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# ARHIMEDE
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*Articles*
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# Articles

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Please, send submittals to: **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to:

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# *Some Applications of Classical Discrete Inequalities*

**José Luis Díaz-Barrero**

**Abstract**

In this note the classical inequalities of Jensen, Hölder, Chebyshev and CBS are applied to obtain new elementary inequalities.

## 1 Introduction

There is no doubt that four of the most famous classical inequalities appeared in the literature are those of Cauchy [1], Chebyshev [6], Jensen [5] and Hölder [4]. These inequalities have been applied to obtain new results in almost all branches of mathematics and since the beginning they were (and currently they still are) the source of a lot of research papers [2, 3]. Our goal in this paper is to use these classical results to derive new elementary cyclic inequalities. Furthermore, some particular cases, when the variables are constrained, are also given.

## 2 The inequalities

In this section we present each of the four mentioned inequalities (usually without a proof) and the particular results we derive for each of them in an orderly fashion.

## 2.1 The CBS inequality

We begin by stating the classical Cauchy-Buniakowski-Schwarz's inequality, CBS for short, that will be used hereafter.

**Theorem 1 (Cauchy-Buniakowski-Schwarz).** *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Then,*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \sum_{k=1}^{n} a_k^2 \cdot \sum_{k=1}^{n} b_k^2$$

*holds.*

*Proof.* Using Lagrange's identity,

$$\sum_{k=1}^{n} a_k^2 \cdot \sum_{k=1}^{n} b_k^2 = \left(\sum_{k=1}^{n} a_k b_k\right)^2 + \sum_{1 \leq k < j \leq n} (a_k b_j - a_j b_k)^2,$$

and the fact that the square of a real number in nonnegative, we get

$$\sum_{k=1}^{n} a_k^2 \cdot \sum_{k=1}^{n} b_k^2 \geq \left(\sum_{k=1}^{n} a_k b_k\right)^2,$$

and the proof is complete. □

Other formulations of CBS are given in the following corollaries.

**Corollary 1.** *Let $a_k$, $b_k > 0$, $1 \leq k \leq n$, be real numbers. Then it is true that*

$$\sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{b_k^2}{a_k} \geq \left(\sum_{k=1}^{n} b_k\right)^2.$$

*Proof.* We have

$$\begin{aligned}\left(\sum_{k=1}^{n} b_k\right)^2 &= \left(\sum_{k=1}^{n} \sqrt{a_k}\cdot\frac{b_k}{\sqrt{a_k}}\right)^2\\ &\le \sum_{k=1}^{n}(\sqrt{a_k})^2\cdot\sum_{k=1}^{n}\left(\frac{b_k}{\sqrt{a_k}}\right)^2\\ &= \sum_{k=1}^{n} a_k\cdot\sum_{k=1}^{n}\frac{b_k^2}{a_k}\end{aligned}$$

on account of CBS. □

**Corollary 2.** *Let $a_k$, $b_k > 0$, $1 \le k \le n$, be real numbers. Then,*

$$\sum_{k=1}^{n} a_k b_k\cdot\sum_{k=1}^{n}\frac{b_k}{a_k} \;\ge\; \left(\sum_{k=1}^{n} b_k\right)^2.$$

*Proof.* On account of CBS, we have

$$\begin{aligned}\sum_{k=1}^{n} a_k b_k\cdot\sum_{k=1}^{n}\frac{b_k}{a_k} &= \sum_{k=1}^{n}\left(\sqrt{a_k b_k}\right)^2\cdot\sum_{k=1}^{n}\left(\sqrt{\frac{b_k}{a_k}}\right)^2\\ &\ge \left(\sum_{k=1}^{n}\sqrt{a_k b_k\cdot\frac{b_k}{a_k}}\right)^2 = \left(\sum_{k=1}^{n} b_k\right)^2.\end{aligned}$$

□

**Corollary 3.** *Let $a_k$, $b_k$, $1 \le k \le n$, be real numbers. Then we have that*

$$\sum_{k=1}^{n} a_k\cdot\sum_{k=1}^{n} a_k b_k^2 \;\ge\; \left(\sum_{k=1}^{n} a_k b_k\right)^2.$$

*Proof.* We have

$$\begin{aligned}\sum_{k=1}^{n} a_k\cdot\sum_{k=1}^{n} a_k b_k^2 &= \sum_{k=1}^{n}(\sqrt{a_k})^2\cdot\sum_{k=1}^{n}(b_k\sqrt{a_k})^2\\ &\ge \left(\sum_{k=1}^{n}\sqrt{a_k}\cdot b_k\sqrt{a_k}\right)^2 = \left(\sum_{k=1}^{n} a_k b_k\right)^2.\end{aligned}$$

□

Using CBS we also obtain the the following cyclic inequality.

**Theorem 2.** *Let $a_1, a_2, \ldots, a_n$ ($n \geq 2$) be positive real numbers. Then,*

$$\sum_{k=1}^{n} \frac{a_k^2}{a_k^2 + a_k a_{k+1} + a_{k+1}^2} \geq \frac{(a_1 + a_2 + \ldots + a_n)^2}{9(a_1^2 + a_2^2 + \ldots + a_n^2)}$$

*holds, where $a_{n+1} = a_1$.*

*Proof.* The claimed inequality is equivalent to

$$\left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} \frac{a_k^2}{a_k^2 + a_k a_{k+1} + a_{k+1}^2}\right)^{1/2} \geq \frac{1}{3} \sum_{k=1}^{n} a_k.$$

Applying CBS inequality to the values $a_1, a_2, \ldots, a_n$ and

$$b_1 = \frac{a_1}{\sqrt{a_1^2 + a_1 a_2 + a_2^2}}, \ldots, b_n = \frac{a_n}{\sqrt{a_n^2 + a_n a_1 + a_1^2}}$$

yields

$$\left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} \frac{a_k^2}{a_k^2 + a_k a_{k+1} + a_{k+1}^2}\right)^{1/2}$$
$$\geq \sum_{k=1}^{n} \frac{a_k^2}{\sqrt{a_k^2 + a_k a_{k+1} + a_{k+1}^2}}.$$

Hence, it suffices to prove that

$$\sum_{k=1}^{n} \frac{a_k}{\sqrt{a_k^2 + a_k a_{k+1} + a_{k+1}^2}} \geq \frac{1}{3} \sum_{k=1}^{n} a_k.$$

The inequality $\dfrac{x^2}{\sqrt{x^2 + xy + y^2}} \geq \dfrac{2x - y}{3}$ is trivial for $2x - y \leq 0$ (using 0 as a refinement), whereas if $2x - y > 0$, then

$$\frac{x^2}{\sqrt{x^2 + xy + y^2}} \geq \frac{x^2}{x + y} = \frac{x^2(2x - y)}{(x + y)(2x - y)}$$
$$\geq \frac{x^2(2x - y)}{\frac{9x^2}{4}} \geq \frac{2x - y}{3},$$

where the second inequality comes from the AM-GM inequality. Setting $x = a_k$ and $y = a_{k+1}$, $1 \leq k \leq n$, we obtain

$$\frac{a_k^2}{\sqrt{a_k^2 + a_k a_{k+1} + a_{k+1}^2}} \geq \frac{2a_k - a_{k+1}}{3}.$$

Adding up the preceding inequalities we get

$$\sum_{k=1}^{n} \frac{a_k^2}{\sqrt{a_k^2 + a_k a_{k+1} + a_{k+1}^2}} \geq \frac{1}{3} \sum_{k=1}^{n} a_k,$$

and the proof is complete. □

**Corollary 4.** *Let $a$, $b$, $c$ be positive real numbers such that $a^2 + b^2 + c^2 = 1$. Then,*

$$\left( \sum_{cyclic} \frac{a^2}{a^2 + ab + b^2} \right)^{1/2} \geq (abc)^{1/3}.$$

*Proof.* Putting $a_1 = a$, $a_2 = b$ and $a_3 = c$ in the preceding result yields

$$\sum_{\text{cyclic}} \frac{a^2}{a^2 + ab + b^2} \geq \frac{(a+b+c)^2}{9\left(a^2 + b^2 + c^2\right)}.$$

Since $a^2 + b^2 + c^2 = 1$, then the above inequality becomes

$$\sum_{\text{cyclic}} \frac{a^2}{a^2 + ab + b^2} \geq \left( \frac{a+b+c}{3} \right)^2 \geq \sqrt[3]{(abc)^2}$$

on account of the AM-GM inequality. The statement follows immediately from the preceding and the proof is complete. □

## 2.2 Chebyshev's inequality

The second discrete inequality that is considered in this paper is Chebyshev's sum inequality.

**Theorem 3 (Chebyshev).** *If $a_1 \le a_2 \le \ldots \le a_n$ and $b_1 \le b_2 \le \ldots \le b_n$ are two nondecreasing sequences of real numbers, then*

$$\sum_{k=1}^{n} a_k b_{n+1-k} \le \frac{1}{n}\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right) \le \sum_{k=1}^{n} a_k b_k.$$

As an application of Chebyshev's sum inequality, we have the following.

**Theorem 4.** *Let $a_1, a_2, \ldots, a_n$ $(n \ge 2)$ be positive real numbers such that $a_1 + a_2 + \ldots + a_n = 1$. Then,*

$$\left(\prod_{i=1}^{n} a_i^{a_i}\right)^{1/(1-a_i)} \ge \prod_{i=1}^{n} a_i^{1/(n-1)}.$$

*Proof.* WLOG we may assume that $a_1 \ge a_2 \ge \ldots \ge a_n > 0$. Then, the finite sequences $\ln a_1 \ge \ln a_2 \ge \ldots \ge \ln a_n$, $\dfrac{a_1}{1-a_1} \ge \dfrac{a_2}{1-a_2} \ge \ldots \ge \dfrac{a_n}{1-a_n}$, and $\dfrac{1}{1-a_1} \ge \dfrac{1}{1-a_2} \ge \ldots \ge \dfrac{1}{1-a_n}$ are sorted in the same way. Applying Chebyshev's inequality twice yields

$$\begin{aligned}
\sum_{i=1}^{n} \frac{a_i \ln a_i}{1-a_i} &= \sum_{i=1}^{n} \frac{a_i}{1-a_i} \ln a_i \ge \frac{1}{n}\left(\sum_{i=1}^{n} \frac{a_i}{1-a_i}\right)\left(\sum_{i=1}^{n} \ln a_i\right) \\
&= \frac{1}{n}\left(\sum_{i=1}^{n} \frac{1}{1-a_i} \cdot a_i\right)\left(\sum_{i=1}^{n} \ln a_i\right) \\
&\ge \frac{1}{n^2}\left(\sum_{i=1}^{n} \frac{1}{1-a_i}\right)\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \ln a_i\right) \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n} \frac{1}{1-a_i}\right)\left(\sum_{i=1}^{n} \ln a_i\right).
\end{aligned}$$

Applying the AM-HM inequality, we have

$$\sum_{i=1}^{n} \frac{1}{1-a_i} \ge n^2 \Big/ \sum_{i=1}^{n} (1-a_i) = \frac{n^2}{n-1}.$$

From the preceding, we get

$$\sum_{i=1}^{n} \frac{a_i \ln a_i}{1-a_i} \geq \frac{1}{n-1} \sum_{i=1}^{n} \ln a_i.$$

Taking into account the properties of the function $f(t) = \ln t$, the statement immediately follows. □

**Corollary 5.** *Let $a$, $b$, $c$ be positive real numbers such that $a+b+c=1$. Then,*

$$(a^a)^{\frac{a}{b+c}} \left(b^b\right)^{\frac{b}{c+a}} (c^c)^{\frac{c}{a+b}} \geq a^{1/2-a}\, b^{1/2-b}\, c^{1/2-c}.$$

*Proof.* Applying the preceding result with $n=3$, we have

$$(a^a)^{1/(1-a)} \left(b^b\right)^{1/(1-b)} (c^c)^{1/(1-c)} \geq (abc)^{1/2}$$

or, equivalently,

$$(a^a)^{1+\frac{a}{b+c}} \left(b^b\right)^{1+\frac{b}{c+a}} (c^c)^{1+\frac{c}{a+b}} \geq (abc)^{1/2}.$$

After rearranging terms, we obtain

$$(a^a)^{\frac{a}{b+c}} \left(b^b\right)^{\frac{b}{c+a}} (c^c)^{\frac{c}{a+b}} \geq a^{1/2-a} b^{1/2-b} c^{1/2-c},$$

and we are done. □

## 2.3 Jensen's inequality

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is convex on $I$ if and only if $f(qx + (1-q)y) \leq qf(x) + (1-q)f(y)$ for all $x$, $y \in I$ and $0 \leq q \leq 1$. If the inequality holds in the opposite direction, then $f$ is said to be concave.

**Theorem 5 (Jensen).** *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function. Let $q_1, q_2, \ldots, q_n$ be nonnegative real numbers such that $q_1 + q_2 + \ldots + q_n = 1$. Then for all $a_k \in I$ $(1 \leq k \leq n)$ we have that*

$$f\left(\sum_{k=1}^{n} q_k a_k\right) \leq \sum_{k=1}^{n} q_k f(a_k).$$

*If $f$ is concave the preceding inequality reverses.*

Notice that when $n = 2$ Jensen's inequality is nothing more than the definition of convexity previously given. A proof and some of its applications can be found in [3].

Next, we present the following cyclic inequality that will be derived using Jensen's inequality.

**Theorem 6.** *Let $a_1, a_2, \ldots, a_n$ be positive numbers lying in the real interval $[1, +\infty)$. Then,*

$$\left(1 - \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{-2}\right)^{1/2} \geq \left(\frac{1}{n}\prod_{k=1}^{n} a_k^{-1}\right)\left(\sum_{k=1}^{n}(a_k^2 - 1)^{1/2}\right).$$

*Proof.* Consider the function $f : (0, 1] \to \mathbb{R}$ defined by $f(x) = \sqrt{1 - x^2}$. We have $f'(x) = \dfrac{-x}{\sqrt{1 - x^2}}$ and $f''(x) = \dfrac{-1}{(1 - x^2)^{3/2}} < 0$ for $0 < x < 1$, so $f$ is concave in $(0, 1)$. Applying Jensen's inequality to the function $f$, namely,

$$f\left(\frac{x_1 + x_2 + \ldots + x_n}{n}\right) \geq \frac{1}{n}\Big(f(x_1) + f(x_2) + \ldots + f(x_n)\Big),$$

and putting $x_k = 1/a_k$, $1 \leq k \leq n$, yields

$$\sqrt{1 - \left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}\right)^2} \geq \frac{1}{n}\sum_{k=1}^{n}\sqrt{1 - \frac{1}{a_k^2}}.$$

Taking into account the AM-HM inequality, we have

$$\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}\right)^2 \geq \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{-2}$$

and

$$\sqrt{1 - \left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^{-2}} \geq \sqrt{1 - \left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}\right)^2} \geq \frac{1}{n}\sum_{k=1}^{n}\frac{\sqrt{a_k^2 - 1}}{a_k}.$$

Since $a_k \geq 1$, $1 \leq k \leq n$, then $1/a_k \leq 1$, and $\dfrac{1}{a_k} \geq \prod_{k=1}^{n} a_k^{-1}$ for $1 \leq k \leq n$. Therefore,

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\sqrt{a_k^2-1}}{a_k} \geq \left(\frac{1}{n}\prod_{k=1}^{n} a_k^{-1}\right)\left(\sum_{k=1}^{n}(a_k^2-1)^{1/2}\right),$$

and the statement follows. Equality holds when $a_1 = a_2 = \ldots = a_n = 1$. This completes the proof. □

**Corollary 6.** *Let $a$, $b$, $c \in [1, +\infty)$ be such that $a + b + c = 6$. Then,*

$$\sqrt{a^2-1} + \sqrt{b^2-1} + \sqrt{c^2-1} < \frac{3\sqrt{3}}{2}\,abc.$$

*Proof.* Applying the preceding result when $n = 3$ and taking into account the constrain condition, we have

$$\begin{aligned}\frac{1}{3abc}\left(\sqrt{a^2-1} + \sqrt{b^2-1} + \sqrt{c^2-1}\right) &\leq \sqrt{1-\left(\frac{3}{a+b+c}\right)^2}\\ &= \frac{\sqrt{3}}{2}.\end{aligned}$$

from which the statement follows. Note that equality is not possible and the proof is complete. □

## 2.4 Hölder's inequality

A generalization of CBS inequality is the well-known inequality of Hölder. It is stated in the following theorem.

**Theorem 7 (Hölder).** *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers and let $p, q$ be positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\sum_{k=1}^{n}|a_k b_k| \leq \left(\sum_{k=1}^{n}|a_k|^p\right)^{1/p}\left(\sum_{k=1}^{n}|b_k|^q\right)^{1/q},$$

*with equality when $|b_k| = c|a_k|^{p-1}$, $1 \leq k \leq n$.*

This result can itself be generalized:

**Theorem 8 (Generalized Hölder).** *Let $p_1, p_2, \ldots, p_k$ be positive numbers such that $\sum p_i^{-1} = 1$, and $a_{i,j}$ for $1 \le i \le k, 1 \le j \le n$ be real numbers. Then*

$$\sum_{j=1}^{n} \left| \prod_{i=1}^{k} a_{i,j} \right| \le \prod_{i=1}^{k} \left( \sum_{j=1}^{n} |a_{i,j}|^{p_i} \right)^{1/p_i}.$$

Finally, we present an inequality that will be derived using the preceding result.

**Theorem 9.** *Let $a$, $b$, $c$, $x$, $y$, $z$ be positive real numbers. Then,*

$$\frac{a^3}{bx} + \frac{b^3}{cy} + \frac{c^3}{az} \ge \frac{(a+b+c)^3}{3(bx+cy+az)}$$

*holds.*

*Proof.* On account of the generalized Hölder's inequality, we have

$$\prod_{i=1}^{3} \left(p_i^3 + q_i^3 + r_i^3\right)^{1/3} \ge p_1p_2p_3 + q_1q_2q_3 + r_1r_2r_3$$

for all positive reals $p_i$, $q_i$, $r_i$, $1 \le i \le 3$.

Putting in the preceding $(p_1, p_2, p_3) = \left(\frac{a}{\sqrt[3]{bx}}, 1, \sqrt[3]{bx}\right)$, $(q_1, q_2, q_3) = \left(\frac{b}{\sqrt[3]{cy}}, 1, \sqrt[3]{cy}\right)$ and $(r_1, r_2, r_3) = \left(\frac{c}{\sqrt[3]{az}}, 1, \sqrt[3]{az}\right)$ yields

$$\sqrt[3]{3} \left(\frac{a^3}{bx} + \frac{b^3}{cy} + \frac{c^3}{az}\right)^{1/3} (bx+cy+az)^{1/3} \ge a+b+c.$$

Cubing both sides and dividing both sides by $3\,(bx+cy+az)$ we obtain

$$\frac{a^3}{bx} + \frac{b^3}{cy} + \frac{c^3}{az} \ge \frac{(a+b+c)^3}{3(bx+cy+az)}.$$

Equality holds when $a = b = c = x = y = z$, and the proof is complete. □

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# References

[1] Cauchy, A.-L. *Cours d'analyse de l'École Royale Polytechnique*. Cambridge Library Collection. Reprint of the 1821 original, Previously published by Cooperativa Libraria Universitaria Editrice Bologna, Bologna, 1992 [MR1267567]. Cambridge University Press, Cambridge, 2009, pp. ii+xvi+576. ISBN: 978-1-108-00208-0. DOI: 10.1017/CBO9780511693328.

[2] Díaz-Barrero, J. L. "Some Discrete Inequalities". *Arhimede math. j. (Barc.)* 1 (1 2014), pp. 1–9. ISSN: 2462-537X.

[3] Díaz-Barrero, J. L. "Hölder's inequality revisited". *Arhimede math. j. (Barc.)* 2 (1 2015), pp. 1–11. ISSN: 2462-537X.

[4] Hölder, O. "Ueber einen Mittelwertsatz." German. *Gött. Nachr.* 2 (1889), pp. 38–47.

[5] Jensen, J. L. W. V. "Sur les fonctions convexes et les inégalités entre les valeurs moyennes". *Acta Math.* 30.1 (1906), pp. 175–193. ISSN: 0001-5962. DOI: 10.1007/BF02418571.

[6] Korkine, A. "Sur une théoreme de M. Tchebychef". *CR Acad. Sci. Paris* 96 (1883), pp. 316–327.

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# Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.
2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to: **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to:

`jose.luis.diaz@upc.edu`

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

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*Solutions to the problems stated in this issue should be posted before*

**Apr 30, 2017**

# Elementary Problems

***E–35.*** *Proposed by Eric Sierra Garzo, CFIS, BarcelonaTech, Barcelona, Spain.* Find the sum of the coefficients of

$$P(x) = 6\left(7x^2 - 9x^3 + 3x^5\right)^{2016} + 3\left(5x^7 - 2x^{11} - 4x^{13}\right)^{2017} - 2x - 1.$$

***E–36.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain, and Mihály Bencze, Braşov, Romania.* Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Show that for all integers $m \geq 1$ there exist $n$ positive reals $b_1, b_2, \ldots, b_n$ such that

$$\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1}{1+a_k} + \frac{1}{1+b_k}\right)^m = 1.$$

***E–37.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $PQRS$ be a square. A straight line through $P$ cuts side $RS$ at $M$ and line $QR$ at $N$. Show that

$$\left(\frac{PQ}{PM}\right)^2 + \left(\frac{PQ}{PN}\right)^2$$

is a positive integer.

***E–38.*** *Proposed by José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.* Let $f(x) = x^2 + 14x + 42$. Solve the equation

$$\underbrace{f(f(f(\ldots(f(x)))))}_{n} = 0.$$

***E–39.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $D$, $E$, $F$ be the points where the angle bisectors of angles $A$, $B$, $C$ of triangle $ABC$ cut the opposite sides, and let $I$ be its incenter. Find the minimum value of

$$\frac{AI}{ID} + \frac{BI}{IE} + \frac{CI}{IF}.$$

***E–40.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In a certain parliament, each member belongs to exactly four committees, and each committee has exactly four members. Prove that the number of members equals the number of committees.

# *Easy–Medium Problems*

***EM–35.*** *Proposed by Mihály Bencze, Braşov, Romania.* Find the value of

$$a_1 + \sum_{k=1}^{n} a_k^2,$$

where $a_k = \dfrac{1}{k} + \dfrac{1}{k+1} + \cdots + \dfrac{1}{n}$.

***EM–36.*** *Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, BarcelonaTech, Spain.* Let $x_1$, $x_2$, $x_3$ be the roots of the equation $x^3 - 3x^2 - 2x + 1 = 0$. Find the equation whose roots are $y_1 = x_1 + \dfrac{1}{x_1}$, $y_2 = x_2 + \dfrac{1}{x_2}$, and $y_3 = x_3 + \dfrac{1}{x_3}$.

***EM–37.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* For all integers $n \geq 1$, show that

$$\sum_{1 \leq i \leq n} i^2 + \sum_{1 \leq i < j \leq n} ij + \sum_{1 \leq j < i \leq n} ji$$

is the square of a positive integer and determine its value.

***EM–38.*** *Proposed by Óscar Rivero Salgado, CFIS, BarcelonaTech, Barcelona, Spain.* Solve in positive integers the following equation

$$(a^2 + b^2 + c^2)^2 = 2(a^4 + b^4 + c^4) + 2016.$$

***EM–39.*** *Proposed by Guillem Alsina Oriol, CFIS, BarcelonaTech, Barcelona, Spain.* Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ have in common two points $X$ and $Y$. We draw a line that cuts $\mathcal{C}_1$ at points $A$ and $B$. Next we draw the lines $AX$, $AY$ which cut $\mathcal{C}_2$ at points $A_X$ and $B_Y$ and lines $BX$, $BY$ which cut $\mathcal{C}_2$ at points $B_X$ and $B_Y$ respectively. Show that the three lines $AB$, $A_XB_Y$ and $A_YB_X$ are parallel.

***EM–40.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $n$ be a nonnegative integer. Compute

$$\sum_{j=1}^{2n+1} \frac{j^3}{(2n+1)^2 - 3(2n+1)j + 3j^2}.$$

# Medium–Hard Problems

***MH–35.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $a$, $b$, $c$ be positive real numbers such that $a+b+c=3$. Prove that

$$\sqrt{\frac{b}{a^2+3}}+\sqrt{\frac{c}{b^2+3}}+\sqrt{\frac{a}{c^2+3}}\leq\frac{3}{2}\sqrt[4]{\frac{1}{abc}}.$$

***MH–36.*** *Proposed by Damià Torres Latorre, CFIS, BarcelonaTech, Barcelona, Spain and Jesús Dueñas Pamplona, Valladolid, Spain.* Let $ABC$ be and acute triangle with circumcircle $\omega$. Let $P$ be a point lying on $\omega$ distinct from $B$ and its diametrically opposite. Let $M$ the intersection point of the tangents to $\omega$ drawn from $P$ and $B$, respectively. Parallel to the tangent to $\omega$ drawn from $A$ and passing for $M$ cuts lines $AB$ and $AC$ in the points $D$ and $E$, respectively. Find the locus of $T$ when $P$ moves on $\omega$.

***MH–37.*** *Proposed by Nicolae Papacu, Slobozia, Romania.* Determine all positive integers $n$ and prime numbers $p$ such that $p^n+8$ is a perfect cube.

***MH–38.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $1<a<b$ be real numbers. Prove that for any $x_1$, $x_2$, $x_3\in[a,b]$ there exists $c\in[a,b]$ such that

$$\frac{1}{\log x_1}+\frac{1}{\log x_2}+\frac{1}{\log x_3}+\frac{1}{\sqrt[3]{\log x_1\log x_2\log x_3}}=\frac{4}{\log c}.$$

***MH–39.*** *Proposed by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom.* Consider the integer grid $\mathbb{Z}^2=\mathbb{Z}\times\mathbb{Z}=\{(a,b)\mid a,b\in\mathbb{Z}\}$. How many different walks can be performed in $2n$ steps of length one starting and ending at the same point and moving in steps only in the directions of the coordinate axes?

***MH–40.*** *Proposed by Mihály Bencze, Braşov, Romania.* Let $ABCD$ be a convex quadrilateral such that $a = AB$, $b = BC$, $c = CD$, $d = DA$, $e = BD$, $f = AC$. Prove that

$$[ABCD] \leq \frac{\sqrt{3}}{12} \min\left\{ \frac{(ab+bf+fa)^2}{a^2+b^2+f^2} + \frac{(cd+df+fc)^2}{c^2+d^2+f^2}, \frac{(ae+ed+da)^2}{a^2+e^2+d^2} + \frac{(bc+ce+eb)^2}{b^2+c^2+e^2} \right\}.$$

# Advanced Problems

***A–35.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Suppose that all the eigenvalues of $A \in M_n(\mathbb{R})$ are positive real numbers. Show that

$$\det(A + A^{-1}) \geq 2^n.$$

***A–36.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $x_k$ $(1 \leq k \leq n)$ be positive numbers and let $\alpha \geq 1$. Prove that

$$\frac{1}{n-1} \sum_{1 \leq i < j \leq n} \frac{x_i^{2\alpha} + x_j^{2\alpha}}{x_i^{2\alpha+2} + x_j^{2\alpha+2}} \leq \frac{1}{2} \sum_{k=1}^{n} \frac{1}{x_k^2}.$$

***A–37.*** *Proposed by Mihály Bencze, Braşov, Romania.* If $a_1 = 1$ and $a_{n+1} = 2a_n + \sqrt{3a_n^2 + 1}$, for all $n \geq 1$, then prove that the sequence $(a_n)_{n \geq 1}$ contains only composite numbers for all $n \geq 2$.

***A–38.*** *Proposed by José Luis Díaz-Barrero and José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.* Let $a$, $b$ be positive real numbers. Prove that

$$\int_0^1 t^{a-1}(1-t)^{b-1}\Gamma(t)\, \mathrm{d}t \geq \frac{a\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)}\, \Gamma\left(\frac{a}{a+b}\right),$$

where $\Gamma(x)$ is the Euler Gamma Function.

***A–39.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Determine all matrices $A \in M_3(\mathbb{R})$ such that

$$\operatorname{adj}(A) = \begin{pmatrix} 2 & -2 & 0 \\ -6 & 9 & -1 \\ 8 & -12 & 2 \end{pmatrix}.$$

***A–40.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $A = (a_{ij})$ be a $3 \times 3$ real orthogonal matrix

with $\det(A) = 1$. Prove that

$$(\mathrm{tr}A - 1)^2 + \sum_{i<j}(a_{ij} - a_{ji})^2$$

is an integer number and determine its value.

# Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to: **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to:

jose.luis.diaz@upc.edu

# *Metric conditions on five collinear points*

**Andrés Sáez-Schwedt**

**Abstract**

Sometimes, the solution of a geometric problem requires proving some metric conditions about four or five collinear points, typically conditions involving the power of a point to a circle, harmonic conjugates, or inverse points with respect to a circle. In this note, many equivalent conditions are given, so that one can pass from either one to any of the other ones. It is shown how to apply this technique to the solution of geometric problems of olympic level.

## 1 The tools

We will refer to the following result informally as the Five Point Lemma.

**Lemma 1 (Five points' lemma).** *Given five collinear points $A$, $O$, $C$, $B$ ,$D$ (in that order), with $O$ the midpoint of $AB$, the following conditions are equivalent:*

$$\frac{DA}{DB} = \frac{CA}{CB}, \tag{1}$$

$$OC \cdot OD = OB^2, \tag{2}$$

$$CO \cdot CD = CA \cdot CB, \tag{3}$$

$$DO \cdot DC = DA \cdot DB, \tag{4}$$

$$OB = \frac{BD \cdot BC}{BD - BC}, \tag{5}$$

$$\frac{OD}{OC} = \left(\frac{BD}{BC}\right)^2. \tag{6}$$

*Proof.* Let $\Gamma$ be the circle of center $O$ and diameter $AB$, and let $T$ be one of the two possible points of $\Gamma$ satisfying $TC \perp AB$.

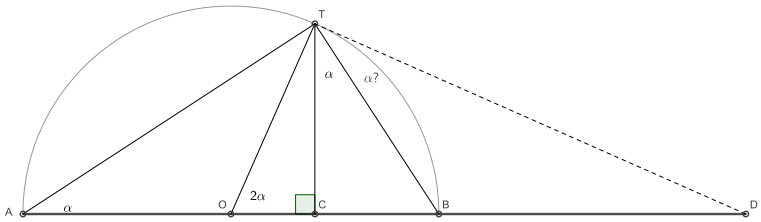


Figure 1: Scheme for the proof of Lemma 1

If we denote $\alpha = \angle BAT$, it is immediate to deduce the other angles indicated in Figure 1 as $\alpha$ and $2\alpha$. We shall prove that all conditions of the lemma are equivalent to $\angle BTD = \alpha$, or equivalently, $\angle OTD = 90°$.

1. If $\angle BTD = \alpha = \angle CTB$, then $TB$ and $TA$ are respectively the internal and external bisectors of $\angle CTD$, so by the bisector's theorem one has that $\frac{BC}{BD}$ and $\frac{AC}{AD}$ are both equal to $\frac{TC}{TD}$, and (1) holds. Conversely, assuming (1), $A, B$ are harmonic conjugate points with respect to $C, D$, and $\Gamma$ is the Apollonius circle of the segment $CD$ and associated ratio $\frac{BC}{BD} = \frac{AC}{AD}$. Since $T$ lies on $\Gamma$, $\frac{TC}{TD}$ must also be equal to the given ratio, so $TB$ must be the bisector of $\angle CTD$, and $\angle BTD = \alpha$.
2. Since triangles $OCT$ and $OTD$ have a common angle with value $2\alpha$, requiring that $OC \cdot OD = OT^2$ (i.e. $\frac{OC}{OT} = \frac{OT}{OD}$) is equivalent to saying that the triangles are similar, which is equivalent to $\angle OTD = \angle OCT = 90°$.
3. Note that $CA \cdot CB = CT^2$, therefore the condition $CO \cdot CD = CT^2$ (i.e. $\frac{CO}{CT} = \frac{CT}{CD}$) is equivalent to the similarity of triangles $COT$ and $CTD$, which occurs if and only if $\angle OTD = 90°$.
4. The circumcircles of triangles $ATB$ and $TOC$ share a common tangent at $T$, therefore (4) is equivalent to saying that $D$

lies on the radical axis of both circles (the common tangent), which occurs if and only if $\angle OTD = 90°$.

5. Substituting $OC = OB - BC$ and $OD = OB + BD$ in $OC \cdot OD = OB^2$ yields $OB = \frac{BD \cdot BC}{BD - BC}$, which establishes the equivalence between (2) and (5).
6. Assuming $\angle BTD = \alpha$ and using the bisector's theorem, we have
$$\frac{OC}{OD} = \frac{OC}{TO} \cdot \frac{TO}{OD} = \cos(2\alpha)^2 = \left(\frac{TC}{TD}\right)^2 = \left(\frac{BC}{BD}\right)^2.$$
The proof that (6) implies any of the other conditions is straightforward. □

By the way, the above construction shows that given three collinear points $A$, $B$, $C$, with $C$ inside segment $AB$, there is a common fourth point $D$ which is at the same time the harmonic conjugate of $C$ with respect to $AB$ (condition (1)), and the inverse of $C$ with respect to the circle of diameter $AB$ (2). This identification of harmonic and inverse points is of great importance in its own, but now we have presented even more equivalent conditions. Remember Figure 1 used in the proof, as a lot of nice properties are hidden there.

**Note 1.** *The previous lemma admitts another short (but somehow ugly) proof. Consider the following distances on the line $AB$:*
$$A - (y+z) - O - (z) - C - (y) - B - (x) - D.$$
*Expressing in terms of $x$, $y$, $z$ one of the studied conditions, e.g.* (2), *leads to*
$$OC \cdot OD = OB^2 \implies z(z+y+x) = (z+y)^2 \implies y^2 + yz - xz = 0.$$
*Straightforward calculations show that* (1), (3), (4) *and* (5) *are also equivalent to the same condition on $x$, $y$, $z$. Condition* (6) *requires some simplification:*
$$\frac{OC}{OD} = \frac{BC^2}{BD^2} \iff \frac{z}{x+y+z} - \frac{y^2}{x^2} = 0$$
$$\iff \frac{(x+y)(xz - zy - y^2)}{(x+y+z)x^2} = 0.$$

# 2 Solved problems

**Problem 1 (Triangles with the same orthocenter).** *In the acute-angled triangle $ABC$ with $AB \neq AC$, let $BE$, $CF$ be altitudes, $M$ the midpoint of $BC$, and $G$ the point of intersection of lines $EF$ and $BC$. Prove that triangles $ABC$ and $AMG$ have the same orthocenter.*

*Solution.* Let $AD$ be the altitude from $A$, and $H$ the orthocenter of $ABC$. Note that by the similarity $DCA \sim DHB$ we have $\frac{DC}{DA} = \frac{DH}{DB}$, or, equivalently, $DC \cdot DB = DA \cdot DH$.

If we were able to prove that $DM \cdot DG = DA \cdot DH$, reversing the above argument would imply that $GH \perp AM$, which will solve the problem. Hence, we have reduced our problem to proving a metric condition among collinear points, namely $DM \cdot DG = DB \cdot DC$, which corresponds to condition (3) of the Five Point Lemma.

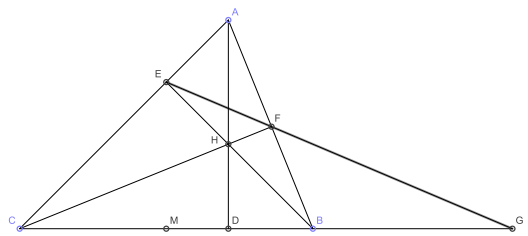


Figure 2: Scheme for the solution to Problem 1

But clearly, we are in the the typical configuration which produces a harmonic range or quadruple, i.e. $\frac{GB}{GC} = \frac{DB}{DC}$, and (1) holds. If suffices to apply the Five Point Lemma, and we are finished. $\square$

**Problem 2 (IMO Shortlist 1997-18).** *$ABC$ is an acute-angled triangle, non isosceles, with altitudes $AD, BE, CF$. The lines $EF$ and $BC$ meet at $P$. The line through $D$ paralell to $EF$ meets $AC$*

*at $Q$ and $AB$ at $R$. Prove that the circumcenter of $PQR$ passes through the midpoint of $BC$.*

*Solution.* We will assume $AC > AB$, the case $AC < AB$ is similar.

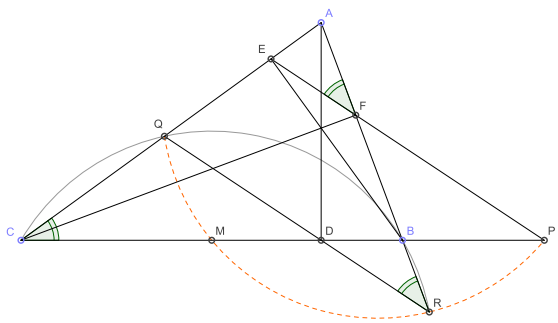


Figure 3: Scheme for the solution to Problem 2

Note that $BCEF$ is cyclic, inscribed in the circle of diameter $BC$, so that $\angle BCA$ is equal to $\angle AFE$, which in turn equals $\angle BRQ$, because $EF||QR$. Therefore, $BRCQ$ is cyclic, and so one has $DB \cdot DC = DQ \cdot DR$.

On the other hand, the problem will be solved if we prove that $DM \cdot DP = DQ \cdot DR$, i.e. if $DM \cdot DP = DB \cdot DC$, a metric condition on five collinear points. Now this is easy: $P$, $D$ are harmonic conjugates with respect to $BC$, hence we can apply (1)$\Rightarrow$(3) in the Five Point Lemma, and we are ready. $\square$

**Problem 3 (IMO Shortlist 2007).** *In the triangle $ABC$, the incircle $\omega$ touches the side $BC$ at $D$, and $M$ is the midpoint of the altitude $AH$. If $DM$ meets $\omega$ again at $N$, prove that $\omega$ and the circumcircle of $BCN$ have a common tangent at $N$.*

*Solution.* If the tangent to $\omega$ at $N$ meets $BC$ at $T$, one has $TN^2 = TD^2$. The problem will be solved once we prove that $TB \cdot TC = TD^2$, a metric condition on collinear points.

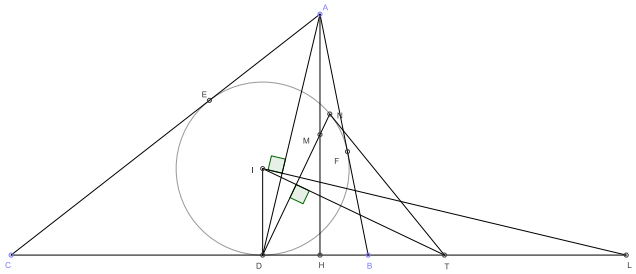


Figure 4: Scheme for the solution to Problem 3

Denote by $f$ the (direct) similarity which sends triangle $DIT$ to $HDM$, and note that any line is transformed under $f$ into an orthogonal line. Construct $L$ such that $T$ is the midpoint of $DL$, i.e. $f$ sends $IL$ to $DA$, so $IL \perp DA$. This means that $DA$ is the polar line of $L$ with respect to $\omega$, hence by reciprocity it follows that $L$ belongs to $EF$, the polar of $A$. Consequently, $L$ is the harmonic conjugate of $D$ with respect to $BC$. Applying (1)$\Rightarrow$(2) in Lemma 1 yields $TB \cdot TC = TD^2$, as we wanted to prove. $\square$

**Problem 4 (IMO Shortlist 2008-G4).** *Let $BE$ and $CF$ be altitudes of an acute-angled triangle $ABC$. Two circles passing through $A$ and $F$ are tangent to $BC$ at $P$ and $Q$, with $B$ lying between $C$ and $Q$. Prove that the lines $PE$ and $QF$ meet on the circumcircle of $AEF$.*

*Solution.* Let $AD$ be the altitude from $A$, $H$ the orthocenter of $ABC$ and $G$ the point of intersection of $PE$ and $QF$. Being $BQ$ tangent to the circumcircle of $AFQ$, one has $\angle BQF = \angle QAF$, which we denote by $\xi$.

In order to solve the problem, we should prove that $\angle QGP = \alpha$, or equivalently, $\angle EPC = \alpha + \xi$, i.e. $AEPQ$ must be a cyclic quadrilateral. One way to prove this is by showing that $CP \cdot CQ = CE \cdot CA$. But $AEDB$ is cyclic, so $CE \cdot CA = CB \cdot CD$, hence we have finally been able to reduce the original problem to a metric

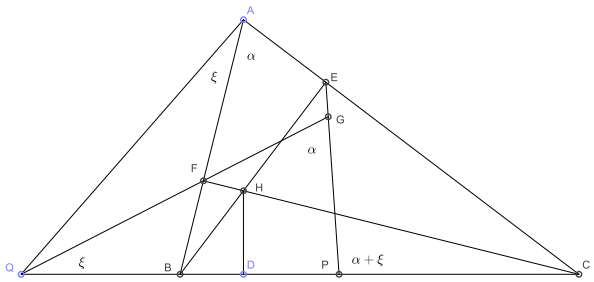


Figure 5: Scheme for the solution to Problem 4

condition on collinear points, namely $CP \cdot CQ = CB \cdot CD$.

Since $AFDC$ is a cyclic quadrilateral, $BD \cdot BC$ is equal to $BF \cdot BA$, which by the tangency conditions is equal to both $BP^2$ and $BQ^2$, i.e. $D$ and $C$ are inverse with respect to the circle of center $B$ and diameter $PQ$. Applying (2)⇒(4) in the Five Point Lemma yields $CD \cdot CB = CP \cdot CQ$, and we are done. □

**Note 2.** *As we have seen, many geometric problems of mathematical olympiads can be solved by inserting a small amount of metric calculations, without violating the spirit of a nice synthetic solution.*

# 3 Proposed exercises

**Exercise 1.** *In triangle $ABC$, the tangent at $A$ to the circumcircle meets line $BC$ at $D$. Prove that $\frac{DB}{DC} = \frac{AB^2}{AC^2}$. (Hint: consider $E$, the foot of the bisector of $A$, and show that $ADE$ is isosceles).*

**Exercise 2.** *$ABC$ is a triangle with $AB = AC$, $\Gamma$ is the circumcircle with center $O$, and $AD$ is a diameter. The tangents to $\Gamma$ at $B$ and $C$ meet at $E$. If $F$ is an arbitrary point of the line $BC$, outside segment $BC$, prove that triangles $FAD$ and $FOE$ have the same orthocenter.*

**Exercise 3.** *Let $AD$, $BE$ and $CF$ be the altitudes of the non isosceles triangle $ABC$, let $M$ be the midpoint of $BC$ and $G$ the intersection point of lines $BC$ and $EF$. Prove that $MF$ is tangent to the circumcircle of $DFG$.*

**Exercise 4.** *The incircle of triangle $ABC$ touches the sides $BC$, $CA$, $AB$ at $D$, $E$, $F$ respectively. Let $G$ be the midpoint of $DE$. The lines $DE$ and $CF$ meet at $H$. Prove that $AB$ is tangent to the circumcircle of $FGH$. (Hint: Construct $J = AB \cap DE$, prove that $F$, $J$ are harmonic conjugates with respect to $A$, $B$, so $H$, $J$ will be conjugates with respect to $E$, $D$, and the Five Point Lemma can be applied.)*

**Exercise 5.** *Triangle $ABC$ is acute-angled and non isosceles, $AD$ is an altitude, $H$ its orthocenter and $M$ the midpoint of $BC$. Construct $I$ with $HI \perp AM$ and $DI || AM$. Prove that $ID$ is the bisector of $\angle BIC$.*

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# *The chinese remainder theorem*

**Óscar Rivero Salgado**

Along the following pages, we start a promenade through some topics from number theory that frequently appear in olympiad problems and that go beyond the introductory notions of divisibility, prime numbers and congruences (Fermat and Wilson's theorem, Euclid's algorithm,...) and whose aim is to give the reader an idea of how powerful these results are with motivational examples. However, a word of caution must be said: most problems in international olympiads (as the IMO) do not require the use of such powerful techniques, just creativity and to have the right idea.

These notes about the Chinese remainder theorem constitute the first part of a series devoted to better understanding some of the cornerstones of basic arithmetic, trying to emphasize its usefulness through some examples taken from olympiad problems. Some problems are discussed with full solutions, in others we just give a useful idea to face them, and the remaining ones are there to encourage the reader to try to practice and enjoy solving them.

Most of the problems appearing here and in the following numbers were presented during the training sessions of the Spanish IMO team that participated in the 57th IMO at Hong-Kong and in the 31st Ibero-American Olympiad at Chile, where I take part as a deputy leader. It was there, in Antofagasta, where I had the idea of writing these notes; I would like to thank the whole Spanish team all the contributions they made to this and the ideas they gave me.

# 1 Statement of the theorem

We frequently have to deal with system of congruences, in which one would like to determine all the integers $x$ that verify, for instance, that they are congruent with $5$ modulo $6$ and with $3$ modulo $7$. Here, one check that $17, 17+42, 17+84, \ldots$ all work. But this cannot always be done. For instance, we cannot find an integer number congruent with $5$ modulo $6$ and with $2$ modulo $8$. The reason is that $6$ and $8$ are not coprime, so forcing a certain congruence modulo $6$ determines the congruence in all its divisors (for instance $2$) so we are saying twice the congruence modulo $2$ and this information can be (and is, in this case) contradictory. But this does not occur if the numbers are coprime.

**Theorem 1 (Chinese remainder).** *Let $n_1, \ldots, n_r$ be a collection of coprime integer numbers. Then, for any integer numbers $a_1, \ldots, a_r$ there is a unique integer $x$ between $0$ and $n_1 \cdot \ldots \cdot n_r - 1$ (or a unique residue class in $\mathbb{Z}/(n_1 \cdot \ldots \cdot n_r)\mathbb{Z}$) such that $x \equiv a_i \mod n_i$ for all $i$. Then, $y \in \mathbb{Z}$ satisfies this congruence if and only it is congruent with $x$ modulo $n_1 \cdot \ldots \cdot n_r$.*

*Proof.* It is enough to prove the theorem when $r = 2$ since then we can proceed by induction, taking $\tilde{n}_1 = n_1 \cdot \ldots \cdot n_{r-1}$ and $\tilde{n}_2 = n_r$, because if $n_r$ is coprime with $n_1, \ldots, n_{r-1}$ it is also coprime with their product.

Then, we can restrict to the following easier result: given $m$ and $n$ relatively prime natural numbers, and $a, b \in \mathbb{Z}$ we must prove that there is one and only one number $x$ between $0$ and $mn - 1$ such that $x \equiv a$ modulo $m$ and $x \equiv b$ modulo $n$. This is exactly the same as saying that $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}/mn\mathbb{Z}$ are in bijection, since if $x$ is a solution, $x + mn$ gives the same residue modulo $m$ and modulo $n$.

Let $\phi : \mathbb{Z}/mn\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ be the application that sends the residue class of $x$ modulo $mn$ to $(x \mod m, x \mod n)$. Suppose that $\phi(x) = \phi(y)$. Then, $\phi(x - y) = 0$, and hence $x - y$ is a multiple of both $m$ and $n$, and consequently, a multiple of $mn$. This implies that $x$ and $y$ are the same element modulo $mn$.

Then, $\phi$ is injective and since both sets have the same cardinality, $\phi$ must be a bijection. $\square$

**Remark 1.** *From the preceding, we may conclude that knowing $z \mod n_1 \cdot \ldots \cdot n_r$ is the same as knowing it modulo each of the $n_i$. In fact, any statement modulo a number $N$ can be reduced to a statement modulo each of its prime powers.*

This proof has the advantage of being very simple, but has the drawback that it is not constructive. However, it is relatively easy to observe that obtaining a solution can be done applying Bézout's identity. For instance, suppose that we want $x$ congruent with $3$ modulo $8$ and with $4$ modulo $5$.

Let us begin constructing a number congruent with $3$ modulo $8$ and with $0$ modulo $5$. For this, just take the Bézout identity

$$8x + 5y = 1,$$

that has $x = 2$ and $y = -3$ as a solution, and observe then that

$$8 \cdot 6 + 5 \cdot (-9) = 3,$$

or alternatively

$$5 \cdot (-9) = 3 - 8 \cdot 6.$$

The number $5 \cdot (-9)$ is clearly a multiple of $5$ and, by construction, it is congruent with $3$ modulo $8$. Then, $-45$ works, and so does $-45 + 40 \cdot 2 = 35$ too.

In the same way we get a number congruent with $0$ modulo $8$ and with $4$ modulo $5$, say $24$. Now, adding up the two numbers, we have $35 + 24 \equiv 19 \mod 40$ and by the properties of congruences, it verifies the two requirements.

The reader interested in some fancy application of this result can read [1]. A general proof of this statement that works for more general rings (for instance, polynomials) can be read in [2].

# 2 Worked examples

**Exercise 1.** *Find the number of solutions of $x^2 = x$ in $\mathbb{Z}/60\mathbb{Z}$.*

**Solution:** We require $x(x-1) \equiv 0 \mod 60$. From $x(x-1) \equiv 0 \mod 4$ and $\gcd(x, x-1) = 1$ we see that either $x$ or $x-1$ must be a multiple of $4$. For the same reason, either $x$ or $x-1$ is a multiple of $3$ and again either $x$ or $x-1$ is a multiple of $5$. We have therefore $8$ possibilities that are independent due to the Chinese remainder theorem. For instance, if $x \equiv 0 \mod 4$, $x \equiv 0 \mod 3, x \equiv 1 \mod 5$ we obtain that $36$ is one possible solution. The other solutions are $25$, $21$, $40$, $16$, $45$, $0$ and $1$ (eight in total).

**Problem 1 (Shortlist Iberoamerican 2009).** *Find the least positive integer $n$ such that for any set of integer numbers $\{a_1, \ldots, a_n\}$ there are two distinct elements $a_i$, $a_j$ such that $2009$ divides either $a_i - a_j$ or $a_i a_j - 1$.*

**Solution:** We are going to build up a set of maximum cardinality such that there is no $a_i$, $a_j$ with the required properties. Observe that we cannot have two elements giving the same residue modulo $2009$. Therefore, work directly over $\mathbb{Z}/2009\mathbb{Z}$. We can always include those elements $a_i$ that are not relatively prime with $2009$ since $a_i x \equiv 1$ will not have a solution. There are $2009 - \phi(2009) = 2009 - 1680 = 329$ such elements. For the remaining $1680$ we can put any number $x$ such that $x^2 \equiv 1$, without creating restrictions. This is equivalent to solving $(x-1)(x+1) \equiv 0$ modulo $7^2 \cdot 41$. It cannot occur that $x-1$ and $x+1$ are both multiple of $7$. Therefore, by the same reasoning as before, there are four possible solutions. The remaining $1676$ are grouped into pairs $(x, y)$ such that $xy \equiv 1 \mod 2009$. We can put an element of each pair. Therefore, we have

$$329 + 4 + 838 = 1171 \text{ possible elements.}$$

Consequently, if there are $1172$, at least two of them are either the same modulo $2009$ or there are elements $x$, $y$ (not necessarily distinct) such that $xy = 1$.

**Problem 2 (IMC2013).** *Let $p$, $q$ be coprime positive integers. Prove that*

$$\sum_{k=0}^{pq-1} (-1)^{\left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{q} \right\rfloor} = \begin{cases} 0 & \text{if } pq \text{ is even,} \\ 1 & \text{if } pq \text{ is odd.} \end{cases}$$

**Solution:** Suppose first that $pq$ is even ($p$ and $q$ opposite parities), and let $a_k = (-1)^{\left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{q} \right\rfloor}$. We will show that $a_k + a_{pq-1-k} = 0$ and hence the terms on the left hand side of the original expression cancel out in pairs. For every positive integer $k$, we have that

$$\left\{\frac{k}{p}\right\} + \left\{\frac{p-1-k}{p}\right\} = \frac{p-1}{p},$$

and therefore we have that

$$\begin{aligned}\left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{pq-1-k}{p} \right\rfloor &= \left(\frac{k}{p} - \left\{\frac{k}{p}\right\}\right) + \left(\frac{pq-1-k}{p} - \left\{\frac{-1-k}{p}\right\}\right) \\ &= \frac{pq-1}{p} - \frac{p-1}{p} = q-1,\end{aligned}$$

and in the same way

$$\left\lfloor \frac{pq-1-k}{q} \right\rfloor + \left\lfloor \frac{k}{q} \right\rfloor = p-1.$$

Since $p$ and $q$ have opposite parities, these two numbers also have opposite parities and hence $a_{pq-1-k} - a_k$.

Now suppose that $pq$ is odd. For every index $k$, let $p_k$ and $q_k$ be the remainders of $k$ modulo $p$ and $q$, respectively (taken in such a way that $0 \le p_k < p$ and $0 \le q_k < k$). Notice that

$$\left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{q} \right\rfloor \equiv p\left\lfloor \frac{k}{p} \right\rfloor + q\left\lfloor \frac{k}{q} \right\rfloor = (k-p_k) + (k-q_k) \equiv p_k + q_k \mod 2.$$

Since $p$ and $q$ are coprime, by the Chinese remainder theorem, the map $k \mapsto (p_k, q_k)$ is a bijection between the sets $\{0, 1, \ldots, pq-1\}$ and $\{0, 1, \ldots, p-1\} \times \{0, 1, \ldots, q-1\}$. Hence,

$$\begin{aligned}\sum_{k=0}^{pq-1} (-1)^{\left\lfloor \frac{k}{p} \right\rfloor + \left\lfloor \frac{k}{q} \right\rfloor} &= \sum_{k=0}^{pq-1} (-1)^{p_k + q_k} = \sum_{i=0}^{p-1} \sum_{j=0}^{q-1} (-1)^{i+j} \\ &= \left(\sum_{i=0}^{p-1} (-1)^i\right) \cdot \left(\sum_{j=0}^{q-1} (-1)^j\right) = 1.\end{aligned}$$

**Problem 3 (Iberoamerican 2012).** *Prove that, for any $n$, there are $n$ consecutive numbers such that any of them is divisible by the sum of its digits.*

**Solution:** Let $s(k)$ be the sum of the digits of $k$. Observe that $s(10^r - a) = 9r - s(a-1)$. We claim that, for a suitable $r$, we can force all the numbers $10^r - a$, where $2 \leq a \leq n+1$, to fulfill this property: for each $a$, there is a prime number $q_a$ such that $q_a | (9r - s(a-1))$ but $q_a \nmid (10^r - a)$. This is enough to solve the problem.

For this purpose, we are going to construct a sequence of $n$ different primes, all bigger than $\max\{5, n+1\}$ (this constrain is always imposed), and then we will construct the desired $r$, forcing that it must satisfy a certain number of congruences that will be compatible thanks to the Chinese remainder theorem. Take first any prime number $q_2$ and impose $q_2 | (9r - s(2-1))$ (or what is the same, $r \equiv 9^{-1}s(2-1) \mod q_2$) and further, $r \equiv 0 \mod q_2 - 1$. This means that $10^r - 2 \equiv 1 - 2 \mod q_2$, and $2 - 1$ cannot be a multiple of $q_2$.

Now, choose $q_3$ in such a way that $q_3$ and $q_3 - 1$ are coprime with both $q_2$ and with all the odd prime factors of $q_2 - 1$ (say that $q_3$ is 2 modulo $q_2$ and modulo the odd primes factors of $q_2 - 1$). Now, impose the conditions $r \equiv 9^{-1}s(3-1) \mod q_3$ and $r \equiv 0 \mod q_3 - 1$. There is no problem with that, since the only possible redundancy is that we put twice a condition over a power of 2, but this condition is always that $r$ is congruent with 0 modulo a power of 2 (same condition always).

In general, $q_i$ is chosen in such a way that $q_i$ and $q_i - 1$ are coprime with the previous $q_j$ and with all the odd prime factors of the previous $q_j - 1$, for instance by letting $q_i \equiv 2$ modulo all these primes. By the Chinese remainder, this is equivalent to the condition $q_i \equiv 2$ the product of all these odd primes and here, the key point here is that there is a very powerful theorem (beyond the scope of olympiad knowledge, but that we can assume without proof), **Dirichlet's theorem**, that states that there are infinitely many primes of the form $an + b$, when $\gcd(a, b) = 1$. Then, impose again $r \equiv 9^{-1}s(i-1) \mod q_i$ and $r \equiv 0 \mod q_i - 1$ and

again the only possible redundance is that we pose more than once that $r$ is 0 modulo a power of 2. This forces $10^r - i \equiv 1 - i \mod q_i$, that cannot be 0 modulo $q_i$ since $0 < i - 1 < n + 1$ and $q_i > n + 1$. Then, by the Chinese remainder theorem again, there is an $r$ verifying all the conditions.

Therefore, beyond the technical parts, the idea is to choose well-behaved prime numbers dividing the sum of the digits but not the number, and do this thanks to the power of Chinese remainder combined with Dirichlet's theorem.

**Problem 4.** *Prove that for every positive integer $n$, there exist integers $a$, $b$ such that $4a^2 + 9b^2 - 1$ is divisible by $n$.*

**Solution:** By the Chinese remainder theorem, it suffices to consider the case of $n$ being a prime power. For $2^k$ take $a \equiv 0 \mod 2^k$ and $b \equiv 3^{-1} \mod 2^k$. For $p^k$ with $p \neq 2$, take $a \equiv 2^{-1} \mod p^k$ and $b \equiv 0 \mod p^k$. Observe that this trick of considering $2^{-1}$ does not represent any problem: we just mean the inverse of 2 modulo a certain prime (not divisible by 2). To check $4a^2 \equiv 1 \mod p^k$ (since $a$ is coprime with $p$), multiply by $a^{-2}$ to each side and this is equivalent to $4 \equiv a^{-2} \equiv 4 \mod p^k$, that is clearly true.

**Problem 5 (USAMO2008).** *Prove that for each positive integer $n$, there are pairwise coprime integers $k_0, k_1, \ldots, k_n$ all strictly greater than 1 such that $k_0 k_1 \cdots k_n - 1$ is the product of two consecutive integers.*

**Solution:** We want to prove that $t(t+1) + 1$ can have as many different prime factors as we want (i.e., for any $n$ there exists a $t$ such that $P(t) = t^2 + t + 1$ has more than $n + 1$ prime divisors).

Observe that if we construct a $t$ such that $P(t) \equiv 0 \mod p_i$ for primes $p_0, \ldots, p_n$ we are done, since we can combine these constructions to have a $T$ such that $P(T) \equiv 0 \mod p_i$ for all $i$ (since it is enough with taking a $T$ such that $T \equiv t_i \mod p_i$).

Therefore, we just have to check that there are infinitely many primes dividing some number of the form $t^2 + t + 1$. If there were finitely may, say $q_1, \ldots, q_n$, we consider $Q = q_1 \cdots q_n$ and observe that $Q^2 + Q + 1$ cannot be divisible by any of the $q_i$.

# 3 Problems

**Problem 6.** *Let $\mathbb{N}$ be the set of positive integers. Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying the following conditions:*

1. $\gcd(f(m), f(n)) = 1$ *when* $\gcd(m, n) = 1$.
2. $n \leq f(n) \leq n + 2012$ *for all* $n$.

*Prove that for any natural number $n$ and any prime $p$, if $p$ divides $f(n)$ then $p$ divides $n$.*

*Hint: Construct large integers $N$ such that $f(N) = N$.*

**Problem 7 (IMO2009).** *Let $n$ be a positive integer and let $a_1, \ldots, a_k$ ($k \geq 2$) be distinct integers in the set $\{1, 2, \ldots, n\}$ such that $n$ divides $a_i(a_{i+1} - 1)$ for $i = 1, \ldots, k-1$. Prove that $n$ does not divide $a_k(a_1 - 1)$.*

**Problem 8 (IMO2016).** *A set of positive integers is called fragrant if it contains at least two elements and each of its elements has a prime factor in common with at least one of the other elements. Let $P(n) = n^2 + n + 1$. What is the least possible positive integer value of $b$ such that there exists a non-negative integer $a$ for which the set*

$$\{P(a+1), P(a+2), \ldots, P(a+b)\}$$

*is fragrant?*

**Problem 9.** *Let $p$ be an odd prime number. Determine all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for all $m$, $n \in \mathbb{Z}$, the following conditions are satisfied:*

1. *If* $m \equiv n \mod p$ *then* $f(m) = f(n)$.
2. $f(mn) = f(m)f(n)$.

# References

[1] Chen, E. "The Chinese Remainder Theorem" (2015). URL: www.mit.edu/~evanchen/olympiad.html.

[2] Lang, S. *Algebra*. third. Vol. 211. Graduate Texts in Mathematics. Springer-Verlag, New York, 2002, pp. xvi+914. ISBN: 0-387-95385-X. DOI: 10.1007/978-1-4613-0041-0.

[3] *The art of problem solving forum (AOPS)*. URL: http://www.artofproblemsolving.com/community.

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# Solutions

No problem is ever permanently closed. We will be very pleased to consider for publication new solutions or comments on the past problems.

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## Elementary Problems

***E–29.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $a$, $b$ be nonnegative real numbers. Prove that

$$a^4 + b^4 + a^2b^2 \geq ab(a^2 + b^2).$$

**Solution 1 by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom.** First, assume that $a \geq b$. In such a case, we have

$$a^4 + b^4 + a^2b^2 = a^2(a^2 + b^2) + b^4 \geq a^2(a^2 + b^2) \geq ab(a^2 + b^2).$$

Otherwise, we have $a < b$, in which case

$$a^4 + b^4 + a^2b^2 = b^2(a^2 + b^2) + a^4 \geq b^2(a^2 + b^2) > ab(a^2 + b^2),$$

thus completing the proof.

**Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.** The conclusion follows by the AM-GM inequality:

$$\begin{aligned} a^4+b^4+a^2b^2 \geq & \frac{a^4+a^2b^2}{2}+\frac{b^4+a^2b^2}{2} \\ \geq & \sqrt{a^6b^2}+\sqrt{a^2b^6} \\ = & ab(a^2+b^2). \end{aligned}$$

**Solution 3 by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain.** Moving everything to the left hand side and multiplying it by $2$, we obtain

$$\begin{aligned} & 2a^4+2b^4+2a^2b^2-2ab(a^2+b^2) \\ = & \left[(a^2+b^2)-ab\right]^2-a^2b^2+a^4+b^4 \\ = & \left[(a^2+b^2)-ab\right]^2+(a^2-b^2)^2+a^2b^2 \geq 0, \end{aligned}$$

as we have a sum of squares.

**Solution 4 by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain.** Let us suppose, without loss of generality (as the inequality is symmetric) that $a \geq b$. Then, by the rearrangement inequality,

$$a^4+b^4 \geq a^3b+ab^3 = ab(a^2+b^2).$$

So, adding a square (that is, a non-negative number) to the left hand side, the inequality remains the same:

$$a^4+b^4 \geq ab(a^2+b^2) \implies a^4+b^4+a^2b^2 \geq ab(a^2+b^2).$$

Note that $a^4+b^4 \geq ab(a^2+b^2)$ would be a more accurate upper bound than that of the statement.

**Solution 5 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.** Let $a$, $b$ be *any* real numbers. We shall use the following inequalites:

$$a^4+b^4 \geq 0, \tag{1}$$

$$a^4+a^2b^2 \geq 2a^3b, \tag{2}$$

and

$$b^4 + a^2b^2 \geq 2ab^3. \tag{3}$$

The validity of (1) is obvious. As for (2) and (3), we have (2)$\Leftrightarrow$ $(a^2 - ab)^2 \geq 0$ and (3)$\Leftrightarrow$ $(b^2 - ab)^2 \geq 0$.

Now adding (1), (2), and (3) and dividing by $2$ yields the given inequality.

**Solution 6 by the proposer.** If $a$, $b$, or both are zero then the statement trivially holds. Suppose that $a$, $b > 0$. Then, $a + b$ and $a^5 + b^5$ are positive and the same occurs with their quotient. That is,

$$a^4 - a^3b + a^2b^2 - ab^3 + b^4 = \frac{a^5 + b^5}{a + b} > 0,$$

and the statement follows.

**Also solved by** *Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain (one more solution); Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain.*

***E–30.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $n \geq 1$ be a positive integer. Find a formula to compute the sum of the $n$ first terms of

$$4 + 10 + 18 + 28 + 40 + 54 + \ldots$$

**Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.** Our task is to compute the sum $\sum_{k=1}^{n} a_k$, where $a_1 = 4$, $a_2 = 10$, $a_3 = 18$, $a_4 = 28$... We accomplish this by first determining a formula for the general terme $a_k$. We are told that

$$\begin{aligned} a_2 - a_1 &= 2 \times 3, \\ a_3 - a_2 &= 2 \times 4, \\ a_4 - a_3 &= 2 \times 5, \\ &\vdots \\ a_k - a_{k-1} &= 2(k+1). \end{aligned}$$

Adding these equalities, we observe that the sum of left sides telescopes to $a_k - a_1$, so we obtain

$$\begin{aligned} a_k &= a_1 + 2(3 + 4 + \cdots + (k+1)) \\ &= 4 + 2(3 + 4 + \cdots + (k+1)). \end{aligned} \tag{4}$$

The sum in the parenthesis is an arithmetic progression with value

$$\frac{1}{2}(3 + (k+1))(k-1) = \frac{1}{2}\big(k^2 + 3k - 4\big).$$

Substituting this in (4), we get

$$a_k = k^2 + 3k.$$

Now we use the summation notation and write

$$\begin{aligned} \sum_{k=1}^{n} a_k &= \sum_{k=1}^{n} \big(k^2 + 3k\big) \\ &= \sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k \\ &= \sum_{k=1}^{n} k^2 + 3 \cdot \frac{1}{2} n(n+1), \end{aligned} \tag{5}$$

since $\sum_{k=1}^{n} k = \frac{1}{2} n(n+1)$ holds by the formula for the sum of any number of consecutive integers.

To arrive at a simple formula for $\sum_{k=1}^{n} k^2$, we start with the identity

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1.$$

Using again the summation notation, we write

$$\sum_{k=1}^{n} \big((k+1)^3 - k^3\big) = \sum_{k=1}^{n} \big(3k^2 + 3k + 1\big).$$

The left side telescopes in such a way that the only remaining quantity is $-1^3+(n+1)^3$. Everything else cancels, and we obtain

$$-1+(n+1)^3 = 3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1.$$

As for the right side, we know that $\sum_{k=1}^{n} 1 = n$ and $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$. So we see that

$$n^3+3n^2+1 = 3\sum_{k=1}^{n} k^2 + 3\cdot\frac{1}{2}n(n+1)+n.$$

Dividing by three, rearranging the terms, combining and factoring yield the result

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1).$$

Substituting this for $\sum_{k=1}^{n} k^2$ into (5), we obtain the desired sum,

$$\sum_{k=1}^{n} a_k = \frac{1}{3}n(n+1)(n+5).$$

**Also solved by** *Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain; Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom; Guillermo Girona San Miguel, Barcelona, Spain, and Isaac Sánchez Barrera, Barcelona Supercomputing Center, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain, and the proposer.*

***E–31.*** *Proposed by José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.* Given a set of nine points with integer coordinates in three-dimensional space, prove that there are two whose midpoint also has integer coordinates.

**Solution by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain.** Let $P(a, b, c)$ be the general form of these nine points, where $a$, $b$, $c$ are integers. If the midpoint of two of them is an integer as well, that means that the $x$-coordinates have the same parity, the $y$-coordinates have the same parity and the $z$-coordinates have the same parity.

The possibilities for the parity of the coordinates of point $P$ are $2 \cdot 2 \cdot 2 = 8$ since each coordinate can be either even or odd. We reason using the Pigeonhole Principle. As we have nine points, there has to be a possibility of parity from those eight that is repeated. That implies that two points will have the same parity for their coordinates and, therefore, the midpoint is going to have integer coordinates.

**Also solved by** *Padraig Condon, University of Birmingham, Birmingham, United Kingdom; Matthew Coulson, University of Birmingham, Birmingham, United Kingdom; Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom; Guillermo Girona San Miguel, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain, and the proposer.*

***E–32.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* The lengths of the sides of a triangle satisfy that the triple of one of them is equal to the sum of the other two. Prove that its inradius is one fourth of one of its altitudes.

**Solution by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom.** Say that the lengths of the sides of the triangle are called $a$, $b$ and $c$. As we know that $3a = b + c$, we easily derive $a + b + c = 4a$. Let $r$ be the inradius of said triangle; it is a well-known result that $r\dfrac{a+b+c}{2} = \mathcal{A}$, where $\mathcal{A}$ is the area of the triangle. Finally, denote the altitude perpendicular to the side of length $a$ by $h_a$. Then, we can write

$$2ra = r\frac{a+b+c}{2} = \mathcal{A} = \frac{a\,h_a}{2} \implies r = \frac{h_a}{4}.$$

**Also solved by** *Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and the proposer.*

***E–33.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* If $x^2 - bx + a = 0$ has two integer roots, then prove that

$$\frac{a(a+b+1)(4a+2b+1)}{36}$$

can be written as the sum of $a$ squares of integer numbers.

**Solution by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.** Let $x_1$, $x_2$ be the roots of the quadratic equation $x^2 - bx + a = 0$. Since $x_1 + x_2 = b$ and $x_1 x_2 = a$, we have

$$a + b + 1 = x_1 x_2 + x_1 + x_2 + 1 = (x_1 + 1)(x_2 + 1)$$

and

$$4a + 2b + 1 = 4x_1 x_2 + 2x_1 + 2x_2 + 1 = (2x_1 + 1)(2x_2 + 1).$$

Hence,

$$\begin{aligned}
&\frac{a(a+b+1)(4a+2b+1)}{36} \\
=&\frac{x_1 x_2 (x_1+1)(x_2+1)(2x_1+1)(2x_2+1)}{36} \\
=&\left(\frac{1}{6}x_1(x_1+1)(2x_1+1)\right)\left(\frac{1}{6}x_2(x_2+1)(2x_2+1)\right).
\end{aligned}$$

Now for positive integers $x_1$, $x_2$ we know that

$$\sum_{i=1}^{x_1} i^2 = \frac{1}{6}x_1(x_1+1)(2x_1+1)$$

and

$$\sum_{j=1}^{x_2} j^2 = \frac{1}{6}x_2(x_2+1)(2x_2+1)$$

(see solution to Problem E–30). So we see that

$$\begin{aligned}\frac{a(a+b+1)(4a+2b+1)}{36} &= \left(\sum_{i=1}^{x_1} i^2\right)\left(\sum_{j=1}^{x_2} j^2\right)\\ &= \sum_{i=1}^{x_1}\sum_{j=1}^{x_2} i^2 j^2\\ &= \sum_{i=1}^{x_i}\sum_{j=1}^{x_2} (ij)^2,\end{aligned}$$

that is, a sum of $x_1x_2 = a$ squares of integers numbers, as desired.

**Also solved by** *Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain, and the proposer.*

***E–34.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Show that there are infinitely many quadruplets of Fibonacci numbers satisfying the equation $2(x^2 + y^2) = z^2 + t^2$.

**Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.** For all $n \geq 2$, we rewrite $F_n = F_{n-1} + F_{n-2}$ in the form $F_{n-2} = F_n - F_{n-1}$, square it and obtain

$$F_{n-2}^2 + 2F_{n-1}F_n = F_{n-1}^2 + F_n^2.$$

Adding $F_{n-1}^2 + F_n^2$ to both sides yields

$$F_{n-2}^2 + (F_{n-1} + F_n)^2 = 2\big(F_{n-1}^2 + F_n^2\big).$$

Substituting $F_{n+1}$ for $F_{n-1} + F_n$, we obtain the Fibonacci identity

$$F_{n-2}^2 + F_{n+1}^2 = 2\big(F_{n-1}^2 + F_n^2\big).$$

Hence,

$$(x, y, z, t) = (F_{n-1}, F_n, F_{n-2}, F_{n+1}) \qquad n = 2, 3, 4, \ldots$$

satisfy the given equation.

**Solution 2 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain.** For a given $n \in \mathbb{N}$, letting $x = F_0 = 0$, $y = z = t = F_n$, $x$, $y$, $z$, $t$ satisfy the equation $2(x^2 + y^2) = z^2 + t^2$, finding this way infinitely many solutions to the equation.

**Also solved by** *Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain, and the proposer.*

# *Easy–Medium Problems*

***EM–29.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $a$, $b$, $c$, $m$ be positive real numbers such that

$$\sqrt{m+a}+\sqrt{m+b}=2\sqrt{m+c}.$$

Prove that $a+b\geq 2c$.

**Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.** We will prove the desired inequality using the inequality

$$2\left(x^2+y^2\right)\geq (x+y)^2, \tag{6}$$

where $x$, $y$ are real numbers and which is equivalent to the obvious $(x-y)^2\geq 0$.

Applying (6) with $x=\sqrt{m+a}$, $y=\sqrt{m+b}$ (note that $\sqrt{m+a}$ and $\sqrt{m+b}$ are real, since $m$, $a$ and $b$ are positive), we obtain

$$\begin{aligned}2((m+a)+(m+b)) &\geq \left(\sqrt{m+a}+\sqrt{m+b}\right)^2\\ &\geq \left(2\sqrt{m+c}\right)^2\\ &= 4(m+c).\end{aligned}$$

Dividing by 2 yields

$$(m+a)+(m+b)\geq 2(m+c),$$

which is equivalent to the proposed inequality.

**Solution 2 by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom.** Consider the function $f:[-m,+\infty)\to\mathbb{R}$ defined by $f(x)=\sqrt{m+x}$. We can compute the first two derivatives,

$$f'(x)=\frac{1}{2\sqrt{m+x}}>0,$$

$$f''(x)=-\frac{1}{4}(m+x)^{-\frac{3}{2}}<0,$$

so $f$ is strictly increasing and concave. By concavity, we can apply Jensen's inequality to write

$$f\left(\frac{a+b}{2}\right) \geq \frac{f(a)+f(b)}{2} = f(c).$$

As $f$ is increasing, this means that

$$\frac{a+b}{2} \geq c,$$

and the statement follows.

**Solution 3 by Ander Lamasion Vidarte, Berlin Mathematical School, Berlin, Germany.** Let $f(x) = x^2 - m$. This function is convex because $f''(x) = 2 > 0$. Therefore we can apply Jensen's inequality:

$$\begin{aligned}\frac{f(\sqrt{m+a})+f(\sqrt{m+b})}{2} &\geq f\left(\frac{\sqrt{m+a}+\sqrt{m+b}}{2}\right)\\ &= f(\sqrt{m+c}).\end{aligned}$$

Substitution of $f(\sqrt{m+x}) = x$ gives $\frac{a+b}{2} \geq c$, or $a+b \geq 2c$.

**Solution 4 by the proposer.** Squaring both sides of $\sqrt{m+a} + \sqrt{m+b} = 2\sqrt{m+c}$ yields

$$2m + a + b + 2\sqrt{(m+a)(m+b)} = 4\,(m+c)$$

or, equivalently, $a + b - 2c = 2m + 2c - 2\sqrt{(m+a)(m+b)}$. To complete the proof it will suffice to see that

$$m + c - \sqrt{(m+a)(m+b)} \geq 0.$$

Indeed, putting $x = \sqrt{m+a}$ and $y = \sqrt{m+b}$ into $\sqrt{m+a} + \sqrt{m+b} = 2\sqrt{m+c}$, we have $x + y = 2\sqrt{m+c}$. Squaring yields $(x+y)^2 = 4(m+c)$, from which $m + c = \dfrac{(x+y)^2}{4}$ follows. Using the preceding we get

$$m + c - \sqrt{(m+a)(m+b)} = \frac{(x+y)^2}{4} - xy = \frac{(x-y)^2}{4} \geq 0,$$

and the statement is proved.

**Also solved by** *Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain; Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain, and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.*

***EM–30.*** *Proposed by José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.* Find all positive solutions of the following system of equations

$$\begin{cases} (x+y+z)^3 = t, \\ (y+z+t)^3 = x, \\ (z+t+x)^3 = y, \\ (t+x+y)^3 = z. \end{cases}$$

**Solution 1 by Alberto Espuny Díaz, University of Birmingham, Birmingham, United Kingdom.** We first look for solutions such that $x = y = z = t$. In such a case we have one single equation, $27x^3 = x$, that is, $x(27x^2 - 1) = 0$. From this, either $x = 0$ or $27x^2 - 1 = 0$, whence the solutions are $x = \pm\dfrac{1}{\sqrt{27}}$. As we are only interested in positive solutions, $\left(\dfrac{1}{\sqrt{27}}, \dfrac{1}{\sqrt{27}}, \dfrac{1}{\sqrt{27}}, \dfrac{1}{\sqrt{27}}\right)$ is a solution.

We now claim that this is the only solution. Indeed, assume otherwise. Since the equations are written in a cyclic way, we may assume without loss of generality that $x \geq y \geq z \geq t$ and $x > t$. By substituting the equations, the previous means that

$$(y+z+t)^3 \geq (z+t+x)^3 \geq (t+x+y)^3 \geq (x+y+z)^3,$$

and as we only consider positive solutions, $y+z+t \geq z+t+x \geq t+x+y \geq x+y+z$. This means that $t \geq z \geq y \geq x$, which contradicts the fact that $x > t$, thus completing the proof.

**Solution 2 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.** Let $s = x+y+z+t$. The first equation can be rewritten as $(s-t)^3 = t$, so $t$ is a positive solution to $(s-\alpha)^3 = \alpha$. Similarly, the same equation holds for $x$, $y$ and $z$. But $(s-\alpha)^3$ is decreasing on $\alpha$ while $\alpha$ is increasing, meaning

that there is at most one solution for $(s-\alpha)^3 = \alpha$. Thus $x = y = z = t$.

The original equation now becomes $(3x)^3 = x$, which has solutions $x = 0$ and $x = \pm\sqrt{\frac{1}{27}}$. Since we are only concerned with positive values, the only solution is $\left(\sqrt{\frac{1}{27}}, \sqrt{\frac{1}{27}}, \sqrt{\frac{1}{27}}, \sqrt{\frac{1}{27}}\right)$.

**Solution 3 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.** More generally, we seek real solutions of the given system, which is symmetric in the four numbers $x$, $y$, $z$, $t$. Denote the sum of all four numbers $x$, $y$, $z$, $t$ by $\sigma$,

$$\sigma = x + y + z + t,$$

and write the equations as

$$(\sigma - t)^3 = t, \tag{7}$$

$$(\sigma - x)^3 = x, \tag{8}$$

$$(\sigma - y)^3 = y, \tag{9}$$

$$(\sigma - z)^3 = z. \tag{10}$$

Probably the easiest way to solve them is to substract each equation from the one following it. Thus the second minus the first is

$$(\sigma - x)^3 - (\sigma - t)^3 = x - t. \tag{11}$$

Now we recall the identity $A^3 - B^3 = (A-B)(A^2 + AB + B^2)$ and apply it with $A = \sigma - x$, $B = \sigma - t$, obtainig

$$\begin{aligned}&(\sigma - x)^3 - (\sigma - t)^3\\ =&(-x+t)\big((\sigma - x)^2 + (\sigma - x)(\sigma - t) + (\sigma - t)^2\big).\end{aligned}$$

When this is substituted into (11), we get

$$(x - t)\big(1 + (\sigma - x)^2 + (\sigma - x)(\sigma - t) + (\sigma - t)^2\big) = 0,$$

and dividing by the positive factor $1 + (\sigma - x)^2 + (\sigma - x)(\sigma - t) + (\sigma - t)^2$, we find

$$x - t = 0.$$

Similarly, (9) minus (8) yields

$$y - x = 0$$

and (10) minus (9) yields

$$z - y = 0.$$

Substituting these results into any of the four original equations we obtain the solutions

$$\begin{aligned}(x, y, z, t) &= \left(\frac{-1}{3\sqrt{3}}, \frac{-1}{3\sqrt{3}}, \frac{-1}{3\sqrt{3}}, \frac{-1}{3\sqrt{3}}\right),\\ (x, y, z, t) &= (0, 0, 0, 0),\\ (x, y, z, t) &= \left(\frac{1}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}, \frac{1}{3\sqrt{3}}\right).\end{aligned}$$

**Also solved by** *the proposer.*

***EM–31.*** *Proposed by Nicolae Papacu, Slobozia, Romania.* If $p$ and $q$ are prime numbers, show that the number $p^{2q} + q^{2p}$ is composite.

**Solution 1 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain.** Consider the case $p,\ q \neq 2$. Since $p,\ q$ will be odd numbers, their powers will be odd, and so $p^{2q} + q^{2p}$ will be an odd number plus an odd number, which is even (and clearly, bigger than 2), so it is composite.

Suppose now $p = q = 2$. Similarly, we will have $p^{2q} + q^{2p}$ will be an even number plus an even number, which is, again, even and bigger than 2, so it is a composite number.

We are left with the case that one of them is 2. Assume WLOG $p = 2,\ q \geq 3$. We will have $p^{2q} + q^{2p} = 4^q + q^4$. If $q \neq 5$, let us calculate the remainder modulo 5, where we will use that if $q$ is odd $(-1)^q = -1$ and Fermat's Little Theorem (if $p$ is a prime, and $p \nmid a$, then $a^{p-1} \equiv 1 \pmod{p}$): $4^q + q^4 \equiv (-1)^q + q^{5-1} \equiv -1 + 1 \equiv 0 \pmod{5}$. So it is divisible by 5, and thus, composite.

Now, if $p = 2,\ q = 5,\ p^{2q} + q^{2p} = 2^{10} + 5^4 = 1024 + 625 = 1649 = 17 \times 97$, which is composite, completing the proof.

**Solution 2 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.** We note that $p$ and $q$ cannot be both odd or both even, as otherwise $N = p^{2q} + q^{2p}$ would be even and greater than 2, so it would be composite. Hence $p = 2$ or $q = 2$. Wlog assume the former, in which case $q$ is odd. We will now show that $N = 4^q + q^4$ is always composite.

Let $a = 2^{\frac{q-1}{2}}$, which is an integer greater than 1. Then $N = 4a^4 + q^4$. Sophie Germain's identity then shows $N = (2a^2 + 2aq + q^2)(2a^2 - 2aq + q^2)$. The first factor is clearly greater than 1, and the other is $2a^2 - 2aq + q^2 = a^2 + (a-q)^2 > 1$. We conclude that $N$ is composite.

**Also solved by** *Miguel Amengual Covas, Cala Figuera, Mallorca, Spain; Ángel Plaza, University of Las Palmas de Gran Canaria, Spain, and the proposer.*

***EM–32.*** *Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Las Palmas de Gran Canaria, Spain.* Let $n$ be a positive integer, $m$ be a real number greater than 1, and $x_k \in (0,1)$ $(1 \le k \le n)$. Prove that

$$\left(\sum_{k=1}^{n} \frac{x_k}{1-x_k^m}\right)\left(n^m - \left(\sum_{k=1}^{n} x_k\right)^m\right) \ge n^m \sum_{k=1}^{n} x_k.$$

**Solution by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.** The sum of all $x_k$ is positive and strictly smaller than $n$, so the proposed inequality is equivalent to

$$\sum_{k=1}^{n} \frac{x_k}{1-x_k^m} \ge \frac{n^m \sum\limits_{k=1}^{n} x_k}{n^m - \left(\sum\limits_{k=1}^{n} x_k\right)^m} = \frac{\sum\limits_{k=1}^{n} x_k}{1 - \left(\frac{1}{n}\sum\limits_{k=1}^{n} x_k\right)^m}.$$

Dividing both sides by $n$, this is equivalent to

$$\frac{1}{n}\sum_{k=1}^{n} \frac{x_k}{1-x_k^m} \ge \frac{\frac{1}{n}\sum\limits_{k=1}^{n} x_k}{1 - \left(\frac{1}{n}\sum\limits_{k=1}^{n} x_k\right)^m},$$

or $\frac{1}{n}\sum_{k=1}^{n} f(x_k) \geq f\left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)$ with $f(x) = \frac{x}{1-x^m}$. If we prove that $f$ is convex in $(0,1)$, then this will follow by Jensen's inequality. But this is true since $f'(x) = \frac{1}{1-x^m} + \frac{mx^m}{(1-x^m)^2}$, which is clearly increasing.

**Also solved by** *the proposer.*

***EM–33.*** *Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $d_a$, $d_b$, and $d_c$ be the distances from the vertices of triangle $ABC$ to its incenter. Show that

$$\frac{d_a^2}{bc} + \frac{d_b^2}{ca} + \frac{d_c^2}{ab}$$

is an integer and determine its value.

**Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.** Let $a$, $b$, $c$ be the sides of $\triangle ABC$ in the usual order. Let $I$ the incentre, $r$ the inradius, and $s$ the semiperimeter of the triangle.

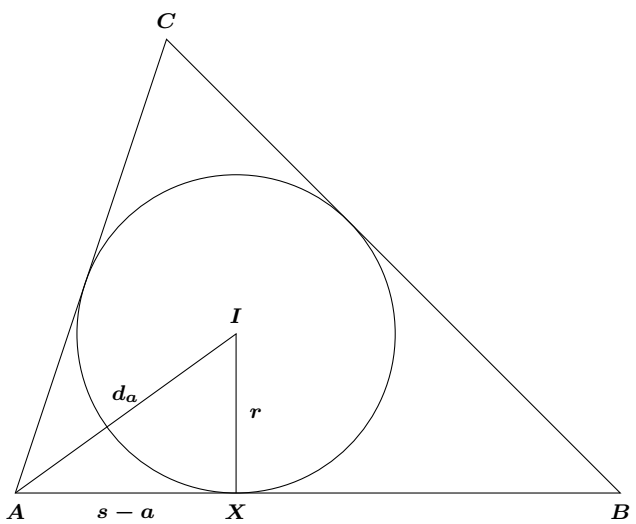


Figure 1: Construction for Solution 1 to Problem EM–33

Figure 1 shows the incircle of $\triangle ABC$ touching the side $BC$ at $X$. In right-triangle $AXI$, the Pythagorean theorem gives

$$d_a^2 = AX^2 + IX^2,$$

and since $AX = s - a$ and $IX = r$, we have

$$d_a^2 = (s-a)^2 + r^2. \tag{12}$$

The area of $\triangle ABC$ may be expressed as $\sqrt{s(s-a)(s-b)(s-c)}$, and also as $rs$. Equating these, squaring and solving for $r^2$, we get

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s}.$$

When this is substituted into (12), we get

$$\begin{aligned} d_a^2 &= (s-a)\cdot\frac{s(s-a)+(s-b)(s-c)}{s} \\ &= (s-a)\cdot\frac{2s^2 - \overbrace{(a+b+c)}^{=2s}s + bc}{s} \\ &= \frac{bc(s-a)}{s}. \end{aligned}$$

Thus

$$\frac{d_a^2}{bc} = \frac{s-a}{s},$$

and the same holds cyclically. We conclude that

$$\frac{d_a^2}{bc} + \frac{d_b^2}{ca} + \frac{d_c^2}{ab} = \frac{3s-(a+b+c)}{s} = \frac{3s-2s}{s} = 1.$$

**Solution 2 by Fernando Ballesta Yagüe, IES Infante Don Juan Manuel, Murcia, Spain.** Consider Figure 2 for the names of the segments. First, let us recall a formula. For any triangle, we have that

$$\sin\frac{\hat{A}}{2} = \sqrt{\frac{(s-c)(s-b)}{bc}}.$$

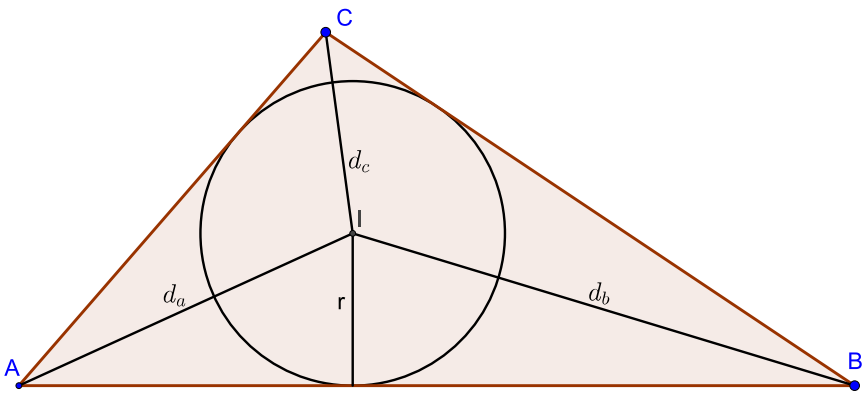


Figure 2: Scheme for Solution 2 to Problem EM–33

Hence, in this problem we have

$$\sin\frac{\hat{A}}{2} = \frac{r}{d_a} = \sqrt{\frac{(s-c)(s-b)}{bc}}$$
$$\Longrightarrow d_a^2 = \frac{r^2}{\frac{(s-c)(s-b)}{bc}} = \frac{r^2bc}{(s-c)(s-b)}$$
$$\Longrightarrow \frac{d_a^2}{bc} = \frac{r^2}{(s-c)(s-b)}.$$

We have a similar expression for $d_b^2$ and $d_c^2$. Substituting in the

given expression, we obtain

$$\begin{aligned}
&\frac{d_a^2}{bc}+\frac{d_b^2}{ac}+\frac{d_c^2}{ab}\\
&=r^2\left[\frac{1}{(s-c)(s-b)}+\frac{1}{(s-a)(s-c)}+\frac{1}{(s-a)(s-b)}\right]\\
&=r^2\frac{(s-a)+(s-b)+(s-c)}{(s-a)(s-b)(s-c)}=r^2\frac{3s-2s}{(s-a)(s-b)(s-c)}\\
&=\frac{r^2s}{(s-a)(s-b)(s-c)}.
\end{aligned}$$

As we can compute the area of a triangle by

$$\mathcal{A}=rs=\sqrt{s(s-a)(s-b)(s-c)},$$

where $r$ is the inradius and $s$ is the semiperimeter, we can substitute above to get

$$\frac{d_a^2}{bc}+\frac{d_b^2}{ac}+\frac{d_c^2}{ab}=\frac{r^2s}{(s-a)(s-b)(s-c)}=\frac{\frac{\mathcal{A}^2}{s}}{\frac{\mathcal{A}^2}{s}}=1.$$

In this way, we have proved that the given expression in an integer equal to $1$.

**Solution 3 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.** Consider the following decomposition of triangle $ABC$:

Let $S$ be the area of triangle $ABC$. Then $S=bc\sin 2\alpha$. If we arrange the two blue triangles with their right angles together, as in the right part of the figure, we form a triangle of area $d_a^2\sin 2\alpha$. If $S_a$, $S_b$ and $S_c$ denote the blue, red and green areas, $\frac{d_a^2}{bc}=\frac{S_a}{S}$. From this we can find

$$\frac{d_a^2}{bc}+\frac{d_b^2}{ca}+\frac{d_c^2}{ab}=\frac{S_a}{S}+\frac{S_b}{S}+\frac{S_c}{S}=\frac{S}{S}=1.$$

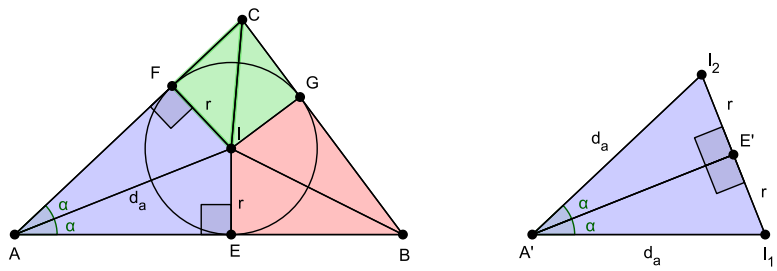


Figure 3: Scheme for Solution 3 to Problem EM–33

**Solution 4 by the proposers.** Let $A$, $B$, $C$ be the angles of $\triangle ABC$ opposite to the sides $a$, $b$, $c$, and let $r$ be the length of its inradius, as usual. Then,

$$\begin{aligned} ad_a^2 + bd_b^2 + cd_c^2 &= r^2\left(\frac{a}{(\sin A/2)^2} + \frac{b}{(\sin B/2)^2} + \frac{c}{(\sin C/2)^2}\right) \\ &= 2r^2\left(\frac{a}{\sin A}\cot\frac{A}{2} + \frac{b}{\sin B}\cot\frac{A}{2} + \frac{c}{\sin C}\cot\frac{C}{2}\right) \\ &= \frac{2ar^2}{\sin A}\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right) \end{aligned}$$

because by the Law of Sines $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Since $r\cot\frac{A}{2} = s - a$, $r\cot\frac{B}{2} = s - b$ and $r\cot\frac{C}{2} = s - c$, as is well-known, then after addition we get

$$s = \frac{a+b+c}{2} = r\left(\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}\right).$$

Then,

$$ad^2a + bd_b^2 + cd_c^2 = \frac{2ars}{\sin A} = \frac{2a}{\sin A}[ABC] = \frac{2a}{\sin A}\left(\frac{bc}{2}\sin A\right) = abc,$$

from which

$$\frac{d_a^2}{bc} + \frac{d_b^2}{ca} + \frac{d_c^2}{ab} = 1$$

follows.

***EM–34.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Compute

$$\sum_{n=1}^{\infty} \arctan\left(\frac{L_{n+1}^2}{1 + L_n L_{n+1}^2 L_{n+2}}\right),$$

where $L_n$ is the $n^{\text{th}}$ Lucas number, defined by $L_0 = 2$, $L_1 = 1$, and for all $n \geq 2$, $L_n = L_{n-1} + L_{n-2}$.

**Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.** We will show that the proposed sum is equal to $\arctan \frac{1}{3}$.

Note that if $\alpha_n = \arctan\left(\frac{L_{n+1}^2}{1+L_n L_{n+1}^2 L_{n+2}}\right)$, then

$$\begin{aligned}
\alpha_n &= \arctan\left(\frac{\frac{1}{L_n L_{n+2}}}{1 + \frac{1}{L_n L_{n+1}} \cdot \frac{1}{L_{n+1} L_{n+2}}}\right) \\
&= \arctan\left(\frac{\frac{1}{L_n L_{n+1}} - \frac{1}{L_{n+1} L_{n+2}}}{1 + \frac{1}{L_n L_{n+1}} \cdot \frac{1}{L_{n+1} L_{n+2}}}\right) \\
&= \arctan\left(\frac{\frac{1}{L_n L_{n+1}} - \frac{1}{L_{n+1} L_{n+2}}}{1 + \frac{1}{L_n L_{n+1}} \cdot \frac{1}{L_{n+1} L_{n+2}}}\right) \\
&= \arctan\left(\frac{1}{L_n L_{n+1}}\right) - \arctan\left(\frac{1}{L_{n+1} L_{n+2}}\right),
\end{aligned}$$

so the proposed series, say $S$, telescopes:

$$S = \sum_{n=1}^{\infty} \left(\arctan\left(\frac{1}{L_n L_{n+1}}\right) - \arctan\left(\frac{1}{L_{n+1} L_{n+2}}\right)\right),$$

and since $\lim_{n\to\infty} \arctan\left(\frac{1}{L_n L_{n+1}}\right) = 0$, then

$$S = \arctan\left(\frac{1}{L_1 L_2}\right) = \arctan \frac{1}{3} \sim 0.321751.$$

**Solution 2 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain.** Let $a_n = \arctan L_n L_{n+1}$. Which means that $\tan a_n = L_n L_{n+1}$. We have $\tan a_{n+1} - \tan a_n = L_{n+2}L_{n+1} - L_{n+1}L_n = L_{n+1}(L_{n+2} - L_n) = L_{n+1}^2$. So,

$$\begin{aligned}\arctan\left(\frac{L_{n+1}^2}{1 + L_n L_{n+1}^2 L_{n+2}}\right) &= \arctan\left(\frac{\tan a_{n+1} - \tan a_n}{1 + \tan a_{n+1} \tan a_n}\right)\\ &= \arctan(\tan(a_{n+1} - \tan a_n))\\ &= a_{n+1} - a_n,\end{aligned}$$

where we used $\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$. We are left with

$$\sum_{n=1}^{\infty} \arctan\left(\frac{L_{n+1}^2}{1 + L_n L_{n+1}^2 L_{n+2}}\right) = \sum_{n=1}^{\infty} a_{n+1} - a_n.$$

This is a telescoping series, and its value is

$$\lim_{n\to\infty} a_n - a_0 = \lim_{n\to\infty} \arctan L_n L_{n+1} - \arctan L_1 L_2 = \frac{\pi}{2} - \arctan 3.$$

**Also solved by** *Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and the proposer.*

# Medium–Hard Problems

***MH–29.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $ABC$ be an acute triangle with circumcenter $O$ and circumradius $R$. If $AO$ cuts the circle $BOC$ again in $A'$, $BO$ cuts circle $COA$ again in $B'$ and $CO$ cuts circle $AOB$ again in $C'$, then find the smallest possible value of

$$\sqrt{\frac{OA'^2 + OB'^2 + OC'^2}{R^2}}.$$

**Solution by the proposer.** First, we observe that when $\triangle ABC$ is equilateral then

$$\sqrt{\frac{OA'^2 + OB'^2 + OC'^2}{R^2}} = 2\sqrt{3}.$$

It suggests to prove that $\sqrt{OA'^2 + OB'^2 + OC'^2} \geq 2\sqrt{3}\,R$. Indeed, let $D$, $E$, $F$ be the intersections of $AO$, $BO$, $CO$ with $BC$, $CA$, $AB$, respectively. We have that $\triangle ODC \sim \triangle OCA'$ because $\angle OBC = \angle OA'C$ (both subtend the same chord $OC = R$ in circle $BOCA'$) and $\angle OCD = \angle OBC$ on account that $\triangle BOC$ is isosceles. Then,

$$\frac{OD}{OC} = \frac{OC}{OA'} \Longleftrightarrow OD \cdot OA' = OC^2 = R^2 \Longleftrightarrow OA' = \frac{R^2}{OD}.$$

Likewise, we get $OB' = \dfrac{R^2}{OE}$ and $OC' = \dfrac{R^2}{OF}$. Since $\dfrac{AD}{OD} = \dfrac{[ABC]}{[BOC]}$, as can be easily checked, and $OA = AD - OD$, we have

$$\frac{R}{OD} = \frac{OA}{OD} = \frac{AD}{OD} - 1 = \frac{[ABC] - [BOC]}{[BOC]} = \frac{[COA] + [AOB]}{BOC}.$$

Setting $[BOC] = x$, $[COA] = y$ and $[BOC] = z$ yields $\dfrac{R}{OD} = \dfrac{y+z}{x}$. Likewise, we get $\dfrac{R}{OE} = \dfrac{z+x}{y}$, and $\dfrac{R}{OF} = \dfrac{x+y}{z}$.

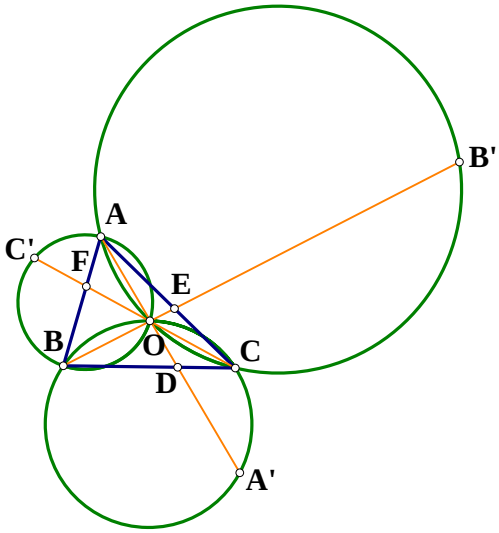


Figure 4: Scheme for problem MH-29

Finally, on account of the preceding and the QM-GM inequality, we have that

$$\begin{aligned}
\sqrt{\frac{OA'^2+OB'^2+OC'^2}{R^2}} &= \sqrt{\frac{R^2}{OD^2}+\frac{R^2}{OE^2}+\frac{R^2}{OF^2}} \\
&= \sqrt{\left(\frac{x+y}{z}\right)^2+\left(\frac{y+z}{x}\right)^2+\left(\frac{z+x}{y}\right)^2} \\
&\geq \sqrt{3}\sqrt[3]{\left(\frac{x+y}{z}\right)\left(\frac{y+z}{x}\right)\left(\frac{z+x}{y}\right)} \\
&\geq \sqrt{3}\sqrt[3]{\left(\frac{2\sqrt{xy}}{z}\right)\left(\frac{2\sqrt{yz}}{x}\right)\left(\frac{2\sqrt{zx}}{y}\right)} \\
&= 2\sqrt{3}
\end{aligned}$$

because $x+y \geq 2\sqrt{xy}$ (cyclic). So, the smallest value of the given expression is $2\sqrt{3}$, and we are done.

***MH–30.*** *Proposed by Andrés Sáez-Schwedt, Universidad de León,*

*León, Spain.* Determine if there exist 2015 prime numbers $p_1, \ldots,$ $p_{2015}$ satisfying

$$p_1 < p_2 < \cdots < p_{2015} \quad \text{and} \quad \frac{p_2}{p_1} > \frac{p_3}{p_2} > \cdots > \frac{p_{2015}}{p_{2014}}.$$

**Solution 1 by the proposer.** Let us assume the following property is known: for every natural number $x$, there exists at least one prime between $x$ and $2x$.

Let $\{t_k\}$ be an increasing sequence of integers (to be determined later on). For each $k$ there exists a prime $p_k$ that verifies $2^{t_k} < p_k < 2^{t_k+1}$. Combining these conditions for $k-1$, $k$ and $k+1$, we may claim that

$$2^{t_{k+1}-t_k-1} < \frac{p_{k+1}}{p_k} < 2^{t_{k+1}-t_k+1}, \quad 2^{t_k-t_{k-1}-1} < \frac{p_k}{p_{k-1}} < 2^{t_k-t_{k-1}+1}.$$

In particular, we can ensure that $\frac{p_{k+1}}{p_k} < \frac{p_k}{p_{k-1}}$ if we take care of choosing the exponents $t_k$ in such a way that they verify $t_{k+1} - t_k + 1 \leq t_k - t_{k-1} - 1$, that is, if the progression of differences $d_{k+1} = t_{k+1} - t_k$ is decreasing, and it decreases "at least by two units": $d_{k+1} \leq d_k - 2$.

A possible sequence of differences might be $d_1 = 2 \cdot 2014$, $d_2 = 2 \cdot 2013$, $\ldots$, $d_{2014} = 2$. Then, choosing an arbitrary $t_1$ and $t_k = t_{k-1} + d_k$ for $t \geq 2$, we obtain a positive answer to the question in the statement.

**Solution 2 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.** We claim the following stronger result: there are prime numbers $p_1 < p_2 < \cdots < p_{2015}$ such that $p_2 - p_1 > p_3 - p_2 > \cdots > p_{2015} - p_{2014}$. This generalizes the result since

$$\frac{p_i}{p_{i-1}} = \frac{p_i^2}{p_{i-1}p_i} \geq \frac{1}{p_{i-1}p_i}\left(\frac{p_{i-1}+p_{i+1}}{2}\right)^2 \geq \frac{1}{p_{i-1}p_i}(p_{i-1}p_{i+1}) = \frac{p_{i+1}}{p_i}$$

Assume that the statement is false. Then we will prove two lemmata. For convenience, let $C = 2015^2$:

**Lemma 1.** *For any positive reals $a$ and $b$, there is a positive integer $1 \leq k \leq C$ such that there is no prime number in $[a + (k - 1)b, a + kb)$.*

*Proof.* If we can take $p_i \in [a + k_i b, a + (k_i + 1)b)$ for $1 \leq i \leq 2015$, then $(k_{i+1} - k_i + 1)b < p_{i+1} - p_i < (k_{i+1} - k_i - 1)b$, by taking the smallest and largest possible values for $p_i$ and $p_{i+1}$. If $k_{i-1} - k_i - 1 \geq k_{i+1} - k_i + 1$, then the sequence $p_{i+1} - p_i$ would be decreasing. This can be achieved by taking $k_i = C - (2016 - i)^2$, contradicting the hypothesis that the statement is false. □

**Lemma 2.** *For any non-negative integer $n$, any interval of $C^n$ positive integers contains at most $(C - 1)^n$ prime numbers. In particular, if $\pi(x) = |\{p | p \leq x, p \text{ prime}\}|$, then $\pi(C^n) \leq (C - 1)^n$.*

*Proof.* Induction on $n$. The result is trivial for $n = 0$. Now suppose that it is true for $n$. Divide any interval of length $C^{n+1}$ into $C$ subintervals of length $C^n$. By Lemma 1, one of those subintervals contains no prime number, and by the induction hypothesis, every other subinterval contains at most $(C-1)^n$ prime numbers. The interval contains at most $(C-1)(C-1)^n = (C-1)^{n+1}$ prime numbers. □

To conclude the proof, note that $\pi(C^n) \leq (C - 1)^n$ implies

$$\begin{aligned}
\sum_{p \text{ prime}} p^{-1} &= \sum_{i=0}^{\infty} \sum_{C^i < p \leq C^{i+1}} p^{-1} \\
&\leq \sum_{i=0}^{\infty} \sum_{C^i < p \leq C^{i+1}} C^{-i} \\
&\leq \sum_{i=0}^{\infty} (C - 1)^{i+1} C^{-i} \\
&\leq \sum_{i=0}^{\infty} C\left(1 - C^{-1}\right)^i,
\end{aligned}$$

which is a convergent sum. It is a well-known consequence of the divergence of the harmonic series that the sum of reciprocals of the prime numbers diverges, so we reach a contradiction.

**Note:** This proof extends to every set $S \subseteq \mathbb{N}$ with $\pi(x) = x^{1-o(1)}$, where $\pi(x) = |\{k|k \leq x, k \in S\}|$.

**Solution 3 by Alberto Espuny-Díaz, University of Birmingham, Birmingham, United Kingdom.** In virtue of the Green-Tao theorem, we know that there exist arbitrarily long arithmetic progressions of primes [1]. Take one such arithmetic progression of primes $p_1 < p_2 < \ldots < p_{2015}$. As a sequence of the form $a_n = \frac{a+(n+1)b}{a+nb}$ for positive $a$ and $b$ is clearly decreasing (and tends to one), the statement holds.

Note that, instead of an arithmetic progression, it is enough to find a sequence of 2015 primes whose differences form a non-increasing sequence.

[1] Green, B. and Tao, T. "The primes contain arbitrarily long arithmetic progressions". *Ann. of Math. (2)* 167 (2 2008), pp. 481–547. ISSN: 0003-486X. DOI: 10.4007/annals.2008.167.481.

***MH–31.*** *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School (BMS), Berlin, Germany.* Let $ABC$ and $ADE$ be two triangles whose altitudes from $A$ have the same length, whose circumcircles are externally tangent and such that $\angle BAC = \angle DAE$, considering them as oriented angles.

- Prove that $BE \parallel CD$.
- Prove that $BD$ and $CE$ intersect on the common tangent to both circumcircles at $A$.

**Solution by the proposer.** Let $r$ be the common length of the altitudes, and let $\Gamma$ be the circle with center $A$ and radius $r$. This circle is tangent to both $BC$ and $DE$. Denote by $\ell$ the common tangent to the circumcircles.

Let the other tangent to $\Gamma$ through $B$ intersect $\ell$ at $F$. If $\alpha$, $\beta$ and $\gamma$ are the angles of $ABC$, as shown in the figure, then $\angle ABF = \beta$ (as $BA$ is the bisector of $\angle CBF$) and $\angle BAF = \gamma$ (by semi-inscribed angle). This leaves $\angle AFB = \alpha$ and $AF = \frac{H}{\sin\alpha}$. Since this depends only on $r$ and $\alpha$, we would obtain the same

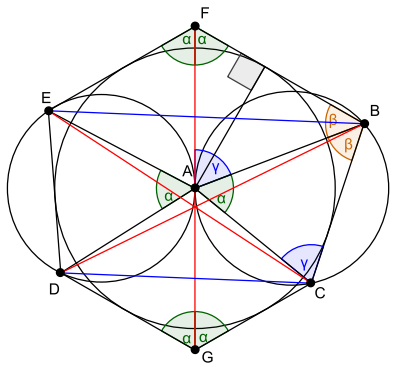


Figure 5: Scheme for problem MH-31

taking the other tangent through $E$. We conclude that $EF$ is tangent to $\Gamma$. Similarly, there is a point $G$ in $\ell$ such that $CG$ and $DG$ are tangent to $\Gamma$.

The sides of hexagon $BCGDEF$ are tangent to a circle. Thus by Brianchon's theorem, lines $BD$, $CE$ and $FG = \ell$ converge. This proves the second part. For the first part, note that $FB$ and $GD$ are parallel (they form an angle $\alpha$ with $\ell$), and so are $FE$ and $GC$. By Desargues' theorem, $BE$ and $CD$ are parallel if and only if $BD$, $CE$ and $FG$ converge, which we already know is true.

***MH–32.*** *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School (BMS), Berlin, Germany.* Find all pairs of positive integers $(a, b)$ satisfying

$$a^7b^2 = (a^2 + b + 2)^3.$$

**Solution by the proposer.** We can write the given equation as

$$ab^2 = \left(1 + \frac{b+2}{a^2}\right)^3.$$

The number $1 + \frac{b+2}{a^2}$ is a rational and its cube is an integer, so it is therefore an integer itself. Hence $b = ka^2 - 2$ for a positive integer $k$.

Plugging this into the given equation we get $a^7(ka^2-2)^2 = a^6(k+1)^3$, and dividing by $a^6$ both sides, $a(ka^2-2)^2 = (k+1)^3$. Taking this equation modulo $k$, the equation becomes $4a \equiv 1 (\text{mod } k)$. This implies $4a \geq k+1$. We use this to obtain the inequality

$$a(a^2-2)^2 \leq a(ka^2-2)^2 = (k+1)^3 \leq (4a)^3$$

which is equivalent to $(a^2-2)^2 \leq 64a^2$, which produces $a^2-2 \leq 8a$ and finally $a \leq 8$ (the positive root of $a^2-8a-2$ is smaller than 9).

If $a$ is odd then $a(ka^2-2)^2$ and $(k+1)^3$ have the parities of $k$ and $k+1$, respectively, so there cannot be a solution. Therefore, $a$ is even, $a = 2c$. The equation is $8c(2kc^2-1)^2 = (k+1)^3$. But $c$ and $(2kc^2-1)^2$ are relatively prime and their product is a cube, so $c$ is a cube, and since $c \leq 4$, we get $c = 1$, $a = 2$, $2(4k-2)^2 = (k+1)^3$, $k = 1$ and $b = ka^2 - 2 = 2$.

The only solution is $(a, b) = (2, 2)$.

***MH–33.*** *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School (BMS), Berlin, Germany.* We have $n$ labelled lamps, each of which can be on or off. Let $X$ be the set of all $2^n$ possible states for the lamps. Find all positive integers $n$ for which there exists a function $f : X \to \{1, 2, \ldots, n+1\}$ with the following property: every two different states $S_1$ and $S_2$ with $f(S_1) = f(S_2)$ differ in at least three lamps.

**Solution by the proposer.** We claim that such a function exists if and only if $n+1$ is a power of two.

First assume that $f$ exists. For every state $S$, consider the state $S_i$ that differs from $S$ in only the $i$-th lamp. Any two states in the set $X_S = \{S, S_1, S_2, \ldots, S_n\}$ differ in one or two lamps, and therefore are mapped by $f$ to two different numbers. But since $|X_S| = n+1 = |\{1, 2, \ldots, n+1\}|$, $f$ is a bijection between those sets. Note also that $S' \in X_S \Leftrightarrow S \in X_{S'}$.

This means that every state $S$ is in exactly one set $X_{S'}$ with $f(S') = 1$. The set $X$, of size $2^n$, is the disjoint union of sets of size $n+1$. Thus $n+1 | 2^n$, and $n+1$ is a power of two.

Now assume that $n = 2^k - 1$. We will construct a function $f : X \to \{1, 2, \ldots, 2^k\}$ with the desired properties, or equivalently, a function $f : X \to \mathcal{P}([k])$ (the set of subsets of $[k] = \{1, 2, \ldots, k\}$). Assign to each of the $2^k - 1$ lamps a nonempty subset of $[k]$, and take $f(S)$ to be the symmetric difference of the sets associated to the lamps that are on (that is, the set of elements from $[k]$ that appear in an odd number of lamps that are on).

Assume that $S_1$ and $S_2$ are two different states such that $f(S_1) = f(S_2)$. Then every element $x \in [k]$ is in an even number of lamps that are different in $S_1$ and $S_2$. Both states cannot differ in just one lamp, otherwise take an $x$ in the set associated to that lamp. $S_1$ and $S_2$ can also not differ in exactly two lamps, as those lamps have different sets associated, and there is some $x$ which is in one but not the other. Hence $S_1$ and $S_2$ differ in at least three lamps, as we wanted to prove.

***MH–34.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $a$, $b$, $c$ denote the lengths of the sides of a triangle, and $r$, its inradius. Prove that

$$\sum_{\text{cyclic}} \frac{a^2(b+c)}{(a+b)(a+c)} \le \frac{a^3+b^3+c^3}{24\,r^2}.$$

**Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.** From the identity

$$\sum_{\text{cyclic}} \frac{2x^2(y+z)}{(x+y)(x+z)} = x+y+z - \sum_{\text{cyclic}} \frac{xy(x-y)^2}{(x+y)(y+z)(z+x)},$$

where $x$, $y$, $z$ are positive real numbers, it follows that

$$\sum_{\text{cyclic}} \frac{2x^2(y+z)}{(x+y)(x+z)} \le x+y+z,$$

with equality just when $x = y = z$.

We apply this inequality with $x = a$, $y = b$, $z = c$ and obtain

$$\sum_{\text{cyclic}} \frac{a^2(b+c)}{(a+b)(a+c)} \le s,$$

where $s = \frac{a+b+c}{2}$ is the semiperimeter of $\triangle ABC$. Hence to prove the proposed inequality, it is sufficient to prove that

$$s \leq \frac{a^3 + b^3 + c^3}{24r^2}$$

or, equivalently,

$$24(s-a)(s-b)(s-c) \leq a^3 + b^3 + c^3,$$

since $r^2 s = (s-a)(s-b)(s-c)$ (Heron's formula).

This last inequality follows easily. From the AM-GM inequality,

$$\begin{aligned}
&a^3 + b^3 + c^3 \geq 3abc \\
=&3\left(\overbrace{(s-b)+(s-c)}^{=a}\right)\left(\overbrace{(s-c)+(s-a)}^{=b}\right)\left(\overbrace{(s-a)+(s-b)}^{=c}\right) \\
\geq&3 \cdot 2\sqrt{(s-b)(s-c)} \cdot 2\sqrt{(s-c)(s-a)} \cdot 2\sqrt{(s-a)(s-b)} \\
=&24(s-a)(s-b)(s-c).
\end{aligned}$$

It is clear that equality holds if and only if the triangle is equilateral.

**Solution 2 by Ander Lamaison Vidarte, Berlin Matiematical School, Berlin, Germany.** Let $s$ be the semiperimeter of the triangle, $x = s - a$, $y = s - b$ and $z = s - c$. Then $r^2 = \frac{xyz}{s}$ (from Hero's formula). We can use the inequality $(s+x)(s+y)(s+z) \geq 64xyz$ (obtained by expanding $s$ as $x+y+z$ and applying AM-GM inside each parenthesis) to bound

$$r^2 \leq \frac{(s+x)(s+y)(s+z)}{64s} = \frac{(a+b)(b+c)(c+a)}{32(a+b+c)}.$$

This substitution reduces our main inequality to

$$\sum_{\text{cyclic}} \frac{a^2(b+c)}{(a+b)(a+c)} \leq \frac{4}{3}\frac{(a^3+b^3+c^3)(a+b+c)}{(a+b)(b+c)(c+a)},$$

or, multiplying both sides by $(a+b)(b+c)(c+a)$,

$$a^2(b+c)^2 + b^2(c+a)^2 + c^2(a+b)^2 \leq \frac{4}{3}(a^3+b^3+c^3)(a+b+c).$$

This can be proved as follows:

$$\begin{aligned}
&a^2(b+c)^2+b^2(c+a)^2+c^2(a+b)^2\\
\le&2a^2(b^2+c^2)+2b^2(c^2+a^2)+2c^2(a^2+b^2)\\
=&4\left(a^2b^2+b^2c^2+c^2a^2\right)\\
\overset{(*)}{\le}&4\left(a^3\frac{b+c}{2}+b^3\frac{c+a}{2}+c^3\frac{a+b}{2}\right)\\
\overset{(**)}{\le}&\frac{4}{3}\left(a^3+b^3+c^3\right)\left(\frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2}\right)\\
=&\frac{4}{3}(a^3+b^3+c^3)(a+b+c),
\end{aligned}$$

where we used Muirhead's inequality in (*) and Chebyshev's sum inequality in (**).

**Solution 3 by the proposer.** To prove the preceding inequality, we insert the term $s=\dfrac{a+b+c}{2}$ (the semiperimeter of the triangle) and we will prove that

$$\sum_{\text{cyclic}}\frac{a^2(b+c)}{(a+b)(a+c)}\le\frac{a+b+c}{2}\le\frac{a^3+b^3+c^3}{24\,r^2}.$$

The RHS inequality immediately follows from the well-known formula $[ABC]=\dfrac{abc}{4R}=sr$, where $[ABC]$ is the area of the triangle and $r$, its inradius. From the preceding, and Euler's inequality $R\ge 2r$, we get

$$s=\frac{abc}{4Rr}\le\frac{abc}{8r^2}\le\frac{1}{8r^2}\left(\frac{a+b+c}{3}\right)^3\le\frac{a^3+b^3+c^3}{24\,r^2}$$

on account of mean inequalities.

To prove the LHS inequality, we will see that

$$\sum_{\text{cyclic}}\frac{2a^2(b+c)}{(a+b)(a+c)}\le a+b+c.$$

Indeed, we have

$$\begin{aligned}\sum_{\text{cyclic}} \frac{2a^2(b+c)}{(a+b)(a+c)} &= \frac{4(a^2b^2+b^2c^2+c^2a^2+a^2bc+ab^2c+abc^2)}{(a+b)(b+c)(c+a)} \\ &= (a+b+c) - \frac{ab(a-b)^2+bc(b-c)^2+ca(c-a)^2}{(a+b)(b+c)(c+a)} \\ &\leq a+b+c.\end{aligned}$$

Equality holds when $a = b = c$, that is, when triangle $ABC$ is equilateral, and we are done.

# Advanced Problems

***A–29.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all integer solutions of the equation

$$\cos\left[\frac{3\pi x}{8}\left(1-\sqrt{1+\frac{160}{9x}+\frac{800}{9x^2}}\right)\right]=1.$$

**Solution by the proposer.** First we write the given equation in the most convenient form

$$\cos\left[\frac{\pi}{8}\left(3x-\sqrt{9x^2+160x+800}\right)\right]=1.$$

Now we suppose that $x$ is a real root of the equation. Then there exists an integer $n$ such that

$$\frac{\pi}{8}\left(3x-\sqrt{9x^2+160x+800}\right)=2\pi n,$$

or

$$3x-16n=\sqrt{9x^2+160x+800}.$$

Squaring both term yields

$$\begin{aligned}(3x-16n)^2&=\left(\pm\sqrt{9x^2+160x+800}\right)^2\\ \iff 9x^2-96nx+256n^2&=9x^2+160x+800,\end{aligned}$$

and we obtain

$$8n^2-25=(3n+5)x.$$

Since

$$(3n+5)x=8n^2-25=8\left(n^2-\frac{25}{9}\right)-\frac{25}{9}=\frac{8}{9}\left(9n^2-25\right)-\frac{25}{9},$$

then we obtain

$$8(3n+5)(3n-5)-9x(3n+5)=25.$$

If $n$, $x$ are integers, then $(3n+5)\mid 25$ and $3n+5\in\{\pm1,\pm5,\pm25\}$. From the preceding we get that $n\in\{-10,-2,0\}$. Substituting in $(3n+5)x=8n^2-25$ we get that $x\in\{-31,-7,-5\}$, but $-7$, $-31$ do not satisfy the equation, and the only solution is $x=-5$.

***A–30.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Compute

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i\le j\le n} \ln\left(\frac{3n-i}{3n+i}\right) \ln\left(\frac{3n-j}{3n+j}\right).$$

**Solution by the proposer.** We begin with a lemma.

**Lemma.** *Let $f : [0,1] \to \mathbb{R}$ be a continuous function. We claim that*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i\le j\le n} f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right) = \frac{1}{2}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2.$$

*Proof.* Indeed, from the identity

$$\left(\sum_{k=1}^{n} x_k\right)^2 = \sum_{k=1}^{n} x_k^2 + 2 \sum_{1\le i\le j\le n} x_i x_j$$

we get

$$\frac{1}{n^2}\left(\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right)^2 = \frac{1}{n^2}\sum_{k=1}^{n} f^2\left(\frac{k}{n}\right) + \frac{2}{n^2} \sum_{1\le i\le j\le n} f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right).$$

Since

$$\lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} f^2\left(\frac{k}{n}\right)\right) = 0,$$

then

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i\le j\le n} f\left(\frac{i}{n}\right) f\left(\frac{j}{n}\right) &= \frac{1}{2}\left[\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)\right]^2 \\ &= \frac{1}{2}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2\end{aligned}$$

as claimed. □

Finally, using $f(x) = \ln\left(\frac{3-x}{3+x}\right)$ in the lemma, we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i\le j\le n} \ln\left(\frac{3n-i}{3n+i}\right) \ln\left(\frac{3n-j}{3n+j}\right) \\
=&\frac{1}{2}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2 = \frac{1}{2}\left[\int_0^1 \ln\left(\frac{3-x}{3+x}\right)\mathrm{d}x\right]^2 \\
=&\frac{1}{2}\left(\left[\ln\left(\frac{6}{3+x}\right)^6 - \ln\left(\frac{3-x}{3+x}\right)^{2(3+x)}\right]_0^1\right)^2 \\
=&\frac{1}{2}\ln^2\left(\frac{729}{1024}\right),
\end{aligned}$$

and we are done.

**Also solved by** *Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.*

***A–31.*** *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School (BMS), Berlin, Germany.* Find all functions $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ such that for all $a, b \in \mathbb{Z}$ the following holds:

$$a + f(af(b)) = f(af(b+1)).$$

**Solution by the proposer.** Setting $a = 1$, $b = k$ we get $1 + f(f(k)) = f(f(k+1))$. This allows us to prove by induction the equality

$$f(f(k)) = f(f(0)) + k. \tag{13}$$

Indeed, this equality is trivially true for $k = 0$, and the equality $1 + f(f(k)) = f(f(k+1))$ implies that (13) holds for $k$ if, and only if, it holds for $k+1$. We conclude that (13) holds for all integers $k$.

The function $f$ must be injective, because

$$\begin{aligned}
f(x) = f(y) &\Longrightarrow f(f(x)) = f(f(y)) \\
&\Longrightarrow f(f(0)) + x = f(f(0)) + y \\
&\Longrightarrow x = y.
\end{aligned}$$

It is also surjective, because setting $k = n - f(f(0))$ we have $f(f(k)) = n$. This means that $f$ is bijective.

Let $z = f^{-1}(0)$. We set $b = z$, $a = f(1) - f(0)$. Then

$$f(1) = a + f(0) = a + f(af(z)) = f(af(z+1)).$$

As $f$ is injective, $1 = af(z+1)$, and $f(z+1) = \pm 1$. Finally, let $t = f(0)$. If $f(z+1) = 1$, we set $a = n$, $b = z$ and get $n + t = n + f(nf(z)) = f(nf(z+1)) = f(n)$. If $f(z+1) = -1$, we set $a = -n$, $b = z$ and get $-n + t = -n + f(-nf(z)) = f(-nf(z+1)) = f(n)$.

We check that both $f(n) = t + n$ and $f(n) = t - n$ are solutions for all integers $t$.

***A–32.*** *Proposed by Dan Popescu, Suceava, Romania.* Let $A$, $B \in M_2(\mathbb{R})$ be such that $AB = BA$. If $\det(A^2 + B^2) = 0$, then prove that $\operatorname{tr}(AB) = \operatorname{tr}(A) \cdot \operatorname{tr}(B)$. Does the same hold for $A$, $B \in M_2(\mathbb{C})$?

**Solution 1 by Miguel Amengual Covas, Cala Figuera, Mallorca, Spain.** Let $A$, $B \in M_2(\mathbb{R})$ be such that $AB = BA$. Since the determinant of a product of two square matrices is the product of their determinants, we have

$$\begin{aligned}\det(A + iB)\det(A - iB) &= \det[(A + iB)(A - iB)] \\ &= \det\left(A^2 + B^2\right) = 0,\end{aligned}$$

where $i$ is the imaginary unit ($i^2 = -1$). Therefore, we know that either $\det(A + iB) = 0$ or $\det(A - iB) = 0$. This implies that at least one of the imaginary parts $\operatorname{Im}(\det(A + iB))$, $\operatorname{Im}(\det(A - iB))$ is zero:

$$\operatorname{Im}(\det(A + iB)) = 0 \quad \text{or} \quad \operatorname{Im}(\det(A - iB)) = 0. \tag{14}$$

By letting $A = \begin{pmatrix} x & y \\ z & t \end{pmatrix}$, $B = \begin{pmatrix} x' & y' \\ z' & t' \end{pmatrix}$ and separating the complex numbers $\det(A + iB)$, $\det(A - iB)$ into real and imaginary parts, we find

$$\det(A \pm iB) = \left( \begin{vmatrix} x & y \\ z & t \end{vmatrix} - \begin{vmatrix} x' & y' \\ z' & t' \end{vmatrix} \right) \pm i \left( \begin{vmatrix} x & y' \\ z & t' \end{vmatrix} + \begin{vmatrix} x' & y \\ z' & t \end{vmatrix} \right).$$

Hence the requeriment (14) implies that

$$\begin{vmatrix} x & y' \\ z & t' \end{vmatrix} + \begin{vmatrix} x' & y \\ z' & t \end{vmatrix} = 0.$$

Expanding these determinants yields

$$xt' - zy' + x't - yz' = 0.$$

It follows that

$$\begin{aligned} \operatorname{tr}(AB) &= (xx' + yz') + (zy' + tt') \\ &= (x+t)(x'+t') + yz' + zy' - xt' - tx' \\ &= (x+t)(x'+t') \\ &= \operatorname{tr}(A) \cdot \operatorname{tr}(B), \end{aligned}$$

which was to be shown.

The same does not hold for $A, B \in M_2(\mathbb{C})$. The matrices $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ provide a counterexample.

**Solution 2 by the proposer.** For all $z \in \mathbb{C}$ and any $A$, $B \in M_2(\mathbb{C})$ we know that

$$\det(A + zB) = \det(A) + (\operatorname{tr}(A) \cdot \operatorname{tr}(B) - \operatorname{tr}(AB))z + \det(B)z^2.$$

On account of the preceding, we have

$$\begin{aligned} \det(A^2 + B^2) &= \det(A + iB) \cdot \det(A - iB) \\ &= (\det(A) - \det(B))^2 + (\operatorname{tr}(A) \cdot \operatorname{tr}(B) - \operatorname{tr}(AB))^2. \end{aligned}$$

If $A$, $B \in M_2(\mathbb{R})$ and $\det(A^2 + B^2) = 0$, then $\det(A) = \det(B)$ and $\operatorname{tr}(AB) = \operatorname{tr}(A) \cdot \operatorname{tr}(B)$.

For the second part, the answer is negative. Consider for instance the matrices

$$A = \begin{pmatrix} 0 & 1+i \\ 1+i & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},$$

for which

$$\det(A^2 + B^2) = \det\begin{pmatrix} (1+i)^2 & 2i \\ 2i & (1+i)^2 \end{pmatrix} = 0,$$

with $\operatorname{tr}(A) \cdot \operatorname{tr}(B) = 0 \cdot 2 = 0$ and $\operatorname{tr}(AB) = 2(i-1) \neq 0$.

***A–33.*** *Proposed by Mihály Bencze, Braşov, Romania.* Prove that

$$\left(\frac{1}{\pi}\int_{2-\sqrt{3}}^{1} e^{-x^2}\mathrm{d}x\right)\left(\frac{1}{\pi}\int_{1}^{2+\sqrt{3}} e^{-x^2}\mathrm{d}x\right) < \frac{1}{36}.$$

**Solution 1 by Víctor Martín Chabrera, FME, BarcelonaTech, Barcelona, Spain.** Using that $e^{-x^2} > 0$ and that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$, and applying AM-GM inequality we have

$$\begin{aligned}
&\left(\frac{1}{\pi}\int_{2-\sqrt{3}}^{1} e^{-x^2}\mathrm{d}x\right)\left(\frac{1}{\pi}\int_{1}^{2+\sqrt{3}} e^{-x^2}\mathrm{d}x\right)\\
\le&\left(\frac{1}{2}\left(\frac{1}{\pi}\int_{2-\sqrt{3}}^{1} e^{-x^2}\mathrm{d}x + \frac{1}{\pi}\int_{1}^{2+\sqrt{3}} e^{-x^2}\mathrm{d}x\right)\right)^2\\
=&\left(\frac{1}{2\pi}\int_{2-\sqrt{3}}^{2+\sqrt{3}} e^{-x^2}\mathrm{d}x\right)^2\\
\le&\left(\frac{1}{2\pi}\int_{0}^{\infty} e^{-x^2}\mathrm{d}x\right)^2 = \left(\frac{1}{2\pi}\frac{\sqrt{\pi}}{2}\right)^2 = \left(\frac{1}{4\sqrt{\pi}}\right)^2 = \frac{1}{16\pi} < \frac{1}{36},
\end{aligned}$$

proving the inequality and obtaining an even better bound.

**Solution 2 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.** Let $S = [2-\sqrt{3}, 1] \times [1, 2+\sqrt{3}] \subset \mathbb{R}^2$. This rectangle is contained in the region $1 \le r \le 4$ and $\pi/4 \le \theta \le \pi/2$, in the usual polar coordinates. Since $e^{-x^2}$ is non-negative, we find

$$\begin{aligned}
\left(\frac{1}{\pi}\int_{2-\sqrt{3}}^{1} e^{-x^2}\mathrm{d}x\right)\left(\frac{1}{\pi}\int_{1}^{2+\sqrt{3}} e^{-x^2}\mathrm{d}x\right) &= \frac{1}{\pi^2}\iint_S e^{-(x^2+y^2)}\mathrm{d}x\mathrm{d}y\\
&\le \frac{1}{\pi^2}\int_{\pi/4}^{\pi/2}\int_1^4 e^{-r^2} r\mathrm{d}r\mathrm{d}\theta\\
&= \frac{1}{8\pi}(e^{-1} - e^{-16})\\
&< \frac{1}{8e\pi} < \frac{1}{36}.
\end{aligned}$$

**Solution 3 by the proposer.** Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \arctan x - \int_0^x e^{-t^2} \mathrm{d}t.$$

Then $f'(x) = \dfrac{1}{1+x^2} - e^{-x^2} = \dfrac{e^{x^2} - (x^2+1)}{(x^2+1)e^{x^2}} \geq 0$ on account that $e^{x^2} \geq x^2 + 1$. Equality holds when $x = 0$ and so, $f$ is increasing. Therefore, if $0 < a < b$ then $f(a) < f(b)$ and

$$\arctan b - \arctan a > \int_a^b e^{-x^2} \mathrm{d}x.$$

Next, we consider both integrals and bound them:

1. If $b = 1$, $a = 2 - \sqrt{3}$, then
$$\int_{2-\sqrt{3}}^{1} e^{-x^2} \mathrm{d}x < \arctan 1 - \arctan\left(2 - \sqrt{3}\right) = \frac{\pi}{6}.$$
2. If $b = 2 + \sqrt{3}$, $a = 1$, then
$$\int_{1}^{2+\sqrt{3}} e^{-x^2} \mathrm{d}x < \arctan\left(2 + \sqrt{3}\right) - \arctan 1 = \frac{\pi}{6}.$$

Multiplying up the preceding expressions and rearranging terms the claimed inequality follows.

***A–34.*** *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $n$ be a positive integer. Compute

$$\left( \sum_{1 \leq j \leq \lfloor \frac{n+1}{2} \rfloor} \frac{1}{2j-1} \binom{n}{2j-1} \right) \Bigg/ \left( \sum_{j=1}^{n} \frac{2^j}{j} \right),$$

where $\lfloor x \rfloor$ denotes the integer part of $x$.

**Solution by the proposer.** Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sum_{j=1}^{n} \frac{1}{j} \binom{n}{j} x^j$. We have that

$$f(1) - f(-1) = 2 \sum_{1 \leq j \leq \lfloor \frac{n+1}{2} \rfloor} \frac{1}{2j-1} \binom{n}{2j-1}.$$

On the other hand,

$$\begin{aligned} f(1)-f(-1) &= \int_{-1}^{1} f'(x)\,\mathrm{d}x = \int_{-1}^{1} \frac{1}{x}\sum_{j=1}^{n}\binom{n}{j}x^j\,\mathrm{d}x \\ &= \int_{-1}^{1} \frac{(1+x)^{n+1}-1}{x}\,\mathrm{d}x = I_n. \end{aligned}$$

Next we compute $I_n$ :

$$\begin{aligned} &I_n \\ =&\int_{-1}^{1} \frac{1}{x}\,\mathrm{d}\left(\frac{(1+x)^{n+1}-(n+1)x-1}{n+1}\right) \\ =&\frac{2^{n+1}-2}{n+1}+\frac{1}{n+1}\int_{-1}^{1}\frac{1}{x}\left(\frac{(1+x)^{n+1}-1}{x}-(n+1)\right)\mathrm{d}x \\ =&\frac{2^{n+1}-2}{n+1}+\frac{1}{n+1}\int_{-1}^{1}\frac{1+(1+x)+\ldots+(1+x)^n-(n+1)}{x}\,\mathrm{d}x \\ =&\frac{2^{n+1}-2}{n+1}+\frac{1}{n+1}(I_1+I_2+\ldots+I_n). \end{aligned}$$

Therefore, $nI_n = (2^{n+1}-2)+I_1+I_2+\ldots+I_{n-1}$ and $(n-1)I_{n-1} = (2^n-2)+I_1+I_2+\ldots+I_{n-2}$. Subtracting the second from the first yields $nI_n-(n-1)I_{n-1} = 2^n+I_{n-1}$, from which

$$I_n = \frac{2^n}{n}+I_{n-1} = \sum_{j=1}^{n}\frac{2^j}{j}.$$

Combining the preceding results, we get

$$\left(\sum_{1\le j\le\lfloor\frac{n+1}{2}\rfloor}\frac{1}{2j-1}\binom{n}{2j-1}\right)\Bigg/\left(\sum_{j=1}^{n}\frac{2^j}{j}\right) = \frac{1}{2},$$

and we are done.

**Also solved by** *Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.*

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