectively, and let H be the intersection point of FI and GJ. Firstly, we prove that H lies over a line which is parallel to the angle bisector of $\angle BAC$ and contains the circumcenter of ABC, that we call O. If $c = |AB|$ and $d = |AD|$, then $DI = \frac{b \quad d}{2}$. This means that when we move D and E and consider new points D', E', H', one has that $DD' = EE'$. Further, considering points H_1 and H_2 lying over $D'H'$ and $H'E'$ in such a way that HH_1 is parallel to DD' and HH_2 is parallel to EE', we see that triangles $H'HH_2$ and $H'HH_1$ are equal and, consequently, HH' is an angle bisector of $\angle DHE$ and $\angle H'HE = \quad = 2 + \quad = 2$. Therefore, H is always over a line which forms an angle of $\quad = 2$ with line AC. Moreover, this line contains O by considering the limit case when $A \quad D \quad E$.

Let A' be the intersection of that line with , and let B' and C' be points over such that $A'B'$ and $A'C'$ are parallel to AB and to AC, respectively. Then, $A'B'$ and $A'C'$ are symmetric lines with respect to the diameter $A'O'$, and F and G are the intersections of with the perpendiculars to $A'B'$ and $A'C'$, drawn from H. The symmetry of the con guration shows that $HF = HG$, and the bisector of segment FG is precisely HO, which is parallel to the angle bisector of $\angle BAC$, as desired.

Solution 3 by Félix Moreno. Let us denote $\angle A = 2$, $\angle B = 2$, $\angle C = 2$, $\angle FBA = \quad$ and $\angle ACG = \quad$. We clearly have that $\angle GBC = \angle GAC = 2$. Since GEC is isosceles, $\angle GEC = \quad$. From here, we deduce that $\angle AEG = \quad$ and $\angle EGA = 2 \quad 2$. Analogously, $\angle DFA = 2 \quad 2$. Now we show that $\angle EGA = \angle DFA$. Using the sine's law in triangles AGE and AGC,

$$\frac{AE}{\sin \angle EGA} = \frac{AG}{\sin \quad} = \frac{AC}{\sin(\quad 2 \quad)} = \frac{AC}{\sin 2 \quad}$$

and, in the same way,

$$\frac{AD}{\sin \angle DFA} = \frac{AB}{\sin 2}.$$

Using the sine's law in ABC together with the fact that $AD = AE$, we conclude that $\sin \angle DFA = \sin \angle EGA$. Now, the equality between the angles follows since both are acute (to see this, we observe that is acute and then $\angle FDA$ is obtuse).

Let X and Y be the intersection points of FG with AB and AC, respectively. Since $\angle FGA =$ and $\angle GAY = 2$ we have that $\angle FYA = 2 +$. Since we have seen that $\angle DFA = \angle EGA$, we conclude that $=$, and then

$$\angle FYA = + = = 2 = \angle DFA,$$

concluding the proof.

Solution 4 by José Pérez. Let D' and E' be the points of such that $AD' = AD = AE = AE'$, with D' in the minor arc of AB. First, we show that E', D and F are collinear. For this purpose, let E'' be the second intersection point of FD and . Then,

$$\angle FBD = \angle FDB = \angle ADE'' = \angle AE''F,$$

where in the last equality we have used that $BFAE''$ is a cyclic quadrilateral and, therefore, $\angle FBD = \angle AE''F$. Then, ADE'' is isosceles and E' E'', as desired. Analogously, we prove that D', E and G are also collinear. Furthermore, $\angle E'DE = \angle E'D'E$. Hence,

$$\angle E'D'E = \angle E'D'G = \angle E'FG,$$

and we see that

$$\angle E'DE = \angle E'FG.$$

From here, we conclude that DE and FG are either the same line or they are parallel.

Solution 5 by Santiago Vázquez. In this solution, let , and denote the angles of triangle ABC. Let O be the center of and P the intersection of the bisectors of segments BD and CE. Let

S and T denote the midpoints of BD and CE, respectively, and F' and G' the second intersection points with of lines PF and PG. The line through O and perpendicular to GG' intersects GG' at Q. Finally, we define H and I as the intersections of DE with FF' and GG', respectively.

Since triangle ADE is isosceles, $\angle DHS = \; /2$. Observe that, since OQ is parallel to CT,

$$OQ = \frac{b}{2} - CT = \frac{b}{2} - \frac{b-d}{2} = \frac{d}{2},$$

where $d = |AD| = |AE|$. This shows that the distance from O to FF' is $d/2$ and, by a symmetric argument, the distance to GG' is the same, and both chords have the same length.

From here, $\angle G'GF = \angle F'FG$, and since $ASPT$ is cyclic, $\; = \angle SPT = \angle FPG$ and then $\angle SFG = \angle PFG = \; /2$. Then, $\angle DHS = \angle GFS$, and therefore HD is parallel to FG, as claimed.

Problem 2. Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots, a_{n+2}$ such that $a_{n+1} = a_1$ and $a_{n+2} = a_2$, and

$$a_i a_{i+1} + 1 = a_{i+2}$$

for $i = 1, 2, \ldots, n$.

Solution by Alejandro Epelde. We begin by observing that when n is a multiple of 3 we can consider sequences of the form

$$(-1, -1, 2, -1, -1, 2, \ldots).$$

Now we prove that it is not possible to obtain a sequence like that when n is not a multiple of 3. We begin with some preliminary observations:

(i) It is not possible to have two consecutive positive numbers: if $a_1, a_2 > 0$, then $a_3 > 1$ and now we can easily prove by induction that the sequence is strictly increasing, which contradicts the statement.

(ii) If $a_1 = 0$, then $a_2; a_3 > 0$ and this is not possible by the previous remark.

(iii) If $a_i; a_{i+1} < 0$, then $a_{i+2} > 0$.

We now prove that if $a_0 > 0$, the next positive number is a_3. In other words, we are saying that if n were not a multiple of 3, in some moment there would be a sequence of positive, negative, positive, and we may assume without loss of generality that $a_0; a_2 > 0$ and $a_1 < 0$. In this case,

$$a_3 = a_1 a_2 + 1 = a_1(a_0 a_1 + 1) + 1 = a_0 a_1^2 + 1 + a_1 < 0,$$

so $a_1 < 1$. Since $a_2 = a_0 a_1 + 1 > 0$, we have that $a_0 < 1$ and also $a_2 < 1$. Then, if we can prove that the sequence of negative number is strictly increasing we are done since this precludes the possibility of having cycles (since $a_{n+1} = a_1$ and $a_{n+2} = a_2$ we can interpret the problem as a sequence of n real numbers). Going backwards (that is, considering the terms $a_1; a_2; \ldots$ with the indexes taken modulo n), an easy induction step shows that, whenever i is even, $0 < a_i < 1$, and whenever i is odd, $a_i < 1$. Then, the sequence of negative numbers is increasing, as shows the inequality

$$a_3 \quad a_1 = a_0 a_1^2 + 1 > 0.$$

This is a contradiction, since the sequence eventually comes back to the starting point.

Problem 4. A site is any point $(x; y)$ in the plane such that x and y are both positive integers less than or equal to 20. Initially, each of the 400 sites is unoccupied. Ana and Beto take turns placing stones with Ana going first. On her turn, Ana places a new red stone on an unoccupied site such that the distance between any two sites occupied by red stones is not equal to $\sqrt{5}$. On his turn, Beto places a new blue stone on any unoccupied site. (A site occupied by a blue stone is allowed to be at any distance from any other occupied site.) They stop as soon as a player cannot place a stone. Find the greatest K such that Ana can ensure that she places at least K red stones, no matter how Beto places his blue stones.

Solution by Rodrigo Marlasca. We first show that 100 is always possible. For that, color the plane with two colors, black and white, as in a chessboard. If Ana always chooses white positions, she can make sure to place at least 100 stones in such a way that no two of them are at a distance $\sqrt{5}$, since positions at a distance $\sqrt{5}$ always belong to different colors.

For the converse, consider first the 4×4 configuration given below.

A	B	C	D
C	D	A	B
B	A	D	C
D	C	B	A

Beto can play in such a way that when Ana selects one letter he takes the "opposite" position with the same letter (with "opposite" we refer to the symmetric one with respect to the center of the square). That way, Ana can place at most 4 stones. Dividing the plane into 4×4 squares and dividing each square like that, we conclude that Ana can place at most 100 stones, as claimed.

Remark. The solution written by the contestant considered a wider casuistic, but we have chosen to write it in a more simplified way.

Problem 5. Let $a_1, a_2, \ldots$ be an infinite sequence of positive integers. Suppose that there is an integer $N > 1$ such that, for each $n \geq N$, the number

$$\frac{a_1}{a_2} + \frac{a_2}{a_3} + \cdots + \frac{a_{n-1}}{a_n} + \frac{a_n}{a_1}$$

is an integer. Prove that there is a positive integer M such that $a_m = a_{m+1}$ for all $m \geq M$.

Solution by Santiago Vázquez. From the conditions of the statement we have that

$$C_n = \frac{a_n}{a_{n+1}} + \frac{a_{n+1}}{a_1} \quad \frac{a_n}{a_1} \quad 1$$

$$= \frac{a_n a_1 + a_{n+1}^2 \quad a_n a_{n+1} \quad a_1 a_{n+1}}{a_1 a_{n+1}}$$

$$= \frac{(a_{n+1} \quad a_1)(a_{n+1} \quad a_n)}{a_1 a_{n+1}}$$

is an integer for all n large enough. Let $v_p(n)$ the greatest non-negative integer such that p divides n. We prove that $v_p(a_n)$ is constant as n goes to in nity. We distinguish two cases: when $v_p(a_1) > 0$ and when $v_p(a_1) = 0$.

In the latter (when p does not divide a_1), we have that $v_p(a_n)$ $v_p(a_{n+1})$ for all n N: if this were not the case, we would have that $v_p(C_n) = v_p(a_n) \quad v_p(a_{n+1}) < 0$. This shows that the sequence $v_p(a_n)$ becomes constant, since it is decreasing and bounded from below. Observe, in fact, that for all but nitely many primes, $v_p(a_N) = 0$ and, therefore, after a nite number of steps we have achieved convergence for the p-adic valuation at all the primes not dividing a_N. In particular, if $v_p(a_N) = 0$, then $v_p(a_n)$ for all n N.

Suppose now that $v_p(a_1) > 0$, which only happens for a nite number of primes. We rst assume that $v_p(a_n)$ $v_p(a_1)$ for some n N. If $v_p(a_{n+1}) > v_p(a_n)$, then

$$v_p(C_n) = v_p(a_1) + v_p(a_n) \quad v_p(a_{n+1}) \quad v_p(a_1) < 0.$$

Similarly, if $v_p(a_{n+1}) < v_p(a_1)$, then

$$v_p(C_n) = 2\, v_p(a_{n+1}) \quad v_p(a_{n+1}) \quad v_p(a_1) < 0.$$

Then, $v_p(a_1)$ $v_p(a_{n+1})$ $v_p(a_n)$ and the sequence eventually becomes constant. If it happens that $v_p(a_k) < v_p(a_1)$ for all $k > N$, then

$$v_p(C_n) = v_p(a_{n+1}) + v_p(a_{n+1} \quad a_n) \quad v_p(a_1) \quad v_p(a_{n+1}),$$

which is necessarily smaller than 0 unless $v_p(a_n) = v_p(a_{n+1})$ (if $v_p(a_n) \neq v_p(a_{n+1})$, then $v_p(a_n \quad a_{n+1}) = \min\{v_p(a_n); v_p(a_{n+1})\}$). Then, $v_p(a_n)$ also becomes constant after some point, and we are done.

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Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

Please, send submittals to José Luis Díaz-Barrero, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Elementary Problems

E–53. Proposed by Marc Felipe Alsina, BarcelonaTech, Barcelona, Spain. Let x, y, z be three positive numbers adding up to 1. Find the minimum value of

$$\frac{4}{x} + \frac{9}{y} + \frac{25}{z}.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY, USA. The minimum value is 100. The Cauchy-Schwarz inequality (Engel's form) yields

$$\frac{4}{x} + \frac{9}{y} + \frac{25}{z} \geq \frac{(2+3+5)^2}{x+y+z} = 100.$$

Equality holds if and only if $2/x = 3/y = 5/z$. With the given condition $x + y + z = 1$, this implies that the minimum value of 100 is attained when $x = 1/5$, $y = 3/10$ and $z = 1/2$.

Solution 2 by José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain. This expression is equivalent to

$$\frac{2}{\frac{x}{2}}+\frac{3}{\frac{y}{3}}+\frac{5}{\frac{z}{5}}.$$

Now, we consider the harmonic mean of two copies of $\frac{x}{2}$, three copies of $\frac{y}{3}$ and five copies of $\frac{z}{5}$, and we get

$$\frac{10}{\frac{2}{\frac{x}{2}}+\frac{3}{\frac{y}{3}}+\frac{5}{\frac{z}{5}}}\le\frac{1}{10}\left(2\frac{x}{2}+3\frac{y}{3}+5\frac{z}{5}\right)=\frac{1}{10}.$$

This shows that $\frac{4}{x}+\frac{9}{y}+\frac{25}{z}\ge 100$. Therefore, the minimum quantity is at least 100. For the bound to be attained, it is necessary that $\frac{x}{2}=\frac{y}{3}=\frac{z}{5}$. Taking into account that $x+y+z=1$, this can only occurs when $x=\frac{2}{10}=\frac{1}{5}$, $y=\frac{3}{10}$ and $z=\frac{5}{10}=\frac{1}{2}$. One can check that indeed we get 100 in the given expression when substituting these numbers.

Also solved by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain, and the proposer.

E–54. Proposed by José Luis Díaz-Barrero and José Gibergans Báguena, BarcelonaTech, Barcelona, Spain. Compute the following sum:

$$\lfloor\sqrt{1}\rfloor+\lfloor\sqrt{2}\rfloor+\lfloor\sqrt{3}\rfloor+\cdots+\lfloor\sqrt{n^2-1}\rfloor.$$

(Here, $\lfloor x\rfloor$ represents the integer part of x).

Solution by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom. For each integer $n\ge 1$ let

$$S(n):=\sum_{i=1}^{n^2-1}\lfloor\sqrt{i}\rfloor,$$

and for each $m\ge 1$ let

$$T(m):=\sum_{i=(m-1)^2}^{m^2-1}\lfloor\sqrt{i}\rfloor.$$

From this definition, it is clear that

$$S(n) = \sum_{m=1}^{n} T(m). \qquad (1)$$

Note, in particular, that $T(1) = 0$.

Recall that the positive $\sqrt{x}$ is an increasing function. We know that $\sqrt{(m-1)^2} = m-1$ and $\sqrt{m^2} = m$, and this means that $\lfloor\sqrt{i}\rfloor = m-1$ for all $i \in \{(m-1)^2, (m-1)^2+1, \ldots, m^2-1\}$. Therefore,

$$T(m) = (m^2 - 1 - (m-1)^2 + 1)(m-1) = (2m-1)(m-1).$$

Substituting this into (1) we conclude that

$$\begin{aligned} S(n) &= \sum_{m=1}^{n} (2m-1)(m-1) = 2\sum_{m=1}^{n} m^2 - 3\sum_{m=1}^{n} m + n \\ &= 2\,\frac{n(n+1)(2n+1)}{6} - 3\frac{n(n+1)}{2} + n \\ &= \frac{4n^3 - 3n^2 - n}{6}. \end{aligned}$$

Also solved by Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain, and the proposer.

E–55. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* If a and b are real numbers for which the equation $x^6 + ax^5 + x^4 + bx^2 + 4 = 0$ has a real root, then prove that $4a^2 + b^2 \geq 16$.

Solution by the proposer. If the given equation has a real root, say x, it must be nonzero, as can be easily checked. Since $x \neq 0$, we may divide by x^4 both terms of the equation and we get

$$x^2 + ax + 1 + \frac{b}{x^2} + \frac{4}{x^4} = 0.$$

Completing squares yields

$$\left(x+\frac{a}{2}\right)^2+\left(\frac{b}{4}+\frac{2}{x^2}\right)^2-\frac{a^2}{4}-\frac{b^2}{16}+1=0,$$

or

$$\left(x+\frac{a}{2}\right)^2+\left(\frac{b}{4}+\frac{2}{x^2}\right)^2=\frac{a^2}{4}+\frac{b^2}{16}-1.$$

On account that the sum of squares is at least zero, we have

$$\frac{a^2}{4}+\frac{b^2}{16}-1\geq 0,$$

from which it follows that $4a^2+b^2\geq 16$.

E–56. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. In a classroom we have n girls and two boys. There are n seats in a row, in which we want to sit n students. We say that two arrangements are equivalent if, for each chair, the gender of the student sitting in that chair in both arrangements is the same. How many non-equivalent arrangements can we find?

Solution by Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain. One way to solve the problem is to count in how many ways we can choose the seats for the boys. We can divide the problem into three cases:

Case 1: 0 boys. Then, there is only one way to choose 0: not choosing any. That is, all the girls sit in whatever order they want.
Case 2: 1 boy. Then, we have $\binom{n}{1}=n$ ways to choose a seat for the boy (he can sit on the first seat, on the second...).
Case 3: 2 boys. We have $\binom{n}{2}=\frac{n(n-1)}{2}$ ways to choose two seats out of the n seats.

Therefore, the number or arrangements in which the n girls and the 2 boys can seat is:

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}=1+n+\frac{n(n-1)}{2}=\frac{n^2+n+2}{2}$$

Also solved by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom, and the proposer

E–57. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Two sides of a triangle have length 7 and 11, respectively. If the median drawn to the third side has length 7, then nd the length of the third side.

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY, USA. Consider triangle ABC, with $b = 7$, $c = 11$, and median $d = 7$ originating from vertex A. We invoke Apollonius's theorem, which relates the length of a median of a triangle to the lengths of its sides. The theorem says that

$$b^2 + c^2 = 2\left(\frac{a}{2}\right)^2 + 2d^2; \text{ or } 7^2 + 11^2 = 2\left(\frac{a}{2}\right)^2 + 2(7)^2.$$

This equation leads to the conclusion that $a = 12$.

Solution 2 by Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain. Let ABC be the triangle of the problem, being AB the side of length 7, AC the side of length 11 and AD the median drawn from A to the third side, that is, BC (see Figure 1).

Figure 1: Scheme for Solution 2 to Problem E–57.

The median divides this last side into two segments of equal length, namely a. By the cosine theorem, we have that

$$\begin{cases} 7^2 = 11^2 + a^2 \quad 2 \quad 11 \quad a \quad \cos \quad , \\ 7^2 = 11^2 + 4a^2 \quad 2 \quad 11 \quad 2a \quad \cos \end{cases}$$

$$\Longrightarrow \begin{cases} 49 = 121 + a^2 \quad 22 \quad a \quad \cos \quad , \\ 49 = 121 + 4a^2 \quad 44 \quad a \quad \cos \quad . \end{cases}$$

If we multiply the rst equation by 2 and subtract from it the second one, we obtain

$$49 = 121 \quad 2a^2 \Longrightarrow 2a^2 = 121 \quad 49 = 72 \Longrightarrow a^2 = 36$$
$$\Longrightarrow a = 6 \Longrightarrow 2a = 12 .$$

Therefore, the length of the third side is 12.

Solution 3 by the proposer. To solve this problem we will use the following result due to Stewart: Let D be a point in the side BC of triangle ABC. Then,

$$b^2 m + c^2 n = a(p^2 + mn),$$

where $m = BD; n = DC;$ and $p = AD$.

Putting $b = 7$, $c = 11$, $m = n = a/2$, and $p = 7$ in the above result yields

$$49\frac{a}{2} + 121\frac{a}{2} = a \quad 49 + \frac{a^2}{4} \quad \Longleftrightarrow \quad 85a = a \quad 49 + \frac{a^2}{4} \quad ,$$

with solutions $\{0; 12; \quad 12\}$. Then, the only valid solution (non degenerate triangle) is $a = 12$, and we are done.

E–58. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let F_n be the n-th Fibonacci number, de ned by $F_1 = F_2 = 1$ and for all $n \quad 1$, $F_{n+2} = F_{n+1} + F_n$. Express in terms of F_n and F_{n+1} the following sum:

$$\sum_{1 \quad i<j \quad n} F_i F_j .$$

Solution by Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain. For our calculations, we need to calculate the sum of the first n Fibonacci numbers and the sum of the squares of the first n Fibonacci numbers.

Let us try to find a formula by induction:

$$F_1 = 1 = 2 - 1 = F_3 - 1,$$
$$F_1 + F_2 = 1 + 1 = 2 = 3 - 1 = F_4 - 1,$$
$$F_1 + F_2 + F_3 = 1 + 1 + 2 = 4 = 5 - 1 = F_5 - 1,$$
$$F_1 + F_2 + F_3 + F_4 = 1 + 1 + 2 + 3 = 7 = 8 - 1 = F_6 - 1.$$

So it seems that the formula will be $\sum_{i=1}^{n} F_i = F_{n+2} - 1$. Let us suppose this is true for $n = k$. Then, for $n = k + 1$,

$$\sum_{i=1}^{k+1} F_i = F_{k+2} - 1 + F_{k+1} = F_{k+3} - 1,$$

which proves the formula.

Now, let us find a formula for the sum of the squares. We have

$$F_1^2 = 1 = 1 \cdot 1 = F_1 \cdot F_2,$$
$$F_1^2 + F_2^2 = 1 + 1 = 2 = 1 \cdot 2 = F_2 \cdot F_3,$$
$$F_1^2 + F_2^2 + F_3^2 = 1 + 1 + 4 = 6 = 2 \cdot 3 = F_3 \cdot F_4,$$
$$F_1^2 + F_2^2 + F_3^2 + F_4^2 = 1 + 1 + 4 + 9 = 15 = 3 \cdot 5 = F_4 \cdot F_5.$$

So it seems like $\sum_{i=1}^{n} F_i^2 = F_n \cdot F_{n+1}$. Let us suppose this is true for $n = k$. Then, for $n = k + 1$,

$$\sum_{i=1}^{k+1} F_i^2 = F_k \cdot F_{k+1} + F_{k+1}^2 = F_{k+1} \cdot (F_k + F_{k+1}) = F_{k+1} \cdot F_{k+2},$$

which fits the formula for $n = k + 1$. Therefore, it is true.

Now,

$$\sum_{1 \le i<j \le n} 2F_iF_j = \left[\left(\sum_{i=1}^{n} F_i\right)^2 - \sum_{i=1}^{n} F_i^2\right]$$

$$= (F_n + F_{n+1} - 1)^2 - F_n F_{n+1}$$

$$= F_n^2 + F_{n+1}^2 + 1 + 2F_nF_{n+1} - 2F_n - 2F_{n+1} - F_nF_{n-1}$$

$$= F_n^2 + F_{n+1}^2 - 2(F_n + F_{n+1}) + 1 .$$

Therefore,

$$\sum_{1 \le i<j \le n} F_iF_j = \frac{1}{2}\left(F_n^2 + F_{n+1}^2 - 2(F_n + F_{n+1}) + 1\right).$$

Also solved by Henry Ricardo, Westchester Area Math Circle, NY, USA, and the proposer.

Easy–Medium Problems

EM–53. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. [Correction] Let a, b, c be distinct positive real numbers such that

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}.$$

Prove that

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} > \frac{9}{2(a+b+c)}.$$

Solution by the proposer. Since a, b, c are distinct reals, then from the conditions given in the statement it follows that $a - b = \frac{1}{c} - \frac{1}{b} = \frac{b-c}{bc}$, $b - c = \frac{c-a}{ca}$ and $c - a = \frac{a-b}{ab}$. Multiplying up the preceding expressions yields

$$(a-b)(b-c)(c-a) = \frac{(b-c)(c-a)(a-b)}{a^2b^2c^2},$$

or $a^2b^2c^2 = 1$, from which it follows that $abc = 1$. Applying the AM-GM inequality, we get

$$a + b + c > 3\sqrt[3]{abc} = 3.$$

Putting

$$\vec{u} = \left(\sqrt{b+c}, \sqrt{c+a}, \sqrt{a+b}\right)$$

and

$$\vec{v} = \left(\frac{a}{\sqrt{b+c}}, \frac{b}{\sqrt{c+a}}, \frac{c}{\sqrt{a+b}}\right)$$

into Cauchy's inequality, we get

$$(a+b+c)^2 \leq 2(a+b+c)\left(\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}\right),$$

that combined with the last inequality yields

$$9 < 2(a+b+c)\left(\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}\right),$$

from which the claimed inequality follows.

Remark. The original statement contained a mistake which was noticed by Miguel Amengual Covas. If the three numbers are allowed to be equal, then the statement is not true, so the condition that a, b, c are different is necessary.

EM–54. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let a and b be two positive integers. Prove that for any odd prime p the equation $\operatorname{lcm}(a; 2a+p) = \operatorname{lcm}(b; 2b+p)$ implies $a = b$.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Let D be the greatest common divisor of a and $2a+p$. We have $D \mid a$ and $D \mid 2a+p$. Hence $D \mid (2a+p) - 2a$, which gives $D \mid p$, and this in view of p being prime, implies $D \in \{1; p\}$.

Analogously, denoting by D' the greatest common divisor of the numbers b and $2b+p$, $D' \in \{1; p\}$.

Since the product of two natural numbers is equal to the product of their least common multiple and their greatest common divisor, we can write the given equation in the equivalent form

$$\frac{a(2a+p)}{D} = \frac{b(2b+p)}{D'}, \tag{1}$$

where $D; D' \in \{1; p\}$.

We claim that the above equation implies $D = D'$. To see this, suppose on the contrary that, say, $D = 1$ and $D' = p$. When these are substituted into (1), we get

$$a(2a+p) - b = \frac{2b^2}{p}. \tag{2}$$

Since p is odd, p and 2 are relatively prime natural numbers and, hence, using both the assumption that p is a prime and (2), we obtain that the number b^2, and hence the number b, is divisible by p. Then, we have

$$b = pm, \tag{3}$$

where m is a natural number.

In (2) we substitute pm for b, obtaining

$$m(2m+1) \quad a = \frac{2a^2}{p}$$

and, by the same reasoning used to derive (3), we obtain $p \mid a$, which implies $p \mid 2a+p$. Thus, p is a common divisor of the numbers a and $2a+p$. This, together with the assumption that a and $2a+p$ are relatively prime natural numbers, implies that $p \mid 1$, contradicting the fact that p is a prime number.

We conclude that $D = D'$.

Multiplying both sides of (1) by $D = D'$, we find that the given equation becomes $a(2a+p) = b(2b+p)$, which is easily brought in the form

$$(a - b)(p + 2(a+b)) = 0 .$$

From here, since $p + 2(a+b) \neq 0$ (because p is odd and $2(a+b)$ is even), we get $a - b = 0$, so

$$a = b,$$

as desired.

Also solved by the proposer.

EM–55. Proposed by Mihaela Berindeanu, Bucharest, Romania. Let $x; y; z; t$ be positive real numbers such that $x+y+z+t = 2$. Show that

$$\left(\frac{4}{x^2} - 1\right)\left(\frac{4}{y^2} - 1\right)\left(\frac{4}{z^2} - 1\right)\left(\frac{4}{t^2} - 1\right) \geq 15^4.$$

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. According to the arithmetic mean-geometric mean inequality,

$$\frac{x+x+y+z+t}{5} \geq \sqrt[5]{x^2yzt}, \qquad \frac{y+z+t}{3} \geq \sqrt[3]{yzt}.$$

Therefore,

$$\frac{4}{x^2} - 1 \geq \frac{15\sqrt[5]{x^2yzt}\sqrt[3]{yzt}}{x^2}. \tag{1}$$

In the same way, we find that the factors $\frac{4}{y^2} - 1$, $\frac{4}{z^2} - 1$, $\frac{4}{t^2} - 1$ on the left side of the proposed inequality satisfy

$$\frac{4}{y^2} - 1 \geq \frac{15\sqrt[5]{xy^2zt}\sqrt[3]{xzt}}{y^2}, \tag{2}$$

$$\frac{4}{z^2} - 1 \geq \frac{15\sqrt[5]{xyz^2t}\sqrt[3]{xyt}}{z^2}, \tag{3}$$

$$\frac{4}{t^2} - 1 \geq \frac{15\sqrt[5]{xyzt^2}\sqrt[3]{xyz}}{t^2}. \tag{4}$$

Multiplying the four inequalities (1) to (4) yields

$$\left(\frac{4}{x^2} - 1\right)\left(\frac{4}{y^2} - 1\right)\left(\frac{4}{z^2} - 1\right)\left(\frac{4}{t^2} - 1\right) \geq 15^4,$$

which is what we set out to prove.

Equality occurs when $x = y = z = t$.

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Let

$$\Pi = \left(\frac{4}{x^2} - 1\right)\left(\frac{4}{y^2} - 1\right)\left(\frac{4}{z^2} - 1\right)\left(\frac{4}{t^2} - 1\right).$$

Clearly,

$$\Pi = \frac{1}{x^2y^2z^2t^2}(2+x)(2-x)(2+y)(2-y)(2+z)(2-z)(2+t)(2-t).$$

Substituting $x+y+z+t$ for 2 in each parenthesis gives

$$\Pi = \frac{(x+x+y+z+t)(y+z+t)(x+y+y+z+t)(x+z+t)\cdot(x+y+z+z+t)(x+y+t)(x+y+z+t+t)(x+y+z)}{x^2y^2z^2t^2}.$$

When multiplied out, the numerator N in this fraction contains $5 \cdot 3 \cdot 5 \cdot 3 \cdot 5 \cdot 3 \cdot 5 \cdot 3 = 50625$ terms, each containing eight factors from $\{x; y; z; t\}$. If the product of these 50625 terms is P, the AM-GM inequality gives

$$\frac{N}{50625} \geq P^{\frac{1}{50625}}.$$

But the symmetry of the factors of N shows that no variable is favored and that, when multiplied out, each of x, y, z, t will occur the same number of times throughout the expression. Since each of the 50625 terms of N contains 8 factors, for a total of 405000, each of x, y, z, t must occurs $405000/4 = 101250$ times, and we have that the product

$$P = x^{101250} y^{101250} z^{101250} t^{101250}.$$

Hence,

$$\frac{N}{50625} \geq P^{\frac{1}{50625}} = \left(x^{101250} y^{101250} z^{101250} t^{101250}\right)^{\frac{1}{50625}} = x^2y^2z^2t^2,$$

that is,

$$\frac{N}{15^4} \geq x^2y^2z^2t^2,$$

which is equivalent to the desired inequality. Equality occurs when $x = y = z = t$.

Also solved by the proposer.

EM–56. Proposed by Nicolae Papacu, Slobozia, Romania. Solve the equation $p + [x] = [px]$, where p is a positive integer. (Here, $[a]$ denotes the integer part of a).

Solution by the proposer. For $p = 1$ the given equation has no solution. Suppose $p \geq 2$. From the fact that $[x] = n$ for some $n \in \mathbb{Z}$ we get $x \in [n; n+1)$. From the fact that $[px] = p + n$ we conclude that $px \in [p+n; p+n+1)$ and $x \in \left[\frac{n+p}{p}; \frac{n+p+1}{p}\right)$. Therefore,

$$x \in [n; n+1) \cap \left[\frac{n+p}{p}; \frac{n+p+1}{p}\right).$$

Now we distinguish the following cases:

(i) For $n \quad 0$ we have $n+1 < \dfrac{n+p}{p}$ and

$$[n; n+1) \ \backslash \ \frac{n+p}{p}; \frac{n+p+1}{p} \ = \ ?\,.$$

(ii) For $n = 1$ we have

$$x \ 2 \ [n; n+1) \ \backslash \ \frac{n+p}{p}; \frac{n+p+1}{p} \ = [1\,; 2) \ \backslash \ \frac{p+1}{p}; \frac{p+2}{p}$$

$$= \ \frac{p+1}{p}; \frac{p+2}{p} \ .$$

(iii) For $n = 2$ we have

$$x \ 2 \ [2; 3) \ \backslash \ \frac{p+2}{p}; \frac{p+3}{p} \ .$$

The last interval becomes $2; \dfrac{5}{2}$ for $p = 2$ and, for $p \quad 3$, we have

$$x \ 2 \ [2; 3) \ \backslash \ \frac{p+2}{p}; \frac{p+3}{p} \ = \ ?\,.$$

(iv) For $n \quad 3$ we have $1 + \dfrac{n+1}{p} \quad 1 + \dfrac{n+1}{2} \quad n$ and

$$x \ 2 \ [2; 3) \ \backslash \ \frac{p+2}{p}; \frac{p+3}{p} \ = \ ?\,.$$

Thus, the only solutions are

$$x \ 2 \ \frac{p+1}{p}; \frac{p+2}{p} \ \text{ for } n = 1 \text{ and } x \ 2 \ \ 2; \frac{5}{2} \ \text{ for } n = 2\,.$$

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona Spain.

EM–57. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. Let S be a set of 17 distinct positive integers. If all prime divisors of each number in S are smaller than 8, then prove that there are two of them whose product is a square.

Solution by the proposer. The form of the elements $x \in S$ is $x = 2^a 3^b 5^c 7^d$. Modulo 2, for the quadruple $(a; b; c; d)$ we have $2^4 = 16$ possibilities. Since we have 17 distinct numbers in S then, by the pigeonhole principle, there are two elements $x_1 = 2^{a_1} 3^{b_1} 5^{c_1} 7^{d_1}$ and $x_2 = 2^{a_2} 3^{b_2} 5^{c_2} 7^{d_2}$ such that $a_1 \equiv a_2 \pmod 2$, $b_1 \equiv b_2 \pmod 2$, $c_1 \equiv c_2 \pmod 2$ and $d_1 \equiv d_2 \pmod 2$. Then, the numbers $a_1 + a_2$, $b_1 + b_2$, $c_1 + c_2$ and $d_1 + d_2$ are even and the number $x_1 x_2$ is a perfect square.

Also solved by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom, and José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain.

EM–58. Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. A circle of radius r is inscribed in a triangle ABC. Tangents to the circle and parallel to the sides of the triangle are drawn. These lines cut three small triangles off $\triangle ABC$. If r_1, r_2 and r_3 are the radii of the inscribed circles to the small triangles, then prove that

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \ge \frac{18}{R},$$

where R is the circumradius of $\triangle ABC$.

Solution 1 by the proposer. On account of Euler's inequality, $R \ge 2r$, we have that

$$\frac{R}{2}\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) \ge r\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right).$$

Hence, the claimed inequality will be established if we prove

$$r\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) \ge 9.$$

Figure 2: Construction for Solution 1 to Problem EM–58.

Indeed, the perimeter of triangle AMN is

$$p_1 = AM + NP + MP = AM + NY + MX = AX + AY.$$

Likewise, the perimeter of triangle BZX is $p_2 = BX + BZ$ and the perimeter of triangle CYZ is $p_3 = CY + CZ$. Thus, if p is the perimeter of $\triangle ABC$, then $p = p_1 + p_2 + p_3$. On the other hand, we have that $\triangle AMN \quad \triangle ABC$ and $\frac{p_1}{p} = \frac{r_1}{r}$ (cyclic). Then,

$$\frac{r_1}{r} + \frac{r_2}{r} + \frac{r_3}{r} = \frac{p_1}{p} + \frac{p_2}{p} + \frac{p_3}{p} = 1,$$

from which it follows that $r = r_1 + r_2 + r_3$.

On account of mean inequalities, we have

$$\frac{r}{3} = \frac{r_1 + r_2 + r_3}{3} = 3 \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right)^{1},$$

or

$$r \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) \quad 9.$$

Equality holds when $r_1 = r_2 = r_3 = r{=}3$, that is, when $\triangle ABC$ is equilateral, and we are done.

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Let B', C' be points on sides AB, CA of $\triangle ABC$, respectively, such that $B'C' \parallel BC$ and $B'C'$ is tangent to the incircle of $\triangle ABC$. Let s be the semiperimeter of triangle ABC and let the sides BC, CA, AB be labelled a, b, c, respectively. Let the incircle touch AB at D. Then,

$$AD = s - a.$$

Since the incircle of $\triangle ABC$ touches one side of $\triangle AB'C'$ internally and the other two sides (extended) externally, it is an excircle of $\triangle AB'C'$. Hence, the tangents from A to the incircle of triangle ABC and the semiperimeter of $\triangle AB'C'$ are equal in length, so

$$AD = \text{semiperimeter of } \triangle AB'C'.$$

Thus,

$$\text{semiperimeter of } \triangle AB'C' = s - a.$$

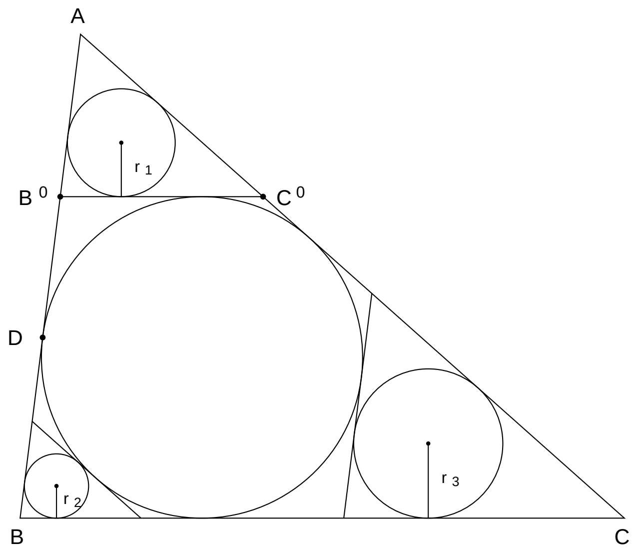

Figure 3: Construction for Solution 2 to Problem EM–58.

Because $B'C' \parallel BC$, triangle ABC is similar to $\triangle AB'C'$ and the inradius of $\triangle ABC$ has the same ratio to the inradius r_1 of $\triangle AB'C'$ as any segment of $\triangle ABC$ has to the corresponding segment of $\triangle AB'C'$. In particular, the perimeter of $\triangle ABC$ and the perimeter of $\triangle AB'C'$ are in this ratio. Therefore,

$$\frac{r}{r_1} = \frac{\text{perimeter of } \triangle ABC}{\text{perimeter of } \triangle AB'C'} = \frac{\text{semiperimeter of } \triangle ABC}{\text{semiperimeter of } \triangle AB'C'} = \frac{s}{s-a}.$$

and

$$\frac{1}{r_1} = \frac{s}{r(s-a)}. \tag{1}$$

In the same way (referring to Figure 3), we find that the radii r_2, r_3 of the incircles of the other cut-off triangles satisfy

$$\frac{1}{r_2} = \frac{s}{r(s-b)} \quad \text{and} \quad \frac{1}{r_3} = \frac{s}{r(s-c)}. \tag{2}$$

Adding, we find

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s}{r}\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right).$$

We shall show that

$$\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \geq \frac{9}{s}, \tag{3}$$

which, together with Euler's inequality ($R \geq 2r$), implies the desired result. Equality occurs only if $\triangle ABC$ is equilateral.

To prove (3), by the definition of s, we have

$$s = (s-a) + (s-b) + (s-c).$$

Hence, the product $s\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right)$ may be written in the form

$$((s-a) + (s-b) + (s-c))\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right).$$

Applying the AM-GM inequality to each factor, we get

$$\begin{aligned} & s\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right) \\ &= ((s-a) + (s-b) + (s-c))\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right) \geq 9, \end{aligned}$$

which is equivalent to (3).

Solution 3 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Rewriting equations (1) and (2) from Solution 2 in the form

$$r_1 = \frac{r(s-a)}{s}, \qquad r_2 = \frac{r(s-b)}{s}, \qquad r_3 = \frac{r(s-c)}{s}$$

and adding, we obtain

$$r_1 + r_2 + r_3 = r.$$

Hence,

$$r\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) = (r_1 + r_2 + r_3)\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) \geq 9$$

or, equivalently,

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \geq \frac{9}{r},$$

and using Euler's inequality we obtain

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \geq \frac{18}{R},$$

where equality holds if and only if $\triangle ABC$ is equilateral.

Also solved by Scott H. Brown, Montgomery, AL, USA.

Medium–Hard Problems

MH–53. Proposed by Pedro Henrique O. Pantoja, University of Campina Grande, Brazil. Let x, y, z be positive real numbers. Prove that

$$\frac{x^2+y^2+z^2}{(x+y+z)^2}+\frac{648(x+y+z)^2}{(x^2+y^2)(y^2+z^2)(z^2+x^2)} \geq 12.$$

Solution by the proposer. By the AM-GM inequality, we have

$$\begin{aligned}
\text{LHS} &= \frac{3(x^2+y^2+z^2)}{3(x+y+z)^2}+\frac{648(x+y+z)^2}{(x^2+y^2)(y^2+z^2)(z^2+x^2)}\\
&\geq 4\sqrt[4]{\frac{648(x^2+y^2+z^2)^3(x+y+z)^2}{27(x+y+z)^2(x^2+y^2)(y^2+z^2)(z^2+x^2)}}\\
&= 4\sqrt[4]{\frac{24(x^2+y^2+z^2)^3}{(x^2+y^2)(y^2+z^2)(z^2+x^2)}}\\
&= 4\sqrt[4]{\frac{3[(x^2+y^2)+(y^2+z^2)+(z^2+x^2)]^3}{(x^2+y^2)(y^2+z^2)(z^2+x^2)}}\\
&\geq 4\sqrt[4]{\frac{3[3\sqrt[3]{(x^2+y^2)(y^2+z^2)(z^2+x^2)}]^3}{(x^2+y^2)(y^2+z^2)(z^2+x^2)}} = 4\sqrt[4]{81} = 12.
\end{aligned}$$

Also solved by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.

MH–54. Proposed by Mihaela Berindeanu, Bucharest, România.
Let ABC be an acute triangle, AP be the bisector from A, AS be the symmedian from A, BD be the median from B and $P, S \in BC$; $D \in AC$. If $AP \cap BD = \{O\}$, $AS \cap BD = \{Q\}$ and $\frac{[BPO]}{[AOD]} = \frac{8}{15}$, calculate $\frac{BQ}{QD}$.

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Suppose the parallel through C to BD meets AS extended at T (Figure 4).

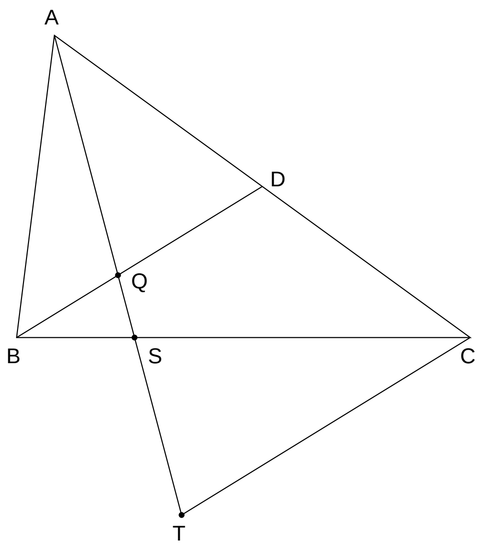

Figure 4: First construction for Solution 1 to Problem MH–54.

From similar triangles QBS and TCS,

$$\frac{BQ}{CT} = \frac{BS}{SC}.$$

But a symmedian of an angle of a triangle divides the opposite side in the ratio of the squares of the sides about the angle, so

$$\frac{BS}{SC} = \frac{AB^2}{CA^2}.$$

Thus,

$$\frac{BQ}{CT} = \frac{AB^2}{CA^2}. \qquad (1)$$

Similar triangles AQD and ATC yield $\frac{QD}{CT} = \frac{DA}{CA}$, and since $CD = DA$, then

$$\frac{QD}{CT} = \frac{1}{2}. \qquad (2)$$

The quotient of equations (1) and (2) is

$$\frac{BQ}{QD} = 2\left(\frac{AB}{CA}\right)^2. \qquad (3)$$

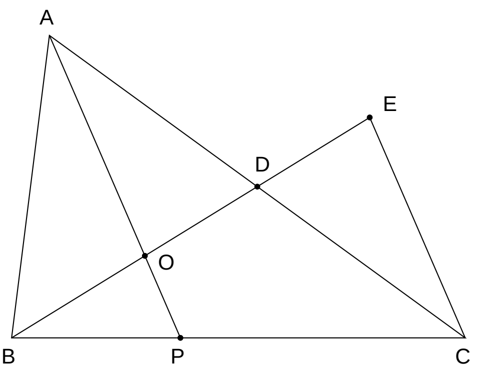

Figure 5: Second construction for Solution 1 to Problem MH–54.

We now suppose the parallel through C to AP meets BD extended at E (Figure 5). We deduce that $4\ AOD$ $4\ CED$; since their corresponding sides AD and DC are equal, triangles AOD and CED are, in fact, congruent, yielding $OD = DE$ and $AO = CE$. Thus, $OE = 2\ \ OD$.

The parallel lines AP and CE also give $4\ BOD$ similar to $4\ BEC$, and we have

$$\frac{OP}{AO} = \frac{OP}{CE} = \frac{BO}{BE} = \frac{BO}{BO + OE} = \frac{BO}{BO + 2\ \ OD} = \frac{\frac{BO}{OD}}{\frac{BO}{OD} + 2}. \quad (4)$$

By the internal angle bisector theorem, applied to $4\ ABD$,

$$\frac{BO}{OD} = \frac{AB}{DA} = \frac{AB}{\frac{1}{2}CA} = 2\ \ \frac{AB}{CA}. \quad (5)$$

We substitute this expression for $\frac{BO}{OD}$ into (4), simplify, and obtain

$$\frac{OP}{AO} = \frac{AB}{AB + CA}. \quad (6)$$

Next we express $[BOP]$ and $[AOD]$ in the form

$$[BPO] = \frac{1}{2}\ BO\ \ OP\ \ \sin(\backslash\ BOP),$$
$$[AOD] = \frac{1}{2}\ AO\ \ OD\ \ \sin(\backslash\ AOD).$$

Since $\angle BOP$ and $\angle AOD$ are vertical angles, we have $\angle BOP = \angle AOD$. Therefore, dividing the above expressions, we obtain

$$\frac{8}{15} = \frac{[BPO]}{[AOD]} = \frac{BO \cdot OP}{AO \cdot OD}.$$

Substituting for $\frac{BO}{OD}$ and $\frac{OP}{AO}$ from (5) and (6), we obtain

$$\frac{4}{15} = \frac{AB^2}{CA(AB + CA)}.$$

We rewrite the last equation in the form

$$15AB^2 - 4AB \cdot CA - 4CA^2 = 0$$

or

$$(3AB - 2CA)(5AB + 2CA) = 0.$$

When $5AB + 2CA = 0$, $\frac{AB}{CA} = -\frac{2}{5}$, which is not admissible. When $3AB - 2CA = 0$, $\frac{AB}{CA} = \frac{2}{3}$ and, by (3),

$$\frac{BQ}{QD} = \frac{8}{9}.$$

Solution 2 by the proposer. Throughout this solution, we use the notation $c := AB$, $b := AC$ and $a := BC$.

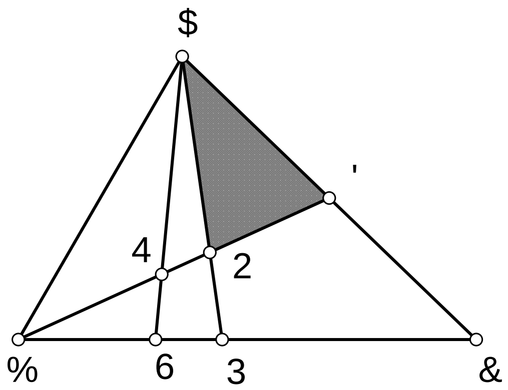

Figure 6: Construction for Solution 2 to Problem MH–54.

Applying the angle bisector theorem, we have

$$\frac{BP}{PC} = \frac{c}{b} \implies \frac{BP}{BC} = \frac{c}{b+c}.$$

Applying Menelaus's theorem in $\triangle BDC$ cut by A, O, P transversal line,

$$\frac{BP}{PC} \cdot \frac{AC}{AD} \cdot \frac{OD}{OB} = 1 \implies \frac{c}{b} \cdot 2 \cdot \frac{OD}{OB} = 1 \implies \frac{OD}{OB} = \frac{b}{2c}.$$

Similarly, applying Menelaus's theorem in $\triangle APC$ cut by B, O, D transversal line,

$$\frac{BP}{BC} \cdot \frac{DC}{DA} \cdot \frac{OA}{OP} = 1 \implies \frac{c}{b+c} \cdot \frac{OA}{OP} = 1 \implies \frac{OA}{OP} = \frac{b+c}{c}.$$

We can now compute the relationship between b and c:

$$\frac{(BOP)}{(AOD)} = \frac{\dfrac{BO \cdot OP \cdot \sin(BOP)}{2}}{\dfrac{AO \cdot OD \cdot \sin(AOD)}{2}} = \frac{OP}{OA} \cdot \frac{OB}{OD} = \frac{c}{b+c} \cdot \frac{2c}{b} = \frac{8}{15}.$$

Therefore,

$$\begin{aligned} \frac{2c^2}{b^2+bc} = \frac{8}{15} &\implies 15c^2 = 4b^2 + 4bc \implies 16c^2 = 4b^2 + 4bc + c^2 \\ &\implies (4c)^2 = (2b+c)^2 \implies 4c = 2b + c \\ &\implies 3c = 2b. \end{aligned}$$

Applying Menelaus's theorem in $\triangle BDC$ cut by A, Q, S transversal line,

$$AS = \text{symmedian} \implies \frac{BS}{SC} = \frac{AB^2}{AC^2} = \frac{c^2}{b^2} = \left(\frac{2}{3}\right)^2 = \frac{4}{9}.$$

Finally,

$$\begin{aligned} \frac{BS}{SC} \cdot \frac{AC}{AD} \cdot \frac{QD}{QB} = 1 &\implies \frac{4}{9} \cdot 2 \cdot \frac{QD}{QB} = 1 \\ &\implies \frac{QD}{QB} = \frac{9}{8} \implies \frac{QB}{QD} = \frac{8}{9}. \end{aligned}$$

Also solved by *José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.*

MH–55. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n and r be positive integers. Prove that

$$\sum_{k=1}^{n}\binom{n}{k}\left[2^{nr}-\binom{n}{k}^{r}\right]\le\frac{nr\,2^{n(r+1)}}{r+1}\sqrt[r]{\frac{1}{r+1}}.$$

Solution by the proposer. Let $x = t(1-t^r)$, $(0\le t\le 1)$. Then, applying the AM-GM inequality yields

$$\begin{aligned}\sqrt[r+1]{rx^r} &= [rt^r(1-t^r)(1-t^r)\dots(1-t^r)]^{1/r+1}\\ &\le \frac{rt^r+(1-t^r)+\dots+(1-t^r)}{r+1}=\frac{r}{r+1}.\end{aligned}$$

That is, $rx^r\le\left(\frac{r}{r+1}\right)^{r+1}$ or, equivalently, $x\le\frac{r}{r+1}\sqrt[r]{\frac{1}{r+1}}$ and

$$t(1-t^r)\le\frac{r}{r+1}\sqrt[r]{\frac{1}{r+1}}.$$

Putting $t=\frac{1}{2^n}\binom{n}{k}$, $k=1,2,\dots,n$, in the last inequality, and adding up the resulting expressions, we obtain

$$\sum_{k=1}^{n}\frac{1}{2^n}\binom{n}{k}\left[1-\frac{1}{2^{nr}}\binom{n}{k}^{r}\right]\le\frac{nr}{r+1}\sqrt[r]{\frac{1}{r+1}},$$

from which the statement follows. Equality holds when $n=r=1$, and we are done.

MH–56. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all positive solutions of the following system of equations:

$$\left.\begin{aligned}\sum_{\text{cyclic}}\frac{a^9+b^7}{b^2+c^2+d^2}&=\frac{8}{3},\\ abcd&=1.\end{aligned}\right\}$$

Solution by the proposer. By inspection we see that $(1,1,1,1)$ is a solution of the given system. Now we claim that it is unique.

Indeed, observe that $(a^9; b^9; c^9; d^9)$ and

$$\frac{1}{b^2+c^2+d^2}; \frac{1}{c^2+d^2+a^2}; \frac{1}{d^2+a^2+b^2}; \frac{1}{a^2+b^2+c^2}$$

are sorted in the same way, as can be easily checked. Then, on account of the rearrangement inequality, we have

$$\frac{a^9}{b^2+c^2+d^2}+\frac{b^9}{c^2+d^2+a^2}+\frac{c^9}{d^2+a^2+b^2}+\frac{d^9}{a^2+b^2+c^2}$$
$$\frac{b^9}{b^2+c^2+d^2}+\frac{c^9}{c^2+d^2+a^2}+\frac{d^9}{d^2+a^2+b^2}+\frac{a^9}{a^2+b^2+c^2}$$

and

$$\frac{a^9+b^7}{b^2+c^2+d^2}+\frac{b^9+c^7}{c^2+d^2+a^2}+\frac{c^9+d^7}{d^2+a^2+b^2}+\frac{d^9+a^7}{a^2+b^2+c^2}$$
$$\frac{b^9+b^7}{b^2+c^2+d^2}+\frac{c^9+c^7}{c^2+d^2+a^2}+\frac{d^9+d^7}{d^2+a^2+b^2}+\frac{a^9+a^7}{a^2+b^2+c^2}.$$

Applying the AM-GM inequality, we have

$$\frac{b^9+b^7}{b^2+c^2+d^2}+\frac{c^9+c^7}{c^2+d^2+a^2}+\frac{d^9+d^7}{d^2+a^2+b^2}+\frac{a^9+a^7}{a^2+b^2+c^2}$$
$$\frac{2b^8}{b^2+c^2+d^2}+\frac{2c^8}{c^2+d^2+a^2}+\frac{2d^8}{d^2+a^2+b^2}+\frac{2a^8}{a^2+b^2+c^2}.$$

Now, we apply Cauchy's inequality to the vectors

$$\vec{u} = \left(\sqrt{\frac{b^4}{b^2+c^2+d^2}}; \sqrt{\frac{c^4}{c^2+d^2+a^2}}; \sqrt{\frac{d^4}{d^2+a^2+b^2}}; \sqrt{\frac{a^4}{a^2+b^2+c^2}}\right)$$

and

$$\vec{v} = \left(\sqrt{b^2+c^2+d^2}; \sqrt{c^2+d^2+a^2}; \sqrt{d^2+a^2+b^2}; \sqrt{a^2+b^2+c^2}\right)$$

to obtain

$$\frac{2b^8}{b^2+c^2+d^2}+\frac{2c^8}{c^2+d^2+a^2}+\frac{2d^8}{d^2+a^2+b^2}+\frac{2a^8}{a^2+b^2+c^2}$$
$$\frac{2(a^4+b^4+c^4+d^4)^2}{3(a^2+b^2+c^2+d^2)}.$$

On account of the AM-QM inequality, we have

$$a^4 + b^4 + c^4 + d^4 \geq \frac{(a^2 + b^2 + c^2 + d^2)^2}{4},$$

and

$$\frac{b^9 + b^7}{b^2 + c^2 + d^2} + \frac{c^9 + c^7}{c^2 + d^2 + a^2} + \frac{d^9 + d^7}{d^2 + a^2 + b^2} + \frac{a^9 + a^7}{a^2 + b^2 + c^2}$$
$$\geq \frac{2(a^4 + b^4 + c^4 + d^4)^2}{3(a^2 + b^2 + c^2 + d^2)} \geq \frac{2(a^2 + b^2 + c^2 + d^2)^4}{3 \cdot 4^2(a^2 + b^2 + c^2 + d^2)}$$
$$= \frac{(a^2 + b^2 + c^2 + d^2)^3}{24} \geq \frac{\left(4\sqrt[4]{a^2b^2c^2d^2}\right)^3}{24} = \frac{8}{3}$$

because $abcd = 1$. Then,

$$\sum_{cyclic} \frac{a^9 + b^7}{b^2 + c^2 + d^2} \geq \frac{8}{3},$$

and $(1, 1, 1, 1)$ is the only positive solution, as claimed.

Also solved by *José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.*

MH–57. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $k \geq 1$ be an integer number and let $u_1, \ldots, u_k$ and $z_1, \ldots, z_k$ be distinct nonzero complex numbers. If the set

$$A = \{a_n \mid a_n = u_1z_1^n + u_2z_2^n + \cdots + u_kz_k^n, n \in \mathbb{N}\}$$

is finite, then prove that there exists a positive integer d such that $a_n = a_{n+d}$ for all $n \in \mathbb{N}$.

Solution by the proposer. If the set A is finite, then the set

$$B = \{(a_n, a_{n+1}, \ldots, a_{n+k-1}) \mid n \in \mathbb{N}\}$$

is also finite. Therefore, there exist $p < q$ such that

$$(a_p, a_{p+1}, \ldots, a_{p+k-1}) = (a_q, a_{q+1}, \ldots, a_{q+k-1})$$

or

$$a_p = a_q,\ a_{p+1} = a_{q+1}, \ldots, a_{p+k-1} = a_{q+k-1}.$$

The last system of equations may be written as

$$\begin{aligned} z_1^p(z_1^d - 1)u_1 + z_2^p(z_2^d - 1)u_2 + \cdots + z_k^p(z_k^d - 1)u_k &= 0, \\ z_1^{p+1}(z_1^d - 1)u_1 + z_2^{p+1}(z_2^d - 1)u_2 + \cdots + z_k^{p+1}(z_k^d - 1)u_k &= 0, \\ &\vdots \\ z_1^{p+k-1}(z_1^d - 1)u_1 + z_2^{p+k-1}(z_2^d - 1)u_2 + \cdots + z_k^{p+k-1}(z_k^d - 1)u_k &= 0. \end{aligned}$$

Putting $x_1 = (z_1^d - 1)u_1, x_2 = (z_2^d - 1)u_2, \ldots, x_k = (z_k^d - 1)u_k$, we get the following homogeneous linear system of equations with unknowns $x_1, x_2, \ldots, x_n$:

$$\begin{aligned} z_1^p x_1 + z_2^p x_2 + \cdots + z_k^p x_k &= 0, \\ z_1^{p+1} x1 + z_2^{p+1} x_2 + \cdots + z_k^{p+1} x_k &= 0, \\ &\vdots \\ z_1^{p+k-1} x + z_2^{p+k-1} x_2 + \cdots + z_k^{p+k-1} x_k &= 0. \end{aligned}$$

Since the determinant of its matrix of coefficients is

$$\begin{vmatrix} z_1^p & z_2^p & \cdots & z_k^p \\ z_1^{p+1} & z_2^{p+1} & \cdots & z_k^{p+1} \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{p+k-1} & z_2^{p+k-1} & \cdots & z_k^{p+k-1} \end{vmatrix} = z_1^p z_2^p \cdots z_k^p V(z_1, z_2, \ldots, z_k) \neq 0,$$

then the system only has the trivial solution $x_1 = x_2 = \cdots = x_k = 0$. Thus, $z_1^d = z_2^d = \cdots = z_k^d = 1$ and $a_n = a_{n+d}$ for all $n \in \mathbb{N}$.

MH–58. *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.* We have four horizontal lines, and the second one from the top is blue. We have a set S of 600 points on those four lines such that each line contains at least two points and, for any five points in S, there are two of them P and Q such that the segment PQ contains a point from S other than P and Q. Prove that the blue line contains at least 100 points from S.

Solution by the proposer. Let A, B, C and D be the lines from top to bottom, and let a, b, c and d be the number of points from S on those lines, respectively. We want to prove that $b \geq 100$.

We will show that $b \geq \frac{a+c}{2} - 1$ and $c \geq \frac{b+d}{2} - 1$. Label the points on line A from left to right as $A_1, A_2, \ldots, A_a$, and label points on the other lines accordingly. For every $1 \leq i \leq a$, and every $1 \leq j < c$, there must be a point from B in either the segment A_iC_j or A_iC_{j+1}, as otherwise the set of five points A_i, B_1, B_2, C_j and C_{j+1} would not satisfy the condition in the statement. The same holds for A_iC_j and $A_{i+1}C_j$.

We consider the segments $A_1C_1, A_1C_2, \ldots, A_1C_{c-1}, A_1C_c, A_2C_c, \ldots, A_aC_c$. This sequence consists of $a + c - 1$ segments and, out of every two consecutive segments, at least one contains a point from S in B, which means that at least $\frac{a+c-2}{2} = \frac{a+c}{2} - 1$ segments contain a point from S. But each segment intersects B at a point to the right to the intersection of the previous one, so the points of intersection are all different, and $b \geq \frac{a+c}{2} - 1$. By the same argument, $c \geq \frac{b+d}{2} - 1$.

From those inequalities we obtain $b + c \geq \frac{a+b+c+d}{2} - 2 = 298$. We also obtain $b \geq \frac{a+c}{2} - 1 \geq \frac{2+(298-b)}{2} - 1 = 149 - \frac{b}{2}$. This means $3b \geq 298$ and, since b is an integer, $b \geq 100$. □

Observation: 100 is indeed the lowest bound. For an example in which line B contains exactly 100 points, consider that the lines are equidistant, and the points in S from each line are the points with integer coordinate x from the following intervals: A contains the points from $[1; 2]$, B from $[1; 100]$, C from $[1; 199]$ and D from $[0; 298]$.

Advanced Problems

A–53. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let $A, B \in M_2(\mathbb{Z})$ be two matrices such that $\det A = \det B = 0$. Show that $\det(A^3 + B^3) \neq 2018$ and $\det(A^3 \quad B^3) \neq 2019$.

Solution by the proposer. We consider the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(x) = \det(A + Bx) = \det A + x \quad + x^2 \det B$$

for some $\quad \in \mathbb{Z}$.

Applying the fact that $\det A = \det B = 0$, we have

$$f(x) = \quad x.$$

Let ε be the root of the equation $\varepsilon^2 + \varepsilon + 1 = 0$. Then ,

$(A^3 + B^3) = (A + B)(A + B\varepsilon)(A + B\varepsilon^2)$, so

$$\begin{aligned} \det A^3 + B^3 &= (\det A + B) \quad \det(A + B\varepsilon) \quad \det A + B\varepsilon^2 \\ &= f(1) \quad f(\varepsilon) \quad f \quad \varepsilon^2 = \quad \varepsilon \quad \varepsilon^2 = \quad ^3\varepsilon^3, \end{aligned}$$

where $\varepsilon^3 = 1$. Thus, $\det(A^3 + B^3) = \quad ^3$, that is, it is a cube number.

$(A^3 \quad B^3) = (A \quad B)(A \quad B\varepsilon)(A \quad B\varepsilon^2)$, so

$$\begin{aligned} \det A^3 \quad B^3 &= (\det A \quad B) \quad \det(A \quad B\varepsilon) \quad \det A \quad B\varepsilon^2 \\ &= f(\quad 1) \quad f(\quad \varepsilon) \quad f \quad \varepsilon^2 \\ &= (\quad) \quad (\quad \varepsilon) \quad \varepsilon^3 = \quad ^3\varepsilon^3, \end{aligned}$$

where $\varepsilon^3 = 1$. So, $\det(A^3 \quad B^3) = \quad ^3$, that is, it is also a cubed number.

From the above we reach the conclusion that $\det(A^3 + B^3) \neq 2018$ and $\det(A^3 \quad B^3) \neq 2019$, because neither 2018 nor 2019 are cube numbers.

A–54. *Proposed by Pedro Henrique O. Pantoja, University of Campina Grande, Brazil.* Prove that

$$\frac{(\tan\frac{\pi}{7})^6 + (\tan\frac{2\pi}{7})^6 + (\tan\frac{3\pi}{7})^6}{(\tan\frac{\pi}{7})^2 + (\tan\frac{2\pi}{7})^2 + (\tan\frac{3\pi}{7})^2}$$

is a prime number.

Solution by the proposer. Observe that $\tan\frac{\pi}{7}$, $\tan\frac{2\pi}{7}$, $\tan\frac{3\pi}{7}$ are roots of the equation $x^6 - 21x^4 + 35x^2 - 7 = 0$ (Chebyshev polynomials). Therefore, $a = \tan^2\frac{\pi}{7}$, $b = \tan^2\frac{2\pi}{7}$, $c = \tan^2\frac{3\pi}{7}$ are roots of the equation $x^3 - 21x^2 + 35x - 7 = 0$. Hence,

$$a + b + c = 21, \qquad ab + bc + ca = 35, \qquad abc = 7.$$

This means that

$$a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca) = 371,$$

which in turn implies that

$$\begin{aligned} a^3 + b^3 + c^3 &= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) + 3abc \\ &= 21(371 - 35) + 21 = 7077. \end{aligned}$$

Finally, we have that

$$\frac{a^3 + b^3 + c^3}{a + b + c} = \frac{7077}{21} = 337$$

is a prime number.

A–55. *Proposed by Leonhard Summerer, University of Vienna, Austria.* Find all integers $n \geq 1$ for which the polynomial

$$P_n(x) = (x - 1)(x - 2)\ldots(x - n) + 1$$

is irreducible over $\mathbb{Q}$.

Solution by the proposer. Let $P_n(x) = A(x)B(x)$ with $\deg(A) < n$ and $\deg(B) < n$. By Gauss's Lemma, we may assume that $A(x), B(x) \in \mathbb{Z}[x]$. Setting $x = k$ for $1 \leq k \leq n$ we obtain

$1 = P_n(k) = A(k)B(k)$, which implies $A(k) = B(k) = 1$ or $A(k) = B(k) = -1$. Then, $C(x) = A(x) - B(x)$, with $\deg(C) < n$, has n zeros and $C(x) \equiv 0$ identically. Since $A(x) = B(x)$, then $P_n(x) = A(x)^2$ and n must be even. Moreover, for $x = 3/2$ we have

$$P_n\left(\frac{3}{2}\right) = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{3-2n}{2}\right) + 1 = A\left(\frac{3}{2}\right)^2 \geq 0$$

only for $n = 2$ and $n = 4$, as can be easily checked. For $n = 2$ we have $P_2(x) = (x-1)(x-2) + 1 = x^2 - 3x + 3$, which is irreducible. For $n = 4$ we have $P_4(x) = (x-1)(x-2)(x-3)(x-4) + 1 = (x^2 - 5x + 5)^2$. So, the answer is $n \neq 4$, and we are done.

A–56. *Proposed by Sergio Falcón and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.* The Fibonacci numbers are defined recursively by $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$, with initial conditions $F_0 = 0$, $F_1 = 1$. Find a closed expression for the following sum, where $r > 1$ and $n \geq r$ are integers:

$$\sum_{j=0}^{n-r}\left(\binom{r+j}{r} - \binom{r+j-1}{r} - \binom{r+j-2}{r}\right)F_{n-(r+j)}.$$

Solution by the proposer. Notice that

$$F_n = \sum_{k\geq 0}\binom{n-1-k}{k}.$$

Then,

$$F_{n-(r+j)} = \sum_{k\geq 0}\binom{n-r-j-k-1}{k}.$$

Let us write the proposed sum as

$$\sum_{j\geq 0}\binom{r+j}{r}F_{n-(r+j)} - \sum_{j\geq 0}\binom{r+j-1}{r}F_{n-(r+j)} - \sum_{j\geq 0}\binom{r+j-2}{r}F_{n-(r+j)}.$$

The first sum, by Vandermonde's Convolution, is

$$\sum_{j=0}^{n-r}\binom{r+j}{r}\sum_{k\ge 0}\binom{n-r-j-k-1}{k}$$

$$=\sum_{k\ge 0}\sum_{j=0}^{n-r}\binom{r+j}{r}\binom{n-r-j-k-1}{k}$$

$$=\sum_{k\ge 0}\binom{n-k}{r+k+1}.$$

Therefore, the proposed sum becomes

$$\sum_{k\ge 0}\left[\binom{n-k}{r+k+1}-\binom{n-k-1}{r+k+1}-\binom{n-k-2}{r+k+1}\right]$$

$$=\sum_{k\ge 0}\left[\binom{n-k-1}{r+k}-\binom{n-k-2}{r+k+1}\right],$$

where we have used Pascal's rule. Now, we may use the snake-oil method to discover the value of the last expression. Let $F(x)$ be its ordinary power generating function, that is, $F(x) = F_1(x) - F_2(x)$, where $F_1(x)$ is the generating function of $\sum_{k\ge 0}\binom{n-k-1}{r+k}$ and $F_2(x)$ is the generating function of $\sum_{k\ge 0}\binom{n-k-2}{r+k+1}$. Then,

$$F_1(x) = \sum_{r\ge 0} x^{r}\sum_{k\ge 0}\binom{n-k-1}{r+k}x^{r+k}$$

$$=\sum_{k\ge 0}x^{-k}\sum_{r\ge 0}\binom{n-k-1}{r+k}x^{r+k}$$

$$=\sum_{k\ge 0}x^{-k}(1+x)^{n-k-1}$$

$$=(1+x)^{n-1}\sum_{k\ge 0}\left(\frac{1}{x+x^2}\right)^{k}$$

$$=(1+x)^{n-1}\frac{1}{1-\frac{1}{x+x^2}}=\frac{x(1+x)^n}{x^2+x-1}.$$

Analogously, $F_2(x) = \dfrac{(1+x)^{n-1}}{x^2+x-1}$ and, therefore,

$$F(x) = \frac{x(1+x)^n}{x^2+x-1} - \frac{(1+x)^{n-1}}{x^2+x-1} = (1+x)^{n-1}.$$

The original sum is now unmasked: it is the coefficient of x^r in the last member above. But that is $\binom{n-1}{r}$, and we have our answer.

A–57. *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.* Let n be a positive integer, and let $f : [0,1] \to \mathbb{R}^n$ be a nonconstant continuous function. Prove that there exist x, $y \in [0,1]$ such that:

- For every $1 \le i \le n$, if $f(x)_i \in \mathbb{Q}$ then $f(x)_i = f(y)_i$.
- There exists an index i such that $f(x)_i \notin \mathbb{Q}$ and $f(y)_i \in \mathbb{Q}$.

(Here, $f(x)_i$ denotes the i-th coordinate of $f(x)$.)

Solution by the proposer. Assume that x and y do not exist. For every $x \in [0,1]$, let $A(x) = \{y \in [0,1] \mid f(x)_i \in \mathbb{Q} \Rightarrow f(x)_i = f(y)_i\}$. This set is the preimage by f of a closed set, so since f is continuous, $A(x)$ is closed. Let $y \in A(x)$. If y had a rational coordinate that x does not have, then (x, y) would be an example for the property in the statement, so by our assumption this does not happen, and $A(x) = A(y)$. This implies that the sets $A(x)$ partition $[0,1]$ into closed sets.

Since f is nonconstant, there exist x, y and i such that $f(x)_i \neq f(y)_i$. By continuity of f, $f(t)_i$ takes every value between $f(x)_i$ and $f(y)_i$, which includes infinitely many rational values, so there are infinitely many sets $A(x)$. However, each set $A(x)$ is of the form $\{y \mid f(x)_i = q_i \; \forall i \in S\}$, where $S \subseteq \{1, 2, \ldots, n\}$. Since there are countably many options for S and each q_i, there are countably many sets $A(x)$. We can label them $A_1, A_2, \ldots$.

$[0,1]$ is the disjoint union of countably many non-empty closed sets A_i. We will construct a sequence of closed intervals $[0,1] = I_0 \supseteq I_1 \supseteq I_2 \supseteq \ldots$, none of which is contained in a set A_j and such that

$I_i \cap A_i = \emptyset$ for all $i \geq 1$. We start with $I_0 = [0, 1]$. Suppose I_{i-1} has been constructed. If $I_{i-1} \cap A_i = \emptyset$, then set $I_i = I_{i-1}$. Otherwise let $x \in I_{i-1} \setminus A_i$ (it exists because $I_{i-1} \not\subseteq A_i$), and $y \in I_{i-1} \cap A_i$. Wlog, assume that $x \leq y$. Let $y' = \min\{t \in A_i | t \geq x\}$ (it exists because A_i is closed). $x \in A_j$ for some $j > i$. Since A_j is closed, it cannot contain $[x, y')$, because it does not contain y', so there is $z \in [x, y')$, $z \notin A_j$. Then we can set $I_i = [x, z]$.

We can now reach a contradiction: $\{I_i\}$ is a sequence of nested closed intervals, so their intersection is non-empty, and contains an element x. For some i, $x \in A_i$. But then $x \in I_i \cap A_i = \emptyset$, contradiction. Thus, x and y as in the statement exist.

A–58. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* **[Correction]** Let n be a positive integer. Compute the following sum:

$$\sum_{k=1}^{n} (-1)^{k+1}\binom{n}{k}\left[\sum_{j=1}^{k}\frac{1}{j^3} + \sum_{1\leq i<j\leq k}\frac{i+j}{i^2j^2} + \sum_{1\leq i<j<\ell\leq k}\frac{1}{ij\ell}\right].$$

Solution by the proposer. First, we observe that the sum stated can be obtained putting $x = 0$ in the more general expression

$$\sum_{k=1}^{n}\left\{(-1)^{k+1}\binom{n}{k}\left[\sum_{j=1}^{k}\frac{1}{(x+j)^3} + \sum_{1\leq i<j\leq k}\frac{2x+i+j}{(x+i)^2(x+j)^2} + \sum_{1\leq i<j<\ell\leq k}\frac{1}{(x+i)(x+j)(x+\ell)}\right]\binom{x+k}{k}^{-1}\right\}.$$

To compute the preceding sum, first we observe that, for all $x \in \mathbb{R}$ and $n > 0$, we have

$$\binom{x+n}{n} = \prod_{k=1}^{n}\frac{x+k}{k}.$$

Differentiating both sides of the preceding identity, we get

$$\frac{d}{dx}\binom{x+n}{n} = \binom{x+n}{n}\sum_{j=1}^{n}\frac{1}{x+j}. \qquad (1)$$

Next, we claim that

$$\sum_{k=0} (\quad 1)^k \binom{n}{k} \binom{x+k}{k}^{1} = \frac{x}{x+n}. \qquad (2)$$

Indeed, we start out with Vandermonde's identity, namely

$$\binom{\ }{n}\binom{n}{j} = \binom{\ }{j}\binom{\ \ j}{n \quad j},$$

and we have

$$\sum_{j=0} (\quad 1)^j \binom{n}{j} \binom{\ }{j}\binom{\ }{n \quad j}^{1} = \sum_{j=0} (\quad 1)^j \binom{\ }{j}\binom{\ \ j}{n \quad j} \qquad (3)$$

Taking into account Vandermonde's convolution, that is,

$$\sum_{k=0} \binom{a}{j}\binom{b}{n \quad j} = \binom{a+b}{n},$$

and the fact that

$$\binom{\ }{n} = (\quad 1)^n \binom{n \qquad 1}{n},$$

then (3) becomes

$$\sum_{j=0} (\quad 1)^j \binom{\ }{j}\binom{\ \ j}{n \quad j} = (\quad 1)^n \sum_{j=0} \binom{\ }{j}\binom{n \qquad 1}{n \quad j}$$

$$= (\quad 1)^n \binom{\ + n \quad 1}{n} = \binom{\ }{n}.$$

If we set $\ = \ (x+1)$ into the preceding expression, we get

$$\sum_{j=0} \binom{n}{j}\binom{\ }{j}\binom{x+j}{j}^{1} = \binom{x + \quad + n}{n}\binom{x+n}{n}^{1},$$

and setting $\ = \ 1$ yields

$$\sum_{j=0} \binom{n \quad\ 1}{j}\binom{\ }{j}\binom{x+j}{j}^{1} = \binom{x+n \quad 1}{n}\binom{x+n}{n}^{1}$$

or, equivalently,

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{x+k}{k}^{-1} = \frac{x}{x+n},$$

as claimed.

Differentiating three times (2) and taking into account (1), we have

$$\sum_{k=1}^{n} \left\{ (-1)^{k+2}\binom{n}{k}\left[\left(\sum_{j=1}^{k} \frac{1}{x+j}\right)^3 + 3\left(\sum_{j=1}^{k}\frac{1}{x+j}\right)\left(\sum_{j=1}^{k} \frac{1}{(x+j)^2}\right) + 2\sum_{j=1}^{k}\frac{1}{(x+j)^3}\right]\binom{x+k}{k}^{-1}\right\} = \frac{6n}{(n+x)^4}.$$

Now we observe that

$$\left(\sum_{j=1}^{k} \frac{1}{x+j}\right)^3 + 3\left(\sum_{j=1}^{k}\frac{1}{x+j}\right)\left(\sum_{j=1}^{k} \frac{1}{(x+j)^2}\right) + 2\sum_{j=1}^{k}\frac{1}{(x+j)^3}$$
$$= 6\left(\sum_{j=1}^{k}\frac{1}{(x+j)^3} + \sum_{1\le i<j\le k}\frac{2x+i+j}{(x+i)^2(x+j)^2} + \sum_{1\le i<j<\ell\le k}\frac{1}{(x+i)(x+j)(x+\ell)}\right),$$

and substituting into the preceding expression, yields

$$\sum_{k=1}^{n} \left\{ (-1)^{k+1}\binom{n}{k}\left[\sum_{j=1}^{k}\frac{1}{(x+j)^3} + \sum_{1\le i<j\le k}\frac{2x+i+j}{(x+i)^2(x+j)^2} + \sum_{1\le i<j<\ell\le k}\frac{1}{(x+i)(x+j)(x+\ell)}\right]\binom{x+k}{k}^{-1}\right\} = \frac{n}{(n+x)^4}.$$

Finally, the given sum is

$$\sum_{k=1}^{n} (-1)^{k+1}\binom{n}{k}\left[\sum_{j=1}^{k}\frac{1}{j^3} + \sum_{1\le i<j\le k}\frac{i+j}{i^2j^2} + \sum_{1\le i<j<\ell\le k}\frac{1}{ij\ell}\right] = \frac{1}{n^3},$$

and we are done.

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