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Theorems about Triangles Similar with the Feuerbach Triangle

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Abstract

We present theorems about triangles similar (but not homothetic) to the Feuerbach triangle. The theorems are discovered by the computer program "Discoverer" created by the authors.

All formulas in this proof refer to Grozdev and Dekov's sheet [3].

Theorem 1 below was published by Krishna [10, Propositions 2 and 4]. Here we give a new simple proof. Note that the forms E_1, E_2 and E_3 are used also by Grozdev, Okumura, and Dekov [6, Theorems 2, 5 and 6].

References for Theorem 1: incentral triangle and perspector can be found in MathWorld [13].

Theorem 1. *The Feuerbach triangle $P_aP_bP_c$ is perspective and similar (but not homothetic) to the incentral triangle $Q_aQ_bQ_c$. The perspector is the Feuerbach point. The ratio of the similarity is*

$$k = \frac{\sqrt{E_1E_2E_3}}{\sqrt{abc}(b+c)(c+a)(a+b)},$$

where

$$\begin{aligned}
E_1 &= a^3 - b^3 - c^3 + 3abc + ba^2 - b^2a + ca^2 - c^2a + c^2b + b^2c,\\
E_2 &= -a^3 + b^3 - c^3 + 3abc - ba^2 + b^2a + ca^2 + c^2a + b^2c - c^2b,\\
E_3 &= -a^3 - b^3 + c^3 + 3abc + ba^2 + b^2a - ca^2 + c^2a - b^2c + c^2b.
\end{aligned}$$

Proof. The incentral triangle is the Cevian triangle of the incenter. The barycentric coordinates of the incenter are $I = (a, b, c)$, so that the barycentric coordinates of the incentral triangle are $Q_a = (0, b, c)$, $Q_b = (a, 0, c)$, $Q_c = (a, b, 0)$.

By using the distance formula (9) we find the lengths of the following segments:

$$P_bP_c = \frac{abc(b+c)}{\sqrt{E_2E_3}}, \qquad P_cP_a = \frac{abc(c+a)}{\sqrt{E_3E_1}},$$
$$P_aP_b = \frac{abc(a+b)}{\sqrt{E_1E_2}}, \qquad Q_bQ_c = \frac{\sqrt{abc}\sqrt{E_1}}{(c+a)(a+b)},$$
$$Q_cQ_a = \frac{\sqrt{abc}\sqrt{E_2}}{(a+b)(b+c)}, \quad Q_aQ_b = \frac{\sqrt{abc}\sqrt{E_3}}{(b+c)(c+a)}.$$

Hence,

$$k = \frac{Q_bQ_c}{P_bP_c} = \frac{Q_cQ_a}{P_cP_a} = \frac{Q_aQ_b}{P_aP_b} = \frac{\sqrt{E_1E_2E_3}}{\sqrt{abc}(b+c)(c+a)(a+b)}.$$

We have proved that the triangles are similar. In order to prove that the triangles are perspective, we use formula (3), and we obtain the equations of the lines P_aQ_a, P_bQ_b and P_cQ_c. Then by using formula (6), we find that these three lines concur in a point. By using formula (5), we find the barycentric coordinates of the intersection point of the lines. We see that the intersection point is the Feuerbach Point.

If the vertices of the triangles are in normalized barycentric coordinates, the infinite points of the lines P_bP_c and Q_bQ_c are as follows:

$$P_b - P_c = ((b+c)(b-c), -2b^2-2bc+a^2-c^2, b^2+2bc-a^2+2c^2),$$
$$Q_b - Q_c = (a(b-c), -b(a+c), c(a+b)).$$

We see that the above points are different. Hence, the segments P_bP_c and Q_bQ_c are not collinear, and we conclude that the triangles are not homothetic. □

References for Theorem 2: Euler reflection point by Pohoata [12]; see also the Encyclopedia of Triangle Centers by Kimberling [9, Point X(110)].

Theorem 2. *Denote by*

- I *the incenter,*
- Q_a *the Euler reflection point of triangle* IBC*,*
- Q_b *the Euler reflection point of triangle* AIC*,*
- Q_c *the Euler reflection point of triangle* ABI*.*

Then the Feuerbach triangle $P_aP_bP_c$ *is similar (but not homothetic) to triangle* $Q_aQ_bQ_c$*. The ratio of similarity is*

$$k = \frac{\sqrt{E_4}}{\sqrt{abc}},$$

where

$$E_4 = a^3 + b^3 + c^3 + 3abc - ba^2 - b^2a - ca^2 - c^2a - c^2b - b^2c.$$

Proof. By using the distance formula (9), we find the lengths of the following segments:

$$IA = \sqrt{\frac{bc(b+c-a)}{a+b+c}},$$
$$IB = \sqrt{\frac{ca(c+a-b)}{a+b+c}},$$
$$IC = \sqrt{\frac{ab(a+b-c)}{a+b+c}}.$$

Hence, the side lengths of triangles IBC, AIC, ABI are as follows:

$$\begin{array}{lll} a_{IBC} = a, & b_{IBC} = IC, & c_{IBC} = IB, \\ a_{AIC} = IC, & b_{AIC} = b, & c_{AIC} = IA, \\ a_{ABI} = IB, & b_{ABI} = IA, & c_{ABI} = c. \end{array}$$

The barycentric coordinates if the Euler reflection point of triangle ABC are as follows:

$$((a^2(a^2-b^2)(a^2-c^2), (b^2(b^2-c^2)(b^2-a^2), (c^2(c^2-a^2)(c^2-b^2)).$$

Now, by using the change of coordinates formula (10) we find the barycentric coordinates of points Q_a, Q_b, Q_c as follows:

$$\begin{aligned}
uQ_a &= a^2(a+b)(a+c),\\
vQ_a &= b(a+b)(a^2-b^2+2ac-ab+c^2),\\
wQ_a &= c(a+c)(a^2+b^2+2ab-ac-c^2),\\
uQ_b &= a(a+b)(b^2-a^2-ab+2bc+c^2),\\
vQ_b &= b^2(b+c)(a+b),\\
wQ_b &= c(b+c)(a^2+b^2+2ab-bc-c^2),\\
uQ_c &= a(a+c)(b^2-a^2-ac+2bc+c^2),\\
vQ_c &= b(b+c)(a^2-b^2+2ac-bc+c^2),\\
wQ_c &= c^2(a+c)(b+c).
\end{aligned}$$

By using the distance formula (9), we find the lengths of the following segments:

$$\begin{aligned}
Q_bQ_c &= \frac{(b+c)\sqrt{abc})\sqrt{E_4}}{\sqrt{E_2E_3}}, \qquad & P_bP_c &= \frac{abc(b+c)}{\sqrt{E_2E_3}},\\
Q_cQ_a &= \frac{(c+a)\sqrt{abc})\sqrt{E_4}}{\sqrt{E_3E_1}}, \qquad & P_bP_c &= \frac{abc(c+a)}{\sqrt{E_3E_1}},\\
Q_aQ_b &= \frac{(a+b)\sqrt{abc})\sqrt{E_4}}{\sqrt{E_1E_2}}, \qquad & P_bP_c &= \frac{abc(a+b)}{\sqrt{E_1E_2}},
\end{aligned}$$

where E_1, E_2, E_3 are as in Theorem 1. Hence

$$k = \frac{Q_bQ_c}{P_bP_c} = \frac{Q_cQ_a}{P_cP_a} = \frac{Q_aQ_b}{P_aP_b} = \frac{\sqrt{E_4}}{\sqrt{abc}}.$$

In order to prove that triangles $P_aP_bP_c$ and $Q_aQ_bQ_c$ are not homothetic, we use the same method as in Theorem 1. □

References for Theorem 3: Kosnita point and excentral triangle in MathWorld [13].

Theorem 3. *Denote by*

- I *the incenter,*
- $J_aJ_bJ_c$ *the excentral triangle,*

- Q_a *the Kosnita point of triangle* J_aBC,
- Q_b *the Kosnita point of triangle* AJ_bC,
- Q_c *the Kosnita point of triangle* $ABJc$.

Then the Feuerbach triangle $P_aP_bP_c$ *is similar (but not homothetic) with triangle* $Q_aQ_bQ_c$. *The ratio of similarity is*

$$k = \frac{Q_bQ_c}{P_bP_c} = \frac{Q_cQ_a}{P_cP_a} = \frac{Q_aQ_b}{P_aP_b} = \frac{\sqrt{E_1E_2E_3}}{E_5\sqrt{abc}},$$

where E_1, E_2, E_3 *are as in Theorem 1, and*

$$E_5 = -a^3 - b^3 - c^3 + abc + ba^2 + b^2a + ca^2 + c^2a + c^2b + b^2c.$$

Proof. The barycentric coordinates of the excentral triangle are as follows:

$$J_a = (-a, b, c), \qquad J_b = (a, -b, c), \qquad J_c = (a, b, -c).$$

By using the distance formula (9), we find the lengths of the following segments:

$$J_aC = \sqrt{\frac{ab(a-b+c)}{b+c-a}}, \qquad J_aB = \sqrt{\frac{ac(a+b-c)}{b+c-a}},$$
$$J_bC = \sqrt{\frac{ab(b+c-a)}{a-b+c}}, \qquad J_bA = \sqrt{\frac{bc(a+b-c)}{a-b+c}},$$
$$J_cB = \sqrt{\frac{ac(b+c-a)}{a+b-c}}, \qquad J_cA = \sqrt{\frac{bc(a-b+c)}{a+b-c}}.$$

The lengths of the sides of triangles J_aBC, AJ_bC, ABJ_c are as follows:

$$\begin{array}{lll} a_{J_aBC} = a, & b_{J_aBC} = J_aC, & c_{J_aBC} = J_aB, \\ a_{AJ_bC} = J_bC, & b_{AJ_bC} = b, & c_{AJ_bC} = J_bA, \\ a_{ABJ_c} = J_cB, & b_{ABJ_c} = J_cA, & c_{ABJ_c} = c. \end{array}$$

The barycentric coordinates of Kosnita point (uK, vK, wK) of triangle ABC are as follows:

$$\begin{aligned} uK &= a^2(a^4+b^4-b^2c^2-a^2(2b^2+c^2))(a^4-b^2c^2+c^4-a^2(b^2+2c^2)), \\ vK &= b^2(b^4+c^4-c^2a^2-b^2(2c^2+a^2))(b^4-c^2a^2+a^4-b^2(c^2+2a^2)), \\ wK &= c^2(c^4+a^4-a^2b^2-c^2(2a^2+b^2))(c^4-a^2b^2+b^4-c^2(a^2+2b^2)). \end{aligned}$$

By using the change of coordinates formula (10) we obtain the barycentric coordinates of points Q_a, Q_b, Q_c as follows:

$$\begin{aligned}
uQ_a &= -a^2(a+b)(a+c),\\
vQ_a &= b(a+b)(a^2-b^2+2ac+ab+c^2),\\
wQ_a &= c(a+c)(a^2+b^2+2ab+ac-c^2),\\
uQ_b &= a(a+b)(b^2-a^2+ab+2bc+c^2),\\
vQ_b &= -b^2(b+c)(a+b),\\
wQ_b &= c(b+c)(b^2+a^2+2ab+bc-c^2),\\
uQ_c &= a(a+c)(b^2-a^2+ac+2bc+c^2),\\
vQ_c &= b(b+c)(a^2-b^2+2ac+bc+c^2),\\
wQ_c &= -c^2(a+c)(b+c).
\end{aligned}$$

By using the distance formula (9), we obtain the lengths of the following segments:

$$\begin{aligned}
Q_bQ_c &= \frac{(b+c)\sqrt{abc}\sqrt{E_1}}{E_5}, \qquad & P_bP_c &= \frac{abc(b+c)}{\sqrt{E_2E_3}},\\
Q_cQ_a &= \frac{(c+a)\sqrt{abc}\sqrt{E_2}}{E_5}, \qquad & P_cP_a &= \frac{abc(c+a)}{\sqrt{E_3E_1}},\\
Q_aQ_b &= \frac{(a+b)\sqrt{abc}\sqrt{E_3}}{E_5}, \qquad & P_aP_b &= \frac{abc(a+b)}{\sqrt{E_1E_2}}.
\end{aligned}$$

Hence

$$k = \frac{Q_bQ_c}{P_bP_c} = \frac{Q_cQ_a}{P_cP_a} = \frac{Q_aQ_b}{P_aP_b} = \frac{\sqrt{E_1E_2E_3}}{E_5\sqrt{abc}}.$$

In order to prove that triangles $P_aP_bP_c$ and $Q_aQ_bQ_c$ are not homothetic, we use the same method as in Theorem 1. □

References for Theorem 4: Grinberg point in Grozdev and Dekov's sheet [3, Section 7] and in the Encyclopedia of Triangle Centers [9, Point X(37)]; anticevian triangle in MathWorld [13] and in the Encyclopedia of Triangle Centers [9, Glossary].

Theorem 4. *Denote by*

- $J_aJ_bJ_c$ *the anticevian triangle of the Grinberg point,*

- *Q_a the circumcenter of triangle J_aBC,*
- *Q_b the circumcenter of triangle AJ_bC,*
- *Q_c the circumcenter of triangle $ABJc$.*

Then the Feuerbach triangle $P_aP_bP_c$ is similar (but not homothetic) with triangle $Q_aQ_bQ_c$. The ratio of similarity is

$$k = \frac{\sqrt{E_1E_2E_3E_6}}{2abc(b+c)(c+a)(a+b)\sqrt{(b+c-a)(c+a-b)(a+b-c)}},$$

where E_1, E_2, E_3 are as in Theorem 1, and

$$\begin{aligned} E_6 =& b^2a^2c^2 + 2a^3c^2b + 2b^3a^2c + 2ba^2c^3 + 2ab^3c^2 + 2a^3b^2c \\ &+ 2ab^2c^3 + a^4b^2 + a^4c^2 - a^5b - a^5c + b^4a^2 + c^4a^2 - c^5a \\ &+ c^2b^4 + c^4b^2 - c^5b - b^5a - b^5c + 2a^3b^3 + 2a^3c^3 + 2c^3b^3 \\ &- c^6 - b^6 - a^6. \end{aligned}$$

Proof. The Grinberg point has barycentric coordinates

$$(u, v, w) = (a(b+c), b(c+a), c(a+b)).$$

Hence, the anticevian triangle of the Grinberg point $J_aJ_bJ_c$ has barycentric coordinates

$$J_a = (-u, v, w), \qquad J_b = (u, -v, w), \qquad J_c = (u, v, -w).$$

We follow the same plan as in the previous Theorem 3.

- By using the distance formula (9) we find the lengths of the following segments: J_aC, J_aB, J_bC, J_bA, J_cB and J_cA. In this way, we find the side lengths of triangles J_aBC, AJ_bC, ABJ_c.
- By using the change of coordinates formula (10) we find the barycentric coordinates of points Q_a, Q_b, Q_c.
- By using the distance formula (9), we find the lengths of the following segments: Q_bQ_c, Q_cQ_a, Q_aQ_b, P_bP_c, P_cP_a and P_aP_b.
- We find the ratio of similarity k.
- By using the method of Theorem 1, we prove that triangles $P_aP_bP_c$ and $Q_aQ_bQ_c$ are not homothetic. □

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Elementary Inequalities

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Abstract

In this paper we will discuss some topics of elementary inequalities, and we will see their application in several interesting problems.

1 Kantorovich Inequality

The Kantorovich inequality is a particular case of the Cauchy-Schwarz inequality, which is itself a generalization of the triangle inequality. The Kantorovich inequality is named after the Soviet economist, mathematician, and Nobel Prize winner Leonid Kantorovich, a pioneer in the field of linear programming. In 1947, G. B. Dantzig implemented a method called *simplex* to resolve problems involving the planning of the U.S. Air Force, marking the beginning of linear programming. The Kantorovich inequality is central to the study of convergence properties of descent methods in optimization.

Theorem 1. *Let $x_1 < x_2 < ... < x_n$ be positive real numbers and $\lambda_i \geq 0$ for all $i = 1, ..., n$, $n \in \mathbb{N}$, with $\lambda_1 + \ldots + \lambda_n = 1$. Then,*

$$\left(\sum_{j=1}^{n} \lambda_j x_j\right)\left(\sum_{j=1}^{n} \lambda_j x_j^{-1}\right) \leq \left(\frac{A}{G}\right)^2, \tag{1}$$

where $A = \frac{x_1+x_n}{2}$ and $G = \sqrt{x_1 x_n}$.

Proof. First, observe that if we replace each x_j by cx_j with $c > 0$ neither the left side nor the right side of the inequality is changed. Therefore, we can assume that $G = 1$. Then $x_n = \frac{1}{x_1}$. Observe now that if $x_1 \leq x_j \leq \frac{1}{x_1}$, then $x_j + \frac{1}{x_j} \leq x_1 + \frac{1}{x_1}$. Hence,

$$\sum_{j=1}^{n} \lambda_j x_j + \sum_{j=1}^{n} \lambda_j \frac{1}{x_j} \leq x_1 + \frac{1}{x_1} = 2A.$$

Now, to obtain our claim we many apply the arithmetic-geometric mean inequality,

$$\begin{aligned} 4\left(\sum_{j=1}^{n} \lambda_j x_j\right)\left(\sum_{j=1}^{n} \lambda_j x_j^{-1}\right) &\leq \left(\sum_{j=1}^{n} \lambda_j x_j + \sum_{j=1}^{n} \lambda_j \frac{1}{x_j}\right)^2 \\ &\leq 4A^2 = 4\left(\frac{A}{G}\right)^2. \end{aligned}$$

□

More formally, the Kantorovich inequality can be expressed this way:

Theorem 2. *Let $p_i \geq 0$, $0 < a \leq x_i \leq b$ for $i = 1, \ldots, n$. Let $A_n = \{1, 2, \ldots, n\}$. Then*

$$\begin{aligned} &\left(\sum_{i=1}^{n} p_i x_i\right)\left(\sum_{i=1}^{n} \frac{p_i}{x_i}\right) \leq \frac{(a+b)^2}{4ab}\left(\sum_{i=1}^{n} p_i\right)^2 \\ &-\frac{(a-b)^2}{4ab} \min\left\{\left(\sum_{i\in X} p_i - \sum_{j\in Y} p_j\right)^2 : X \cup Y = A_n, X \cap Y = \varnothing\right\}. \end{aligned} \tag{2}$$

The following well-known lemma will be important to solve the following problems.

Lemma 1 (Nesbitt). *Let a, b, c be positive real numbers. Then,*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}. \tag{3}$$

Proof. Straightforward. (Hint: Use the Cauchy-Schwarz inequality). □

In the following example we will use the Kantorovich inequality.

Example 1 (Roberto Bosch Cabrera). *Let* $0 < x < y < z$. *Show that*

$$(x+y+z)(xy+yz+zx) < \left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)^2 y(x+z)^2.$$

Solution. By Theorem 1,

$$\left(\frac{x}{3}+\frac{y}{3}+\frac{z}{3}\right)\left(\frac{1}{3x}+\frac{1}{3y}+\frac{1}{3z}\right) \leq \left(\frac{z+x}{2}\right)^2 \cdot \frac{1}{zx}.$$

Simplifying we have that

$$(x+y+z)(xy+yz+zx) < \frac{9}{4} \cdot (z+x)^2 y.$$

Using Lemma 1,

$$(x+y+z)(xy+yz+zx) < \left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)^2 y(x+z)^2. \quad \square$$

The following example was ontained from the book of Gelca and Andreescu [4]. Note that if $t_1 + \ldots + t_n = 1$ then the result is summarized in Theorem 1. Here we will give another proof.

Example 2. *Let* $0 < a < b$ *and* $t_i \geq 0$, $i = 1, \ldots, n$. *Prove that for any* $x_1, x_2, ..., x_n \in [a, b]$,

$$\left(\sum_{i=1}^{n} t_i x_i\right)\left(\sum_{i=1}^{n} \frac{t_i}{x_i}\right) \leq \frac{(a+b)^2}{4ab}\left(\sum_{i=1}^{n} t_i\right)^2.$$

Solution. The left-hand side of the inequality under discussion is a convex function in each x_i. Hence in order to maximize this expression we must choose some of the x_i's equal to a and the

others equal to b. For such a choice, denote by u the sum of the t_i's for which $x_i = a$ and by v the sum of the t_i's for which $x_i = b$. It remains to prove the simpler inequality

$$(ua + bv)\left(\frac{u}{a} + \frac{v}{b}\right) \leq \frac{(a+b)^2}{4ab}(u+v)^2.$$

This is equivalent to

$$4(ua + vb)(ub + va) \leq (ua + vb + ub + va)^2,$$

which is the AM–GM inequality applied to $ua + vb$ and $ub + va$. □

Example 3. *Let $x_1 < x_2 < \ldots < x_n$ be positive real numbers and $\lambda_1 \geq 0$ for all $i = 1, \ldots, n$, $n \in \mathbb{N}$ with $\lambda_1 + \ldots + \lambda_n = 1$. Prove that*

$$\left(\sum_{j=1}^{n} \lambda_j x_j^2\right)\left(\sum_{j=1}^{n} \lambda_j x_j^{-2}\right) \leq \left(\frac{MQ}{G}\right)^4,$$

where $MQ = \sqrt{\frac{x_1^2 + x_n^2}{2}}$ and $G = \sqrt{x_1 x_n}$.

Solution. As in Theorem 1, we can replace each x_j by cx_j with $c > 0$. Therefore, we can assume that $G = 1$. Then $x_n = \frac{1}{x_1}$. Observe now that if $x_1 \leq x_j \leq \frac{1}{x_1}$, then $x_j^2 + \frac{1}{x_j^2} \leq x_1^2 + \frac{1}{x_1^2}$. Hence,

$$\sum_{j=1}^{n} \lambda_j x_j^2 + \sum_{j=1}^{n} \lambda_j \frac{1}{x_j^2} \leq x_1^2 + \frac{1}{x_1^2} = 2 \cdot MQ^2.$$

Now, to obtain our claim we many apply the arithmetic-geometric mean inequality,

$$\begin{aligned} 4\left(\sum_{j=1}^{n} \lambda_j x_j^2\right)\left(\sum_{j=1}^{n} \lambda_j x_j^{-2}\right) &\leq \left(\sum_{j=1}^{n} \lambda_j x_j^2 + \sum_{j=1}^{n} \lambda_j \frac{1}{x_j^2}\right)^2 \\ &\leq 4 \cdot MQ^4 = 4\left(\frac{MQ}{G}\right)^4. \end{aligned}$$

□

This problem can be used to motivate students to investigate under what conditions equality is fulfilled in the inequality of Kantorovich.

2 Hölder's inequality

Hölder's inequality is a generalization of the Cauchy-Schwarz's inequality. It is well known and useful for solving problems. We will see some of its applications below.

Theorem 3. *Let a_{ij}, $1 \le i \le m$, $1 \le j \le n$, be positive real numbers. Then the following inequality holds:*

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}\right) \ge \left(\sum_{j=1}^{n}\left(\prod_{i=1}^{m} a_{ij}\right)^{\frac{1}{m}}\right)^{m}. \tag{4}$$

The following problem was proposed in the International Mathematical Olympiad in 2001.

Example 4 (IMO-2001). *Let a, b, c be positive numbers. Prove that*

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \ge 1.$$

Solution. By Hölder's inequality

$$\left(\sum_{\text{cyc}} \frac{a}{\sqrt{a^2+8bc}}\right)\left(\sum_{\text{cyc}} \frac{a}{\sqrt{a^2+8bc}}\right)\left(\sum_{\text{cyc}} a(a^2+8bc)\right) \ge (a+b+c)^3,$$

thus we need only show that

$$(a+b+c)^3 \ge a^3+b^3+c^3+24abc.$$

Using the identity

$$(a+b+c)^3 = a^3+b^3+c^3+3(a+b)(b+c)(c+a)$$

the last inequality is equivalent to

$$(a+b)(b+c)(c+a) \ge 8abc,$$

which is true by the AM-MG inequality, with equality if only if $a=b=c$. □

The following example shows that is not always obvious to use Hölder's inequality.

Example 5 (USAMO 2004). *Let a, b, c be positive real numbers. Prove that*

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

Solution. Observe that

$$\begin{aligned} a^5 - a^2 + 3 &= a^5 - a^2 + 1 - a^3 + 2 + a^3 \\ &= 2 + a^3 + a^2(a^3 - 1) - (a^3 - 1) \\ &= 2 + a^3 + (a^2 - 1)(a^3 - 1). \end{aligned}$$

Then by Hölder's inequality we have that

$$\begin{aligned} \prod_{\text{cyc}}(a^5 - a^2 + 3) &= \prod_{\text{cyc}}(2 + a^3 + (a^2 - 1)(a^3 - 1)) \\ &\geq \prod_{\text{cyc}}(2 + a^3) \\ &= (a^3 + 1 + 1)(1 + b^3 + 1)(1 + 1 + c^3) \\ &\geq (a + b + c)^3. \end{aligned}$$

Equality occurs, for example, when $a = b = c = 1$. □

Example 6 (Marius Stanean and Mircea Lascu, MR J209). *Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that*

$$\frac{(b + c)^5}{a} + \frac{(c + a)^5}{b} + \frac{(a + b)^5}{c} \geq \frac{32}{9}(ab + bc + ca).$$

Solution. By Hölder's inequality

$$\begin{aligned} (a + b + c)\left(\frac{(b+c)^5}{a} + \frac{(c+a)^5}{b} + \frac{(a+b)^5}{c}\right)\prod_{i=1}^{3}(1 + 1 + 1) \\ \geq ((b + c) + (c + a) + (a + b))^5, \end{aligned}$$

hence

$$\frac{(b + c)^5}{a} + \frac{(c + a)^5}{b} + \frac{(a + b)^5}{c} \geq \frac{32}{27}.$$

Because $a+b+c=1$ we have that

$$1=(a+b+c)^2 \geq 3(ab+bc+ca) \Longrightarrow \frac{1}{3} \geq ab+bc+ca.$$

Then,

$$\frac{(b+c)^5}{a}+\frac{(c+a)^5}{b}+\frac{(a+b)^5}{c} \geq \frac{32}{9}\cdot\frac{1}{3} \geq \frac{32}{9}(ab+bc+ca),$$

with equality if only if $a=b=c=1/3$. □

Example 7 (Pham Huu Duc). *If a, b, c are positive real numbers, show that*

$$\frac{a}{(2a+b+c)(b+c)}+\frac{b}{(2b+c+a)(c+a)}+\frac{c}{(2c+a+b)(a+b)} \geq \frac{9}{8\sqrt{3(ab+bc+ca)}}.$$

Solution. By Hölder's Inequality, we have that

$$\left[\sum_{\text{cyc}}\frac{a}{(2a+b+c)(b+c)}\right]\left[\sum_{\text{cyc}}a(b+c)\right]\left[\sum_{\text{cyc}}a(2a+b+c)\right] \geq (a+b+c)^3.$$

Therefore,

$$\sum_{\text{cyc}}\frac{a}{(2a+b+c)(b+c)} \geq \frac{(a+b+c)^3}{4(ab+bc+ca)(a^2+b^2+c^2+ab+bc+ca)}.$$

It suffices to prove that

$$\frac{(a+b+c)^3}{4(ab+bc+ca)(a^2+b^2+c^2+ab+bc+ca)} \geq \frac{9}{8\sqrt{3(ab+bc+ca)}},$$

or

$$4(a+b+c)^6 \geq 27(ab+bc+ca)(a^2+b^2+c^2+ab+bc+ca)^2.$$

Setting $x = ab + bc + ca$ and $y = a^2 + b^2 + c^2$ we have

$$4(y + 2x)^3 \geq 27x(x + y)^2 \Leftrightarrow (5x + 4y)(x - y)^2 \geq 0.$$

The proof is completed. Equality holds if and only if $a = b = c$. □

The last example shows how a problem that is hard to solve using other tools, can be simple to solve with Hölder's inequality.

Example 8 (Japan 2001). *Three nonnegative real numbers a, b, c satisfy $a^2 \leq b^2 + c^2$, $b^2 \leq c^2 + a^2$ and $c^2 \leq a^2 + b^2$. Prove the inequality*

$$(a + b + c)(a^2 + b^2 + c^2)(a^3 + b^3 + c^3) \geq 4(a^6 + b^6 + c^6).$$

When does equality hold?

Solution. By Hölder's inequality,

$$(a + b + c)(a^2 + b^2 + c^2)(a^3 + b^3 + c^3) \geq (a^2 + b^2 + c^2)^3.$$

So it is sufficient to show that

$$(a^2 + b^2 + c^2)^3 \geq 4(a^6 + b^6 + c^6),$$

which is equivalent to

$$a^4(b^2 + c^2) + b^4(c^2 + a^2) + c^4(a^2 + b^2) + 2a^2b^2c^2 \geq a^6 + b^6 + c^6,$$

which is immediately to our hypotheses, and since $2a^2b^2c^2 \geq 0$. Obviously, equality occurs if and only if $a = b = c = 0$. □

3 A useful lemma

We will solve some basic problems of inequality using a small lemma.

Lemma 2. *If a and b be positive real numbers, then*

$$\frac{a + b}{2} \leq \sqrt[3]{\frac{a^3 + b^3}{2}} \leq \frac{a^2 + b^2}{a + b}. \tag{5}$$

Proof. We will prove that $\frac{a+b}{2} \le \sqrt[3]{\frac{a^3+b^3}{2}}$. In fact,

$$\left(\frac{a+b}{2}\right)^3 \le \frac{a^3+b^3}{2} \iff 4(a^3+b^3) \ge (a+b)^3$$
$$\iff (1+1)(1+1)(a^3+b^3) \ge (a+b)^3,$$

which is true by Hölder's inequality. Now, we will prove that $\sqrt[3]{\frac{a^3+b^3}{2}} \le \frac{a^2+b^2}{a+b}$. In fact,

$$\frac{a^3+b^3}{2} \le \left(\frac{a^2+b^2}{a+b}\right)^3$$
$$\iff (a^3+3a^2b+3ab^2+b^3)(a^3+b^3) \le 2(a^6+3a^4b^2+3a^2b^4+b^6)$$
$$\iff (a-b)^4(a^2+ab+b^2) \ge 0,$$

with equality if and only if $a=b$. □

We will use the lemma in some mathematical olympiad problems. The first example is from the mathematical olympiad Poland second round 2007.

Example 9 (Poland 2007). *Let a, b, c, d be positive real numbers satisfying the following condition:*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}=4.$$

Prove that

$$\sqrt[3]{\frac{a^3+b^3}{2}}+\sqrt[3]{\frac{b^3+c^3}{2}}+\sqrt[3]{\frac{c^3+d^3}{2}}+\sqrt[3]{\frac{d^3+a^3}{2}} \le 2(a+b+c+d)-4.$$

Solution. By Lemma 2,

$$\sum_{\text{cyc}} \sqrt[3]{\frac{a^3+b^3}{2}} \le \sum_{\text{cyc}} \frac{a^2+b^2}{a+b} = \sum_{\text{cyc}}(a+b) - \sum_{\text{cyc}} \frac{2ab}{a+b}$$
$$= 2(a+b+c+d) - 2\sum_{\text{cyc}} \frac{1}{\frac{1}{a}+\frac{1}{b}}.$$

By the Cauchy-Schwarz inequality,

$$\sum_{\text{cyc}} \frac{1}{\frac{1}{a}+\frac{1}{b}} \geq \frac{(1+1+1+1)^2}{2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)} = \frac{8}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)} = 2$$

$$\Rightarrow -2\sum_{\text{cyc}} \frac{1}{\frac{1}{a}+\frac{1}{b}} \leq -4$$

$$\Rightarrow \sum_{\text{cyc}} \sqrt[3]{\frac{a^3+b^3}{2}} \leq 2(a+b+c+d)-4.$$

Equality occurs, for example, when $a=b=c=d=1$. □

The next example was used in the selection test for IMO 2010 in Turkey.

Example 10 (Turkey TST 2010). *Show that*

$$\sum_{cyc} \sqrt[4]{\frac{(a^2+b^2)(a^2-ab+b^2)}{2}}$$
$$\leq \frac{2}{3}(a^2+b^2+c^2)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

for all positive real numbers a, b, c.

Solution. By Lemma 2,

$$\sum_{\text{cyc}} \sqrt[4]{\frac{(a^2+b^2)(a^2-ab+b^2)}{2}} = \sum_{\text{cyc}} \sqrt[4]{\frac{(a^2+b^2)(a^3+b^3)}{2(a+b)}}$$
$$\leq \sum_{\text{cyc}} \frac{a^2+b^2}{a+b}$$

because

$$\frac{a^3+b^3}{2} \leq \left(\frac{a^2+b^2}{a+b}\right)^3.$$

Then it is enough to prove that

$$\sum_{\text{cyc}} \frac{a^2+b^2}{a+b} \le \frac{2}{3}(a^2+b^2+c^2)\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$
$$\Longrightarrow \sum_{\text{cyc}} \frac{2a^2-b^2-c^2}{b+c} \ge 0,$$

which is true by Chebyshev's inequality, with equality if and only if $a=b=c$. □

To conclude this section, we have the following.

Example 11 (Pedro Pantoja MR J181). *Let a, b, c, d be positive real numbers. Prove that*

$$\left(\frac{a+b}{2}\right)^3+\left(\frac{c+d}{2}\right)^3 \le \left(\frac{a^2+d^2}{a+d}\right)^3+\left(\frac{b^2+c^2}{b+c}\right)^3.$$

Solution. By Lemma 2,

$$\left(\frac{a+b}{2}\right)^3+\left(\frac{c+d}{2}\right)^3 \le \frac{a^3+b^3}{2}+\frac{c^3+d^3}{2},$$

with equality if and only if $a=b$ and $c=d$. Again by Lemma 2,

$$\frac{a^3+b^3}{2}+\frac{c^3+d^3}{2} \le \left(\frac{a^2+d^2}{a+d}\right)^3+\left(\frac{b^2+c^2}{b+c}\right)^3.$$

with equality if and only if $a=b=c=d$. By transitivity we obtain the desired inequality. □

4 A problem of the Mathematical Olympiad Iran 1996

With the problem below will solve two problems that are difficult to solve in other ways. It is a difficult problem, proposed in Mathematical Olympiad of Iran in 1996.

Lemma 3 (Iran 1996). *Prove the following inequality for positive real numbers x, y, z:*

$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\geq\frac{9}{4}. \qquad (6)$$

Proof (Iurie Boreico [1, page 108, problem 104]). With the substitution $x+y=c$, $y+z=a$, $z+x=b$, we have that $ab+bc+ac=3(xy+yz+zx)+x^2+y^2+z^2$ and $a^2+b^2+c^2=2(x^2+y^2+z^2)+2(xy+yz+zx)$. Then

$$xy+yz+zx=\frac{1}{2}(ab+bc+ca)-\frac{1}{4}(a^2+b^2+c^2),$$

hence

$$\frac{1}{4}(2ab+2bc+2ca-a^2-b^2-c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\geq\frac{9}{4}$$
$$\Longrightarrow 2\left(\sum_{\text{cyc}}\frac{a}{b}+\frac{b}{a}\right)+\sum_{\text{cyc}}\frac{2ab}{c^2}-\sum_{\text{cyc}}\left(\frac{a^2}{c^2}+\frac{b^2}{c^2}\right)-12\geq 0.$$

So the proposed inequality becomes

$$\sum_{\text{cyc}}\left(\frac{2}{ab}-\frac{1}{c^2}\right)(a-b)^2\geq 0.$$

Let $a\geq b\geq c$. If $2c^2\geq ab$, each term in the above expression is positive and we are done. So, let $2c^2<ab$. First, we prove that $2b^2\geq ac$ and $2a^2\geq bc$. Suppose that $2b^2<ac$. Then $(b+c)^2\leq 2(b^2+c^2)<a(b+c)$ and so $b+c<a\Longrightarrow 2x<0$, false. Clearly, we can write the inequality like that

$$\left(\frac{2}{ac}-\frac{1}{b^2}\right)(a-c)^2+\left(\frac{2}{bc}-\frac{1}{a^2}\right)(b-c)^2\geq\left(\frac{1}{c^2}-\frac{2}{ab}\right)(a-b)^2.$$

We can immediately see that the inequality $(a-c)^2\geq(a-b)^2+(b-c)^2$ holds and thus if suffices to prove that

$$\left(\frac{2}{ac}+\frac{2}{bc}-\frac{1}{a^2}-\frac{1}{b^2}\right)(b-c)^2\geq\left(\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{ab}-\frac{1}{ac}\right)(a-b)^2.$$

But it is clear that $\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{ab}-\frac{1}{ac}<\left(\frac{1}{b}-\frac{1}{c}\right)^2$ and so the right hand side is at most $\frac{(a-b)^2(b-c)^2}{b^2c^2}$. Also, it is easy to see that

$$\frac{2}{ac}+\frac{2}{bc}-\frac{1}{a^2}-\frac{1}{b^2}\geq\frac{1}{ac}+\frac{1}{bc}>\frac{(a-b)^2}{b^2c^2},$$

which shows that the left hand side is at least $\frac{(a-b)^2(b-c)^2}{b^2c^2}$, and this finish the proof. □

A second proof of this problem can be found in the book of Andreescu et al. [1, pages 108-109]. Here an application:

Example 12 (Marius Olteanu). *Prove that in any triangle ABC the inequality*

$$\frac{1}{\sin^2 A(\sin B+\sin C)^2}+\frac{1}{\sin^2 B(\sin C+\sin A)^2}+\frac{1}{\sin^2 C(\sin A+\sin B)^2}\geq\frac{4}{3}$$

holds.

Solution. Making $x=\sin A\sin B$, $y=\sin C\sin A$, $z=\sin B\sin C$ in the previous example, we have

$$\sum_{\text{cyc}}\frac{1}{\sin^2 A(\sin B+\sin C)^2}\geq\frac{9}{4\sin A\sin B\sin C(\sin A+\sin B+\sin C)}.$$

By the AM-MG inequality,

$$\sin A\sin B\sin C\leq\left(\frac{\sin A+\sin B+\sin C}{3}\right)^3,$$

and by Jensen's inequality applied to the sine function in the interval $[0,\pi]$, we have that

$$\frac{\sin A+\sin B+\sin C}{3}\leq\sin\left(\frac{A+B+C}{3}\right)=\frac{\sqrt{3}}{2}.$$

The result follows immediately. □

The next example was used in the selection test for IMO 2007 in Vietnam.

Example 13 (Vietnam TST 2007). *Given a triangle ABC, find the minimum of*

$$\frac{\cos^2\frac{A}{2}\cos^2\frac{B}{2}}{\cos^2\frac{C}{2}}+\frac{\cos^2\frac{B}{2}\cos^2\frac{C}{2}}{\cos^2\frac{A}{2}}+\frac{\cos^2\frac{C}{2}\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}}.$$

Solution. Let

$$L=\sum_{\text{cyc}}\frac{\cos^2\frac{B}{2}\cos^2\frac{C}{2}}{\cos^2\frac{A}{2}}.$$

Let $a=\tan\frac{A}{2}$, $b=\tan\frac{B}{2}$, $c=\tan\frac{C}{2}$ so $ab+bc+ca=1$. Since $1+a^2=\sec^2\frac{A}{2}$, $1+b^2=\sec^2\frac{B}{2}$ and $1+c^2=\sec^2\frac{C}{2}$, we have that

$$\begin{aligned}L&=\sum_{\text{cyc}}\frac{(1+a^2)}{(1+b^2)(1+c^2)}=\sum_{\text{cyc}}\frac{1}{\frac{(1+b^2)(1+c^2)}{(1+a^2)}}\\&=\sum_{\text{cyc}}\frac{1}{\frac{(ab+bc+ca+b^2)(ab+bc+ca+c^2)}{(ab+bc+ca+a^2)}}\\&=\sum_{\text{cyc}}\frac{1}{\frac{(a+b)(c+b)(a+c)(b+c)}{(b+a)(a+c)}}=\sum_{\text{cyc}}\frac{1}{(b+c)^2}.\end{aligned}$$

By Lemma 2,

$$\sum_{\text{cyc}}\frac{1}{(b+c)^2}\geq\frac{9}{4(ab+bc+ca)}=\frac{9}{4}.$$

Thus $L\geq\frac{9}{4}$. Equality holds when ABC is equilateral. The minimum value is equal to $\frac{9}{4}$. □

5 Independent study problems

This last section is devoted to stating some problems for the reader. All these can be solved with the techniques covered in this text.

Problem 1. *Let* $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ *be positive real numbers and* $m = \min_{1\le i\le n}\left\{\frac{a_i}{b_i}\right\}$, $M = \max_{1\le i\le n}\left\{\frac{a_i}{b_i}\right\}$. *Prove that*

$$\begin{aligned}&(a_1^2 + a_2^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + \ldots + b_n^2)\\ &\le \frac{(M+m)^2}{4Mm}(a_1b_1 + a_2b_2 + \ldots + a_nb_n).\end{aligned}$$

Problem 2 (José Luis Díaz-Barrero). *Let* $x_1, \ldots, x_n$ *be real numbers such that* $x_i \in [a, b]$, $1 \le i \le n$, $0 < a < b$. *Prove that*

$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i^3\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}\right) \le \frac{(a^2+b^2)^3}{6a^2b^2}.$$

Problem 3 (IMO 04, Shortlisted). *Let* a, b, c *be positive numbers such that* $ab + bc + ca = 1$. *Prove that*

$$\sqrt[3]{\frac{1}{a} + 6b} + \sqrt[3]{\frac{1}{b} + 6c} + \sqrt[3]{\frac{1}{c} + 6a} \le \frac{1}{abc}.$$

Problem 4 (Bielorussian). *Let* a, b, c, x, y, z *be positive real numbers. Prove that*

$$\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} \ge \frac{(a+b+c)^3}{3(x+y+z)}.$$

Problem 5 (Iran 1996/alternative solution). *Let* x, y, $z > 0$. *Prove that*

- $\dfrac{x}{y+z} + \dfrac{y}{x+z} + \dfrac{z}{x+y} \ge \dfrac{3(x^2+y^2+z^2)}{(x+y+z)^2} + \dfrac{1}{2}$,
- $\dfrac{xy}{(x+y)^2} + \dfrac{zx}{(z+x)^2} + \dfrac{yz}{(y+z)^2} + \dfrac{3(x^2+y^2+z^2)}{(x+y+z)^2} \ge \dfrac{7}{4}$.

Adding up these last two inequalities we have that

$$(xy + yz + zx)\left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right) \ge \frac{9}{4}.$$

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Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.
2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

October 31, 2018

Elementary Problems

E–53. *Proposed by Marc Felipe Alsina, BarcelonaTech, Barcelona, Spain.* Let x, y, z be three positive numbers adding up to 1. Find the minimum value of

$$\frac{4}{x}+\frac{9}{y}+\frac{25}{z}.$$

E–54. *Proposed by José Luis Díaz-Barrero and José Gibergans Báguena, BarcelonaTech, Barcelona, Spain.* Compute the following sum:

$$\lfloor\sqrt{1}\rfloor+\lfloor\sqrt{2}\rfloor+\lfloor\sqrt{3}\rfloor+\ldots+\lfloor\sqrt{n^2-1}\rfloor$$

(here, $\lfloor x\rfloor$ represents the integer part of x).

E–55. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* If a and b are real numbers for which the equation $x^6+ax^5+x^4+bx^2+4=0$ has a real root, then prove that $4a^2+b^2\geq 16$.

E–56. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In a classroom we have n girls and two boys. There are n seats in a row, in which we want to sit n students. We say that two arrangements are equivalent if, for each chair, the gender of the student sitting in that chair in both arrangements is the same. How many non-equivalent arrangements can we find?

E–57. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Two sides of a triangle have length 7 and 11, respectively. If the median drawn to the third side has length 7, then find the length of the third side.

E–58. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let F_n be the n-th Fibonacci number defined by $F_1=F_2=1$ and for all $n\geq 1$, $F_{n+2}=F_{n+1}+F_n$. Express in terms of F_n and F_{n+1} the following sum :

$$\sum_{1\leq i<j\leq n} F_iF_j.$$

Easy–Medium Problems

EM–53. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be positive real numbers such that

$$a + \frac{1}{b} = b + \frac{1}{c} = c + \frac{1}{a}.$$

Prove that

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{9}{2\,(a+b+c)}.$$

EM–54. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a and b be two positive integers. Prove that for any odd prime p the equation $\operatorname{lcm}(a, 2a+p) = \operatorname{lcm}(b, 2b+p)$ implies $a = b$.

EM–55. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let x, y, z, t be positive real numbers such that $x+y+z+t = 2$. Show that

$$\left(\frac{4}{x^2} - 1\right)\left(\frac{4}{y^2} - 1\right)\left(\frac{4}{z^2} - 1\right)\left(\frac{4}{t^2} - 1\right) \geq 15^4.$$

EM–56. *Proposed by Nicolae Papacu, Slobozia, Romania.* Solve the equation $p + [x] = [px]$, where p is a positive integer (here, $[a]$ denotes the integer part of a).

EM–57. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let S be a set of 17 distinct positive integers. If all prime divisors of each number in S are smaller than 8, then prove that there are two of them whose product is a square.

EM–58. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* A circle of radius r is inscribed in a triangle ABC. Tangents to the circle and parallel to the sides of the triangle are drawn. This lines cut three small triangles off $\triangle ABC$. If r_1, r_2

and r_3 are the radii of the inscribed circles to small triangles, then prove that

$$\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\geq\frac{18}{R},$$

where R is the circumradius of $\triangle ABC$.

Medium–Hard Problems

MH–53. *Proposed by Pedro Henrique O. Pantoja, University of Campina Grande, Brazil.* Let x, y, z be positive real numbers. Prove that

$$\frac{x^2+y^2+z^2}{(x+y+z)^2}+\frac{648(x+y+z)^2}{(x^2+y^2)(y^2+z^2)(z^2+x^2)}\geq 12.$$

MH–54. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let ABC be an acute triangle, AP be the bisector from A, AS be the symmedian from A, BD be the median from B and P, $S\in BC$, $D\in AC$. If $AP\cap BD=\{O\}$, $AS\cap BD=\{Q\}$ and $\dfrac{[BPO]}{[AOD]}=\dfrac{8}{15}$, calculate $\dfrac{BQ}{QD}$.

MH–55. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n and r be positive integers. Prove that

$$\sum_{k=1}^{n}\binom{n}{k}\left(2^{nr}-\binom{n}{k}^r\right)\leq\frac{nr\,2^{n(r+1)}}{r+1}\sqrt[r]{\frac{1}{r+1}}.$$

MH–56. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all positive solutions of the following system of equations:

$$\left.\begin{aligned}\sum_{\text{cyclic}}\frac{a^9+b^7}{b^2+c^2+d^2}&=\frac{8}{3},\\ abcd&=1.\end{aligned}\right\}$$

MH–57. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $k\geq 1$ be an integer number and let $u_1,\ldots,u_k$ and $z_1,\ldots,z_k$ be distinct nonzero complex numbers. If the set

$$A=\{a_n\mid a_n=u_1z_1^n+u_2z_2^n+\ldots+u_kz_k^n,n\in\mathbb{N}\}$$

is finite, then prove that there exists a positive integer d such that $a_n=a_{n+d}$ for all $n\in\mathbb{N}$.

MH–58. *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.* We have four horizontal lines, and the second one from the top is blue. We have a set S of 600 points on those four lines such that each line contains at least two points and, for any five points in S, there are two of them P and Q such that the segment PQ contains a point from S other than P and Q. Prove that the blue line contains at least 100 points from S.

Advanced Problems

A–53. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let A, $B \in M_2(\mathbb{Z})$ be two matrices such that $\det A = \det B = 0$. Show that $\det(A^3 + B^3) \neq 2018$ and $\det(A^3 - B^3) \neq 2019$.

A–54. *Proposed by Pedro Henrique O. Pantoja, University of Campina Grande, Brazil.* Prove that

$$\frac{(\tan\frac{\pi}{7})^6 + (\tan\frac{2\pi}{7})^6 + (\tan\frac{3\pi}{7})^6}{(\tan\frac{\pi}{7})^2 + (\tan\frac{2\pi}{7})^2 + (\tan\frac{3\pi}{7})^2}$$

is a prime number.

A–55. *Proposed by Leonhard Summerer, University of Vienna, Austria.* Find all integers $n \geq 1$ for which the polynomial

$$P_n(x) = (x-1)(x-2)\dots(x-n) + 1$$

is irreducible over $\mathbb{Q}$.

A–56. *Proposed by Sergio Falcón and Ángel Plaza, both at University of Las Palmas de Gran Canaria, Spain.* The Fibonacci numbers are defined recursively by $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$, with initial conditions $F_0 = 0$, $F_1 = 1$. Find a closed expression for the following sum, where $r > 1$ and $n \geq r$ are integers:

$$\sum_{j=0}^{n-r}\left(\binom{r+j}{r} - \binom{r+j-1}{r} - \binom{r+j-2}{r}\right)F_{n-(r+j)}.$$

A–57. *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.* Let n be a positive integer, and let $f\colon [0,1] \to \mathbb{R}^n$ be a nonconstant continuous function. Prove that there exist x, $y \in [0,1]$ such that:

- For every $1 \leq i \leq n$, if $f(x)_i \in \mathbb{Q}$ then $f(x)_i = f(y)_i$.
- There exists an index i such that $f(x)_i \notin \mathbb{Q}$ and $f(y)_i \in \mathbb{Q}$.

(Here, $f(x)_i$ denotes the i-th coordinate of $f(x)$.)

A–58. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Compute the following sum:

$$\sum_{k=1}^{n}(-1)^{k+1}\binom{n}{k}\left[\sum_{j=1}^{k}\frac{1}{j^3}+\sum_{1\le i<j\le k}\frac{1}{ij(i+j)}+\sum_{1\le i<j<\ell\le k}\frac{1}{ij\ell}\right].$$

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Mach Ado about Pigeonhole Principle

José Luis Díaz-Barrero and Marc Felipe Alsina

1 Introduction

It is obvious that no function from a set of cardinal at least $n+1$ to a set of cardinal n can be one-to-one because some two elements of the domain have the same image. The Pigeonhole Principle, for short PHP, in its simplest form, is a rephrasing of this statement.

The Pigeonhole Principle gets its name from the idea of a grid of little boxes that might be used, for example, to sort mail, or as mailboxes for a pool of people in an office. The boxes in such grids are sometimes called pigeonholes in analogy with stacks of boxes used to house homing pigeons when these were used to carry messages.

The first formalization of this idea is believed to have been made by Peter Gustav Lejeune Dirichlet in 1834 under the name **drawer principle**. For this reason it is also commonly called **Dirichlet's box principle** or **Dirichlet's drawer principle**.

The PHP is an extremely useful tool, not just in combinatorics, but in about almost every branch of mathematics as well. To solve problems using the Pigeonhole Principle there are several steps that must be taken into account:

- To decide what the pigeons are. Usually, they are the objects with a special property.

- To set up the pigeonholes. You want to do this so that when you get two pigeons in the same pigeonhole, they have the property you want. To use the PHP, it is necessary to set things up so that there are fewer pigeonholes than pigeons.
- To give a rule for assigning the pigeons to the pigeonholes. Note that the conclusion of the Pigeonhole Principle holds for any assignment of pigeons to pigeonholes.
- To apply the PHP to your setup and get the desired conclusion.

Our goal in this Mathlesson is to give some examples that will be discussed and completely solved using the PHP, as we will see later on.

2 Pigeonhole Principle

The simplest case of the pigeonhole principle states that if we partition a set with more than n elements into n pairwise disjoint parts, then at least one part has more than one element. This easily generalizes to the following result.

Theorem 1 (Pigeonhole Principle). *Let n, k be two positive integers. Suppose that we place at least $kn + 1$ objects into n boxes. Then some box must contain at least $k + 1$ objects.*

Proof. We argue by contradiction. Assume that the answer is not. Then every box has at most k objects, so that the total number of objects is at most kn. This is a contradiction, since we have at least $kn + 1$ objects. □

3 Problems

Hereafter, some more or less well-known examples where one solution is obtained by using the PHP are given. We begin with the following.

Problem 1. *Let a, b, c, d be four integer numbers. Prove that*

$$(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

is divisible by 12.

Solution. First, we observe that the difference of two even numbers is even and the same occurs if they are odd. Now, we organize in cases depending on the number of even and odd integers among a, b, c, d. If we represent them as $(\#Even, \#Odd)$, we have the following possible situations:

$$(4,0),(3,1),(2,2),(1,3),(0,4).$$

For the first case, we have that all the six factors are even and the factor 2^6 divides the product. For the remaining pairs, the powers of two that appear are 2^3, 2^2, 2^3, 2^6, respectively. So, the number is a multiple of $2^2 = 4$. Now, we claim that it is a multiple of 3. Indeed, dividing an integer by 3 the possible remainders are 0, 1 and 2. Since we have four integers a, b, c, d and three possible remainders, by the PHP two of them must be equal and the difference of these number is a multiple of three, as claimed. This completes the proof. □

Another example that uses the PHP to solve problems in arithmetics is the following [1].

Problem 2 (IMAC 2013). *Show that in any set of three distinct integers there are two of them, say a and b, such that the number $a^5b^3 - a^3b^5$ is a multiple of* 10.

Solution. First we observe that the statement holds if the set includes $a = 0$ or $b = 0$. Let us denote $N(a,b) = a^5b^3 - a^3b^5$. Since $N(-a,-b) = N(a,b)$ and $N(-a,b) = N(a,-b) = -N(a,b)$, then WLOG we may assume that the three distinct integers are all positive. Now, it is easy to check that $a^5b^3 - a^3b^5$ is even, and it suffices to prove that $N(a,b)$ is a multiple of 5, which will certainly occur if either a or b is a multiple of 5. Since

$$N(a,b) = a^3b^3(a^2 - b^2) = a^3b^3(a-b)(a+b),$$

what we have to prove is the following claim:

Claim. *Given any three positive integers none of which is multiple of* 5*, the sum or difference of two of them is a multiple of* 5*.*

Indeed, the last digit of any number not multiple of 5 lies in the set

$$\{1, 2, 3, 4, 6, 7, 8, 9\}.$$

Let $A = \{1, 4, 6, 9\}$ and $B = \{2, 3, 7, 8\}$ (pigeonholes). Of the three integers (pigeons) in our set, by the PHP, at least two belong to A or at least two belong to B. In any case, either their sum or their difference is a multiple of 5, as can be easily checked, and we are done. □

The next problem appeared in Spanish Mathematical Olympiad 2018.

Problem 3. *Let* S *be a sphere of radius* 4*. Suppose that the surface of* S *is coloured with* 4 *distinct colours. Prove that there exist two points* X*,* $Y \in S$ *of the same colour with* $|XY| \in \{4\sqrt{3}, 2\sqrt{6}\}$*.*

Solution. Since the points of the sphere are coloured with four different colors, to solve the problem it will suffice to find five different points that meet the conditions of the statement and apply the Pigeonhole Principle (PHP).

This can be done, for example, observing that if we inscribe an equilateral triangle in a circumference of maximum diameter of the sphere, then the side of this triangle measures $4\sqrt{3}$, and that taking two such triangles, contained in perpendicular planes and having a common vertex, the uncommon vertices are $2\sqrt{6}$ apart. Indeed, in a cartesian coordinate system with origin at the center of S, its surface contains all points (x, y, z) that satisfy the equation $x^2+y^2+z^2 = 16$. Now, consider the points of the sphere $A(4, 0, 0)$, $B(-2, 2\sqrt{3}, 0)$, $C(-2, -2\sqrt{3}, 0)$ that lie on the circle

$$\Gamma_1 = \{(x, y, z) \mid x^2 + y^2 = 16, z = 0\}$$

and are the vertices of an inscribed equilateral triangle of side length $4\sqrt{3}$. Likewise, the points $A(4, 0, 0)$, $D(-2, 0, 2\sqrt{3})$ and $E(-2, 0, -2\sqrt{3})$ are the vertices of an equilateral triangle of side length $4\sqrt{3}$ inscribed in the circle

$$\Gamma_2 = \{(x, y, z) \mid x^2 + z^2 = 16, y = 0\}.$$

Note that the planes of circles Γ_1 and Γ_2 are perpendicular, that the equilateral triangles have a common vertex A, and that

$$AB = AC = AD = AE = BC = DE = 4\sqrt{3}$$

and

$$BD = BE = CD = CE = 2\sqrt{6}.$$

Thus, each of these five points A, B, C, D, E is at distance $4\sqrt{3}$ or $2\sqrt{6}$ from each of the remaining four, and by the PHP, there are two of them of the same colour. □

Problem 4. *Each one of the six contestants and the leader of a Team attending to a Mathematical Contest and sitting around a table writes a number in a piece of paper. Show that there must be two people for which either the sum or the difference in their numbers is a multiple of* 10.

Solution. A multiple of 10 is easy to identify by the digit 0 in the units position. If two people have written numbers ending with the same digit, the difference in their numbers is a multiple of 10. Also, if someone sitting around the table has a number ending in the digit 2 and another team member wrote a number ending with the digit 8, their sum of numbers ends with the digit 0. Continuing in this way suggests to consider the following pigeonholes:

$$\{1,9\} \quad \{2,8\} \quad \{3,7\} \quad \{4,6\} \quad \{5\} \quad \{0\}.$$

The pigeons are the seven numbers. When these are placed in the box labeled with the set containing the last digit of the number, the pigeonhole principle guarantees that at least one of the six boxes contains at least two numbers. If these two numbers occur to have the same units digit, then their difference is a multiple of 10. If the two numbers have different last digits, these must be 1 and 9, or 2 and 8, or 3 and 7, or 4 and 6. In each case, the sum of the two numbers is a multiple of 10. □

An algebra application of the PHP is illustrated in the following example [2].

Problem 5. *Let $a_1, a_2, \ldots, a_{13}$ be distinct real numbers. Prove that there exists two of them, say a and b, such that*

$$0 < \frac{a-b}{1+ab} \leq 2 - \sqrt{3}.$$

Solution. For each $i \in \{1, 2, \ldots, 13\}$ there exists a real number $x_i \in (-\pi/2, \pi/2)$ such that $\tan(x_i) = a_i$. If we divide the interval $(-\pi/2, \pi/2)$ into 12 parts of length $\pi/12$, then by the Pigeonhole Principle, at least two of the numbers x_i and x_j lie in the same subinterval. Then, we have that

$$0 < x_i - x_j \leq \frac{\pi}{12} \implies \tan(0) < \tan(x_i - x_j) \leq \tan\left(\frac{\pi}{12}\right).$$

Calling $a = \tan(x_i)$ and $b = \tan(x_j)$, from the preceding we have that

$$0 < \frac{a-b}{1+ab} \leq \tan\left(\frac{\pi}{12}\right) = 2 - \sqrt{3}. \qquad \square$$

An application of PHP to analysis is the following example.

Problem 6. *Let a be an irrational number. Prove that there exist infinitely many rational numbers $r = p/q$ such that*

$$|a - r| < \frac{1}{q^2}.$$

Solution. Let Q be a positive integer. Assume also that $a > 0$. Consider the fractional parts $\{0\}, \{a\}, \{2a\}, \ldots, \{Qa\}$ of the first $(Q+1)$ multiples of a. By the pigeonhole principle, two of these must fall into one of the Q (semiopen) intervals

$$\left[0, \frac{1}{Q}\right), \left[\frac{1}{Q}, \frac{2}{Q}\right), \ldots, \left[\frac{Q-1}{Q}, 1\right),$$

where, as usual, $[A, B) = \{x : A \leq x < B\}$. In other words, there are integers s, q_1, and q_2 such that $\{q_1 a\} \in [s/Q, (s+1)/Q)$ and $\{q_2 a\} \in [s/Q, (s+1)/Q)$. Taking $q = |q_1 - q_2|$ we obtain, for

some integer p, $|qa - p| < 1/Q$. Dividing by q we get $|a - p/q| < 1/(Qq) \leq 1/q^2$ since, by definition, $0 < q \leq Q$.

We have yet to prove that the number of such pairs (p, q) is infinite. Assume, on the contrary, that the inequality $|a - r_i| < q_i^{-2}$ only holds for a finite number of $r_i = p_i/q_i$, $i = 1, 2, \ldots, N$. Since none of the differences is exactly 0, there exists an integer Q such that $|a - r_i| > 1/Q$ for all $i = 1, \ldots, N$. Apply our starting argument to this Q to produce $r = p/q$ such that $|a - r| < 1/(Qq) \leq 1/Q$. Hence r cannot be one of the r_i, $i = 1, \ldots, N$. On the other hand, as before, $|a - r| < 1/q^2$, contradicting the assumption that the fractions r_i, $i = 1, \ldots, N$, were all the fractions with this property. This completes the proof. □

Problem 7. *In Ancient Greece,* 17 *authors liked to write each other letters about theatre plays. Each pair of authors discussed a unique play, which could be a comedy, a tragedy or a drama. Show that there were* 3 *among those* 17 *such that they wrote to each other about the same genre.*

Solution. Focus on one author. He (by the context, we may assume all authors are male) discussed a total of 16 plays, which should fit in one of the three genres: comedy, tragedy or drama. Therefore, by the Pigeonhole Principle, there are $6 = 5 + 1$ of them lying in the same genre $(16 = 5 \cdot 3 + 1)$. The 6 corresponding authors wrote to the first author about one genre, so if any pair of them talked about it as well, we have found the three authors we were looking for. Otherwise, those 6 authors can only write to each other about two of the genres.

Ignore all authors but those 6. We have reduced the problem from 17 authors and 3 genres to just 6 authors and 2 genres. Focus on one of those authors. He discussed 5 plays that must fit into two genres, so again, by the PHP, three of them coincide in genre. If any pair among the three corresponding authors also studied that genre, we found the trio. Otherwise, those three must write to each other about the remaining genre, and we are done. □

References

[1] Alexandrescu, P. and Díaz-Barrero, J. L. *International Mathematical Arhimede Contest*. Editura Nomina, 2017.

[2] Díaz-Barrero, J. L. and Bencze, M. *Mathematical Contest Notebook*. STUDIS Publishing House, 2015.

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Contests

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to find interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Mathcontest 2018

Óscar Rivero Salgado

1 Introduction

The fifth edition of the BarcelonaTech Mathcontest took place on Saturday 17th February 2018 at the Faculty of Mathematics and Statistics of Universitat Politècnica de Catalunya. This is a one-day competition addressed to students who have won or have qualified in good positions at the corresponding local stages of the Spanish Mathematical Olympiad. At the same time, and since 2016, it serves to select the Spanish team for the European Girls' Mathematical Olympiad (EGMO), which this year took place in Florence (Italy).

In this competition, students should solve, in a maximum time of four hours, four problems about the different topics that usually appear in mathematical olympiads (algebra, combinatorics, geometry and number theory). In this fifth edition, there were thirty nine contestants, and the three gold medals were for Alejandro Epelde (Madrid), Jose Pérez (Jaén) and Félix Moreno (València).

2 Problems and solutions

We now present the four problems that appeared in the paper, as well as some solutions and comments to them.

Problem 1. Let f be a non-constant function such that

$$f(x) = \frac{f(x-1) + f(x+1)}{\sqrt{2}}.$$

1. Determine the smallest positive integer k such that $f(x+k) = f(x)$.
2. Find an example of such a function.

Solution 1 for the first part (official, slightly modified). We clearly have that

$$f(x+2) = \sqrt{2}f(x+1) - f(x),$$

so

$$f(x+3) = f(x+1) - \sqrt{2}f(x)$$

and

$$f(x+4) = \sqrt{2}(f(x+1) - \sqrt{2}f(x)) - \sqrt{2}f(x+1) + f(x) = -f(x).$$

Consequently, the equality

$$f(x+8) = f(x)$$

holds. We have to check now that $k = 8$ is in fact the minimum value. For that, we observe that if f is k_1-periodic and k_2-periodic, it is also $\gcd(k_1, k_2)$-periodic: to see this, choose integer numbers α and β (by Bézout's theorem) such that

$$\alpha \cdot k_1 + \beta \cdot k_2 = \gcd(k_1, k_2).$$

Then,

$$f(x + \gcd(k_1, k_2)) = f(x + \alpha \cdot k_1 + \beta \cdot k_2) = f(x),$$

as desired. Then, if there exists a small k (say k') with $f(x+k') = f(x)$, then $\gcd(8, k')$ also works, but this quantity divides 8 and it is strictly smaller than 8. Then, it must be a divisor of 4 and the function is therefore 4-periodic. However,

$$-f(x) = f(x+4) = f(x)$$

implies that $f(x)$ is identically zero, and this contradicts the statement.

Solution 2 for the first part (by the jury members). Consider the sequence of functions (a_n) given by $a_n = f(x+n)$. We clearly have that

$$a_{n+2} = \sqrt{2}a_{n+1} - a_n,$$

and this is a second-order recurrence whose general expression is given by

$$f(x+n) = \alpha \cdot \sin\left(\frac{\pi n}{4}\right) + \beta \cdot \cos\left(\frac{\pi n}{4}\right),$$

where α and β only depend on the initial conditions. In particular,

$$\beta = f(x), \qquad \alpha = \sqrt{2}f(x+1) - f(x).$$

Then,

$$f(x+k) = f(x)\Big(\cos\left(\frac{\pi k}{4}\right) - \sin\left(\frac{\pi k}{4}\right)\Big) + \sqrt{2}f(x+1)\sin\left(\frac{\pi k}{4}\right).$$

We clearly have that $k = 8$ works and by the above argument we conclude that it must be the minimum one.

Solution 1 for the second part (official). We have that

$$f(x) = \sin\left(\frac{\pi \cdot x}{4}\right)$$

clearly works: by using the standard trigonometric formulas,

$$\sin\left(\frac{\pi \cdot x}{4} - \frac{\pi}{4}\right) + \sin\left(\frac{\pi \cdot x}{4} + \frac{\pi}{4}\right) = \sqrt{2} \cdot \sin\left(\frac{\pi \cdot x}{4}\right),$$

as desired.

Solution 2 for the second part (by the jury members). Observe that the given equation relates the values at $f(x)$, $f(x+1)$ and $f(x+2)$. Hence, real numbers x, y such that $x - y$ is not an integer number are not related in any way by the functional equation. Moreover, for each real number α, the set of values $\{f(\alpha+n)\}$, where $n \in \mathbb{Z}$, is completely determined after fixing the value of two of them. In particular, any function like the one described in the statement is obtained by fixing its values in the interval $[0,2)$ and then applying the recursion. If moreover we impose the condition that not all the values in $[0,2)$ are zero, we have all the possible solutions.

Remark. *One of the contestants presented as an example the function that is zero in $\mathbb{R}\backslash\mathbb{Z}$ and in the integers satisfies that*

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is a multiple of } 4, \\ 1 & \text{if } n \text{ is congruent with } 2 \mod 8, \\ -1 & \text{if } n \text{ is congruent with } 6 \mod 8, \\ 1/\sqrt{2} & \text{if } n \text{ is congruent with } 1,3 \mod 8, \\ -1/\sqrt{2} & \text{if } n \text{ is congruent with } 5,7 \mod 8. \end{cases}$$

Problem 2. For any positive number n, let $\tau(n)$ denote the number of positive divisors of n. We say that a positive number is *nice* if

$$\frac{\tau(10n)}{\tau(n)} < \frac{5}{2}.$$

Determine how many positive numbers smaller or equal than 2018 are *nice*.

Solution. Let

$$n = 2^a \cdot 5^b \cdot \prod_{i=1}^{r} p_i^{k_i}$$

be the prime decomposition of a positive number n, with a, $b \geq 0$, $p_i \neq 2$, 5 and $k_i \geq 1$. Then,

$$10n = 2^{a+1} \cdot 5^{b+1} \cdot \prod_{i=1}^{r} p_i^{k_i},$$

and consequently

$$\frac{\tau(10n)}{\tau(n)} = \frac{(a+2)(b+2)\prod_{i=1}^{r}(k_i+1)}{(a+1)(b+1)\prod_{i=1}^{r}(k_i+1)} = \Big(1+\frac{1}{a+1}\Big)\Big(1+\frac{1}{b+1}\Big).$$

If both a, $b \geq 1$, the quotient is smaller or equal than $9/4 < 5/2$. On the other hand, if one of them is 0 (say a) then it must hold that

$$1 + \frac{1}{b+1} < \frac{5}{4},$$

or what is the same $b \geq 4$. Hence, the condition holds if and only if at least one of the following conditions holds:

- The number is a multiple of 10.
- The number is a multiple of 16.
- The number is a multiple of 625.

Fix now a positive number M. By the inclusion-exclusion principle, the amount of numbers that are smaller or equal than M and satisfy some of the conditions is

$$\begin{aligned}&\left\lfloor\frac{N}{10}\right\rfloor+\left\lfloor\frac{N}{16}\right\rfloor+\left\lfloor\frac{N}{625}\right\rfloor-\left\lfloor\frac{N}{80}\right\rfloor-\left\lfloor\frac{N}{1250}\right\rfloor-\left\lfloor\frac{N}{10000}\right\rfloor+\left\lfloor\frac{N}{10000}\right\rfloor\\=&\left\lfloor\frac{N}{10}\right\rfloor+\left\lfloor\frac{N}{16}\right\rfloor+\left\lfloor\frac{N}{625}\right\rfloor-\left\lfloor\frac{N}{80}\right\rfloor-\left\lfloor\frac{N}{1250}\right\rfloor.\end{aligned}$$

Hence, substituting N by 2018 we get

$$201+126+3-25-1=304.$$

Remark. *All the correct solutions presented during the contest were essentially like this one. Some of them proceed through a more involved case by case analysis, in which the key idea was that τ is weakly multiplicative, that is, $\tau(mn)=\tau(m)\tau(n)$ for any integers m,n such that $\gcd(m,n)=1$.*

Problem 3. Let ABC be a triangle, with $AB<BC$, and let I be its incenter. The incircle of this triangle is tangent to AC at D and to BC in E. Let F be the point in segment BC such that $AB=FB$. Prove that AF, BI and DE are concurrent.

We present now five different approaches to the problem. Solutions 1 and 2 are essentially the same; both of them, together with solutions 3 and 4, are based on defining a point M in terms of the intersection of two of the lines AF, BI and DE, and then showing that M belongs to the remaining line.

Solution 1 (official). The triangle ABF is isosceles with $AB=FB$, and BI is the angle bisector of $\angle ABF$, so it is also the median, and intersects AF at its midpoint. We need to prove that DE intersects AF at its midpoint.

Let G be the point at which the incircle of ABC touches AB. Then,

$$AD=AG=AB-GB=FB-EB=EF.$$

Let R be the point at which the parallel to BC through A intersects DE. Then, triangles EDC and RDA are similar, because their sides are parallel. Moreover, $CD = CE$ implies $AR = AD = EF$. $ARFE$ is a parallelogram, and its diagonals meet at the midpoint of each of them, so the midpoint of AF is in DE.

Solution 2. Let $M = DE \cap AF$. We must prove that B, I and M are collinear. It is enough with proving that M is the midpoint of the side AF in triangle ABF, since this would imply that BM is an angle bisector and consequently goes through I. As in the previous solution, we have $AD = EF$. Now, applying Menelaus' theorem in triangle AFC to the line DME,

$$\frac{AM}{MF} \cdot \frac{FE}{EC} \cdot \frac{CD}{DA} = \frac{AM}{MF} = 1,$$

as desired.

Solution 3 by Marc Felipe, CFIS, BarcelonaTech, Barcelona, Spain. Let $M = AF \cap BI$. We must prove that E, M and D are collinear, and for that it is enough to prove that

$$\angle BEM = \angle BED.$$

On the one hand,

$$\angle BED = \pi - \angle CED = \pi/2 + \angle C/2,$$

where we have used that CED is an isosceles triangle. On the other hand, taking into account that $FEIM$ is a cyclic quadrilateral (since $\angle FEI = \angle FMI = \pi/2$),

$$\begin{aligned}\angle BEM &= \pi - \angle FEM = \pi - \angle FIM = \pi - \angle AIM \\ &= \angle AIB = \pi - \angle A/2 - \angle B/2 = \pi/2 + \angle C/2,\end{aligned}$$

as desired.

Solution 4 by Gerard Orriols, CFIS, BarcelonaTech, Barcelona, Spain. Let $M = BI \cap DE$; we would have to prove that AM intersects BC in a point F' such that $AB = BF'$, and this is

equivalent to the angle bisector BI be an altitude of triangle ABF', i.e., $\angle BMA = \pi/2$. For that, it is enough to prove that triangles AMB and IEB are similar, and since $\angle ABI = \angle IBE$, we need

$$\frac{BA}{BM} = \frac{BI}{BE},$$

or alternatively

$$\frac{BA}{BI} = \frac{BM}{BE},$$

so if we prove that ABI and BME are similar we are done. Indeed, $\angle ABI = \angle MBE$ and

$$\angle MEB = \pi - \angle DEC = 90 + \angle C/2 = \angle AIB.$$

Solution 5 by Óscar Rivero, BarcelonaTech, Barcelona, Spain. We will use barycentric coordinates. As usual, let

$$A = (1:0:0), \quad B = (0:1:0), \quad C = (0:0:1),$$

and therefore

$$\begin{array}{ll} I = (a:b:c), & D = (p-c:0:p-a), \\ E = (0:p-c, p-b), & F = (0:a-c:c). \end{array}$$

Now we can compute the different lines. Firstly, the equation of BI is given by

$$cx - az = 0.$$

Then, the equation of AF is

$$-cy + (a-c)z = 0.$$

Finally, the line DE has equation

$$-(p-a)x - (p-b)y + (p-c)z = 0.$$

Now, a straightforward computation shows that

$$\begin{vmatrix} c & 0 & -a \\ 0 & -c & a-c \\ a-b-c & b-a-c & a+b-c \end{vmatrix} = 0,$$

and the colinearity of the three lines follows.

Problem 4. Let S be a finite set. Consider three partitions of S, each one with n elements:

$$A_1, A_2, \ldots, A_n; \quad B_1, B_2, \ldots, B_n; \quad C_1, C_2, \ldots, C_n.$$

If for all $1 \leq i, j, k \leq n$ the inequality

$$|A_i \cap B_j| + |B_j \cap C_k| + |C_k \cap A_i| \geq n$$

holds, prove that $|S| \geq \frac{n^3}{3}$, and determine when equality holds.

Solution. We have that

$$\sum_{j=1}^{n} |A_i \cap B_j| = |A_i|,$$

and similar equalities hold after cyclic permutation. Adding up the given condition for $1 \leq i, j, k \leq n$, we get that

$$\begin{aligned} n^4 &\leq \sum_{i,j,k=1}^{n} (|A_i \cap B_j| + |B_j \cap C_k| + |C_k \cap A_i|) \\ &= \sum_{i,k=1}^{n} (|A_i| + |C_k| + n|A_i \cap C_k|) \\ &= n|S| + n|S| + n|S| = 3n|S|, \end{aligned}$$

and consequently $|S| \geq \frac{n^3}{3}$.

We now claim that equality holds if and only if n is a multiple of 3. On the one hand, if equality is achieved, $\frac{n^3}{3}$ must be an integer number and hence n must be a multiple of 3.

Suppose now that n is a multiple of 3, and let S be a set of cardinal $\frac{n^3}{3}$. We partition it into n^2 subsets, each one with $n/3$ elements, that we index by A_{ij}, with $1 \leq i, j \leq n$.

For $1 \leq i \leq n$, we define

$$A_i = \bigcup_{j=1}^{n} A_{ij}, \qquad B_i = \bigcup_{j=1}^{n} A_{ji}, \qquad C_i = \bigcup_{j+k \equiv i \pmod{n}} A_{jk}.$$

Then, for all i, j, k we have that

$$|A_i \cap B_j| = |B_j \cap C_k| = |C_k \cap A_i| = \frac{n}{3},$$

and we are done.

Remark. *While the proof of the inequality seems quite natural, the equality case may look trickier. A natural guess is suggested by looking for subsets satisfying* $|A_i \cap B_j| = \frac{n}{3}$, *from where one may try to group the* $n^3/3$ *elements in* n^2 *groups of* $n/3$ *elements. Then, the construction of the* A_i, B_i *and* C_i *can be inspired from classical arguments coming from finite geometries, considering three families of parallel lines in the plane.*

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Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

jose.luis.diaz@upc.edu

Elementary Problems

E–47. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Show that among 2018 distinct positive integers there are two of them whose sum is at least 4035.

Solution 1 by Arminda Trujillo, Colegio Heidelberg, and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. Since the positive integers are distinct, if they are all consecutive from 1, the last two will be 2017 and 2018, whose sum is precisely 4035. In any other case, one of the first 2018 positive integer is not included in the set, and so another positive integer greater than 2018 has to be included. Therefore there are two of them whose sum is greater or equal than 4035.

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. Given a set $\{a_i\}$ of 2018 positive integers, we may, without loss of generality, relabel the integers if necessary and arrange them in increasing order of magnitude:

$$1 \leq a_1 < a_2 < a_3 < \cdots < a_{2018}.$$

It is obvious that $a_k \geq k$ for $k = 1, 2, \ldots, 2018$. In particular, $a_{2017} \geq 2017$ and $a_{2018} \geq 2018$, so that $a_{2017} + a_{2018} \geq 2017 + 2018 = 4035$.

Also solved by *Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain; Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom; José Gibergans Báguena, BarcelonaTech, Barcelona, Spain; Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain; Isaac Sánchez Barrera, Barcelona Supercomputing Center (BSC) and Universitat Politècnica de Catalunya (UPC), Barcelona, Spain, and the proposer.*

E–48. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Without the aid of a computer, show that

$$\cot 36^\circ \cdot \cot 72^\circ = \frac{\sqrt{5}}{5}.$$

Solution 1 by Arpon Basu, AECS-4, Mumbai, India. We have

$$\begin{aligned}\cot 36^\circ \cdot \cot 72^\circ &= \frac{1}{\tan 36^\circ \cdot \tan 72^\circ} = \frac{1}{\tan(2\cdot 18^\circ)\cdot\tan(90^\circ - 18^\circ)} \\ &= \frac{1}{\frac{2\tan 18^\circ}{(1-\tan^2 18^\circ)}\cot 18^\circ} = \frac{(1-\tan^2 18^\circ)}{2}.\end{aligned}$$

Let $A = 18^\circ$. Therefore

$$\begin{aligned}5A = 90^\circ &\implies 3A = 90^\circ - 2A \\ &\implies \sin(3A) = \sin(90^\circ - 2A) = \cos(2A).\end{aligned}$$

Thus,

$$\sin(3A) = \cos(2A) \Rightarrow 3\sin A - 4\sin^3 A = 1 - 2\sin^2 A,$$

or

$$\begin{aligned}&4\sin^3 A - 2\sin^2 A - 3\sin A + 1 = 0 \\ \iff &(\sin A - 1)(4\sin^2 A + 2\sin A - 1) = 0.\end{aligned}$$

The three solutions of the above equation are

$$\sin A = 1, \quad \sin A = \frac{\sqrt{5}-1}{4}, \quad \sin A = -\frac{\sqrt{5}+1}{4}.$$

As $A = 18^\circ$, the only valid solution is $\sin A = \frac{\sqrt{5}-1}{4}$. Then,

$$\tan^2 A = \frac{\sin^2 A}{\cos^2 A} = \frac{\sin^2 A}{1-\sin^2 A} = \frac{(\frac{\sqrt{5}-1}{4})^2}{(1-(\frac{\sqrt{5}-1}{4})^2)} = \frac{(3-\sqrt{5})}{(5+\sqrt{5})}.$$

Therefore,

$$(1-\tan^2 A) = (1-\tan^2 18^\circ) = (1 - \frac{(3-\sqrt{5})}{(5+\sqrt{5})}) = \frac{2}{\sqrt{5}}.$$

Finally, we have

$$\cot 36^\circ \cdot \cot 72^\circ = \frac{(1-\tan^2 18^\circ)}{2} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}.$$

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. We will prove the desired equality in the equivalent form

$$\frac{1-\tan^2 36^\circ}{2\tan^2 36^\circ} = \frac{\sqrt{5}}{5} \tag{1}$$

using the identity

$$\cot 72^\circ = \frac{1-\tan^2 36^\circ}{2\tan 36^\circ}.$$

Now we express $\tan^2 36^\circ$ in terms of $\cos 72^\circ$ by means of the half angle formula:

$$\tan^2 36^\circ = \frac{1-\cos 72^\circ}{1+\cos 72^\circ}.$$

When this is substituted into (1), we get

$$\frac{\cos 72^\circ}{1-\cos 72^\circ} = \frac{\sqrt{5}}{5},$$

which we write in the form

$$\cos 72^\circ = \frac{1}{1+\sqrt{5}}.$$

We accomplish this by considering an isosceles $\triangle ABC$ with $AB = AC = 1$ and $\angle CAB = 36^\circ$. Hence

$$\angle ABC = 2 \cdot \angle CAB.$$

This is equivalent to the equality

$$CA^2 = BC(BC + AB).$$

(One can also arrive at an equivalent expression by using the fact that the ratio of the radius of the circumcircle of a regular decagon to a side is the golden ratio).

Substituting 1 for CA and AB and solving for BC, we obtain

$$BC = \frac{2}{1+\sqrt{5}}.$$

But then

$$\cos 72^\circ = \frac{\frac{1}{2}BC}{AB} = \frac{1}{1+\sqrt{5}}, \tag{2}$$

which was to be proved.

We give another proof of (2) in a more routine manner. Applying the identity

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

with $\alpha \in \{18^\circ, 36^\circ\}$, we obtain

$$\sin 36^\circ = 2\sin 18^\circ \cos 18^\circ \quad \text{and} \quad \sin 72^\circ = 2\sin 36^\circ \cos 36^\circ.$$

Their product is

$$\sin 36^\circ \sin 72^\circ = 4\sin 18^\circ \cos 18^\circ \sin 36^\circ \cos 36^\circ,$$

and since $\sin 72^\circ = \cos 18^\circ$, we have

$$4\cos 36^\circ \sin 18^\circ = 1. \tag{3}$$

Now we recall a corollary of the addition formula for the sine, $\sin(x+y) - \sin(x-y) = 2\cos x \sin y$, and apply it with $x = 36^\circ$, $y = 18^\circ$, obtaining

$$\sin 54^\circ - \sin 18^\circ = 2\cos 36^\circ \sin 18^\circ.$$

When this is substituted into (3), we get

$$\sin 54^\circ - \sin 18^\circ = \frac{1}{2},$$

which is equivalent to

$$\cos 36^\circ - \sin 18^\circ = \frac{1}{2}, \tag{4}$$

and therefore

$$\begin{aligned}(\cos 36^\circ + \sin 18^\circ)^2 &= (\cos 36^\circ - \sin 18^\circ)^2 + 4\cos 36^\circ \sin 18^\circ \\ &= \left(\frac{1}{2}\right)^2 + 1 = \frac{5}{4}.\end{aligned}$$

This, in turn, is equivalent to

$$\cos 36^\circ + \sin 18^\circ = \frac{\sqrt{5}}{2}. \tag{5}$$

We add (4) and (5); then divide both sides by 2. The result is

$$\cos 36^\circ = \frac{1+\sqrt{5}}{4}.$$

Substituting this into

$$\cos 72^\circ = 2\cos^2 36^\circ - 1$$

and simplifying, we get

$$\cos 72^\circ = \frac{1}{1+\sqrt{5}}.$$

Solution 3 by Scott H. Brown, Auburn University Montgomery, AL, USA. We found the following exact values for $\sin 36^\circ$, $\cos 36^\circ$ and $\sin 72^\circ$, $\cos 72^\circ$ from [1]:

$$\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}, \qquad \cos 36^\circ = \frac{\sqrt{5}+1}{4}$$

and

$$\sin 72^\circ = \frac{\sqrt{10+2\sqrt{5}}}{4}, \qquad \cos 72^\circ = \frac{\sqrt{5}-1}{4}.$$

We can use these to find the exact values for $\cot 36^\circ$ and $\cot 72^\circ$ as follows:

$$\cot 36^\circ = \frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}, \qquad \cot 72^\circ = \frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}.$$

Multiplying the two preceding values and simplifying proves the equality claimed holds true.

[1] Alexander Bogomolny, Cosine of 36 degrees from *Interactive Mathematics Miscellany and Puzzles*. `https://www.cut-the-knot.org/pythagoras/cos36.shtml`.

Also solved by *Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain; José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain; Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona; Julio Cesar Mohnsam, Instituto Federal Sul Rio Grandense - IF SUL, Brazil; Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA, and the proposer.*

E–49. *Proposed by José Gibergans Báguena, BarcelonaTech, Barcelona, Spain.* Solve in $\mathbb{R}$ the following system of equations:

$$\left.\begin{aligned} x &= 5y\sqrt{1+z^2}, \\ y &= 5z\sqrt{1+x^2}, \\ z &= 5x\sqrt{1+y^2}. \end{aligned}\right\}$$

Solution 1 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. It is trivial to see that $x = y = z = 0$ is a solution, and that, if (x, y, z) is a solution, on the one hand, $(-x, -y, -z)$, and on the other, (y, z, x), (z, x, y), are also solutions. Assume there is some non-trivial solution. By the previous observation, we can assume WLOG that $x > 0$. So, we have

$$\begin{aligned} x = 5y\sqrt{1+z^2} &= 5 \cdot 5z\sqrt{1+x^2}\sqrt{1+z^2} \\ &= 5 \cdot 5 \cdot 5x\sqrt{1+y^2}\sqrt{1+x^2}\sqrt{1+z^2} \\ &= 125x\sqrt{1+y^2}\sqrt{1+x^2}\sqrt{1+z^2} > 125x, \end{aligned}$$

which leads to a contradiction since, as $x > 0$, $x > 125x$. Therefore the trivial solution $x = y = z = 0$ is the only solution.

Solution 2 by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. First, we observe that $x = y = z = 0$ is a solution. Furthermore, if $x \neq 0$, then $y \neq 0$ and $z \neq 0$. Multiplying up the preceding equations, we get

$$xyz = 125xyz\sqrt{(1+x^2)(1+y^2)(1+z^2)}.$$

Dividing both sides by $125xyz$ we obtain

$$\frac{1}{125} = \sqrt{(1+x^2)(1+y^2)(1+z^2)} > 1,$$

which is impossible. Thus, the only solution of the given system is $(0, 0, 0)$, and we are done.

Solution 3 by Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain.

$$\begin{cases} x &= 5y\sqrt{1+z^2}, \\ y &= 5z\sqrt{1+x^2}, \\ z &= 5x\sqrt{1+y^2}, \end{cases} \implies \begin{cases} x^2 &= 25y^2(1+z^2), \\ y^2 &= 25z^2(1+x^2), \\ z^2 &= 25x^2(1+y^2), \end{cases}$$

$$\implies x^2+y^2+z^2 = 25y^2\big(1+z^2\big) + 25z^2\big(1+x^2\big) + 25x^2\big(1+y^2\big)$$

$$\implies 24\big(x^2+y^2+z^2\big) + 25\big(x^2z^2+x^2y^2+y^2z^2\big) = 0.$$

As on the left hand side we have a sum of squares, it is always going to be greater than or equal to 0, where the equality can only be obtained if every square in the sum is equal to 0. Therefore,

$$x^2 = y^2 = z^2 = x^2z^2 = x^2y^2 = y^2z^2 = 0 \implies x = y = z = 0.$$

Therefore, the only possible solution is that the three unknowns are 0.

Also solved by *Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA, and the proposer.*

E–50. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let h_a, h_b, h_c be the altitudes of triangle ABC. Let P be a point inside $\triangle ABC$. Find the maximum value of

$$\frac{d_a \cdot d_b \cdot d_c}{h_a \cdot h_b \cdot h_c},$$

where d_a, d_b, d_c are the distances from P to the sides BC, CA and AB, respectively.

Solution 1 by David Suárez, IES Politécnico, and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. Since the point P is inside ΔABC, then $\frac{d_a}{h_a}$, $\frac{d_b}{h_b}$, $\frac{d_d}{h_d}$ are respectively equal to the quotient of the areas of the subtriangles with vertex P and each of the sides a, b and c divided by the area of triangle ΔABC. Since these quotients are the barycentric coordinates of point P, which add up to one, by the AM-GM inequality the product at hand will be maximum where $\frac{d_a}{h_a} = \frac{d_b}{h_b} = \frac{d_d}{h_d} = \frac{1}{3}$, so the maximum value is $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$.

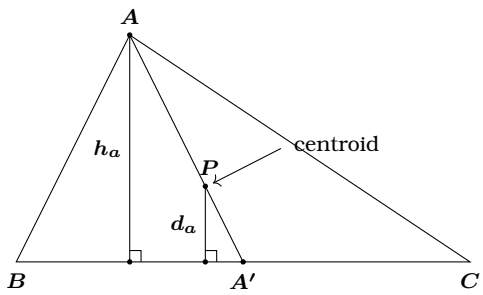

Figure 1: Sketch for Solution 2 of Problem E–50.

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. The answer is $\frac{1}{27}$. Indeed, on the one hand, we suppose that P is the centroid of $\triangle ABC$. Referring to Figure 1 and denoting the midpoint of side BC by A', we have

$$\frac{d_a}{h_a} = \frac{PA'}{AA'} = \frac{1}{3}.$$

Similarly,

$$\frac{d_b}{h_b} = \frac{d_c}{h_c} = \frac{1}{3}.$$

Hence

$$\frac{d_a \cdot d_b \cdot d_c}{h_a \cdot h_b \cdot h_c} = \frac{1}{27}.$$

On the other hand, let a, b, c be the sides of $\triangle ABC$. Since P is a point inside the triangle, the area of $\triangle ABC$ may be expressed as $\frac{1}{2}ad_a + \frac{1}{2}bd_b + \frac{1}{2}cd_c$, and also as $\frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$.

Equating these, we find that

$$ad_a + bd_b + cd_c = ah_a = bh_b = ch_c.$$

If we multiply the numerator and denominator of the given expression by abc, then apply the arithmetic-geometric inequality, we obtain

$$\frac{d_a \cdot d_b \cdot d_c}{h_a \cdot h_b \cdot h_c} = \frac{(ad_a)(bd_b)(cd_c)}{(ah_a)(bh_b)(ch_c)} \leq \frac{\left[\frac{1}{3}(ad_a + bd_b + cd_c)\right]^3}{(ah_a)(bh_b)(ch_c)}.$$

From the preceding we see that

$$(ad_a + bd_b + cd_c)^3 = (ah_a)(bh_b)(ch_c),$$

and therefore

$$\frac{d_a \cdot d_b \cdot d_c}{h_a \cdot h_b \cdot h_c} \leq \frac{1}{27},$$

and we are done.

Also solved by *José Gibergans Báguena, BarcelonaTech, Barcelona, Spain, and the proposer.*

E–51. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Without the aid of a computer, show that the coefficient a_n of the monomial $a_n\, x^n$ in the expression

$$\sum_{n\geq 0} a_n\, x^n = \frac{2x}{1 - 12x + 35x^2}, \quad \text{for } x \in \left(-\frac{1}{7}, \frac{1}{7}\right),$$

is a nonnegative integer and determine its value.

Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Using the method of partial fractions, the rational function

$$\frac{2x}{1 - 12x + 35x^2}$$

may be written in the form

$$\frac{1}{1 - 7x} - \frac{1}{1 - 5x}.$$

Since $x \in \left(\frac{-1}{7}, \frac{1}{7}\right)$, we have $|7x| < 1$ and $|5x| < 1$. Hence $\frac{1}{1-7x}$ and $\frac{1}{1-5x}$ are the sums of the infinite geometric progressions

$$1 + 7x + (7x)^2 + \cdots \qquad \text{and} \qquad 1 + 5x + (5x)^2 + \cdots,$$

respectively. Therefore,

$$\begin{aligned}\sum_{n\geq 0} a_n x^n &= \frac{1}{1 - 7x} - \frac{1}{1 - 5x} \\ &= \sum_{n\geq 0} (7x)^n - \sum_{n\geq 0} (5x)^n = \sum_{n\geq 0} (7^n - 5^n) x^n.\end{aligned}$$

From this it follows that

$$a_n = 7^n - 5^n,$$

a nonnegative integer.

Also solved by *Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain; José Gibergans Báguena, BarcelonaTech, Barcelona, Spain; Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain; Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA, and the proposer.*

E–52. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let O be the center of the circumcircle γ of triangle ABC. If it lies outside of $\triangle ABC$ and $\angle ABC = 36°$, then find the value of the angles of triangle AOC.

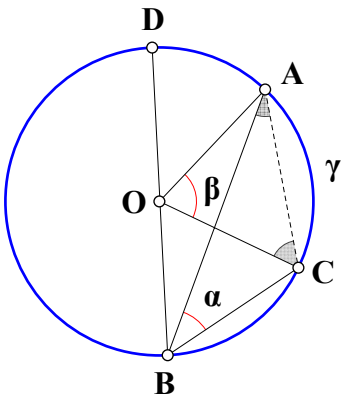

Figure 2: Sketch for Solution 1 of Problem E–52.

Solution 1 by the proposer. First, we draw the diameter BD. Let $\alpha_1 = \angle ABD$ and $\alpha_2 = \angle CBD$. Then, on the one hand we have $\alpha = \angle ABC = \alpha_2 - \alpha_1$ and, on the other hand, $\angle AOD = 2\alpha_1$ and $\angle COD = 2\alpha_2$. Thus,

$$\beta = \angle COA = \angle COD - \angle AOD = 2\alpha_2 - 2\alpha_1 = 2\alpha = 72°$$

and the angles of the isosceles triangle AOC measure $72°$, $54°$ and $54°$, respectively.

Solution 2 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona. If O lies outside the triangle, this means ABC is an obtuse triangle. As $\angle ABC$ is an acute angle, this means B lies in the longest arc of the two defined by A and C on the circumcircle. We can now apply the inscribed central theorem to get $\angle AOC = 2\angle ABC = 72°$. Thus, the angles of the isosceles triangle AOC measure $72°$, $54°$ and $54°$.

Also solved by *José Gibergans Báguena, BarcelonaTech, Barcelona, Spain.*

Easy–Medium Problems

EM–47. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* **[Correction]** Let $A_1, A_2, \ldots, A_n$ be the vertices of an n-gon inscribed in a circle of center O. If O and the A_i's are lattice points, then prove that the sum of the squares of the sides of the n-gon is an even number.

Solution by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona. For this problem we do not even have to care about the polygon being inscribed in a circle having a center which is a lattice point. Let $A_i = (x_i, y_i)$ with $x_i, y_i \in \mathbb{Z}$ for all $i \in \{1, \ldots, n\}$. The sum of the squares of the sides, considering $A_0 \equiv A_n$, $x_0 \equiv x_n$, $y_0 \equiv y_n$, will be

$$\begin{aligned}\sum_{i=1}^{n} \|A_i - A_{i-1}\|^2 &= \sum_{i=1}^{n} (x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 \\ &= \sum_{i=1}^{n} x_i^2 + x_{i-1}^2 + 2x_i x_{i-1} + y_i^2 + y_{i-1}^2 + 2y_i y_{i-1} \\ &= 2\sum_{i=1}^{n} (x_i^2 + y_i^2) + 2\sum_{i=1}^{n} (x_i x_{i-1} + y_i y_{i-1}),\end{aligned}$$

which is an integer multiple of 2 since all values x_i and y_i are integers, as we wanted to prove.

Also solved by *Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom; José Gibergans Báguena, BarcelonaTech, Barcelona, Spain, and the proposer.*

EM–48. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let ABC be an acute triangle with orthocenter H and incenter I. If A' is the midpoint of side BC, A'' is the midpoint of AH, M is the midpoint of AI and $A''M \perp A'M$, then calculate $\measuredangle BAC$.

Solution 1 by the proposer. Let B', C' be midpoints of AC and AB, respectively.

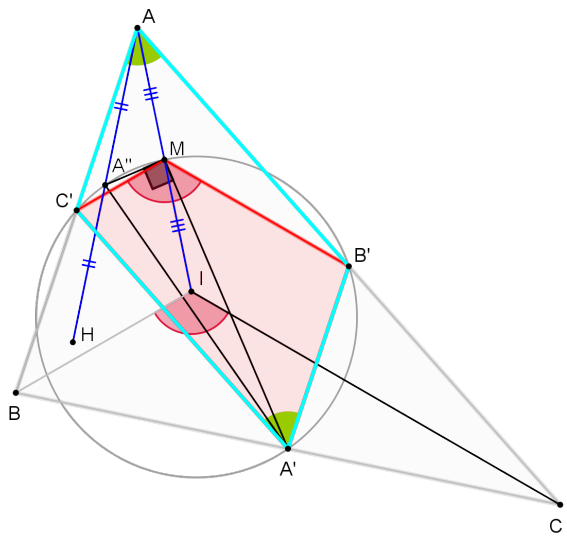

Figure 3: Scheme for Solution 1 of Problem EM–48.

The circle with diameter $A'A''$ is the nine-point circle for $\triangle ABC$. $A''M \perp A'M$, so $\triangle A'A''M$ is a right triangle and M is concyclic with A'' and A', B', C'.

In $\triangle AIC$,

$$\left.\begin{array}{l} AB' = B'C \\ AM = MI \end{array}\right\} \implies MB' = \text{ midsegment } \implies MB' \parallel IC.$$

In $\triangle ABI$,

$$\left.\begin{array}{l} AC' = C'B \\ AM = MI \end{array}\right\} \implies MC' = \text{ midsegment } \implies MC' \parallel IB.$$

As angles with parallel sides,

$$\measuredangle C'MB' \equiv \measuredangle BIC.$$

IB and IC are bisectors, so

$$\measuredangle C'MB' = \measuredangle BIC = 180^\circ - \frac{\measuredangle ABC}{2} - \frac{\measuredangle BCA}{2} = 90^\circ + \frac{\measuredangle BAC}{2}.$$

In $\triangle\, ABC$,

$$\left.\begin{array}{l} AC' = C'B \\ BA' = A'C \end{array}\right\} \implies A'C' = \text{midsegment} \implies C'A' \parallel AC,$$
$$\left.\begin{array}{l} AB' = B'C \\ BA' = A'C \end{array}\right\} \implies AC' = \text{midsegment} \implies A'B' \parallel AC'.$$

Therefore, $AC'A'B'A$ is a parallelogram and so $\measuredangle C'A'B' = \measuredangle BAC$. $A'B'MC'$ is a cyclic quadrilateral, so $\measuredangle C'A'B' + \measuredangle C'MB' = 180°$. We conclude that

$$\begin{aligned} \measuredangle BAC + 90° + \frac{\measuredangle BAC}{2} = 180° &\implies \frac{3}{2}\measuredangle BAC = 90° \\ &\implies \measuredangle BAC = 60°. \end{aligned}$$

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Let R, r, s represent respectively the circumradius, the inradius and the semiperimeter of $\triangle ABC$ with sides a, b, c opposite A, B, C. Since $\angle A''MA'$ is a right angle, the circle on $A'A''$ as diameter passes through M. This circle is the nine-point circle of $\triangle ABC$, of radius $\frac{1}{2}R$. Hence

$$A'A'' = R. \tag{1}$$

Now we recall the fact that a segment joining the midpoints of two sides of a triangle is half as long as the third side and apply it to $\triangle AHI$, obtaining

$$A''M = \frac{1}{2}HI. \tag{2}$$

Next we note that $A'M$ is a median of $\triangle AIA'$ (see Figure 4), so

$$4 \cdot A'M^2 = 2 \cdot AA'^2 + 2 \cdot IA'^2 - AI^2. \tag{3}$$

Since AA' is a median of $\triangle ABC$, we have

$$AA'^2 = \frac{1}{4}\left(2b^2 + 2c^2 - a^2\right).$$

Referring to Figure 5, let D and E denote the feet of the perpendiculars from I to BC and CA, respectively. By the Pythagorean

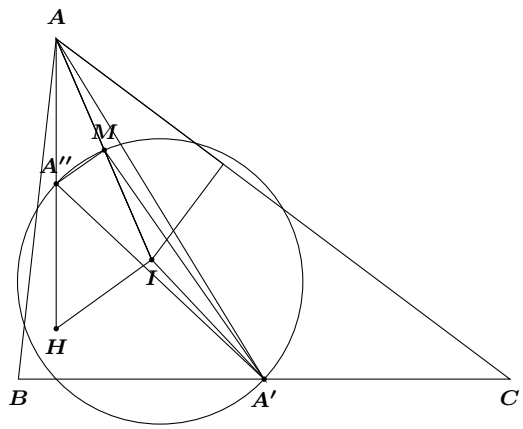

Figure 4: First sketch for Solution 2 of Problem EM–48.

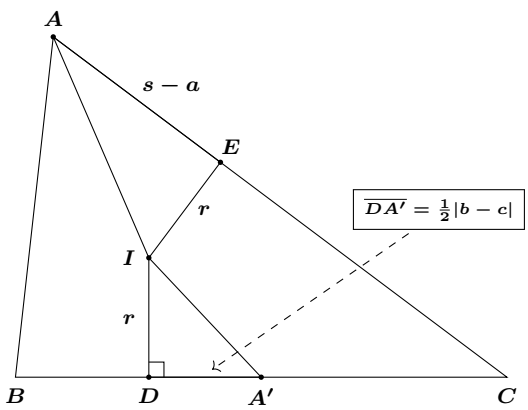

Figure 5: Second sketch for Solution 2 of Problem EM–48.

theorem, applied to $\triangle IDA'$, where $DA' = \frac{|b-c|}{2}$,

$$IA'^2 = r^2 + \frac{(b-c)^2}{4},$$

and applied to $\triangle IEA$,

$$AI^2 = r^2 + (s-a)^2.$$

When these are substituted into (3), it becomes

$$4 \cdot A'M^2 = r^2 + \frac{-3a^2 + 5b^2 + 5c^2 + 2ab - 6bc + 2ca}{4}. \qquad (4)$$

From right $\triangle A''MA'$, we obtain

$$A''M^2 + MA'^2 = A'A''^2,$$

and using (1), (2) and (4) yields

$$\frac{1}{4}HI^2 + \frac{1}{4}\left(r^2 + \frac{-3a^2 + 5b^2 + 5c^2 + 2ab - 6bc + 2ca}{4}\right) = R^2.$$

Then, using the formula

$$HI^2 = 4R^2 + 4Rr + 3r^2 - s^2$$

we have

$$Rr + r^2 + \frac{b^2 + c^2 - 2bc - a^2}{4} = 0,$$

which can be rewritten in the form

$$Rr + r^2 - (s-b)(s-c) = 0.$$

Multiply all terms by s, substitute $\dfrac{abc}{4}$ for Rrs and use Heron's formula to write $r^2 s = (s-a)(s-b)(s-c)$. This yields

$$bc = 4(s-b)(s-c)$$

or, equivalently,

$$\sqrt{\frac{(s-b)(s-c)}{bc}} = \frac{1}{2},$$

which, in turn, is equivalent to

$$\sin\left(\frac{A}{2}\right) = \frac{1}{2}.$$

This is satisfied when $\frac{A}{2} = 30^\circ$, i.e. $\angle BAC = 60^\circ$.

Also solved by *José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.*

EM–49. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Inside a square of side 1 there are several circumferences. If the sum of their perimeters is 100, prove that there is a line perpendicular to one side of the square intersecting 32 of them.

Solution by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom. Let n be the number of circumferences and enumerate them with an index i. Consider the projection of each circumference to the base of the square and let the endpoints of each projection be $0 \le a_i < b_i \le 1$. Note that $b_i - a_i$ is the diameter of the i-th circumference. Define a function $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \sum_{i=1}^{n} \mathbb{1}_{[a_i,b_i]}(x),$$

where $\mathbb{1}_{[a,b]}(x)$ is the characteristic function of the interval $[a,b]$.

Suppose that there is no line perpendicular to the base of the square that intersects 32 circumferences. Then $f(x) < 31$ for all $x \in [0,1]$, which means that $\int_0^1 f(x)\,\mathrm{d}x \le 31$. However, by the statement hypothesis we know that $\int_0^1 f(x)\,\mathrm{d}x = 100/\pi > 31$, thus reaching a contradiction.

Also solved by *Marc Felipe Alsina, BarcelonaTech, Barcelona, Spain; Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain, and the proposer.*

EM–50. *Proposed by Nicolae Papacu, Slobozia, Romania.* Find all real solutions of the system of equations

$$\left.\begin{aligned} [x]\,[y] &= x + y, \\ [x] + [y] &= [x\,y], \end{aligned}\right\}$$

where $[a]$ represents the integer part of the real number a.

Solution by the proposer. Let $x = [x] + \{x\} = n + \alpha$ with $n \in \mathbb{Z}$, $\alpha \in [0,1)$, and let $y = [y] + \{y\} = p + \beta$ with $p \in \mathbb{Z}$, $\beta \in [0,1)$.

Then the given system becomes

$$\begin{cases} np = n + p + \alpha + \beta, \\ n + p = [np + \alpha p + \beta p + \alpha\beta] = np + [\alpha p + \beta p + \alpha\beta]. \end{cases}$$

On account of the first equation, $\alpha + \beta \in \mathbb{Z}$. As $\alpha + \beta \in [0, 2)$, we consider the following cases:

1. Let $\alpha + \beta = 0$. Then, $\alpha = \beta = 0$ and the system becomes

$$\begin{cases} n \cdot p = n + p, \\ n + p = n \cdot p, \end{cases}$$

   or $(n-1)(p-1) = 1$. Then, $n - 1 = -1$, $p - 1 = -1$ or $n - 1 = 1$, $p - 1 = 1$, from which it follows that $n = p = 0$ or $n = p = 2$ and the solutions are $(x, y) = (0, 0)$ and $(x, y) = (2, 2)$.
2. Let $\alpha + \beta = 1$. Then the first equation becomes $np = n + p + 1$ or $(n-1)(p-1) = 2$ and we get that

$$(n, p) \in \{(0, -1), (-1, 0), (2, 3), (3, 2)\}.$$

   - If $(n, p) = (0, -1)$, then $x = \alpha$, $y = -1 + (1 - \alpha) = -\alpha$ and the second equation is also satisfied: $-1 = [-\alpha^2]$ when $\alpha \neq 0$. Therefore, we obtain the solution $(x, y) = (\alpha, -\alpha)$ for $\alpha \in (0, 1)$.
   - If $(n, p) = (-1, 0)$ then with a similar reasoning we get the solution $(x, y) = (-\beta, \beta)$ with $\beta \in (0, 1)$.
   - If $(n, p) = (2, 3)$, then from $n + p = np + [\alpha p + \beta p + \alpha\beta]$ we get $[3\alpha + 2\beta + \alpha\beta] = -1$, which is impossible because $3\alpha + 2\beta + \alpha\beta \geq 0$ and there is no solution.
   - If $(n, p) = (3, 2)$, then as in the previous case the system does not have solution.

Also solved by *Víctor Martín Chabrera, FME, Barcelona, Spain and José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.*

EM–51. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In the convex quadrilateral $ABCD$, choose points A, B, C and D in the interior of the sides AB, BC, CD and DA, respectively, such that

$$\frac{AA'}{A'B} = \frac{BB'}{B'C} = \frac{CC'}{C'D} = \frac{DD'}{D'A} = r.$$

Compute $\dfrac{[A'B'C'D']}{[ABCD]}$ and express it as function of r. Here, the expression $[XYZT]$ represents the area of quadrilateral $XYZT$.

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Clearly we have that

$$[A'B'C'D'] = [ABCD] - ([A'BB'] + [B'CC'] + [C'DD'] + [D'AA']). \tag{1}$$

Adding 1 to each side of $\frac{AA'}{A'B} = r$ gives $\frac{AB}{A'B} = r + 1$. Hence

$$A'B = \frac{1}{r+1}AB,$$

and since $AA' = AB - A'B$, we have

$$AA' = \frac{r}{r+1}AB.$$

Similarly,

$$\begin{aligned} BB' &= \tfrac{r}{r+1}BC, & B'C &= \tfrac{1}{r+1}BC, \\ CC' &= \tfrac{r}{r+1}CD, & C'D &= \tfrac{1}{r+1}BC, \\ DD' &= \tfrac{r}{r+1}DA, & D'A &= \tfrac{1}{r+1}DA. \end{aligned}$$

Writing $[A'BB'] = \frac{1}{2}A'B \cdot BB' \sin B$ and substituting $\frac{r}{(r+1)^2}AB \cdot BC$ for $A'B \cdot BB'$ from above, we get

$$[A'BB'] = \frac{r}{(r+1)^2}[ABC].$$

Similarly,

$$[B'CC'] = \frac{r}{(r+1)^2}[BCD],$$
$$[C'DD'] = \frac{r}{(r+1)^2}[CDA],$$
$$[D'AA'] = \frac{r}{(r+1)^2}[DAB].$$

We can therefore rewrite (1) in the form

$$\begin{aligned}&[A'B'C'D']\\ &= [ABCD] - \frac{r}{(r+1)^2}([ABC] + [BCD] + [CDA] + [DAB]). \quad (2)\end{aligned}$$

The area of $ABCD$ is $[ABC] + [CDA]$ and also $[BCD] + [DAB]$; hence

$$2[ABCD] = [ABC] + [BCD] + [CDA] + [DAB].$$

The equation (2) now reads

$$[A'B'C'D'] = [ABCD] - \frac{2r}{(r+1)^2}[ABCD],$$

so

$$\frac{[A'B'C'D']}{[ABCD]} = \frac{r^2+1}{(r+1)^2}.$$

Solution 2 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona. Let us try to write A' as a convex combination of A and B (that is, $A' = \lambda_1 A + \lambda_2 B$, with $\lambda_1,\ \lambda_2 \geq 0,\ \lambda_1 + \lambda_2 = 1$). Setting $\lambda_1 = \frac{1}{r+1} \geq 0$ and $\lambda_2 = \frac{r}{r+1} \geq 0,\ \lambda_1 + \lambda_2 = 1$ and

$$\begin{aligned}\frac{AA'}{A'B} = \frac{\|A' - A\|}{\|B - A'\|} &= \frac{\|\frac{1}{r+1}A + \frac{r}{r+1}B - A\|}{\|B - \frac{1}{r+1}A - \frac{r}{r+1}B\|}\\ &= \frac{\|-\frac{r}{r+1}A + \frac{r}{r+1}B\|}{\|-\frac{1}{r+1}A + \frac{1}{r+1}B\|} = r.\end{aligned}$$

So, we have found a convex combination for A':

$$A' = \frac{1}{r+1}A + \frac{r}{r+1}B.$$

Analogously with the rest of points, we obtain

$$\begin{aligned} B' &= \frac{1}{r+1}B + \frac{r}{r+1}C, \\ C' &= \frac{1}{r+1}C + \frac{r}{r+1}D, \\ D' &= \frac{1}{r+1}D + \frac{r}{r+1}A. \end{aligned}$$

We will use the well-known formula $[ABCD] = \frac{1}{2}\|\overrightarrow{AC} \times \overrightarrow{BD}\|$ and the basic properties of the cross product (bilinearity and anticommutativity) to get the following:

$$\begin{aligned} &[A'B'C'D'] \\ &= \frac{1}{2}\left\|\overrightarrow{A'C'} \times \overrightarrow{B'D'}\right\| = \frac{1}{2}\|(C'-A') \times (D'-B')\| \\ &= \frac{1}{2}\left\|\left(\frac{1}{r+1}C + \frac{r}{r+1}D - \frac{1}{r+1}A - \frac{r}{r+1}B\right)\right. \\ &\quad \left.\times \left(\frac{1}{r+1}D + \frac{r}{r+1}A - \frac{1}{r+1}B - \frac{r}{r+1}C\right)\right\| \\ &= \frac{1}{2}\left\|\left(\frac{1}{r+1}(C-A) + \frac{r}{r+1}(D-B)\right)\right. \\ &\quad \left.\times \left(\frac{1}{r+1}(D-B) + \frac{r}{r+1}(A-C)\right)\right\| \\ &= \frac{1}{2}\left\|\frac{1}{(r+1)^2}(C-A) \times (D-B) + \frac{r}{(r+1)^2}(C-A) \times (A-C)\right. \\ &\quad \left. + \frac{r}{(r+1)^2}(D-B) \times (D-B) + \frac{r^2}{(r+1)^2}(D-B) \times (A-C)\right\| \\ &= \frac{1}{2}\left\|\frac{1}{(r+1)^2}(C-A) \times (D-B) + \frac{r^2}{(r+1)^2}(D-B) \times (A-C)\right\| \\ &= \frac{1}{2}\left\|\frac{r^2+1}{(r+1)^2}(C-A) \times (D-B)\right\| \end{aligned}$$

$$= \frac{1}{2}\frac{r^2+1}{(r+1)^2}\|\overrightarrow{AC} \times \overrightarrow{BD}\| = \frac{r^2+1}{(r+1)^2}[ABCD].$$

So, $\dfrac{[A'B'C'D']}{[ABCD]} = \dfrac{r^2+1}{(r+1)^2}$.

Also solved by *the proposer.*

EM–52. *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.* Let ABC be an acute triangle with orthocenter H and circumcenter O. BO and CO intersect AH at P and Q, respectively. Prove that the areas of BCH, BCP and BCQ add up to the area of ABC.

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Let D be the foot of the altitude from A to BC. Since $\triangle ABC$ is acute-angled, D is an interior point of side BC and

$$BD + DC = BC. \tag{1}$$

Now triangles BCP, BCQ, BCH and ABC have the same base BC and their altitudes are PD, QD, HD and AD, respectively. Since the areas of triangles with equal bases are proportional to the altitudes of the triangles, it suffices to prove that

$$PD + QD + HD = AD.$$

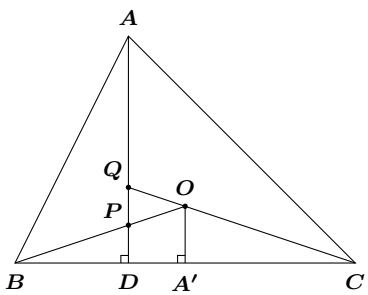

Figure 6: First sketch for Solution 1 of Problem EM–52.

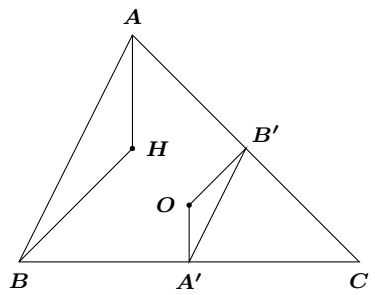

Figure 7: Second sketch for Solution 1 of Problem EM–52.

Let A' be the midpoint side BC. From similar right-angled triangles PBD and OBA' (Figure 6),

$$\frac{PD}{BD} = \frac{OA'}{BA'}$$

and

$$PD \cdot BA' = BD \cdot OA'.$$

Now, the distance along an altitude from a vertex to the orthocenter is twice the distance from the circumcenter to the opposite side of the triangle. This is not difficult to prove and we shall do so shortly. Then $OA' = \frac{1}{2}AH$, and since $BA' = \frac{1}{2}BC$, we have

$$PD \cdot BC = BD \cdot AH.$$

Similarly,

$$QD \cdot BC = DC \cdot AH.$$

Adding the last two expressions, we obtain

$$(PD + QD)BC = (BD + DC)AH.$$

We substitute BC for $BD + DC$ from (1), obtaining

$$PD + QD = AH.$$

Adding HD to each side gives

$$PD + QD + HD = AD,$$

as desired.

To prove that $OA' = \frac{1}{2}AH$ (Figure 7), let B' be the midpoint of side CA of $\triangle ABC$. First, $\triangle OA'B'$ has its sides parallel to those of the $\triangle ABC$, so the two triangles are similar. Next, $A'B' = \frac{1}{2}AB$, so the ratio between the two corresponding segments OA' and AH will be $1:2$. That is, $OA' = \frac{1}{2}AH$.

Solution 2 by the proposer. We shall show that triangles ABH and BCP have the same area, as well as ACH and BCQ. This obviously proves the statement, as $S_{ABC} = S_{BCH} + S_{ABH} + S_{ACH}$ (S_{XYZ} denotes the area of triangle XYZ). We start noting that, due to AOB being an isosceles triangle, $\angle ABP = \angle ABO = \frac{180^\circ - \angle AOB}{2} = \frac{180^\circ - 2\angle ACB}{2} = 90^\circ - \angle ACB = \angle CBH$. Equivalently, $\angle ABH = \angle CBP$. In addition, $\angle BAP = 90^\circ - \angle ABC = \angle BCH$.

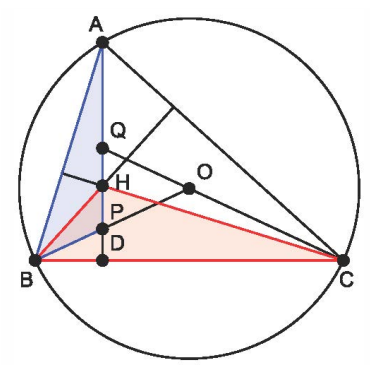

Figure 8: Sketch for Solution 1 of Problem EM–52.

The triangles ABP and CBH are similar because they have two equal angles. Therefore, $\frac{BA}{BP} = \frac{BC}{BH}$, or $BA \cdot BH = BC \cdot BP$. We conclude that $S_{ABH} = \frac{BA \cdot BH \sin \angle ABH}{2} = \frac{BC \cdot BP \sin \angle CBP}{2} = S_{BCP}$. Analogously, $S_{ACH} = S_{BCQ}$.

Medium–Hard Problems

MH–47. *Proposed by Ismael Morales López, Universidad Complutense de Madrid, Madrid, Spain.* There are 2018 students on a mathematics competition. We say the pair of students (A, B) is friendly if A knows B. This relation is naturally supposed to be symmetric. Find the greatest integer n such that at least one of the following conditions always holds:

(i) There exists a student that knows at least other n students.
(ii) There exists a set of $2n$ students that does not contain any friendly pair.

Solution by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom. We claim that the maximum n for which the statement holds is 32. Indeed, to see that it does not hold for $n \geq 33$, consider the following configuration of friends. Divide the set of students into groups of 33 students (and one group of 5) and suppose that all students within a group are friendly, and that no other pair of students is friendly. Then, as each student is friendly only with other members of their group, of which there are at most 32, there is no student who is friendly with n other students, so (i) does not hold. On the other hand, any set of students that does not contain any friendly pair can only have at most one student from each of the groups, hence the largest such set has size $\lceil \frac{2018}{33} \rceil = 62 < 2n$, so (ii) does not hold either.

Now we must prove that at least one of (i) or (ii) must hold for $n = 32$. If (i) holds we are done, so assume this is not the case. We must then show that (ii) holds. We argue by contradiction: assume that it does not hold, that is, the largest set of students such that no two of them are friendly has size at most 63 (say that this set is A). Partition the set of all students into this set (of size $|A| \leq 63$) and the set of all the other students (say B, $|B| \geq 2018 - 63 = 1955$). Note that every student in B must be friendly with some student in A, as otherwise they could be added to A and (i) would hold, a contradiction. But this means that, overall, there are at least 1955 friendly pairs such that one of their

members is in A. But as $|A| \leq 63$, by the pigeonhole principle this means that there is some student in A who is friendly with at least $\lceil \frac{1955}{63} \rceil = 32$ students in B, a contradiction. Therefore, at least one of (i) or (ii) must hold.

Also solved by *the proposer.*

MH–48. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Prove that

$$\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^{2k}C(n,k)}\right)^{n} > e^{(n+1)-5^n}.$$

Here, $C(n,k)$ represents the binomial coefficient $\binom{n}{k}$.

Solution by Henry Ricardo, Westchester Area Math Circle, NY, USA. Noting that $f(x) = \ln x$ is strictly concave ($f''(x) = -1/x^2 < 0$ for $x > 0$), we take the logarithm of the LHS of the proposed inequality and apply Jensen's inequality to see that

$$\begin{aligned} n\ln\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^{2k}C(n,k)}\right) &> n\cdot\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{1}{2^{2k}C(n,k)}\right) \\ &= -\sum_{k=1}^{n}\ln\big(2^{2k}C(n,k)\big). \end{aligned} \tag{1}$$

Using the known inequality $\ln x \leq x - 1$ for $x > 0$ in the form $-\ln x \geq 1 - x$, we have that

$$\begin{aligned} -\sum_{k=1}^{n}\ln\big(2^{2k}C(n,k)\big) &\geq \sum_{k=1}^{n}\big(1 - 2^{2k}C(n,k)\big) \\ &= n - \sum_{k=1}^{n}2^{2k}C(n,k) \\ &= n - (5^n - 1) = (n+1) - 5^n. \end{aligned} \tag{2}$$

Putting (1) and (2) together, we see that

$$\ln\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^{2k}C(n,k)}\right)^{n} > (n+1) - 5^n,$$

and exponentiation yields the desired inequality.

Also solved by *Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain, and the proposer.*

MH–49. *Proposed by Nicolae Papacu, Slobozia, Romania.* Let $x_1, x_2, \ldots, x_n$ be $n \geq 2$ positive numbers other than one such that $x_1^2 + x_2^2 + \ldots + x_n^2 = n^3$. Prove that

$$\frac{\log_{x_1}^4 x_2}{x_1 + x_2} + \frac{\log_{x_2}^4 x_3}{x_2 + x_3} + \ldots + \frac{\log_{x_n}^4 x_1}{x_n + x_1} \geq \frac{1}{2}.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY, USA. In our proof, we use the following well-known or easily derived results:

1. $\log_a b \cdot \log_b c = \log_a c$;
2. $(x_1 + x_2 + \cdots + x_n)^2 \leq n(x_1^2 + x_2^2 + \cdots + x_n^2) = n^4$, or $1/(x_1 + x_2 + \cdots + x_n) \geq 1/n^2$ [Cauchy-Schwarz].

The Engels form of the Cauchy-Schwarz inequality gives us

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{\log_{x_1}^4 x_2}{x_1 + x_2} &\geq \frac{\left(\sum_{\text{cyclic}} \log_{x_1}^2 x_2\right)^2}{\sum_{\text{cyclic}} (x_1 + x_2)} \\
&= \frac{\left(\sum_{\text{cyclic}} \log_{x_1}^2 x_2\right)^2}{2\sum_{\text{cyclic}} x_1} \\
&\overset{\text{AM-GM}}{\geq} \frac{\left[n\left(\prod_{\text{cyclic}} \log_{x_1} x_2\right)^{2/n}\right]^2}{2\sum_{\text{cyclic}} x_1} \\
&\overset{1,\,2}{\geq} \frac{n^2}{2n^2} = \frac{1}{2}.
\end{aligned}$$

Solution 2 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. We have

$$\frac{\log_{x_1}^4 x_2}{x_1 + x_2} + \frac{\log_{x_2}^4 x_3}{x_2 + x_3} + \ldots + \frac{\log_{x_n}^4 x_1}{x_n + x_1}$$

$$
\begin{aligned}
&\geq \frac{\log_{x_1}^4 x_2}{\sqrt{2}\sqrt{x_1^2+x_2^2}} + \frac{\log_{x_2}^4 x_3}{\sqrt{2}\sqrt{x_1^2+x_2^2}} + \ldots + \frac{\log_{x_n}^4 x_1}{\sqrt{2}\sqrt{x_n^2+x_1^2}} \\
&\geq \frac{n}{\sqrt{2}} \sqrt[n]{\frac{\log_{x_1}^4 x_2}{\sqrt{x_1^2+x_2^2}} \frac{\log_{x_2}^4 x_3}{\sqrt{x_2^2+x_3^2}} \cdots \frac{\log_{x_n}^4 x_1}{\sqrt{x_n^2+x_1^2}}} \\
&= \frac{n}{\sqrt{2}} \sqrt[n]{\frac{\left(\frac{\log x_2}{\log x_1}\right)^4}{\sqrt{x_1^2+x_2^2}} \frac{\left(\frac{\log x_3}{\log x_2}\right)^4}{\sqrt{x_2^2+x_3^2}} \cdots \frac{\left(\frac{\log x_1}{\log x_n}\right)^4}{\sqrt{x_n^2+x_1^2}}} \\
&= \frac{n}{\sqrt{2}} \frac{1}{\sqrt[n]{\sqrt{x_1^2+x_2^2}\sqrt{x_2^2+x_3^2}\cdots\sqrt{x_n^2+x_1^2}}} \\
&\geq \frac{n}{\sqrt{2}} \frac{1}{\sqrt{\frac{x_1^2+x_2^2+x_2^2+x_3^2+\ldots x_n^2+x_1^2}{n}}} = \frac{n}{\sqrt{2}} \frac{1}{\sqrt{\frac{2(x_1^2+x_2^2+\ldots x_n^2)}{n}}} \\
&= \frac{n}{\sqrt{2}} \frac{1}{\sqrt{\frac{2n^3}{n}}} = \frac{n}{\sqrt{2}} \frac{1}{\sqrt{2n^2}} = \frac{1}{2},
\end{aligned}
$$

where the inequalities we used are the AM-QM inequality (if $a_1, \ldots, a_n$ are positive, $\frac{a_1+\ldots+a_n}{n} \leq \sqrt{\frac{a_1^2+\ldots+a_n^2}{2}}$) on the denominators, the AM-GM inequality (if $a_1, \ldots, a_n$ are positive, $\frac{a_1+\ldots+a_n}{n} \geq \sqrt[n]{a_1^2 \cdots a_n^2}$), and the GM-QM inequality (if $a_1, \ldots, a_n$ are positive, $\sqrt{\frac{a_1^2+\ldots+a_n^2}{n}} \geq \sqrt[n]{a_1 \cdots a_n}$), respectively.

Also solved by *the proposer.*

MH–50. *Proposed by Óscar Rivero Salgado, BarcelonaTech, Barcelona, Spain.* Let a be an integer and $p \geq 3$ be a prime number. Prove that

$$a^p + (a+1)^p + \ldots + (a+p-1)^p$$

is a multiple of p^2.

Solution by the proposer. In the set of p numbers $a, a+1, \ldots, a+p-1$ there is only one that it is multiple of p and its p-th power is a multiple of p^2. The remaining can be grouped in pairs, say $\{i, j\}$, such that $i + j \equiv 0 \pmod{p}$ because among p consecutive integers there is exactly one of each class modulo p. Then

$i + j = kp$ or $i = -j + kp$. Then, $i^p = (-j + kp)^p = -j^p + hp^2$ and $i^p + j^p = hp^2$. Finally, adding up these expressions we get that the total sum is multiple of p^2.

Solution by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona. Before we begin, let us make a couple of remarks for the sake of the proof. First, it is easy to check that $\frac{(p-1)!}{r!(p-r)!} = \frac{1}{p}\frac{p!}{r!(p-r)!} = \frac{1}{p}\binom{p}{r}$ is an integer, since $\binom{p}{r}$ is a multiple of p if $r \neq 0, p$. Second, if $b = ak$, and $b \not\equiv 0 \pmod{p}$, $\frac{b}{a} \equiv k \equiv ba^{-1} \pmod{p}$, where a^{-1} is the inverse of a modulo p. Therefore, there is no difference between working with quotients and working with inverses.

Lemma 1. *Let $p \geq 3$ be a prime number and k be an integer. Then, $p | \sum_{r=1}^{p-1} \frac{(p-1)!}{(p-r)!r!}(p-k)^r k^{p-r}$.*

Proof. Let us work modulo p. By Wilson's theorem $(p-1)! \equiv -1 \pmod{p}$, and so

$$\sum_{r=1}^{p-1} \frac{(p-1)!}{(p-r)!r!}(p-k)^r k^{p-r} \equiv \sum_{r=1}^{p-1} \frac{-1}{(p-r)!r!}(p-k)^r k^{p-r}$$

If a is a number that has an inverse b modulo p (i.e. $ab \equiv 1 \pmod{p}$), the inverse of $-a$ modulo p will be $-b$, since $(-a)(-b) \equiv ab \equiv 1 \pmod{p}$. Using this fact alongside with $p - k \equiv -k \pmod{p}$,

$$\begin{aligned}
&\sum_{r=1}^{p-1} \frac{-1}{r!(p-r)!}(p-k)^r k^{p-r} \equiv \sum_{r=1}^{p-1} \frac{-1}{1\cdots r \cdot 1 \cdots (p-r)}(-k)^r k^{p-r} \\
&\equiv \sum_{r=1}^{p-1} \frac{-1}{(-1)^{p-r} 1 \cdots r \cdot (p-1)(p-2)\cdots r}(-1)^r k^r k^p k^{-r} \\
&\equiv \sum_{r=1}^{p-1} \frac{1}{(-1)^{p-r} r(p-1)!}(-1)^{r+1} k^p \equiv \sum_{r=1}^{p-1} \frac{(-1)^{p+1}}{r(p-1)!} k^p \\
&\equiv \sum_{r=1}^{p-1} \frac{(-1)^{p+1}}{-r} k^p \equiv \sum_{r=1}^{p-1} \frac{(-1)^p}{r} k^p \pmod{p}.
\end{aligned}$$

As there is a one-to-one correspondence between $\{1, \ldots, p-1\}$

and their inverses modulo p, which also stay in $\{1, \ldots, p-1\}$,

$$\sum_{r=1}^{p-1} \frac{(-1)^p}{r} k^p \equiv (-1)^p k^p \sum_{r=1}^{p-1} \frac{1}{r} \equiv (-k)^p \sum_{a=1}^{p-1} a$$
$$\equiv (-k)^p \frac{(p-1)p}{2} \equiv 0 \pmod{p}$$

since it is a multiple of p and 2 is invertible modulo a prime $p \geq 3$. □

Lemma 2. *$p^2 | (p-k)^p + k^p$ if $p \geq 3$ is a prime number and k is an integer.*

Proof. Let us expand $p^p = ((p-k)+k)^p$

$$p^p = ((p-k)+k)^p = \sum_{r=0}^{p} \binom{p}{r} (p-k)^r k^{p-r}$$
$$= (p-k)^p + k^p + \sum_{r=1}^{p-1} \frac{p!}{(p-r)!r!} (p-k)^r k^{p-r}$$
$$= (p-k)^p + k^p + p\left(\sum_{r=1}^{p-1} \frac{(p-1)!}{(p-r)!r!} (p-k)^r k^{p-r}\right).$$

By Lemma 1, $\sum_{r=1}^{p-1} \frac{(p-1)!}{(p-r)!r!} (p-k)^r k^{p-r}$ is a multiple of p. Therefore, as $p^2 | p^p$, reducing modulo p^2 we get $0 \equiv (p-k)^p + k^p \pmod{p^2}$, completing the proof. □

Lemma 3. *$p^2 | 1^p + 2^p + \ldots (p-1)^p$ if $p \geq 3$ is a prime number.*

Proof. Reordering the terms, $1^p + 2^p + \ldots + (p-1)^p = \sum_{k=1}^{p-1} k^p = \sum_{k=1}^{\frac{p-1}{2}} (k^p + (p-k)^p)$, since p must be odd. By Lemma 2 each of this summands is a multiple of p^2, and so, the total sum $1^p + 2^p + \ldots (p-1)^p$ is also a multiple of p^2. □

Lemma 4. *$(a+p)^p - a^p$ is a multiple of p^2 for every prime p and $a \in \mathbb{Z}$.*

Proof. We have that

$$\begin{aligned}(a+p)^p - a^p &= -a^p + \sum_{k=0}^{p} \binom{p}{k} a^{p-k} p^k \\ &= -a^p + a^p + p^p + \sum_{k=1}^{p-1} \binom{p}{k} a^{p-k} p^k \\ &= p^p + p \sum_{k=1}^{p-1} \frac{(p-1)!}{(p-k)!k!} a^{p-k} p^k \\ &= p^p + p^2 \sum_{k=1}^{p-1} \frac{(p-1)!}{(p-k)!k!} a^{p-k} p^{k-1},\end{aligned}$$

which is a multiple of p^2. □

Now we are ready for the final step. Let us proceed by induction on a. If $a = 0$, by Lemma 3, $0^p + 1^p + \ldots + (p-1)^p \equiv 1^p + \ldots + (p-1)^p \equiv 0 \pmod{p^2}$.

If $a^p + (a+1)^p + \ldots + (a+p-1)^p \equiv 0 \pmod{p^2}$, applying Lemma 4 we have that

$$\begin{aligned}&(a+1)^p + \ldots + (a+p-1)^p + (a+p)^p \\ \equiv\, &(a^p + (a+1)^p + \ldots + (a+p-1)^p) + ((a+p)^p - a^p) \\ \equiv\, &0 \pmod{p^2}\end{aligned}$$

and

$$\begin{aligned}&(a-1)^p + a^p + \ldots + (a+p-2)^p \\ \equiv\, &(a^p + (a+1)^p + \ldots + (a+p-1)^p) - (((a-1)+p)^p - (a-1)^p) \\ \equiv\, &0 \pmod{p^2}.\end{aligned}$$

Therefore, $p^2 | a^p + (a+1)^p + \ldots (a+p-1)^p$ for every $a \in \mathbb{Z}$ and $p \geq 3$ prime, as we wanted to prove.

MH–51. *Proposed by Ismael Morales López, Universidad Complutense de Madrid, Madrid, Spain.* Find all strictly increasing sequences $\{a_n\}_{n\geq 1}$ of positive integers such that

(i) $a_{2018n} - a_n \leq 2017n$.
(ii) If a_k is the sum of two squares then k is, too.

Solution by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. As $\{a_n\}_{n\geq 1} \subset \mathbb{N}_{\geq 1}$ is strictly increasing, that means that $a_{n+k} - a_n \geq k$, for all $k \geq 1$. This is really easy to check. By induction, for $k = 1$ this is true, and if this holds for a some k, it will also hold for $k+1$: $a_{n+k+1} - a_n = (a_{n+k-1} - a_{n+k}) + (a_{n+k} - a_n) \geq 1+k$. Now, fixing a positive integer n and setting $k = 2017n$, $2017n \leq a_{n+k} - a_n = a_{2018n} - a_n \leq 2017n$, which clearly means that, for every $k \geq 1$, $a_{n+1} = a_n + 1, \forall n \in \{k, \ldots, 2018k - 1\}$. Now it is trivial to see that $a_{m+1} = a_m + 1$ for all $m \geq 1$. If this is false for some m, the numbers $a_m, \ldots, a_{2018m}$ are consecutive as we have just seen, which is a contradiction since $m < m+1 < 2018m$.

So, all the numbers of the sequence are consecutive and, therefore, $a_n = n + k$ for some integer $k \geq 0$. It is well-known that a natural n is the sum of two squares if and only if its prime decomposition does not contain a prime p raised to an odd power such that $p \equiv 3 \pmod 4$. We will prove that such a sequence $a_n = n + k$ satisfies the second condition of the statement if and only if $k = 0$. Clearly for $k = 0$ it is true. If $k > 0$ let us distinguish between two cases. If k is itself a sum of squares (its prime decomposition does not contain a prime p raised to an odd power such that $p \equiv 3 \pmod 4$), $4k$ will also be a sum of two squares for the same reason, but $3k$ will not, since 3 is a prime, $3 \equiv 3 \pmod 4$, and, if the prime 3 did not appear in the prime decomposition of k or it appeared but raised to an even power, in the prime decomposition of $3k$ it will appear raised to an odd power. So, $a_{3k} = k + 3k = 4k$ which violates condition .

Assume now that $k > 0$ is not a sum of two squares. In this case $k > 1$ since 1 is the sum of two squares. Then, there exists some integer $b > 1$ such that bk is the sum of two squares (for example, setting $b = k$). Thus, let $c > 1$ be the minimum integer greater than 1 such that ck is the sum of two squares. Then,

$a_{(c-1)k} = (c-1)k + k = ck$, which also violates condition as $(c-1)k$ is not the sum of two squares by the minimality of c and ck is.

In conclusion, the only strictly increasing sequence $\{a_n\}_{n\geq 1}$ satisfying the demanded requirements is the trivial sequence $a_n = n$.

Also solved by *the proposer.*

MH–52. *Proposed by Ángel Plaza and Sergio Falcón, Universidad Las Palmas de Gran Canaria, Las Palmas de Gran Canaria, Spain.* The Fibonacci numbers are defined recursively by $F_n = F_{n-1}+F_{n-2}$ with initial values $F_0 = 0$, $F_1 = 1$. Prove that

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1}^2 5^{2i} \geq \frac{2^{2n-2}F_n^2}{1+\lfloor \frac{n-1}{2} \rfloor},$$

where $\lfloor \cdot \rfloor$ denotes the integer part.

Solution 1 by Henry Ricardo, Westchester Area Math Circle, NY, USA. In the proof that follows, we use the formula (Catalan, 1857)

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1} 5^i = 2^{n-1}F_n.$$

(This is easily proved by expanding φ^n and ϕ^n in Binet's formula $F_n = (\varphi^n - \phi^n)/\sqrt{5}$ via the binomial theorem.)

We apply the Cauchy-Schwarz inequality to the pairs $(x_i, y_i) = \left(\binom{n}{2i+1}5^i, 1\right)$ to get

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\binom{n}{2i+1}5^i\right)^2 \cdot \sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} 1^2 \geq \left(\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1}5^i\right)^2$$
$$= (2^{n-1}F_n)^2 = 2^{2n-2}F_n^2,$$

or

$$\sum_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2i+1}^2 5^{2i} \geq \frac{2^{2n-2}F_n^2}{1+\lfloor \frac{n-1}{2} \rfloor}.$$

Solution 2 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. From the QM-AM inequality,

$$\sqrt{\frac{x_1^2+\ldots+x_n^2}{n}} \geq \frac{x_1+\ldots+x_n}{n}.$$

We can get the equivalent form

$$x_1^2+\ldots+x_n^2 \geq \frac{(x_1+\ldots+x_n)^2}{n}.$$

Therefore,

$$\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2i+1}^2 5^{2i} \geq \frac{\left(\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2i+1}5^i\right)^2}{1+\left\lfloor\frac{n-1}{2}\right\rfloor}.$$

We can see that the sum $\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2i+1}5^i$ can be rewritten as

$$\begin{aligned}\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2i+1}5^i &= \sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2i+1}\sqrt{5}^{2i}\\ &= \frac{1}{\sqrt{5}}\sum_{i=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n}{2i+1}\sqrt{5}^{2i+1}\\ &= \frac{1}{\sqrt{5}}\sum_{\substack{i=0\\ k\text{ odd}}}^{n}\binom{n}{k}\sqrt{5}^{k}.\end{aligned}$$

As, for a polynomial $P(x)$, the expression $\frac{P(x)-P(-x)}{2}$ gives the same polynomial but removing the even part, for the polynomial $P(x) = (1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$, we have that $\frac{P(x)-P(-x)}{2} = \frac{(1+x)^n-(1-x)^n}{2} = \sum_{\substack{k=0\\ k\text{ odd}}}^{n}\binom{n}{k}x^k$. Plugging $x=\sqrt{5}$, we will get

$$\begin{aligned}\sum_{\substack{i=0\\ k\text{ odd}}}^{n}\binom{n}{k}\sqrt{5}^{k} &= \frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2}\\ &= 2^n\frac{\sqrt{5}}{2}\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} = 2^{n-1}\sqrt{5}F_n.\end{aligned}$$

Here, we have used the closed form of the Fibonacci sequence, $F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$. So,

$$\sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2i+1} 5^i = \frac{1}{\sqrt{5}} \sum_{\substack{i=0 \\ k\text{odd}}}^{n} \binom{n}{k} \sqrt{5}^k = 2^{n-1} F_n.$$

Finally, we have that

$$\sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2i+1}^2 5^{2i} \geq \frac{\left(\sum_{i=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \binom{n}{2i+1} 5^i\right)^2}{1 + \left\lfloor \frac{n-1}{2} \right\rfloor} = \frac{\left(2^{n-1} F_n\right)^2}{1 + \left\lfloor \frac{n-1}{2} \right\rfloor} = \frac{2^{2n-2} F_n^2}{1 + \left\lfloor \frac{n-1}{2} \right\rfloor},$$

as we wanted to prove.

Also solved by *the proposers.*

Advanced Problems

A–47. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let A and B be 2×2 matrices with integer entries such that $AB = BA$, $\det(A+B) = 2$, and $\det(A^3 + B^3) = 2^3$. Compute $\det(A^2 + B^2)$.

Solution by the proposer. Let us denote $\det A = a$ and $\det B = b$. Let $f\colon \mathbb{C} \to \mathbb{C}$ be defined by

$$f(x) = \det(A + xB) = \det A + x\alpha + x^2 \det B = a + \alpha x + bx^2.$$

Note that

$$f(1) = \det(A+B) = 2 \implies a + \alpha + b = 2. \tag{1}$$

Since matrices A and B commute, we have that

$$A^3 + B^3 = (A+B)(A+\varepsilon B)\big(A + \varepsilon^2 B\big),$$

where $\varepsilon^2 + \varepsilon + 1 = 0$. Therefore,

$$\det\big(A^3 + B^3\big) = 2^3 \implies f(1)f(\varepsilon)f\big(\varepsilon^2\big) = 8.$$

As $f(1) = 2$,

$$f(\varepsilon)f\big(\varepsilon^2\big) = \frac{8}{2} = 4,$$

that is,

$$\big(a + \alpha\varepsilon + b\varepsilon^2\big)\big(a + \alpha\varepsilon^2 + b\varepsilon\big) = 4.$$

After some calculations, we obtain

$$a^2 + b^2 + \alpha^2 - ab - a\alpha - b\alpha = 4. \tag{2}$$

From (1) we get $\alpha = 2 - a - b$. Replacing it in (2) yields

$$a^2 + b^2 + (2-a-b)^2 - ab - (2-a-b)(a+b) = 4.$$

After collecting similar terms, we have that

$$\begin{aligned} 3a^2 + 3b^2 - 6a - 6b + 3ab &= 0, \\ a^2 + b^2 - 2a - 2b + ab &= 0, \\ a^2 + a(b-2) + b^2 - 2b &= 0. \end{aligned}$$

Since $a \in \mathbb{R}$, we must have that

$$\begin{aligned} & (b-2)^2 - 4b(b-2) \geq 0 \\ \implies & (b-2)(b-2-4b) \geq 0 \\ \implies & (b-2)(-3b-2) \geq 0 \implies b \in \left[-\frac{2}{3}, 2\right]. \end{aligned}$$

As $b \in \mathbb{Z}$, we must have $b \in \{0, 1, 2\}$. Now study these three cases.

- $b = 0 \implies a^2 - 2a = 0 \implies a = 0$ or $a = 2$. In the first case,
$$b = 0,\ a = 0 \Rightarrow \alpha = 2.$$
In the second,
$$b = 0,\ a = 2 \Rightarrow \alpha = 0.$$
- $b = 1 \implies a^2 - a - 1 = 0 \implies a \notin \mathbb{Z}$. This is an inadmissible solution.
- $b = 2 \implies a^2 = 0$, and
$$b = 2, a = 0 \implies \alpha = 0.$$
So we obtain three functions,
$$\begin{aligned} f_1 &: \mathbb{C} \to \mathbb{C}, \quad f_1(x) = 2x, \\ f_2 &: \mathbb{C} \to \mathbb{C}, \quad f_2(x) = 2x^2, \\ f_3 &: \mathbb{C} \to \mathbb{C}, \quad f_3(x) = 2. \end{aligned}$$

Note that

$$\det\left(A^2 + B^2\right) = \det(A + iB)\det(A - iB) = f(i)f(-i) = 4.$$

Since for each one of the three preceding functions $f(i)f(-i) = 4$, we conclude that $\det(A^2 + B^2) = 4$.

Also solved by *José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.*

A–48. *Proposed by José Luis Díaz-Barrero, Barcelonatech, Barcelona, Spain.* Without using the series expansion of the hyperbolic functions, show that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sinh^2 x$ is not a polynomial.

Solution by Marc Felipe i Alsina, BarcelonaTech, Barcelona, Spain. First, we claim that the only non-constant polynomials $A(x)$ such that $A(2x)$ can be written as

$$A(2x) = B(A(x)),$$

for a polynomial $B(x)$, are of the form $A(x) = a_n x^n + a_0$, for a_0, $a_n \in \mathbb{R}$ and $B(x) = 2^n x + (1-2^n)a_0$. Indeed, if $A(2x) = B(A(x))$, then

$$\deg(A(x)) = \deg(A(2x)) = \deg(B(x)) \cdot \deg(A(x)).$$

Since the degree of a nonzero polynomial is a nonnegative integer, then we have that either $A(x)$ is a constant k and $B(k) = k$ or else $B(x)$ is linear. Namely, $B(x) = ax + b$, for$a$, $b \in \mathbb{R}$.

Assume that $A(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n$, $a_n \neq 0$. Then, we have that

$$A(2x) = a_0 + 2a_1 x + \ldots + 2^{n-1}a_{n-1}x^{n-1} + 2^n a_n x^n$$

and

$$B(A(x)) = (aa_0 + b) + aa_1 x + \ldots + aa_{n-1}x^{n-1} + aa_n x^n.$$

Equating the preceding polynomials yields

$$\left.\begin{array}{rcl} 2^n a_n & = & aa_n, \\ 2^{n-1}a_{n-1} & = & aa_{n-1}, \\ & \vdots & \\ 2a_1 & = & aa_1, \\ a_0 & = & aa_0 + b, \end{array}\right\}$$

from which it follows that $a = 2^n$, $a_k = 0$ for $1 \leq k \leq n-1$ and $b = (1-2^n)a_0$. So, $A(x) = a_n x^n + a_0$ and $B(x) = 2^n x + (1-2^n)a_0$, a_0, $a_n \in \mathbb{R}$, as claimed.

To prove that $f(x) = \sinh^2 x$ is not a polynomial we argue by contradiction. Suppose that f is a polynomial. Then, $f(2x) = \sinh^2 2x = 4\sinh^2 x(1+\sinh^2 x) = B(\sinh^2 x) = B(f(x))$, where $B(x) = 4x(1+x)$. But $B(x)$ is a quadratic polynomial; this, jointly with $f(2x) = B(f(x))$, leads us to a contradiction on account of the previous claim. Hence, $f(x) = \sinh^2 x$ is not a polynomial and we are done.

Also solved by *the proposer.*

A–49. *Proposed by Nicolae Papacu, Slobozia, Romania.* Let A, $B \in M_2(\mathbb{Q})$ be matrices such that

$$\det(A^2 - pI_2) = \det(B^2 - qI_2) = \det((AB)^2 - pqI_2) = 0,$$

where p, q are prime numbers.

a) If $p \neq q$, then prove that $AB = BA$.
b) If $p = q$, then prove that $(AB)^2 + (BA)^2 = 2p^2 I_2$.

Solution by the proposer. a) Let

$$P(X) = \det(A - XI_2) = X^2 - \operatorname{tr}(A)X + \det A \in \mathbb{Q}[X].$$

Since

$$\det(A^2 - pI_2) = \det(A - \sqrt{p}I_2) \cdot \det(A + \sqrt{p}I_2) = 0,$$

it follows that $P(\sqrt{p}) = 0$ or $P(-\sqrt{p}) = 0$. On account that $P(X) \in \mathbb{Q}[X]$ and $\pm\sqrt{p} \notin \mathbb{Q}$, then $P(\sqrt{p}) = P(-\sqrt{p}) = 0$. So, $\operatorname{tr}(A) = 0$ and $\det A = -p$. Then by the Cayley-Hamilton formula, we get that $A^2 = pI_2$. Thus, matrix A is invertible with $A^{-1} = \dfrac{1}{p}A$. Likewise, $B^2 = qI_2$ and $B^{-1} = \dfrac{1}{q}B$ and $(AB)^2 = pqI_2$ because $\sqrt{pq} \notin \mathbb{Q}$. We have that $(AB)^2 = pqI_2 = A^2B^2$ or $ABAB = AABB \iff AB = BA$.

b) If $p = q$ we have that $A^2 = pI_2$ and $B^2 = pI_2$. Since

$$(AB)^2 - p^2 I_2 = (AB)^2 - A^2B^2 = A(BA - AB)B,$$

then we have that

$$\det((AB)^2 - p^2 I_2) = \det A \cdot \det(BA - AB) \cdot \det B = 0,$$

from which it follows that $\det(BA - AB) = 0$ because A and B are invertible. Since $\operatorname{tr}(BA - AB) = 0$ then, on account of the Cayley-Hamilton theorem, we have that $(BA - AB)^2 = O_2$ or $(BA)^2 - BA^2B - AB^2A + (AB)^2 = O_2$. From the fact that $A^2 = B^2 = pI_2$ we get that $BA^2B = BpI_2B = pB^2 = p^2I_2$. Likewise, $AB^2A = p^2I_2$. Adding up the preceding expressions yields $(AB)^2 + (BA)^2 = 2p^2I_2$.

Also solved by *José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.*

A–50. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* [**Correction**] Compute

$$\int_1^\infty \frac{2}{[t]^3 + 6[t]^2 + 11[t] + 6}\,\mathrm{d}t,$$

where $[x]$ represents the integer part of x.

Solution by the proposer. Let $f(t) = \dfrac{2}{[t]^3 + 6[t]^2 + 11[t] + 6}$. If $n \le t < n+1$, then $[t] = n$ and

$$\begin{aligned}\int_n^{n+1} f(t)\,\mathrm{d}t &= \int_n^{n+1} \frac{2}{6 + 11n + 6n^2 + n^3}\,\mathrm{d}t \\ &= \frac{2}{(n+1)(n+2)(n+3)} \int_n^{n+1} \mathrm{d}t \\ &= \frac{2}{(n+1)(n+2)(n+3)}.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_1^n f(t)\,\mathrm{d}t &= \sum_{k=1}^{n-1}\int_k^{k+1} f(t)\,\mathrm{d}t = \sum_{k=1}^{n-1}\frac{2}{(k+1)(k+2)(k+3)}\\ &= \sum_{k=1}^{n-1}\left(\frac{1}{(k+1)(k+2)} - \frac{1}{(k+2)(k+3)}\right)\\ &= \frac{1}{6} - \frac{1}{(n+1)(n+2)}.\end{aligned}$$

Taking limits when $n \to \infty$, we have that

$$\begin{aligned}\int_1^\infty \frac{2}{[t]^3+6[t]^2+11[t]+6}\,\mathrm{d}t &= \lim_{n\to\infty}\int_1^n f(t)\,\mathrm{d}t\\ &= \lim_{n\to\infty}\left(\frac{1}{6} - \frac{1}{(n+1)(n+2)}\right) = \frac{1}{6}.\end{aligned}$$

A–51. *Proposed by Mihály Bencze, Braşov, Romania.* Let a, b be complex numbers which satisfy that $|a^k + b^k| \le 2$ for any positive integer n and for all $k \in \{3,5,7,\ldots,2n+1\}$. If $|ab| \le 1$, then prove that $|a+b| \le 2$.

Solution 1 by the proposer. We have

$$\begin{aligned}(a+b)^{2n+1} &= a^{2n+1} + b^{2n+1} + \binom{2n+1}{1}(a^{2n-1}+b^{2n-1})ab\\ &\quad + \ldots + \binom{2n+1}{n}a^n b^n (a+b),\end{aligned}$$

therefore

$$\begin{aligned}|a+b|^{2n+1} &\le |a^{2n+1}+b^{2n+1}| + \binom{2n+1}{1}|ab||a^{2n-1}+b^{2n-1}|\\ &\quad + \ldots + \binom{2n+1}{n}|ab|^n|a+b|.\end{aligned}$$

If $|a+b| = r > 0$, then

$$r^{2n+1} \le 2 + 2\binom{2n+1}{1} + \ldots + 2\binom{2n+1}{n-1} + \binom{2n+1}{n}r$$

or

$$r^{2n+1} - \binom{2n+1}{n} r - 2\left(2^{2n} - \binom{2n+1}{n}\right) \leq 0$$

because

$$\binom{2n+1}{0} + \binom{2n+1}{1} + \ldots + \binom{2n+1}{n} = 2^{2n}.$$

The inequality can be written as

$$(r-2)(r^{2n} + 2r^{2n-1} + 4r^{2n-2} + 8r^{2n-3} + \ldots + 2^{2n-1}r \\ + \left(2^{2n} - \binom{2n+1}{n}\right) \leq 0,$$

from which we get that $r \leq 2$, and therefore $|a+b| \leq 2$.

Solution 2 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. Let us try to solve the problem without using the hypothesis $|ab| \leq 1$. To start, it is easy to see that the statement "for any positive integer n and for all $k \in \{3, 5, 7, \ldots, 2n+1\}$" means "for every odd $n \geq 3$". Let us consider different cases. If $|a| \leq 1$, $|b| \leq 1$, using the triangle inequality, $|a^k + b^k| \leq |a|^k + |b|^k \leq 2$ and $|a+b| \leq |a| + |b| \leq 2$. If $|b| > 1$ and $|b| > |a|$, let $c = \frac{b}{a}$, so $|c| > 1$. Then, $|a^k + b^k| \geq |b|^k - |a|^k = |b|^k\left(1 - \frac{1}{|c|^k}\right) \geq |b|^k\left(1 - \frac{1}{|c|}\right)$, which goes to infinity as k goes to infinity, which violates the hypothesis $|a^k + b^k| \leq 2 \ \ \forall\, n \in \{3, 5, 7, \ldots\}$. The same happens if $|a| > 1$ and $|a| > |b|$.

Now we are left with the case $|a| = |b| > 1$. Let $c = \frac{b}{a}$, so $|c| = 1$ and therefore can be written as $c = e^{i\theta}$. Then, $|a^k + b^k| \leq 2$ is equivalent to $|a^k(1 + e^{ik\theta})| \leq 2$. We can see that

$$\begin{aligned} |1 + e^{ik\theta}| &= \sqrt{(1+\cos(k\theta))^2 + \sin^2(k\theta)} \\ &= \sqrt{1 + 2\cos(k\theta) + \cos(k\theta)^2 + \sin(k\theta)^2} \\ &= \sqrt{2 + 2\cos(k\theta)} = 2\sqrt{\frac{1+\cos(k\theta)}{2}} \\ &= 2\left|\cos\left(\frac{k\theta}{2}\right)\right|. \end{aligned}$$

Now let $k = 2m + 1$. If we want

$$|a^k(1 + e^{ik\theta})| = 2|a|^{2m+1}\left|\cos\left(\left(m + \frac{1}{2}\right)\theta\right)\right|$$

to be at most 2 for every $m \leq 1$, we will need $\cos\left(\left(m + \frac{1}{2}\right)\theta\right)$ to tend to zero as m goes to infinity. It is clear that this will only happen if $\theta = (2n + 1)\pi$ for some $n \in \mathbb{Z}$, that is, if it becomes a zero sequence. Otherwise, $\cos\left(\left(m + \frac{1}{2}\right)\theta\right)$ would, as m grows to infinity, oscillate around zero but without tending towards zero. In that case,

$$\begin{aligned}
|a|^k 2\left|\cos\left(\left(m + \frac{1}{2}\right)\theta\right)\right| &= |a|^k 2\left|\cos\left(\left(m + \frac{1}{2}\right)(2n + 1)\pi\right)\right| \\
&= |a|^k 2\left|\cos\left(\left((2nm + m + n + \frac{1}{2}\right)\pi\right)\right| \\
&= 0,
\end{aligned}$$

since $\cos\alpha = 0$ if and only if $\alpha = \left(k + \frac{1}{2}\right)\pi$ for some integer k. If $\theta = (2n + 1)\pi$, as $b = ae^{i\theta}$, $b = ae^{i(2n+1)\pi} = ae^{2n\pi i}e^{\pi i} = a \cdot 1 \cdot (-1) = -a$. In that case $|a + b| = |a - a| = 0 \leq 2$, as we wanted to show.

Also solved by *José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.*

A–52. *Proposed by Mihály Bencze, Braşov, Romania.* Find the general term of the sequence $\{a_n\}_{n\geq 1}$ if $a_1 = \frac{1}{2}$ and for all $n \geq 1$, we have that $(n^3 + 3n^2 + 2n)\, a_{n+1} = a_n + n^4 + 4n^3 + 5n^2 + n$.

Solution by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain. Let $b_n = n - a_n$ (or, equivalently, $a_n = n - b_n$). We can write the recurrence formula as:

$$\begin{aligned}
&(n^3+3n^2+2n)(n+1-b_{n+1}) = n-b_n+n^4+4n^3+5n^2+n \\
\iff &n^4 + 3n^3 + 2n^2 + n^3 + 3n^2 + 2n - (n^3 + 3n^2 + 2n)b_{n+1} \\
&= -b_n + n^4 + 4n^3 + 5n^2 + 2n \\
\iff &-(n^3 + 3n^2 + 2n)b_{n+1} = -b_n \\
\iff &n(n + 1)(n + 2)b_{n+1} = b_n.
\end{aligned}$$

From here we can deduce that $b_n = \frac{1}{(n-1)!n!(n+1)!}$. Let us prove this by induction. For $n = 1$, this is true, and assuming induction hypothesis for some positive integer n,

$$\begin{aligned} b_{n+1} &= \frac{b_n}{n(n+1)(n+2)} \\ &= \frac{1}{(n-1)!n!(n+1)!n(n+1)(n+2)} \\ &= \frac{1}{n!(n+1)(n+2)!}, \end{aligned}$$

as we wanted to see. So, the general term for $a_n = n - b_n$ is $a_n = n - \dfrac{1}{(n-1)!n!(n+1)!}$.

Also solved by *José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain, and the proposer.*

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