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Articles

Arhimede Mathematical Journal aims to publish interesting and attractive papers with elegant mathematical exposition. Articles should include examples, applications and illustrations, whenever possible. Manuscripts submitted should not be currently submitted to or accepted for publication in another journal.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Inequalities and Limits

José Luis Díaz-Barrero

Abstract

In this paper, we use inequalities and identities to obtain some results involving Riemann sums and integrals that are applied to compute limits.

1 Introduction

The following limit was published by Díaz-Barrero [3]:

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\arctan\left(\frac{k}{n^2}\right).$$

In order to compute it, a technique involving Riemann sums like the ones appeared in [2, 4] may be utilized. In this paper, we use some inequalities involving real functions to derive results about Riemann sums that let us calculate limits of some numerical sequences by means of integrals [1]. Moreover, some limits are also computed applying this technique.

2 Main results

Hereafter, several results involving continuous functions that will be used to compute limits of sequences of real numbers are stated and proven. We begin with the following.

Theorem 1. *Let* $f\colon [0,1] \to (0,+\infty)$ *be a bounded integrable function. Then,*

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left[1+\frac{1}{n}f\left(\frac{k}{n}\right)\right]=\int_0^1 f(x)\,\mathrm{d}x.$$

Proof. Since f is bounded, there exists $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in [0,1]$. Now we will use the well-known inequality for the function $f(t) = \ln(t+1)$,

$$x-\frac{x^2}{2}\leq \ln(1+x)\leq x,$$

valid for all $x \in [0,1]$. We put $x = \frac{1}{n}f\left(\frac{k}{n}\right)$, $1 \leq k \leq n$, into the inequality (notice that, since f is bounded, $x \in [0,1]$ for all k for n large enough). Adding the resulting inequalities yields

$$\sum_{k=1}^{n}\frac{1}{n}f\left(\frac{k}{n}\right)-\sum_{k=1}^{n}\frac{1}{2n^2}f^2\left(\frac{k}{n}\right)\leq\sum_{k=1}^{n}\ln\left[1+\frac{1}{n}f\left(\frac{k}{n}\right)\right]\leq\sum_{k=1}^{n}\frac{1}{n}f\left(\frac{k}{n}\right).$$

Taking limits when $n\to\infty$, we get

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x-\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2n^2}f^2\left(\frac{k}{n}\right)&\leq\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left[1+\frac{1}{n}f\left(\frac{k}{n}\right)\right]\\&\leq\int_0^1 f(x)\,\mathrm{d}x.\end{aligned}$$

Since $0\leq\sum_{k=1}^{n}\frac{1}{2n^2}f^2\left(\frac{k}{n}\right)\leq\frac{M^2}{2n}$, then when $n\to\infty$ we obtain that

$$0\leq\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{2n^2}f^2\left(\frac{k}{n}\right)\leq\lim_{n\to\infty}\frac{M^2}{2n}=0$$

and

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left[1+\frac{1}{n}f\left(\frac{k}{n}\right)\right]=\int_0^1 f(x)\,\mathrm{d}x,$$

and this completes the proof. □

Corollary 1. *Let $f\colon [0,1] \to (0,+\infty)$ be a bounded integrable function. Then,*

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left[1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right] = e^{\int_0^1 f(x)\,\mathrm{d}x}.$$

Proof. Since the function $\ln(t)$ is continuous in $(0,+\infty)$, then we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left[1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right] = \lim_{n\to\infty} \ln \prod_{k=1}^{n} \left[1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right] = \int_0^1 f(x)\,\mathrm{d}x$$

or

$$\ln\left(\lim_{n\to\infty} \prod_{k=1}^{n} \left[1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right]\right) = \int_0^1 f(x)\,\mathrm{d}x,$$

from which the statement follows. □

Next we present some applications of the above result similar to the ones appeared in [5].

Problem 1. *Compute*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left[1 + \frac{1}{n}\left(\frac{n^2-k^2}{n^2+k^2}\right)\right].$$

Solution. Applying Theorem 1 to $f(x) = \dfrac{1-x^2}{1+x^2}$ we have

$$\begin{aligned}\lim_{n\to\infty} \sum_{k=1}^{n} \ln\left[1 + \frac{1}{n}\left(\frac{n^2-k^2}{n^2+k^2}\right)\right] &= \int_0^1 \frac{1-x^2}{1+x^2}\,\mathrm{d}x \\ &= 2\arctan(x) - x\Big|_0^1 \\ &= \frac{\pi-2}{2}.\end{aligned}$$

Problem 2. *Compute*

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left[1+\frac{1}{n}\left(\frac{n-k}{n+k}\right)\right].$$

Solution. We have

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left[1+\frac{1}{n}\left(\frac{k-n}{k+n}\right)\right]=\lim_{n\to\infty}\prod_{k=1}^{n}\left[1+\frac{1}{n}\left(\frac{1-k/n}{1+k/n}\right)\right].$$

Putting $f(x)=\dfrac{1-x}{1+x}$ in Corollary 1 and taking into account that

$$\int_0^1\frac{1-x}{1+x}\,\mathrm{d}x=2\,\ln(1+x)-x\Big|_0^1=\ln(4)-1,$$

we obtain

$$\lim_{n\to\infty}\prod_{k=1}^{n}\left[1+\frac{1}{n}\left(\frac{n-k}{n+k}\right)\right]=e^{-1+\ln(4)}=\frac{4}{e}.$$

Another result involving limits and integrals is the following.

Theorem 2. *Let* $f\colon[0,1]\to(0,+\infty)$ *be a bounded integrable function. Then,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{f\left(\frac{k}{n}\right)}{1+2\sqrt{\frac{1}{n}f\left(\frac{k}{n}\right)+1}}=\frac{1}{3}\int_0^1 f(x)\,\mathrm{d}x.$$

Proof. Putting $x=\dfrac{1}{n}f\left(\dfrac{k}{n}\right)$ in the following inequality

$$x-\frac{1}{3}x^2\le\frac{3x}{1+2\sqrt{x+1}}\le x,\quad\text{for all}\quad x\in[0,1]$$

(we can do this because f is bounded, so $x \in [0,1]$ for f large enough), we obtain

$$\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) - \sum_{k=1}^{n} \frac{1}{3n^2} f^2\left(\frac{k}{n}\right) \leq \sum_{k=1}^{n} \frac{\frac{3}{n} f\left(\frac{k}{n}\right)}{1 + 2\sqrt{\frac{1}{n} f\left(\frac{k}{n}\right) + 1}} \leq \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right).$$

Taking limits when $n \to \infty$, we get

$$\int_0^1 f(x)\,\mathrm{d}x - \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{3n^2} f^2\left(\frac{k}{n}\right) \leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\frac{3}{n} f\left(\frac{k}{n}\right)}{1 + 2\sqrt{\frac{1}{n} f\left(\frac{k}{n}\right) + 1}} \leq \int_0^1 f(x)\,\mathrm{d}x.$$

Since f is bounded, there exists $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in [0,1]$ and $0 \leq \sum_{k=1}^{n} \frac{1}{3n^2} f^2\left(\frac{k}{n}\right) \leq \frac{M^2}{3n}$. Therefore, when $n \to \infty$, we have that

$$0 \leq \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{3n^2} f^2\left(\frac{k}{n}\right) \leq \lim_{n\to\infty} \frac{M^2}{3n} = 0$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\frac{3}{n} f\left(\frac{k}{n}\right)}{1 + 2\sqrt{\frac{1}{n} f\left(\frac{k}{n}\right) + 1}} = \int_0^1 f(x)\,\mathrm{d}x,$$

from which the statement follows. □

Next we apply the preceding result in the following problem.

Problem 3. *Compute the limit*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k+n}{n+2\sqrt{n^2+n+k}}.$$

Solution. We have

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k+n}{n+2\sqrt{n^2+n+k}} &= \lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n} \frac{k+n}{1+2\sqrt{\frac{n^2+n+k}{n^2}}} \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{\frac{k}{n}+1}{1+2\sqrt{\frac{1}{n}\left(\frac{k}{n}+1\right)+1}}.\end{aligned}$$

Now putting $f(x) = x+1$ in Theorem 2, we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{k+n}{n+2\sqrt{n^2+n+k}} = \frac{1}{3}\int_0^1 (x+1)\,\mathrm{d}x = \frac{1}{2}.$$

Theorem 3. *Let* $f\colon [0,1] \to (0,+\infty)$ *be a bounded integrable function. Then,*

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[e^{-\frac{1}{n}f\left(\frac{k}{n}\right)} \sin\left(\frac{1}{n} f\left(\frac{k}{n}\right)\right)\right] = \int_0^1 f(x)\,\mathrm{d}x.$$

Proof. Putting $x = \dfrac{1}{n} f\left(\dfrac{k}{n}\right)$ $(1 \le k \le n)$ in the inequality

$$x - x^2 \le e^{-x}\sin x \le x, \text{ valid for } x \in [0,1]$$

(we can do this because f is bounded, so $x \in [0,1]$ for f large enough), and adding up the resulting inequalities we get

$$\sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right) - \sum_{k=1}^{n} \frac{1}{n^2} f^2\left(\frac{k}{n}\right) \le \sum_{k=1}^{n} e^{-f\left(\frac{k}{n}\right)} \sin\left(\frac{k}{n}\right) \le \sum_{k=1}^{n} \frac{1}{n} f\left(\frac{k}{n}\right).$$

Taking limits when $n \to \infty$, we get

$$\int_0^1 f(x)\,\mathrm{d}x - \lim_{n\to\infty} \sum_{k=1}^n \frac{1}{n^2} f^2\left(\frac{k}{n}\right) \le \lim_{n\to\infty} \sum_{k=1}^n e^{-f\left(\frac{k}{n}\right)} \sin\left(\frac{k}{n}\right)$$
$$\le \int_0^1 f(x)\,\mathrm{d}x.$$

Since f is bounded, there exists $M \ge 0$ such that $|f(x)| \le M$ for all $x \in [0,1]$ and $0 \le \sum_{k=1}^n \frac{1}{n^2} f^2\left(\frac{k}{n}\right) \le \frac{M^2}{n}$. Thus, when $n \to \infty$,

$$0 \le \lim_{n\to\infty} \sum_{k=1}^n \frac{1}{n^2} f^2\left(\frac{k}{n}\right) \le \lim_{n\to\infty} \frac{M^2}{n} = 0$$

and

$$\lim_{n\to\infty} \sum_{k=1}^n \left[e^{-\frac{1}{n} f\left(\frac{k}{n}\right)} \sin\left(\frac{1}{n} f\left(\frac{k}{n}\right)\right)\right] = \int_0^1 f(x)\,\mathrm{d}x,$$

as claimed. □

Problem 4. *Compute the following limit:*

$$\lim_{n\to\infty} \sum_{k=1}^n \left[e^{-k^2/n^3} \sin\left(\frac{k^2}{n^3}\right)\right].$$

Solution. Putting $f(x) = x^2$ in Theorem 3, we have

$$\lim_{n\to\infty} \sum_{k=1}^n \left[e^{-k^2/n^3} \sin\left(\frac{k^2}{n^3}\right)\right] = \int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}.$$

Finally, another result for computing limits using an identity and integrals is the following.

Theorem 4. *Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Then,*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right) = \frac{1}{2}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2.$$

Proof. From the identity

$$\left(\sum_{k=1}^{n} x_k\right)^2 = \sum_{k=1}^{n} x_k^2 + 2 \sum_{1\le i<j\le n} x_i x_j,$$

we get

$$\frac{1}{n^2}\left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right)^2 = \frac{1}{n^2}\sum_{k=1}^{n} f^2\left(\frac{k}{n}\right) + \frac{2}{n^2}\sum_{1\le i<j\le n} f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right).$$

Since

$$\lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} f^2\left(\frac{k}{n}\right)\right) = 0,$$

then

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{1\le i<j\le n} f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right) = \frac{1}{2}\left[\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right]^2$$
$$= \frac{1}{2}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2,$$

and the proof is complete. □

As an application, we compute the following limit.

Problem 5. *Determine*

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{1\le i<j\le n}\left(\frac{n^4-(i^2+j^2)\,n^2+i^2j^2}{n^4+(i^2+j^2)\,n^2+i^2j^2}\right)^2.$$

Solution. Applying Theorem 4 to $f(x) = \left(\dfrac{1-x^2}{1+x^2}\right)^2$ we have

$$\lim_{n\to\infty} \frac{1}{n^2}\sum_{1\le i<j\le n}\left(\frac{n^4-(i^2+j^2)\,n^2+i^2j^2}{n^4+(i^2+j^2)\,n^2+i^2j^2}\right)^2$$
$$= \lim_{n\to\infty} \frac{1}{n^2}\sum_{1\le i<j\le n}\left(\frac{(n^2-i^2)(n^2-j^2)}{(n^2+i^2)(n^2+j^2)}\right)^2$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\int_0^1 \left(\frac{1-x^2}{1+x^2}\right)^2 \mathrm{d}x\right)^2 \\
&= \frac{1}{2}\left(x - 2\arctan(x) + \frac{2x}{x^2+1}\Big|_0^1\right)^2 = \frac{(4-\pi)^2}{32}.
\end{aligned}
$$

References

[1] Apostol, T. M. *Mathematical analysis*. Second. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1974, pp. xvii+492.

[2] Díaz-Barrero, J. L. “Problem A-23”. *Arhimede math. j.* 2.2 (2015), p. 103.

[3] Díaz-Barrero, J. L. “Problem A-46”. *Arhimede math. j.* 4.1 (2017), p. 23.

[4] Díaz-Barrero, J. L. and Gibergans Báguena, J. “On Limits Computed by Integrals”. *Foaie Matematica* 7.1 (2005), pp. 1–6.

[5] Kaczor, W. J. and Nowak, M. T. *Problems in mathematical analysis. III. Integration*. Vol. 21. Student Mathematical Library. American Mathematical Society, Providence, RI, 2003, pp. x+356. ISBN: 0-8218-3298-0. URL: https://doi.org/10.1090/stml/021.

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Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.

2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

May 31, 2018

Elementary Problems

E–47. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Show that among 2018 distinct positive integers there are two of them whose sum is at least 4035.

E–48. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Without the aid of a computer, show that

$$\cot 36^\circ \cdot \cot 72^\circ = \frac{\sqrt{5}}{5}.$$

E–49. *Proposed by José Gibergans Báguena, BarcelonaTech, Barcelona, Spain.* Solve in $\mathbb{R}$ the following system of equations:

$$\left.\begin{aligned} x &= 5y\sqrt{1+z^2}, \\ y &= 5z\sqrt{1+x^2}, \\ z &= 5x\sqrt{1+y^2}. \end{aligned}\right\}$$

E–50. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let h_a, h_b, h_c be the altitudes of triangle ABC. Let P be a point inside $\triangle ABC$. Find the maximum value of

$$\frac{d_a \cdot d_b \cdot d_c}{h_a \cdot h_b \cdot h_c},$$

where d_a, d_b, d_c are the distances from P to the sides BC, CA and AB, respectively.

E–51. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Without the aid of a computer, show that the coefficient a_n of the monomial $a_n x^n$ in the expression

$$\sum_{n\geq 0} a_n x^n = \frac{2x}{1-12x+35x^2}, \quad \text{for } x \in \left(-\frac{1}{7}, \frac{1}{7}\right),$$

is a nonnegative integer and determine its value.

E–52. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let O be the center of the circumcircle γ of triangle ABC. If it lies outside of $\triangle ABC$ and $\angle ABC = 36^\circ$, then find the value of the angles of triangle AOC.

Easy–Medium Problems

EM–47. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* **[Correction]** Let $A_1, A_2, \ldots, A_n$ be the vertices of an n-gon inscribed in a circle of center O. If O and the A_i's are lattice points, then prove that the sum of the squares of the sides of the n-gon is an even number.

EM–48. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let ABC be an acute triangle with orthocenter H and incenter I. If A' is the midpoint of side BC, A'' is the midpoint of AH, M is the midpoint of AI and $A''M \perp A'M$, then calculate $\measuredangle BAC$.

EM–49. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Inside a square of side 1 there are several circumferences. If the sum of their perimeters is 100, prove that there is a line perpendicular to one side of the square intersecting 32 of them.

EM–50. *Proposed by Nicolae Papacu, Slobozia, Romania.* Find all real solutions of the system of equations

$$\left.\begin{array}{r}[x]\,[y] = x + y, \\ [x] + [y] = [x\,y],\end{array}\right\}$$

where $[a]$ represents the integer part of the real number a.

EM–51. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In the convex quadrilateral $ABCD$, choose points A, B, C and D in the interior of the sides AB, BC, CD and DA, respectively, such that

$$\frac{AA'}{A'B} = \frac{BB'}{B'C} = \frac{CC'}{C'D} = \frac{DD'}{D'A} = r.$$

Compute $\dfrac{[A'B'C'D']}{[ABCD]}$ and express it as function of r. Here, the expression $[XYZT]$ represents the area of quadrilateral $XYZT$.

EM–52. *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.* Let ABC be an acute triangle with orthocenter H and circumcenter O. BO and CO intersect AH at P and Q, respectively. Prove that the areas of BCH, BCP and BCQ add up to the area of ABC.

Medium–Hard Problems

MH–47. *Proposed by Ismael Morales López, Universidad Complutense de Madrid, Madrid, Spain.* There are 2018 students on a mathematics competition. We say the pair of students (A, B) is friendly if A knows B. This relation is naturally supposed to be symmetric. Find the greatest integer n such that at least one of the following conditions always holds:

i) There exists a student that knows at least other n students.
ii) There exists a set of $2n$ students that does not contain any friendly pair.

MH–48. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Prove that

$$\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^{2k}C(n,k)}\right)^{n} > e^{(n+1)-5^{n}}.$$

Here, $C(n, k)$ represents the binomial coefficient $\binom{n}{k}$.

MH–49. *Proposed by Nicolae Papacu, Slobozia, Romania.* Let $x_1,x_2,\ldots,x_n$ be $n \geq 2$ positive numbers other than one such that $x_1^2 + x_2^2 + \ldots + x_n^2 = n^3$. Prove that

$$\frac{\log_{x_1}^4 x_2}{x_1 + x_2} + \frac{\log_{x_2}^4 x_3}{x_2 + x_3} + \ldots + \frac{\log_{x_n}^4 x_1}{x_n + x_1} \geq \frac{1}{2}.$$

MH–50. *Proposed by Óscar Rivero Salgado, BarcelonaTech, Barcelona, Spain.* Let a be an integer and $p \geq 3$ be a prime number. Prove that

$$a^p + (a+1)^p + \ldots + (a+p-1)^p$$

is a multiple of p^2.

MH–51. *Proposed by Ismael Morales López, Universidad Complutense de Madrid, Madrid, Spain.* Find all strictly increasing sequences $\{a_n\}_{n\geq 1}$ of positive integers such that:

i) $a_{2018n} - a_n \leq 2017n$.
ii) If a_k is the sum of two squares then k is, too.

MH–52. *Proposed by Ángel Plaza and Sergio Falcón, Universidad Las Palmas de Gran Canaria, Las Palmas de Gran Canaria, Spain.* The Fibonacci numbers are defined recursively by $F_n = F_{n-1}+F_{n-2}$ with initial values $F_0 = 0,\ F_1 = 1$. Prove that

$$\sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} \binom{n}{2i+1}^2 5^{2i} \geq \frac{2^{2n-2}F_n^2}{1+\lfloor\frac{n-1}{2}\rfloor},$$

where $\lfloor\cdot\rfloor$ denotes the integer part.

Advanced Problems

A–47. *Proposed by Mihaela Berindeanu, Bucharest, Romania.* Let A and B be 2×2 matrices with integer entries such that $AB = BA$, $\det(A + B) = 2$, and $\det(A^3 + B^3) = 2^3$. Compute $\det(A^2 + B^2)$.

A–48. *Proposed by José Luis Díaz-Barrero, Barcelonatech, Barcelona, Spain.* Without using the series expansion of the hyperbolic functions, show that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sinh^2 x$ is not a polynomial.

A–49. *Proposed by Nicolae Papacu, Slobozia, Romania.* Let A, $B \in M_2(\mathbb{Q})$ be matrices such that

$$\det(A^2 - pI_2) = \det(B^2 - qI_2) = \det((AB)^2 - pqI_2) = 0,$$

where p, q are prime numbers.

a) If $p \neq q$, then prove that $AB = BA$.
b) If $p = q$, then prove that $(AB)^2 + (BA)^2 = 2p^2 I_2$.

A–50. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Compute

$$\int_1^\infty \frac{2}{[t]^3 + 6[t^2] + 11[t] + 6}\,\mathrm{d}t,$$

where $[x]$ represents the integer part of x.

A–51. *Proposed by Mihály Bencze, Braşov, Romania.* Let a, b be complex numbers which satisfy that $|a^k + b^k| \leq 2$ for any positive integer n and for all $k \in \{3,5,7,\ldots,2n+1\}$. If $|ab| \leq 1$, then prove that $|a + b| \leq 2$.

A–52. *Proposed by Mihály Bencze, Braşov, Romania.* Find the general term of the sequence $\{a_n\}_{n\geq 1}$ if $a_1 = \frac{1}{2}$ and for all $n \geq 1$, we have that $(n^3 + 3n^2 + 2n)\, a_{n+1} = a_n + n^4 + 4n^3 + 5n^2 + n$.

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

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Conjecture and Proof

José Luis Díaz-Barrero and Alberto Espuny Díaz

1 Introduction

When working with mathematical statements a natural question arises: how do we prove a claim in mathematics? That is, how do we establish the correctness of a mathematical statement? This question was first answered by various Greek scholars well over two thousand years ago. Interestingly, their basic idea of what a mathematical proof should be has been accepted, with relatively minor modifications, right up until this day. This is in contrast to the situation in other sciences, where even in the last three hundred years there have been tremendous changes, advances, and controversy about what constitutes a proof. In part, this is because the range of methods allowed in mathematical proofs is quite a bit more specific and narrow than in other fields. As is the case of a hypothesis in other sciences, a conjecture is a statement that has not been proved yet, although there is usually evidence for believing it.

When solving a problem, a usual technique consists in the following: first, we begin by analyzing particular cases and obtaining data that allows us to observe a pattern and to make a conjecture about its solution. After the conjecture is made, the next step is to prove it by choosing an appropriate technique of proof and obtain a full solution. The goal of this lesson is to solve some problems using this procedure.

2 Problems

Hereafter, some more or less well-known examples where the solution is conjectured are discussed and completely solved.

Problem 1 (The Tower of Hanoi). *There are n circular disks of decreasing radii, each with a hole at the center, and three pegs A, B, and C fixed vertically on a table so that the distance between the feet of any two of them is greater than the diameter of the largest disk. Initially these disks are slipped onto peg A with the largest disk at the bottom and the others on top of this, in decreasing order of size. A legal move is defined as the transfer of the top disk from one of the three pegs to the top of the stack on one of the other two pegs where it rests on a larger disk. Determine the number of legal moves needed to transfer all n disks from peg A to another peg.*

Solution. Let $f(n)$ be the number of legal moves. Clearly, $f(1) = 1$. Furthermore, if $n = 2$, then, in order to transfer the second ring to the second peg, we must first transfer the first ring from the first peg to the auxiliary peg (the third peg); then we place the second ring on the second peg and transfer the first (smallest) ring to the second peg. Thus, $f(2) = 2\,f(1) + 1 = 3$. If $n = 3$, then to transfer the lowest (largest) ring to the second peg, in the necessary arrangement, we must first move the top two rings to the third peg (using the second peg as auxiliary). This requires $f(2)$ moves, and $f(2)$ moves will be required again to place the rings on the second peg after the largest ring is moved from the first peg to the second peg (now using the first peg as auxiliary). Thus, $f(3) = 2\,f(2) + 1 = 7$. Now we make the following conjecture:

Conjecture. $f(n) = 2\,f(n-1) + 1 = 2^n - 1,\quad \textit{for}\ n \geq 2.$

We now prove the conjecture. The base cases for the recurrence are given by the examples above. In general, assume that we want to prove the recurrence for any value of $n \geq 2$. If we ignore the bottom ring, we can move all the other $n-1$ rings to the second peg in $f(n-1)$ moves. Then, we move the bottom ring to the third peg, and move all others to the third peg in $f(n-1)$ moves again, which shows that $f(n) \leq 2\,f(n-1) + 1$. To show that equality

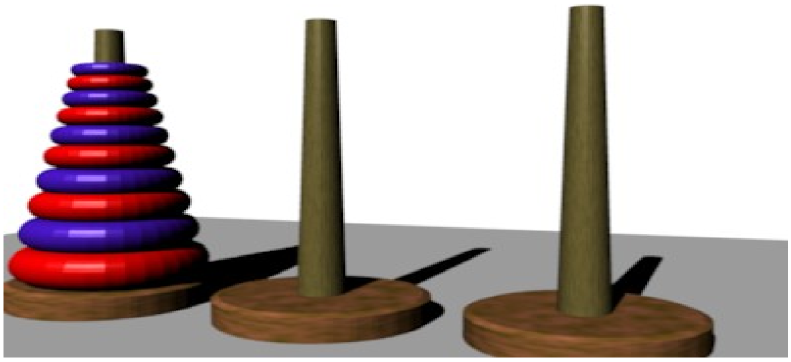

Figure 1: The Tower of Hanoi.

must hold, it suffices to observe that moving the smallest $n-1$ disks to another peg before moving the largest disk is unavoidable, as this is the only position in which the largest disk can be moved. The same must be done after the largest disk has been moved. Therefore, at least $2\,f(n-1)+1$ steps are required. This proves the recurrence holds for all $n \geq 2$.

Now let us solve the recurrence by mathematical induction. Suppose that $f(n-1) = 2^{n-1}-1$ then $f(n) = 2\,f(n-1)+1 = 2\,(2^{n-1}-1)+1 = 2^n-1$. Thus, by the principle of finite mathematical induction, it follows that $f(n) = 2^n - 1$ for all n.

We may also obtain the explicit value of $f(n)$ as follows. Subtracting $f(n-1) = 2\,f(n-2)+1$ from $f(n) = 2\,f(n-1)+1$ yields

$$f(n) = 3f(n-1) - 2f(n-2).$$

To solve the preceding homogeneous recursion we try with $f(n) = t^n$. Then, we have $t^n = 3\,t^{n-1} - 2\,t^{n-2}$, or $t^{n-2}\,(t^2-3t+2) = 0$. Since $t = 0$ is not a solution, then we have $t = 1$ and $t = 2$ and their linear combinations as candidates. That is, the general solution is

$$f(n) = A \cdot 2^n + B \cdot 1^n = A \cdot 2^n + B.$$

On account that $f(1) = 1$ and $f(2) = 3$ we get $A = 1$ and $B = -1$. Thus, $f(n) = 2^n - 1$, and we are done.

Problem 2. *Suppose that there are n lines in the plane, no two parallel and no three intersecting at a point. Into how many regions is the plane divided by these lines? What is the maximum number of regions into which the space $\mathbb{R}^3$ can be divided by n planes?*

Solution. In order to make a conjecture, first we generate some data. Let $f(n)$ be the number of regions into which the plane is divided by n lines in general position. With some simple sketches we see that $f(1) = 2$, $f(2) = 4$, $f(3) = 7$ and $f(4) = 11$. What can we say about the progression of numbers $2,4,7,11,\dots$? The successive differences of these numbers are $2,3,4,\dots$ Thus, it seems that $f(n) - f(n-1) = n$, or, equivalently, $f(n) = f(n-1) + n$, for $n \geq 2$. Now we make the following conjecture:

Conjecture. $f(n) = f(n-1) + n,$ *for* $n \geq 2$.

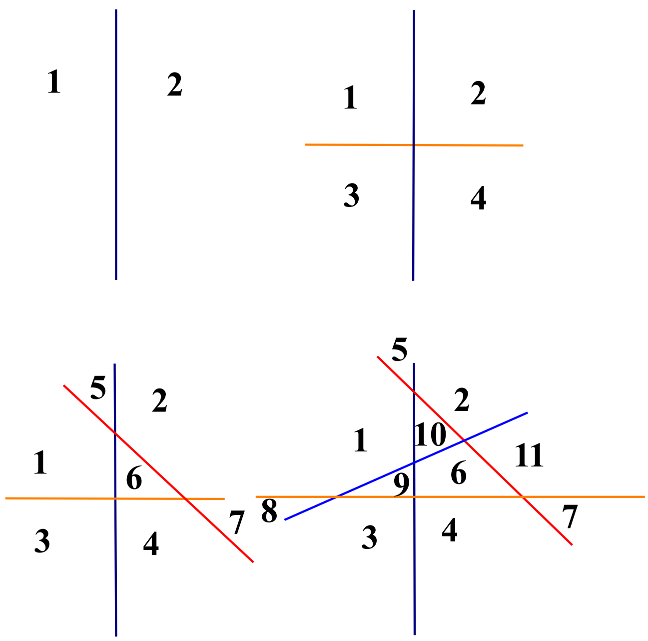

Figure 2: One, two, three, and four lines dividing the plane into two, four, seven, and eleven regions, respectively.

In order to prove the conjectured expression, consider the following.

Assume that the plane is divided into $f(n-1)$ regions by $n-1$ lines and we draw a new line (with the conditions from the statement). This line intersectes each of the previous $n-1$ lines, and as each point of intersection is always between two regions (never more, as no three lines intersect in a point) it must cross n of the $f(n-1)$ regions. The new line divides each of these regions into two, meaning that the n regions become $2n$ or, equivalently, that there are n more regions than before.

The expression $f(n) - f(n-1) = n$ is a recurrence relation for $f(n)$. To obtain the value of $f(n) = f(n-1) + n$ explicitly we will argue by *working backwards*:

$$\begin{aligned} f(n) &= f(n-1) + n = f(n-2) + (n-1) + n \\ &= f(n-3) + (n-2) + (n-1) + n \\ &= \dots \\ &= f(1) + (2 + 3 + 4 + \dots + n) \\ &= 2 + \frac{n(n+1)}{2} - 1 = \frac{n^2+n+2}{2}. \end{aligned}$$

Now we face the space problem. The number of spacial regions will be largest when no four planes intersect in a single point, and when the intersection of any three planes are non-parallel lines. We will assume that these two conditions are satisfied in the following argument, and denote the maximum number of spacial regions by $g(n)$.

Thus, we suppose that $\mathbb{R}^3$ is divided by n planes into $g(n)$ regions. We now add an additional plane. By the conditions of the last paragraph, this plane is cut by the original n planes in n lines, no three of which are concurrent, and no two of which are parallel (that is, they are in general position). The new $(n+1)$-th plane is therefore divided by the n lines into $f(n)$ plane regions. Each one of these $f(n)$ regions divides each spacial region it traverses into two, so that the addition of the $(n+1)$-th plane increases $g(n)$ by $f(n)$. This proves the following conjecture.

Conjecture. $g(n+1) = g(n) + f(n)$, *for* $n \geq 0$.

That is,

$$\begin{aligned} g(1) &= g(0) + f(0), \\ g(2) &= g(1) + f(1), \\ g(3) &= g(2) + f(2), \\ &\vdots \\ g(n) &= g(n-1) + f(n-1). \end{aligned}$$

Now we can solve this recurrence to compute the number of regions. Adding up the preceding expressions and taking into account that $f(n) = \dfrac{n^2+n+2}{2}$ yields

$$\begin{aligned} g(n) &= 2 + f(1) + f(2) + \ldots + f(n-1) \\ &= 2 + \sum_{k=1}^{n-1}\left(1 + \frac{k(k+1)}{2}\right) \\ &= \frac{n^3+5n+6}{2}, \end{aligned}$$

which is the maximum number of regions into which the space $\mathbb{R}^3$ can be divided by n planes.

Problem 3. *Into how many regions do n circles divide the plane, if each two circles intersect in two points, and no three of the circles pass through the same point? What is the maximum number of regions into which the space $\mathbb{R}^3$ can be divided by n spheres?*

Solution. As in the previous problems, we begin by obtaining some first values. We denote the maximum number of regions into which the plane is divided by n circles in general position by $f(n)$. Thus, we suppose that the plane is divided by n circles into $f(n)$ regions. We study the cases when there are 1, 2, 3 and 4 circles in the plain; we observe that the plane is divided into 2, 4, 8 and 14 regions, respectively (see Figure 3). That is, $f(1) = 2$, $f(2) = 4$, $f(3) = 8$ and $f(4) = 14$. By observing the progression so far, we observe that the successive differences of these numbers are $2,4,6,\ldots$ Thus, it seems that $f(n) - f(n-1) = 2(n-1)$, or, equivalently, $f(n) = f(n-1) + 2(n-1)$, for $n \geq 2$. Now we make the following conjecture:

Conjecture. $f(n+1) = f(n) + 2n, \quad \text{for } n \geq 1.$

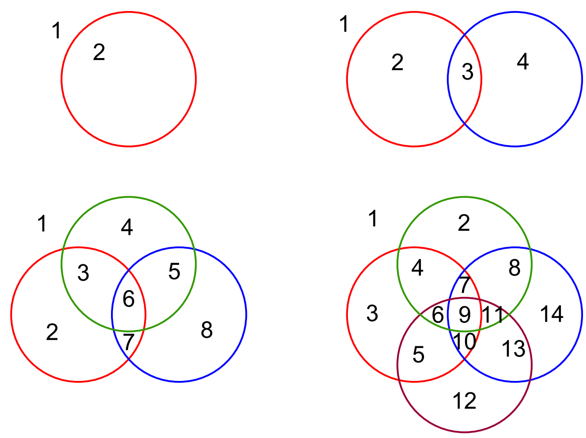

Figure 3: One, two, three, and four circles dividing the plane into two, four, eight, and fourteen regions, respectively.

In order to show that this holds for all values of n, suppose n circles are given in the plane. They will divide the plane into a maximum number of pieces if every two of them intersect (that is, if no two of them are tangent and none of them lies entirely within or outside of another) and no three of them are concurrent. Such sets of circles always exist. In fact it is possible to draw infinitely many circles in the plane in such a way that any two of them intersect in two points, but no three of them are concurrent. For example, construct two intersecting circles of the same radius r with centers A and B. Then draw all circles of radius r whose centers are on the line segment AB. This family clearly has the desired properties.

Suppose now that the plane is divided by n circles into $f(n)$ regions. We add an additional circle. This $(n+1)$-th circle intersects each of the first n circles in two points. These $2n$ points divide the $(n+1)$-th circle into $2n$ arcs. Each of these arcs divides in two one of the regions formed by the first n circles. That is, there are n more regions than there were before. This concludes the proof.

To compute the number of regions, we have that

$$\begin{aligned} f(1) &= 2, \\ f(2) &= f(1) + 2 \cdot 1, \\ f(3) &= f(2) + 2 \cdot 2, \\ f(4) &= f(3) + 2 \cdot 3, \\ &\vdots \\ f(n) &= f(n-1) + 2 \cdot (n-1). \end{aligned}$$

Adding up the preceding expressions yields

$$f(n) = 2 + 2 \cdot 1 + 2 \cdot 2 + 2 \cdot 3 + \ldots + 2 \cdot (n-1) = n^2 - n + 2,$$

which is the maximum number of regions into which the plane can be divided by n circles.

We now consider the problem of spheres in $\mathbb{R}^3$. For brevity, we will not state the hypotheses we must impose in order to insure a maximum number of pieces but will tacitly assume them throughout. We denote the maximum number of spacial regions into which n spheres divide the space by $g(n)$. Suppose that n spheres have been drawn. Let us see by how much the $(n+1)$-th sphere increases the number of pieces. The $(n+1)$-th sphere meets each of the first n spheres in a circle. The circles of intersection will all be different, no two of them will be tangent, and —viewed as curves on the $(n+1)$-th sphere— none of them will lie inside or outside another.

Previously we proved that, under these conditions, n circles in a plane will divide the plane into $n^2 - n + 2$ pieces. The same argument can be used to prove the corresponding theorem for circles on a sphere. Therefore the surface of the $(n+1)$-th sphere is divided into n^2-n+2 regions by the circles at which it intersects the first n spheres. Each of these regions splits into two one of the pieces into which the first n spheres had divided space. The $(n+1)$-th sphere thus increases the number of pieces by $f(n) = n^2-n+2$. That is,

$$g(1) = g(0) + f(0),$$

$$\begin{aligned}
g(2) &= g(1) + f(1),\\
g(3) &= g(2) + f(2),\\
&\vdots\\
g(n) &= g(n-1) + f(n-1),
\end{aligned}$$

and consequently the total number of pieces is

$$\begin{aligned}
g(n) &= 2 + f(1) + f(2) + \ldots + f(n-1)\\
&= 2 + (1^2 - 1 + 2) + (2^2 - 2 + 2) + \ldots\\
&\quad + ((n-1)^2 - (n-1) + 2)\\
&= 2 + (1^2 + 2^2 + \ldots + (n-1)^2 - (1 + 2 + \ldots + (n-1))\\
&\quad + \underbrace{(2 + 2 + \ldots + 2)}_{n-1}\\
&= \frac{n(n^2 - 3n + 8)}{3}.
\end{aligned}$$

Problem 4. *Let $n \geq 2$ be a positive integer and $S = \{1, 2, \ldots, n\}$. For every $k \in \{1, 2, \ldots, n-1\}$, prove that*

$$x_k = \frac{1}{n+1} \sum_{\substack{A \subset S\\ |A|=k}} (\min A + \max A)$$

is an integer number and determine its value.

Solution. Observing the sequence

$$\underbrace{1, 2, \ldots, i-1}_{i-1}, i, \underbrace{i+1, i+2, \ldots, n}_{n-i}$$

we may **conjecture** that, for each $i \in \{1, 2, \ldots, n\}$, the number of subsets each with k elements and having i as its minimum element is $\binom{n-i}{k-1}$. Likewise, there are $\binom{i-1}{k-1}$ subsets each with k elements and having i as its maximum element. Indeed, from

$$\underbrace{1, 2, \ldots, i-1}_{i-1}, i, \underbrace{i+1, i+2, \ldots, n}_{n-i}$$

we have that any one of the $\binom{n-i}{k-1}$ subsets of numbers greater than i together with $\{i\}$ forms a subset of k elements of S whose minimum element is i. Likewise, any one of the $\binom{i-1}{k-1}$ subsets of numbers smaller than i together with $\{i\}$ becomes a subset of k elements of S whose maximum element is i. Thus,

$$\begin{aligned} x_k &= \frac{1}{n+1} \sum_{\substack{A \subset S \\ |A|=k}} (\min A + \max A) \\ &= \frac{1}{n+1}\left[\binom{n-1}{k-1} + 2\binom{n-2}{k-1} + \ldots + (n-k+1)\binom{k-1}{k-1}\right. \\ &\quad \left. + n\binom{n-1}{k-1} + (n-1)\binom{n-2}{k-1} + \ldots + k\binom{k-1}{k-1}\right] \\ &= \binom{n-1}{k-1} + \binom{n-2}{k-1} + \ldots + \binom{k-1}{k-1}. \end{aligned}$$

Finally, for each value of k we prove by induction on $n \geq k$ that

$$P(n): \binom{n-1}{k-1} + \binom{n-2}{k-1} + \ldots + \binom{k-1}{k-1} = \binom{n}{k}.$$

For $n = k$, $P(k)$ trivially holds. Now we assume that $P(n)$ holds and we have to prove that $P(n+1)$ also holds. Indeed,

$$\binom{n}{k-1} + \binom{n-1}{k-1} + \ldots + \binom{k-1}{k-1} = \binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k},$$

and we are done.

We close this section with the following problem.

Problem 5. *Let n be a positive integer. Compute*

$$\sum_{k=1}^{n} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \frac{2^k}{(i_1+1)(i_2+1)\ldots(i_k+1)}.$$

Solution. For every function f for which $f(k) \neq 0$ $(1 \leq k \leq n)$ we have

$$\begin{aligned}
&\prod_{k=1}^{n}\left(1+\frac{1}{f(k)}\right)\\
=&1+\sum_{k=1}^{n}\frac{1}{f(k)}+\sum_{1\le i_1<i_2\le n}\frac{1}{f(i_1)f(i_2)}+\ldots+\frac{1}{f(1)f(2)\cdots f(n)}\\
=&1+\sum_{k=1}^{n}\sum_{1\le i_1<\ldots<i_k\le n}\frac{1}{f(i_1)f(i_2)\cdots f(i_k)}.
\end{aligned}$$

Putting $f(x)=\dfrac{x+1}{2}$ into the preceding expression, we get

$$1+\sum_{k=1}^{n}\sum_{1\le i_1<\ldots<i_k\le n}\frac{2^k}{(i_1+1)(i_2+1)\ldots(i_k+1)}=\prod_{k=1}^{n}\left(1+\frac{2}{k+1}\right).$$

Let us denote $P(n)=\prod_{k=1}^{n}\left(1+\dfrac{2}{k+1}\right)$ and we get some particular values. For $n=1$, we have

$$P(1)=2=\frac{(1+2)\,(1+3)}{6};$$

for $n=2$, we obtain

$$P(2)=\frac{10}{3}=\frac{(2+2)\,(2+3)}{6},$$

and for $n=3$, we obtain

$$P(3)=5=\frac{(3+2)\,(3+3)}{6}.$$

The above suggest to **conjecture** that

$$P(n)=\prod_{k=1}^{n}\left(1+\frac{2}{k+1}\right)=\frac{(n+2)(n+3)}{6}.$$

To prove this, we argue by induction. The base case when has already been proved. Assume that the identity holds for n. We have to prove that

$$\prod_{k=1}^{n+1}\left(1+\frac{2}{k+1}\right)=\frac{(n+3)(n+4)}{6}.$$

Indeed,

$$\begin{aligned}\prod_{k=1}^{n+1}\left(1+\frac{2}{k+1}\right) &= \prod_{k=1}^{n}\left(1+\frac{2}{k+1}\right)\left(1+\frac{2}{n+2}\right)\\ &= \left(\frac{(n+2)(n+3)}{6}\right)\left(1+\frac{2}{n+2}\right)\\ &= \frac{(n+3)(n+4)}{6}.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{k=1}^{n}\sum_{1\le i_1<\ldots<i_k\le n}\frac{2^k}{(i_1+1)(i_2+1)\ldots(i_k+1)} &= \frac{(n+2)(n+3)}{6}-1\\ &= \frac{n(n+5)}{6},\end{aligned}$$

and we are done.

References

[1] Steiner, J. “Einige Gesetze über die Theilung der Ebene und des Raumes.” *Journal für die reine und angewandte Mathematik* 1 (1826), pp. 349–364.

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Contests

In this section, the Journal offers sets of problems appeared in different mathematical contests over the world, as well as their solutions. This gives readers an opportunity to find interesting problems and develop their own solutions.

No problem is permanently closed. We will be very pleased to consider new solutions to problems posted in this section for publication. Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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XXXII Iberoamerican Mathematical Olympiad

Óscar Rivero Salgado

1 Introduction

The XXXII edition of the Iberoamerican Mathematical Olympiad (Olimpiada Iberoamericana de Matemáticas, OIM) was held between the 15th and the 23rd of September 2017 in Puerto Iguazú (Argentina). As is customary in this type of events, one of the main goals is to promote the study of sciences in general and, in particular, mathematics. Another goal is to support scientific talent and initiative among different countries' youth. However, the olympic experience goes further: throughout the contest, participants had a chance to share experiences and deepen the friendship among themselves and the countries they represent.

More information about this competition can be found in the web
`http://www.oma.org.ar/ibero2017/`

The Spanish team for this competition was formed by Alberto Acosta Reche (Toledo), Rafah Hajjar Muñoz (Valencia), Aitor Iribar López (León) and Jordi Rodríguez Manso (Barcelona). The deputy leader was Óscar Rivero Salgado, and the leader, José Luis Díaz Barrero.

2 Problems and Solutions

In the following, we present the statements given to the contestants in the competition with some solutions of Spaniard participants, slightly modified by the deputy leader.

Problem 1. For every positive integer n, let $S(n)$ be the sum of its digits. We say that n has property P if all the terms in the infinite sequence $n, S(n), S(S(n)), \dots$ are even numbers; we say that n has property I if all the terms of the sequence are odd. Show that in the interval $[1, 2017]$ there are more n with property I than with property P.

Solution by Rafah Hajjar, CFIS, BarcelonaTech, Barcelona, Spain. First of all, we will prove that

$$S(2k+1) = S(2k) + 1,$$

for any positive integer k. To see this, observe that the last digit of $2k$ belongs to $\{0, 2, 4, 6, 8\}$, and hence the only digit of $2k+1$ that differs from $2k$ is the last one, that is exactly one unit greater, as desired.

From this result, we deduce that if n has property P, then $n+1$ has property I. This holds because

$$S(n+1) = S(n) + 1, \quad S(S(n+1)) = S(S(n)+1) = S(S(n)) + 1,$$

from which it follows that $S^k(n+1) = S^k(n) + 1$ since $S^k(n)$ is always even.

Hence, for each n with property P we can construct $n' := n + 1$ with property I, and this mapping is injective. Moreover, 1 has property I and is not in the image of the mapping, so

$$|I| \geq |P| + 1 \quad \text{and consequently} \quad |I| > |P|.$$

Problem 2. Let ABC be an acute triangle and let Γ be its circumcircle. Let D be a point on segment BC, different from B and C, and let M be the midpoint of AD. The line perpendicular to AB through D intersects AB in E and Γ in F, with point D between E and F. Lines FC and EM intersect at point X. If $\angle DAE = \angle AFE$, show that line AX is tangent to Γ.

Solution 1 by Rafah Hajjar, CFIS, BarcelonaTech, Barcelona, Spain. Let F' be the second intersection of DE with Γ ($F \neq F'$). Observe that $F'B$ is parallel to AD, since

$$\angle ABF' = \angle AFF' = \angle AFE = \angle BAD,$$

where the first equality holds because $AFBF'$ is cyclic, and the last one, because of the statement of the problem.

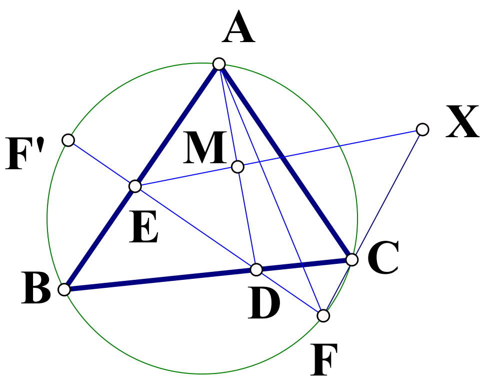

Figure 1: Scheme for Problem Problem 2.

Let $K = AF' \cap BC$. Observe that the previous fact gives the equality

$$\frac{AM}{MD} \cdot \frac{DB}{BK} \cdot \frac{KF'}{F'A} = 1,$$

and hence applying Ceva's theorem to triangle AKD, we see that lines AB, $F'D$ and KM meet at point E (and consequently K, E and M lie in the same line).

We now apply Pascal's theorem to $AAF'FCB$, and defining $X' = AA \cap CF$ (where AA is the tangent line to Γ through A), we see that K, E and X' are collinear, so they are in the same line as M. Therefore, $X' = EM \cap CF = X$, and since X' is in the tangent to Γ through A, so is X, and we conclude that XA is tangent to Γ, as desired.

Solution 2 by Alberto Acosta, Universidad Complutense de Madrid, Madrid, Spain. Since $\angle DAE = \angle AFE$ and $\angle AED = \angle AEF$, triangles ADE and FAE are similar. This implies that

$$EA^2 = ED \cdot EF,$$

and in particular EA is tangent to the circumcircle of triangle ADF.

On the other hand, $\angle DEA = 90^\circ$ and M is the midpoint of AD, so M is the center of the circumcircle of DEA. It turns out that $\angle MEA = \angle EAM$ and, consequently,

$$\angle EAF = 90^\circ - \angle DAE = 90^\circ - \angle MAE = 90^\circ - \angle AEM.$$

Hence, EM is perpendicular to AF.

Using the previous results,

$$\begin{aligned} \angle CXE = \angle FXE &= 90^\circ - \angle AFX = 90^\circ - \angle AFC \\ &= 90^\circ - \angle ABC = 90^\circ - \angle EBD = \angle BDE. \end{aligned}$$

Consequently,

$$\angle CDE + \angle EXC = 180^\circ$$

and E, D, C, X lie on the same circle. In particular, by power of a point,

$$FC \cdot FX = FD \cdot FE.$$

Since AF is perpendicular to EX,

$$\begin{aligned} XA^2 &= EA^2 + XF^2 - EF^2 = ED \cdot EF - EF^2 + XF^2 \\ &= -EF \cdot FD + XF^2 = -XF \cdot FC + XF^2 \\ &= -XF \cdot CF + XF^2 = XF(XF - CF) = XC \cdot XF. \end{aligned}$$

Using power of a point from X to the circumcircle of ABC, we have that XA is tangent to Γ in A.

Remark. *In the solution written by the contestant in the exam (Solution 2), he used oriented angles and lengths; here, for ease of notation, we have directly written his solution in the language of metric geometry.*

Problem 3. Consider the configurations of integer numbers

$$\begin{array}{ccccc} a_{1,1} & & & & \\ a_{2,1} & a_{2,2} & & & \\ a_{3,1} & a_{3,2} & a_{3,3} & & \\ \cdots & \cdots & \cdots & & \\ a_{2017,1} & a_{2017,2} & a_{2017,3} & \cdots & a_{2017,2017} \end{array}$$

with $a_{i,j} = a_{i+1,j} + a_{i+1,j+1}$ for all i, j such that $1 \leq j \leq i \leq 2016$. Determine the maximum amount of odd integers that one such configuration can contain.

Solution by Rafah Hajjar, CFIS, BarcelonaTech, Barcelona, Spain. We will work modulo two and consider that all the entries are either 0 or 1. Consider the configuration in which

$$a_{ij} = \begin{cases} 0 & \text{if } i+j \equiv 1 \mod 3, \\ 1 & \text{elsewhere.} \end{cases}$$

This configuration satisfies the requirement of the statement and the total number of ones is

$$1 + 2(3 + 6 + \ldots + 2016) = 1 + 2016 \cdot \left(\frac{2016}{3} + 1 \right) = 1356769.$$

We will prove that this bound cannot be improved. For that, consider the six numbers

$$a_{i,j}, a_{i+1,j}, a_{i+1,j+1}, a_{i+2,j}, a_{i+2,j+1}, a_{i+2,j+2},$$

for $1 \leq j \leq i \leq 2015$. A trivial check shows that one must have at least two zeros between these six numbers. In particular, if $a_{i,j}$, $a_{i+1,j}$ and $a_{i+2,j}$ are one, then both $a_{i+1,j+1}$ and $a_{i+2,j+1}$ should be zero. With this observation in mind, let c_n be the number of zeros in row n. We will prove by induction that

$$c_{n+1} + c_n + c_{n-1} \geq n,$$

and from here we will get that the given bound cannot be improved (since base cases are trivial).

Suppose that the result holds for a board with $n-2$ rows and we will prove it for a board with $n+1$ rows. For $1 \leq i \leq n+1$, let d_i be the number of zeros in column i and rows $n-1$, n or $n+1$. Fix a certain $1 \leq i \leq n-1$; if $d_i = 0$, this means that $d_{i+1} \geq 2$. Furthermore, if $d_n = 0$, then $d_{n+1} = 1$. From here, we deduce that

$$c_{n-1} + c_n + c_{n+1} = d_1 + d_2 + \ldots + d_n + d_{n+1} \geq n,$$

as claimed.

Problem 4. Let ABC be an acute triangle with $AC > AB$ and O its circumcenter. Let D be a point on segment BC such that O is in the interior of triangle ADC and $\angle DAO + \angle ADB = \angle ADC$. Let P, Q be the circumcenters of triangles ABD and ACD, respectively, and M the intersection point of lines BP and CQ. Show that lines AM, PQ and BC are concurrent.

Solution by Alberto Acosta, Universidad Complutense de Madrid, Madrid, Spain. We will proceed through three steps: first we show that M is in the circumcircle of ABC; then, we show that $BPQC$ is cyclic; and we finish by showing that $PAMQ$. Then, by concurrency of radical axis, we will be done.

1. Proof that M is in the circumcircle of ABC.

$$\angle MBC = \angle PBD = 90^\circ - \angle BPD/2 = 90^\circ - \angle BAD;$$
$$\angle BCM = \angle DCQ = 90^\circ - \angle DQC/2 = 90^\circ - \angle DAC.$$

Hence,

$$\begin{aligned}\angle CMB &= 180^\circ - \angle MBC - \angle BCM \\ &= \angle BAD + \angle DAC = \angle BAC.\end{aligned}$$

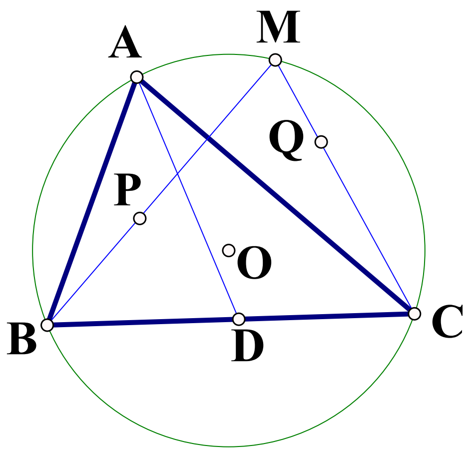

Figure 2: Scheme for Problem Problem 4.

2. Proof that $BPQC$ is cyclic. Let $T = BP \cap AD$. Since AD is the radical axis of the circumcircles of both BDA and ADC, we see that AD is perpendicular to PQ. Then,

$$\begin{aligned}
\angle BPQ &= 180^\circ - \angle TPQ \\
&= 180^\circ - (180^\circ - 90^\circ - \angle DTB) \\
&= 90^\circ + \angle DTB \\
&= 90^\circ + (180^\circ - \angle PBD - \angle BDA) \\
&= 270^\circ - (90^\circ - \angle BAD) - \angle BDA \\
&= 180^\circ + \angle BAD - \angle BDA \\
&= 180^\circ + \angle BAD + \angle DAO - \angle ADC \\
&= 180^\circ + \angle BAO - \angle ADC \\
&= 180^\circ + 90^\circ - \angle ACD - \angle ADC \\
&= 90^\circ + \angle DAC.
\end{aligned}$$

On the other hand, by the properties of the central angle,

$$\angle BCQ = 90^\circ - \angle DQC/2 = 90^\circ - \angle DAC,$$

and, consequently,

$$\angle BPQ + \angle BCQ = 180^\circ.$$

3. Proof that $PAMQ$ is cyclic.

$$\angle AMP = \angle AMB = \angle ACB;$$

$$\angle AQP = 90 - \angle DAQ = \angle AQD/2 = \angle ACD = \angle ACB.$$

Since $\angle AMP = \angle AQP$, we are done.

Problem 5. Given a positive integer n, we write all its divisors in the blackboard. Ana and Beto play the following game: in turns, each one colors the divisors of n of either red or blue. They can choose the color they prefer in each turn, but they can only color numbers that have not been colored before. The game finishes when all the numbers have been colored. If the product of all numbers painted in red is a perfect square, Ana wins. In any other case, Beto wins. If Ana begins playing, determine, for each n, who has the winning strategy.

Solution by Alberto Acosta, Universidad Complutense de Madrid, Madrid, Spain, and Rafah Hajjar, CFIS, BarcelonaTech, Barcelona, Spain. If $n = k^2$, in the first turn Ana colors k in blue. Then, if in a given turn Beto colors number d of a certain color, Ana uses the same color for n/d. Since the number of divisors of a perfect square is odd and $\sqrt{n}$ has already been chosen, Ana can follow this strategy. In each pair of turns, either we do not add red numbers or we add a pair of number whose product is a perfect square. Hence, once the game is finished, the product is a perfect square and Ana wins.

Assume now that n is not a perfect square. If n is a prime number, then Ana clearly wins just by painting n in blue. Elsewhere, since n is not a perfect square, there is a prime divisor that appears with odd exponent; if it appears with odd exponent in k divisors and Ana colors the penultimate of these, then Beto can win by choosing the color in such a way that in the product of all red numbers the parity of the exponent of p is odd. Then, since n has an even number of divisors, Beto will finish the game, and since n is not a prime number there exists at least a prime p with odd exponent appearing at least twice, in such a way that Beto can force Ana to color the penultimate of these divisors, winning the game.

Problem 6. Let $n > 2$ be an even positive integer and $a_1 < a_2 < \ldots < a_n$ real numbers such that $a_{k+1} - a_k \leq 1$ for all k with $1 \leq k \leq n-1$. Let A be the set of pairs (i,j) with $1 \leq i < j \leq n$ and $j-i$ even, and let B be the set of pairs (i,j) with $1 \leq i < j \leq n$ and $j-i$ odd. Show that

$$\prod_{(i,j)\in A} (a_j - a_i) > \prod_{(i,j)\in B} (a_j - a_i).$$

Solution by Jordi Rodríguez, CFIS, BarcelonaTech, Barcelona, Spain. We will proceed by induction. We first prove the result for $n = 4$. In particular, we show that

$$\begin{aligned}(a_4 - a_2)(a_3 - a_1) &> (a_4 - a_1)(a_3 - a_2)\\ &\geq (a_4 - a_1)(a_3 - a_2)(a_4 - a_3)(a_2 - a_1).\end{aligned}$$

The last inequality is obvious by the constraints in the numbers. The first one is equivalent to

$$-a_4a_1 - a_2a_3 + a_1a_3 + a_2a_4 = (a_4 - a_3)(a_2 - a_1) > 0,$$

which is clearly true.

Assume now that the statement is true for $2n$ and show it for $2n+2$; if we are able to show that

$$\begin{aligned}&\prod_{i=1}^{n}((a_{2n+2} - a_{2i})(a_{2n+1} - a_{2i-1}))\\ \geq &\prod_{i=1}^{n}((a_{2n+2} - a_{2i-1})(a_{2n+1} - a_{2i}))\end{aligned}$$

we will be done by using the induction hypothesis.

However, it would be enough to show independently that, after renaming $a = a_{2n+2}$, $b = a_{2n+1}$, $c = a_{2i}$ and $d = a_{2i-1}$,

$$(a - c)(b - d) \geq (a - d)(b - c),$$

or what is the same

$$-ad - cb + bd + ac = (a - b)(c - d) \geq 0,$$

and this follows directly because $a \geq b$ and $c \geq d$.

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Solutions

No problem is ever permanently closed. We will be very pleased to consider new solutions or comments on past problems for publication.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

`jose.luis.diaz@upc.edu`

Elementary Problems

E–41. *Proposed by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom.* Prove that the square of the perimeter of a rectangle is at least 16 times its area.

Solution 1 by Laura Cánovas i Hidalgo, Universitat de Barcelona, Barcelona, Spain. Let the sides of the rectangle be a and b, the perimeter $2a + 2b$ and the area $a \cdot b$. Then,

$$\begin{aligned} & (2a+2b)^2 \geq 16ab \\ \iff & 4a^2 + 8ab + 4b^2 \geq 16ab \\ \iff & 4a^2 - 8ab + 4b^2 \geq 0 \\ \iff & (2a-2b)^2 \geq 0. \end{aligned}$$

The last inequality holds because the squares are never negative.

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. Denoting the length, width, perimeter, and area of a rectangle by l, w, P, and A, respectively, we apply the AM-GM inequality to see that

$$P^2 = 4(l+w)^2 \geq 4\left(2\sqrt{lw}\right)^2 = 16\,lw = 16A.$$

Equality holds if and only if the rectangle is in fact a square.

Solution 3 by Jose Pérez Cano, IES Alfonso XI, Alcalá la Real, Jaén, Spain. Geometrically, if we take a rectangle and put four of them like in one side of the picture we get the perimeter. Hence, the side of the square of Figure 1 is the perimeter of the rectangle.

Computing the total area, we have sixteen rectangles minus the four shading squares of the corners plus the big square in the center and four other squares. As we see, the total is 16 original rectangles plus some 4 other squares of area $(b-a)^2$. Therefore, the equality is given if, and only if, the original rectangle is a square.

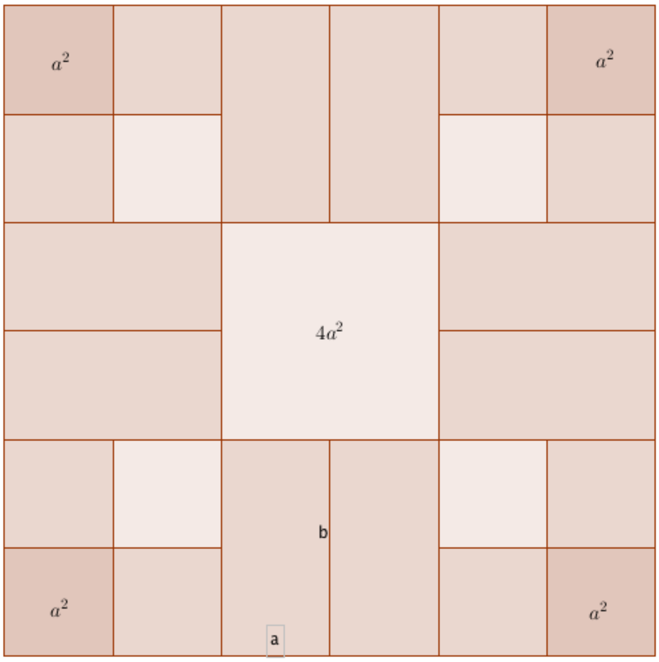

Figure 1: Sketch for Solution 3 of Problem E–41.

Solution 4 by the proposer. Let us prove that the inequality holds for any rectangle of area $\mathcal{A}$. Let one of the sides be x, which means that the other side is $\frac{\mathcal{A}}{x}$, and the perimeter is $p = 2x + 2\frac{\mathcal{A}}{x}$. By differentiating, we can find the point in which this expression takes a minimum value. Indeed, by considering the perimeter as a function of x we have that

$$p'(x) = 2 - 2\frac{\mathcal{A}}{x^2},$$

which equals 0 when $x = \sqrt{\mathcal{A}}$ (here we only consider the positive square root). This means that the minimum perimeter is $p_{\min} = 4\sqrt{\mathcal{A}}$, and $p \geq 4\sqrt{\mathcal{A}}$. Squaring this expression yields the result.

Also solved by *Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain; Padraig Condon, University of Birmingham, Birm-*

ingham, United Kingdom; Guillermo Girona San Miguel, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Jose Pérez Cano, IES Alfonso XI, Alcalá la Real, Jaén, Spain (one more solution); Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA (one more solution), and Isaac Sánchez Barrera, Barcelona Supercomputing Center (BSC).

E–42. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* For every integer $n \geq 1$ let t_n denote the n-th triangular number, defined by $t_n = \frac{n(n+1)}{2}$. Find the values of n for which

$$\frac{1^2 + 2^2 + \ldots + n^2}{t_1 + t_2 + \ldots + t_n}$$

is an integer number.

Solution 1 by Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain. We have that

$$1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Now, we have to find out what is the value of the denominator:

$$\begin{aligned}
\sum_{i=1}^{n} \frac{i(i+1)}{2} &= \frac{1}{2}\sum_{i=1}^{n}(i^2 + i) = \frac{1}{2}\left(\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right) \\
&= \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] \\
&= \frac{1}{12}[n(n+1)(2n+1) + 3n(n+1)] \\
&= \frac{1}{12}n(n+1)[(2n+1) + 3] \\
&= \frac{1}{12}n(n+1)(2n+4).
\end{aligned}$$

Therefore, we have that

$$\frac{1^2 + 2^2 + \ldots + n^2}{t_1 + t_2 + \ldots + t_n} = \frac{\frac{n(n+1)(2n+1)}{6}}{\frac{n(n+1)(2n+4)}{12}} = \frac{2(2n+1)}{2n+4} = \frac{2n+1}{n+2}.$$

We want this number to be an integer. However, at first sight it is difficult to say for which value of n that expression is an integer. In order to deduce it better, let's divide:

$$\frac{2n+1}{n+2} = 2 - \frac{3}{n+2}.$$

Now, it is more clear that that expression will be an integer if and only if $\frac{3}{n+2}$ is an integer. That is going to happen when $n+2$ divides 3, that is, when $n+2$ is equal to 1, 3, -1 or -3. Since n is a positive integer, the only two possibilities are that $n+2=1$ or $n+2=3$. If $n+2=1$, then $n=-1$, which cannot be our solution since n is positive. If $n+2=3$, then $n=1$.

Hence, the only solution is $n=1$.

Solution 2 by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom. We have that

$$\begin{aligned}
\frac{1^2+2^2+\ldots+n^2}{t_1+t_2+\ldots+t_n} &= \frac{\sum_{i=1}^{n} i^2}{\sum_{i=1}^{n} \frac{i(i+1)}{2}} = \frac{2\sum_{i=1}^{n} i^2}{\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i} \\
&= \frac{2\left(\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right)}{\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i} - 2\frac{\sum_{i=1}^{n} i}{\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i} \\
&= 2 - 2\frac{\sum_{i=1}^{n} i}{\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i}.
\end{aligned}$$

Now, this expression can only be an integer if $A_n := \dfrac{\sum_{i=1}^{n} i}{\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i}$

is half of an integer. Notice that $\sum_{i=1}^{n}(i^2+i) > \sum_{i=1}^{n} i$ for all $n \geq 1$ but A_n is always positive, so the only possible solution must be given when $A_n = \frac{1}{2}$. And this only happens if $\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} i$, which is obviously only true when $n = 1$, as the squares of natural numbers are bigger than the original numbers except for 1.

Also solved by *Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Jose Pérez Cano, IES Alfonso XI, Alcalá la Real, Jaén, Spain; Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA, and the proposer.*

E–43. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In each square of a 2017×2017 chessboard either a $+1$ or a -1 is written. Let r_i be the product of the numbers lying on the i-th row, and let c_j be the product of the numbers lying on the j-th column. Show that $r_1+r_2+\ldots+r_{2017}+c_1+c_2+\ldots+c_{2017} \neq 0$.

Solution by Fernando Ballesta Yagüe, Universidad de Murcia, Murcia, Spain. We are going to consider the chessboard as a matrix whose entries are the squares of the chessboard. Let's begin with the extreme case where all the entries of the matrix are 1's. Then, the result of summing all the row products and all the column products is:

$$\begin{aligned} & r_1 + r_2 + \ldots + r_{2017} + c_1 + c_2 + \ldots + c_{2017} \\ = & \overbrace{1+1+\ldots+1}^{2017\ 1's} + \overbrace{1+1+\ldots+1}^{2017\ 1's} = 2 \cdot 2017 = 4034. \end{aligned}$$

Now, let us see what happens when an entry is changed by its opposite number (that is, when a 1 is changed by a -1 or viceversa). Imagine we have a specific entry $a_{i,j}$. The values of the product of the numbers on the row and the ones on the column of this number are r_i and c_j. If they both are 1, then if we change the sign of $a_{i,j}$, now both are -1. So, the total result would decrease by 4 units (we had that $r_i + c_j = 1 + 1 = 2$ and now we have $r_i + c_j = -1 + (-1) = -2$). If both r_i and c_j were -1, then the case would be the opposite: by changing the sign, their value would be 1, so their sum would have passed from -2 to 2 (similarly, it would have been increased by 4 units). The only case that is left to consider is if either r_i or c_j is equal to 1 and the other to -1. By changing the sign of $a_{i,j}$ we would change the sign of r_i and c_j. As they were -1 and 1, now they would be 1 and -1, so the total result would still be 0 (in this case, $r_i + c_j$ does not change).

Therefore, we can conclude that there are only three possibilities: to increase the total value of the sum $r_1 + r_2 + \ldots + r_{2017} + c_1 + c_2 + \ldots + c_{2017}$ by 4, to decrease it by 4, or to leave it as it is. That means that, given the initial value 4034, by switching the sign of cells we can only add to it an integer multiple of 4. In particular, we can reach any possible state by switching the sign of cells. However, 4034 is not divisible by 4, so we are never going to be able to make the total sum 0 by subtracting 4 repeatedly.

Also solved by *Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom, and Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.*

E–44. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all positive integers that are divisible by 385 and have exactly 385 distinct positive divisors.

Solution by Jose Pérez Cano, IES Alfonso XI, Alcalá la Real, Jaén, Spain. We have that $385 = 5 \cdot 7 \cdot 11$. In order for any number to be divisible by 385 it must be of the form $N = 5^a 7^b 11^c q$ for some positive integers a, b, c and q. Now let us define $\sigma\colon \mathbb{N} \to \mathbb{N}$ such that, for any natural number $A = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, $\sigma(A) = (\alpha_1 + 1)(\alpha_2 + 1) \cdots (\alpha_n + 1)$. Then, $\sigma(A)$ is the number of divisors of A. We now have $\sigma(N) = 385 = 5 \cdot 7 \cdot 11$, so we can deduce from this that, in order for N to have exactly 385 divisors, $q = 1$ and a, b, c are some permutation of the values 4, 6, 10. Hence, all the solutions are

$$5^4 7^6 11^{10},\ 5^6 7^4 11^{10},\ 5^4 7^{10} 11^6,\ 5^{10} 7^4 11^6,\ 5^6 7^{10} 11^4,\ 5^{10} 7^6 11^4.$$

Also solved by *Alberto Espuny Díaz, University of Birmingham,*

Birmingham, United Kingdom; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA, and the proposer.

E–45. *Proposed by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom.* Given a regular pentagon of side length 1, find the triangle with the largest area contained inside it.

Solution by the proposer. First of all, all three vertices of the triangle will have to lay on the sides of the pentagon. Indeed, assume that were not the case, and that we have a triangle with maximum area, one of whose vertices is not on a side of the pentagon. Then, we can move this vertex perpendicularly to the opposite side of the triangle, which we consider the base, until it reaches a side, thus increasing the altitude of the triangle and, hence, its area. So the previous triangle did not have maximum area.

Furthermore, in order to compute the area we may assume that all three vertices of the triangle lay in vertices of the pentagon. Indeed, assume that we have a triangle of maximum area, one of whose vertices lays on a side of the pentagon, but not on a vertex. There are two possible cases. If the triangle side opposite this vertex is parallel to the side of the pentagon where the vertex lays, then the area does not change if the vertex is moved to one of the vertices of the pentagon side, so we may assume that it is in a vertex. If, on the contrary, they are not parallel, then the area will increase by taking the triangle vertex to one of the endpoints of the pentagon side, so the original triangle was not of maximum area.

As the triangle has three vertices and the pentagon has five, in any distribution of the vertices of the triangle among those of the pentagon there will be at least two adjacent vertices, that is, one of the sides of the triangle will coincide with one of the sides of the pentagon. We may think of this side as the base of the triangle. It is now clear that the area will be maximized for the vertex opposite this side, as it maximizes the triangles's altitude.

Note that this isosceles triangle is of maximum area, but not the only one. Any of the longer sides is parallel to one side of the pentagon, so moving the opposite vertex along that pentagon side will yield different triangles with the same area.

Also solved by *Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.*

E–46. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $ABCD$ be a trapezium with bases $AB = a$ and $CD = b$, respectively. Let M be a point on AD such that $\beta\, MA = \alpha\, MD$ for some reals α and β. If the parallel to the bases drawn from M meets BC at N, then show that

$$MN \geq a^{\frac{\beta}{\alpha+\beta}} \cdot b^{\frac{\alpha}{\alpha+\beta}}.$$

Solution 1 by the proposer. Since in $\triangle ACD$ we have $\dfrac{MA}{MD} = \dfrac{\alpha}{\beta}$, then

$$\frac{MA}{AD} = \frac{MA}{MA+MD} = \frac{MA}{MA+\frac{\beta}{\alpha}MA} = \frac{\alpha}{\alpha+\beta}.$$

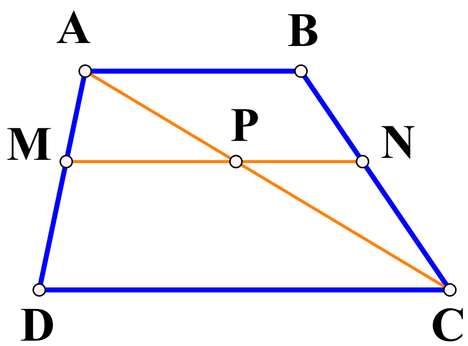

Figure 2: Sketch for Solution 1 of Problem E–46.

On the other hand, on account of Thales theorem's, we have

$$\frac{MA}{AD} = \frac{MP}{DC} = \frac{MP}{b} = \frac{\alpha}{\alpha+\beta},$$

from which it follows that $MP = \dfrac{\alpha\, b}{\alpha+\beta}$. Likewise, from triangle ABC we obtain $PN = \dfrac{\beta\, a}{\alpha+\beta}$ and

$$MN = MP + PN = \frac{\alpha\, b}{\alpha+\beta} + \frac{\beta\, a}{\alpha+\beta} = \frac{\alpha\, b + \beta\, a}{\alpha+\beta}.$$

Now applying powered AM-GM inequality to the numbers b and a with powers $w_1 = \dfrac{\alpha}{\alpha+\beta}$ and $w_2 = \dfrac{\beta}{\alpha+\beta}$ yields

$$MN = \frac{\alpha\, b + \beta\, a}{\alpha+\beta} \geq a^{\frac{\beta}{\alpha+\beta}} \cdot b^{\frac{\alpha}{\alpha+\beta}}.$$

Solution 2 by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom. Let us define a function $\ell(x)$ that gives the length of a parallel to the base at any point x in the line defined by AD. We have that $\ell(0) = a$ and $\ell(1) = b$, where we have taken x to be a variable that takes value 0 for AB and 1 for DC. By similarities, the function that measures the distance between two points in lines AD and BC in a parallel to AB must be linear. The only linear function that fulfills these conditions is $\ell(x) = a + x(b-a)$.

For the point M we have $x = \frac{AM}{AD} = \frac{\alpha}{\alpha+\beta}$. Hence,

$$MN = \frac{\beta}{\alpha+\beta}a + \frac{\alpha}{\alpha+\beta}b.$$

To complete the proof, consider the weighted AM-GM inequality with weights $\frac{\beta}{\alpha+\beta}$ and $\frac{\alpha}{\alpha+\beta}$, which directly yields

$$MN \geq a^{\frac{\beta}{\alpha+\beta}} \cdot b^{\frac{\alpha}{\alpha+\beta}}.$$

Also solved by *Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.*

Easy–Medium Problems

EM–41. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all solutions of the equation $p(x) = q(x)$, where $p(x)$ and $q(x)$ are polynomials of degree 2 with leading coefficient one such that

$$\sum_{k=1}^{n} p(k) = \sum_{k=1}^{n} q(k),$$

and n is a positive integer.

Solution by the proposer. Let $p(x) = x^2 + ax + b$ and $q(x) = x^2 + cx + d$. Then equation $p(x) - q(x) = 0$ or $p(x) = q(x)$ can be written as

$$ax + b = cx + d$$

If the numbers a and c are distinct, then the solution is unique. Indeed, since

$$\sum_{k=1}^{n} p(k) = (1 + 2 + \ldots + n)\, a + nb = \frac{n(n+1)}{2}\, a + nb$$

and

$$\sum_{k=1}^{n} q(k) = (1 + 2 + \ldots + n)\, c + nd = \frac{n(n+1)}{2}\, c + nd,$$

then we get

$$\frac{n(n+1)}{2}\, a + nb = \frac{n(n+1)}{2}\, c + nd \Leftrightarrow \frac{n+1}{2}\, a + b = \frac{n+1}{2}\, c + d$$

From the preceding we conclude that the only solution of $p(x) - q(x) = 0$ is $x = \dfrac{n+1}{2}$. If $p(x) = q(x)$ then any real number x is a solution. On the other hand, if $a = c$ and $b \neq d$ there are no solutions, and we are done.

Also solved by *Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Jose Pérez Cano, IES Alfonso XI, Alcalá la Real, Jaén, Spain, and Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA.*

EM–42. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* If 2017 points on a circle are joined by straight lines in all possible ways and no three of these lines meet at a single point inside the circle, then find the number of triangles that can be formed.

Solution by the proposer. We solve the problem for the general case of n points on the circle. We distinguish the following cases:

1. We have triangles with all vertices lying inside the circle. These are formed by exactly three chords, which in turn are formed by exactly six points on the circumference.

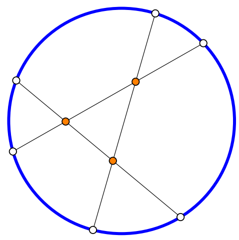

Figure 3: Illustration of case 1.

   Therefore, the number of triangles with all three vertices inside the circle is the number of subsets of six points from the n given points. That is, $\binom{n}{6}$.
2. We also have triangles with exactly two vertices lying inside the circle. These are formed by chords emanating from exactly five points on the circumference.
   However, each set of five circumference points forms with its chords exactly five such triangles. Thus, the number of triangles with exactly two vertices inside the circle is five times the number of subsets of fives points from the n points. That is, $5\binom{n}{5}$.
3. Other triangles are those with exactly one vertex lying inside the circle. These are formed by chords emanating from exactly four points on the circumference. However, each group of four circumference points forms with its chords exactly four such triangles.

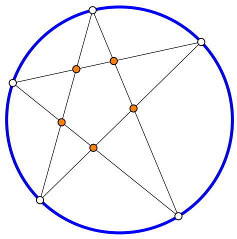

Figure 4: Illustration of case 2.

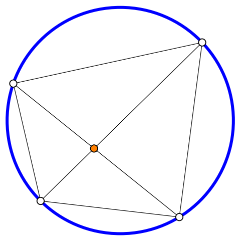

Figure 5: Illustration of case 3.

Therefore, the number of triangles with exactly one vertex inside the circle is four times the number of subsets of four points from the n points. That is, $4\binom{n}{4}$.

4. Finally, there remain only the triangles with all three vertices on the circumference. The number of such triangles is the number of subsets of three points from the n points. That is, $\binom{n}{3}$.

Thus the total number of triangles is

$$N(n) = \binom{n}{6} + 5\binom{n}{5} + 4\binom{n}{4} + \binom{n}{3},$$

and in the particular case when $n = 2017$ we have

$$N(2017) = 94212958848915816.$$

Also solved by *Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.*

EM–43. *Proposed by Nicolae Papacu, Slobozia, Romania.* Let a, b, c be three positive real numbers such that $\sqrt{a}+\sqrt{b}+\sqrt{c}=1$. Prove that

$$\frac{\sqrt{a}}{a^2+2bc}+\frac{\sqrt{b}}{b^2+2ca}+\frac{\sqrt{c}}{c^2+2ab}\le\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Solution 1 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. By the AM-GM inequality we have

$$\begin{aligned}\sum_{\text{cyclic}}\frac{\sqrt{a}}{a^2+2bc}&\le\sum_{\text{cyclic}}\frac{\sqrt{a}}{a^2+2((b^2+c^2)/2)}\\&=\frac{1}{a^2+b^2+c^2}\sum_{\text{cyclic}}\sqrt{a}\\&=\frac{1}{a^2+b^2+c^2}.\end{aligned}$$

But the Cauchy-Schwarz inequality gives us

$$\begin{aligned}1=\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2&=\left(a\cdot\frac{1}{\sqrt{a}}+b\cdot\frac{1}{\sqrt{b}}+c\cdot\frac{1}{\sqrt{c}}\right)^2\\&\le(a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),\end{aligned}$$

which implies that

$$a^2+b^2+c^2\ \ge\ \frac{1}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}},\quad\text{or}\quad\frac{1}{a^2+b^2+c^2}\ \le\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

and finishes the proof.

Solution 2 by the proposer. Applying Cauchy's inequality to the vectors $\vec{u}=(a,\sqrt{bc},\sqrt{bc})$ and $\vec{v}=\left(\frac{1}{\sqrt{a}},\frac{1}{\sqrt{b}},\frac{1}{\sqrt{c}}\right)$, we have

$$\begin{aligned}1=(\sqrt{a}+\sqrt{b}+\sqrt{c})^2&=\left(a\cdot\frac{1}{\sqrt{a}}+\sqrt{bc}\cdot\frac{1}{\sqrt{b}}+\sqrt{bc}\cdot\frac{1}{\sqrt{c}}\right)^2\\&\le(a^2+bc+bc)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),\end{aligned}$$

from which it follows that

$$\frac{\sqrt{a}}{a^2+2bc}\sqrt{a}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \quad \text{(cyclic)}.$$

Adding up the preceding inequalities yields

$$\frac{\sqrt{a}}{a^2+2bc}+\frac{\sqrt{b}}{b^2+2ca}+\frac{\sqrt{c}}{c^2+2ab} \le (\sqrt{a}+\sqrt{b}+\sqrt{c})\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Equality holds when $a = b = c = 1/9$, and we are done.

EM–44. *Proposed by Andrés Sáez-Schwedt, Universidad de León, León, Spain.* Let ABC be a triangle with $AB = AC > BC$, and let O be the center of its circumcircle Γ. The tangent to Γ at C meets the line AB at D. In the minor arc AC of Γ, consider the point E such that

$$\angle EOC + 2\angle DOA = 360^\circ.$$

If BE meets CD at F, show that $FA = FC$.

Solution 1 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany. Let P be the point of intersection of the tangents to Γ through A and B. Let $\alpha = \angle BCA = \angle PBA$. Since the triangle ABP is isosceles, we have $AB = 2BP\cos\alpha$, and $AD = BD + 2BP\cos\alpha$. We obtain the following identities:

$$CD^2 = BD\cdot AD = BD^2 + 2BD\cdot BP\cos\alpha,$$

$$AP^2 + CD^2 = BP^2 + BD^2 - 2BD\cdot BP\cos(180^\circ - \alpha) = DP^2,$$

$$OP^2 + CD^2 = OA^2 + AP^2 + CD^2 = OC^2 + DP^2.$$

From this last equation we conclude that $OD \perp CP$. The angles $\angle PCB$ and $\angle DOA$ add up to 180°, since the sides are perpendicular. Thus $\angle DOA = \frac{1}{2}\angle EOC = \angle EBC = \angle FBC$.

We claim that P and F are symmetric with respect to OA. Indeed, F is in the tangent to Γ through C, so its symmetric must be in the tangent to Γ through B. Moreover, it must be the point F' on that tangent satisfying $\angle F'CB = \angle FBC$. That point is P. Then $FA = FC$ follows from $PA = PB$.

Solution 2 by the proposer. Since $\angle EOC + 2\angle DOA = 360°$, there exists a point E' on the minor arc AB of Γ such that

$$\angle E'OA = \angle AOE \qquad \text{and} \qquad \angle DOE' = \angle COD,$$

that is, E' is at the same time the symmetric point of E with respect to AO and the reflection of C with respect to DO. Thus, DE' is the second tangent from D to Γ.

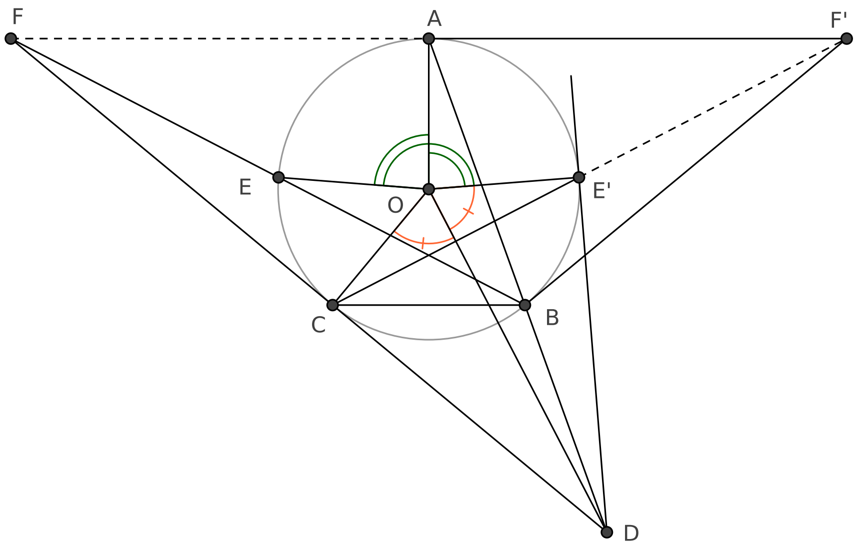

Figure 6: Scheme for problem EM-44.

Let F' be the point of intersection of the tangents to Γ at A and B. Now, D is a point of AB, the polar line of F' with respect to the circle Γ, so F' must lie on the polar of D, which is line CE', i.e. C, E', F' are collinear.

But then, F and F' must be symmetric points with respect to AO, because they are defined as the intersection of symmetric lines. Hence, $FC = F'B = F'A = FA$.

EM–45. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $A_1, A_2, \ldots, A_n$ be the vertices of an n-gon inscribed in a circle of center O. If O and the A_i's are lattice points, then prove that the square of the perimeter of the n-gon is an even number.

Solution. The statement is not correct. It has been corrected and published as Problem EM–47 [Correction].

EM–46. *Proposed by Ángel Plaza de la Hoz, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain.* Prove that for any positive integer n the chain of inequalities

$$F_n^{1/L_n} \le F_{2n}^{1/(2L_n)} \le \frac{F_n^{1/F_n} + L_n^{1/L_n}}{2} \le F_{2n}^{1/(2F_n)} \le L_n^{1/F_n}$$

holds, where F_n is the n-th Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and for $n \ge 2$, $F_n = F_{n-1} + F_{n-2}$, and L_n is the n-th Lucas number, defined by $L_0 = 2$, $L_1 = 1$, and for $n \ge 2$, $L_n = L_{n-1} + L_{n-2}$.

Solution 1 by Jose Pérez Cano, IES Alfonso XI, Alcalá la Real, Jaén, Spain. Let's go step by step

1. $F_n^{1/L_n} \le F_{2n}^{1/(2L_n)} \iff F_n^2 \le F_{2n}$. In order to prove the latter, the next result is needed:

$$F_{2n} = F_{n+1} \cdot F_n + F_n \cdot F_{n-1}.$$

By induction, for $n = 2$, $F_4 = F_3 \cdot F_2 + F_2 \cdot F_1 = 2 \cdot 1 + 1 \cdot 1 = 3$. Now, considering it true up to some n,

$$F_{2(n+1)} = F_{2n+1} + F_{2n} = 2F_{2n} + F_{2n-1} = 3F_{2n} - F_{2(n-1)}.$$

And now, applying the hypothesis of induction,

$$\begin{aligned} 3F_{2n} - F_{2(n-1)} &= 3(F_{n+1}F_n + F_nF_{n-1}) - (F_nF_{n-1} + F_{n-1}F_{n-2}) \\ &= 3F_{n+1}F_n + 2F_nF_{n-1} - (F_{n+1} - F_n)(2F_n - F_{n+1}) \\ &= 2F_nF_{n-1} + F_{n+1}^2 + 2F_n^2 = 2F_nF_{n+1} + F_{n+1}^2 \\ &= F_{n+2}F_{n+1} + F_{n+1}F_n. \end{aligned}$$

Using this result, the statement to prove is equivalent to

$$F_n^2 \le F_n(F_{n+1} + F_{n-1}) \iff F_n \le F_{n+1} + F_{n-1} = F_n + 2F_{n-1},$$

which is self-evident since every term is positive. Finally, equality holds if and only if $F_{n-1} = 0$ or, similarly, if and only if $n = 1$.

2. $F_{2n}^{\frac{1}{2L_n}} \le \frac{F_n^{\frac{1}{F_n}} + L_n^{\frac{1}{L_n}}}{2}$. Applying the AM-GM inequality is enough to prove that

$$F_{2n}^{\frac{1}{L_n}} \le F_n^{\frac{1}{F_n}} L_n^{\frac{1}{L_n}} \iff L_n \ge \frac{F_{2n}}{F_n^{\frac{F_n}{L_n}}}.$$

We get now that

$$\begin{aligned} \frac{F_{2n}}{F_n^{\frac{F_n}{L_n}}} &= \frac{F_n(F_{n+1} + F_{n-1})}{F_n^{\frac{F_n}{L_n}}} = \frac{F_n + 2F_{n-1}}{F_n^{\frac{F_n}{L_n}-1}} \\ &\le \frac{F_n + 2F_{n-1}}{F_n} < 3 < L_n, \end{aligned}$$

where in the last term we use the fact that $L_n \ge 2F_n$ for $n \ne 1$, which can be proven by induction. The cases where $L_n \le 3$ can be checked one by one since they are just $n = 0$, 1 and 2 and, as before, equality only holds for $n = 1$.

3. $\frac{F_n^{\frac{1}{F_n}} + L_n^{\frac{1}{L_n}}}{2} \le F_{2n}^{\frac{1}{2F_n}}$. Using the first inequality I have proven, it is enough with proving

$$\frac{F_n^{\frac{1}{F_n}} + L_n^{\frac{1}{L_n}}}{2} \le F_n^{\frac{1}{F_n}} \iff F_n^{\frac{1}{F_n}} \ge L_n^{\frac{1}{L_n}}.$$

Consider now the function $f(x) = \frac{ln(x)}{x}$. When differentiating, we notice that it is strictly decreasing for $x > e$, as $f'(x) = \frac{1-ln(x)}{x^2}$. Since $L_n > F_n$, $f(F_n) > f(L_n)$ and taking exponentials at both sides we are done. The remaining cases $n = 0$, 1 and 2 can be tested one by one, and equality only holds for $n = 1$.

4. $F_{2n}^{\frac{1}{2F_n}} \le L_n^{\frac{1}{F_n}} \iff F_{2n} \le L_n^2 \Leftrightarrow F_n(F_{n+1} + F_{n-1}) = F_n^2 + 2F_nF_{n-1} \le 3F_n^2 < (2F_n)^2 \le L_n^2$. Since I have used the fact that $L_n \ge 2F_n$, which only holds for $n \ne 1$, equality will only hold for $n = 1$.

Solution 2 by the proposer. The third inequality follows by the AM-GM inequality and by the fact that $L_n \geq F_n$ for $n \geq 1$, so

$$\frac{F_n^{1/F_n} + L_n^{1/L_n}}{2} \leq \sqrt{F_n^{1/F_n} L_n^{1/L_n}} \leq \sqrt{(F_n L_n)^{1/F_n}} = F_{2n}^{1/(2F_n)}$$

since $F_n L_n = F_{2n}$.

The fourth inequality follows since $\sqrt{F_{2n}} = \sqrt{F_n L_n} \leq L_n$.

The first inequality follows similarly, because $F_n \leq \sqrt{F_n L_n} = \sqrt{F_{2n}}$.

Finally, the second inequality follows also by the AM-GM inequality since

$$F_{2n}^{1/(2L_n)} = \sqrt{F_n^{1/L_n} \cdot L_n^{1/L_n}} \leq \frac{F_n^{1/L_n} + L_n^{1/L_n}}{2} \leq \frac{F_n^{1/F_n} + L_n^{1/L_n}}{2}.$$

Medium–Hard Problems

MH–41. *Proposed by Mihály Bencze, Braşov, Romania.* Determine

$$\left[\sum_{k=1}^{n}(k+1)^{\frac{1}{k^4+k^2+1}}\right],$$

where $[x]$ denotes the integer part of x.

Solution by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. We show that the integer part is n.

First of all, we have

$$\sum_{k=1}^{n}(k+1)^{\frac{1}{k^4+k^2+1}} > \sum_{k=1}^{n} 1 = n. \tag{1}$$

Next, we apply Bernoulli's inequality —$(1+x)^r \leq 1+rx$ for $x \geq 1$ and $r \in [0,1]$— to see that

$$\begin{aligned}\sum_{k=1}^{n}(k+1)^{\frac{1}{k^4+k^2+1}} &\leq \sum_{k=1}^{n}\left(1+\frac{k}{k^4+k^2+1}\right) = n + \sum_{k=1}^{n}\frac{k}{k^4+k^2+1}\\ &= n + \frac{1}{2}\sum_{k=1}^{n}\left(\frac{1}{k^2-k+1} - \frac{1}{k^2+k+1}\right)\\ &= n + \frac{1}{2}\left(1 - \frac{1}{n^2+n+1}\right) < n + \frac{1}{2}. \end{aligned} \tag{2}$$

Now we see that (1) and (2) imply that the integer part of the sum is n.

Also solved by *the proposer.*

MH–42. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In Mathcontestland there are 2017 towns. Every pair of towns is either connected by a single road, or is not connected. If we consider any subset of 2015 towns, the total number of roads connecting these towns to each other is a constant. If there are R roads in Mathcontestland, then find all possible values of R.

Solution 1 by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom. Two possible solutions are when there are no roads ($R = 0$) and when there is a road between every two towns, in which case $R = \frac{2017 \cdot 2018}{2} = 2033136$ and there are 2029105 roads in any subset of 2015 towns. We claim that these are the two unique solutions to this problem.

We work in general with n towns (the solutions, then, are given by $R = 0$ and $R = \binom{n}{2}$). For ease of notation, we label the towns as $a_1, a_2, \ldots, a_n$. Let d_i, $i \in \{1, \ldots, n\}$, be the number of roads that start from a_i. Notice that $\sum_{i=1}^{n} d_i = 2R$, as we count each road twice, once from each city it connects. Given that R is a value fixed by the statement, the condition that the number of roads connecting any subset of $n-2$ cities is constant is equivalent to saying that the number of roads incident to any pair of cities (that is, the number of roads that go from each of the cities to the remaining $n-2$ cities plus the possible road between the two cities) is constant too. We will refer to this number as $d_{i,j}$.

Now assume that there are two towns a_i and a_j such that $d_i \leq d_j - 2$. Then, $d_{i,k} \leq d_i + d_k < d_j + d_k - 1 \leq d_{j,k}$ for any $k \neq i, j$, which is a contradiction of the statement hypothesis. So all towns must have d_i equal to d or $d+1$, for some fixed value $d \in \{0, 1, \ldots, n-2\}$.

Assume that there are at least two towns with d roads reaching them (say, a_i and a_j) and at least two with $d+1$ (a_k and a_ℓ). Then, $d_{i,j} \leq 2d < 2d+1 \leq d_{k,\ell}$, so we reach a new contradiction.

If there is only one town with $d+1$ roads incident to it (say, a_i), then choose a town a_j it is connected to and a town a_k that is connected to a_j. Then, $d_{j,k} = 2d - 1 < 2d = d_{i,j}$, so we reach a new contradiction. If there are no such towns a_j and a_k, that means that there is only one road reaching a_j, so $d_j = d = 1$. In such a case, choose any two towns connected by a road, say a_ℓ and a_m. Then $d_{\ell,m} = 0 < 1 \leq d_{i,k}$ for any k, a contradiction again.

If, on the contrary, there is only one town with d roads reaching it (say, a_i), then choose a town a_j that is connected to a_i. Then, for any $k \neq i$, j, $d_{i,j} = 2d < 2d + 1 \leq d_{j,k}$, a new contradiction. If no such a_j exists, that must mean $d_i = 0$, but then, taking any two towns a_ℓ and a_m which are not connected (which must exist, as there is at most one road reaching each town) we have $d_{i,\ell} = 1 < 2 = d_{\ell,m}$, a new contradiction. Hence, we must have that all towns have the same number of roads incident to them, d.

Now, the cases $d = 0$ and $d = n - 1$ give the two solutions we already discussed, so assume $d \in \{2, 3, \dots, n-2\}$. Then, no town is connected to every other town, and every town is connected to some town. Choose any town a_i, and choose a town a_j to which it is connected by a road and a town a_k to which it is not. Then, $d_{i,j} = 2d - 1 < 2d = d_{i,k}$, so we reach one last contradiction. We have exhausted all possible cases, so there can be no other solution to the problem.

Solution 2 by Sophie Tandonnet, University of Warwick, Coventry, United Kingdom, and Tássio Naia, University of Birmingham, Birmingham, United Kingdom. Either every pair of towns is connected, or there are no roads (i.e., $R \in \{0, 2017 \times 2016/2\}$). Our solution relies on two claims, which are more general than strictly necessary to solve the problem. In particular, they allow us to understand what happens to R when 2017 and 2015 are replaced, respectively, by positive integers n and s, where $3 < s < n$.

We begin by introducing some some notation. Let V be a set of towns. Instead of thinking of towns as connected (or not) by a road, let us say that each town is connected to all others, but some of the roads are blocked and cannot be used. For all distinct towns $x, y \in V$, we write xy to refer to the road connecting x and y; let us say that x is a *red neighbour* of y (or vice-versa) if xy is blocked; we say that x is a *green neighbour* of y (or vice-versa) otherwise. In the same spirit, we say that a road is *red* if it is blocked and *green* otherwise, and we say that a subset $S \subseteq V$ is *monochromatic* if all roads between towns in S have the same colour. Also, if s is an integer, we shall say that V is *s-uniform* if there exists a constant m such that for all $S \subseteq V$ with $|S| = s$ there are precisely m green roads among towns in S. (In the problem V is 2015-uniform.) Finally, for all $S \subseteq V$, we say that S is *d-regular* if every town in S has precisely d green neighbours (and $|S| - d - 1$ red neighbours) in S; we say that S is *regular* if it is d-regular for some integer d.

Claim 1. *If $s = |V| - 1 > 2$ and V is s-uniform, then V is regular.*

Proof. Fix $x \in V$. Let g be the number of green roads in V, let k be the number of green roads in $V - x$, and, for each $x \in V$, let $d(x)$ be the number of green neighbours of x. Note that for all $x, y \in V$ we have $g = d(x) + k = d(y) + k$, so $d(x) = d(y)$ and thus V is regular. $\square$

Claim 2. *If $|V| > 3$ and $V \setminus \{x\}$ is regular for all $x \in V$, then V is monochromatic.*

Proof. We begin by observing that if all roads leading to some town x have the same colour c, then all roads of V must have colour c, so V is monochromatic. (Indeed, choose $y \in V \setminus \{x\}$ and note that $V \setminus \{y\}$ is monochromatic with colour c; therefore, for all $w, z \in V \setminus \{x, y\}$ the road wz has colour c, and so do all roads in $V \setminus \{x\}$ by the choice of y.) So we may assume (looking for a contradiction) that every town has at least one green neighbour and one red neighbour. Let $a \in V$ be an arbitrarily chosen town. Then a has both a green neighbour v_g and a red neighbour v_r.

We now show that for each $b \in V \setminus \{a, v_g, v_r\}$, the road bv_g is green. Let d be the number of green neighbours of v_g in $V \setminus \{b\}$. Since $V \setminus \{b\}$ is regular, it follows that v_r has d green neighbours in $V \setminus \{b\}$. Furthermore, v_g has $d - 1$ green neighbours in $V \setminus \{a, b\}$ and v_r has d green neighbours in $V \setminus \{a, b\}$; therefore, since $V \setminus \{a\}$ is regular, v_g is a green neighbour of b.

To conclude, recall that $V \setminus \{b\}$ is regular, and let d be the number of green neighbours of each town in $V \setminus \{b\}$. By our asumption, every town has a red neighbour, so v_r is the only red neighbour of v_g. This in turn implies that $d \geq |V| - 2$; however, both a and v_g are red neighbours of v_r, so $d < |V| - 3$, a contradiction. $\square$

We now combine the claims. Let V be a set of 2017 towns. Since V is 2015-uniform, every subset of V with 2016 elements is regular (by Claim 1) and therefore V is monochromatic (by Claim 2). Therefore, either every road is green (unblocked) or every road is red (blocked).

Solution 3 by the proposer. We consider the general case with n towns. Let K denote the **constant** number of roads connecting any subset of $n-2$ towns and let $c_{ij} \in \{0,1\}$ denote the number of roads connecting town i and town j. Finally, for $i = 1, 2, \ldots, n$ let d_i denote the total number of roads connected to town i. Note that

$$R \leq \binom{n}{2} = \frac{n(n-1)}{2}.$$

Clearly,

$$\sum_{i=1}^{n} d_i = 2R \quad \text{and} \quad \sum c_{ij} = R,$$

where the latter sum is over all $2-$element subsets $\{i, j\}$ of the set $\{1, 2, \ldots, n\}$. The number of roads connected to at least one of the towns with number i or j is equal to $d_i + d_j - c_{ij}$. Thus, for any $2-$element subsets $\{i, j\} \subset \{1, 2, \ldots, n\}$, we have

$$K = R - d_i - d_j + c_{ij}.$$

Adding all these equations for every $2-$element subset $\{i, j\}$ yields

$$\binom{n}{2} K = \binom{n}{2} R - 2(n-1)\, R + R,$$

which may be written as

$$n(n-1)\, K = (n-2)(n-3)\, R.$$

Note that both $n(n-1)$ and $(n-2)(n-3)$ are divisible by 2, and that the only integer $k > 2$ which divides both $n(n-1)$ and $(n-2)(n-3)$ is 3, this latter case occurring if and only if n is divisible by 3. Since 3 does not divide 2017, in the situation of the given problem $n(n-1)/2$ and $(n-2)(n-3)/2$ are coprime. Hence, R is a multiple of $n(n-1)/2$. As $R \leq n(n-1)/2$ with equality when all the pairs of towns a are connected, the only possibilities are $R = n(n-1)/2$ or $R = 0$. Therefore, the total number of roads is

$$R = 2017 \cdot 1008 = 2033136 \quad \text{or} \quad R = 0.$$

Also solved by *Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.*

MH–43. *Proposed by Andrés Sáez-Schwedt, Universidad de León, León, Spain.* Points A, B, C and D are collinear in that order. On a circle ω through B and C, two new points E, F are chosen, such that lines AE and DF meet on ω. The tangents to ω at B and C meet at G (possibly at infinity). The tangents to ω at E and F meet GA and GD at P and Q, respectively. Prove that line PQ is tangent to ω.

Solution by the proposer. Define $H = AE \cap DF$ and let PH' (with $H' \neq E$) be the second tangent from P to ω. Also, $I = EH' \cap BC$ and E' is the second point of intersection of HI with ω. Note that I is the intersection of BC, the polar line of G with respect to the circle ω, and EH', the polar of P, so that PG is the polar of I. Being A, P, G collinear, the polar of A must also pass through I, and also through G, because A lies on the polar of G. Therefore, AIG is a self-polar triangle.

Now, since I is one of the diagonal points of the cyclic quadrilateral $EHH'E'$, the other two diagonal points must form a self-polar triangle with I. Line $E'H'$ must cut EH at a point in the polar of I, a point which must necessarily be A, i.e. A, E', H' are collinear. Similarly, the third diagonal point $EE' \cap HH'$ must lie on the polars of A and I, so it can only be G, the pole of AI, which means that EE' and HH' pass through G.

What we have proved up to now is the following: the second tangent from P to ω is PH', where H' is the second point of intersection of GH with ω. Repeating the same reasoning with Q, the second tangent from Q to ω is QH', for the same point H' as before. Thus, PH' and QH' are the same line, tangent to ω.

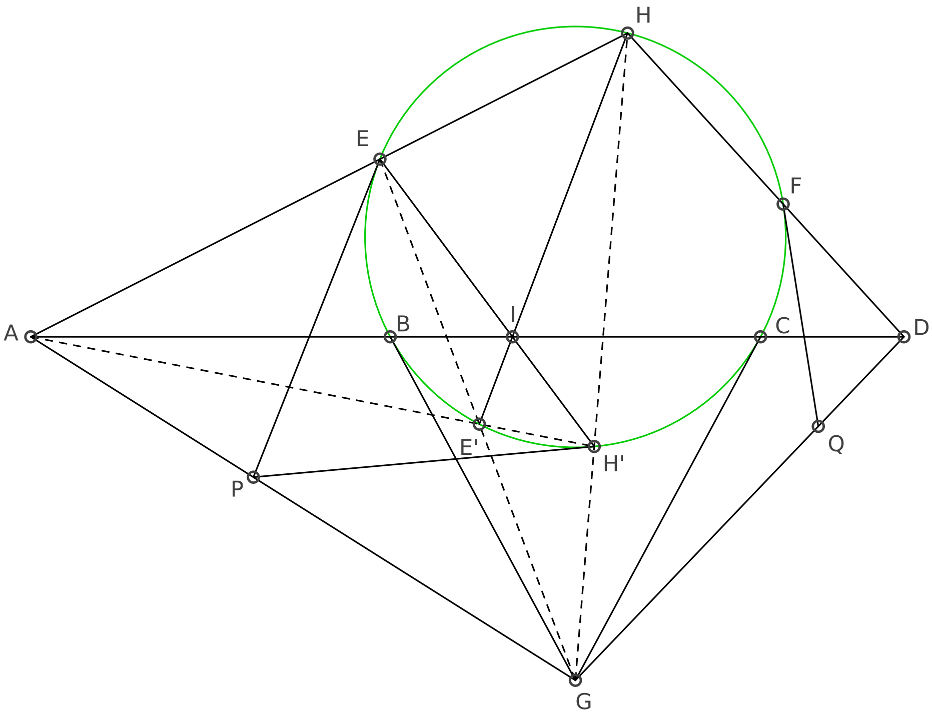

Figure 7: Scheme for problem MH-45

MH–44. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let x, y and z be be three nonzero real numbers such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{xyz}$. For all positive reals a, b, c, prove that

$$\frac{(a+b)x+(b+c)y+(c+a)z}{\sqrt{ab+bc+ca}} \geq 2.$$

Solution by the proposer. We have

$$\begin{aligned}
&(a+b)x+(b+c)y+(c+a)z\\
&=(a+b+c)\,(x+y+z)-(ax+by+cz)\\
&=\sqrt{(a+b+c)^2\,(x+y+z)^2}-(ax+by+cz)\\
&=\sqrt{(a^2+b^2+c^2+2(ab+bc+ca))(x^2+y^2+z^2+2(xy+yz+zx))}\\
&\quad-(ax+by+cz).
\end{aligned}$$

Applying CBS to the vectors $\vec{u} = (\sqrt{2(ab+bc+ca)}, \sqrt{a^2+b^2+c^2})$ and $\vec{v} = (\sqrt{2(xy+yz+zx)}, \sqrt{x^2+y^2+z^2})$ yields

$$\begin{aligned} &2\sqrt{(ab+bc+ca)(xy+yz+zx)}+\sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)} \\ \le &\sqrt{(a^2+b^2+c^2+2(ab+bc+ca))(x^2+y^2+z^2+2(xy+yz+zx))}. \end{aligned}$$

Writing the constrain in the form $xy + yz + zx = 1$ we get

$$\begin{aligned} &(a+b)x + (b+c)y + (c+a)z \\ = &(a+b+c)\,(x+y+z) - (ax+by+cz) \\ \ge &2\sqrt{(ab+bc+ca)}+\sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)}-(ax+by+cz). \end{aligned}$$

Again, applying CBS to the vectors $\vec{r} = (a, b, c)$ and $\vec{s} = (x, y, z)$ we have

$$\sqrt{(a^2+b^2+c^2)\,(x^2+y^2+z^2)} - (ax+by+cz) \ge 0,$$

and

$$(a+b)x + (b+c)y + (c+a)z \ge 2\,\sqrt{(ab+bc+ca)}$$

follows. Equality holds when $a = b = c$ and $x = y = z = \sqrt{3}/3$, and we are done.

Also solved by *Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.*

MH–45. *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.* We say that a set of positive integers S is good if there is a function $f : \mathbb{N} \to S$ such that no integer k with $2 \le k \le 2017$ can be written as $\frac{xf(x)}{yf(y)}$. Find the smallest positive integer n such that $S = \{1, 2017, 2017^2, \dots, 2017^n\}$ is good, or prove that such an integer does not exist.

Solution by the proposer. We claim that $n = 10$ is the smallest value. To see that the set S cannot be good with $n \le 9$, we note that in this case $|S| = n + 1 \le 10$. By pigeonhole principle, two of the numbers $1, 2, 2^2, \dots, 2^{10}$ have the same image by f. If these values are 2^i and 2^j with $i < j$, setting $x = 2^j$ and $y = 2^i$ we obtain $\frac{xf(x)}{yf(y)} = \frac{2^j f(2^j)}{2^i f(2^i)} = 2^{j-i}$, which is an integer between 2 and $2^{10} = 1024$.

Now we want to construct a function f for $n = 10$. Let $g(k)$ denote the number of prime numbers in the factorization of k, counted with multiplicity (for example, $g(2^a 3^b) = a + b$). Then we see that the only positive integer with $g(k) = 0$ is $k = 1$, and the smallest k with $g(k) \geq 11$ is $k = 2^{11} = 2048$. Hence $1 \leq g(k) \leq 10$ for every $2 \leq k \leq 2017$.

Notice that $g(2017) = 1$ and $g(2017^k) = k$. Since 1 and 11 are relatively prime, the numbers $g(1), g(2017), \ldots, g(2017^{10})$ have different residues when dividing by 11. For every n, choose $f(n)$ from S so that $g(n) + g(f(n))$ is divisible by 11. That way, if $yf(y)$ divides $xf(x)$ for some x and y, then we have

$$g\left(\frac{xf(x)}{yf(y)}\right) = \underbrace{g(x) + g(f(x))}_{\text{divisible by 11}} - \underbrace{(g(y) + g(f(y)))}_{\text{divisible by11}}$$

which is divisible by 11. In particular, $\frac{xf(x)}{yf(y)}$ cannot be an integer between 2 and 2017. This concludes our proof.

MH–46. *Proposed by Ismael Morales López, Universidad Autónoma de Madrid, Madrid, Spain.* Let a, b, c be the lengths of the sides of a given triangle ABC, and m_a, m_b, m_c its medians. Prove that it is true that

$$\sum_{\text{cyclic}} \frac{a+b}{c^2 + 2ab + 3bc + 3ca} \geq \frac{1}{\sqrt{m_a^2 + m_b^2 + m_c^2}}.$$

Solution by the proposer. First, we use the rearrangement inequality on the sequences a, b, c and $\frac{1}{a^2+2bc+3ca+3ab}$, $\frac{1}{b^2+2ca+3ab+3bc}$, $\frac{1}{c^2+2ab+3bc+3ca}$. Without loss of generality, we may assume that $a \geq b \geq c$, and from this we get

$$\frac{1}{a^2+2bc+3ca+3ab} \leq \frac{1}{b^2+2ca+3ab+3bc} \leq \frac{1}{c^2+2ab+3bc+3ca}.$$

The last inequality can be proved as follows:

$$\frac{1}{b^2+2ca+3ab+3bc}-\frac{1}{a^2+2bc+3ca+3ab}$$
$$=\frac{(a-b)(a+b+c)}{(a^2+2bc+3ca+3ab)(b^2+2ca+3ab+3bc)}\geq 0;$$

the other one is completely analogous. So, we know that

$$\sum_{\text{cyclic}}\frac{b}{a^2+2bc+3ca+3ab}\geq\sum_{\text{cyclic}}\frac{a}{a^2+2bc+3ca+3ab},$$
$$\sum_{\text{cyclic}}\frac{c}{a^2+2bc+3ca+3ab}\geq\sum_{\text{cyclic}}\frac{a}{a^2+2bc+3ca+3ab},$$

and adding up both yields

$$\sum_{\text{cyclic}}\frac{b+c}{a^2+2bc+3ca+3ab}\geq\sum_{\text{cyclic}}\frac{2a}{a^2+2bc+3ca+3ab}.$$

So it is enough to prove that

$$\frac{a}{a^2+2bc+3ca+3ab}+\frac{b}{b^2+2ca+3ab+3bc}+\frac{c}{c^2+2ab+3bc+3ca}$$
$$\geq\frac{1}{2\sqrt{m_a^2+m_b^2+m_c^2}}.$$

It is obvious that

$$(a-b)^2+(b-c)^2+(c-a)^2\geq 0\Rightarrow 3(a^2+b^2+c^2)\geq(a+b+c)^2$$

and it is a well known fact that $m_a=\sqrt{\frac{2b^2+2c^2-a^2}{4}}$. Therefore, adding up the three equations, we obtain $3(a^2+b^2+c^2)=4(m_a^2+m_b^2+m_c^2)$. Now, we will use CBS inequality in the form of Arthur

Engel:

$$\begin{aligned}
\sum_{\text{cyclic}} \frac{a}{a^2+2bc+3ca+3ab} &= \sum_{\text{cyclic}} \frac{a^2}{a^3+2abc+3ca^2+3a^2b} \\
&\geq \frac{(a+b+c)^2}{\sum_{cyclic} a^3+2abc+3ca^2+3a^2b} \\
&= \frac{(a+b+c)^2}{(a+b+c)^3} = \frac{1}{(a+b+c)} \\
&\geq \frac{1}{\sqrt{3(a^2+b^2+c^2)}} = \frac{1}{2\sqrt{m_a^2+m_b^2+m_c^2}},
\end{aligned}$$

and we are done.

Advanced Problems

A–41. *Proposed by Marcin J. Zygmunt, AGH University of Science and Technology, Kraków, Poland.* Let $\gamma_1 < \gamma_2 < \ldots < \gamma_n$ be real numbers,

$$f(x) = x + \frac{1}{\gamma_1 - x} + \frac{1}{\gamma_2 - x} + \cdots + \frac{1}{\gamma_n - x},$$

and let $\alpha,\ \beta \in \mathbb{R},\ \alpha < \beta$. Compute the total length of the preimage $f^{-1}([\alpha, \beta])$ (the total length of a set consisting of intervals is the sum of their lengths.)

Solution by the proposer. First, we observe that f is continuous and strictly increasing in its domain, i.e. in

$$(-\infty, \gamma_1) \cup (\gamma_1, \gamma_2) \cup \cdots \cup (\gamma_n, +\infty).$$

Moreover the image of each one of these intervals is the whole real line. Hence the preimage of the interval $[\alpha, \beta]$ consists in $n+1$ disjoint intervals. That is,

$$f^{-1}([\alpha, \beta]) = (\alpha_1, \beta_1) \cup (\alpha_2, \beta_2) \cup \cdots \cup (\alpha_{n+1}, \beta_{n+1}),$$

where $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$ and $\beta_1, \beta_2, \ldots, \beta_{n+1}$ are solutions to the equations $f(x) = \alpha$ and $f(x) = \beta$, respectively. Then, the total length of $f^{-1}([\alpha, \beta])$ is

$$L = \sum_{i=1}^{n+1} (\beta_i - \alpha_i) = \sum_{i=1}^{n+1} \beta_i - \sum_{i=1}^{n+1} \alpha_i.$$

On the other hand, every solution of the equation $f(x) = \alpha$ is also a zero of the polynomial

$$\begin{aligned} p(x) &= (x-\alpha)(x-\gamma_1)\cdots(x-\gamma_n) + a_{n-1}x^{n-1} + \cdots \\ &= x^{n+1} - (\alpha + \gamma_1 + \gamma_2 + \cdots + \gamma_n)x^n + a_{n-1}x^{n-1} + \cdots, \end{aligned}$$

from which it follows that

$$\alpha_1 + \alpha_2 + \cdots + \alpha_{n+1} = \alpha + \gamma_1 + \gamma_2 + \cdots + \gamma_n$$

on account of Viète's formulae. Likewise, from $f(x) = \beta$ we get

$$\beta_1 + \beta_2 + \cdots + \beta_{n+1} = \beta + \gamma_1 + \gamma_2 + \cdots + \gamma_n.$$

Subtracting the first from the second expression we obtain that the total length is

$$L = \sum_{i=1}^{n+1} \beta_i - \sum_{i=1}^{n+1} \alpha_i = \beta - \alpha.$$

Also solved by *Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.*

A–42. *Proposed by Óscar Rivero Salgado, BarcelonaTech, Barcelona, Spain.* Let p be a prime number and consider A a $p \times p$ matrix with complex entries, satisfying that $\mathrm{Tr}(A) = \mathrm{Tr}(A^2) = \ldots = \mathrm{Tr}(A^{p-1}) = 0$ and $\mathrm{Tr}(A^p) = p$. Find $\det(A^i + j\,\mathrm{Id})$, for i, $j \in \mathbb{Z}$.

Solution by the proposer. Let $\lambda_1, \ldots, \lambda_p$ the eigenvalues of A. Then, we know that $\sum_{i=1}^{p} \lambda_i^k = 0$ for $1 \leq k \leq p-1$ and $\sum_{i=1}^{p} \lambda_i^p = p$. From the theory of symmetric polynomials, this determines completely (up to order) the λ_i, which must be given by $\lambda_i = \zeta_p^i$, where ζ_p is a primitive p-th root of unit.

Now, clearly

$$\det(A^i + j\,\mathrm{Id}) = \prod_{k=1}^{p} (\zeta_p^{ki} + j).$$

If i is a multiple of p, then we directly get that the result is $(1+j)^p$. Elsewhere,

$$\det(A^i + j\,\mathrm{Id}) = \prod_{k=1}^{p} (\zeta_p^{k} + j).$$

Observe that

$$X^p - 1 = \prod_{k=1}^{p} (X - \zeta_p^k) = (-1)^p \prod_{k=1}^{p} (\zeta_p^k - X),$$

and taking now $X = -j$, it yields that

$$(-j)^p - 1 = (-1)^p \prod_{k=1}^{p} (\zeta_p^k + j).$$

Hence

$$\det(A^i + j\,\mathrm{Id}) = \prod_{k=1}^{p} (\zeta_p^k + j) = j^p + (-1)^{p+1}.$$

Also solved by *Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.*

A–43. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all real solutions of the following system of equations

$$\left.\begin{array}{l} x^4 + x^2 + (y^4+1)\sqrt{y^4+1} = (2+\sqrt{2})\, z^2\sqrt{z^4+1}, \\ y^4 + y^2 + (z^4+1)\sqrt{z^4+1} = (2+\sqrt{2})\, x^2\sqrt{x^4+1}, \\ z^4 + z^2 + (x^4+1)\sqrt{x^4+1} = (2+\sqrt{2})\, y^2\sqrt{y^4+1}. \end{array}\right\}$$

Solution by the proposer. Adding up the given equations yields

$$\sum_{\text{cyclic}} \left[x^4 + x^2 + (x^4+1)\sqrt{x^4+1} - (2+\sqrt{2})\, x^2\sqrt{x^4+1} \right] = 0.$$

Now we claim that for all nonzero real x,

$$x^4 + x^2 + (x^4+1)\sqrt{x^4+1} - (2+\sqrt{2})\, x^2\sqrt{x^4+1} \geq 0$$

holds. Indeed, the claimed inequality may be written in the more convenient form

$$x^4 + x^2 + (x^4+1)\sqrt{x^4+1} \geq (2+\sqrt{2})\, x^2\sqrt{x^4+1}.$$

After diving both terms by $x^2\sqrt{x^4+1}$, the expression becomes

$$x^2 + \frac{1}{x^2} + \frac{x^2+1}{\sqrt{x^4+1}} \geq 2 + \sqrt{2}.$$

To prove the above inequality we consider the function $f\colon \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by

$$f(x) = x^2 + \frac{1}{x^2} + \frac{x^2+1}{\sqrt{x^4+1}}.$$

We have

$$\begin{aligned} f'(x) &= 2x - \frac{2}{x^3} + \frac{2x}{\sqrt{x^4+1}} - \frac{2(x^2+1)x^3}{\sqrt{(x^4+1)^3}} \\ &= \frac{2(x-1)\,(x+1)\,(x^2\sqrt{(x^4+1)^3} - x^4 + \sqrt{(x^4+1)^3})}{x^3\sqrt{(x^4+1)^3}} \end{aligned}$$

and the only real roots of $f'(x) = 0$ are $x = \pm 1$.

On the other hand,

$$f''(x) = 2 + \frac{6}{x^4} + \frac{2}{\sqrt{x^4+1}} - \frac{6x^2+14x^4}{\sqrt{(x^4+1)^3}} + \frac{12(x^2+1)x^6}{\sqrt{(x^4+1)^5}}$$

and $f''(\pm 1) = 8 - \sqrt{2} > 0$. Therefore, f has two minimum points at $x = \pm 1$. This means that $f(x) \geq f(\pm 1)$ for all nonzero real number x. That is,

$$f(x) = x^2 + \frac{1}{x^2} + \frac{x^2+1}{\sqrt{x^4+1}} \geq 2 + \sqrt{2} = f(\pm 1).$$

Equality holds when $x = \pm 1$.

Finally, on account of the preceding we conclude that the only solutions of the given system are

$$(-1,-1,-1), (-1,-1,1), (-1,1,-1), (1,-1-1),$$

$$(-1,1,1), (1,-1,1), (1,1,-1), \text{ and } (1,1,1),$$

and we are done.

A–44. *Proposed by Nicolae Papacu, Slobozia, Romania.* Let a and x_0 be real numbers and let $\{x_n\}_{n\geq 0}$ be the sequence defined by $x_{n+1} = ax_n^2 - (2a-1)x_n$.

(a) Prove that the sequence $\{x_n\}_{n\geq 0}$ is convergent if, and only if, $ax_0 \in [a-1, a]$.

(b) If $ax_0 \in [a-1, a]$, compute $\lim\limits_{n\to+\infty} n(x_n - \ell)$, where $\ell = \lim\limits_{n\to+\infty} x_n$.

Solution by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany. The statement of the problem is false. Let $a = 2$ and $x_0 = 0$. Then the sequence $x_n = 0$ satisfies the recurrence and converges, but $ax_0 = 0 \notin [a-1, a]$.

Part (b) does not have a solution, since we can also have that $ax_0 \in [a-1, a]$ and $\{x_n\}_{n\geq 0}$ does not converge. For example, let $a = 3$ and $x_0 = 1$. A simple computation gives $x_1 = -2$ and $x_2 = 22$. We now claim that, for $n \geq 3$ we have $x_n > 2x_{n-1}$, implying that the sequence does not converge. Indeed, let $n \geq 3$ be the smallest value for which $x_n \leq 2x_{n-1}$. Then in particular $x_{n-1} \geq x_2 = 22$ and so $x_n = x_{n-1}(3x_{n-1} - 5) > 2x_{n-1}$, contradiction. We conclude that the sequence does not converge.

A–45. *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School (BMS), Berlin, Germany.* Let n be a positive integer, and let A be a $n \times (n+1)$ matrix with integer entries. Assume that, for every prime p, the matrix A considered as a matrix with entries in $\mathbb{Z}/p\mathbb{Z}$ has rank n. Prove that there exists a matrix B of size $(n+1) \times n$ with integer entries such that $AB = \mathrm{Id}_n$.

Solution 1 by the proposer. Let A_i be the minor obtained by removing the $i-$th column and let $d_i = \det(A_i)$. Then from the condition on the statement we know that $\gcd(d_1, d_2, ..., d_{n+1}) = 1$ because for each prime p at least one minor is not divisible by p. We claim that there exist integers $c_1, c_2, ..., c_{n+1}$ such that $c_1d_1 + c_2d_2 + ... + c_{n+1}d_{n+1} = 1$. In fact, let $g_i = \gcd(d_1, d_2, ..., d_i)$. We claim that there are $c_1, c_2, ..., c_i$ such that $c_1d_1 + c_2d_2 + ... + c_id_i = g_i$. This can be proved by induction. For $i = 2$, this is

Bézout's identity. If those coefficients exist for some i, then using Bézout's identity, $g_{i+1} = \gcd(g_i, d_{i+1}) = ug_i + vd_{i+1} = uc_1d_1 + uc_2d_2 + ... + uc_id_i + vd_{i+1}$, which means that these coefficients exist for $i+1$. By induction, the claim follows.

Construct matrix C by placing an extra row on top of matrix A, with $c_{1i} = (-1)^i c_i$. Expanding the determinant by the first row we see that $\det(C) = 1$. Because its entries are integers, its inverse D has also integer entries. Remove the first column of D to produce a matrix B of size $(n+1)\times n$ with $AB = \mathrm{Id}_n$ (this follows from $CD = \mathrm{Id}_{n+1}$, removing the first row and the first column).

Solution 2 by the proposer. We construct d_i and c_i as in Solution 1. Now we will construct matrices B_i of size $(n+1)\times n$ such that $AB_i = d_i\mathrm{Id}_n$. If $d_i = 0$, then we can make $B_i = 0$. If $d_i \neq 0$, then the entries of A_i^{-1} are integers divided by d_i. This means that $d_iA_i^{-1}$ has integer entries and $A_i(d_iA_i^{-1}) = d_i\mathrm{Id}_n$. Inserting a row of zeroes in A_i^{-1} as the i-th row produces an $(n+1)\times n$ matrix B_i with integer entries such that $AB_i = d_i\mathrm{Id}_n$. Finally, making $B = c_1B_1 + c_2B_2 + ...c_{n+1}B_{n+1}$ we find

$$\begin{aligned} AB &= A(c_1B_1 + c_2B_2 + ... + c_{n+1}B_{n+1}) \\ &= c_1AB_1 + c_2AB_2 + ... + c_{n+1}AB_{n+1} \\ &= (c_1d_1 + c_2d_2 + ... + c_{n+1}d_{n+1})\mathrm{Id}_n = \mathrm{Id}_n. \end{aligned}$$

A–46. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Compute

$$\lim_{n\to+\infty} \sum_{k=1}^{n} \arctan\left(\frac{k}{n^2}\right).$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. Let $x = \arctan\left(\frac{k}{n^2}\right)$. As $\sin x < x < \tan x$ for $x \in (0, \pi/2)$ it follows that

$$\frac{k}{\sqrt{n^4+k^2}} < \arctan\left(\frac{k}{n^2}\right) < \frac{k}{n^2}.$$

We have that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}=\lim_{n\to\infty}\frac{\frac{n(n+1)}{2}}{n^2}=\frac{1}{2}.$$

On the other hand, since for $1\le k\le n$, $n^4+1\le n^4+k^2\le n^4+n^2$, then

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{\sqrt{n^4+n^2}}\le\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{\sqrt{n^4+k^2}}\le\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{\sqrt{n^4+1}}\\ \lim_{n\to\infty}\frac{\frac{n(n+1)}{2}}{\sqrt{n^4+n^2}}\le\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{\sqrt{n^4+k^2}}\le\lim_{n\to\infty}\frac{\frac{n(n+1)}{2}}{\sqrt{n^4+1}}\\ \frac{1}{2}\le\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{\sqrt{n^4+k^2}}\le\frac{1}{2}.\end{aligned}$$

Therefore, also from sandwich lemma the proposed limit is

$$\lim_{n\to\infty}\sum_{k=1}^{n}\arctan\left(\frac{k}{n^2}\right)=\frac{1}{2}.$$

Solution 2 by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA. Putting $x=k/n^2$ in the known inequality

$$x-\frac{x^3}{3}\ <\ \arctan x\ <\ x\qquad\text{valid for } x>0$$

and summing, we have

$$\frac{1}{n^2}\sum_{k=1}^{n}k-\frac{1}{3n^6}\sum_{k=1}^{n}k^3\ <\ \sum_{k=1}^{n}\arctan\left(\frac{k}{n^2}\right)\ <\ \frac{1}{n^2}\sum_{k=1}^{n}k,$$

or, after using familiar algebraic formulas and simplifying,

$$\frac{1}{2}+\frac{1}{2n}-\frac{(n+1)^2}{12n^4}\ <\ \sum_{k=1}^{n}\arctan\left(\frac{k}{n^2}\right)\ <\ \frac{1}{2}+\frac{1}{2n}.$$

Thus the sum tends to $1/2$ as $n\to\infty$.

Solution 3 by the proposer. We begin with the following lemma.

Lemma 1. *Let $\alpha > 0$ be a real number. If $f\colon[-\alpha,\alpha]\to\mathbb{R}$ is a continuous function two times differentiable in $(-\alpha,\alpha)$ such that $f(0)=0$ and f'' is bounded in $(-\alpha,\alpha)$, then the sequence $\{x_n\}_{n\geq 1}$ defined by*

$$x_n = \begin{cases} \displaystyle\sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) & \text{for } n > \frac{1}{\alpha}; \\ 0 & \text{for } n \leq \frac{1}{\alpha} \end{cases}$$

is convergent.

Proof. First, we observe that if $n > \frac{1}{\alpha}$, then $\frac{k}{n^2} \leq \frac{1}{n} < \alpha$ for $1 \leq k \leq n$, and $[0, \frac{k}{n^2}] \subset (-\alpha,\alpha)$. Applying Taylor's formula, we get

$$f\left(\frac{k}{n^2}\right) = f(0) + \frac{f'(0)}{1!}\left(\frac{k}{n^2}\right) + \frac{f''(c_k)}{2!}\left(\frac{k}{n^2}\right)^2, \quad \left(0 < c_k < \frac{k}{n^2}\right)$$

and

$$x_n = \sum_{k=1}^{n} f\left(\frac{k}{n^2}\right) = f'(0)\sum_{k=1}^{n}\frac{k}{n^2} + f''(c_k)\sum_{k=1}^{n}\frac{k^2}{2n^4}.$$

From the above, it immediately follows that

$$\begin{aligned}\left|x_n - f'(0)\sum_{k=1}^{n}\frac{k}{n^2}\right| &= \left|\sum_{k=1}^{n}\frac{k^2}{2n^4}f''(c_k)\right| \\ &\leq \sum_{k=1}^{n}\frac{k^2}{2n^4}|f''(c_k)| \leq M\sum_{k=1}^{n}\frac{k^2}{2n^4},\end{aligned}$$

where $0 < M < +\infty$. Taking into account the well-known closed form of the sums of the first and second powers of positive integers yields

$$\left|x_n - f'(0)\frac{n(n+1)}{2n^2}\right| \leq \frac{Mn(n+1)(2n+1)}{12n^4}.$$

When $n \to \infty$, from the preceding we obtain

$$\left|x_n - \frac{1}{2}f'(0)\right| < \epsilon, \quad (\epsilon > 0)$$

and

$$\lim_{n\to\infty} x_n = \frac{1}{2}f'(0).$$

This completes the proof. $\square$

Now we apply the Lemma to the function $f\colon [-\alpha, \alpha] \to \mathbb{R}$ defined by $f(x) = \arctan x$. We have $f'(x) = \dfrac{1}{1+x^2}$ and $f''(x) = \dfrac{-2x}{(1+x^2)^2}$, which is bounded, as can be easily checked. Then,

$$\lim_{n\to\infty} x_n = \frac{1}{2} f'(0) = \frac{1}{2}.$$

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