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On Some Diophantine Equations

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Abstract

In this note we give integer solutions of generalizations of well-known diophantine equations, such as $11^x - 2^x = y^2$, $27^x - 2^x = y^2$, $3^x - 2^x = y^2$ or $3^x - 2^y = z^2$. Actually, equations of the form $(p^2+2)^x \pm 2^x = y^2$ and $(p^2+2)^x \pm 2^y = z^2$ are studied and, whenever possible, solved in nonnegative integers.

1 The Equations

Hereafter, we present some nonstandard diophantine equations and discuss their solutions in nonnegative integers. We begin with the following.

Theorem 1. *If $p \geq 1$ is an odd integer, then the only solutions to the equation*

$$(p^2 + 2)^x - 2^x = y^2$$

in nonnegative integers are $(x, y) = (0, 0)$ and $(x, y) = (1, p)$.

Proof. Since p is an odd positive integer, then $p^2 \equiv 1 \pmod 8$ and $p^2 + 2 \equiv 3 \pmod 8$. Then, $(p^2 + 2)^x \equiv 1 \pmod 8$ when x is even and $(p^2 + 2)^x \equiv 3 \pmod 8$ when x is odd. By inspection, we observe that $(x, y) = (0, 0)$ and $(x, y) = (1, p)$ are solutions of the given equation in positive integers. We claim that there are no other solutions in positive integers. Indeed, for $x \geq 2$ we have that y is an odd positive integer. Next we will see that x is even.

Suppose that $x \geq 3$ is odd, then we have $(p^2+2)^x \equiv 3 \pmod 8$. On the other hand, $2^x \equiv 0 \pmod 8$, $(p^2+2)^x - 2^x \equiv 3 \pmod 8$ and $y^2 \equiv 1 \pmod 8$. As a consequence, we obtain

$$(p^2+2)^x - 2^x \neq y^2.$$

Therefore, $x = 2t$, where $t \geq 1$ is an integer, and the given equation becomes

$$(p^2+2)^{2t} - 2^{2t} = y^2 \quad \text{or} \quad \big((p^2+2)^t - y\big)\big((p^2+2)^t + y\big) = 2^{2t}.$$

On account of the well-known fact that for a, b odd integers with gcd (a,b) it holds that $(a-b, a+b) = 2(a,b)$, we have

$$d = \big((p^2+2)^t - y, (p^2+2)^t + y\big) = 2\,\big((p^2+2)^t, y\big)$$

and $d \mid 2y$. Since $d^2 \mid 2^{2t}$, d is even and y is odd, we conclude that $d = 2$.

The facts that $((p^2+2)^t - y)\,((p^2+2)^t + y) = 2^{2t}$ and $d = 2$ mean that

$$\begin{aligned} (p^2+2)^t - y &= 2, \\ (p^2+2)^t + y &= 2^{2t-1}. \end{aligned}$$

Adding up the above equations, we obtain

$$(p^2+2)^t = 2^{2t-2} + 1.$$

Now we claim that the preceding equation does not have positive integer solutions. Indeed, for $t = 1$ we have $p = 0$ which is not possible, and for $t \geq 2$ we have $2^{2t-1} + 1 \equiv 1 \pmod 4$, or $(p^2+2)^t \equiv 1 \pmod 4$, which means that t is even. Putting $t = 2w$ in the last equation, we get $(p^2+2)^{2w} = 2^{4w-2} + 1$ or

$$\big((p^2+2)^w - 2^{2w-1}\big)\,\big((p^2+2)^w + 2^{2w-1}\big) = 1.$$

From the preceding it immediately follows that $(p^2+2)^w - 2^{2w-1} = (p^2+2)^w + 2^{2w-1} = 1$. But this is not possible, and the proof is complete. $\square$

Theorem 2. *If $p \geq 1$ is an odd integer, then the only solutions to the equation*

$$(p^2 - 2)^x + 2^x = y^2$$

in nonnegative integers are $(x, y) = (1, p)$.

Proof. We distinguish the following cases:

1. If $p = 1$ then the equation becomes $(-1)^x + 2^x = y^2$. For $x = 1$ we have $y = 1$, and for $x \geq 2$ we consider two cases:
   (a) If $x = 2t$, then $1 + 2^{2t} = y^2$ or, equivalently, $(y - 2^t)(y + 2^t) = 1$, from which $y - 2^t = y + 2^t = 1$, which is impossible.
   (b) If $x = 2t + 1$ then $-1 + 2^{2t+1} = y^2$, which is impossible on account that $-1 + 2^{2t+1} \equiv 7 \pmod 8$ and $y^2 \equiv 1 \pmod 8$. So, $-1 + 2^{2t+1} \neq y^2$.
2. If $p \geq 3$ the given equation has no solutions for $x = 0$, and for $x = 1$ we have $y = p$. Suppose that $x \geq 2$ and observe that y is odd. We will prove that x is even. Indeed, if $x \geq 3$ is odd, then $(p^2 - 2)^x \equiv 7 \pmod 8$. Since $2^x \equiv 0 \pmod 8$, then $(p^2 - 2)^x + 2^x \equiv 7 \pmod 8$, and since $y^2 \equiv 1 \pmod 8$, then $(p^2 - 2)^x + 2^x \neq y^2$. So $x = 2t$ and the equation becomes

$$(p^2 - 2)^{2t} + 2^{2t} = y^2.$$

   Now we will see that the above equation has no solutions in positive integers. Indeed, for $p \geq 3$ we have

$$\left((p^2 - 2)^z\right)^2 < (p^2 - 2)^{2z} + 2^{2z} < \left((p^2 - 2)^z + 1\right)^2.$$

   The LHS inequality holds trivially. For the RHS inequality, we have

$$(p^2 - 2)^{2z} + 2^{2z} < \left((p^2 - 2)^z + 1\right)^2 \iff 2^{2z} < 2\,(p^2 - 2)^z + 1$$

   because $(p^2 - 2)^z + 1 > 7^z > 2^{2z}$. For $p = 1$ the equation becomes $1 + 2^{2t} = y^2$. It does not have positive integer solutions on account that $(y - 2^t)(y + 2^t) = 1 \iff (y - 2^t) = (y + 2^t) = 1$, which is impossible. Thus, the only solutions are $(x, y) = (1, p)$, and this completes the proof. □

Next we study the solutions to an equation in three variables.

Theorem 3. *If $p \geq 3$ is an odd integer, then the only solutions to the equation*

$$(p^2+2)^x - 2^y = z^2$$

in nonnegative integers are $(x, y, z) = (0, 0, 0)$ and $(x, y, z) = (1, 1, p)$. If $p = 1$, there are three additional solutions.

Proof. First, we consider some particular cases. For $x = 0$ the equation becomes $1 - 2^y = z^2$ with solutions $y = z = 0$. For $x = 1$ we have $p^2 + 2 - 2^y = z^2$. From $p^2 \equiv 1 \pmod 8$ and $z^2 \equiv 1 \pmod 8$ it follows that $2^y - 2 \equiv 0 \pmod 8$, with solutions $y = 1$ and $z = p$. Therefore, we have the solutions $(x, y, z) = (0, 0, 0)$ and $(x, y, z) = (1, 1, p)$.

For $y = 0$ the given equation becomes $(p^2 + 2)^x - 1 = z^2$. We distinguish two cases:

1. If $x \geq 2$ is odd, then we have $(p^2 + 2)^x \equiv 3 \pmod 8$ or $(p^2 + 2)^x - 1 \equiv 2 \pmod 8$ and, therefore, $z^2 \equiv 2 \pmod 8$, which is impossible.
2. If x is even, say $x = 2t$, $t \in \mathbb{N}$, then we have $(p^2+2)^{2t}-1 = z^2$ or
$$\left((p^2 + 2)^t - z\right)\left((p^2 + 2)^t + z\right) = 1.$$
From the preceding, $(p^2 + 2)^t - z = (p^2 + 2)^t + z = 1$, which is impossible.

For $y = 1$ the given equation becomes $(p^2 + 2)^x - 2 = z^2$. If $x \geq 2$ is even then the equation has no solutions because $(p^2 + 2)^x \equiv 1 \pmod 8$ and $z^2 + 2 \equiv 3 \pmod 8$. For $x = 1$ we have $z = p$ and then the solution is $(x, y, z) = (1, 1, p)$. If $x \geq 3$ is an odd integer, then it has at least one odd prime factor, say q. Thus, $x = qt$ and then $(p^2 + 2)^x = w^q$, where $w = (p^2 + 2)^t$. Under this hypothesis, the equation $(p^2 + 2)^x - 2 = z^2$ becomes $z^2 + 2 = w^q$ with q an odd prime. This equation was solved in [1] and has the solution $(z, q, w) = (5, 3, 3)$. Then, $w = (p^2 + 2)^t = 3$, from which we get $t = p = 1$. So, for $p = 1$ we have the solution $(x, y, z) = (3, 1, 5)$.

If $x, y \geq 2$ then z is an odd number. We claim that x is even. Indeed, if $x \geq 3$ is odd we have $(p^2 + 2)^x \equiv 3 \pmod 8$ and $2^y \equiv 0$

$\pmod 8$, from which it follows that $(p^2+2)^x - 2^y \equiv 3 \pmod 8$. On the other hand, $y^2 \equiv 1 \pmod 8$ and, therefore, $(p^2+2)^x - 2^y \neq y^2$. Hence, x is even, as claimed. Putting $x = 2t$, $t \in \mathbb{N}$, in $(p^2+2)^x - 2^y = z^2$ yields $(p^2+2)^{2t} - 2^y = z^2$. The last equation may be written as

$$\left((p^2+2)^t - z\right)\left((p^2+2)^t + z\right) = 2^y.$$

Since

$$d = \left((p^2+2)^t - z\right), \left((p^2+2)^t + z\right) = 2\left((p^2+2)^t, z\right),$$

then $d \mid 2z$. From $d \mid ((p^2+2)^t - z)$, $d \mid ((p^2+2)^t + z)$ and $((p^2+2)^t - z)((p^2+2)^t + z) = 2^y$ it results that $d \mid 2^y$. Hence, we have $d \mid 2z$ and $d^2 \mid 2^y$, from which we get $d = 2$ on account that d is even and z is odd. From $((p^2+2)^t - z)((p^2+2)^t + z) = 2^y$ and $d = 2$ we obtain

$$\begin{aligned} (p^2+2)^t - z &= 2, \\ (p^2+2)^t + z &= 2^{y-1}. \end{aligned}$$

Subtracting the above equations we get $z = 2^{y-2} - 1$, and adding them up yields $(p^2+2)^t = 2^{y-2} + 1$. Now we distinguish the following cases:

1. If $y = 2$ we obtain $(p^2+2)^t = 2$, which is impossible because $p \geq 1$.
2. If $y = 3$ we have $(p^2+2)^t = 3$, from which $p = 1$, $t = 1$, $x = 2t = 2$ and $z = 2^{y-2} - 1 = 1$. So $(x, y, z) = (2, 3, 1)$ is a solution.
3. If $y = 4$ the equation $(p^2+2)^t = 5$ has no solutions.
4. For $y \geq 5$ we have $2^{y-2} + 1 \equiv 1 \pmod 8$ and then $(p^2+2)^t \equiv 1 \pmod 8$, so t is even, say $t = 2v$, $v \in \mathbb{N}$. Substituting in the equation $(p^2+2)^t = 2^{y-2}+1$ we obtain $(p^2+2)^{2v} = 2^{y-2}+1$ or $(p^2+2)^{2v} - 1 = 2^{y-2}$ and $((p^2+2)^v - 1)((p^2+2)^v + 1) = 2^{y-2}$. Since $((p^2+2)^v - 1)((p^2+2)^v + 1) = 2$, then we have

$$\begin{aligned} (p^2+2)^v - 1 &= 2, \\ (p^2+2)^v + 1 &= 2^{y-3}. \end{aligned}$$

From the first equation we obtain $p = 1$ and $v = 1$, and the equation $(p^2 + 2)^{2v} = 2^{y-2} + 1$ becomes $9 = 2^{y-2} + 1$, from which we get $y = 5$. Combining the preceding we have $t = 2$, $x = 4$ and $z = 2^{y-2} - 1 = 7$.

So for $p = 1$ we have the solution $(x, y, z) = (4, 4, 7)$. Finally, the equation $(p^2 + 2)^x - 2^y = z^2$ has the solutions $(x, y, z) = (0, 0, 0)$ and $(x, y, z) = (1, 1, p)$, as we wanted to prove. In the particular case when $p = 1$, equation $3^x - 2^y = z^2$ has the solutions

$$\{(0,0,0), \quad (1,1,1), \quad (3,1,5), \quad (2,3,1), \quad (4,5,7)\}. \qquad \square$$

Finally, we have the following question.

Question (Open problem). *Let $p \geq 1$ be an odd integer. What are the solutions to*

$$(p^2 - 2)^x + 2^y = z^2$$

in nonnegative integers?

We will now solve all cases except the case where $x \geq 3$ is odd and $y = 1$. The only solutions then are $(x, y, z) = (0, 3, 3)$ and $(x, y, z) = (1, 1, p)$.

Proof. First, we consider the cases when $x \in \{0, 1\}$ and $y \in \{0, 1\}$. If $x = 0$, we have $1 + 2^y = z^2$ or $(z - 1)(z + 1) = 2^y$. Since $(z - 1, z + 1) = 2$ and $z - 1 < z + 1$, then we have $z - 1 = 2$ and $z + 1 = 2^{y-1}$, from which $y = 3$, $z = 3$ follow. For $x = 1$, we have $p^2 - 2 + 2^y = z^2$. Since $p^2 \equiv 1 \pmod 8$ and $z^2 \equiv 1 \pmod 8$, then $2^2 - 2 \equiv 0 \pmod 8$, so $y = 1$ and $z = p$. Hence, $(1, 1, p)$ is a solution. For $y = 0$ the given equation becomes $(p^2 - 2)^x + 1 = z^2$, or $z^2 - (p^2 - 2)^x = 1$. Then, $z = 3$, $p^2 - 2 = 2$, and $x = 3$. This means that $p = 2$ on account of Catalan-Mihailescu Theorem, which states that the equation $x^y - z^t = 1$, where x, y, z and t are positive integers bigger than one, has the unique solution $(x, y, z, t) = (3, 2, 2, 3)$. The value $p = 2$ is not possible because p is odd. For $y = 1$ we have $(p^2 - 2)^x + 2 = z^2$. If $x = 1$ we have $z = p$, and $(x, y, z) = (1, 1, p)$ are solutions. If x is even, then $(p^2 - 2)^x + 2 \equiv 3 \pmod 8$ and $z^2 \equiv 1 \pmod 8$. Therefore, the given equation has no solutions. For x, $y \geq 2$ we have that

z is odd. If we suppose that $x \geq 3$, then $(p^2-2)^x \equiv 7 \pmod 8$, but $2^y \equiv 0 \pmod 4$. Thus, $(p^2-2)^x + 2^y \equiv 3 \pmod 4$ and $y^2 \equiv 1 \pmod 8$, from which $(p^2-2)^x + 2^y \neq z^2$ follows. From the preceding we conclude that x is even, say $x = 2t$, and the equation may be written as $(p^2-2)2^t + 2^y = z^2$ or

$$\left(z - (p^2-2)^t\right)\left(z + (p^2-2)^t\right) = 2^y.$$

Let $d = (z - (p^2-2)^t, z + (p^2-2)^t)$. It is an even number sucht-hat $d \mid 2z$. On the other hand, from $d \mid (z - (p^2-2)^t)$ and $d \mid (z + (p^2-2)^t)$ and the equation results that $d^2 \mid 2^y$. Since $d \mid 2z$ and $d^2 \mid 2^y$ we get that $d = 2$ on account that z is odd. $\square$

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A note on a problem from the 2012 Spanish Mathematical Olympiad

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Abstract

This note gives some different solutions to a problem stated in the 2012 Spanish Mathematical Olympiad. In addition, some generalizations and related results are proven.

1 Introduction

The problem was originally stated in the XLVIII Spanish Mathematical Olympiad that took place in Santander in 2012. The original statement reads as follows:

Problem. *Consider the sequence of numbers given by*

$$a_0 = 1, \quad a_1 = 5, \quad a_{n+2} = \frac{a_{n+1}^2 + 4}{a_n}.$$

Prove that $a_n \in \mathbb{Z}$, $\forall\, n \in \mathbb{Z}$, and give an explicit formula for a_n.

We will study the generalized sequence

$$a_0 = 1 \quad a_1 = k \quad a_{n+2} = \frac{a_{n+1}^2 + k - 1}{a_n}, \tag{1}$$

where $k > 1$ since, for $k = 1$, the sequence is a constant sequence. In Section 2 we will give two proofs of the fact that $a_n \in \mathbb{Z}$, $\forall\, n \in \mathbb{Z}$. Section 3 contains an explicit formula for a_n. Finally, in Section 4, some other interesting assertions for this sequence will be proven.

2 Proof of the main statement

Two proofs for the main statement will be given. The first one is based on finding a linear relationship for the sequence of numbers a_n. The second one uses the fact that two consecutive members of the sequence are coprime.

2.1 First proof

First proof. Indeed, it will be proven that the generalized sequence (1) fulfills a linear relationship, from where it is straightforward to see that a_n are all integers. The linear relationship that will be proven is

$$a_{n+2} = (k+1)a_{n+1} - a_n \quad \forall\, n \geq 0. \tag{2}$$

The proof will be performed by induction. For $n = 0$,

$$a_2 = \frac{a_1^2 + (k-1)}{a_0} = k^2 + k - 1 = k(k+1) - 1 = a_1(k+1) - a_0,$$

which is the linear relationship we wanted to prove. Similarly, for $n = 1$,

$$\begin{aligned} a_3 &= \frac{a_2^2 + (k-1)}{a_1} = k^3 + 2k^2 - k - 1 \\ &= (k^2 + k - 1)(k+1) - k = a_2(k+1) - a_1. \end{aligned}$$

Now, assume that the linear relationship is held for a_n and a_{n+1}, that is,

$$a_{n+1} = (k+1)a_n - a_{n-1} \tag{3}$$

and

$$a_n = (k+1)a_{n-1} - a_{n-2}. \tag{4}$$

On substituting

$$a_{n+2} = \frac{a_{n+1}^2 + k - 1}{a_n}$$

in (2), it is equivalent to prove (2) and to prove that

$$\frac{a_{n+1}^2 + k - 1}{a_n} = (k+1)a_{n+1} - a_n,$$

which, in turns, is equivalent to see that

$$a_{n+1}^2 + k - 1 = (k+1)a_n a_{n+1} - a_n^2,$$

which, by substitution at both sides of (3) and expansion, is equivalent to

$$\begin{aligned}(k+1)^2 a_n^2 - 2(k+1)a_n a_{n-1} + a_{n-1}^2 + k - 1 \\ =(k+1)^2 a_n^2 - (k+1)a_n a_{n-1} - a_n^2.\end{aligned}$$

Simplification and substitution of $a_{n-1}^2 + k - 1 = a_n a_{n-2}$ at the left hand side of the equality, and collecting all the terms on the same side, leads to

$$-(k+1)a_n a_{n-1} + a_n^2 + a_n a_{n-2} = 0.$$

Here, a_n can be simplified since it does not vanish (otherwise the sequence would not be well defined). Therefore, the last equality holds if, and only if,

$$-(k+1)a_{n-1} + a_n + a_{n-2} = 0$$

is fulfilled. And this equality is satisfied by the induction hypothesis (4). □

2.2 Second proof

We will state and prove two lemmas that will help us in the proof of the main result. We assume that $a_1, a_2, \ldots, a_{n+1}$ are integers, and we will show that a_{n+2} is also an integer.

Lemma 1. *For $3 \leq \ell \leq n+1$, a_ℓ and $k-1$ are coprime.*

Proof. By induction on ℓ.

Since $a_3 = k^2 + k - 1$, $a_3 - (k-1) = k^2$ is an integer. So, if p divides both a_3 and $k-1$, it also divides k^2 and, therefore, it divides k. But this is impossible since it is very well-known that k and $k-1$ are coprime.

Now, assume that the statement is true for a_ℓ with $\ell \leq n$. We are going to prove that the statement also holds for a_{n+1}.

From $a_{n+1}a_{n-1} = a_n^2 + k - 1$, if a_{n+1} and $k-1$ have a common factor p, then this factor p also divides a_n^2 and, hence, a_n, which contradicts the induction hypothesis. □

Lemma 2. *For $2 \leq \ell \leq n+1$, a_ℓ and $a_{\ell-1}$ are coprime.*

Proof. By induction on ℓ.

It is an straightforward computation to check that the statement is true for $\ell = 2$.

Now, assume the statement holds for $\ell \leq n$. Let us prove the case $\ell = n+1$. Assume a_{n+1} and a_n have a common factor $p \neq 1$. Then, from the equality $a_{n+1}a_{n-1} - a_n^2 = k - 1$ it can be inferred that p also divides $k-1$. Therefore p divides a_n and $k-1$, which enters in contradiction with the previous lemma. □

Second proof of the main statement. Assume that, for $\ell \leq n+1$, a_ℓ are integers. We are going to prove that a_{n+2} is also an integer.

Substitute a_{n+1} as a function of a_n and a_{n-1} from its definition in the equation

$$a_{n+2} = \frac{a_{n+1}^2 + k - 1}{a_n}.$$

This leads to

$$\begin{aligned} a_{n+2} &= \frac{(a_n^2 + k - 1)^2 + a_{n-1}^2(k-1)}{a_{n-1}^2\, a_n} \\ &= \frac{a_n^4 + 2(k-1)a_n^2 + (a_{n-1}^2 + k - 1)(k-1)}{a_{n-1}^2\, a_n} \\ &= \frac{a_n^4 + 2(k-1)a_n^2 + a_n a_{n-2}(k-1)}{a_{n-1}^2\, a_n}. \end{aligned} \tag{5}$$

Let Q be the numerator of the previous equalities, $Q = a_n^4 + 2(k-1)a_n^2 + a_n a_{n-2}(k-1)$. By induction hypothesis, Q is an integer and a_n divides Q clearly. On the other hand, the numerator in (5) can be written as

$$Q = a_{n+1}^2 a_{n-1}^2 + a_{n-1}^2(k-1),$$

which is an integer and a multiple of a_{n-1}^2. Since by the previous lemma a_n and a_{n-1} are coprime, Then Q is a multiple of $a_n a_{n-1}^2$. □

3 More properties for this sequence

This section is devoted to stating and proving interesting relationships for this sequence of numbers.

1. $a_n = (k-1)c_n + 1$, where c_n fulfills

$$c_{n+1} = (k+1)c_n - c_{n-1} + 1 \tag{6}$$

with $c_0 = 0$ and $c_1 = 1$.
Proof. Since $a_0 = 1$, c_0 must be 0, while $a_1 = k$ leads to $c_1 = 1$. We are going to prove the statement by induction. For $n = 2$, from $a_2 = k^2 + k - 1 = (k-1)(k+2) + 1$, $c_2 = k + 2$. Let us check the equality (6) for $n = 2$:

$$c_2 = k + 2 = (k+1) \cdot 1 - 0 + 1 = 6c_1 - c_0 + 1,$$

as desired. Now, by induction,

$$\begin{aligned} a_{n+1} &= (k+1)a_n - a_{n-1} \\ &= (k+1)((k-1)c_n + 1) - ((k-1)c_{n-1} + 1) \\ &= (k-1)((k+1)c_n - c_{n-1} + 1) + 1, \end{aligned}$$

from where $a_{n+1} = (k-1)c_{n+1} + 1$. □
In other words, not only $a_n \in \mathbb{Z}$, but they are also congruent with 1 modulo $k - 1$.

2. For all $k \geq 1$ and $n \geq 0$,

$$\begin{aligned} a_n &= \tfrac{1}{2}\left(1 + \tfrac{k-1}{\sqrt{(k+1)^2-4}}\right)\left(\tfrac{k+1+\sqrt{(k+1)^2-4}}{2}\right)^n \\ &+ \tfrac{1}{2}\left(1 - \tfrac{k-1}{\sqrt{(k+1)^2-4}}\right)\left(\tfrac{k+1-\sqrt{(k+1)^2-4}}{2}\right)^n. \end{aligned}$$

Proof. Consider the characteristic polynomial of the linear recurrence found in Section 2. That is, for $a_{n+2} = (k+1)a_{n+1} - a_n$, $p_c(x) = x^2 - (k+1)x + 1$. Since its roots are

$$x = \frac{k+1 \pm \sqrt{(k+1)^2 - 4}}{2},$$

the solution for a_n is written as

$$a_n = A\left(\frac{k+1+\sqrt{(k+1)^2-4}}{2}\right)^n + B\left(\frac{k+1-\sqrt{(k+1)^2-4}}{2}\right)^n.$$

On substituting the initial conditions $a_0 = 1$, $a_1 = k$, a linear system of two equations and two unknowns is obtained:

$$\begin{cases} 1 = A + B, \\ k = A\dfrac{k+1+\sqrt{(k+1)^2-4}}{2} + B\dfrac{k+1-\sqrt{(k+1)^2-4}}{2}. \end{cases}$$

The solution for this system is

$$A = \frac{1}{2}\left(1 + \frac{k-1}{\sqrt{(k+1)^2-4}}\right) \qquad B = \frac{1}{2}\left(1 - \frac{k-1}{\sqrt{(k+1)^2-4}}\right).$$

Hence, the explicit formulae for the elements of the sequence is

$$\begin{aligned} a_n &= \tfrac{1}{2}\left(1 + \tfrac{k-1}{\sqrt{(k+1)^2-4}}\right)\left(\tfrac{k+1+\sqrt{(k+1)^2-4}}{2}\right)^n \\ &+ \tfrac{1}{2}\left(1 - \tfrac{k-1}{\sqrt{(k+1)^2-4}}\right)\left(\tfrac{k+1-\sqrt{(k+1)^2-4}}{2}\right)^n. \quad \square \end{aligned}$$

4 Conclusions

In this note, the statement of a problem from the 2012 Spanish Mathematical Olympiad has been generalized. Two proofs of this statement have been given, one finding a linear relationship for the nonlinear sequence defined in the statement, and another one using primarily some algebraic notions. Additionally, we have proven that the elements of the sequence are congruent with 1 modulo $k-1$.

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[2] Kelley, W. G. and Peterson, A. C. *Difference equations*. Second. An introduction with applications. Harcourt/Academic Press, San Diego, CA, 2001, pp. x+403. ISBN: 0-12-403330-X.

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Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.
2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

Nov 30, 2017

Elementary Problems

E–41. *Proposed by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom.* Prove that the square of the perimeter of a rectangle is at least 16 times its area.

E–42. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* For every integer $n \geq 1$ let t_n denote the n-th triangular number, defined by $t_n = \frac{n(n+1)}{2}$. Find the values of n for which

$$\frac{1^2 + 2^2 + \ldots + n^2}{t_1 + t_2 + \ldots + t_n}$$

is an integer number.

E–43. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In each square of a 2017×2017 chessboard either a $+1$ or a -1 is written. Let r_i be the product of the numbers lying on the i-th row, and let c_j be the product of the numbers lying on the j-th column. Show that $r_1+r_2+\ldots+r_{2017}+c_1+c_2+\ldots+c_{2017} \neq 0$.

E–44. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all positive integers that are divisible by 385 and have exactly 385 distinct positive divisors.

E–45. *Proposed by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom.* Given a regular pentagon of side length 1, find the triangle with the largest area contained inside it.

E–46. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $ABCD$ be a trapezium with bases $AB = a$ and $CD = b$, respectively. Let M be a point on AD such that $\beta\, MA = \alpha\, MD$ for some reals α and β. If the parallel to the bases drawn from M meets BC at N, then show that

$$MN \geq a^{\frac{\beta}{\alpha+\beta}} \cdot b^{\frac{\alpha}{\alpha+\beta}}.$$

Easy–Medium Problems

EM–41. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all solutions of the equation $p(x) = q(x)$, where $p(x)$ and $q(x)$ are polynomials of degree 2 with leading coefficient one such that

$$\sum_{k=1}^{n} p(k) = \sum_{k=1}^{n} q(k),$$

and n is a positive integer.

EM–42. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* If 2017 points on a circle are joined by straight lines in all possible ways and no three of these lines meet at a single point inside the circle, then find the number of triangles that can be formed.

EM–43. *Proposed by Nicolae Papacu, Slobozia, Romania.* Let a, b, c be three positive real numbers such that $\sqrt{a} + \sqrt{b} + \sqrt{c} = 1$. Prove that

$$\frac{\sqrt{a}}{a^2 + 2bc} + \frac{\sqrt{b}}{b^2 + 2ca} + \frac{\sqrt{c}}{c^2 + 2ab} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

EM–44. *Proposed by Andrés Sáez-Schwedt, Universidad de León, León, Spain.* Let ABC be a triangle with $AB = AC > BC$, and let O be the center of its circumcircle Γ. The tangent to Γ at C meets the line AB at D. In the minor arc AC of Γ, consider the point E such that

$$\angle EOC + 2\angle DOA = 360^\circ.$$

If BE meets CD at F, show that $FA = FC$.

EM–45. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $A_1, A_2, \ldots, A_n$ be the vertices of an n-gon inscribed in a circle of center O. If O and the A_i's are lattice points, then prove that the square of the perimeter of the n-gon is an even number.

EM–46. *Proposed by Ángel Plaza de la Hoz, Department of Mathematics, University of Las Palmas de Gran Canaria, Spain.* Prove that for any positive integer n the chain of inequalities

$$F_n^{1/L_n} \leq F_{2n}^{1/(2L_n)} \leq \frac{F_n^{1/F_n} + L_n^{1/L_n}}{2} \leq F_{2n}^{1/(2F_n)} \leq L_n^{1/F_n},$$

holds, where F_n is the n-th Fibonacci number, defined by $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$, and L_n is the n-th Lucas number, defined by $L_0 = 2$, $L_1 = 1$, and for $n \geq 2$, $L_n = L_{n-1} + L_{n-2}$.

Medium–Hard Problems

MH–41. *Proposed by Mihály Bencze, Braşov, Romania.* Determine

$$\left[\sum_{k=1}^{n}(k+1)^{\frac{1}{k^4+k^2+1}}\right],$$

where $[x]$ denotes the integer part of x.

MH–42. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In Mathcontestland there are 2017 towns. Every pair of towns is either connected by a single road, or is not connected. If we consider any subset of 2015 towns, the total number of roads connecting these towns to each other is a constant. If there are R roads in Mathcontestland, then find all possible values of R.

MH–43. *Proposed by Andrés Sáez-Schwedt, Universidad de León, León, Spain.* Points A, B, C and D are collinear in that order. On a circle ω through B and C, two new points E, F are chosen, such that lines AE and DF meet on ω. The tangents to ω at B and C meet at G (possibly at infinity). The tangents to ω at E and F meet GA and GD at P and Q, respectively. Prove that line PQ is tangent to ω.

MH–44. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let x, y and z be be three nonzero real numbers such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{xyz}$. For all positive reals a, b, c, prove that

$$\frac{(a+b)x+(b+c)y+(c+a)z}{\sqrt{ab+bc+ca}}\geq 2.$$

MH–45. *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.* We say that a set of positive integers S is good if there is a function $f:\mathbb{N}\to S$ such that no integer k

with $2 \leq k \leq 2017$ can be written as $\frac{xf(x)}{yf(y)}$. Find the smallest positive integer n such that $S = \{1, 2017, 2017^2, \ldots, 2017^n\}$ is good, or prove that such an integer does not exist.

MH–46. *Proposed by Ismael Morales López, Universidad Autónoma de Madrid, Madrid, Spain.* Let a, b, c be the lengths of the sides of a given triangle ABC, and m_a, m_b, m_c its medians. Prove that it is true that

$$\sum_{\text{cyclic}} \frac{a+b}{c^2+2ab+3bc+3ca} \geq \frac{1}{\sqrt{m_a^2+m_b^2+m_c^2}}.$$

Advanced Problems

A–41. *Proposed by Marcin J. Zygmunt, AGH University of Science and Technology, Kraków, Poland.* Let $\gamma_1 < \gamma_2 < \ldots < \gamma_n$ be real numbers,

$$f(x) = x + \frac{1}{\gamma_1 - x} + \frac{1}{\gamma_2 - x} + \cdots + \frac{1}{\gamma_n - x},$$

and let α, $\beta \in \mathbb{R}$, $\alpha < \beta$. Compute the total length of the preimage $f^{-1}([\alpha, \beta])$ (the total length of a set consisting of intervals is the sum of their lengths.)

A–42. *Proposed by Óscar Rivero Salgado, BarcelonaTech, Barcelona, Spain.* Let p be a prime number and consider A a $p \times p$ matrix with complex entries, satisfying that $\mathrm{Tr}(A) = \mathrm{Tr}(A^2) = \ldots = \mathrm{Tr}(A^{p-1}) = 0$ and $\mathrm{Tr}(A^p) = p$. Find $\det(A^i + j\,\mathrm{Id})$, for i, $j \in \mathbb{Z}$.

A–43. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all real solutions of the following system of equations

$$\left.\begin{aligned} x^4 + x^2 + (y^4+1)\sqrt{y^4+1} &= (2+\sqrt{2})\, z^2\sqrt{z^4+1}, \\ y^4 + y^2 + (z^4+1)\sqrt{z^4+1} &= (2+\sqrt{2})\, x^2\sqrt{x^4+1}, \\ z^4 + z^2 + (x^4+1)\sqrt{x^4+1} &= (2+\sqrt{2})\, y^2\sqrt{y^4+1}. \end{aligned}\right\}$$

A–44. *Proposed by Nicolae Papacu, Slobozia, Romania.* Let a and x_0 be real numbers and let $\{x_n\}_{n\geq 0}$ be the sequence defined by $x_{n+1} = ax_n^2 - (2a-1)x_n$.

(a) Prove that the sequence $\{x_n\}_{n\geq 0}$ is convergent if, and only if, $ax_0 \in [a-1, a]$.

(b) If $ax_0 \in [a-1, a]$, compute $\lim\limits_{n\to+\infty} n(x-\ell)$, where $\ell = \lim\limits_{n\to+\infty} x_n$.

A–45. *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School (BMS), Berlin, Germany.* Let n be a positive integer, and let

A be a $n \times (n+1)$ matrix with integer entries. Assume that, for every prime p, the matrix A considered as a matrix with entries in $\mathbb{Z}/p\mathbb{Z}$ has rank n. Prove that there exists a matrix B of size $(n+1) \times n$ with integer entries such that $AB = Id_n$.

A–46. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Compute

$$\lim_{n \to +\infty} \sum_{k=1}^{n} \arctan\left(\frac{k}{n^2}\right).$$

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Forward-Backward Induction

J. L. Díaz-Barrero

1 Introduction

When trying to prove a statement $P(n)$ involving the positive integer n and a forward inductive argument is difficult for every n, one can use the following strategy: first, prove the statement for infinitely many values of n, and then, prove $P(n)$ for the gaps. The proof for the gaps can either be by forward induction, or backward induction.

Downward inductive arguments have been around a long time. Many important authors, including Cauchy (1759-1857) and Weierstrass (1815-1897, see [3, p. 19]), have used them. Forward-backward induction, also known as Cauchy's induction, is a variant of the well-known principle of mathematical induction. The main difference between this type of induction and the classical one is the inductive step. It can be stated as follows:

Theorem 1. *Let $\{P(n)\}_{n\geq 1}$ be a sequence of propositions involving the positive integer n. If it holds that*

- **Step 1:** *$P(n_0)$ is true for a positive integer n_0.*
- **Step 2:** *This is made out of two parts.*
  Forward induction: *$P(k) \Rightarrow P(2k)$ for every integer $k \geq n_0$.*
  Backward induction: *$P(k) \Rightarrow P(k-1)$ for every integer $k \geq n_0 + 2$.*

Then, $P(n)$ is true for all integers $n \geq n_0$.

In particular when $n_0 = 2$ the statement becomes

Theorem 2. *Let $P(n)$ be a proposition involving $n \in \mathbb{N}$. If $P(2)$ holds, $P(n) \Rightarrow P(2n)$ for $n \geq 2$ and $P(n) \Rightarrow P(n-1)$ for all $n \geq 4$, then $P(n)$ holds for all $n \geq 2$.*

In the following section, we will use the forward-backward induction to prove the AM-GM inequality and Huygens's inequality.

2 Applications

Hereafter, some examples where forward-backward induction may be used are presented. We begin with the proof given by Cauchy [1] of the AM-GM inequality.

Theorem 3. *Let $a_1, a_2, \ldots, a_n$ be $n \geq 1$ positive real numbers. Then,*

$$\left(\prod_{j=1}^{n} a_j\right)^{1/n} \leq \frac{1}{n}\sum_{j=1}^{n} a_j.$$

Proof. The case when $n = 1$ trivially holds, so we will start considering the base case when $n = 2$. That is, we have to prove that for any positive reals a_1, a_2 the inequality $\dfrac{a_1 + a_2}{2} \geq \sqrt{a_1 a_2}$ holds. Indeed, the preceding inequality is equivalent to $a_1 + a_2 \geq 2\sqrt{a_1 a_2}$ or $(\sqrt{a_1} - \sqrt{a_2})^2 \geq 0$, which trivially holds. This is the base step.

Next we prove the forward induction step assuming the inequality holds for some integer k and we prove it for $2k$. That is, we suppose that

$$\left(\prod_{j=1}^{k} a_j\right)^{1/k} \leq \frac{1}{k}\sum_{j=1}^{k} a_j$$

holds and we will prove that

$$\left(\prod_{j=1}^{2k} a_j\right)^{1/2k} \leq \frac{1}{2k}\sum_{j=1}^{2k} a_j$$

also holds. Indeed, we have

$$\begin{aligned}
\frac{a_1 + a_2 + \ldots + a_{2k}}{2k} &= \frac{1}{2}\left(\frac{a_1 + \ldots + a_k}{k} + \frac{a_{k+1} + \ldots + a_{2k}}{k}\right) \\
&\geq \frac{1}{2}\left(\sqrt[k]{a_1 a_2 \ldots a_k} + \sqrt[k]{a_{k+1} a_{k+2} \ldots a_{2k}}\right) \\
&\geq \sqrt{\sqrt[k]{a_1 a_2 \ldots a_k} \cdot \sqrt[k]{a_{k+1} a_{k+2} \ldots a_{2k}}} \\
&= \sqrt[2k]{a_1 a_2 \ldots a_{2k}}.
\end{aligned}$$

The first inequality follows from k variable AM-GM, which is true because of our inductive hypothesis, and the second inequality follows from the 2 variable AM-GM, which we just proved above (base step).

Now we prove the backward induction step. Since the AM-GM inequality holds for any positive reals, in particular it also holds when $a_k = \dfrac{a_1 + a_2 + \ldots + a_{k-1}}{k-1}$. Then, assuming that

$$\frac{a_1 + a_2 + \ldots + a_k}{k} \geq \sqrt[k]{a_1 a_2 \ldots a_k}$$

holds, we have

$$\begin{aligned}
&\frac{1}{k}\left(a_1 + \ldots + a_{k-1} + \frac{a_1 + \ldots + a_{k-1}}{k-1}\right) \geq \sqrt[k]{a_1 \ldots a_{k-1} \cdot \frac{a_1 + \ldots + a_{k-1}}{k-1}} \\
&\iff \frac{a_1 + \ldots + a_{k-1}}{k-1} \geq \sqrt[k]{a_1 \ldots a_{k-1} \cdot \frac{a_1 + \ldots + a_{k-1}}{k-1}} \\
&\iff \left(\frac{a_1 + \ldots + a_{k-1}}{k-1}\right)^k \geq a_1 \ldots a_{k-1} \cdot \frac{a_1 + \ldots + a_{k-1}}{k-1} \\
&\iff \left(\frac{a_1 + \ldots + a_{k-1}}{k-1}\right)^{k-1} \geq a_1 a_2 \ldots a_{k-1} \\
&\iff \frac{a_1 + \ldots + a_{k-1}}{k-1} \geq \sqrt[k-1]{a_1 a_2 \ldots a_{k-1}}
\end{aligned}$$

Since we have shown that the base case and the inductive hypothesis steps hold, then by the forward-backward induction (FBI) principle, the AM-GM inequality is proven. □

Another interesting example due to Huygens [4] is the following.

Theorem 4. *For $n \geq 2$ and $a_1, \ldots, a_n, b_1, \ldots, b_n > 0$,*

$$\sqrt[n]{(a_1+b_1)\ldots(a_n+b_n)} \geq \sqrt[n]{a_1\ldots a_n} + \sqrt[n]{b_1\ldots b_n}$$

holds, with equality if, and only if, the vectors $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are linearly dependent.

Proof. If we say $P(n)$ is the property that

$$\sqrt[n]{(a_1+b_1)\cdots(a_n+b_n)} \geq \sqrt[n]{a_1\cdots a_n} + \sqrt[n]{b_1\cdots b_n},$$

then when $n = 2$ we have

$$\sqrt{(a_1+b_1)(a_2+b_2)} \geq \sqrt{a_1a_2} + \sqrt{b_1b_2}.$$

That is,

$$\begin{aligned} (a_1+b_1)(a_2+b_2) &\geq a_1a_2 + b_1b_2 + 2\sqrt{a_1a_2b_1b_2}, \\ \Longleftrightarrow (a_1b_2+a_2b_1)^2 &\geq 4a_1a_2b_1b_2, \\ \Longleftrightarrow (a_1b_2-a_2b_1)^2 &\geq 0, \end{aligned}$$

and $P(2)$ holds.

For $n \geq 2$, suppose that $P(n)$ holds. We have to see that $P(2n)$ also holds. In fact,

$$\begin{aligned} &\sqrt[2n]{(a_1+b_1)\cdots(a_{2n}+b_{2n})} \\ &= \left\{\sqrt[n]{(a_1+b_1)\cdots(a_n+b_n)}\,\sqrt[n]{(a_{n+1}+b_{n+1})\cdots(a_{2n}+b_{2n})}\right\}^{1/2} \\ &\geq \left\{\left(\sqrt[n]{a_1\cdots a_n} + \sqrt[n]{b_1\cdots b_n}\right)\left(\sqrt[n]{a_{n+1}\cdots a_{2n}} + \sqrt[2n]{b_{n+1}\cdots b_{2n}}\right)\right\}^{1/2} \\ &\geq \left\{\sqrt[n]{a_1\cdots a_n}\,\sqrt[n]{a_{n+1}\cdots a_{2n}}\right\}^{1/2} + \left\{\sqrt[n]{b_1\cdots b_n}\,\sqrt[2n]{b_{n+1}\cdots b_{2n}}\right\}^{1/2} \\ &= \sqrt[2n]{a_1\cdots a_{2n}} + \sqrt[2n]{b_1\cdots b_{2n}}. \end{aligned}$$

Finally, for $n \geq 4$, suppose that $P(n)$ holds and we have to see that $P(n-1)$ also holds. Indeed,

$$\begin{aligned} &\left\{(a_1+b_1)\cdots(a_{n-1}+b_{n-1})\left(\sqrt[n-1]{a_1\cdots a_{n-1}} + \sqrt[n-1]{b_1\cdots b_{n-1}}\right)\right\}^{1/n} \\ &\geq \left\{a_1\cdots a_{n-1}\,\sqrt[n-1]{a_1\cdots a_{n-1}}\right\}^{1/n} + \left\{b_1\cdots b_{n-1}\,\sqrt[n-1]{b_1\cdots b_{n-1}}\right\}^{1/n} \\ &= \sqrt[n-1]{a_1\cdots a_{n-1}} + \sqrt[n-1]{b_1\cdots b_{n-1}}. \end{aligned}$$

After some straightforward algebra, we get

$$\sqrt[n-1]{(a_1+b_1)\cdots(a_{n-1}+b_{n-1})} \geq \sqrt[n-1]{a_1\cdots a_{n-1}} + \sqrt[n-1]{b_1\cdots b_{n-1}}.$$

Equality holds when the vectors of a's and b's are proportional. Therefore, by Cauchy's induction $P(n)$ holds for all $n \geq 2$, and this completes the proof. □

An immediate consequence of the preceding result is the following.

Corollary 1. *Suppose that $x_1, x_2, \cdots, x_n$ and α are positive real numbers. Then,*

$$\frac{1}{n}\sum_{k=1}^{n}\left(x_k+\alpha^k\right)^2 \geq \left(\alpha^{(n+1)/2} + \prod_{k=1}^{n} x_k^{1/n}\right)^2$$

holds.

Proof. First, we write the inequality claimed as

$$\sqrt{\frac{(x_1+\alpha)^2+\cdots+(x_n+\alpha^n)^2}{n}} \geq \alpha^{(n+1)/2} + \sqrt[n]{x_1x_2\cdots x_n}.$$

Taking into account the QM-GM inequality, we have

$$\sqrt{\frac{(x_1+\alpha)^2+\cdots+(x_n+\alpha^n)^2}{n}} \geq \sqrt[n]{(x_1+\alpha)\cdots(x_n+\alpha^n)}.$$

So, to prove our inequality it will suffice to establish that

$$\sqrt[n]{(x_1+\alpha)\cdots(x_n+\alpha^n)} \geq \alpha^{(n+1)/2} + \sqrt[n]{x_1\cdots x_n}.$$

Now, putting $a_k = x_k$ and $b_k = \alpha$, $1 \leq k \leq n$, in the last theorem, we get

$$\begin{aligned}\sqrt[n]{(x_1+\alpha)\cdots(x_n+\alpha)} &\geq \sqrt[n]{\alpha^{1+2+\cdots+n}} + \sqrt[n]{x_1\cdots x_n}\\ &= \alpha^{(n+1)/2} + \sqrt[n]{x_1\cdots x_n},\end{aligned}$$

and the proof is complete. □

Finally, we close this note with a nice result that appeared in [2].

Theorem 5. *Let $a_1, a_2, \ldots, a_n$ be positive integers such that $a_1 \leq a_2 \leq \ldots \leq a_n$. If*

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1,$$

then $a_n < 2^{n!}$.

To prove the statement we need the following lemma.

Lemma 1. *Let $n \geq 2$, and let $a_1, a_2, \ldots, a_n$ be positive integers such that*

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1.$$

For $m < n$, it holds that

$$\frac{1}{a_1 a_2 \ldots a_m} \leq 1 - \left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_m}\right).$$

Proof. Let r be a positive rational number such that

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_m} + \frac{1}{r} = 1.$$

Then, we have

$$\begin{aligned} r &= \frac{1}{1 - \left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_m}\right)} \\ &= \frac{a_1 a_2 \ldots a_m}{a_1 a_2 \ldots a_m - \left(\sum_{i=1}^{m} \prod_{j \neq i} a_j\right)} \\ &\leq a_1 a_2 \ldots a_m \end{aligned} \tag{1}$$

on account that since both sides of the above expression are positive, then the last denominator is a positive integer. That is, it is at least 1 and the last inequality immediately follows. Now, inverting the expression in (1) we obtain

$$\frac{1}{a_1 a_2 \ldots a_m} \leq 1 - \left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_m}\right),$$

and this completes the proof. □

Proof of the theorem. We argue by contradiction. For $n \geq 2$, let $A(n)$ be the statement that for positive integers satisfying $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n$, if $\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} = 1$, then $a_n < 2^{n!}$. Assume, for some fixed $n \geq 2$, that $A(n)$ fails. That is, suppose the statement

$$P(n) \,:\, a_n \geq 2^{n!}$$

holds. By strong backward induction, we prove that $a_j \geq 2^{j!}$ for $1 \leq j \leq n$. Suppose the assumption is proved for $n \geq j \geq m+1$. It remains to prove $P(m)$. To this goal,

$$\begin{aligned} \frac{1}{a_m} &\leq \sqrt[m]{\frac{1}{a_1 a_2 \ldots a_m}} \leq \sqrt[m]{\left(1 - \frac{1}{a_1} - \frac{1}{a_2} - \ldots - \frac{1}{a_m}\right)} \\ &= \sqrt[m]{\left(\frac{1}{a_{m+1}} + \ldots + \frac{1}{a_n}\right)} \leq \sqrt[m]{\sum_{j=m+1}^{n} \frac{1}{2^{j!}}} \leq \frac{1}{2^{m!}}. \end{aligned}$$

But in this case we have

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n} \leq \sum_{j=1}^{n} \frac{1}{2^{j!}} < \sum_{j=1}^{+\infty} \frac{1}{2^j} = 1.$$

Contradiction. Therefore, $A(n)$ is true and this completes the proof. □

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Solutions

No problem is ever permanently closed. We will be very pleased to consider for publication new solutions or comments on the past problems.

Please, send submittals to **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to

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Elementary Problems

E–35. *Proposed by Eric Sierra Garzo, CFIS, BarcelonaTech, Barcelona, Spain.* Find the sum of the coefficients of

$$P(x) = 6\left(7x^2 - 9x^3 + 3x^5\right)^{2016} + 3\left(5x^7 - 2x^{11} - 4x^{13}\right)^{2017} - 2x - 1.$$

Solution by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain. Let us consider a general polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. We have that $P(1) = a_n + a_{n-1} + \ldots + a_1 + a_0$, that is, $P(1)$ is the sum of the coefficients of the polynomial. Therefore, to compute the sum we just need to substitute $x = 1$ in the given expression:

$$\begin{aligned} P(1) &= 6 \cdot (7 - 9 + 3)^{2016} + 3 \cdot (5 - 2 - 4)^{2017} - 2 - 1 \\ &= 6 \cdot 1^{2016} + 3 \cdot (-1)^{2017} - 3 = 6 - 3 - 3 = 0. \end{aligned}$$

Also solved by *Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Andrew J. Turner, University of Birmingham, Birmingham, United Kingdom, and the proposer.*

E–36. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain, and Mihály Bencze, Braşov, Romania.* Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Show that for all integers $m \geq 1$ there exist n positive reals $b_1, b_2, \ldots, b_n$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{1+a_k}+\frac{1}{1+b_k}\right)^m = 1.$$

Solution 1 by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom. Observe that one possible solution would be to make each of the sums $\dfrac{1}{1+a_k}+\dfrac{1}{1+b_k}$ equal 1. Indeed, in such a case, no matter which value of m we take, we would have that each of the summands equals 1, and as there are exactly n summands and we divide by n, the desired equality holds. Now let us prove that

$$\frac{1}{1+a_k}+\frac{1}{1+b_k} = 1$$

may hold for any positive value of a_k. Indeed, we have that

$$\begin{aligned}\frac{1}{1+a_k}+\frac{1}{1+b_k} = 1 &\iff \frac{1}{1+b_k} = \frac{1+a_k-1}{1+a_k}\\ &\iff 1+b_k = \frac{1+a_k}{a_k}\\ &\iff b_k = \frac{1}{a_k},\end{aligned}$$

which is a positive number. This completes the proof.

Solution 2 by Henry Ricardo, Westchester Math Circle, New York, USA. Let $b_k = 1/a_k$ for $k = 1, 2, \ldots, n$. Then, for any

integer $m \geq 1$,

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{1+a_k}+\frac{1}{1+b_k}\right)^m &= \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{1+a_k}+\frac{1}{1+1/a_k}\right)^m \\ &= \frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{1+a_k}+\frac{a_k}{a_k+1}\right)^m \\ &= \frac{1}{n}\sum_{k=1}^{n}(1)^m = 1.\end{aligned}$$

Also solved by *Miguel Cidrás Senra, CFIS, BarcelonaTech, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain, and the proposer.*

E–37. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $PQRS$ be a square. A straight line through P cuts side RS at M and line QR at N. Show that

$$\left(\frac{PQ}{PM}\right)^2+\left(\frac{PQ}{PN}\right)^2$$

is a positive integer.

Solution 1 by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain. Let $\angle PNQ = \alpha$. Then

$$\frac{PQ}{PN} = \sin\alpha.$$

Looking at the angles we have

$$\begin{aligned}\angle NRM = \angle PSM = 90^\circ &\implies \angle RMN = 90^\circ - \alpha = \angle PMS \\ &\implies \angle SPM = \alpha.\end{aligned}$$

As $PQRS$ is a square, $PQ = PS$, so we have that

$$\cos\alpha = \frac{PS}{PM} = \frac{PQ}{PM}.$$

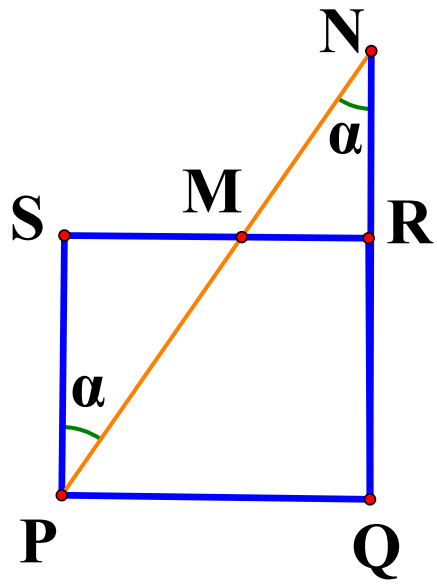

Figure 1: Construction for Solution 1 of Problem E–37.

Therefore,

$$\left(\frac{PQ}{PM}\right)^2 + \left(\frac{PQ}{PN}\right)^2 = \sin^2\alpha + \cos^2\alpha = 1.$$

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. Let X be the point of side PQ of square $PQRS$ such that XM is parallel to QR. Since PQ is parallel to RS, we have $XM = QR$. Thus,

$$XM = PQ. \tag{1}$$

Since $XM \parallel QR$ and $QR \perp PQ$, we have $XM \perp PQ$. From the similar right triangles PXM and PQN, we get $\frac{PQ}{PN} = \frac{PX}{PM}$; hence

$$\begin{aligned}\left(\frac{PQ}{PM}\right)^2 + \left(\frac{PQ}{PN}\right)^2 &= \left(\frac{PQ}{PM}\right)^2 + \left(\frac{PX}{PM}\right)^2 \\ &= \frac{PQ^2 + PX^2}{PM^2} \\ &= \frac{XM^2 + PX^2}{PM^2} \qquad \text{(from (1))} \\ &= 1,\end{aligned}$$

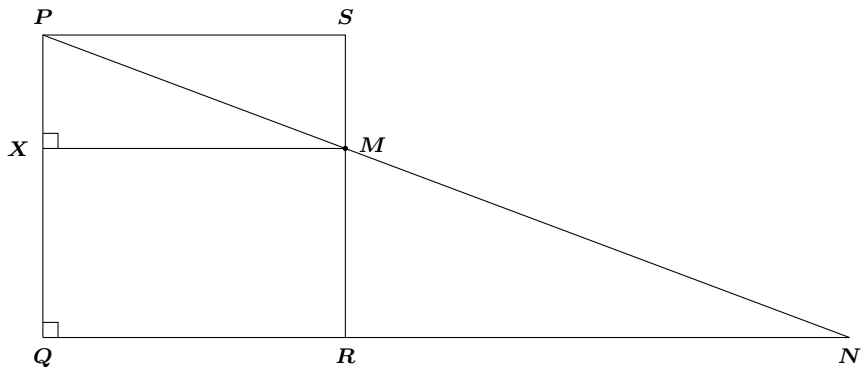

Figure 2: Construction for Solution 2 of Problem E–37.

since $XM^2 + PX^2 = PM^2$ holds by the Pythagorean theorem applied to $\triangle PXM$.

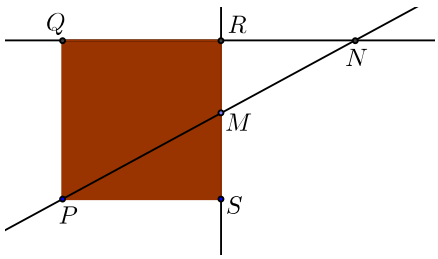

Figure 3: Scheme for Solution 3 of Problem E–37.

Solution 3 by Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain. Let l be the length of the square. It is clear that at least one of M, N is touching the square. By symmetry, we can assume WLOG that M is touching the square. From Figure 3 we have that, if we center the origin at P and put S at $(l, 0)$ in the plane, M will have coordinates (l, h). The line has a slope $\frac{h}{l}$, so, as N has second coordinate equal to h and N lies on the line,

$N = \left(\frac{l^2}{h}, l\right)$. Therefore,

$$\begin{aligned}\left(\frac{PQ}{PM}\right)^2 + \left(\frac{PQ}{PN}\right)^2 &= \left(\frac{l}{\sqrt{l^2+h^2}}\right)^2 + \left(\frac{l}{\sqrt{\frac{l^4}{h^2}+l^2}}\right)^2 \\ &= \frac{l^2}{l^2+h^2} + \frac{l^2}{\frac{l^4}{h^2}+l^2} \\ &= \frac{l^2}{l^2+h^2} + \frac{h^2}{l^2+h^2} = \frac{l^2+h^2}{l^2+h^2} = 1.\end{aligned}$$

Also solved by *Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain, and the proposer.*

E–38. *Proposed by José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.* Let $f(x) = x^2 + 14x + 42$. Solve the equation

$$\underbrace{f(f(f(\ldots(f(x)))))}_{n} = 0.$$

Solution 1 by the proposer. Since $f(x) = (x+7)^2 - 7$, then $f(f(x)) = (x+7)^4 - 7$, $f(f(f(x))) = (x+7)^8 - 7$ and so on. By induction, we easily establish that

$$\underbrace{f(f(f(\ldots(f(x)))))}_{n} = (x+7)^{2^n} - 7.$$

Then the solutions of the equation $(x+7)^{2^n} - 7 = 0$ are $-7 \pm \sqrt[2^n]{7}$, and we are done.

Solution 2 by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain. For $n = 1$ we have

$$\begin{aligned} f(x) = x^2 + 14x + 42 &= 0 \\ \Longrightarrow x = \frac{-14 \pm \sqrt{196 - 4 \cdot 42}}{2} &= -7 \pm \sqrt{7}. \end{aligned}$$

For $n = 2$ we have

$$f(f(x)) = 0 \implies f(x) = -7 \pm \sqrt{7}.$$

However, $f(x) = x^2 + 14x + 42 = (x+7)^2 - 7 \geq -7$, so

$$\begin{aligned} f(f(x)) = 0 &\implies f(x) = -7 + \sqrt{7} \\ &\implies x^2 + 14x + 49 - \sqrt{7} = 0 \\ &\implies x = \frac{-14 \pm \sqrt{196 - 196 + 4\sqrt{7}}}{2} = -7 \pm \sqrt[4]{7}. \end{aligned}$$

It looks like the general solution is going to be

$$\underbrace{f(f(f(\ldots(f(x)))))}_{n} = 0 \implies x = -7 \pm \sqrt[2^n]{7}.$$

Let us prove this by induction. We have already proved the base case. Now, we need to prove that if for some $n = k$ the property holds, then it also holds for $n = k+1$. Indeed,

$$\underbrace{f(f(f(\ldots(f(x)))))}_{k+1} = 0 \implies f(x) = -7 + \sqrt[2^k]{7}$$

by induction hypothesis. Notice that we only consider the solution greater than -7 since, as we already stated, the image of f is in $[-7, +\infty)$. Hence,

$$\begin{aligned} & f(x) = -7 + \sqrt[2^k]{7} \\ \implies & x^2 + 14x + 49 - \sqrt[2^k]{7} = 0 \\ \implies & x = \frac{-14 \pm \sqrt{196 - 196 + 4\sqrt[2^k]{7}}}{2} = -7 \pm \sqrt[2^{k+1}]{7}. \end{aligned}$$

Therefore, the induction hypothesis is true, and for any $n \in \mathbb{N}$ the solution of the equation is going to be $x = -7 \pm \sqrt[2^n]{7}$.

Also solved by *Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain, and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.*

E–39. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let D, E, F be the points where the angle bisectors of angles A, B, C of triangle ABC cut the opposite sides, and let I be its incenter. Find the minimum value of

$$\frac{AI}{ID}+\frac{BI}{IE}+\frac{AI}{IF}.$$

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. By the angle bisector theorem, applied to $\triangle ABD$,

$$\frac{AI}{ID}=\frac{AB}{BD},$$

and applied to $\triangle ABC$,

$$\frac{AB}{BD}=\frac{CA}{DC}=\frac{AB+CA}{BD+DC}=\frac{AB+CA}{BC}.$$

Hence

$$\frac{AI}{ID}=\frac{AB+CA}{BC}.$$

Similarly,

$$\frac{BI}{IE}=\frac{BC+AB}{CA} \qquad \text{and} \qquad \frac{CI}{IF}=\frac{CA+BC}{AB}.$$

Hence,

$$\begin{aligned}\frac{AI}{ID}+\frac{BI}{IE}+\frac{CI}{IF}&=\frac{AB+CA}{BC}+\frac{BC+AB}{CA}+\frac{CA+BC}{AB}\\&=\left(\frac{AB}{BC}+\frac{BC}{CA}\right)+\left(\frac{BC}{CA}+\frac{CA}{BC}\right)+\left(\frac{CA}{AB}+\frac{AB}{CA}\right)\\&\geq 2+2+2\\&=6.\end{aligned}$$

Note that equality holds if, and only if, $\triangle ABC$ is equilateral. We therefore conclude that the minimum value of the given expression is 6 and it is achieved only for equilateral $\triangle ABC$.

Solution 2 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany. Let S, S_A, S_B and S_C be the areas of ABC, IBC, ICA and IAB, respectively. Let P be the point where the incircle touches BC and let Q be the foot of the altitude from A. Triangles DIP and DAQ have parallel sides, so

$$\frac{ID}{AD} = \frac{IP}{AQ} = \frac{\frac{BC \cdot IP}{2}}{\frac{BC \cdot AQ}{2}} = \frac{S_A}{S}.$$

From this, $\dfrac{AI}{ID} = \dfrac{AD - ID}{ID} = \dfrac{S - S_A}{S_A} = \dfrac{S_B + S_C}{S_A}$. We can do the same for the other sides. Thus

$$\begin{aligned}\frac{AI}{ID} + \frac{BI}{IE} + \frac{CI}{IF} &= \frac{S_A + S_B}{S_C} + \frac{S_B + S_C}{S_A} + \frac{S_C + S_A}{S_B} \\ &= \left(\frac{S_A}{S_B} + \frac{S_B}{S_A}\right) + \left(\frac{S_B}{S_C} + \frac{S_C}{S_B}\right) + \left(\frac{S_C}{S_A} + \frac{S_A}{S_C}\right) \\ &\geq 2 + 2 + 2 = 6.\end{aligned}$$

Note that this bound is tight, since for an equilateral triangle we have $S_A = S_B = S_C$.

Remark. Next, for ease of reference, we state and prove the bisector theorem.

Theorem (Bisector theorem). *Let D be the point where the bisector of angle A in triangle ABC cuts the side BC. Then,*

$$\frac{AB}{BD} = \frac{AC}{CD}.$$

Proof. On account of the Sine Law, we have

$$\frac{x}{\sin \alpha/2} = \frac{c}{\sin \gamma}$$

in $\triangle ABD$, and

$$\frac{y}{\sin \alpha/2} = \frac{c}{\sin(\pi - \gamma)}$$

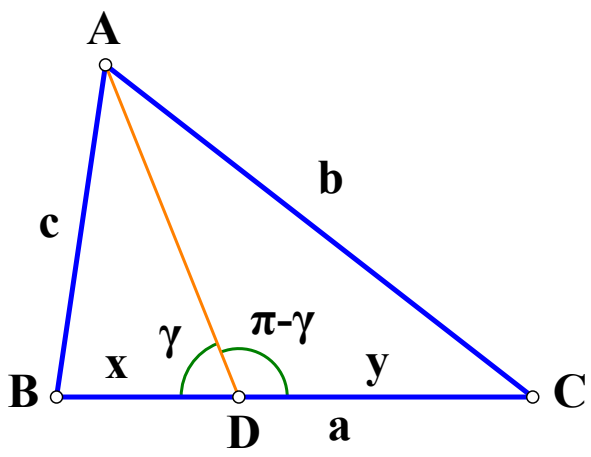

Figure 4: Construction for the proof of the Bisector Theorem.

in $\triangle ADC$, from which

$$\frac{x}{y} = \frac{c}{b} \Longleftrightarrow \frac{AB}{BD} = \frac{c}{x} = \frac{b}{y} = \frac{AC}{CD}$$

follows. □

Also solved by *Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain; Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain, and the proposer.*

E–40. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In a certain parliament, each member belongs to exactly four committees, and each committee has exactly four members. Prove that the number of members equals the number of committees.

Solution 1 by Miguel Cidrás Senra, CFIS, BarcelonaTech, Barcelona, Spain. Let m be the number of members and c, the number of committees. We can label the members and the comittees and define a variable a_{ij} such that $a_{ij} = 1$ if, and only if, the i-th member belongs to the j-th committee, and $a_{ij} = 0$

otherwise. We can observe that $\sum_{i=1}^{m} a_{ij} = 4$ and $\sum_{j=1}^{c} a_{ij} = 4$. Then,

$$4 \cdot c = \sum_{j=1}^{c} \sum_{i=1}^{m} a_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{c} a_{ij} = 4 \cdot m,$$

and finally, we obtain $c = m$.

Solution 2 by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom. Consider a graph defined as follows. Consider two sets of vertices, one of them consisting on the members of the parliament (call this set A), and the other one consisting on the committees (call this set B). We join a member of the parliament to a committee with an edge if, and only if, the member belongs to the committee. This graph is clearly bipartite (that is, all the edges join A to B, and there are no edges joining two vertices in A or two vertices in B). The conditions of the statement tell us that the degree of each vertex (the number of edges incident to that vertex) is 4, both for the vertices in A and in B. Now assume that $|A| = n$ and $|B| = m$. As each edge joins a vertex in A with a vertex in B, we may use a double-counting argument: the overall number of edges can be counted as the number of edges incident to A, or as the number of edges incident to B. These are, respectively, $4n$ and $4m$. We then have that $4n = 4m$, which immediately implies the claim.

Solution 3 by the proposer. Consider a matrix (called the incidence matrix) in which each row represents an individual, and each column represents a committee. An entry is 1 if the individual corresponding to its row belongs to the committee corresponding to its column; otherwise, the entry is 0. Of course, the roles of rows and columns may be interchanged. An example of such a configuration is shown below.

To solve the problem, we will count how many 1's there are in the incidence matrix. Suppose that there are n committees and m members. Then the incidence matrix is an $m \times n$ matrix. The given conditions tell us that each row contains 4 ones, so there are $4m$ ones in total. On the other hand, each column contains 4 ones, so there are $4n$ ones in total. Equating the two counts we see that $4m = 4n$, so $m = n$, which is what we wanted to prove.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Also solved by *Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain; Padraig Condon, University of Birmingham, Birmingham, United Kingdom; Matthew Coulson, University of Birmingham, Birmingham, United Kingdom; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain, and Patrick Morris, Berlin Mathematical School, Berlin, Germany.*

Easy–Medium Problems

EM–35. *Proposed by Mihály Bencze, Braşov, Romania.* Find the value of

$$a_1 + \sum_{k=1}^{n} a_k^2,$$

where $a_k = \dfrac{1}{k} + \dfrac{1}{k+1} + \cdots + \dfrac{1}{n}$.

Solution 1 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany. Let $a_{k,n} = \dfrac{1}{k} + \dfrac{1}{k+1} + \cdots + \dfrac{1}{n}$. We want to find the value of $s_n = a_{1,n} + \sum\limits_{k=1}^{n} a_{k,n}^2$. We claim that $s_n = 2n$.

We proceed by induction. For $n = 1$ it holds because $s_1 = a_{1,1} + a_{1,1}^2 = 2$. Now we prove the induction step:

$$\begin{aligned}
s_n - s_{n-1} &= a_{1,n} - a_{1,n-1} + a_{n,n}^2 + \sum_{k=1}^{n-1}(a_{k,n}^2 - a_{k,n-1}^2) \\
&= \frac{1}{n} + \frac{1}{n^2} + \sum_{k=1}^{n-1}(a_{k,n} - a_{k,n-1})(a_{k,n} + a_{k,n-1}) \\
&= \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n}\sum_{k=1}^{n-1}\left(\frac{1}{n} + 2a_{k,n-1}\right) \\
&= \frac{1}{n} + \frac{1}{n^2} + \frac{n-1}{n^2} + \frac{2}{n}\sum_{k=1}^{n-1}\sum_{i=k}^{n-1}\frac{1}{i} \\
&= \frac{2}{n} + \frac{2}{n}\sum_{i=1}^{n-1}\sum_{k=1}^{i}\frac{1}{i} \\
&= \frac{2}{n} + \frac{2}{n}\sum_{i=1}^{n-1} 1 \\
&= \frac{2}{n} + \frac{2(n-1)}{n} = 2.
\end{aligned}$$

Solution 2 by Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain. We have $a_k = H_n - H_{k-1}$, where $H_k = \sum_{i=1}^{k} \frac{1}{i}$ is the k-th harmonic number.

$$\begin{aligned}
H_n + \sum_{k=1}^{n}\left(\sum_{j=k}^{n}\frac{1}{j}\right)^2 =& H_n + \sum_{k=1}^{n}\sum_{j=k}^{n}\frac{1}{j^2} + 2\sum_{k\le i<j\le n}\frac{1}{ij}\\
=& H_n + \sum_{j=1}^{n}\frac{|\{k\in\mathbb{N}: 1\le k\le j\}|}{j^2}\\
&+ 2\sum_{1\le i<j\le n}\frac{|\{k\in\mathbb{N}: 1\le k\le i\}|}{ij}\\
=& H_n + \sum_{j=1}^{n}\frac{j}{j^2} + 2\sum_{1\le i<j\le n}\frac{i}{ij}\\
=& H_n + \sum_{j=1}^{n}\frac{1}{j} + 2\sum_{1\le i<j\le n}\frac{1}{j}\\
=& 2H_n + 2\sum_{j=1}^{n}\frac{|\{i\in\mathbb{N}: 1\le i<j\}|}{j}\\
=& 2H_n + 2\sum_{j=1}^{n}\frac{j-1}{j}\\
=& 2H_n + 2n - 2H_n = 2n.
\end{aligned}$$

Solution 3 by the proposer. Squaring the a_k's we obtain k terms of the form $\frac{1}{k^2}$ in $a_1^2, a_2^2, \ldots, a_k^2$. Adding them up, we get $k\,\frac{1}{k^2} = \frac{1}{k}$ and the sum of all of them is $\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = a_1$.

Now we take the two-term products. If $j < i$, then $2\,\frac{1}{ij}$ appears in all a_k^2 for which $k \le j$, therefore their sum is $j\,\frac{2}{ji} = \frac{2}{i}$.

For all i there exist $i-1$ terms such that $j < i$. Therefore, for each fixed i, all terms of the form $\frac{2}{ij}$ have sum $(i-1)\frac{2}{i} = 2 - \frac{2}{i}$.

Computing all of these sums we get

$$\begin{aligned}\left(2-\frac{2}{1}\right)+\left(2-\frac{2}{2}\right)+\cdots+\left(2-\frac{2}{n}\right) &= 2n-2\left(\frac{1}{1}+\cdots+\frac{1}{n}\right)\\ &= 2n-2a_1.\end{aligned}$$

Finally,

$$a_1+\sum_{k=1}^{n} a_k^2 = a_1+a_1+2n-2a_1 = 2n,$$

and we are done.

Also solved by *Henry Ricardo, Westchester Math Circle, New York, USA.*

EM–36. *Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let x_1, x_2, x_3 be the roots of the equation $x^3-3x^2-2x+1=0$. Find the equation whose roots are $y_1=x_1+\dfrac{1}{x_1}$, $y_2=x_2+\dfrac{1}{x_2}$, and $y_3=x_3+\dfrac{1}{x_3}$.

Solution by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain. The equation $x^3-3x^2-2x+1=0$ has x_1, x_2 and x_3 as roots. Therefore, according to Cardano-Vieta's formulas,

$$\begin{aligned}x_1+x_2+x_3 &= 3,\\ x_1x_2+x_2x_3+x_3x_1 &= -2,\\ x_1x_2x_3 &= -1.\end{aligned}$$

We can find, by the same formulas, the equation which has y_1, y_2 and y_3 as roots by knowing that $y_1=x_1+\dfrac{1}{x_1}$, $y_2=x_2+\dfrac{1}{x_2}$, and $y_3=x_3+\dfrac{1}{x_3}$. Therefore, we have to find the values of $y_1+y_2+y_3$, $y_1y_2+y_2y_3+y_3y_1$ and $y_1y_2y_3$. But first, we need to calculate some

expressions that are going to be necessary:

$$\begin{aligned}
\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3} &= \frac{x_1x_2+x_1x_3+x_2x_3}{x_1x_2x_3} = \frac{-2}{-1} = 2,\\
x_1^2+x_2^2+x_3^2 &= (x_1+x_2+x_3)^2 - 2(x_1x_2+x_1x_3+x_2x_3)\\
&= 9-2(-2) = 13,\\
x_1^2x_2^2+x_1^2x_3^2+x_2^2x_3^2 &= (x_1x_2+x_1x_3+x_2x_3) - 2x_1x_2x_3(x_1+x_2+x_3)\\
&= 4+6 = 10,
\end{aligned}$$

and

$$\begin{aligned}
&(x_1+x_2+x_3)(x_1x_2+x_1x_3+x_2x_3)\\
&= x_1^2x_2+x_1x_2x_3+x_1^2x_3+x_2^2x_1+x_2^2x_3\\
&\quad + x_1x_2x_3+x_1x_2x_3+x_3^2x_1+x_3^2x_2\\
\Longrightarrow\ & x_1^2x_2+x_1^2x_3+x_2^2x_1+x_2^2x_3+x_3^2x_1+x_3^2x_2\\
&= (x_1+x_2+x_3)(x_1x_2+x_1x_3+x_2x_3) - 3x_1x_2x_3\\
&= 3(-2)-3(-1) = -3.
\end{aligned}$$

Now, we can find out the values of $y_1+y_2+y_3$, $y_1y_2+y_1y_3+y_2y_3$ and $y_1y_2y_3$:

$$\begin{aligned}
y_1+y_2+y_3 &= x_1+x_2+x_3+\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3} = 3+2 = 5,\\
y_1y_2+y_1y_3+y_2y_3 &= \left(x_1+\frac{1}{x_1}\right)\left(x_2+\frac{1}{x_2}\right)+\left(x_1+\frac{1}{x_1}\right)\left(x_3+\frac{1}{x_3}\right)\\
&\quad + \left(x_2+\frac{1}{x_2}\right)\left(x_3+\frac{1}{x_3}\right)\\
&= x_1x_2+\frac{x_1}{x_2}+\frac{x_2}{x_1}+\frac{1}{x_1x_2}+x_1x_3+\frac{x_1}{x_3}+\frac{x_3}{x_1}\\
&\quad + \frac{1}{x_1x_3}+x_2x_3+\frac{x_2}{x_3}+\frac{x_3}{x_2}+\frac{1}{x_2x_3}\\
&= x_1x_2+x_2x_3+x_3x_1\\
&\quad + \left(\frac{x_1}{x_2}+\frac{x_2}{x_1}+\frac{x_1}{x_3}+\frac{x_3}{x_1}+\frac{x_2}{x_3}+\frac{x_3}{x_1}\right)\\
&\quad + \left(\frac{1}{x_1x_2}+\frac{1}{x_1x_3}+\frac{1}{x_2x_3}\right)
\end{aligned}$$

$$
\begin{aligned}
&= -2 + \frac{x_1 + x_2 + x_3}{x_1x_2x_3} \\
&\quad + \frac{x_1^2x_2 + x_1^2x_3 + x_2^2x_1 + x_2^2x_3 + x_3^2x_1 + x_3^2x_2}{x_1x_2x_3} \\
&= -2 - 3 + 3 = -2, \\
y_1y_2y_3 &= \left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\left(x_3 + \frac{1}{x_3}\right) \\
&= x_1x_2x_3 + \frac{x_1x_2}{x_3} + \frac{x_1x_3}{x_2} + \frac{x_1}{x_2x_3} + \frac{x_2x_3}{x_1} \\
&\quad + \frac{x_2}{x_1x_3} + \frac{x_3}{x_1x_2} + \frac{1}{x_1x_2x_3} \\
&= x_1x_2x_3 + \frac{1}{x_1x_2x_3} + \left(\frac{x_1x_2}{x_3} + \frac{x_1x_3}{x_2} + \frac{x_2x_3}{x_1}\right) \\
&\quad + \left(\frac{x_1}{x_2x_3} + \frac{x_2}{x_1x_3} + \frac{x_3}{x_1x_2}\right) \\
&= -1 - 1 + \frac{x_1^2x_3^2 + x_1^2x_2^2 + x_2^2x_3^2}{x_1x_2x_3} + \frac{x_1^2 + x_2^2 + x_3^2}{x_1x_2x_3} \\
&= -2 - 10 - 13 = -25.
\end{aligned}
$$

To sum up, we have

$$
\begin{aligned}
y_1 + y_2 + y_3 &= 5, \\
y_1y_2 + y_2y_3 + y_3y_1 &= -2, \\
y_1y_2y_3 &= -25.
\end{aligned}
$$

According to Cardano-Vieta's formulas, this system of equations corresponds to the polynomial $p(y) = y^3 - 5y^2 - 2y + 25$ whose roots are y_1, y_2 and y_3.

Note that this is just one possible solution. Actually, there are infinitely many polynomials of degree 3 that have these roots, and they are of the form

$$p(y) = ay^3 - 5ay^2 - 2ay + 25a \quad \forall\, a \in \mathbb{R} \setminus \{0\}.$$

Also solved by *Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and the proposers.*

EM–37. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* For all integers $n \geq 1$, show that

$$\sum_{1\leq i\leq n} i^2 + \sum_{1\leq i<j\leq n} ij + \sum_{1\leq j<i\leq n} ji$$

is the square of a positive integer and determine its value.

Solution 1 by Henry Ricardo, Westchester Math Circle, New York, USA. The expansion

$$\begin{aligned}\left(\sum_{1\leq i\leq n} x_i\right)^2 &= \sum_{1\leq i\leq n} x_i^2 + 2\sum_{1\leq i,j\leq n} x_i x_j \\ &= \sum_{1\leq i\leq n} x_i^2 + \sum_{1\leq i<j\leq n} x_i x_j + \sum_{1\leq j<i\leq n} x_j x_i\end{aligned}$$

is well known.

Replacing x_i by i gives us

$$\left(\sum_{1\leq i\leq n} i\right)^2 = \sum_{1\leq i\leq n} i^2 + \sum_{1\leq i<j\leq n} ij + \sum_{1\leq j<i\leq n} ji,$$

so that the given sum is the square of the positive integer $\sum_{1\leq i\leq n} i = n(n+1)/2$.

Solution 2 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. We add all entries of the following table

$$\begin{array}{ccccc} 1\cdot 2 & 1\cdot 3 & \ldots & 1\cdot(n-1) & 1\cdot n \\ & 2\cdot 3 & \ldots & 2\cdot(n-1) & 2\cdot n \\ & & \ddots & \vdots & \vdots \\ & & & & (n-1)n \end{array}$$

by adding the entries of each column and then adding the column sums. We obtain

$$1\cdot 2 + (1+2)\cdot 3 + \ldots + (1+2+\ldots+(n-1))\cdot n.$$

Since each parenthesis consists of a sum of consecutive integers starting with 1, this expression may be rewritten as

$$1 \cdot 2 + \frac{2(1+2)}{2} \cdot 3 + \ldots + \frac{(n-1)(1+(n-1))}{2} \cdot n$$

or, equivalently,

$$\frac{1}{2}\left(1 \cdot 2^2 + 2 \cdot 3^2 + \ldots + (n-1) \cdot n^2\right).$$

Thus,

$$\sum_{1 \le i < j \le n} ij = \frac{1}{2}\left(1 \cdot 2^2 + 2 \cdot 3^2 + \ldots + (n-1) \cdot n^2\right).$$

Since $\displaystyle\sum_{1 \le i < j \le n} ij = \sum_{1 \le j < i \le n} ji$, we have

$$\begin{aligned}
&\sum_{1 \le i \le n} i^2 + \sum_{1 \le i < j \le n} ij + \sum_{1 \le j < i \le n} ji \\
=&\sum_{1 \le i \le n} i^2 + 2 \sum_{1 \le i < j \le n} ij \\
=&\left(1^2 + 2^2 + \ldots n^2\right) + \left(1 \cdot 2^2 + 2 \cdot 3^2 + \ldots + (n-1) \cdot n^2\right) \\
=&1^2 + \underbrace{\left(2^2 + 1 \cdot 2^2\right)}_{=2^3} + \ldots + \underbrace{\left(n^2 + (n-1)n^2\right)}_{=n^3} \\
=&1^3 + 2^3 + \ldots + n^3 \\
=&\left(\frac{n(n+1)}{2}\right)^2,
\end{aligned}$$

where the last equality may be easily established by mathematical induction on n.

Now, since $\frac{n(n+1)}{2}$ is a natural number for any natural n, the proof is complete.

Also solved by *Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain, and the proposer.*

EM–38. *Proposed by Óscar Rivero Salgado, CFIS, BarcelonaTech, Barcelona, Spain.* Solve in positive integers the following equation:

$$(a^2+b^2+c^2)^2 = 2(a^4+b^4+c^4)+2016.$$

Solution by the proposer. Assume that $a \le b \le c$ and factor this as

$$(a+b+c)(b+c-a)(a+c-b)(a+b-c) = 2016 = 2^5 \cdot 3^2 \cdot 7.$$

None of the factors can be negative, since only one of them could have this property and, then, the resulting product would be negative too. Furthermore, observe that the four factors have the same parity, that must be even, so we can write

$$\frac{a+b+c}{2} \cdot \frac{b+c-a}{2} \cdot \frac{a+c-b}{2} \cdot \frac{a+b-c}{2} = 2 \cdot 3 \cdot 3 \cdot 7.$$

Write

$$X = \frac{b+c-a}{2}, \qquad Y = \frac{a+c-b}{2}, \qquad Z = \frac{a+b-c}{2},$$

and observe that the condition says that

$$XYZ(X+Y+Z) = 2 \cdot 3 \cdot 3 \cdot 7.$$

We must have that exactly one factor is even. If X, Y, Z are all odd, so is their sum. If more than one is even we have a contradiction. Then, exactly one is even and the other two are odd, but in that case the sum is even, a contradiction. Hence, there are no solutions to the equation.

EM–39. *Proposed by Guillem Alsina Oriol, CFIS, BarcelonaTech, Barcelona, Spain.* Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ have in common two points X and Y. We draw a line that cuts $\mathcal{C}_1$ at points A and B. Next we draw the lines AX, AY which cut $\mathcal{C}_2$ at points A_X and A_Y and lines BX, BY which cut $\mathcal{C}_2$ at points B_X and B_Y, respectively. Show that the three lines AB, A_XB_Y and A_YB_X are parallel.

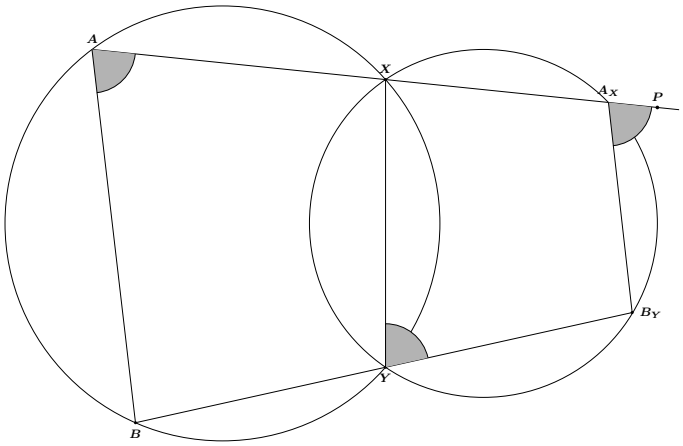

Figure 5: First scheme for Solution 1 of Problem EM–39.

Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain. As in Figure 5, we have taken the point P to lie on AA_X extended.

The cyclic quadrilaterals $ABYX$ and XYB_YA_X tell us that

$$\angle PAB = \angle XAB = \angle XYB_Y = \angle PA_XB_Y.$$

Therefore the line AB is parallel to the line A_XB_Y.

Similarly (see Figure 6), since points A, B, Y, X lie on a circle,

$$\angle A_YAB = \angle YAB = \angle YXB,$$

and since B_XA_YYX is a cyclic quadrilateral,

$$\angle YXB_X = \angle AA_YB_X.$$

Thus

$$\angle A_YAB = \angle AA_YB_X,$$

so that AB and A_YB_X are parallel. The conclusion follows.

Solution 2 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany. We will show that A_YB_X is parallel to AB. To prove the same for A_XB_Y, just exchange the roles of A and B.

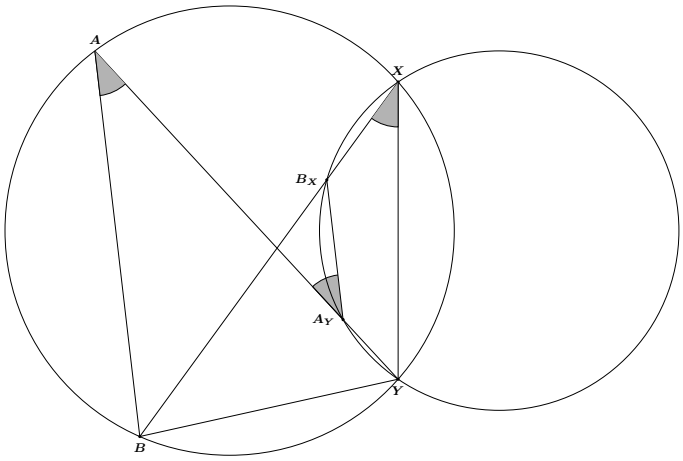

Figure 6: Second scheme for Solution 1 of Problem EM–39.

Suppose first that the relative positions of the points are the same as in the figure (that is, X lies between B and B_X, B and Y are on the same arc AX in $\mathcal{C}_1$, etc.). Then,

$$\begin{aligned}\angle ABB_X &= \angle ABX = \angle AYX = \angle A_YYX \\ &= \angle A_YB_XX = \angle A_YB_XB,\end{aligned}$$

which proves that AB and A_YB_X are parallel.

In order to prove a more general case, we would need to consider other cases, like when B and Y are on different arcs AX. Checking these cases, we might need to take supplementary angles. Instead, we can reduce all cases to this one by considering all angles above as oriented and taking modulo $180°$. For example, modulo $180°$, if A, B, C and D lie on a circle then $\angle ABD = \angle ACD$ regardless of their relative position. Similarly, if A, B and C are colinear, then $\angle ABD = \angle CBD$. Finally, AB is parallel to CD if, and only if, $\angle ABC = \angle DCB$. With this, the proof of the particular case also works for the general case.

Also solved by *the proposer.*

EM–40. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Bar-*

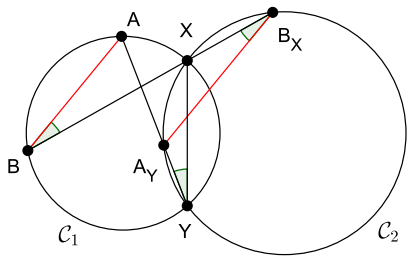

Figure 7: Scheme for Solution 2 of Problem EM–39.

celona, Spain. Let n be a nonnegative integer. Compute

$$\sum_{j=1}^{2n+1} \frac{j^3}{(2n+1)^2 - 3(2n+1)j + 3j^2}.$$

Solution by Miguel Cidrás Senra, CFIS, BarcelonaTech, Barcelona, Spain. Let a and b be two different real numbers. Then,

$$\begin{aligned}
&\frac{b^3}{a^2 - 3ab + 3b^2} + \frac{(a-b)^3}{a^2 - 3a(a-b) + 3(a-b)^2} \\
=&\frac{b^3}{a^2 - 3ab + 3b^2} + \frac{(a-b)^3}{a^2 - 3a^2 + 3ab + 3a^2 - 6ab + 3b^2} \\
=&\frac{b^3}{a^2 - 3ab + 3b^2} + \frac{(a-b)^3}{a^2 - 3ab + 3b^2} \\
=&\frac{(a-b)^3 + b^3}{a^2 - 3ab + 3b^2} = \frac{a^3 - 3a^2b + 3ab^2 - b^3 + b^3}{a^2 - 3ab + 3b^2} = a.
\end{aligned}$$

We can observe that the initial expression is equivalent to

$$\begin{aligned}
&\sum_{j=1}^{2n+1} \frac{j^3}{(2n+1)^2 - 3(2n+1)j + 3j^2} \\
=&\sum_{j=0}^{2n+1} \frac{j^3}{(2n+1)^2 - 3(2n+1)j + 3j^2}
\end{aligned}$$

$$= \sum_{j=0}^{n} \left(\frac{j^3}{(2n+1)^2 - 3(2n+1)j + 3j^2} + \right.$$
$$\left. + \frac{(2n+1-j)^3}{(2n+1)^2 - 3(2n+1)(2n+1-j) + 3(2n+1-j)^2} \right).$$

Finally, using the first identity with $a = 2n+1$ and $b = j$, we obtain

$$\sum_{j=1}^{2n+1} \frac{j^3}{(2n+1)^2 - 3(2n+1)j + 3j^2} = \sum_{j=0}^{n} 2n+1 = (n+1)(2n+1)$$
$$= 2n^2 + 3n + 1.$$

Also solved by *Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain; Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain, and the proposer.*

Medium–Hard Problems

MH–35. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

$$\sqrt{\frac{b}{a^2+3}}+\sqrt{\frac{c}{b^2+3}}+\sqrt{\frac{a}{c^2+3}}\le\frac{3}{2}\sqrt[4]{\frac{1}{abc}}.$$

Solution 1 by Miguel Cidrás Senra, CFIS, BarcelonaTech, Barcelona, Spain. First, we observe that

$$\begin{aligned}
&\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}\\
=&\frac{1}{a^2+1+1+1}+\frac{1}{b^2+1+1+1}+\frac{1}{c^2+1+1+1}\\
\le&\frac{1}{4\sqrt[4]{a^2}}+\frac{1}{4\sqrt[4]{b^2}}+\frac{1}{4\sqrt[4]{c^2}}=\frac{\sqrt{bc}+\sqrt{ca}+\sqrt{ab}}{4\sqrt{abc}} &(2)\\
\le&\frac{\sqrt{b^2}+\sqrt{c^2}+\sqrt{a^2}}{4\sqrt{abc}}=\frac{a+b+c}{4\sqrt{abc}}=\frac{3}{4\sqrt{abc}}, &(3)
\end{aligned}$$

where the inequality in (2) comes from the AM-GM inequality, and that in (3) is a consequence of the rearrangement inequality. On the other hand, applying the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
&\sqrt{\frac{b}{a^2+3}}+\sqrt{\frac{c}{b^2+3}}+\sqrt{\frac{a}{c^2+3}}\\
\le&\sqrt{(b+c+a)\left(\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}\right)}\\
=&\sqrt{3\left(\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}\right)}\\
\le&\sqrt{3\cdot\frac{3}{4\sqrt{abc}}}=\frac{3}{2}\sqrt[4]{\frac{1}{abc}},
\end{aligned}$$

where the last inequality is a consequence of (2) and (3).

Solution 2 by Ander Lamaison Vidarte. Berlin Mathematical School, Berlin, Germany. Consider the chain of inequalities

$$\begin{aligned}
&\sqrt{\frac{b}{a^2+3}}+\sqrt{\frac{c}{b^2+3}}+\sqrt{\frac{a}{c^2+3}}\\
\le&\sqrt{\frac{b}{4\sqrt{a}}}+\sqrt{\frac{c}{4\sqrt{b}}}+\sqrt{\frac{a}{4\sqrt{c}}} && (4)\\
=&\frac{1}{2\sqrt[4]{abc}}\left(\sqrt[4]{b^3c}+\sqrt[4]{c^3a}+\sqrt[4]{a^3b}\right)\\
\le&\frac{1}{2\sqrt[4]{abc}}\left(\frac{3b+c}{4}+\frac{3c+a}{4}+\frac{3a+b}{4}\right) && (5)\\
=&\frac{3}{2}\sqrt[4]{\frac{1}{abc}},
\end{aligned}$$

where in (4) we apply AM-GM to $(a^2,1,1,1)$ to obtain $a^2+3\ge 4\sqrt{a}$, and in (5) we apply AM-GM to (b,b,b,c) to obtain $\sqrt[4]{b^3c}\le\frac{3b+c}{4}$.

Solution 3 by Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain. We can apply the Cauchy-Schwarz inequality to vectors $\left(\sqrt{b},\sqrt{c},\sqrt{a}\right)$ and $\left(\frac{1}{\sqrt{a^2+3}},\frac{1}{\sqrt{b^2+3}},\frac{1}{\sqrt{c^2+3}}\right)$ to get

$$\begin{aligned}
&\sqrt{\frac{b}{a^2+3}}+\sqrt{\frac{c}{b^2+3}}+\sqrt{\frac{a}{c^2+3}}\\
\le&\sqrt{a+b+c}\sqrt{\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}}\\
=&\sqrt{3}\sqrt{\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}}.
\end{aligned}$$

Applying AM-GM inequality we see that

$$\frac{1}{x^2+3}=\frac{1}{x^2+1+1+1}\le\frac{1}{4\sqrt[4]{x^2}}=\frac{1}{4\sqrt{x}}.$$

Therefore,

$$\begin{aligned}\sqrt{3}\sqrt{\sum_{\text{cyc}}\frac{1}{a^2+3}} &\le \sqrt{3}\sqrt{\sum_{\text{cyc}}\frac{1}{4\sqrt{a}}}\\ &= \sqrt{3}\sqrt{\sum_{\text{cyc}}\frac{\sqrt{bc}}{4\sqrt{abc}}}\\ &= \frac{\sqrt{3}}{2}\sqrt[4]{\frac{1}{abc}}\sqrt{\sum_{\text{cyc}}\sqrt{bc}}\\ &= \frac{\sqrt{3}}{2}\sqrt[4]{\frac{1}{abc}}\sqrt{\sqrt{bc}+\sqrt{ca}+\sqrt{ab}}.\end{aligned}$$

Now, we can apply again the Cauchy-Schwarz inequality to vectors $\left(\sqrt{b},\sqrt{c},\sqrt{a}\right)$ and $\left(\sqrt{c},\sqrt{a},\sqrt{b}\right)$ to obtain

$$\begin{aligned}&\frac{\sqrt{3}}{2}\sqrt[4]{\frac{1}{abc}}\sqrt{\sqrt{bc}+\sqrt{ca}+\sqrt{ab}}\\ \le&\frac{\sqrt{3}}{2}\sqrt[4]{\frac{1}{abc}}\sqrt{\sqrt{b+c+a}\sqrt{c+a+b}}\\ =&\frac{\sqrt{3}}{2}\sqrt[4]{\frac{1}{abc}}\sqrt{\sqrt{3}\sqrt{3}} = \frac{\sqrt{3}}{2}\sqrt[4]{\frac{1}{abc}}\sqrt{3} = \frac{3}{2}\sqrt[4]{\frac{1}{abc}},\end{aligned}$$

thus completing the proof. Equality is reached if, and only if, $a=b=c=1$.

Also solved by *the proposer.*

MH–36. *Proposed by Damià Torres Latorre, CFIS, BarcelonaTech, Barcelona, Spain, and Jesús Dueñas Pamplona, Valladolid, Spain.* Let ABC be an acute triangle with circumcircle ω. Let P be a point lying on ω distinct from B and its diametrically opposite. Let M the intersection point of the tangents to ω drawn from P and B, respectively. Parallel to the tangent to ω drawn from A and passing through M cuts lines AB and AC in the points D and E, respectively. The perpendicular line to AB through D and the perpendicular to AC through E intersect at T. Find the locus of T when P moves on ω.

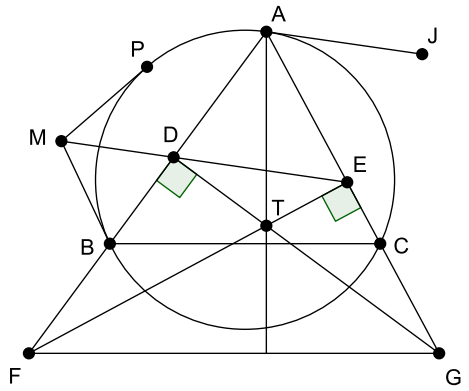

Figure 8: Construction for the solution of problem MH-36.

Solution by Ander Lamaison Vidarte, BMS, Berlin, Germany. Let ET intersect AB at F, DT intersect AC at G, and let J be a point on the tangent to ω at A such that AJ and DE have the same orientation. Because ABC is an acute triangle, the angles $\angle ADE$ and $\angle AED$ are acute, and thus D lies between A and F, and E lies between A and G. Since $\angle FDG = \angle FEG = 90°$, $FDEG$ is a cyclic quadrilateral. Thus

$$\begin{aligned}\angle AFG =& \angle DFG = 180° - \angle DEG = \angle AED \\ =& \angle AEM = \angle EAJ = \angle ABC,\end{aligned}$$

so FG is parallel to BC. Since $FT \perp AG$ and $GT \perp AF$, T is the orthocenter of AFG, and $AT \perp FG \parallel BC$. We conclude that T is on the altitude from A in ABC.

We will now reverse the construction to check if every point in the altitude can be reached in this way. Let T be a point in the altitude. Project T on AB to obtain D, and on AC to obtain E. By similarity of all figures drawn this way, DE is always parallel to some direction, which must be the tangent to ω through A. Let M be the intersection of DE and the tangent to ω through B. If $M \neq B$, then there is another tangent to ω through M, and let P be the point of tangency. If $M = B$, no such P can be found, because $P = B$ is explicitly forbidden. If $M = B$, then $D = B$, and $\angle ABT = 90°$. Therefore the locus of T is the altitude from A in ABC, excluding the point with $\angle ABT = 90°$.

Also solved by *the proposers.*

MH–37. *Proposed by Nicolae Papacu, Slobozia, Romania.* Determine all positive integers n and prime numbers p such that p^n+8 is a perfect cube.

Solution by Miguel Cidrás Senra, CFIS, BarcelonaTech, Barcelona, Spain. Observe that

$$\begin{aligned} p^n + 8 \text{ is a perfect cube} &\iff p^n + 8 = k^3 \\ &\iff p^n = (k-2)(k^2+2k+4) \\ &\iff p^n = (k-2)((k+4)(k-2)+12). \end{aligned}$$

Therefore, either $k-2=1$ or $k-2=p^a$ for some a.

In the first case, we have $k=3$, so $p^n = 3^3 - 8 = 19^1$, and $(n,p)=(1,19)$ is one solution.

In the second case, $k-2=p^a$ for some a, so $p^{n-2a} = k+4+\frac{12}{k-2}$. As $k-2$ is a power of a primer number and divides 12, $k-2=2$, 3 or 4, so $k=4$, 5 or 6. If $k=4$, $4^3-8=56=7\cdot 8$. If $k=5$, $5^3-8=117=9\cdot 13$. If $k=6$, $6^3-8=208=13\cdot 16$. And none of them is a power of a primer number.

So, there is only one natural number n and one primer number p such that p^n+8 is a perfect cube.

Also solved by *Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain, and the proposer.*

MH–38. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $1<a<b$ be real numbers. Prove that for any x_1, x_2, $x_3 \in [a,b]$ there exists $c\in[a,b]$ such that

$$\frac{1}{\log x_1}+\frac{1}{\log x_2}+\frac{1}{\log x_3}+\frac{1}{\sqrt[3]{\log x_1 \log x_2 \log x_3}} = \frac{4}{\log c}.$$

Solution by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. It is enough to consider

$$\log c = \frac{4}{\frac{1}{\log x_1}+\frac{1}{\log x_2}+\frac{1}{\log x_3}+\frac{1}{\sqrt[3]{\log x_1 \log x_2 \log x_3}}}.$$

That is, $\log c$ is the harmonic mean of the numbers $\log x_1$, $\log x_2$, $\log x_3$, and $\sqrt[3]{\log x_1 \log x_2 \log x_3}$. Therefore,

$$\min(\log x_1, \log x_2, \log x_3) \leq \log c \leq \max(\log x_1, \log x_2, \log x_3),$$

and since the function $\log x$ is positive, increasing and continuous in $[a, b]$, then $c \in [a, b]$.

Also solved by *Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain; Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain; Henry Ricardo, Westchester Area Math Circle, New York, USA, and the proposer.*

MH–39. *Proposed by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom.* Consider the integer grid $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} = \{(a, b) \mid a, b \in \mathbb{Z}\}$. How many different walks can be performed in $2n$ steps of length one starting and ending at the same point and moving in steps only in the directions of the coordinate axes?

Solution 1 by Padraig Condon, University of Birmingham, Birmingham, United Kingdom, and the proposer. We claim that the answer is $\binom{2n}{n}^2$. Indeed, divide the walk into $2n$ steps and, among those, choose n (for which there are $\binom{2n}{n}$ choices). Colour these steps, say, red. Independently of this, choose n steps among the $2n$, and colour them blue.

Now consider the following. We are going to say that if a step is coloured both blue and red, it is an "up" step, if it has no colour it is "down", if it is only red it goes "right", and if it is only blue it goes "left". It is very easy to check that there are as many left steps as right steps, and as many up steps as down steps, which means that the resulting walk given by any such colouring is closed and ends back at the origin. Furthermore, each of these colourings clearly gives a different walk. All that remains is to check that these cover all possible closed walks. But for any given closed

walk, one can colour each step according to the described colouring, so it could be the result of one of our choices. This completes the proof.

Solution 2 by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain. In order to start and end at the same point, the number of steps to the right has to be the same as the number of steps to the left, and the number of steps upwards has to be the same as the number of steps downwards. Therefore, the number of vertical steps has to be even, just like the number of horizontal steps.

To begin with, let x be the number of horizontal steps and y the number of vertical steps, so that $x + y = 2n$. To solve the counting, we can divide the counting in several disjoint cases.

For $x = 0$, $y = 2n$ we have n steps upwards and n downwards. We can think of it as the number of possible permutations of n arrows pointing upwards ($\uparrow$) and n arrows pointing downwards ($\downarrow$). So, the number of possible different walks is

$$\frac{(2n)!}{n!n!}.$$

For $x = 2$, $y = 2n - 2$ we have $n - 1$ steps upwards, $n - 1$ downwards, 1 step to the left and 1 to the right, which corresponds to the arrows that we need to permute. The number of possible different walks is

$$\frac{(2n)!}{1!1!(n-1)!(n-1)!}.$$

In general, for $x = 2k$, $y = 2n - 2k$ we have $n - k$ steps upwards, $n - k$ downwards, k steps to the left and k to the right, which corresponds to the arrows that we need to permute. The number of possible different walks is

$$\frac{(2n)!}{k!k!(n-k)!(n-k)!}.$$

Reasoning in the same way until $x = 2n$ and $y = 0$ and adding all the disjoint cases, we have that the number of possible paths

is

$$\sum_{k=0}^{n} \frac{(2n)!}{k!k!(n-k)!(n-k)!}.$$

Now the problem is to calculate this sum. Let us consider the general term

$$\frac{(2n)!}{k!k!(n-k)!(n-k)!}.$$

We have that

$$\frac{2n!}{k!k!(n-k)!(n-k)!} = \frac{n!}{k!(n-k)!}\frac{n!}{k!(n-k)!}\frac{(2n)!}{n!n!} = \binom{n}{k}^2\binom{2n}{n}.$$

Therefore,

$$\sum_{k=0}^{n} \frac{(2n)!}{k!k!(n-k)!(n-k)!} = \binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2.$$

In order to compute this last sum, consider the binomial theorem applied to $(1+x)^{2n}$. A direct application yieds

$$(1+x)^{2n} = \binom{2n}{0} + \binom{2n}{1}x + \ldots + \binom{2n}{n}x^n + \ldots + \binom{2n}{2n}x^{2n}.$$

However, we can also evaluate it as

$$\begin{aligned}(1+x)^{2n} =& (1+x)^n(1+x)^n = \left[\binom{n}{0} + \binom{n}{1}x + \ldots + \binom{n}{n}x^n\right]^2 \\ =& \binom{n}{0}\binom{n}{0} + \left[\binom{n}{0}\binom{n}{1} + \binom{n}{1}\binom{n}{0}\right]x + \ldots \\ &+ \left[\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \ldots + \binom{n}{n}\binom{n}{0}\right]x^n \\ &+ \ldots + \binom{n}{n}\binom{n}{n}x^{2n}.\end{aligned}$$

By equating the coefficients of x^n we obtain

$$\binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \ldots + \binom{n}{n}\binom{n}{0} = \binom{2n}{n},$$

and knowing that $\binom{n}{k} = \binom{n}{n-k}$ we have

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \binom{2n}{n}.$$

So, overall, the number of walks is $\binom{2n}{n}^2$.

Solution 3 by Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain. Let $a_{r,s}$ be the number of ways of getting from $(0,0)$ to (r,s) after $2n$ steps. Let $f_n(x,y) = \sum_{r,s\in\mathbb{Z}} a_{r,s}x^r y^s$ be the generating function of $a_{r,s}$. We have $f(x,y) = (x + x^{-1} + y + y^{-1})^{2n}$ (here, each summand represents moving east, west, north and south, respectively). Hence, the number of different walks consisting of $2n$ steps which start and end at the same point (which we can assume to be the origin WLOG) will be $[x^0, y^0]f_n(x,y)$ (here, $[x^r, y^s]f_n(x,y)$ means the coefficient of the term $x^r y^s$ of the series). We will need to prove the following lemma:

Lemma. $\displaystyle\sum_{m=0}^{n} \binom{n}{m}^2 = \binom{2n}{n}.$

Proof. On the one hand we have that

$$(1+x)^{2n} = \sum_{m=0}^{2n} \binom{n}{m} x^m.$$

On the other,

$$\begin{aligned}(1+x)^{2n} = ((1+x)^n)^2 &= \left(\sum_{m=0}^{n} \binom{n}{m} x^m\right)\left(\sum_{k=0}^{n} \binom{n}{k} x^k\right) \\ &= \sum_{0\le m,k\le n} \binom{n}{m}\binom{n}{k} x^{m+k}.\end{aligned}$$

We see that the coefficient of x^n is $\binom{2n}{n}$ in the first case, and $\sum_{\substack{0\le m,k\le n\\ m+k=n}} \binom{n}{m}\binom{n}{k}$ in the second. Expanding this last expression

we have

$$\begin{aligned}\sum_{\substack{0\le m,k\le n\\ m+k=n}}\binom{n}{m}\binom{n}{k} &= \sum_{m=0}^{n}\binom{n}{m}\binom{n}{n-m}\\ &= \sum_{m=0}^{n}\binom{n}{m}\binom{n}{m}\\ &= \sum_{m=0}^{n}\binom{n}{m}^2,\end{aligned}$$

as we wanted to show. □

Having proven this lemma, we can find the value of $[x^0,y^0]f_n(x,y)$:

$$\begin{aligned}&[x^0,y^0]f_n(x,y)\\ =&[x^0,y^0](x+x^{-1}+y+y^{-1})^{2n}\\ =&[x^0,y^0]\sum_{m=0}^{2n}\binom{2n}{m}\sum_{i=0}^{m}\binom{m}{i}x^ix^{-(m-i)}\sum_{j=0}^{2n-m}\binom{2n-m}{j}y^jy^{-(2n-m-j)}\\ =&[x^0,y^0]\sum_{m=0}^{2n}\binom{2n}{m}\sum_{i=0}^{m}\binom{m}{i}x^{2i-m}\sum_{j=0}^{2n-m}\binom{2n-m}{j}y^{2j+m-2n}.\end{aligned}$$

To get the coefficient of x^0y^0 we will need to have $i=\frac{m}{2}$ and $j=n-\frac{m}{2}$. So we only need to sum over even values of m:

$$\begin{aligned}&[x^0,y^0]\sum_{m=0}^{2n}\binom{2n}{m}\sum_{i=0}^{m}\binom{m}{i}x^{2i-m}\sum_{j=0}^{2n-m}\binom{2n-m}{j}y^{2j+m-2n)}\\ =&\sum_{k=0}^{n}\binom{2n}{2k}\binom{2k}{k}\binom{2n-2k}{n-k}\\ =&\sum_{k=0}^{n}\frac{(2n)!}{(2k)!(2n-2k)!}\frac{(2k)!}{k!k!}\frac{(2n-2k)!}{(n-k)!(n-k)!}\\ =&\sum_{k=0}^{n}(2n)!\frac{1}{k!k!}\frac{1}{(n-k)!(n-k)!}\\ =&\sum_{k=0}^{n}\frac{(2n)!}{n!n!}\frac{n!}{k!(n-k)!}\frac{n!}{k!(n-k)!}\end{aligned}$$

$$=\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n}{k}$$
$$=\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2=\binom{2n}{n}\binom{2n}{n}=\binom{2n}{n}^2.$$

Solution 4 by the proposer. In any such walk, there must be as many steps to the right as steps to the left, and as many steps upwards as downwards (otherwise, we would not reach the same point). As this is the only condition, in order to completely determine such a walk, one must choose the steps in which he moves in each direction. Assume that in a particular walk we know that, overall, there are i steps upwards. Then, naturally, there must be i steps downwards, and $n-i$ to the right and to the left. The number of walks with these steps is

$$\binom{2n}{i}\binom{2n-i}{i}\binom{2(n-i)}{n-i}\binom{n-i}{n-i},$$

corresponding to choosing the steps in which we move upwards, then those in which we move downwards, then those among the remaining ones in which we move to the right, and finally, those in which we move to the left.

In our problem we do not know how many steps upwards there are, so we must compute all the possibilities. Clearly, as there are $2n$ steps overall, the number of upwards steps is upper bounded by n, so the number of such closed walks is

$$\sum_{i=0}^{n}\binom{2n}{i}\binom{2n-i}{i}\binom{2(n-i)}{n-i}\binom{n-i}{n-i}.$$

Let us compute this number. Using the usual formula for combinatorial numbers we have

$$\sum_{i=0}^{n}\binom{2n}{i}\binom{2n-i}{i}\binom{2(n-i)}{n-i}\binom{n-i}{n-i}$$
$$=\sum_{i=0}^{n}\frac{(2n)!}{i!(2n-i)!}\frac{(2n-i)!}{i!(2n-2i)!}\frac{(2n-2i)!}{(n-i)!(n-i)!}$$

$$=\sum_{i=0}^{n}\frac{(2n)!}{(i!)^2((n-i)!)^2}=\sum_{i=0}^{n}\frac{(2n)!(n!)^2}{(n!)^2(i!)^2((n-i)!)^2}$$
$$=\frac{(2n)!}{(n!)^2}\sum_{i=0}^{n}\frac{(n!)^2}{(i!)^2((n-i)!)^2}=\binom{2n}{n}\sum_{i=0}^{n}\binom{n}{i}^2.$$

Now consider the sum $\sum_{i=0}^{n}\binom{n}{i}^2=\sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}$. Combinatorially speaking, we have two sets of n elements and we choose i from the first one and $n-i$ from the second one. We do this for every possible value of i. For each of these values, what we are doing is choosing n elements from a set of $2n$ in such a way that i of them belong to a certain subset, and as we add over all values of i, we are obtaining all possible ways too choose n elements from a set of $2n$ That is, we have $\sum_{i=0}^{n}\binom{n}{i}^2=\binom{2n}{n}$, so overall, the number of closed walks of length $2n$ is $\binom{2n}{n}^2$.

Also solved by *Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain (one more solution); Matthew Coulson, University of Birmingham, Birmingham, United Kingdom; José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain, and Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.*

MH–40. *Proposed by Mihály Bencze, Braşov, Romania.* Let $ABCD$ be a convex quadrilateral such that $a=AB$, $b=BC$, $c=CD$, $d=DA$, $e=BD$, and $f=AC$. Prove that

$$[ABCD]\le\frac{\sqrt{3}}{12}\min\left\{\frac{(ab+bf+fa)^2}{a^2+b^2+f^2}+\frac{(cd+df+fc)^2}{c^2+d^2+f^2},\right.$$
$$\left.\frac{(ae+ed+da)^2}{a^2+e^2+d^2}+\frac{(bc+ce+eb)^2}{b^2+c^2+e^2}\right\}.$$

Solution by the proposer. First, we claim that in any triangle XYZ with the usual notations, it is true that

$$x^2+y^2+z^2\ge 4\sqrt{3}\mathcal{A}+(x-y)^2+(y-z)^2+(z-x)^2,$$

where $\mathcal{A} = [XYZ]$ (Hadwiger-Finsler's inequality).

Indeed, on account of the cosines law we get

$$x^2 = y^2 + z^2 - 2yz\cos X,$$

X being the angle between y and z. This expression is equivalent to

$$x^2 = (y-z)^2 + 2yz(1-\cos X).$$

Since $\mathcal{A} = \dfrac{1}{2}yz\sin X$, then $x^2 = (y-z)^2 + 4\,\mathcal{A}\dfrac{1-\cos X}{\sin X}$. On account that

$$1-\cos X = 2\sin^2\frac{X}{2} \quad \text{and} \quad \sin X = 2\sin\frac{X}{2}\cos\frac{X}{2},$$

we have

$$x^2 = (y-z)^2 + 4\mathcal{A}\tan\frac{X}{2} \quad \text{(cyclic)}.$$

Adding up these expressions yields

$$\begin{aligned} &x^2+y^2+z^2 \\ =&(x-y)^2+(y-z)^2+(z-x)^2+4\mathcal{A}\left(\tan\frac{X}{2}+\tan\frac{Y}{2}+\tan\frac{Z}{2}\right). \end{aligned}$$

Now, since the halves of the triangle's angles are less than $\frac{\pi}{2}$, the function $\tan$ is convex, and we have

$$\tan\frac{X}{2}+\tan\frac{Y}{2}+\tan\frac{Z}{2} \geq 3\tan\frac{X+Y+Z}{6} = 3\tan\frac{\pi}{6} = \sqrt{3}$$

on account of Jensen's inequality. From the preceding it immediately follows that

$$x^2+y^2+z^2 \geq 4\sqrt{3}\mathcal{A} + (x-y)^2+(y-z)^2+(z-x)^2,$$

as claimed.

Since

$$\begin{aligned} \left(\sum xy\right)^2 &= \left(\sum x^2 - \sum xy\right)^2 - \left(\sum x^2 - 2\sum xy\right)\left(\sum x^2\right) \\ &= \frac{1}{2}\left(\sum (x-y)^2\right)^2 + \left(2\sum xy - \sum x^2\right)\left(\sum x^2\right) \\ &\geq \left(2\sum xy - \sum x^2\right)\left(\sum x^2\right) \end{aligned}$$

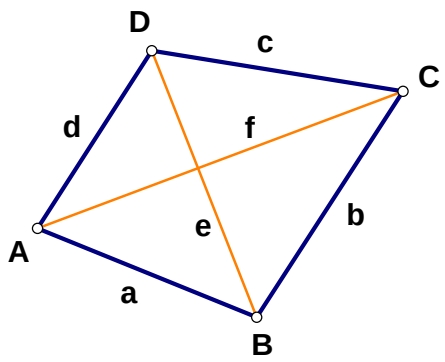

Figure 9: Scheme for the solution of Problem MH–40.

and

$$2\sum xy - \sum x^2 \geq 4\sqrt{3}\,\mathcal{A},$$

then

$$\left(\sum xy\right)^2 \geq 4\sqrt{3}\mathcal{A}\sum x^2$$

or

$$4\sqrt{3}\mathcal{A} \leq \frac{\left(\sum xy\right)^2}{\sum x^2}.$$

Using the last inequality for the triangles ABC and ACD, we get

$$\begin{aligned} 4\sqrt{3}\,[ABCD] &= 4\sqrt{3}\,[ABC] + 4\sqrt{3}\,[ACD] \\ &\leq \frac{(ab+bf+fa)^2}{a^2+b^2+f^2} + \frac{(cd+df+fc)^2}{c^2+d^2+f^2}. \end{aligned}$$

Using again the above inequality for the triangles ABD and BCD, we get

$$\begin{aligned} 4\sqrt{3}\,[ABCD] &= 4\sqrt{3}\,[ABD] + 4\sqrt{3}\,[BCD] \\ &\leq \frac{(ae+ed+da)^2}{a^2+e^2+d^2} + \frac{(bc+ce+eb)^2}{b^2+c^2+e^2}, \end{aligned}$$

and the statement follows.

Advanced Problems

A–35. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Suppose that all the eigenvalues of $A \in M_n(\mathbb{R})$ are positive real numbers. Show that

$$\det(A + A^{-1}) \geq 2^n.$$

Solution 1 by Alberto Espuny-Díaz, University of Birmingham, Birmingham, United Kingdom. Let the eigenvalues of A be $\lambda_1, \lambda_2, \ldots, \lambda_n$. Consider the Jordan normal form of A; this Jordan form is an upper triangular matrix that has the eigenvalues of A in the main diagonal. Let this matrix be called J. Furthermore, $\det A = \det J$, as a matrix and its Jordan normal form are similar. As J is upper triangular, its inverse is given by an upper triangular matrix whose diagonal entries are the inverses of the diagonal entries of J. That is,

$$J + J^{-1} = \begin{pmatrix} \lambda_1 + \lambda_1^{-1} & & & & \\ 0 & \lambda_2 + \lambda_2^{-1} & & U & \\ 0 & 0 & \lambda_3 + \lambda_3^{-1} & & \\ \vdots & \vdots & \vdots & \ddots & x \\ 0 & 0 & 0 & \cdots & \lambda_n + \lambda_n^{-1} \end{pmatrix}$$

is an upper-triangular matrix, and its determinant can be computed simply as the product of the elements of the diagonal. Hence,

$$\det(A + A^{-1}) = \prod_{i=1}^{n} \left(\lambda_i + \frac{1}{\lambda_i} \right) \geq 2^n,$$

on account of the well-known fact that $x + \frac{1}{x} \geq 2$ (which can be proved simply by convexity or by the AM-GM inequality).

Solution 2 by Henry Ricardo, Westchester Math Circle, New York, USA. The determinant of a matrix is the product of its eigenvalues. Also, if the eigenvalues of $A \in M_n(\mathbb{R})$ are the positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, then the eigenvalues of A^{-1}

are $1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_n$, and the eigenvalues of $A + A^{-1}$ are $\lambda_1 + 1/\lambda_1, \lambda_2 + 1/\lambda_2, \ldots, \lambda_n + 1/\lambda_n$. Therefore

$$\begin{aligned}\det(A + A^{-1}) &= \left(\lambda_1 + \frac{1}{\lambda_1}\right) \cdot \left(\lambda_2 + \frac{1}{\lambda_2}\right) \cdot \cdots \cdot \left(\lambda_n + \frac{1}{\lambda_n}\right) \\ &\geq \underbrace{2 \cdot 2 \cdot \cdots \cdot 2}_{n \text{ factors}} = 2^n\end{aligned}$$

because $a + 1/a \geq 2$ for any positive value of a, a consequence of the AM-GM inequality. Equality holds in the determinant inequality if and only if $\lambda_i = 1$ for $1 \leq i \leq n$.

Solution 3 by Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain. Let $B \in M_n(\mathbb{R})$ and let $b_1, \ldots, b_n$ be its eigenvalues. We know that

- the eigenvalues of $(B + \lambda I_n)$ are $b_1 + \lambda, \ldots, b_n + \lambda$;
- the eigenvalues of B^m, where $m \in \mathbb{Z}$, are $b_1^m, \ldots, b_n^m$;
- the determinant of a square matrix is the product of its eigenvalues.

Let $a_1, \ldots, a_n$ be the eigenvalues (remember all of them are positive) of A. We have that

$$\begin{aligned}\det(A + A^{-1}) &= \det(A^{-1}(A^2 + I_n)) = \det(A^{-1}) \det(A^2 + I) \\ &= \prod_{k=1}^{n} a_k^{-1} \prod_{k=1}^{n} a_k^2 + 1 = \prod_{k=1}^{n} \frac{a_k^2 + 1}{a_k} \\ &= \prod_{k=1}^{n} a_k + \frac{1}{a_k} \geq \prod_{k=1}^{n} 2 = 2^n,\end{aligned}$$

where we have used AM-GM inequality to see that if $x > 0$, $x + \frac{1}{x} \geq 2\sqrt{x\frac{1}{x}} = 2$. Therefore, we reach equality if, and only if, $a_1 = \ldots = a_n = 1$.

Also solved by *Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and the proposer.*

A–36. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let x_k $(1 \le k \le n)$ be positive numbers and let $\alpha \ge 1$. Prove that

$$\frac{1}{n-1} \sum_{1 \le i < j \le n} \frac{x_i^{2\alpha} + x_j^{2\alpha}}{x_i^{2\alpha+2} + x_j^{2\alpha+2}} \le \frac{1}{2} \sum_{k=1}^{n} \frac{1}{x_k^2}.$$

Solution 1 by Henry Ricardo, Westchester Math Circle, New York, USA. First we apply the power means inequality to see that

$$\left(\frac{(x_i^2)^\alpha + (x_j^2)^\alpha}{2} \right)^{\frac{1}{\alpha}} \le \left(\frac{(x_i^2)^{\alpha+1} + (x_j^2)^{\alpha+1}}{2} \right)^{\frac{1}{\alpha+1}},$$

or

$$x_i^{2\alpha} + x_j^{2\alpha} \le 2^{\frac{1}{\alpha+1}} \cdot \left(x_i^{2\alpha+2} + x_j^{2\alpha+2} \right)^{\frac{\alpha}{\alpha+1}}. \tag{6}$$

Therefore, using (6) and the AM-GM inequality (twice), we have

$$\begin{aligned}
\sum_{1 \le i < j \le n} \frac{x_i^{2\alpha} + x_j^{2\alpha}}{x_i^{2\alpha+2} + x_j^{2\alpha+2}} &\le 2^{\frac{1}{\alpha+1}} \cdot \sum_{1 \le i < j \le n} \frac{\left(x_i^{2\alpha+2} + x_j^{2\alpha+2} \right)^{\frac{\alpha}{\alpha+1}}}{x_i^{2\alpha+2} + x_j^{2\alpha+2}} \\
&= 2^{\frac{1}{\alpha+1}} \cdot \sum_{1 \le i < j \le n} \frac{1}{\left(x_i^{2\alpha+2} + x_j^{2\alpha+2} \right)^{\frac{1}{\alpha+1}}} \\
&\le 2^{\frac{1}{\alpha+1}} \cdot \sum_{1 \le i < j \le n} \frac{1}{\left(2\sqrt{x_i^{2\alpha+2} \cdot x_j^{2\alpha+2}} \right)^{\frac{1}{\alpha+1}}} \\
&= \sum_{1 \le i < j \le n} \frac{1}{x_i x_j} = \sum_{1 \le i < j \le n} \frac{x_i x_j}{x_i^2 x_j^2} \\
&\le \sum_{1 \le i < j \le n} \frac{(x_i^2 + x_j^2)/2}{x_i^2 x_j^2} \\
&= \frac{1}{2} \cdot \sum_{1 \le i < j \le n} \left(\frac{1}{x_i^2} + \frac{1}{x_j^2} \right) \\
&= \frac{n-1}{2} \cdot \sum_{k=1}^{n} \frac{1}{x_k^2}.
\end{aligned}$$

Dividing by $(n-1)$ gives us the desired result.

Solution 2 by the proposer. First we write the given inequality in the most convenient form

$$\sum_{1\le i<j\le n}\frac{x_i^{2\alpha}+x_j^{2\alpha}}{x_i^{2\alpha+2}+x_j^{2\alpha+2}}\le\frac{n-1}{2}\sum_{k=1}^{n}\frac{1}{x_k^2}.$$

Since the function $f:[0,+\infty)\to\mathbb{R}$ defined by $f(x)=x^{2\alpha+1}$ is nonnegative and increasing, then for all $a,b\in[0,+\infty)$ we have

$$(a-b)\left(a^{2\alpha+1}-b^{2\alpha+1}\right)\ge 0.$$

From the preceding we immediately have that

$$a^{2\alpha+2}+b^{2\alpha+2}\ge ba^{2\alpha+1}+ab^{2\alpha+1},\quad\text{or}\quad\frac{a^{2\alpha+2}+b^{2\alpha+2}}{a^{2\alpha}+a^{2\alpha}}\ge ab.$$

Setting $a=x_i$ and $b=x_j$ into the preceding inequality yields

$$\frac{x_i^{2\alpha+2}+x_j^{2\alpha+2}}{x_i^{2\alpha}+x_j^{2\alpha}}\ge x_ix_j\iff\frac{x_i^{2\alpha}+x_j^{2\alpha}}{x_i^{2\alpha+2}+x_j^{2\alpha+2}}\le\frac{1}{x_ix_j}.$$

Taking into account the GH-HM inequality, we have

$$\frac{1}{x_ix_j}\le\frac{1}{2}\left(\frac{1}{x_i^2}+\frac{1}{x_j^2}\right)$$

and

$$\sum_{1\le i<j\le n}\frac{x_i^{2\alpha}+x_j^{2\alpha}}{x_i^{2\alpha+2}+x_j^{2\alpha+2}}\le\sum_{1\le i<j\le n}\frac{1}{x_ix_j}$$
$$\le\frac{1}{2}\sum_{1\le i<j\le n}\left(\frac{1}{x_i^2}+\frac{1}{x_j^2}\right)=\frac{n-1}{2}\sum_{k=1}^{n}\frac{1}{x_k^2},$$

and we are done.

Also solved by *Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.*

A–37. *Proposed by Mihály Bencze, Braşov, Romania.* If $a_1 = 1$ and $a_{n+1} = 2a_n + \sqrt{3a_n^2 + 1}$, for all $n \geq 1$, then prove that the sequence $(a_n)_{n\geq 1}$ contains only composite numbers for all $n \geq 2$.

Solution 1 by the proposer. Let $u(n) = \dfrac{a^n + b^n}{2}$, $v(n) = \dfrac{a^n - b^n}{2\sqrt{3}}$, $a = 2+\sqrt{3}$, $b = 2-\sqrt{3}$. Then, $(u, v) = (u(n), v(n))$ $(n = 1, 2, \dots)$ are all solutions of the equation $u^2 - 3v^2 = 1$ $(u, v \in \mathbb{N})$.

$$\begin{aligned} 2v(n) + \sqrt{3v^2(n) + 1} = 2v(n) + u(n) &= 2\frac{a^n - b^n}{a - b} + 2\frac{a^n + b^n}{a + b} \\ &= \frac{a^{n+1} - b^{n+1}}{2\sqrt{3}} = v(n + 1) \Rightarrow a_n = v(n). \end{aligned}$$

If n is a composite number, then n has a divisor m with $1 < m < n$. But $v(m)$ is a positive integer satisfying $1 < v(m) < v(n)$ and $v(m)|v(n)$. This implies that $v(n)$ is not prime. Therefore, if $v(n)$ is a prime, then n must be prime. Furthermore, since $v(2) = 4$, n must be an odd prime.

Let $A(k) = \dfrac{c^k + d^k}{\sqrt{2}}$, $B(k) = \dfrac{c^k - d^k}{\sqrt{6}}$ when $c = \dfrac{1 + \sqrt{3}}{\sqrt{2}}$, $d = \dfrac{1 - \sqrt{3}}{\sqrt{2}}$. Then $(A, B) = (A(k), B(k))$ $(k = 1, 3, \dots)$ are all solutions of the equation $A^2 - 3B^2 = -2$ $(A, B \in \mathbb{N})$.

Since $a = c^2$, $b = d^2$ we get $v(k) = A(k)B(k)$. If n is an odd prime, then $v(n) = A(n)B(n) > 1$, $v(n) > 1$ for $n > 1$, so $v(n)$ is composite.

Solution 2 by Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain. We may start by observing that $a_n > 0$ $\forall n \in \mathbb{N}_{>0}$. In fact, the recurrence implies that $a_{n+1} > 2a_n$. Also, we can find a value for a_0 which is compatible with the recurrence:

$$\begin{aligned} a_1 = 1 = 2a_0 + \sqrt{3a_0 + 1} &\implies 1 + 4a_0^2 - 4a_0 = 3a_0^2 + 1 \\ &\iff a_0(a_0 - 4) = 0. \end{aligned}$$

We get two solutions, $a_0 = 0$ and $a_0 = 4$, from which only the first one is compatible with the recurrence. So, for every non-negative

integer n,

$$\begin{aligned} a_{n+1} = 2a_n + \sqrt{3a_n^2+1} &\Longleftrightarrow a_{n+1} - 2a_n = \sqrt{3a_n^2+1} \\ &\Longleftrightarrow a_{n+1}^2 + 4a_n^2 - 4a_n a_{n+1} = 3a_n^2 + 1 \\ &\Longleftrightarrow a_{n+1}^2 + a_n^2 - 4a_n a_{n+1} = 1. \end{aligned}$$

Computing the first terms $1, 4, 15, 56, 209, \ldots$ we may observe that $a_{n+1} = 4a_n - a_{n-1}$. Let us prove this:

$$\begin{aligned} a_{n+1} = 4a_n - a_{n-1} &\Longleftrightarrow 2a_n + \sqrt{3a_n^2+1} = 4a_n - a_{n-1} \\ &\Longleftrightarrow \sqrt{3a_n^2+1} = 2a_n - a_{n-1} \\ &\Longleftrightarrow 3a_n^2 + 1 = 4a_n^2 + a_{n-1}^2 - 4a_{n-1}a_n \\ &\Longleftrightarrow a_n^2 + a_{n-1}^2 - 4a_{n-1}a_n = 1, \end{aligned}$$

which is true as we have previously shown. In particular, this means a_n is integer $\forall n \geq 0$.

Solving $x^2 = 4x - 1$ we get two solutions, $x_1 = 2 + \sqrt{3}$, $x_2 = 2 - \sqrt{3}$. Hence, the general formula of (a_n) will be of the form $a_n = A(2+\sqrt{3})^n + B(2-\sqrt{3})^n$. We can determine the constants A, B by solving, for example for $a_0 = 0$, $a_1 = 1$, from where we get $A = \frac{1}{2\sqrt{3}}$, $B = -\frac{1}{2\sqrt{3}}$.

From $a_{n+1} = 4a_n - a_{n-1}$, we get $a_{n+1} \equiv a_{n-1} \pmod 2$. Therefore, as $a_2 = 4$, we inductively find that $a_n \equiv 0 \pmod 2$ if n is even. Since $a_{n+1} = 2a_n + \sqrt{3a_n^2+1} > (2+\sqrt{3})a_n > 3a_n$, $(a_n)_{n\in\mathbb{N}}$ is increasing, and therefore $a_n \geq 4$ if $n \geq 2$, which means a_n is composite for all even numbers $n \geq 2$.

We are left to prove that a_n is composite for all odd $n \geq 3$. We will prove that $a_{n+1}^2 - a_n^2 = a_{2n+1}$ for every $n \geq 1$, but first we will see that $a_{n+1}^2 - a_n^2$ is composite if $n \geq 1$.

We have $a_{n+1}^2 - a_n^2 = (a_{n+1}+a_n)(a_{n+1}-a_n)$. As $a_{n+1}+a_n > a_{n+1} \geq a_2 = 4$ and $a_{n+1} - a_n = 2a_n + \sqrt{3a_n^2+1} - a_n > (1+\sqrt{3})a_n > 2a_1 = 2$, we have that $a_{n+1}^2 - a_n^2$ is composite. We will take into

account that $(2+\sqrt{3})(2-\sqrt{3}) = 4-3 = 1$ to see that, from the formula $a_n = \frac{1}{2\sqrt{3}}\Big((2+\sqrt{3})^n - (2-\sqrt{3})^n\Big)$,

$$\begin{aligned}
&a_{n+1}^2 - a_n^2\\
=&\frac{1}{(2\sqrt{3})^2}\left(\left((2+\sqrt{3})^{n+1}-(2-\sqrt{3})^{n+1}\right)^2-\left((2+\sqrt{3})^n-(2-\sqrt{3})^n\right)^2\right)\\
=&\frac{1}{12}\left((2+\sqrt{3})^{2n+2}+(2-\sqrt{3})^{2n+2}-2-(2+\sqrt{3})^{2n}-(2-\sqrt{3})^{2n}+2\right)\\
=&\frac{1}{12}\Big((2+\sqrt{3})^{2n}((2+\sqrt{3})^2-1)+(2-\sqrt{3})^{2n}((2-\sqrt{3})^2-1)\Big).
\end{aligned}$$

We can see that $(2+\sqrt{3})^2 - 1 = 4+4\sqrt{3}+3-1 = 6+4\sqrt{3} = \frac{12}{2\sqrt{3}}(\sqrt{3}+2)$ and that $(2-\sqrt{3})^2-1 = 4-4\sqrt{3}+3-1 = 6-4\sqrt{3} = -\frac{12}{2\sqrt{3}}(-\sqrt{3}+2)$. Therefore,

$$\begin{aligned}
&a_{n+1}^2 - a_n^2\\
=&\frac{1}{12}\Big((2+\sqrt{3})^{2n}((2+\sqrt{3})^2-1)+(2-\sqrt{3})^{2n}((2-\sqrt{3})^2-1)\Big)\\
=&\frac{1}{12}\left((2+\sqrt{3})^{2n+1}\frac{12}{2\sqrt{3}}-(2-\sqrt{3})^{2n+1}\frac{12}{2\sqrt{3}}\right)\\
=&\frac{1}{2\sqrt{3}}\Big((2+\sqrt{3})^{2n+1}-(2-\sqrt{3})^{2n+1}\Big) = a_{2n+1}.
\end{aligned}$$

Therefore, all odd numbers that can be written as $2n+1$, $n \geq 1$, which are the odd numbers equal or greater than 3, are composite as well, thus completing the proof.

A–38. *Proposed by José Luis Díaz-Barrero and José Gibergans–Báguena, BarcelonaTech, Barcelona, Spain.* Let a, b be positive real numbers. Prove that

$$\int_0^1 t^{a-1}(1-t)^{b-1}\Gamma(t)\,\mathrm{d}t \geq \frac{a\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)}\,\Gamma\left(\frac{a}{a+b}\right),$$

where $\Gamma(x)$ is the Euler Gamma Function.

Solution 1 by Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain. We can observe several things in the inequality.

First, we have something very similar to the beta function $B(a,b)$, which is defined as

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t,$$

and has the following well-known remarkable property: $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$.

The gamma function $\Gamma(x)$, which is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t,$$

has the following well-known properties: it is continuous, $\Gamma(x) > 0$ if $x > 0$, $\Gamma(x+1) = x\Gamma(x)$, and, if n is a strictly positive integer, $\Gamma(n) = (n-1)!$.

We will now prove that $\Gamma(x)$ is decreasing in $(0,1)$. We will use the digamma function $\psi(x)$, which relates to the gamma function in the following way: $\Gamma'(x) = \psi(x)\Gamma(x)$. It is enough to prove that, $\forall\, x \in (0,1)$, $\psi(x) < 0$, since, by the positivity of $\Gamma(x)$, this would imply $\Gamma'(x) < 0$, $\forall\, x \in (0,1)$. The integral representation of the digamma function is also well-known, and it is the following:

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right)\mathrm{d}t.$$

As $f(s) = e^s$ is an increasing function, for $x \in (0,1)$, $t > 0$, we will have $tx < t$, which means that $-tx > -t$, which implies that $e^{-xt} > e^{-t}$. Therefore, since t, e^{-t}, $1-e^{-t}$ are positive functions over the strictly positive real numbers, we will have $\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}} < \frac{e^{-t}}{t} - \frac{e^{-t}}{1-e^{-t}}$, and so,

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right)\mathrm{d}t < \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-t}}{1-e^{-t}}\right)\mathrm{d}t = \psi(1).$$

We see that $\frac{e^{-t}}{t} - \frac{e^{-t}}{1-e^{-t}} = \frac{e^{-t}}{t(1-e^{-t})}(1-e^{-t}-t)$. Since $e^x \geq 1 + x$, $\forall\, x \in \mathbb{R}$, we will have $1-e^{-t}-t \leq 1-(1-t)-t = 0$. Therefore,

by the positivity of $\frac{e^{-t}}{t(1-e^{-t})}$, we have $\frac{e^{-t}}{t} - \frac{e^{-t}}{1-e^{-t}} \leq 0$, $\forall\, t > 0$, and so, $\forall\, x \in (0,1)$, we have

$$\psi(x) < \psi(1) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-t}}{1-e^{-t}}\right) \mathrm{d}t \leq 0,$$

as we wanted to see.

Let us prove that $\Gamma(x)$ is a convex function over the strictly positive real numbers. As $\Gamma(x)$ is continous (and well defined) over the strictly positive real numbers, we have

$$\begin{aligned}\frac{d^2}{dx^2}\Gamma(x) =& \frac{d^2}{dx^2}\int_0^\infty t^{x-1}e^{-t}\,\mathrm{d}t = \int_0^\infty \frac{\partial^2}{\partial^2 x}(t^{x-1}e^{-t}\,\mathrm{d}t)\\ =& \int_0^\infty (\ln(t))^2(t^{x-1}e^{-t}\,\mathrm{d}t) > 0\end{aligned}$$

since $\ln^2(t)t^{x-1}e^{-t} > 0\ \forall\, t \in (0,\infty)$.

Therefore, since $\Gamma(x)$ is positive and decreasing in $(0,1)$, we have that, on the one hand,

$$\begin{aligned}\int_0^1 t^{a-1}(1-t)^{b-1}\Gamma(t)\,\mathrm{d}t \geq& \int_0^1 t^{a-1}(1-t)^{b-1}\min_{0<s<1}\Gamma(s)\,\mathrm{d}t\\ =& \int_0^1 t^{a-1}(1-t)^{b-1}\Gamma(1)\,\mathrm{d}t\\ =& 0!\int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t\\ =& B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\end{aligned}$$

and, on the other hand, as $a,\ b > 0$, $1 < \frac{a}{a+b} + 1 < 2$. By the convexity of $\Gamma(x)$, as $\Gamma(1) = 0! = 1 = 1! = \Gamma(2)$, we have that, $\Gamma(c) \leq 1$, $\forall\, c \in (1,2)$, and so, since $x\Gamma(x) = \Gamma(x+1)$,

$$0 \leq \frac{a}{a+b}\Gamma\left(\frac{a}{a+b}\right) = \Gamma\left(\frac{a}{a+b}+1\right) \leq 1,$$

and therefore

$$\int_0^1 t^{a-1}(1-t)^{b-1}\Gamma(t)dt \geq \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \geq \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}\frac{a}{a+b}\Gamma\left(\frac{a}{a+b}\right),$$

thus proving the inequality.

Solution 2 by the proposer. To prove the statement we will need to apply Jensen's inequality for integrals. Namely, if $f : [a, b] \to \mathbb{R}$ is a convex function and $h : [a, b] \to \mathbb{R}_+^*$ and $u : [a, b] \to \mathbb{R}_+$ are integrable functions, then

$$f\left(\frac{\displaystyle\int_a^b h(x)u(x)\,\mathrm{d}x}{\displaystyle\int_a^b h(x)\,\mathrm{d}x}\right) \leq \frac{\displaystyle\int_a^b h(x)f(u(x))\,\mathrm{d}x}{\displaystyle\int_a^b h(x)u(x)\,\mathrm{d}x}.$$

Let X be a Beta's random variable, whose probability density function is

$$\begin{aligned} h_X(t) &= \frac{1}{B(a,b)} t^{a-1}(1-t)^{b-1} I_{\{0<t<1\}} \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} t^{a-1}(1-t)^{b-1} I_{\{0<t<1\}} \end{aligned}$$

with expectation given by $\mathbb{E}(X) = \frac{a}{a+b}$. Setting $u(t) = t$ and $f(t) = \Gamma(t)$, that is convex in $(0, 1)$, into Jensen's inequality, we have

$$\begin{aligned} \frac{a}{a+b}\,\Gamma\left(\frac{a}{a+b}\right) &\leq \frac{1}{B(a,b)} \int_0^1 t^{a-1}(1-t)^{b-1}\Gamma(t)\,\mathrm{d}t \\ &= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 t^{a-1}(1-t)^{b-1}\Gamma(t)\,\mathrm{d}t. \end{aligned}$$

from which the statement immediately follows.

A–39. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Determine all matrices $A \in M_3(\mathbb{R})$ such that

$$\operatorname{adj}(A) = \begin{pmatrix} 2 & -2 & 0 \\ -6 & 9 & -1 \\ 8 & -12 & 2 \end{pmatrix}.$$

Solution 1 by Xavier Ros Oton, Barcelona, Spain. It is well known that

$$A\operatorname{adj}(A) = \det A \cdot I,$$

where I is the identity matrix.

Then, since

$$\operatorname{adj}(A)^{-1} = \begin{pmatrix} 2 & -2 & 0 \\ -6 & 9 & -1 \\ 8 & -12 & 2 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix},$$

we have that

$$A = \det A \operatorname{adj}(A)^{-1} = \frac{\det A}{2}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix} = \lambda\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix}.$$

But, after some calculations, we get that

$$\operatorname{adj}\left[\lambda\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix}\right] = \lambda^2\begin{pmatrix} 2 & -2 & 0 \\ -6 & 9 & -1 \\ 8 & -12 & 2 \end{pmatrix},$$

so $\lambda = \pm 1$, and

$$A = \pm\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix}.$$

Solution 2 by Miguel Cidrás Senra, CFIS, BarcelonaTech, Barcelona, Spain. The adjugate of $B \in M_n$ is defined as $B \cdot \operatorname{adj}(B) = \det B \cdot I_n$. In our case, $n = 3$.

We can calculate $\det A$, as

$$\begin{aligned} A \cdot \operatorname{adj}(A) = \det A \cdot I_3 &\implies \det A \cdot \det \operatorname{adj}(A) = \det(\det A \cdot I_n) \\ &\implies \det A \cdot 4 = (\det A)^3 \cdot \det I_3 = (\det A)^3 \\ &\implies \det A^3 - 4 \det A = 0 \\ &\implies \det A = 0,\ 2 \text{ or } -2. \end{aligned}$$

We can calculate $\operatorname{adj}(\operatorname{adj}(A))$, as

$$\begin{aligned} &\operatorname{adj}(A) \cdot \operatorname{adj}(\operatorname{adj}(A)) = \det \operatorname{adj}(A) \cdot I_3 \\ \implies &\operatorname{adj}(A) \cdot \operatorname{adj}(\operatorname{adj}(A)) = 4 \cdot I_3 \\ \implies &A \cdot \operatorname{adj}(A) \cdot \operatorname{adj}(\operatorname{adj}(A)) = 4 \cdot A \\ \implies &\det A \cdot \operatorname{adj}(\operatorname{adj}(A)) = 4 \cdot A. \end{aligned}$$

If $\det A = 0$, then $4 \cdot A = 0 \cdot \operatorname{adj}(\operatorname{adj}(A)) = 0_3$ and $A = 0_3$. However, $\operatorname{adj}(0_3) = 0_3$, and we reach a contradiction.

If $\det A = 2$, then $4 \cdot A = 2 \cdot \operatorname{adj}(\operatorname{adj}(A))$, so $A = \dfrac{1}{2}\operatorname{adj}(\operatorname{adj}(A))$.

If $\det A = -2$, then $4A = -2\operatorname{adj}(\operatorname{adj}(A))$, so $A = \dfrac{-1}{2}\operatorname{adj}(\operatorname{adj}(A))$.

So, there are only two matrices: $\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix}$ and $\begin{pmatrix} -3 & -2 & -1 \\ -2 & -2 & -1 \\ 0 & -4 & -3 \end{pmatrix}$.

Solution 3 by Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain. Using that, if A is invertible, $\det(A^{-1}) = \frac{1}{\det(A)}$ and that, if $A \in M_n$, $\det(\lambda A) = \lambda^n \det(A)$, where $\lambda \in \mathbb{C}$, we can use the well-known formula $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$ to see that

$$\begin{aligned} \frac{1}{\det(A)} &= \det(A^{-1}) = \det\left(\frac{1}{\det(A)}\operatorname{adj}(A)\right) \\ &= \frac{1}{(\det(A))^n}\det(\operatorname{adj}(A)), \end{aligned}$$

which leads to $(\det(A))^{n-1} = \det(\operatorname{adj}(A))$. In this case we have that $\det(\operatorname{adj}(A)) = 4$ and $n = 3$, and so, $\det(A) = \pm 2$.

We can go back again to the formula $A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A)$ and take the inverse to see that $A = \det(A)(\operatorname{adj}(A))^{-1} = \pm 2(\operatorname{adj}(A))^{-1}$. If we compute $\operatorname{adj}(A)^{-1}$, we get:

$$(\operatorname{adj}(A))^{-1} = \frac{1}{2}\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix},$$

from where we will get

$$A = \pm\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix}.$$

It is easy to check that for both signs the adjugate of A is the matrix given in the statement.

Solution 4 by the proposer. For every matrix $A \in M_n(\mathbb{R})$ we have

$$|\mathrm{adj}(A)| = \left|(|A|)A^{-1}\right| = |A|^{n-1} \tag{7}$$

and

$$\mathrm{adj}(\mathrm{adj}(A)) = |\mathrm{adj}(A)|(\mathrm{adj}(A))^{-1} = |A|^{n-1}(|A|A^{-1})^{-1} = |A|^{n-2}A. \tag{8}$$

Since

$$|\mathrm{adj}(A)| = \begin{vmatrix} 2 & -2 & 0 \\ -6 & 9 & -1 \\ 8 & -12 & 2 \end{vmatrix} = 4,$$

then $|A| = \pm 2$. On the other hand, applying (8) we get

$$\mathrm{adj}(\mathrm{adj}(A)) = \begin{pmatrix} 6 & 4 & 2 \\ 4 & 4 & 2 \\ 0 & 8 & 6 \end{pmatrix},$$

from which it immediately follows that

$$A = \pm \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix}.$$

Also solved by *Miguel Amengual Covas, Cala Figuera, Mallorca, Spain, and Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.*

A–40. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $A = (a_{ij})$ be a 3×3 real orthogonal matrix with $\det(A) = 1$. Prove that

$$(\mathrm{tr}A - 1)^2 + \sum_{i<j}(a_{ij} - a_{ji})^2$$

is an integer number and determine its value.

Solution 1 by Eva Elduque, Zaragoza, Spain, and David Alfaya Sánchez, Madrid, Spain. We have

$$\begin{aligned}
&(\mathrm{tr}A-1)^2+\sum_{i<j}(a_{ij}-a_{ji})^2\\
=&(a_{11}+a_{22}+a_{33})^2-2(a_{11}+a_{22}+a_{33})+1+\sum_{i<j}(a_{ij}^2+a_{ji}^2-2a_{ij}a_{ji})\\
=&\sum_{i,j=1}^{3}a_{ij}^2-2(a_{11}+a_{22}+a_{33})\\
&+2\left(\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}+\begin{vmatrix} a_{11} & a_{13}\\ a_{31} & a_{33}\end{vmatrix}+\begin{vmatrix} a_{22} & a_{23}\\ a_{23} & a_{33}\end{vmatrix}\right)+1.
\end{aligned}$$

Since A is orthogonal, then $A^t=A^{-1}$, Taking into account that $\det A=1$, we have

$$a_{ii}=\frac{\det C_{ii}}{\det A}=\det C_{ii},$$

where C_{ij} denotes the matrix cofactor of a_{ij}. In this case,

$$a_{33}=\begin{vmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix};\quad a_{22}=\begin{vmatrix} a_{11} & a_{13}\\ a_{31} & a_{33}\end{vmatrix};\quad a_{11}=\begin{vmatrix} a_{22} & a_{23}\\ a_{23} & a_{33}\end{vmatrix}.$$

So,

$$(\mathrm{tr}A-1)^2+\sum_{i<j}(a_{ij}-a_{ji})^2=\sum_{i,j=1}^{3}a_{ij}^2+1=\mathrm{tr}(AA^t)+1=\mathrm{tr}(I)+1=4,$$

and we are done.

Solution 2 by the proposer. Expanding the characteristic polynomial of A as

$$\begin{aligned}
\det(A-tI)&=t^3-(a_{11}+a_{22}+a_{33})t^2\\
&+(a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}-a_{12}a_{21}-a_{13}a_{31}-a_{23}a_{32})t-1,
\end{aligned}$$

multiplying out $\det(A-tI)=(t-t_1)(t-t_2)(t-t_3)$ and comparing both expressions, we get

$$\begin{aligned}
t_1+t_2+t_3&=a_{11}+a_{22}+a_{33},\\
t_1t_2+t_2t_3+t_3t_1&=a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}\\
&\quad -a_{12}a_{21}-a_{13}a_{31}-a_{23}a_{32}.
\end{aligned}$$

When $\det A = 1$ then A has one eigenvalue equal to 1, as is well-known. WLOG, we can assume that $t_1 = 1$. Now from $t_1 t_2 t_3 = 1$ it follows that $t_2 t_3 = 1$, and

$$\begin{aligned}
&(\mathrm{tr} A - 1)^2 + \sum_{i<j} (a_{ij} - a_{ji})^2 \\
=&(a_{11}+a_{22}+a_{33} - 1)^2 + (a_{12} - a_{21})^2 + (a_{13} - a_{31})^2 + (a_{23} - a_{32})^2 \\
=&(a_{11}^2 + a_{12}^2 + a_{13}^2) + (a_{21}^2 + a_{22}^2 + a_{23}^2) + (a_{31}^2 + a_{32}^2 + a_{33}^2) + 1 \\
&+ 2\left(\sum_{1 \le i < j \le 3} a_{ii} a_{jj} - \sum_{1 \le i < j \le 3} a_{ij} a_{ji} - \sum_{i=1}^{3} a_{ii} \right) \\
=&1 + 1 + 1 + 1 + 2(t_1 t_2 + t_2 t_3 + t_3 t_1) - 2(t_1 + t_2 + t_3) = 4.
\end{aligned}$$

This completes the proof.

Also solved by *Víctor Martín Chabrera, BarcelonaTech, Barcelona, Spain.*

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