No. 1
2016

ARHIMEDE
MATHEMATICAL JOURNAL

Articles
Problems
Mathlessons

ARHIMEDE ASSOCIATION

CONTENTS

Articles

Arhimede Mathematical Journal aims to publish interesting and attractive papers with elegant mathematical exposition. Articles should include examples, applications and illustrations, whenever possible. Manuscripts submitted should not be currently submitted to or accepted for publication in another journal.

Please, send submittals to: **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to:

jose.luis.diaz@upc.edu

A powerful tool for the study of systems of equations: Bolzano type theorems

A. Cañada and S. Villegas

1 Introduction and motivation

In secondary school, students are familiar with the study of equations. If they are of a particular kind, such as polynomials of degree two or special types of trigonometric, logarithmic or exponential equations, the teacher provides methods and ideas for obtaining the solutions explicitly. For example,

- The equation $2^{x-1} + 2^x + 2^{x+1} = 28$ has a unique solution, $x = 3$.
- The equation $x^4 - 5x^2 + 6 = 0$ has four solutions: $\sqrt{2}$, $-\sqrt{2}$, $\sqrt{3}$, and $-\sqrt{3}$.
- The equation $4\sin x - \cos 2x + 1 = 0$ has infinitely many solutions: $x = k\pi,\ k \in \mathbb{Z}$ (the set of integer numbers).

In these trivial examples, some basic properties of the considered elementary functions are used to explicitly obtain the solutions: the change of variable $2^{x-1} = y$ in the first case, the change of variable $x^2 = y$ in the second case, and the use of the formula $-\cos 2x = 2\sin^2 x - 1$, in the third one.

However, too many equations which arise in applied sciences cannot be solved explicitly. For example,

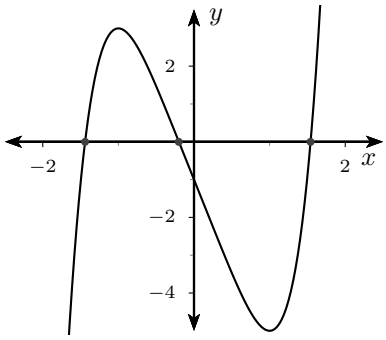

Figure 1: $f(x) = x^5 - 5x - 1$

1. The simple equation $x^5 - 5x - 1 = 0$ has three real but not rational solutions, i.e., solutions of the type

$$x = \frac{a}{b},\ a \in \mathbb{Z},\ b \in \mathbb{Z} \setminus \{0\},$$

since according to Fubini's rule, the only possible rational solutions of the previous equation are 1 and -1, and none of them is a solution of that equation.

2. For any given real number a, the equation $e^x + x^3 + x + \cos x = a$, has a unique solution, but it cannot be obtained explicitly.

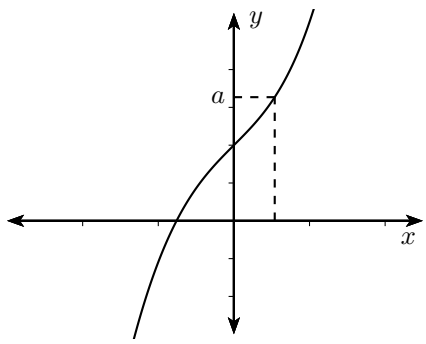

Figure 2: $f(x) = e^x + x^3 + x + \cos x$

In these situations, Bolzano's theorem provides a good method to prove the existence of solutions. This theorem, together with an additional study of the monotonicity of the given function, can

provide an adequate and complete study on the solutions of the considered equation.

Bolzano's theorem contains all the conditions of a very good theorem: simple statement, affordable proof, and a large and wide applicability in the scientific world.

Throughout the paper, $\mathbb{R}$ will denote the set of real numbers.

Theorem 1 (Bolzano, 1817). *If for some real numbers $a < b$, $f : [a, b] \to \mathbb{R}$ is a continuous functions such that $f(a) < 0 < f(b)$, then there exists some point $c \in (a, b)$ satisfying the equation $f(x) = 0$.*

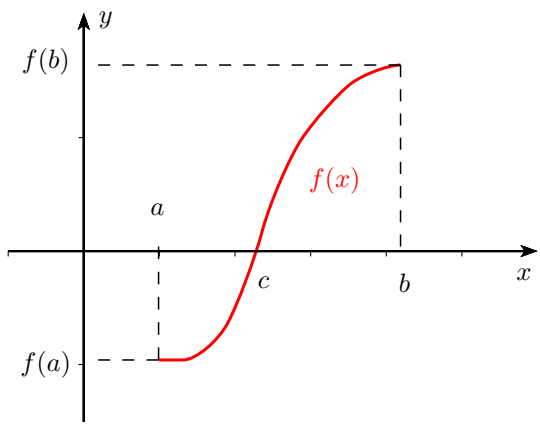

Figure 3

For example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^k + h(x)$, with k an odd natural number and $h : \mathbb{R} \to \mathbb{R}$ continuous and satisfying $\lim\limits_{|x|\to+\infty} \dfrac{h(x)}{|x|^k} = 0$, then the equation $f(x) = 0$ has a solution. This is the case of a polynomial equation of odd degree $x^k + a_{k-1}x^{k-1} + \ldots + a_1x + a_0 = 0$ (where $a_{k-1}, \ldots, a_1, a_0$ are given real numbers), as well as the case where the function h is continuous and bounded as in the equation $x^k + \sin^3(e^{x^2} + 7) = 0$.

In regard to the existence of solutions, similar ideas can be used if we consider scalar equations with several variables, i.e., f :

$\mathbb{R}^n \to \mathbb{R}$. If f is continuous and there exist some points $a, b \in \mathbb{R}^n$ such that $f(a) < 0 < f(b)$, then the equation in several variables $f(x_1, \ldots, x_n) = 0$ has at least one solution in the "open segment" of $\mathbb{R}^n$ defined as

$$(a, b)_{\mathbb{R}^n} = \{(1 - \lambda)a + \lambda b,\ \lambda \in (0, 1)\}.$$

The proof of this fact is trivial if we consider the continuous function $g : [0, 1] \to \mathbb{R}$, defined as $g(\lambda) = f((1 - \lambda)a + \lambda b)$ and we observe that $g(0) = f(a) < 0 < f(b) = g(1)$, and finally we apply Bolzano's theorem.

As an example, if $h : \mathbb{R}^3 \to \mathbb{R}$ is continuous and bounded, the equation $x_1 e^{x_2} + h(x_1, x_2, x_3) = a$ has a solution for each $a \in \mathbb{R}$. The proof of this fact is very easy, since

$$\lim_{x_1 \to +\infty} x_1 e^{x_2} + h(x_1, x_2, x_3) = +\infty$$

and

$$\lim_{x_1 \to -\infty} x_1 e^{x_2} + h(x_1, x_2, x_3) = -\infty.$$

For example, this is the case of the equation $x_1 e^{x_2} + x_1^2 e^{-x_1^2} + \sin(x_1 x_2^5 + \ln(1 + x_1^2)) = a$.

At this point, we should note that if we are considering not only the existence, but also the multiplicity of solutions, the situation may be completely different from the scalar case ($n = 1$). Let us clarify this statement: if $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function with $f'(x) > 0$, $\forall\ x \in \mathbb{R}$, then the equation $f(x) = 0$, has at most one solution, because f is strictly increasing. However if $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x, y) = x + y$, the equation $f(x, y) = 0$ has infinitely many solutions, although both partial derivatives, $f_x(x, y) = 1$ and $f_y(x, y) = 1$, are positive in $\mathbb{R}^2$. This example must not be surprising, since if $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable function, then its derivative f' is a function from $\mathbb{R}^n$ into $\mathbb{R}^n$ and we can not define, in an appropriate way, what $f'(x)$ being positive for $x \in \mathbb{R}^n$ means.

The situation is much more complicated in the case of systems of equations

$$f_1(x_1, \ldots, x_n) = 0, \ldots, f_n(x_1, \ldots, x_n) = 0,$$

i.e., the case where $f = (f_1, \ldots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$.

Let us consider, for instance, the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f(x, y) = ((e^y + 1)\sin x, (e^y + 1)\cos x)$. The function f is continuous and its image, $f(\mathbb{R}^2)$, contains points of the four quadrants of $\mathbb{R}^2$, but the equation $f(x, y) = 0$ has no solutions, since $f(\mathbb{R}^2) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1^2 + y_2^2 > 1\}$ (see Figure 4).

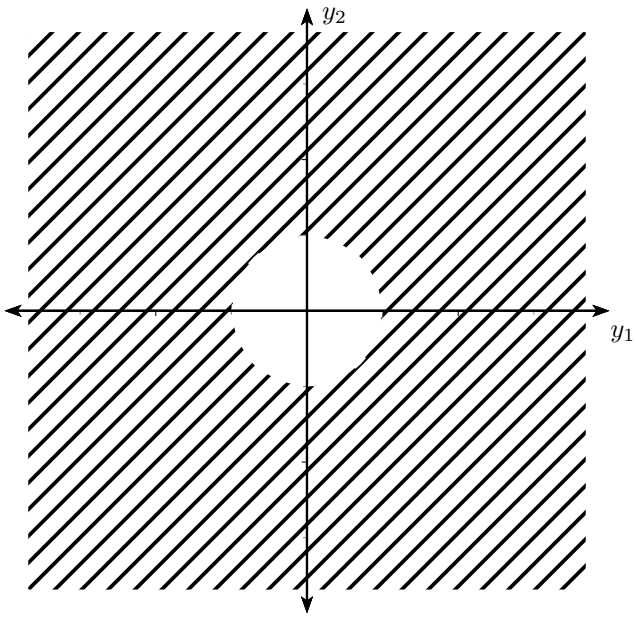

Figure 4

In Bolzano's theorem, the main hypothesis (besides the continuity of the considered function) is that the image of the function f takes values into the two sets $\mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}$ and $\mathbb{R}_- = \{x \in \mathbb{R} : x < 0\}$. But it is clear from the previous example that the key idea to study systems of equations is not that the image

of the function f takes values into the 2^n subsets

$$\begin{gathered}\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0, x_2 > 0, \ldots, x_n > 0\},\\ \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0, x_2 < 0, \ldots, x_n > 0\},\\ \cdots\\ \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 < 0, x_2 < 0, \ldots, x_n < 0\}.\end{gathered}$$

Just to prove the existence of solutions for systems of equations, the key idea is that the function f has an appropriate behavior at the topological boundary of the considered domain. Let us recall this topological concept.

For x, $y \in \mathbb{R}^n$, $d(x, y)$ denotes their euclidean distance. If Ω is a given subset of $\mathbb{R}^n$, the topological boundary of Ω, $\partial\Omega$, is defined as the set of points $x \in \mathbb{R}^n$ such that for each $r > 0$, the open euclidean ball of center x and radius r, $B_{\mathbb{R}^n}(x; r) = \{y \in \mathbb{R}^n : d(x, y) < r\}$, contains points of Ω and points of the complementary set $\mathbb{R}^n \setminus \Omega$.

Turning to Bolzano's theorem, think that the hypotheses are given in terms of the behavior of the function f on the topological boundary of $\Omega = (a, b)$, since in this simple case, $\partial\Omega = \{a, b\}$ is a set with two points.

We finish this section with several reflections and questions:

1. Bolzano's theorem is stated for the case when $n = 1$ and $\Omega = (a, b)$ is an interval of real numbers. If, for example, $n = 2$, the most simple generalization is, perhaps, $\Omega = (a, b) \times (c, d)$, a rectangle. Then, $\partial\Omega$ is the set given by the union of its four sides:

   $$\partial\Omega = \{a\} \times [c, d] \cup \{b\} \times [c, d] \cup [a, b] \times \{c\} \cup [a, b] \times \{d\}.$$

   If $f : [a, b] \times [c, d] \to \mathbb{R}^2$, $(x, y) \to (f_1(x, y), f_2(x, y))$, is a continuous function, can we provide sign type conditions on the components f_1, f_2 on the corresponding opposite sides of Ω such that the system of equations $(f_1(x, y), f_2(x, y)) = (0, 0)$ has a solution in Ω?

2. If Ω is an open euclidean ball in $\mathbb{R}^2$, of center x and radius r, then there are no sides. Now, $\partial\Omega$ is a circumference. Is it possible to give sufficient conditions in terms of the behavior of the continuous function $f : \overline{\Omega} \to \mathbb{R}^2$ on $\partial\Omega$, such that the system of equations $f(x, y) = 0$ has a solution in Ω? (here $\overline{\Omega}$ is the closed euclidean ball of center x and radius r, $\overline{B}_{\mathbb{R}^n}(x; r) = \{y \in \mathbb{R}^n : d(x, y) \leq r\}$).
3. If Ω is a given "general" subset of $\mathbb{R}^n$, with n an arbitrary natural number, how can we prove that the system of n equations $f(x) = 0$ has a solution in Ω?
4. Is there some concept or theory that unifies all these previous cases?

The answers are given in the next section.

2 Systems of equations in rectangles, balls and...

2.1 The case of a rectangle

If we are considering systems of equations in a rectangle, the so called Poincaré-Miranda's theorem is an appropriate generalization of Bolzano's theorem. Roughly speaking, it can be stated as follows: *if each component function of the given system has opposite signs on the corresponding opposite sides of some rectangle, then the system of equations has at least one solution inside of such rectangle.* More precisely,

Theorem 2 (Poincaré-Miranda). *If*

$$f : [a, b] \times [c, d] \to \mathbb{R}^2,\ (x, y) \to (f_1(x, y), f_2(x, y)),$$

is a continuous function and

$$f_1(a, y) < 0 < f_1(b, y),\ \forall\ y \in [c, d],$$
$$f_2(x, c) < 0 < f_2(x, d),\ \forall\ x \in [a, b],$$

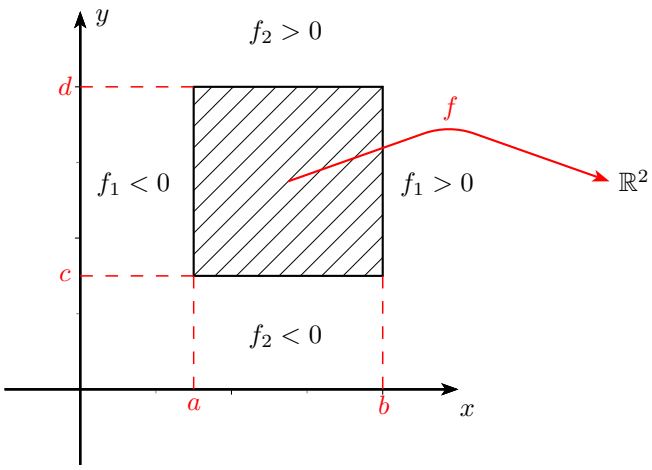

Figure 5

then the system of two equations

$$f_1(x, y) = 0,\ f_2(x, y) = 0,$$

has at least one solution in $(a, b) \times (c, d)$.

As a nontrivial example, if $h,\ g : \mathbb{R}^2 \to \mathbb{R}^2$ are bounded and continuous functions and $(a, b) \in \mathbb{R}^2$ is given, the system of equations

$$x^5 + h(x, y) = a, \quad \frac{y}{1 + |y|} e^{y^2} + g(x, y) = b,$$

has at least one solution in the rectangle $[-r, r] \times [-r, r]$ for r a sufficiently large positive real number. To prove this fact, let us note that

$$\lim_{x \to +\infty} x^5 + h(x, y) = +\infty, \ \lim_{x \to -\infty} x^5 + h(x, y) = -\infty$$

and that

$$\lim_{y \to +\infty} \frac{y}{1 + |y|} e^{y^2} + g(x, y) = +\infty, \ \lim_{y \to -\infty} \frac{y}{1 + |y|} e^{y^2} + g(x, y) = -\infty.$$

As a consequence, the image of the function

$$f : \mathbb{R}^2 \to \mathbb{R}^2,\ (x, y) \to (x^5 + h(x, y), \frac{y}{1 + |y|} e^{y^2} + g(x, y))$$

is the whole space $\mathbb{R}^2$.

The previous theorem was proved by Poincaré in 1886 and, obviously, there is a formulation for the general case of n equations in a rectangle contained in $\mathbb{R}^n$ (see [6]). In 1940 Miranda proved that it is equivalent to Brouwer's fixed point theorem [5].

2.2 The case of a euclidean ball

It can be affirmed that Poincaré-Miranda's theorem is very intuitive, since, on the one hand, the natural generalization of an interval of real numbers $[a, b]$ is a box in the euclidean space $\mathbb{R}^n$ given by $[a_1, b_1] \times \ldots \times [a_n, b_n]$ and, on the other hand, the sign type hypothesis of Bolzano's theorem $f(a) < 0 < f(b)$ is replaced by an appropriate sign type hypothesis on the component functions of $f = (f_1, \ldots, f_n)$ on the corresponding opposite sides of the n-dimensional rectangle. But, what happens if we are dealing with subsets of $\mathbb{R}^n$ which are not supposed to have sides? For instance, a ball. In this case we have the following result.

Theorem 3 (Systems of equations in euclidean balls). *Let $f : \overline{B}_{\mathbb{R}^n}(0; r) \to \mathbb{R}^n$ be a continuous function such that*

$$\langle f(x), x\rangle > 0, \ \forall\, x \in \partial\overline{B}_{\mathbb{R}^n}(0; r) \tag{1}$$

(where $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in $\mathbb{R}^n$). Then the equation $f(x) = 0$ has a solution in the open ball $B_{\mathbb{R}^n}(0; r)$.

An example: if $h_1, h_2, h_3 : \mathbb{R}^2 \to \mathbb{R}$ are bounded and continuous functions, the system of equations

$$x + h_1(y, z) = 0, \ y + h_2(x, z) = 0, \ z + h_3(x, y) = 0$$

has a solution in $B_{\mathbb{R}^3}(0; r)$ for sufficiently large r. To see this,

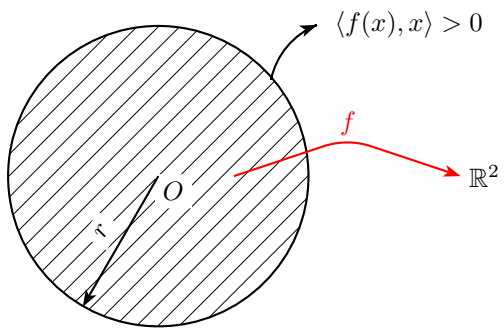

Figure 6

take into account that

$$\begin{aligned}
&\lim_{x^2+y^2+z^2\to+\infty} \langle (x,y,z), (x+h_1(y,z), y+h_2(x,z), z+h_3(x,y))\rangle \\
&= \lim_{x^2+y^2+z^2\to+\infty} x(x+h_1(y,z)) + y(y+h_2(x,z)) + z(z+h_3(x,y)) \\
&= \lim_{x^2+y^2+z^2\to+\infty} (x^2+y^2+z^2) + xh_1(y,z) + yh_2(x,z) + zh_3(x,y) \\
&= +\infty.
\end{aligned}$$

Let us think that, as in Poincaré-Miranda's theorem, Theorem 3 is also a generalization of the classical Bolzano's theorem, which is obtained if $n = 1$. To clarify this claim, take into account that, in Bolzano's theorem, it is clearly not restrictive to assume that $a = -r$, $b = r$, where r is a positive real number. Then, we can formulate Bolzano's theorem (Theorem 1) in the following equivalent manner:

Theorem 4 (Bolzano, revisited). *If $f : [-r, r] \to \mathbb{R}$ is continuous and*

$$f(-r)(-r) > 0, f(r)r > 0$$

(which is equivalent to $f(-r) < 0 < f(r)$), then the equation $f(x) = 0$ has a solution in $(-r, r)$, the open ball of center zero and radius r in $\mathbb{R}$.

In the previous lines, we have stated two generalizations of Bolzano's theorem which are, apparently, very different. It seems that they

are not related. Nothing is further from reality, since Bolzano's theorem, Poincaré-Miranda's theorem and Theorem 3, on systems of equations on an euclidean ball, can be viewed from a unified point of view by using a powerful tool called the Brouwer degree theory (see [4]).

3 Some applications of Bolzano type theorems

The previous Bolzano type theorems are not only of interest to mathematicians. In this section, we briefly discuss some of their elementary applications. The first one uses the classical Bolzano's theorem in one and two variables to prove the simultaneous bisection of two given polygons [3]. The second one is about how to use Poincaré-Miranda's theorem to prove the existence of fixed points of a pair of functions of two variables. The fixed point theory has been of fundamental importance in the development of general equilibrium theory in Economy [7].

In the next theorem, we consider for simplicity the case of two polygons, but the same ideas may be used to deal with two bounded subsets of $\mathbb{R}^2$ with well defined area.

Theorem 5. *For any pair of given convex polygons P_1 and P_2, there exists a line $Ax + By = C$ which bisects them simultaneously, i.e., if*

$$\begin{aligned} P_1^+ &= P_1 \cap \{(x,y) \in \mathbb{R}^2 : Ax + By > C\}, \\ P_1^- &= P_1 \cap \{(x,y) \in \mathbb{R}^2 : Ax + By < C\}, \\ P_2^+ &= P_2 \cap \{(x,y) \in \mathbb{R}^2 : Ax + By > C\}, \\ P_2^- &= P_2 \cap \{(x,y) \in \mathbb{R}^2 : Ax + By < C\}, \end{aligned}$$

then $area(P_1^+) = area(P_1^-)$, $area(P_2^+) = area(P_2^-)$.

Main ideas of the proof. In the first step, we apply Theorem 1 for functions of one variable. It is clear that there exists a unique line

in each direction which bisects the polygon P_1. That is, for each given $(a,b) \in S^1 = \{(a,b) \in \mathbb{R}^2 : a^2 + b^2 = 1\}$ there exists a unique $c = c(a,b)$ such that

$$area(P_1^+(a,b,c)) = area(P_1^-(a,b,c)),$$

where

$$P_1^+(a,b,c) = P_1 \cap \{(x,y) \in \mathbb{R}^2 : ax + by > c\},$$
$$P_1^-(a,b,c) = P_1 \cap \{(x,y) \in \mathbb{R}^2 : ax + by < c\}.$$

In fact, if $(a,b) \in S^1$ is fixed, then the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(c) = area(P_1^+(a,b,c)) - area(P_1^-(a,b,c))$$

is continuous and takes positive and negative values.

In the second step, we apply Bolzano's theorem for functions of two variables. More precisely, the function $H : S^1 \to \mathbb{R}$ defined by

$$H(a,b) = area(P_2^+(a,b,c(a,b))) - area(P_2^-(a,b,c(a,b)))$$

is continuous and takes positive and negative values. Therefore, there exists $(A,B) \in S^1$ such that $H(A,B) = 0$, i.e., the line $Ax + By = c(A,B)$ bisects P_2, and this line also bisects P_1, by the definition of $c(A,B)$. □

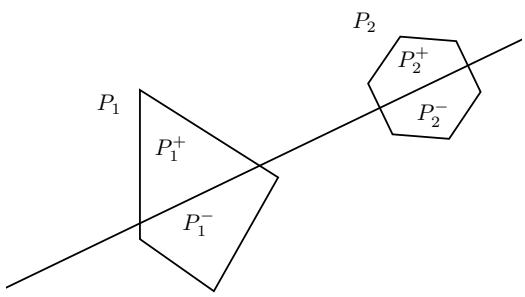

Figure 7

Finally, general equilibrium is a unified framework for studying the general interdependence of economic activities: consumption,

production, exchange. Traditionally, proofs of the existence of equilibrium rely on fixed-point theorems such as Brouwer's fixed-point theorem [1]. In this regard, we can say that it is very intuitive that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and bounded, then the fixed point equation $x = f(x)$ has at least one solution, since we can apply Bolzano's Theorem 1 to the function $F : \mathbb{R} \to \mathbb{R}$ defined as $F(x) = x - f(x)$.

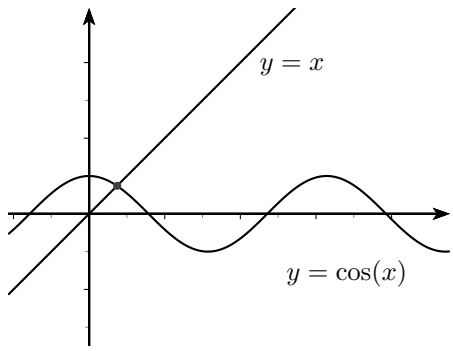

Figure 8

Perhaps, the intuition is difficult to use when we are in the case of systems of equations. In this case we show how to use the Poincaré-Miranda Theorem 2 to prove, very easily, the existence of a fixed point. Indeed, if $(f, g) : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \to (f(x, y), g(x, y))$ are continuous and bounded functions, then for some sufficiently large $r \in \mathbb{R}^+$, we can apply Poincaré-Miranda's theorem 2 to the function $(F, G) : [-r, r] \times [-r, r] \to \mathbb{R}^2$ defined as $(F, G)(x, y) = (x - f(x, y), y - g(x, y))$. As a consequence, $(F, G)(x_0, y_0) = (0, 0)$, for some $(x_0, y_0) \in [-r, r] \times [-r, r]$, and therefore $(x_0, y_0) = (f(x_0, y_0), g(x_0, y_0))$, for some $(x_0, y_0) \in [-r, r] \times [-r, r]$.

Finally, we comment that Theorem 3 is only a special case of the celebrated Gale-Nikaido-Debreu lemma [1], which, according to many authors, has been of great interest in the development of general equilibrium theory (especially in market equilibrium).

Acknowledgment
The authors thank Miguel Navarro for his collaboration in graphics editing.

References

[1] Ben-El-Mechaiekh, H., Bich, P., and Florenzano, M. "General equilibrium and fixed point theory: a partial survey". *J. Fixed Point Theory Appl.* 6.2 (2009), pp. 207–226. ISSN: 1661-7738. DOI: 10.1007/s11784-009-0129-2.

[2] Cañada, A. and Villegas, S. "¿El Teorema de Bolzano en varias variables?" *Gac. R. Soc. Mat. Esp.* 7.1 (2004), pp. 101–121. ISSN: 1138-8927.

[3] Courant, R. and John, F. *Introduction to calculus and analysis. Vol. I.* Classics in Mathematics. Reprint of the 1989 edition. Springer-Verlag, Berlin, 1999, pp. xxiv+661. ISBN: 3-540-65058-X. DOI: 10.1007/978-3-642-58604-0.

[4] Deimling, K. *Nonlinear functional analysis.* Springer-Verlag, Berlin, 1985, pp. xiv+450. ISBN: 3-540-13928-1. DOI: 10.1007/978-3-662-00547-7.

[5] Miranda, C. "Un'osservazione su un teorema di Brouwer". *Boll. Un. Mat. Ital. (2)* 3 (1940), pp. 5–7.

[6] Poincaré, H. "Sur les courbes définies par une équation différentielle IV". *J. Math. Pures Appl.* 85 (1886), pp. 151–217.

[7] Urai, K. *Fixed Points and Economic Equilibria.* Vol. 5. World Scientific, 2010.

A. Cañada
Department of Mathematical Analysis
University of Granada, Granada, Spain
acanada@ugr.es

S. Villegas
Department of Mathematical Analysis
University of Granada, Granada, Spain
svillega@ugr.es

Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction.
2. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

Please, send submittals to: **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to:

`jose.luis.diaz@upc.edu`

The section is divided into four subsections: Elementary Problems, Easy–Medium High School Problems, Medium–Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

Nov 30, 2016

Elementary Problems

E–29. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b be nonnegative real numbers. Prove that

$$a^4 + b^4 + a^2b^2 \geq ab(a^2 + b^2).$$

E–30. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $n \geq 1$ be a positive integer. Find a formula to compute the sum of the n first terms of

$$4 + 10 + 18 + 28 + 40 + 54 + \ldots$$

E–31. *Proposed by José Gibergnas-Báguena, BarcelonaTech, Barcelona, Spain.* Given a set of nine points with integer coordinates in three-dimensional space, prove that there are two whose midpoint also has integer coordinates.

E–32. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* The lengths of the sides of a triangle satisfy that the triple of one of them is equal to the sum of the other two. Prove that its inradius is one fourth of one of its altitudes.

E–33. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* If $x^2 - bx + a = 0$ has two integer roots, then prove that

$$\frac{a(a+b+1)(4a+2b+1)}{36}$$

can be written as the sum of a squares of integer numbers.

E–34. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers defined by $F_0 = 0$, $F_1 = 1$, and for all $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$. Show that there are infinitely many quadruplets of Fibonacci numbers satisfying the equation $2(x^2 + y^2) = z^2 + t^2$.

Easy–Medium Problems

EM–29. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c, m be positive real numbers such that

$$\sqrt{m+a}+\sqrt{m+b}=2\sqrt{m+c}.$$

Prove that $a+b\geq 2c$.

EM–30. *Proposed by José Gibergans-Báguena, BarcelonaTech, Barcelona, Spain.* Find all positive solutions of the following system of equations

$$\begin{cases}(x+y+z)^3=t,\\(y+z+t)^3=x,\\(z+t+x)^3=y,\\(t+x+y)^3=z.\end{cases}$$

EM–31. *Proposed by Nicolae Papacu, Slobozia, Romania.* If p and q are prime numbers, show that the number $p^{2q}+q^{2p}$ is composite.

EM–32. *Proposed by Ángel Plaza, University of Las Palmas de Gran Canaria, Las Palmas de Gran Canaria, Spain.* Let n be a positive integer, m be a real number greater that 1, and $x_k\in(0,1)$ $(1\leq k\leq n)$. Prove that

$$\left(\sum_{k=1}^{n}\frac{x_k}{1-x_k^m}\right)\left(n^m-\left(\sum_{k=1}^{n}x_k\right)^m\right)\geq n^m\sum_{k=1}^{n}x_k.$$

EM–33. *Proposed by José Gibergans-Báguena and José Luis Díaz-Barrero BarcelonaTech, Barcelona, Spain.* Let d_a, d_b, and d_c be the distances from the vertices of triangle ABC to its incenter. Show that

$$\frac{d_a^2}{bc}+\frac{d_b^2}{ca}+\frac{d_c^2}{ab}$$

is an integer and determine its value.

EM–34. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Compute

$$\sum_{n=1}^{\infty} \arctan\left(\frac{L_{n+1}^2}{1 + L_n L_{n+1}^2 L_{n+2}}\right),$$

where L_n is the nth Lucas number, defined by $L_0 = 2$, $L_1 = 1$, and for all $n \geq 2$, $L_n = L_{n-1} + L_{n-2}$.

Medium–Hard Problems

MH–29. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let ABC be an acute triangle with circumcenter O and circumradius R. If AO cuts the circle BOC again in A', BO cuts circle COA again in B' and CO cuts circle AOB again in C', then find the smallest possible value of

$$\sqrt{\frac{OA'^2 + OB'^2 + OC'^2}{R^2}}.$$

MH–30. *Proposed by Andrés Sáez-Schwedt, Universidad de León, León, Spain.* Determine if there exist 2015 prime numbers $p_1, \ldots,$ p_{2015} satisfying

$$p_1 < p_2 < \cdots < p_{2015} \quad \text{and} \quad \frac{p_2}{p_1} > \frac{p_3}{p_2} > \cdots > \frac{p_{2015}}{p_{2014}}.$$

MH–31. *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School (BMS), Berlin, Germany.* Let ABC and ADE be two triangles whose altitudes from A have the same length, whose circumcircles are externally tangent and such that $\angle BAC = \angle DAE$, considering them as oriented angles.

- Prove that $BE \parallel CD$.
- Prove that BD and CE intersect on the common tangent to both circumcircles at A.

MH–32. *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School (BMS), Berlin, Germany.* Find all pairs of positive integers (a, b) satisfying

$$a^7 b^2 = (a^2 + b + 2)^3.$$

MH–33. *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School (BMS), Berlin, Germany.* We have n labelled lamps, each of which can be on or off. Let X be the set of all 2^n possible

states for the lamps. Find all positive integers n for which there exists a function $f : X \to \{1, 2, \ldots, n+1\}$ with the following property: every two different states S_1 and S_2 with $f(S_1) = f(S_2)$ differ in at least three lamps.

MH–34. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c denote the lengths of the sides of a triangle, and r, its inradius. Prove that

$$\sum_{\text{cyclic}} \frac{a^2(b+c)}{(a+b)(a+c)} \leq \frac{a^3+b^3+c^3}{24\,r^2}.$$

Advanced Problems

A–29. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find all integer solutions of the equation

$$\cos\left[\frac{3\pi x}{8}\left(1-\sqrt{1+\frac{160}{9x}+\frac{800}{9x^2}}\right)\right]=1.$$

A–30. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Compute

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{1\le i\le j\le n}\ln\left(\frac{3n-i}{3n+i}\right)\ln\left(\frac{3n-j}{3n+j}\right).$$

A–31. *Proposed by Ander Lamaison Vidarte, Berlin Mathematical School (BMS), Berlin, Germany.* Find all functions $f:\mathbb{Z}\longrightarrow\mathbb{Z}$ such that for all $a,\ b\in\mathbb{Z}$ the following holds:

$$a+f(af(b))=f(af(b+1)).$$

A–32. *Proposed by Dan Popescu, Suceava, Romania.* Let $A,\ B\in M_2(\mathbb{R})$ be such that $AB=BA$. If $\det(A^2+B^2)=0$, then prove that $tr(AB)=tr(A)\cdot tr(B)$. Does the same hold for $A,\ B\in M_2(\mathbb{C})$?

A–33. *Proposed by Mihály Bencze, Braşov, Romania.* Prove that

$$\left(\frac{1}{\pi}\int_{2-\sqrt{3}}^{1}e^{-x^2}dx\right)\left(\frac{1}{\pi}\int_{1}^{2+\sqrt{3}}e^{-x^2}dx\right)<\frac{1}{36}.$$

A–34. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Compute

$$\left(\sum_{1\le j\le\lfloor\frac{n+1}{2}\rfloor}\frac{1}{2j-1}\binom{n}{2j-1}\right)\Bigg/\left(\sum_{j=1}^{n}\frac{2^j}{j}\right),$$

where $\lfloor x\rfloor$ denotes the integer part of x.

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons with material useful to solve mathematical problems.

Please, send submittals to: **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to:
jose.luis.diaz@upc.edu

The sum of digits function

A. Lamaison Vidarte

1 Introduction

In number theory olympiad problems, one often finds conditions on the digits of a non-negative integer, and has to prove certain results based on that information. A standard tool to deal with these problems is the sum of the digits of the number, which we will denote by $\sigma(n)$ for $n \geq 0$.

2 Main results

The function σ satisfies many properties which are easily checked, either from the definition or by induction. A few examples are:

- If a, b and n are non-negative integers with $10^n > b$, then $\sigma(10^n a + b) = \sigma(a) + \sigma(b)$.
- $\sigma(n) - \sigma(n-1) = 1 - 9k$, where k is the greatest integer such that $10^k | n$. In particular, $\sigma(n) \leq \sigma(n-1) + 1$.
- $\sigma(n) \leq 9 \log_{10}(n+1)$ (Hint: what is the smallest possible value of n for a fixed value of $\sigma(n)$?).

The proof of the first two results should be very simple. Also, note that the second property, together with $\sigma(0) = 0$, completely determines the value of $\sigma(n)$ for all $n \geq 0$.

In this lesson we will prove three of the most useful properties of the function σ. The first one is a well-known rule to find the residue modulo 9 of a positive integer:

Theorem 1. *For any non-negative integer n, we have $\sigma(n) \equiv n \pmod 9$.*

Proof. We proceed by induction. For $n = 0$ the result holds because $\sigma(0) = 0$. Now suppose that the result is true for some $n - 1$. By the second property above, $\sigma(n) - \sigma(n-1) = 1 - 9k$, where k is a non-negative integer. Hence

$$\sigma(n) \equiv \sigma(n-1)+1-9k \equiv \sigma(n-1)+1 \equiv n-1+1 \equiv n \pmod 9.$$

The statement follows by induction. □

This produces an algorithm to find the residue modulo 9 of a positive integer:

- If $n \leq 9$, then the residue modulo 9 is n.
- If $n \geq 10$, repeat the algorithm for $\sigma(n)$.

This algorithm converges very quickly for large numbers: the smallest number for which four iterations are required is 23 digits long, and the smallest number requiring five iterations is over two sextillion digits long!

An idea for an alternative proof of Theorem 1 would work like this: the decimal expansion of n corresponds to an expression n in terms of powers of ten. For example, $3802 = 3\cdot10^3+8\cdot10^2+2\cdot10^0$. Now, since $10 \equiv 1 \pmod 9$, we have $10^k \equiv 1 \pmod 9$. From this,

$$3802 \equiv 3\cdot10^3+8\cdot10^2+2\cdot10^0 \equiv 3+8+0+2 \equiv \sigma(3802) \pmod 9.$$

This alternative proof links to the second result that we will prove. We can write 3802 as the sum of 13 powers of ten, and in general, n can be written as the sum of $\sigma(n)$ powers of ten. We call this sum the *decimal sum* of n. Can we obtain n as the sum of fewer powers of ten?

Theorem 2. *No positive integer n can be written as the sum of fewer than $\sigma(n)$ powers of ten. The only expression of n as the sum of $\sigma(n)$ powers of ten is the decimal sum of n.*

Proof. Consider an expression of n as a minimal sum of powers of ten. By minimality, this sum does not include ten equal summands 10^a, as we could replace them by a single summand 10^{a+1}. This means that every summand appears nine times or fewer. But sums in which each power appears at most nine times are precisely decimal sums, and exactly one decimal sum adds up to n. We conclude that this is the only minimal sum of powers of ten adding up to n. □

Finally, some simple applications of Theorem 2 are the following two inequalities involving the function σ. We say that $m+n$ equals m plus n *digit by digit* if the units digit of $m+n$ equals the units digit of m plus the units digit of n, the tens digit of $m+n$ is the sum of the tens digit of m plus the tens digit of n, and so on. For example, 6789 equals 681 plus 6108 digit by digit, but 3000 does not equal 1999 plus 1001 digit by digit (because $9+1 \neq 0$ in the units digit, for example).

Theorem 3. *Let m and n be two non-negative integers. Then*

$$\sigma(m+n) \leq \sigma(m) + \sigma(n),$$
$$\sigma(mn) \leq \sigma(m)\sigma(n).$$

Moreover, in the first inequality, equality holds if and only if $m+n$ equals m plus n digit by digit.

Proof. Let $m = \sum_{i=1}^{\sigma(m)} 10^{a_i}$ and $n = \sum_{j=1}^{\sigma(n)} 10^{b_j}$ be the decimal sums of m and n. Together, they form $m+n = \sum_{i=1}^{\sigma(m)} 10^{a_i} + \sum_{j=1}^{\sigma(n)} 10^{b_j}$, which is the sum of $\sigma(m)+\sigma(n)$ powers of ten. By Theorem 2, the first inequality follows.

If $m+n$ is digit by digit m plus n, then trivially $\sigma(m+n) = \sigma(m)+\sigma(n)$. On the other hand, if $\sigma(m+n) = \sigma(m)+\sigma(n)$, then $m+n = \sum_{i=1}^{\sigma(m)} 10^{a_i} + \sum_{j=1}^{\sigma(n)} 10^{b_j}$ is the sum of $\sigma(m+n)$ powers of ten. By the equality case of Theorem 2, this must be the decimal sum of $m+n$. We can then check that this implies that $m+n$ equals m plus n digit by digit.

For the second inequality, we use again Theorem 2 for

$$mn = \left(\sum_{i=1}^{\sigma(m)} 10^{a_i}\right)\left(\sum_{j=1}^{\sigma(n)} 10^{b_j}\right) = \sum_{i=1}^{\sigma(m)}\sum_{j=1}^{\sigma(n)} 10^{a_i+b_j},$$

which is the sum of $\sigma(m)\sigma(n)$ powers of ten. $\square$

We will see another proof of the first inequality in order to answer another question. Let m and n be two non-negative integers. Consider the number $\frac{\sigma(m)+\sigma(n)-\sigma(m+n)}{9}$. By Theorem 1, this number is an integer. By Theorem 3, this number is non-negative. Could it be that this number is counting something? To provide a little bit of insight, Theorem 3 also says that this number is 0 if and only if $m+n$ equals m plus n digit by digit.

The answer to this question lies in the basic algorithm for pencil-and-paper addition. We write the decimal expressions of m and n, with the units digit of m over the units digit of n, the tens digit of m over the tens digit of n, and so on. Then we add column by column, right to left. Whenever the sum of a column is greater than 9, we only write the units digit of the result and we add a one in the next column. This is called 'carrying the one'. The number $\frac{\sigma(m)+\sigma(n)-\sigma(m+n)}{9}$ counts precisely how many times we carry the one in the pencil-and-paper addition of m and n.

Let us see why. Consider the example from Figure 1, in which we carry the one five times, and $\frac{\sigma(m)+\sigma(n)-\sigma(m+n)}{9} = \frac{35+36-24}{9} = 5$.

	+1	+1		+1			+1	+1	
		6	3	0	9	5	1	3	8
+		8	7	4	2	1	5	6	3
	1	5	0	5	1	6	7	0	1

Figure 1: Example of sum with pencil-and-paper algorithm.

The value of $\sigma(m)+\sigma(n)-\sigma(m+n)$ is equal to the sum of the contributions of each column in the sum. We can see that, if no

one is carried from or to a column (purple column), then the sum of the digits of m and n in that column minus the corresponding digit of $m+n$ equals 0. If a one is carried from a column, but not to that column (red columns), then it equals 10. If it is carried to a column but not from that column (green), then it equals -1. Finally, if it is carried from and to a column (blue), then it equals 9.

The net effect of a carry is that the contribution of the column to $\sigma(m)+\sigma(n)-\sigma(m+n)$ increases by 10 in that column, and decreases by 1 in the next, hence the number of carries equals $\frac{\sigma(m)+\sigma(n)-\sigma(m+n)}{9}$.

A similar argument can be used to prove the second inequality from Theorem 3 using the pencil-and-paper algorithm for multiplication, but this time the argument is more subtle. Remember that the algorithm has two steps, in the first we multiply a by each digit of b, and in the second step we add those numbers. To prove the inequality, first bound the sum of digits of each of the numbers produced in the first step by $\sigma(a)$ times each digit of b, then use the first inequality from Theorem 3.

The results seen here are not unique to base 10. We can define analogous results in any base b. If σ_b denotes the sum of digits in base b, then:

- $\sigma_b(n) \equiv n \pmod{b-1}$.
- n cannot be written as the sum of fewer than $\sigma_b(n)$ powers of b. The only way of writing n as the sum of $\sigma_b(n)$ powers of b is the base b sum of n.
- $\sigma_b(m+n) \leq \sigma_b(m)+\sigma_b(n)$, with equality if and only if $m+n$ equals m plus n digit by digit in base b.
- $\sigma_b(mn) \leq \sigma_b(m)\sigma_b(n)$.

Before seeing applications of these results, it is worth mentioning that there are two other categories of olympiad problems which involve the function σ. The first is problems in which a certain function is defined by a recurrence, and then one is asked to prove a property of this function. It is common for this function to be a function related to σ_b for some b.

A common pattern in this kind of problems is that the recurrence usually gives the value of $f(bk), f(bk+1)\ldots, f(bk+(b-1))$ as a function of $f(k)$. See Spain 2010, Problem 2 for an example of this.

The other kind of problem which involves the function σ is one in which one has to prove the existence of n, usually of infinitely many such n, for which some equation involving n holds. For example, in the problem MH-27 in this journal, one is asked to prove that for every positive integer m there is n such that $\sigma(n^2) = 4^m$.

The key to problems like this is often finding an adequate pattern for the digits of n so that the equation is satisfied. For example, the problem mentioned above can be solved by taking n of the form $333\ldots32$, or by taking n of the form $1101000100\ldots001$, where the 1s are in the positions which are powers of two. To find this kind of patterns, it often helps to think of σ using the interpretation given in Theorem 2.

3 Applications

Problem 1 (Spain 2001). *The integers from 1 to 9 are arranged in a 3×3 grid. The following six three-digit numbers are then added: the three numbers produced by reading each row left to right, and the three columns read top to bottom. Can the resulting sum be 2001?*

Solution. Call the numbers in the rows a, b and c, and the numbers in the columns d, e and f. Then $\sigma(a)+\sigma(b)+\sigma(c) = 1+2+\cdots+9 = 45$. Similarly, $\sigma(d)+\sigma(e)+\sigma(f) = 45$. Then

$$\begin{aligned} a+b+c+d+e+f \equiv& \sigma(a)+\sigma(b)+\sigma(c)+\sigma(d)+\sigma(e)+\sigma(f) \\ \equiv& 45+45 \\ \equiv& 0 \pmod 9. \end{aligned}$$

But $2001 \equiv \sigma(2001) \equiv 3 \pmod 9$, so we cannot have $a+b+c+d+e+f = 2001$. $\square$

Problem 2 (Canada 2011). *Consider* 70*-digit numbers* n*, with the property that each of the digits* $1, 2, 3, \ldots, 7$ *appears in the decimal expansion of* n *ten times (and* 8*,* 9 *and* 0 *do not appear). Show that no number of this form can divide another number of this form.*

Solution. Every number of this form has $\sigma(n) = 10(1 + 2 + \cdots + 7) = 280$, which is congruent to 1 modulo 9. If a number of this form n_1 divides another n_2, then $\frac{n_2}{n_1} \equiv 1 \pmod 9$. By Theorem 1, $n \equiv 1 \pmod 9$. Since we cannot have $\frac{n_2}{n_1} = 1$ (as $n_1 \neq n_2$), we must have $\frac{n_2}{n_1} \geq 10$.

In addition, n_1 and n_2 have 70 digits and the leading digits are nonzero, so $10^{69} \leq n_1, n_2 < 10^{70}$. But then

$$n_2 < 10^{70} = 10 \cdot 10^{69} \leq 10n_1 \leq n_2,$$

and we reach a contradiction. □

Problem 3. *Let* n *be a positive integer, and* $a = 10^n - 1$*. Prove that for any positive integer* b*,* $\sigma(ab) \geq 9n$*.*

Solution. Assume the opposite. Consider the smallest positive integer m divisible by a with $\sigma(m) < 9n$. By Theorem 2, this is the smallest multiple of a which can be written as the sum of fewer than $9n$ powers of ten. None of the powers 10^k in its decimal sum are greater than or equal to 10^n, since otherwise we could replace them by 10^{k-n} and the result would not change modulo a (and would still consist of fewer than $9n$ powers of 10). But this means $m < 10^n$, and the only multiple of a smaller than 10^n is a itself, which has $\sigma(a) = \sigma(99\ldots99) = 9n$. Contradiction. □

Problem 4 (Ibero 2014). *Find the smallest positive integer* k *such that*

$$\sigma(k) = \sigma(2k) = \sigma(3k) = \cdots = \sigma(2013k) = \sigma(2014k).$$

Solution. Since $9|\sigma(9k) = \sigma(k)$, we have $9|k$. This means that $9999|1111k$, and by Problem 3, $\sigma(k) = \sigma(1111k) \geq 36$, which means $k \geq 9999$. We claim that this is the minimal value. Indeed,

if $k = 9999$, then for any $1 \leq a \leq 2014$,

$$\begin{aligned}\sigma(9999a) =& \sigma(10000a - a)\\ =& \sigma(10000(a-1) + (10000 - a))\\ =& \sigma(a-1) + \sigma(10000 - a)\\ =& \sigma(9999).\end{aligned}$$

□

Problem 5 (AMJ MH-13). *Let a, b and c be three positive integers, each of them without repeated digits in their decimal expression. If $a + b + c = 58888$, prove that there is a digit, other than 0, which appears in at least two of the three numbers.*

Solution. If no digit other than 0 appears in two of the three numbers, then each number appears at most once, and $\sigma(a) + \sigma(b) + \sigma(c) \leq 1 + 2 + \cdots + 9 = 45$. We also have $\sigma(a) + \sigma(b) + \sigma(c) \equiv a + b + c \equiv \sigma(a + b + c) \equiv 37 \pmod 9$. Combining both results, $\sigma(a)+\sigma(b)+\sigma(c) \leq 37$. Finally, $\sigma(a)+\sigma(b)+\sigma(c) \geq \sigma(a+b+c) = 37$. We conclude that $\sigma(a) + \sigma(b) + \sigma(c) = 37$.

$\sigma(a + b + c) = \sigma(a) + \sigma(b) + \sigma(c)$, so $a + b + c$ equals a plus b plus c digit by digit. $a + b + c = 58888$ does not have any 9, so neither a nor b nor c can have a 9. The digits are then between 0 and 8. But then $37 = \sigma(a) + \sigma(b) + \sigma(c) \leq 1 + 2 + \cdots + 8 = 36$, contradiction. □

4 Exercises

Problem 6 (Spain 2001). *Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying $f(1) = f(2^n) = 1$ and, if $n < 2^s$, then $f(2^s + n) = f(n) + 1$.*

- *Find the maximum value of $f(n)$ for $n \leq 2001$.*
- *Find the smallest n such that $f(n) = 2001$.*

Problem 7 (Poland 2012). *Prove that there exist infinitely many positive integers n such that $\sigma(2^n + n) < \sigma(2^n)$.*

Problem 8 (IMO 1975). *Let* $A = \sigma(4444^{4444})$ *and* $B = \sigma(A)$. *Find* $\sigma(B)$.

Problem 9 (Brazil 2014). *Find all integers* n, $n > 1$, *with the following property: for all* k, $0 \leq k < n$, *there exists a multiple of* n *whose digits sum leaves a remainder of* k *when divided by* n.

Problem 10 (Argentina 1999). *Let* a, b, c *be positive integers. Assume that* $\max\{\sigma(a+b), \sigma(b+c), \sigma(c+a)\} < 5$. *Prove that we can have* $\sigma(a+b+c) > 50$, *but not* $\sigma(a+b+c) > 60$.

References

[1] Andreescu, T., Andrica, D., and Feng, Z. *104 Number Theory Problems*. Birkhäuser, Boston, 2006, pp. 49–50. ISBN: 978-0-8176-4527-4.

Ander Lamaison Vidarte
Berlin Mathematical School (BMS)
Berlin, Germany
alamaisonv@gmail.com

Solutions

No problem is ever permanently closed. We will be very pleased to consider for publication new solutions or comments on the past problems.

Please, send submittals to: **José Luis Díaz-Barrero**, Enginyeria Civil i Ambiental, UPC BARCELONATECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain, or by e-mail to:
jose.luis.diaz@upc.edu

Elementary Problems

E–23. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let M be the midpoint of side BC in a triangle ABC. Show that

$$\frac{AB^2 + AC^2}{MA^2 + MB^2}$$

is an integer and determine its value.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. Let $D = A + 2\overrightarrow{AM}$. Then, $ABDC$ is a parallelogram. The parallelogram identity states that the sum of the squares of the lengths of the four sides of a parallelogram

equals the sum of the squares of the lengths of the two diagonals. In our case, $2(AB^2 + AC^2) = (2MA)^2 + (2MB)^2$. Therefore $AB^2 + AC^2 = 2(MA^2 + MB^2)$, so

$$\frac{AB^2 + AC^2}{MA^2 + MB^2} = 2.$$

Solution 2 by the proposer. On account of Pythagoras theorem, from the figure, we have

$$\begin{aligned} AB^2 &= AH^2 + BH^2 = AH^2 + (BM - HM)^2 \\ &= AH^2 + BM^2 + HM^2 - 2 \cdot BM \cdot HM \\ &= AM^2 + BM^2 - 2 \cdot BM \cdot HM \end{aligned}$$

and

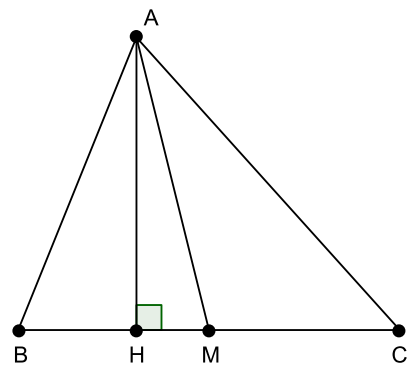

Figure 1: Construction for the solution of Problem E–23.

$$\begin{aligned} AC^2 &= AH^2 + HC^2 = AH^2 + (HM + MC)^2 \\ &= AH^2 + HM^2 + MC^2 + 2 \cdot HM \cdot MC \\ &= AM^2 + MC^2 + 2 \cdot HM \cdot MC. \end{aligned}$$

Adding up the preceding expressions yields

$$AB^2 + AC^2 = 2(MA^2 + MB^2),$$

so

$$\frac{AB^2 + AC^2}{MA^2 + MB^2} = 2.$$

Solution 3 by Alberto Espuny-Díaz, BarcelonaTech, Barcelona, Spain. Apply de Law of cosines for MA. We have that

$$MA^2 = MB^2 + AB^2 - 2\,MB \cdot AB\,\cos B.$$

Taking into accout that M is the midpoint of BC, we can write

$$MA^2 = \frac{1}{4}BC^2 + AB^2 - BC \cdot AB\,\cos B.$$

Then, again by the Law of cosines,

$$-BC \cdot AB\,\cos B = \frac{1}{2}(AC^2 - AB^2 - BC^2).$$

Hence,

$$\begin{aligned} MA^2 + MB^2 &= \frac{1}{4}BC^2 + AB^2 + \frac{1}{2}(AC^2 - AB^2 - BC^2) + \frac{1}{4}BC^2 \\ &= \frac{1}{2}(AB^2 + AC^2). \end{aligned}$$

Substituting this in the statement yields

$$\frac{AB^2 + AC^2}{MA^2 + MB^2} = 2.$$

Solution 4 by Fernando Ballesta Yagüe, I.E.S. Infante don Juan Manuel, Murcia, Spain. With the notation of Figure 2, we want to find

$$\frac{AB^2 + AC^2}{MA^2 + MB^2} = \frac{b^2 + c^2}{m_a^2 + \frac{a^2}{4}}.$$

By Stewart's theorem,

$$m_a^2 = \frac{\frac{a}{2}b^2 + \frac{a}{2}c^2}{a} - \frac{a^2}{4} = \frac{b^2 + c^2}{2} - \frac{a^2}{4} \implies m_a^2 + \frac{a^2}{4} = \frac{b^2 + c^2}{2}.$$

Therefore,

$$\frac{AB^2 + AC^2}{MA^2 + MB^2} = \frac{b^2 + c^2}{m_a^2 + \frac{a^2}{4}} = \frac{b^2 + c^2}{\frac{b^2+c^2}{2}} = 2.$$

Also solved by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and Victor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain.

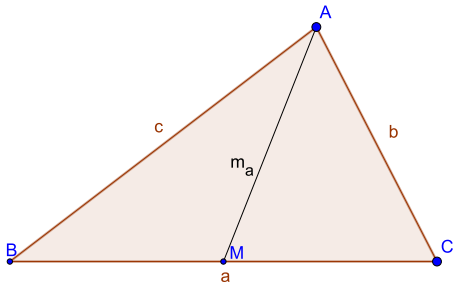

Figure 2: Construction for the solution of Problem E–23.

E–24. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $a_1, a_2, \ldots, a_n$ be positive numbers such that $a_1 a_2 \ldots a_n = 1$. Prove that

$$\frac{1}{n}\left[\left(1+\frac{1}{a_1 a_n}\right)^2+\left(1+\frac{1}{a_2 a_{n-1}}\right)^2+\ldots+\left(1+\frac{1}{a_n a_1}\right)^2\right] \geq 4.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. The conclusion follows by the AM-GM inequality. Notice that $\left(1+\frac{1}{a_i a_j}\right)^2 = 1+\frac{2}{a_i a_j}+\frac{1}{a_i^2 a_j^2}$, and therefore the left-hand side of the inequality is

$$\begin{aligned} LHS &= 1+2\frac{\sum_{i=1}^{n}\frac{1}{a_i a_{n+1-i}}}{n}+\frac{\sum_{i=1}^{n}\frac{1}{a_i^2 a_{n+1-i}^2}}{n} \\ &\geq 1+2\frac{1}{\sqrt[n]{\prod_{i=1}^{n} a_i^2}}+\frac{1}{\sqrt[n]{\prod_{i=1}^{n} a_i^4}} = 4. \end{aligned}$$

Solution 2 by Fernando Ballesta Yagüe, I.E.S. Infante don Juan Manuel, Murcia, Spain. We can use the AM-GM inequality

because every a_i is positive. Then,

$$1+\frac{1}{a_1a_n}\geq 2\sqrt{\frac{1}{a_1a_n}} \implies \left(1+\frac{1}{a_1a_n}\right)^2\geq\frac{4}{a_1a_n}.$$

Applying this to all the summands yields

$$\left(1+\frac{1}{a_1a_n}\right)^2+\ldots+\left(1+\frac{1}{a_na_1}\right)^2\geq\frac{4}{a_1a_n}+\ldots+\frac{4}{a_na_1}.$$

By the AM-GM inequality,

$$\frac{4}{a_1a_n}+\ldots+\frac{4}{a_na_1}\geq n\sqrt[n]{\frac{4}{a_1a_n}\cdot\ldots\cdot\frac{4}{a_na_1}}=n\sqrt[n]{\frac{4^n}{(a_1a_2\ldots a_n)^2}}.$$

As $a_1a_2\ldots a_n=1$, we have that

$$\left(1+\frac{1}{a_1a_n}\right)^2+\left(1+\frac{1}{a_2a_{n-1}}\right)^2+\ldots+\left(1+\frac{1}{a_na_1}\right)^2\geq 4n$$

and the result follows directly.

Solution 3 by the proposer. Computing the square root of both sides of the claimed inequality, we get

$$\sqrt{\frac{1}{n}\left[\left(1+\frac{1}{a_1a_n}\right)^2+\left(1+\frac{1}{a_2a_{n-1}}\right)^2+\ldots+\left(1+\frac{1}{a_na_1}\right)^2\right]}\geq 2.$$

On account of QM-AM-GM inequalities, we have

$$\begin{aligned}
&\sqrt{\frac{1}{n}\left[\left(1+\frac{1}{a_1a_n}\right)^2+\left(1+\frac{1}{a_2a_{n-1}}\right)^2+\ldots+\left(1+\frac{1}{a_na_1}\right)^2\right]}\\
&\geq\frac{1}{n}\left[\left(1+\frac{1}{a_1a_n}\right)+\left(1+\frac{1}{a_2a_{n-1}}\right)+\ldots+\left(1+\frac{1}{a_na_1}\right)\right]\\
&\geq\sqrt[n]{\left(1+\frac{1}{a_1a_n}\right)\left(1+\frac{1}{a_2a_{n-1}}\right)\ldots\left(1+\frac{1}{a_na_1}\right)}\\
&\geq\sqrt[n]{2^n\sqrt{\frac{1}{(a_1a_2\ldots a_n)^2}}}=2
\end{aligned}$$

on account of the constraint and the fact that, for $1 \leq i \leq n$, it is true that

$$\left(1 + \frac{1}{a_i a_{n+1-i}}\right) \geq 2\sqrt{\frac{1}{a_i a_{n+1-i}}}.$$

Equality occurs when $a_1 = a_2 = \ldots = a_n = 1$, and we are done.

Also solved by Alberto Espuny-Díaz, BarcelonaTech, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain.

E–25. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Find the smallest positive integer whose remainder is 2 when divided by 5, 3 when divided by 7, and 4 when divided by 11.

Solution 1 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. We have to find the smallest positive integer x that can be written as $x = 2 + 5a = 3 + 7b = 4 + 11c$ for some integers a, b, c. The Chinese Remainder Theorem tells us that the system above has a unique solution modulo $5 \times 7 \times 11 = 385$.

From the equation $3 + 7b = 4 + 11c$, we have $b = \frac{1+11c}{7}$. In order for b to be an integer, it is easy to see that c must be of the form $5 + 7k$, from where we get $b = 8 + 11k$.

From $2 + 5a = 3 + 7b$, we also have $a = \frac{2+11c}{5} = \frac{2+55+77k}{5}$. If we want a to be an integer, k must satisfy $k = 4+5m$ for some integer m. Therefore, $a = \frac{2+55+308+385m}{5}$, and so $x = 2+5a = 367+385m$. Setting $m = 0$, we have the smallest positive solution, $x = 367$.

Solution 2 by Fernando Ballesta Yagüe, I.E.S. Infante don Juan Manuel, Murcia, Spain. Let this number be x. As its remainder when divided by 5 is 2, $x = 5k_1 + 2$. When divided by 7

its remainder is 3, so

$$\begin{aligned} x = 5k_1 + 2 \equiv 3 \pmod 7 &\implies 5k_1 \equiv 1 \pmod 7 \\ &\implies k_1 \equiv 3 \pmod 7 \\ &\implies k_1 = 7k_2 + 3 \\ &\implies x = 5(7k_2 + 3) + 2 = 35k_2 + 17. \end{aligned}$$

Finally, as its remainder is 4 when divided by 11,

$$\begin{aligned} x = 35k_2 + 17 \equiv 4 \pmod{11} &\implies 2k_2 + 6 \equiv 4 \pmod{11} \\ &\implies 2k_2 \equiv 9 \pmod{11} \\ &\implies k_2 \equiv 10 \pmod{11} \\ &\implies k_2 = 11k_3 + 10 \\ &\implies x = 35(11k_3 + 10) + 17 \\ &\implies x = 385k_3 + 367. \end{aligned}$$

Therefore, the smallest positive integer that fulfills the conditions is 367, obtained by taking $k_3 = 0$.

Also solved by Alberto Espuny-Díaz, BarcelonaTech, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and the proposer.

E–26. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Compute

$$\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\cdots\left(1 - \frac{1}{2015^2}\right)\left(1 - \frac{1}{2016^2}\right).$$

Solution 1 by Alberto Espuny-Díaz, BarcelonaTech, Barcelona, Spain. We can write each of the terms we are multiplying as $1 - \frac{1}{i^2}$, for $2 \le i \le 2016$. This can be rewritten as $\frac{i^2 - 1}{i^2}$. The numerator can then be written as the product of $i-1$ and $i+1$, so each of the terms is $\frac{(i-1)(i+1)}{i \cdot i}$. If all of these are put together,

then, we have that

$$\begin{aligned}
&\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{2015^2}\right)\left(1-\frac{1}{2016^2}\right)\\
=&\frac{1\cdot 3}{2\cdot 2}\frac{2\cdot 4}{3\cdot 3}\frac{3\cdot 5}{4\cdot 4}\cdots\frac{2014\cdot 2016}{2015\cdot 2015}\frac{2015\cdot 2017}{2016\cdot 2016}\\
=&\frac{1}{2}\frac{3\cdot 2}{2\cdot 3}\frac{4\cdot 3}{3\cdot 4}\cdots\frac{2016\cdot 2015}{2015\cdot 2016}\frac{2017}{2016}=\frac{2017}{2\cdot 2016}=\frac{2017}{4032}.
\end{aligned}$$

Solution 2 by the proposer. Factoring terms, we have

$$\begin{aligned}
&\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{2015^2}\right)\left(1-\frac{1}{2016^2}\right)\\
=&\left(1+\frac{1}{2}\right)\left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{3}\right)\cdots\left(1+\frac{1}{2016}\right)\left(1-\frac{1}{2016}\right)\\
=&\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\cdots\left(1+\frac{1}{2015}\right)\left(1+\frac{1}{2016}\right)\\
&\cdot\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\cdots\left(1-\frac{1}{2015}\right)\left(1-\frac{1}{2016}\right)\\
=&\left(\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{2017}{2016}\right)\cdot\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{2015}{2016}\right)\\
=&\frac{2017}{2}\cdot\frac{1}{2016}=\frac{2017}{4032}.
\end{aligned}$$

Solution 3 by Lucía Ma Li, IES Isabel de España, and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. Let $\{a_n\}_{n\geq 1}$ be the sequence defined by $a_n=\prod_{k=2}^{n+1}\left(1-\frac{1}{k^2}\right)$, so the problem asks for the value of a_{2015}. It may be proved by induction that for n even, $a_n=\frac{\frac{n}{2}+1}{n+1}$, while if n is odd, then $a_n=\frac{n+2}{2(n+1)}$. For $n=1$ and $n=2$ we have $a_1=\frac{3}{4}$ and $a_2=\frac{2}{3}$ which verify the claimed formulas. Let us assume the a_n also follows our formulas. Then for a_{n+1} we have to consider two cases:

1) If n is even then

$$\begin{aligned} a_{n+1} &= a_n\left(1-\frac{1}{(n+2)^2}\right) \\ &= \frac{\frac{n}{2}+1}{n+1}\cdot\frac{(n+2)^2-1}{(n+2)^2} \\ &= \frac{(n+2)(n+1)(n+3)}{2(n+1)(n+2)^2} \\ &= \frac{n+3}{2(n+2)}. \end{aligned}$$

2) On the other hand, if n is odd then

$$\begin{aligned} a_{n+1} &= a_n\left(1-\frac{1}{(n+2)^2}\right) \\ &= \frac{n+2}{2(n+1)}\cdot\frac{(n+2)^2-1}{(n+2)^2} \\ &= \frac{(n+1)(n+3)}{2(n+1)(n+2)} \\ &= \frac{\frac{n+1}{2}+1}{n+2}. \end{aligned}$$

In our problem, $a_{2015} = \dfrac{2017}{4032}$.

Also solved by Fernando Ballesta Yagüe, I.E.S. Infante don Juan Manuel, Murcia, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain.

E–27. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $n \geq 1$ be an integer. Show that the polynomial

$$(x-1)\left[(x-3)^{2(n+1)}+3\right]-x^2$$

is a multiple of $x-2$.

Solution 1 by Fernando Ballesta Yagüe, I.E.S. Infante don Juan Manuel, Murcia, Spain. By the factor theorem, the given polynomial is divisible by $x-2$ if, and only if, 2 is a root of the polynomial, that is $P(2)=0$. By substituting, we find that

$$\begin{aligned} P(2) &= (2-1)\big[(2-3)^{2(n+1)}+3\big]-2^2 \\ &= \big[(-1)^{2(n+1)}+3\big]-4 = 1+3-4=0, \end{aligned}$$

and therefore, the polynomial is a multiple of $x-2$.

Solution 2 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany. Denote the polynomial from the statement as $P_n(x)$. We will prove that $x-2$ divides all polynomial P_n for $n \geq -1$. We will proceed by induction. For $n=-1$, we obtain $P_{-1}(x)=(x-1)[(x-3)^0+3]-x^2=-(x-2)^2$, which is divisible by $x-2$. Now, suppose that the statement is true for n, and we want to prove it for $n+1$. We just need to see that $P_{n+1}(x)-P_n(x)$ is divisible by $x-2$. But this is true because

$$\begin{aligned} P_{n+1}(x)-P_n(x) &= (x-1)(x-3)^{2(n+2)}-(x-1)(x-3)^{2(n+1)} \\ &= (x-1)(x-3)^{2(n+1)}\big((x-3)^2-1\big) \\ &= (x-1)(x-3)^{2(n+1)}(x-4)(x-2). \end{aligned}$$

Solution 3 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. Let $y=x-2$. We have to prove that for some polynomial $p(y)$

$$(y+1)\big[(y-1)^{2(n+1)}+3\big]-(y+2)^2 = yp(y).$$

Consider the polynomial $q(y) = -\sum_{k=1}^{2n+2}\binom{2n+2}{k}(-y)^{k-1}$. Using

binomial expansion:

$$\begin{aligned}
&(y+1)\big[(y-1)^{2(n+1)}+3\big]-(y+2)^2\\
=&(y+1)\left(1+\sum_{k=1}^{2n+2}\binom{2n+2}{k}y^k(-1)^{2n+2-k}\right)\\
&+3(y+1)-y^2-4y-4\\
=&y+1+(y+1)\sum_{k=1}^{2n+2}\binom{2n+2}{k}(-y)^k-y^2-y-1\\
=&-y(y+1)\sum_{k=1}^{2n+2}\binom{2n+2}{k}(-y)^{k-1}-y^2\\
=&y(y+1)q(y)-y^2=y[(y+1)q(y)-y].
\end{aligned}$$

Letting $p(y)=(y+1)q(y)-y$ we obtain the desired result.

Also solved by Alberto Espuny-Díaz, BarcelonaTech, Barcelona, Spain; Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain, and the proposer.

E–28. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let x,y,z,t be positive real numbers such that $xyzt=1$. Prove that

$$t(x^2+yz)(y+z)\geq 4.$$

Solution 1 by Lucía Ma Li, IES Isabel de España, and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. The conclusion follows immediately by the AM-GM inequality:

$$t\left(\frac{x^2+yz}{2}\right)\left(\frac{y+z}{2}\right)\geq t\sqrt{x^2yz}\sqrt{yz}=txyz=1.$$

Solution 2 by Fernando Ballesta Yagüe, I.E.S. Infante don Juan Manuel, Murcia, Spain. We can expand the left side of the inequality,

$$t(x^2+yz)(y+z)=t(x^2y+x^2z+y^2z+z^2y)=tx^2y+tx^2z+ty^2z+tz^2y.$$

By the AM-GM inequality (which we can apply because x, y, z and t are positive) we have that

$$tx^2y + tx^2z + ty^2z + tz^2y \geq 4\sqrt[4]{tx^2y \cdot tx^2z \cdot ty^2z \cdot tz^2y} = 4xyzt.$$

The result follows, as $xyzt = 1$.

Solution 3 by the proposer. On account of the constraint, we have

$$\begin{aligned} t(x^2+yz)(y+z) &= t(x^2(y+z)+yz(y+z)) \\ &= zt(x^2+y^2)+yt(x^2+z^2) \\ &= \frac{zt(x^2+y^2)+yt(x^2+z^2)}{xyzt} \\ &= \frac{x^2+y^2}{xy}+\frac{x^2+z^2}{xz} \\ &= \frac{x}{y}+\frac{y}{x}+\frac{x}{z}+\frac{z}{x}. \end{aligned}$$

Since $t + \dfrac{1}{t} \geq 2$ holds for all real number $t > 0$ (as can be easily proved), then

$$t(x^2+yz)(y+z) = \frac{x}{y}+\frac{y}{x}+\frac{x}{z}+\frac{z}{x} \geq 2+2 = 4.$$

Equality holds when $x = y = z = t = 1$, and we are done.

Also solved by Alberto Espuny-Díaz, BarcelonaTech, Barcelona, Spain; Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and the proposer.

Easy–Medium Problems

EM–23. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* In any triangle ABC, with the usual notations, prove that the identity

$$\sum_{\text{cyclic}} \sqrt{\frac{\sin A}{\sin B \sin C}} = \sqrt{\frac{2R}{r}\left(\sum_{\text{cyclic}} \sin A\right)}$$

holds.

Solution 1 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. We have to take into account the Law of Sines: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, and that $r\frac{a+b+c}{2} = \mathcal{A} = \frac{abc}{4R}$, where $\mathcal{A}$ denotes the area of the triangle. We have

$$\begin{aligned}\sum_{\text{cyc}} \sqrt{\frac{\sin A}{\sin B \sin C}} &= \sum_{\text{cyc}} \sqrt{\frac{a}{2R}\frac{2R}{b}\frac{2R}{c}} = \sum_{\text{cyc}} \sqrt{\frac{a}{bc}2R} = \sum_{\text{cyc}} \sqrt{\frac{a}{bc}\frac{abc}{2\mathcal{A}}} \\ &= \frac{a+b+c}{\sqrt{2\mathcal{A}}} = \sqrt{\frac{a+b+c}{\frac{2\mathcal{A}}{a+b+c}}} = \sqrt{\frac{a+b+c}{r}} \\ &= \sqrt{\frac{1}{r}\sum_{\text{cyc}} a} = \sqrt{\frac{2R}{r}\sum_{\text{cyc}} \frac{a}{2R}} = \sqrt{\frac{2R}{r}\sum_{\text{cyc}} \sin A},\end{aligned}$$

proving thus the identity.

Solution 2 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany. Denote $x = \sin A$, $y = \sin B$ and $z = \sin C$. From the Law of Sines, we have $a = 2Rx$, $b = 2Ry$ and $c = 2Rz$. Moreover, if h_A is the length of the altitude from A, then $h_A = bz$, so if $[ABC]$ is the area of the triangle, then we can write the following identity:

$$2[ABC] = ah_A = abz = R^2xyz.$$

Another way of expressing $2[ABC]$ is the following:

$$2[ABC] = r(a+b+c) = Rr(x+y+z).$$

Dividing one expression by the other,

$$\frac{R}{r} = \frac{x+y+z}{xyz}.$$

This is enough to deduce

$$\begin{aligned}\sqrt{\frac{x}{yz}} + \sqrt{\frac{y}{zx}} + \sqrt{\frac{z}{xy}} &= \sqrt{\frac{x^2}{xyz}} + \sqrt{\frac{y^2}{xyz}} + \sqrt{\frac{z^2}{xyz}} \\ &= \frac{x+y+z}{\sqrt{xyz}} \\ &= \sqrt{\frac{x+y+z}{xyz}(x+y+z)} \\ &= \sqrt{\frac{R}{r}(x+y+z)},\end{aligned}$$

which is indeed the proposed identity.

Also solved by Fernando Ballesta Yagüe, I.E.S. Infante don Juan Manuel, Murcia, Spain, and the proposer.

EM–24. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $p \neq 5$ be an odd prime number. Show that there exists a positive integer k for which the number

$$N = \underbrace{1111\ldots111}_{k}$$

satisfies that $N \equiv 0 \pmod{p}$.

Solution 1 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany. Let a_n be the number with n digits, all of them 1. The residue modulo p of a_n is in $\{0, 1, \ldots, p-1\}$

so, by pigeonhole principle, two numbers $a_i < a_j$ from the set $\{a_1, a_2, \ldots, a_{p+1}\}$ have the same residue modulo p. Then

$$p|a_j - a_i = \underbrace{11\ldots11}_{j} - \underbrace{11\ldots11}_{i} = \underbrace{11\ldots11}_{j-i}\underbrace{00\ldots00}_{i} = 10^i a_{j-i}.$$

Since p does not divide 10^i (as $p \neq 2$ and $p \neq 5$), p must divide a_{j-i}.

Solution 2 by Fernando Ballesta Yagüe, I.E.S. Infante don Juan Manuel, Murcia, Spain. We have

$$N = 11 \overset{k)}{\ldots} 1 = \frac{99 \overset{k)}{\ldots} 9}{9} = \frac{10^k - 1}{9}.$$

We need to find some value k such that $\dfrac{10^k - 1}{9}$ is divisible by p. We need to consider two cases.

First, consider the case $p = 3$. This case is different because it is not sufficient to find a value of k such that $10^k - 1$ is divisible by $p = 3$. As $10^k - 1$ is divided by 9, we need to find some k such that $10^k - 1$ is divisible by 27. However, there is a shorter solution: a number is divisible by 3 if the sum of its digits is divisible by 3. Therefore, in this case k has to be a multiple of 3, because the sum of the digits of N is k (e.g. 111).

Now consider the general case $p \neq 3$. In this case we just need to focus on the numerator.

$$10^k - 1 \equiv 0 \pmod{p} \implies 10^k \equiv 1 \pmod{p},$$

so as $\varphi(p) = p - 1$, by Euler's theorem $k = (p-1)t$ satisfies the condition for all $t \in \mathbb{Z}^+$.

Solution 3 by the proposer. First, we claim that if $p \nmid n$ and $p \nmid (n-1)$ then $p \,|\, (1 + n + n^2 + \ldots + n^{p-2})$. Indeed, on account of Fermat's little theorem, if $p \nmid n$ and $p \nmid (n-1)$, then we have

$$1 + n + n^2 + \ldots + n^{p-2} = \frac{n^{p-1} - 1}{n - 1} \equiv 0 \pmod{p}.$$

That is, $p \mid (1 + n + n^2 + \ldots + n^{p-2})$ as claimed.

For $n = 10$ we observe that all odd primes distinct from 5 do not divide n, and the only prime that divides $n - 1 = 10 - 1 = 9$ is 3, that also divides $N = 111$. For all the remaining odd primes p, on account of the claim, we have that

$$p \mid (1 + 10 + 10^2 + \ldots + 10^{p-2}) = \underbrace{1111\ldots111}_{p-1},$$

and we are done.

Also solved by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain.

EM–25. *Proposed by Alberto Espuny-Díaz, CFIS, BarcelonaTech, Barcelona, Spain.* Consider a rhombus. Let one of its diagonals be called a_1, and the other, b_1. Each of these diagonals divides the rhombus in two isosceles triangles, for each of which the incenter can be obtained. The four incenters determine a new rhombus of diagonals a_2 and b_2. Prove that by repeating this process, the figure tends to a square.

Solution 1 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany. The shape of a rhombus is determined by the ratio of its two diagonals. If we can prove that this ratio tends to 1, then the figure will tend to a square. Let $ABCD$ be a rhombus, E the point where the diagonals meet, and $A'B'C'D'$ be the rhombus produced from $ABCD$ after one iteration. By the symmetry of the rhombus, A' is in the segment EA, B' is in EB', etc.

Suppose WLOG that $EC = EA \geq EB = ED$. Denote $r = \frac{EA}{EB}$, and $r' = \frac{EA'}{EB'}$. Now BA' is the bisector of $\angle ABE$, which means that $\frac{AA'}{EA'} = \frac{AB}{EB}$. Adding 1 on both sides, $\frac{EA}{EA'} = \frac{AB+EB}{EB}$, or equivalently,

$$EA' = \frac{EA \cdot EB}{AB + EB}.$$

Analogously, $EB' = \frac{EA \cdot EB}{AB+EA}$. Dividing one expression by the other,

$$r' = \frac{EA'}{EB'} = \frac{\frac{EA \cdot EB}{AB+EB}}{\frac{EA \cdot EB}{AB+EA}} = \frac{AB+EA}{AB+EB}.$$

Now notice that $\frac{AB}{EB} = \frac{\sqrt{EA^2+EB^2}}{EB} = \sqrt{\frac{EA^2}{EB^2}+1} = \sqrt{r^2+1}$. Then

$$r' = \frac{AB+EA}{AB+EB} = \frac{\frac{AB+EA}{EB}}{\frac{AB+EB}{EB}} = \frac{\sqrt{r^2+1}+r}{\sqrt{r^2+1}+1} =: f(r).$$

We can observe that

$$1 = \frac{\sqrt{r^2+1}+1}{\sqrt{r^2+1}+1} < f(r) < \frac{r\sqrt{r^2+1}+r}{\sqrt{r^2+1}+1} = r, \tag{1}$$

which means that the ratio of the diagonals is decreasing and bounded by 1. As a consequence the limit of the ratio exists and is at least 1. This limit must be a fixed point of f, since f is continuous. But by (1), there is no fixed point of f in $(1, +\infty)$, meaning that the limit is 1.

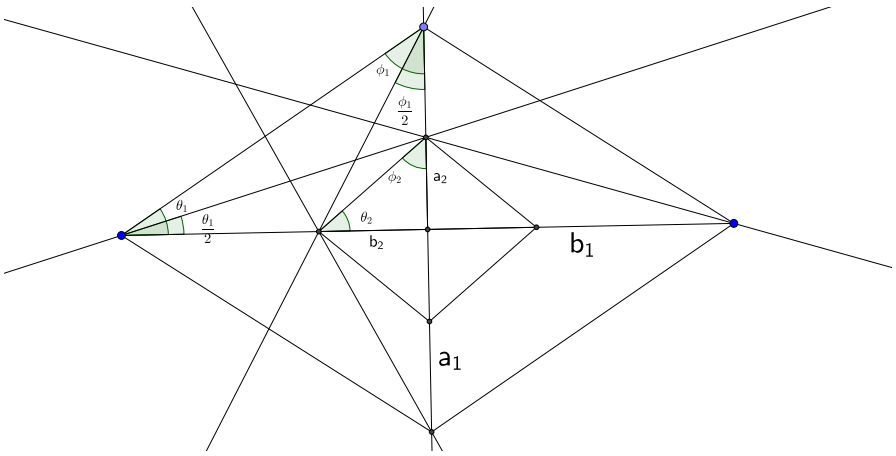

Figure 3: Construction for problem EM-25

Solution 2 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. From Figure 3, let $\tan\theta_n = \frac{a_n}{b_n}$, $\tan\phi_n = \frac{b_n}{a_n} = \tan\left(\frac{\pi}{2} - \theta_n\right) = \cot\theta_n$. If the figure tends to a square, it will be

satisfied that $\lim_{n\to\infty}\tan\theta_n = \lim_{n\to\infty}\tan\phi_n = \tan\frac{\pi}{4} = 1$. As $\theta_n = \frac{\pi}{2} - \phi_n$ for all n, the following trigonometric relations hold: $\sin\theta_n = \cos\phi_n$, $\cos\theta_n = \sin\phi_n$. We can assume WLOG that $\theta_1 \le \phi_1$, which means $\theta_1 \in \left(0, \frac{\pi}{4}\right]$. Since the incenter of a triangle is the intersection of its bisectors,

$$\begin{aligned}
\tan\theta_{n+1} &= \frac{a_{n+1}}{b_{n+1}} = \frac{\frac{a_{n+1}}{b_n}\frac{b_n}{a_n}}{\frac{b_{n+1}}{a_n}} = \frac{\tan\left(\frac{\theta_n}{2}\right)\tan\phi_n}{\tan\left(\frac{\phi_n}{2}\right)} \\
&= \sqrt{\frac{1-\cos\theta_n}{1+\cos\theta_n}\frac{1+\cos\phi_n}{1-\cos\phi_n}}\frac{\sin\phi_n}{\cos\phi_n} \\
&= \sqrt{\frac{1-\cos\theta_n}{1+\cos\theta_n}\frac{1+\sin\theta_n}{1-\sin\theta_n}}\frac{\cos\theta_n}{\sin\theta_n} \\
&= \sqrt{\frac{\sin^2\theta_n}{(1+\cos\theta_n)^2}\frac{(1+\sin\theta_n)^2}{\cos^2\theta_n}}\frac{\cos\theta_n}{\sin\theta_n} = \frac{1+\sin\theta_n}{1+\cos\theta_n}.
\end{aligned}$$

It is immediate to see that, if $0 < \theta_n \le \frac{\pi}{4}$, as $\cos\theta_n \ge \sin\theta_n$, $\tan\theta_{n+1} \le 1$, which means that the following property is satisfied as well: $0 < \theta_{n+1} \le \frac{\pi}{4}$, which, by induction, means that $0 < \theta_n \le \frac{\pi}{4}$, $\forall n$. It is also easy to see that, if $\theta_1 \in \left(0, \frac{\pi}{4}\right]$, the relation $\frac{1+\sin\theta_n}{1+\cos\theta_n} \ge \tan\theta_n$ holds. Indeed,

$$\begin{aligned}
&\frac{1+\sin\theta_n}{1+\cos\theta_n} \ge \tan\theta_n \\
\iff &(1+\sin\theta_n)\cos\theta_n \ge (1+\cos\theta_n)\sin\theta_n \\
\iff &\cos\theta_n + \sin\theta_n\cos\theta_n \ge \sin\theta_n + \sin\theta_n\cos\theta_n \\
\iff &\cos\theta_n \ge \sin\theta_n,
\end{aligned}$$

which is true if $0 < \theta_n \le \frac{\pi}{4}$. Therefore, the sequence $\{\tan\theta_n\}_{n\in\mathbb{N}}$ is monotonically increasing and is upper bounded by 1, and so, it has a limit. At the limit, $\tan\theta_{n+1} = \frac{1+\sin\theta_n}{1+\cos\theta_n}$, which means $\sin\theta_n = \cos\theta_n$, and thus $\lim_{n\to\infty}\theta_n = \frac{\pi}{4}$, as we wanted to show.

Also solved by Fernando Ballesta Yagüe, I.E.S. Infante don Juan Manuel, Murcia, Spain, and the proposer.

EM–26. *Proposed by Damià Torres Latorre, CFIS, BarcelonaTech, Barcelona, Spain.* A new bookstore sells one or two books per day, and the day it opened it sold exactly one book. Prove that, for any positive integer a, there exists a period of consecutive days in which a books were sold.

Solution 1 by Alberto Espuny-Díaz, BarcelonaTech, Barcelona, Spain. Let the number of books that are sold each day be x_i, for $i \geq 1$. We have that $x_1 = 1$, $x_i \in \{1, 2\}$ for all $i > 1$. Assume after n days they have sold at least a books, and on the $(n-1)$-th they still had not. There are two possibilities:

- After n days, they have sold exactly a books. In this case, the right period of days is from the first to the n-th.
- After n days, they have sold exactly $a+1$ books (they cannot have sold more, as they only sell one or two books per day, and one day before they still had not reached a). As the first day they sold one book, they must have sold a during the rest of the days, that is, the correct period of days is from the second to the n-th.

Solution 2 by Francesc Gispert Sánchez, CFIS, BarcelonaTech, Barcelona, Spain. Let x_i be the number of books sold in the i-th day for $i \geq 1$. In particular, $x_1 = 1$. For a given a, consider the minimum integer $n \geq 2$ such that

$$\sum_{i=2}^{n} x_i \geq a\,.$$

By definition of n, we have that

$$\sum_{i=2}^{n-1} x_i < a$$

and x_n is either 1 or 2. Therefore, we distinguish two cases.

If

$$\sum_{i=2}^{n} x_i = a\,,$$

we can choose the period of days comprised between the second and the n-th days (both included) and the problem is solved.

Otherwise, it must be

$$\sum_{i=2}^{n} x_i = a + 1$$

and $x_n = 2$. Consequently,

$$\sum_{i=1}^{n-1} x_i = -1 + \sum_{i=2}^{n} x_i = a$$

and we can choose the period of days comprised between the first and the $(n-1)$-th days (both included).

Solution 3 by Fernando Ballesta Yagüe, I.E.S. Infante don Juan Manuel, Murcia, Spain. Let a_i be the number of books sold on the i-th day, with $a_1 = 1$, and let a be any positive integer. We want to see that for every a there exists a period of consecutive days where a books were sold.

We are going to prove it by induction. For $a = 1$, we have that $a_1 = 1$. For $a = 2$, there are two possibilities:

- If $a_2 = 1$, then $a_1 + a_2 = 2$.
- If $a_2 = 2$, then $a_2 = 2$.

Let us suppose that the property holds for $a = n$. Then, there is at least a period of consecutive days where n books were sold. Let $a_k + a_{k+1} + \ldots + a_l = n$ with the least possible value k (since by the well ordering principle there exists a least positive integer for which the property is true). We need to prove that the property stands for $a = n + 1$, and for that, we need to consider several possibilities:

- If $a_{l+1} = 1$, then $a_k + a_{k+1} + \ldots + a_{l+1} = n + 1$.
- If $a_{l+1} = 2$,
  - If $a_k = 1$, then $a_{k+1} + a_{k+2} + \ldots + a_{l+1} = n + 1$.
  - If $a_k = 2$,
    - If $a_{k-1} = 1$, then $a_{k-1} + a_k + \ldots + a_l = n + 1$.

* If $a_{k-1} = 2$ and $a_l = 1$, then $a_{k-1} + a_k + \ldots + a_{l-1} = n + 1$.
* If $a_{k-1} = 2$ and $a_l = 2$, then $a_{k-1} + a_k + \ldots + a_{l-1} = n$. But this cannot happen, as we have said that k was the least value for which there is a period of consecutive days in which n books were sold.

Therefore, the property also holds for $a = n + 1$, which completes the proof.

Solution 4 by the proposer. Consider the sequences of consecutive days in which two books are sold. We are going to distinguish two cases.

If there exist arbitrarily large periods such that the bookstore sells two books a day, there exists one such period with length $l > \left\lfloor \frac{a}{2} \right\rfloor$. We can take now the $\left\lfloor \frac{a}{2} \right\rfloor$ first days of this period if a is even, or these days and the immediately preceding one (in which one book was sold because it is not in the selected period) if a is odd. Notice that we can always do this, because these periods cannot contain the day the bookstore opened.

If these periods are not arbitrarily large, then there exists N such that, among N consecutive days, there is one in which only one book was sold. So we have that among N consecutive days, the maximum possible number of books sold is $2N - 1$. Now we look at the first $(a + 1)N$ days the bookstore is open. Define x_i the number of books sold between the first and the i-th day. Trivially all the x_i are different. Observe that if there exist i, j such that $x_j - x_i = a$, we take the period between the days $i + 1$ and j and the problem is solved. So let us consider the set $S = \{x_1, \ldots, x_{(a+1)N}, x_1 + a, \ldots, x_{(a+1)N} + a\}$. This set has $2(a + 1)N$ elements, and each of them is less than or equal to $(a + 1)(2N - 1) + a$. So we have a set of $2(a + 1)N$ positive integers, each of them not greater than $2(a + 1)N - 1$, which means that there are, at least, two identical elements. Finally, as the x_i are different among them, and so are the $x_i + a$, having two identical elements means we have $x_j = x_i + a$, for some i, j, and this is equivalent to the statement.

Also solved by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany; Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain, and Isaac Sánchez Barrera, BarcelonaTech, Barcelona, Spain.

EM–27. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let a, b, c be three positive numbers such that $a+b+c=1$. Determine the maximum value of

$$(a+b)^c(b+c)^a(c+a)^b.$$

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. Let E be the given expression. Since $a+b+c=1$, by the weighted AM–GM inequality, we have

$$E \leq \frac{c(a+b)+a(b+c)+b(c+a)}{a+b+c} = c(1-c)+a(1-a)+b(1-b).$$

Now, since function $f(x)=x(1-x)$ is concave because $f''(x)=-2$, by Jensen's inequality $E \leq 3\cdot\dfrac{a+b+c}{3}\left(1-\dfrac{a+b+c}{3}\right)=\dfrac{2}{3}$. Notice that equality holds when $a=b=c=\frac{1}{3}$.

Solution 2 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. We have $(a+b)^c(b+c)^a(c+a)^b=(1-c)^c(1-a)^a(1-b)^b$. We see that if any variable is 1, the expression equals 0. Taking the logarithm of the expression, we get $a\ln(1-a)+b\ln(1-b)+c\ln(1-c)$. As $\ln x$ is an increasing, convex function in the interval $(0,+\infty)$, we can use the weighted version of Jensen's inequality and power means inequality to see that:

$$\begin{aligned} a\ln(1-a)+b\ln(1-b)+c\ln(1-c) &\leq \ln(1-(a^2+b^2+c^2)) \\ &\leq \ln\left(1-\frac{(a+b+c)^2}{3}\right) \\ &= \ln(2/3), \end{aligned}$$

reaching equality if, and only if, $a=b=c=\frac{1}{3}$. Taking the exponential (or substituting in the main inequality) we see that the maximum value of the expression is $\frac{2}{3}$.

Solution 3 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany. When $a = b = c = \frac{1}{3}$, the expression equals $\frac{2}{3}$. We will show that this is the maximum value. Indeed, assume WLOG that $a \geq b \geq c$. Then $b + c \leq a + c \leq a + b$ and, since the logarithm is an increasing function, $\log(b + c) \leq \log(a + c) \leq \log(a + b)$. Using Chebyshev's sum inequality,

$$\begin{aligned}
& c\log(a+b) + a\log(b+c) + b\log(c+a) \\
\leq\ & \frac{(a+b+c)(\log(a+b) + \log(b+c) + \log(c+a))}{3} \\
=\ & \log\left(\sqrt[3]{(a+b)(b+c)(c+a)}\right) \\
\leq\ & \log\left(\frac{(a+b)+(b+c)+(c+a)}{3}\right) \\
=\ & \log\frac{2}{3}.
\end{aligned}$$

Taking exponents on both ends of the equation we obtain the desired bound.

Also solved by Fernando Ballesta Yagüe, I.E.S. Infante don Juan Manuel, Murcia, Spain, and the proposer.

EM–28. *Proposed by Alberto Espuny-Díaz, BarcelonaTech, Barcelona, Spain.* Compute the value of the series

$$\sum_{n=1}^{\infty} 2\sin^4\frac{\pi}{2^n} - 6\cos^2\frac{\pi}{2^n}\sin^2\frac{\pi}{2^n}.$$

Solution 1 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany. Denoting $x = \frac{\pi}{2^n}$, each term of the sum is $2\sin^4 x - 6\sin^2 x\cos^2 x$. Since we take the sum on powers of two, in order to write this as a telescopic sum we need to find an expression of $2\sin^4 x - 6\sin^2 x\cos^2 x$ as $f(2x) - f(x)$.

Using the identity $\sin^2 x + \cos^2 x = 1$, we can rewrite the expression as $2\sin^2 x(\sin^2 x + \cos^2 x) - 8\sin^2 x\cos^2 x = 2\sin^2 x -$

$8\sin^2 x\cos^2 x$. Now remember that $\sin 2x = 2\sin x\cos x$, so taking squares, $\sin^2 2x = 4\sin^2 x\cos^2 x$. Thus,

$$\begin{aligned} 2\sin^4 x - 6\sin^2 x\cos^2 x &= 2\sin^2 x - 8\sin^2 x\cos^2 x \\ &= 2\sin^2 x - 2\sin^2 2x. \end{aligned}$$

From this we find that $f(x) = -2\sin^2 x$ works. Now the sum that we want to compute is

$$\begin{aligned} \sum_{n=1}^{\infty} 2\sin^4\frac{\pi}{2^n} - 6\cos^2\frac{\pi}{2^n}\sin^2\frac{\pi}{2^n} &= \sum_{n=1}^{\infty} f\left(\frac{\pi}{2^{n-1}}\right) - f\left(\frac{\pi}{2^n}\right) \\ &= f(\pi) - \lim_{n\to+\infty} f\left(\frac{\pi}{2^n}\right) \\ &= 0. \end{aligned}$$

Solution 2 by the proposer. Let S be the value of the sum. First, take one of the sines as a common factor, obtaining

$$S = \sum_{n=1}^{\infty} 2\sin\frac{\pi}{2^n}\left(\sin^3\frac{\pi}{2^n} - 3\cos^2\frac{\pi}{2^n}\sin\frac{\pi}{2^n}\right).$$

Observe now that what we have inside the parenthesis is the formula of the sine of a triple angle, with a change in its sign. This means that

$$S = \sum_{n=1}^{\infty} -2\sin\frac{\pi}{2^n}\sin\frac{3\pi}{2^n}.$$

Using the product-to-sum formula for the sine we have

$$S = \sum_{n=1}^{\infty} \cos\frac{2\pi}{2^{n-1}} - \cos\frac{2\pi}{2^n}.$$

Now this is a telescopic sum, so, for any fixed N, all the terms in the sum

$$S_N = \sum_{n=1}^{N} \cos\frac{2\pi}{2^{n-1}} - \cos\frac{2\pi}{2^n}$$

cancel out except the first and the last. We then have that

$$S_N = \cos 2\pi - \cos\frac{2\pi}{2^N}.$$

Finally,

$$S = \lim_{N\to\infty} S_N = \cos 2\pi - \lim_{N\to\infty} \cos\frac{2\pi}{2^N} = 1 - 1 = 0.$$

Also solved by Fernando Ballesta Yagüe, I.E.S. Infante don Juan Manuel, Murcia, Spain; Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain, and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

Medium–Hard Problems

MH–23. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Prove that

$$\frac{1}{n}\left[\sum_{k=1}^{n}\left(\frac{1+a^2}{a}\right)^{2k}\prod_{\substack{j=1\\ j\neq k}}^{n}\left(\frac{1+a^2}{a}\right)^{j}\right] \geq 2^{\frac{(n+1)^2}{2}}$$

holds for all $a > 0$.

Solution 1 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. Note that $\dfrac{1+a^2}{a} = a + \dfrac{1}{a} \geq 2$ for all $a > 0$. Therefore, the left-hand side of the proposed inequality is

$$\begin{aligned} LHS &\geq \frac{1}{n}\left(\sum_{k=1}^{n} 2^{2k}\prod_{\substack{j\neq k\\ j=1}}^{n} 2^{j}\right) \\ &\geq \left(\prod_{j=1}^{n} 2^{j}\right)\frac{1}{n}\sum_{k=1}^{n} 2^{k} \\ &\geq 2^{\sum_{k=1}^{n} k}\sqrt[n]{2^{\sum_{k=1}^{n} k}} \\ &= 2^{\frac{(n+1)n}{2}}\cdot 2^{\frac{n+1}{2}} \\ &= 2^{\frac{(n+1)^2}{2}}. \end{aligned}$$

Solution 2 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. Let $x = \frac{1+a^2}{a} = \frac{1}{a} + a \geq 2$ We have:

$$\begin{aligned} \frac{1}{n}\left[\sum_{k=1}^{n} x^{2k}\prod_{\substack{j=1\\ j\neq k}}^{n} x^{j}\right] &= \frac{1}{n}\left[\sum_{k=1}^{n} x^{k}\prod_{j=1}^{n} x^{j}\right] \\ &= \frac{1}{n}\left[\sum_{k=1}^{n} x^{k}x^{\frac{n(n+1)}{2}}\right] = x^{\frac{n(n+1)}{2}}\frac{1}{n}\sum_{k=1}^{n} x^{k}. \end{aligned}$$

Applying AM-GM inequality:

$$x^{\frac{n(n+1)}{2}}\frac{1}{n}\sum_{k=1}^{n}x^k \geq x^{\frac{n(n+1)}{2}}\sqrt[n]{\prod_{k=1}^{n}x^k} = x^{\frac{n(n+1)}{2}}\sqrt[n]{x^{\frac{n(n+1)}{2}}}$$
$$= x^{\frac{n(n+1)}{2}}x^{\frac{(n+1)}{2}} = x^{\frac{(n+1)^2}{2}} \geq 2^{\frac{(n+1)^2}{2}},$$

reaching equality if and only if $n = a = 1$.

Also solved by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and the proposer.

MH–24. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $A(z) = z^3 - a_2z^2 + a_1z - a_0$ be a polynomial with real coefficients. If all the zeros of $A(z)$ are positive, then prove that

$$9a_0^2 + a_1^2a_2^2 \geq \frac{4}{3}a_1^3 + 6a_0a_1a_2.$$

Solution 1 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. We can rewrite our inequality as

$$9a_0^2 - 6a_0a_1a_2 + a_1^2a_2^2 \geq \frac{4}{3}a_1^3.$$

Or even simpler,

$$(3a_0 - a_1a_2)^2 \geq \frac{4}{3}a_1^3.$$

Let α, β, γ be the three positive zeros of $A(z)$. Using Vieta's formulas, we have $a_0 = \alpha\beta\gamma$; $a_1 = \alpha\beta + \alpha\gamma + \beta\gamma$; $a_2 = \alpha + \beta + \gamma$. Applying AM-GM inequality to a_1, we obtain

$$a_1 = \alpha\beta + \alpha\gamma + \beta\gamma \geq 3\sqrt[3]{\alpha^2\beta^2\gamma^2} = 3a_0^{2/3}.$$

Applying Cauchy-Schwarz to $\vec{u} = (\alpha, \beta, \gamma)$ and $\vec{v} = (\beta, \gamma, \alpha)$ yields

$$a_2^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\alpha\gamma + 2\beta\gamma \geq 3(\alpha\beta + \alpha\gamma + \beta\gamma) = 3a_1,$$

from where we get

$$a_1a_2 \geq 3a_0^{2/3} \cdot \sqrt{3}a_1^{1/2} \geq 3a_0^{2/3} \cdot \sqrt{3}\sqrt{3}a_0^{1/3} = 9a_0.$$

Thus, $a_1a_2 \geq 9a_0 \geq 3a_0$, and taking square roots, we can rewrite our inequality as

$$a_1a_2 - 3a_0 \geq \frac{2}{\sqrt{3}}a_1^{3/2}.$$

Finally,

$$a_1a_2 - 3a_0 \geq a_1a_2 - \frac{1}{3}a_1a_2 = \frac{2}{3}a_1a_2 \geq \frac{2\sqrt{3}}{3}a_1a_1^{1/2} = \frac{2}{\sqrt{3}}a_1^{3/2},$$

reaching equality if and only if $\alpha = \beta = \gamma$.

Solution 2 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany. Let x, y, z be the roots of $A(z)$. Then Vieta's formulas tell us that $a_0 = xyz$, $a_1 = xy + yz + zx$ and $a_2 = x + y + z$, all of them positive. From this we can obtain the following inequalities:

$$a_1a_2 = (xy + yz + zx)(x + y + z) \geq \left(3\sqrt[3]{xyz}^2\right)(3\sqrt[3]{xyz}) = 9a_0,$$

$$a_2^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) \geq 3(xy + yz + zx) = 3a_1.$$

This is enough to tackle our main inequality:

$$\begin{aligned} 0 &\leq \frac{(a_1a_2 - 9a_0)(5a_1a_2 - 9a_0)}{9} \\ &= 9a_0^2 - 6a_0a_1a_2 + \frac{5}{9}a_1^2a_2^2 \\ &\leq 9a_0^2 - 6a_0a_1a_2 + \frac{5}{9}a_1^2a_2^2 + \frac{4}{9}a_1^2(a_2^2 - 3a_1) \\ &= 9a_0^2 - 6a_0a_1a_2 + a_1^2a_2^2 - \frac{4}{3}a_1^3. \end{aligned}$$

Rearranging the terms in the last inequality we obtain the one in the statement.

Also solved by the proposer.

MH–25. *Proposed by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.* Show that for any numbers a, b, $c > 0$ such that $a + b + c = 3$ we have

$$\left(a^3 + 2a^2 + 2a + 1\right)\left(b^3 + 2b^2 + 2b + 1\right)\left(c^3 + 2c^2 + 2c + 1\right) \leq 6^3.$$

Solution 1 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. Taking the logarithm of both sides, we want to prove that

$$\ln(a^3+2a^2+2a+1)+\ln(b^3+2b^2+2b+1)+\ln(c^3+2c^2+2c+1) \leq 3\ln 6.$$

We have that, if $f(x) > 0$ is twice-differentiable, $g(x) = \ln(f(x))$ is also twice-differentiable and

$$\frac{d^2}{dx^2}g(x) = \frac{d^2}{dx^2}\ln(f(x)) = \frac{d}{dx}\left(\frac{f'(x)}{f(x)}\right) = \left(\frac{f''(x)f(x) - [f'(x)]^2}{[f(x)]^2}\right),$$

meaning that $g(x)$ is concave if and only if $[f'(x)]^2 \geq f(x)f''(x)$, $f(x) \neq 0$. Taking $f(x) = x^3 + 2x^2 + 2x + 1$, $f(x) > 0\ \forall x \in [0, 3]$, and:

$$\begin{aligned}[f'(x)]^2 - f(x)f''(x) =&(3x^2 + 4x + 2)^2\\ &- (x^3 + 2x^2 + 2x + 1)(6x + 4)\\ =&(9x^4 + 24x^3 + 28x^2 + 16x + 4)\\ &- (6x^4 + 16x^3 + 20x^2 + 14x + 4)\\ =&3x^4 + 8x^3 + 8x^2 + 2x,\end{aligned}$$

which is greater or equal than 0 for all $x \in [0, 3]$. This way, we have proved that $g(x) = \ln(x^3 + 2x^2 + 2x + 1)$ is concave, and so, by Jensen's inequality,

$$g(a) + g(b) + g(c) \leq 3g\left(\frac{a + b + c}{3}\right) = 3g(1) = 3\ln 6,$$

as we wanted to show. Equality is reached only when $a = b = c = 1$.

Solution 2 by the proposer. By taking logarithms, the proposed inequality may be written as

$$\begin{aligned}\frac{\ln(a^3 + 2a^2 + 2a + 1)}{3} + \frac{\ln(b^3 + 2b^2 + 2b + 1)}{3}&\\ + \frac{\ln(c^2 + 2c^2 + 2c + 1)}{3} &\leq \ln 6.\end{aligned}$$

Let us consider the function $f : (0,3) \to \mathbb{R}$ defined by $f(x) = \ln(x^3 + 2x^2 + 2x + 1)$ for $x \in (0,3)$. Then,

$$f''(x) = \frac{-3x^4 - 8x^3 - 8x^2 - 2x}{(x^3 + 2x^2 + 2x + 1)^2},$$

and since $f''(x) < 0$ for $x \in (0, M)$, f is concave. By Jensen's inequality

$$\frac{f(a) + f(b) + f(c)}{3} \leq f\left(\frac{a+b+c}{3}\right) = f(1) = \ln 6.$$

The same argument applies for inequalities of the form

$$f(a) \cdot f(b) \cdot f(c) \leq M^3$$

for any numbers a, b, $c > 0$ such that $a + b + c = 3$, with $f(x)$ a positive function such that $f(x)f''(x) - (f'(x))^2 < 0$ for $x \in (0,3)$, and $f(1) = M$.

Also solved by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.

MH–26. *Proposed by Mihály Bencze, Braşov, Romania.* Find all real solutions of the following system of equations:

$$\begin{cases} (x_1^2+1)(x_1+1)^2 + 7x_2^2 = 5x_3(x_3^2 + x_3 + 1), \\ (x_2^2+1)(x_2+1)^2 + 7x_3^2 = 5x_4(x_4^2 + x_4 + 1), \\ \qquad\qquad\qquad\qquad \vdots \\ (x_n^2+1)(x_n+1)^2 + 7x_1^2 = 5x_2(x_2^2 + x_2 + 1). \end{cases}$$

Solution by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. Summing the n equations yields

$$\sum_{i=1}^{n} (x_i^2+1)(x_i+1)^2 + 7x_i^2 = \sum_{i=1}^{n} 5x_i(x_i^2 + x_i + 1)$$

$$\Longrightarrow \sum_{i=1}^{n} (x_i^2+1)(x_i+1)^2 + 7x_i^2 - 5x_i(x_i^2 + x_i + 1) = 0.$$

Expanding,

$$\begin{aligned}
0 &= \sum_{i=1}^{n} (x_i^2+1)(x_i+1)^2 + 7x_i^2 - 5x_i(x_i^2+x_i+1) \\
&= \sum_{i=1}^{n} x_i^4 + 2x_i^3 + 2x_i^2 + 2x_i + 1 + 7x_i - 5x_i^3 - 5x_i^2 - 5x_i \\
&= \sum_{i=1}^{n} x_i^4 - 3x_i^3 + 4x_i^2 - 3x_i + 1 \\
&= \sum_{i=1}^{n} (x_i-1)^2(x_i^2 - x_i + 1) \\
&= \sum_{i=1}^{n} (x_i-1)^2((x_i-1/2)^2 + 3/4)
\end{aligned}$$

As we can see, $\sum_{i=1}^{n} (x_i-1)^2((x_i-1/2)^2+3/4)$ will be positive if any of the x_i's is different from 1. One can easily check that $x_1 = \ldots = x_n = 1$ is a solution of the system of equations, and so, it is the only one.

Also solved by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and the proposer.

MH–27. *Proposed by Mihály Bencze, Braşov, Romania.* Let n be a positive integer. Show that there exists a positive integer x such that the sum of the digits of x^2 is 4^n.

Solution 1 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany. Consider the number

$$x = \sum_{i=1}^{k} 10^{2^i}.$$

This number satisfies

$$x^2 = \left(\sum_{i=1}^{k} 10^{2^i}\right)\left(\sum_{j=1}^{k} 10^{2^j}\right) = \sum_{i=1}^{k}\sum_{j=1}^{k} 10^{2^i+2^j}.$$

Since this is a sum of powers of ten in which there are at most two equal exponents (fewer than ten), this is the decimal sum of x^2. The sum of digits of x^2 is the number of powers of ten in the sum, which is k^2. To complete the problem, set $k = 2^n$.

Solution 2 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. We claim that, for every perfect square $k = m^2$, $m \in \mathbb{N}$, there exists a positive integer x such that the sum of the digits of x^2 is k. We will show that for every $n \geq 1$, the sum of the digits of the square of the number $a_n = \sum_{k=1}^{n} 10^{2^{k-1}-1}$ is n^2. Indeed, if we proceed by induction, $a_1^2 = 1^2$, and, for every $n \geq 2$:

$$a_{n+1}^2 = (a_n + 10^{2^n-1})^2 = a_n^2 + 2a_n \cdot 10^{2^n-1} + 10^{2^{n+1}-2}.$$

If we denote as $s(n)$ the sum of the digits of n, in base 10, we clearly have $s(a_n) = n$. Since no two different summands out of the previous three have a non-zero digit at the same position, the sum of the digits of a_{n+1}^2 will be the sum of the sum of digits of the three summands:

$$\begin{aligned} s(a_{n+1}^2) &= s(a_n^2) + s(2a_n \cdot 10^{2^n-1}) + s(10^{2^{n+1}-2}) \\ &= n^2 + 2n + 1 = (n+1)^2. \end{aligned}$$

As $4^n = (2^n)^2$, 4^n is a square number, and so, there exists x (letting $x = a_{2^n}$) such that $s(x^2) = 4^n$, completing the proof.

Solution 3 by the proposer. We consider the sequence of integers $\{x(k)\}_{k\geq 1}$ defined by

$$x(k) = \overline{\underbrace{33\ldots3}_{(k-1)\,times}\ 2} = \frac{10^k - 4}{3}.$$

Then,

$$\begin{aligned} x^2(k) = \left(\frac{10^k - 4}{3}\right)^2 &= \frac{1}{9}\left(10^{2k} - 1\right) - \frac{8}{9}\left(10^k - 1\right) + 1 \\ &= \overline{\underbrace{11\ldots1}_{(k-1)\,times}\ 0\ \underbrace{22\ldots2}_{(k-1)\,times}\ 4}. \end{aligned}$$

The sum of the digits of $x^2(k)$ is $(k-1)+2(k-1)+4=3k+1$. Since $4^n \equiv 1 \pmod 3$ and $3k+1$ increases when k increases, there exists a positive integer m such that $4^n = 3m+1$, and we are done.

MH–28. *Proposed by Damià Torres Latorre, CFIS, BarcelonaTech, Barcelona, Spain.* Let $\tau(n)$ be the number of positive divisors of the positive integer n, and let $\sigma(n)$ be the sum of these divisors. Prove that

$$\frac{\tau(n)^2}{n} < \frac{2\sigma(n)}{n} \leq \ln(n)+4.$$

Solution 1 by the proposer. Define $\sigma(n)$ as the sum of the divisors of the positive integer n. As they are all different, $\sigma(n) \geq 1+2+\ldots+\tau(n) = \dfrac{\tau(n)\,(\tau(n)+1)}{2}$. Hence $\tau(n)^2 < 2\sigma(n)$. Now, as if d is a divisor of n, so is $\frac{n}{d}$, $\sigma(n) = \sum\limits_{d|n} \frac{n}{d} = n \sum\limits_{d|n} \frac{1}{d}$. We can split the divisors into two groups, as for each divisor before $\sqrt{n}$, there is one after, so

$$\begin{aligned}\sigma(n) &\leq n\left(\sum_{d|n, d\leq\sqrt{n}} \frac{1}{d} + \sum_{d|n, d\leq\sqrt{n}} \frac{d}{n}\right) \leq n\left(\sum_{d=1}^{\sqrt{n}} \frac{1}{d} + \sum_{d=1}^{\sqrt{n}} \frac{d}{n}\right)\\ &\leq n\left(1+\ln(\sqrt{n}) + \frac{\sqrt{n}(\sqrt{n}+1)}{2n}\right) \leq n\left(1+\frac{\ln n}{2} + \frac{1+\frac{1}{\sqrt{n}}}{2}\right)\\ &\leq n\left(2+\frac{\ln n}{2}\right).\end{aligned}$$

Putting together these two inequalities, we get

$$\frac{\tau(n)^2}{n} < \frac{2\sigma(n)}{n} \leq \frac{4n+n\ln n}{n} = \ln n + 4.$$

Solution 2 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. Let $n = p_1^{\alpha_1}\cdots p_k^{\alpha_k}$, $\alpha_1 \geq 0$, $\alpha_2, \ldots \alpha_k \geq 1$ (we will consider the factor $p_1 = 2$ even if n is odd) be the prime factorization of n. We know m is a divisor of n if and only if m has the following prime decomposition: $m = p_1^{\beta_1}\cdots p_k^{\beta_k}$, $0 \leq$

$\beta_1 \leq \alpha_1, \ldots, 0 \leq \beta_k \leq \alpha_k$, meaning we have a total of $\tau(n) = (\alpha_1 + 1) \cdots (\alpha_k + 1)$ divisors for n. The sum of all these divisors will be the following:

$$\begin{aligned}
&\sum_{\beta_1=0}^{\alpha_1} \cdots \sum_{\beta_k=0}^{\alpha_k} p_1^{\beta_1} \cdots p_k^{\beta_k} \\
=&\sum_{\beta_1=0}^{\alpha_1} \cdots \sum_{\beta_{k-1}=0}^{\alpha_{k-1}} p_1^{\beta_1} \cdots p_{k-1}^{\beta_{k-1}} (p_k^0 + \ldots + p_k^{\alpha_k}) \\
=&\frac{p_k^{\alpha_k+1} - 1}{p_k - 1} \sum_{\beta_1=0}^{\alpha_1} \cdots \sum_{\beta_{k-1}=0}^{\alpha_{k-1}} p_1^{\beta_1} \cdots p_{k-1}^{\beta_{k-1}} \\
=&\frac{p_1^{\alpha_1+1} - 1}{p_1 - 1} \cdots \frac{p_k^{\alpha_k+1} - 1}{p_k - 1} = \prod_{m=1}^{k} \frac{p_m^{\alpha_m+1} - 1}{p_m - 1}.
\end{aligned}$$

Now, proving the LHS inequality will be equivalent to proving that

$$\prod_{m=1}^{k} (\alpha_m + 1)^2 < 2 \prod_{m=1}^{k} \frac{p_m^{\alpha_m+1} - 1}{p_m - 1} \tag{2}$$

We will prove that $(\alpha_1 + 1)^2 < 2\frac{2^{\alpha_1+1}-1}{2-1} = 2^{\alpha_1+2} - 2$, $\forall \alpha_1 \in \mathbb{N}$.

Inequality holds for $\alpha_1 = 0$, 1, 2, and, for $\alpha_1 \geq 3$, $2^{\alpha_1+2} = \binom{\alpha_1+2}{0} + \ldots + \binom{\alpha_1+2}{\alpha_1+2} > \binom{\alpha_1+2}{0} + \binom{\alpha_1+2}{2} + \binom{\alpha_1+2}{\alpha_1} + \binom{\alpha_1+2}{\alpha_1+2} = 1 + 2\frac{(\alpha_1+2)(\alpha_1+1)}{2} + 1 > 2 + (\alpha_1 + 1)^2$, as we wanted to show.

Now we will prove that, for all $\alpha_j \geq 1$, $\alpha_j \in \mathbb{N}$, with $j \geq 2$ (and so, $p_j \geq 3$), $(\alpha_j + 1)^2 \leq \frac{p_j^{\alpha_j+1}-1}{p_j-1}$ holds.

Let us start by proving that, if $p \geq 3$, the inequality $p^x \geq xp$ holds $\forall x \geq 1$.

Let $g_p(x) = p^x - xp$. We have $g_p(1) = p - p = 0$. g_p is infinitely differentiable and $g_p'(x) = (\ln p)p^x - p > p^x - p \geq p - p = 0$, and so $g_p(x) \geq 0$, $\forall x \geq 1$ (here, we have used that $\ln p \geq \ln 3 > \ln e = 1$).

Consider the function $f_p(x) = p^{x+1} - 1 - (x + 1)^2(p - 1)$, $p \geq 3$.

For $x = 1$, $f_p(1) = p^2 - 4p + 4 - 1 = (p-2)^2 - 1 \geq 1 - 1 = 0$.

For $x = 2$, $f_p(2) = p^3 - 1 - 9p + 9 = p^2 p - 9p + 8 \geq 9p - 9p + 8 > 0$.

Now, as $f_p(x)$ is differentiable over the real line, if $x \geq 2$, $f_p'(x) = (\ln p)p^{x+1} - 1 - 2(x+1)(p-1) > p^{x+1} - 2p(x+1) - 1 + 2(x+1) > p^{x+1} - 2p(x+1) = p(p^x - 2x - 2) = p(p^x - xp + px - 2x - 2) \geq p(x(p-2) - 2) \geq p(2(3-2) - 2) = 0$.

As $f_p(2) > 0$, $f_p'(x) > 0$, $\forall x \geq 2$, we conclude that $f_p(x) > 0$, $\forall x \geq 2$. This way we have proven the inequality.

As $(\alpha_1 + 1)^2 < 2\frac{p_1^{\alpha_1+1}-1}{p_1-1}$, and $(\alpha_j + 1)^2 \leq \frac{p_j^{\alpha_j+1}-1}{p_j-1}, \forall j = 2, \ldots, k$, we have proven equation (2).

We see that the RHS inequality $\frac{2\sigma(n)}{n} < \ln n + 4$ is equivalent to:

$$2\frac{\prod_{m=1}^{k} \frac{p_m^{\alpha_m+1}-1}{p_m-1}}{\prod_{m=1}^{k} p^{\alpha_m}} = 2\prod_{m=1}^{k} \frac{p_m^{\alpha_m+1}-1}{p_m^{\alpha_m+1} - p_m^{\alpha_m}} < 4 + \sum_{m=1}^{k} \alpha_m \ln p_m.$$

We will find an upper bound and a lower bound for $\frac{p_m^{\alpha_m+1}-1}{p_m^{\alpha_m+1}-p_m^{\alpha_m}}$:

$$\frac{p_m^{\alpha_m+1}-1}{p_m^{\alpha_m+1} - p_m^{\alpha_m}} = 1 + \frac{p_m^{\alpha_m}-1}{(p_m-1)p_m^{\alpha_m}} = 1 + \frac{1}{p_m-1}\left(1 - \frac{1}{p_m^{\alpha_m}}\right).$$

With this, we find the upper bound

$$1 + \frac{1}{p_m-1}\left(1 - \frac{1}{p_m^{\alpha_m}}\right) < 1 + \frac{1}{p_m-1} = \frac{p_m}{p_m-1}$$

and the lower bound

$$\begin{aligned} 1 + \frac{1}{p_m-1}\left(1 - \frac{1}{p_m^{\alpha_m}}\right) &\geq 1 + \frac{1}{p_m-1}\left(1 - \frac{1}{p_m^{1}}\right) \\ &= 1 + \frac{1}{p_m-1}\frac{p_m-1}{p_m} \\ &= 1 + \frac{1}{p_m} = \frac{p_m+1}{p_m}, \end{aligned}$$

reaching equality if and only if $\alpha_m = 1$.

From now on we will consider $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $\alpha_1, \ldots \alpha_k \geq 1$ (that is, we will not consider the factor of 2 in odd numbers).

With the previous bounds, we can prove that, for $k = 0, 1, 2, 3, 4$, the desired inequality will hold. As $p_1 \geq 2$, $p_2 \geq 3$, $p_3 \geq 5$, $p_4 \geq 7$:

For $k = 0$, $n = 1$ and $\frac{2\sigma(1)}{1} = 2 < 4 = 4 + \ln 1$.

For $k = 1$, $n = p_1^{\alpha_1}$ and

$$\frac{2\sigma(n)}{n} = 2\frac{p_1^{\alpha_1+1} - 1}{p_1^{\alpha_1+1} - p_1^{\alpha_1}} < 2\frac{p_1}{p_1 - 1} \leq 2\frac{2}{1} = 4 < 4 + \ln n.$$

For $k = 2$ we will distinguish between two cases:

If $\alpha_1 = \alpha_2 = 1$:

$$2\prod_{m=1}^{2} \frac{p_m^{\alpha_m+1} - 1}{p_m^{\alpha_m+1} - p_m^{\alpha_m}} = 2\prod_{m=1}^{2} \frac{p_m + 1}{p_m} \leq 2\frac{3}{2}\frac{4}{3} = 4 < 4 + \ln n.$$

Otherwise:

$$2\prod_{m=1}^{2} \frac{p_m^{\alpha_m+1} - 1}{p_m^{\alpha_m+1} - p_m^{\alpha_m}} < 2\prod_{m=1}^{2} \frac{p_m}{p_m - 1} \leq 2\frac{2}{1}\frac{3}{2} = 6.$$

As there is a value of α_m greater that 1, by rearrangement inequality:

$$4 + \alpha_1 \ln p_1 + \alpha_2 \ln p_2 \geq 4 + 2\ln p_1 + \ln p_2 \geq 4 + 2\ln 2 + \ln 3 > 6.$$

Similar case for $k = 3$:

If $\alpha_1 = \alpha_2 = \alpha_3 = 1$:

$$2\prod_{m=1}^{3} \frac{p_m^{\alpha_m+1} - 1}{p_m^{\alpha_m+1} - p_m^{\alpha_m}} = 2\prod_{m=1}^{3} \frac{p_m + 1}{p_m} \leq 2\frac{3}{2}\frac{4}{3}\frac{6}{5} = \frac{24}{5} < 5$$

and

$$4 + \ln n \geq 4 + \ln 2 + \ln 3 + \ln 5 > 5.$$

Otherwise:

$$2\prod_{m=1}^{3} \frac{p_m^{\alpha_m+1} - 1}{p_m^{\alpha_m+1} - p_m^{\alpha_m}} < 2\prod_{m=1}^{3} \frac{p_m}{p_m - 1} \leq 2\frac{2}{1}\frac{3}{2}\frac{5}{4} = \frac{15}{2}.$$

As there is a value of α_m greater that 1, by rearrangement inequality:

$$\begin{aligned} 4 + \alpha_1 \ln p_1 + \alpha_2 \ln p_2 + \alpha_3 \ln p_3 &\geq 4 + 2\ln p_1 + \ln p_2 + \ln p_3 \\ &\geq 4 + 2\ln 2 + \ln 3 + \ln 5 > \frac{15}{2}. \end{aligned}$$

Finally, for $k = 4$,

$$2\prod_{m=1}^{4} \frac{p_m^{\alpha_m+1} - 1}{p_m^{\alpha_m+1} - p_m^{\alpha_m}} < 2\prod_{m=1}^{4} \frac{p_m}{p_m - 1} < 2\frac{2}{1}\frac{3}{2}\frac{5}{4}\frac{7}{6} = \frac{35}{4} < 9$$

and

$$4 + \ln n \geq 4 + \ln p_1 + \ldots + \ln p_4 \geq 4 + \ln(2 \times 3 \times 5 \times 7) > 9.$$

On the one hand, we see that, for $k = 5$, as $p_5 \geq 11$,

$$\prod_{m=1}^{k} \frac{p_m^{\alpha_m+1} - 1}{p_m^{\alpha_m+1} - p_m^{\alpha_m}} < \prod_{m=1}^{k} \frac{p_m}{p_m - 1} \leq \frac{2}{1}\frac{3}{2}\frac{5}{4}\frac{7}{6}\frac{11}{10} = \frac{385}{80} < 10 = 2k,$$

and, for $k > 5$, as, if $m > 5$, $p_m > p_5 + (m - 5) \geq m + 6$,

$$\begin{aligned} \prod_{m=1}^{k} \frac{p_m^{\alpha_m+1} - 1}{p_m^{\alpha_m+1} - p_m^{\alpha_m}} &= \left(\prod_{m=1}^{5} \frac{p_m^{\alpha_m+1} - 1}{p_m^{\alpha_m+1} - p_m^{\alpha_m}}\right)\left(\prod_{m=6}^{k} \frac{p_m^{\alpha_m+1} - 1}{p_m^{\alpha_m+1} - p_m^{\alpha_m}}\right) \\ &< 10\prod_{m=6}^{k} \frac{p_m^{\alpha_m+1} - 1}{p_m^{\alpha_m+1} - p_m^{\alpha_m}} < 10\prod_{m=6}^{k} \frac{p_m}{p_m - 1} \\ &\leq 10\frac{12}{11}\frac{13}{12}\cdots\frac{k+6}{k+5} = 10\frac{k+6}{11} < k + 6 \leq 2k. \end{aligned}$$

On the other hand, as, if $m \geq 5$, $\ln p_m \geq \ln 11 > \ln e^2 = 2$, for $k \geq 5$

$$4+\ln n \geq 4+\sum_{m=1}^{k} \ln p_m = 4+\sum_{m=1}^{4} \ln p_m + \sum_{m=5}^{k} \ln p_m > 9+2(k-4) > 2k,$$

thus completing the proof.

Also solved by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.

Advanced Problems

A–23. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Compute

$$\lim_{n\to+\infty}\frac{1}{n^3}\sum_{1\le i<j\le n}\frac{n+\sqrt{n(i+j)}+ij}{n^2+n(i+j)+ij}.$$

Solution by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. We can prove this limit is 0 by using the sandwich theorem:

$$\begin{aligned}0 &\le \lim_{x\to\infty}\frac{1}{n^3}\sum_{1\le i<j\le n}\frac{n+\sqrt{n(i+j)}+ij}{n^2+n(i+j)+ij}\\ &\le \lim_{x\to\infty}\frac{1}{n^3}\sum_{1\le i<j\le n}\frac{n^2+n(i+j)+ij}{n^2+n(i+j)+ij}\\ &\le \lim_{x\to\infty}\frac{1}{n^3}\sum_{1\le i<j\le n}1 \le \lim_{x\to\infty}\frac{1}{n^3}\sum_{1\le i,j\le n}1 = \lim_{x\to\infty}\frac{n^2}{n^3}=0.\end{aligned}$$

Also solved by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany, and the proposer.

A–24. *Proposed by Mihály Bencze, Braşov, Romania.* Let $ABCD$ be a quadrilateral. Let us denote by E the midpoint of side AB, F the centroid of triangle ABC, K the centroid of triangle BCD, and G the centroid of quadrilateral $ABCD$. Prove that

$$\frac{6\,MB}{MA\cdot ME}+\frac{2\,MC}{ME\cdot MF}+\frac{MD}{MF\cdot MG}\ge\frac{9\,MK}{MA\cdot MG}$$

holds for all points M in the plane of $ABCD$ different from A, E, F, G.

Solution 1 by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain. From the following identity involving complex numbers,

$$\frac{z_2}{z_1(z_1+z_2)}+\frac{z_3}{(z_1+z_2)(z_1+z_2+z_3)}$$
$$+\frac{z_4}{(z_1+z_2+z_3)(z_1+z_2+z_3+z_4)}=\frac{z_2+z_3+z_4}{z_1(z_1+z_2+z_3+z_4)},$$

we get the inequality

$$\frac{|z_2|}{|z_1|\cdot|z_1+z_2|}+\frac{|z_3|}{|z_1+z_2|\cdot|z_1+z_2+z_3|}$$
$$+\frac{|z_4|}{|z_1+z_2+z_3|\cdot|z_1+z_2+z_3+z_4|}\geq\frac{|z_2+z_3+z_4|}{|z_1|\cdot|z_1+z_2+z_3+z_4|}.$$

Setting $A(a)$, $B(b)$, $C(c)$, $D(d)$, $E\left(\frac{a+b}{2}\right)$, $F\left(\frac{a+b+c}{3}\right)$, $K\left(\frac{b+c+d}{3}\right)$, $G\left(\frac{a+b+c+d}{4}\right)$, $M(z)$, $z_1=z-a$, $z_2=z-b$, $z_3=z-c$, and $z_4=z-d$ in the preceding inequality yields

$$\frac{|z-b|}{2\cdot|z-a|\cdot\left|z-\frac{a+b}{2}\right|}+\frac{|z-c|}{6\cdot\left|z-\frac{a+b}{2}\right|\cdot\left|z-\frac{a+b+c}{3}\right|}$$
$$+\frac{|z-d|}{12\cdot\left|z-\frac{a+b+c}{3}\right|\cdot\left|z-\frac{a+b+c+d}{4}\right|}$$
$$\geq\frac{3\cdot\left|z-\frac{b+c+d}{3}\right|}{4\cdot|z-a|\cdot\left|z-\frac{a+b+c+d}{4}\right|},$$

or equivalently,

$$\frac{3\cdot MK}{4\cdot MA\cdot MG}\leq\frac{MB}{2\cdot MA\cdot ME}+\frac{MC}{6\cdot ME\cdot MF}+\frac{MD}{12\cdot MF\cdot MG},$$

from which we obtain

$$\frac{9\,MK}{MA\cdot MG}\leq\frac{6\,MB}{MA\cdot ME}+\frac{2\,MC}{ME\cdot MF}+\frac{MD}{MF\cdot MG},$$

and we are done.

Solution 2 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. Let

$$a = M - A,\ b = M - B,\ c = M - C,\ d = M - D$$

be vectors of $\mathbb{R}^2$. We have that

$$E = \frac{A+B}{2}, F = \frac{A+B+C}{3}, G = \frac{A+B+C+D}{4}, K = \frac{B+C+D}{3}.$$

So, $\overrightarrow{EM} = M - \frac{A+B}{2} = \frac{M-A+M-B}{2} = \frac{a+b}{2}$, and so, $ME = \|\frac{a+b}{2}\|$. Analogously for F, G and K, $MF = \|\frac{a+b+c}{3}\|$, $MG = \|\frac{a+b+c+d}{4}\|$, $MK = \|\frac{b+c+d}{3}\|$.

Thus, we can rewrite our inequality as:

$$\frac{6\|b\|}{\|a\|\|\frac{a+b}{2}\|} + \frac{2\|c\|}{\|\frac{a+b}{2}\|\|\frac{a+b+c}{3}\|} + \frac{\|d\|}{\|\frac{a+b+c}{3}\|\|\frac{a+b+c+d}{4}\|} \geq \frac{9\|\frac{b+c+d}{3}\|}{\|a\|\|\frac{a+b+c+d}{4}\|}$$

$$\frac{12\|b\|}{\|a\|\|a+b\|} + \frac{12\|c\|}{\|a+b\|\|a+b+c\|} + \frac{12\|d\|}{\|a+b+c\|\|a+b+c+d\|}$$
$$\geq \frac{12\|b+c+d\|}{\|a\|\|a+b+c+d\|}$$

$$\frac{\|b\|}{\|a\|\|a+b\|} + \frac{\|c\|}{\|a+b\|\|a+b+c\|} + \frac{\|d\|}{\|a+b+c\|\|a+b+c+d\|}$$
$$\geq \frac{\|b+c+d\|}{\|a\|\|a+b+c+d\|}.$$

We will now prove the following lemma:

Lemma. *Let x, y, $z \in \mathbb{R}^2$ be vectors, such that $x \neq 0$, $x+y \neq 0$, $x+y+z \neq 0$. Then the following inequality holds:*

$$\frac{\|y\|}{\|x\|\|x+y\|} + \frac{\|z\|}{\|x+y\|\|x+y+z\|} \geq \frac{\|y+z\|}{\|x\|\|x+y+z\|}.$$

Proof. We can rewrite our inequality as

$$\|y\|\|x+y+z\| + \|x\|\|z\| \geq \|y+z\|\|x+y\|.$$

We will associate a complex number to a vector of $\mathbb{R}^2$ via the following application:

$$\begin{array}{rccc} f: & \mathbb{R}^2 & \to & \mathbb{C} \\ & (a,b) & \mapsto & a+bi. \end{array}$$

Let $X = f(x)$, $Y = f(y)$, $Z = f(z)$. It is immediate to see that f is linear and that $\|x\| = |f(x)|$. Thus, by these properties of f, proving the last inequality will be equivalent to proving the following:

$$|Y||X+Y+Z| + |X||Z| \geq |Y+Z||X+Y|.$$

By the properties of complex numbers and using the triangle inequality,

$$\begin{aligned} |Y||X+Y+Z| + |X||Z| &= |XY + Y^2 + YZ| + |XZ| \\ &\geq |XY + Y^2 + YZ + XZ| \\ &= |(X+Y)(Y+Z)| = |X+Y||Y+Z|. \end{aligned}$$

□

Using this lemma, we can complete the proof:

$$\begin{aligned} &\frac{\|b\|}{\|a\|\|a+b\|} + \frac{\|c\|}{\|a+b\|\|a+b+c\|} + \frac{\|d\|}{\|a+b+c\|\|a+b+c+d\|} \\ \geq &\frac{\|(b+c)\|}{\|a\|\|a+(b+c)\|} + \frac{\|d\|}{\|a+(b+c)\|\|a+(b+c)+d\|} \\ \geq &\frac{\|b+c+d\|}{\|a\|\|a+b+c+d\|}. \end{aligned}$$

A–25. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let $z_1, z_2, \ldots, z_n$ be the zeros of the polynomial with

complex coefficients $A(z) = \sum_{k=0}^{n} a_k z^k$ ($a_n = 1$). Prove that z_i^2, $1 \le i \le n$, lie in the disk $\mathcal{C} = \{z \in \mathbb{C} : |z| \le r\}$, where

$$r = 2 \max_{0 \le k \le n-1} \left\{ \frac{|a_k|^2 + \sum_{j=1}^{n-k+1} |a_{k-j} a_{k+j}|}{|a_{k+1}|^2} \right\}$$

with $a_\ell = 0$ if $\ell < 0$ or $\ell > n$.

Solution 1 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. Let $y \in \{x_1, \ldots, x_n\}$ be one of the roots of $A(z)$ satisfying $|y| = \max_{1 \le i \le n} |z_i|$. We want to prove the following:

$$|y|^2 \le r = 2 \max_{0 \le k \le n-1} \left\{ \frac{|a_k|^2 + \sum_{j=1}^{n-k+1} |a_{k-j} a_{k+j}|}{|a_{k+1}|^2} \right\}.$$

For the special case $y = 0$, the inequality holds. Otherwise, we consider the polynomial $B(z) = A(yz) = \sum_{k=0}^{n} a_k y^k z^k = \sum_{k=0}^{n} b_k z^k$, with $b_j = a_j y^j$, $\forall j \in \{0, \ldots, n\}$. We clearly have that, if $A(z_i) = 0$, $B(\frac{z_i}{y}) = 0$, so all the roots of $B(z)$ have modulus smaller or equal than 1, and 1 is a root of $B(z)$.

With this, we see that the inequality is equivalent to:

$$\begin{aligned}
1 &\le 2 \max_{0 \le k \le n-1} \left\{ \frac{1}{|y|^2} \frac{|a_k|^2 + \sum_{j=1}^{n-k+1} |a_{k-j} a_{k+j}|}{|a_{k+1}|^2} \right\} \\
&= 2 \max_{0 \le k \le n-1} \left\{ \frac{|y|^{2k}}{|y|^{2k+2}} \frac{|a_k|^2 + \sum_{j=1}^{n-k+1} |a_{k-j} a_{k+j}|}{|a_{k+1}|^2} \right\} \\
&= 2 \max_{0 \le k \le n-1} \left\{ \frac{|a_k y^k|^2 + \sum_{j=1}^{n-k+1} |(a_{k-j} y^{k-j})(a_{k+j} y^{k+j})|}{|a_{k+1} y^{k+1}|^2} \right\} \\
&= 2 \max_{0 \le k \le n-1} \left\{ \frac{|b_k|^2 + \sum_{j=1}^{n-k+1} |b_{k-j} b_{k+j}|}{|b_{k+1}|^2} \right\}.
\end{aligned}$$

We will prove the inequality holds by contradiction. Assume that, for every k satisfying $0 \le k \le n-1$, letting $c_i = |b_i|$, $\forall i \in \{0, \ldots, n\}$

$$1 > 2 \left\{ \frac{c_k^2 + \sum_{j=1}^{n-k+1} c_{k-j} c_{k+j}}{c_{k+1}^2} \right\}.$$

This means that

$$\frac{c_{k+1}^2}{2} > c_k^2 + \sum_{j=1}^{n-k+1} c_{k-j}c_{k+j}.$$

Summing the inequalities from $k = 0$ to $k = n - 1$,

$$\frac{\sum_{k=1}^{n} c_k^2}{2} > \sum_{k=0}^{n-1} c_k^2 + \sum_{k=0}^{n-1}\sum_{j=1}^{n-k+1} c_{k-j}c_{k+j},$$

$$\sum_{k=0}^{n} c_k^2 - c_0^2 > 2\sum_{k=0}^{n} c_k^2 + 2\sum_{k=0}^{n}\sum_{j=1}^{n-k+1} c_{k-j}c_{k+j} - 2c_n^2.$$

Rearranging terms:

$$\sum_{k=0}^{n} c_k^2 + 2\sum_{k=0}^{n}\sum_{j=1}^{n-k+1} c_{k-j}c_{k+j} < 2c_n^2 - c_0^2 \leq 2c_n^2.$$

As $B(1) = 0$, $b_0 + \ldots + b_n = 0$, and so, by the triangular inequality, $c_0 + \ldots + c_{n-1} \geq c_n$.

We will prove that, $\forall s, t,\ 0 \leq s < t \leq n,\ s \equiv t \pmod 2$, there is exactly one term $c_s c_t$ in the sum $\sum_{k=0}^{n}\sum_{j=1}^{n-k+1} c_{k-j}c_{k+j}$.

We will start by proving that the term $c_s c_t$ appears at most once. Suppose that $k_1 - j_1 = s = k_2 - j_2$, $k_1 + j_1 = t = k_2 + j_2$. Summing the two equations we get $2k_1 = 2k_2$ and subtracting the first to the second, $2j_1 = 2j_2$, from where we get $(k_1, j_1) = (k_2, j_2)$.

Clearly, we have that $(k, j) = f(s, t) = \left(\frac{t+s}{2}, \frac{t-s}{2}\right)$. It is immediate to see that f is bijective, with inverse $f^{-1}(k, j) = (k - j, k + j)$. We want to see that, by this bijection, $0 \leq s < t \leq n$ implies that k and j are inside the limits of the sum. If we take into account that $t \geq s + 2$, for k it is trivial: $0 \leq s < k = \frac{s+t}{2} < t \leq n$, and for j: $1 \leq j = \frac{t-s}{2} = t - \frac{t+s}{2} = t - j \leq n - j < n - j + 1$. With this we proved that the term $c_s c_t$ appears at least once, and therefore, exactly once.

Having proved this, by the identity

$$\left(\sum_{k=0}^{n} a_k\right)^2 = \sum_{k=0}^{n} a_k^2 + 2 \sum_{0\le k<m\le n} a_k a_m$$

we get that

$$\sum_{k=0}^{n} c_k^2 + 2\sum_{k=0}^{n}\sum_{j=1}^{n-k+1} c_{k-j}c_{k+j} = \left(\sum_{\substack{0\le k\le n\\ k \text{ odd}}} c_k\right)^2 + \left(\sum_{\substack{0\le k\le n\\ k \text{ even}}} c_k\right)^2.$$

Thus we are only left with this:

$$\left(\sum_{\substack{0\le k\le n\\ k \text{ odd}}} c_k\right)^2 + \left(\sum_{\substack{0\le k\le n\\ k \text{ even}}} c_k\right)^2 < 2c_n^2.$$

As $f(x) = x^2$ is a convex function over the real line, applying Jensen's inequality:

$$\begin{aligned}\left(\sum_{\substack{0\le k\le n\\ k \text{ odd}}} c_k\right)^2 + \left(\sum_{\substack{0\le k\le n\\ k \text{ even}}} c_k\right)^2 &\ge 2\left(\frac{c_0 + \ldots + c_{n+1} + c_n}{2}\right)^2\\ &\ge 2\left(\frac{2c_n}{2}\right)^2 = 2c_n^2,\end{aligned}$$

leading to a contradiction.

Solution 2 by the proposer. The statement is a consequence of the following well known result of Gershgorin [1], which for ease of reference we state here:

Theorem 1. *Let $A = (a_{ij})$ be an $n \times n$ complex matrix, and let R_i be the sum of the moduli of the off-diagonal elements in the i-th row. Then each eigenvalue of A lays on the union of the circles $|z - a_{ii}| \le R_i$, $i = 1, 2, \cdots, n$. The analogous result holds if columns of A are considered.*

In proving our statement we will use the larger circles

$$|z| \leq |a_{ii}| + R_i, \quad i = 1, 2, \cdots, n.$$

Let $B(z) = \sum_{k=0}^{n} b_k z^k$, $b_n = 1$, be the monic complex polynomial whose zeros are $z_1^2, z_2^2, \cdots, z_n^2$, and let

$$F(B) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -b_0 \\ 1 & 0 & \cdots & 0 & -b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -b_{n-1} \end{pmatrix}$$

be its companion matrix.

Now, we take $D = \operatorname{diag}(|a_1|^2, |a_2|^2, \cdots, |a_{n-1}|^2, |a_n|^2)$ and we form the matrix

$$D^{-1}FD = \begin{pmatrix} 0 & 0 & \cdots & 0 & -\frac{b_0|a_n|^2}{|a_1|^2} \\ \dfrac{|a_1|^2}{|a_2|^2} & 0 & \cdots & 0 & -\frac{b_1|a_n|^2}{|a_2|^2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \dfrac{|a_{n-1}|^2}{|a_n|^2} & -\frac{b_{n-1}|a_n|^2}{|a_n|^2} \end{pmatrix}.$$

Since the eigenvalues of $D^{-1}FD$ are the same as those of F (that is, the zeros of $B(z)$), by direct application of Gershgorin's theorem we have that

$$|z_i^2| \leq \max_{1 \leq k \leq n-1} \left\{ \frac{|b_0|}{|a_1|^2}, \frac{|a_k|^2}{|a_{k+1}|^2} + \frac{|b_k|}{|a_{k+1}|^2} \right\}, \quad i = 1, 2, \cdots, n. \tag{3}$$

On the other hand, in [2] we established that coefficients of $B(z)$ are related to coefficients of $A(z)$ by the expressions

$$b_k = (-1)^{n-k} \left(a_k^2 + 2 \sum_{j=1}^{n-k+1} (-1)^j a_{k-j} a_{k+j} \right), \quad k = 0, 1 \cdots, n, \tag{4}$$

with $a_\ell = 0$ if $\ell < 0$ or $\ell > n$.

Substituting (4) into (3) we have

$$\begin{aligned}
|z_i^2| &\le \max_{1\le k\le n-1}\left\{\frac{|b_0|}{|a_1|^2}, \frac{|a_k|^2+|b_k|}{|a_{k+1}|^2}\right\}\\
&= \max_{1\le k\le n-1}\left\{\frac{|a_0|^2}{|a_1|^2}, \frac{|a_k|^2+|(-1)^{n-k}\left(a_k^2+2\sum_{j=1}^{n-k+1}(-1)^j a_{k-j}a_{k+j}\right)|}{|a_{k+1}|^2}\right\}\\
&\le \max_{1\le k\le n-1}\left\{\frac{|a_0|^2}{|a_1|^2}, \frac{2|a_k^2|+2\sum_{j=1}^{n-k+1}|a_{k-j}a_{k+j}|}{|a_{k+1}|^2}\right\}\\
&\le 2\max_{0\le k\le n-1}\left\{\frac{|a_k^2|+\sum_{j=1}^{n-k+1}|a_{k-j}a_{k+j}|}{|a_{k+1}|^2}\right\}, \quad i=1,2,\cdots,n,
\end{aligned}$$

and we are done.

REFERENCES

1. Gershgorin, S., Über die Abgrenzung der Eigenwerte einer Matrix. *Bull. Acad. Sc. Leningrad* (1931) 749–754.

2. Díaz-Barrero, J. L., Answer to Problem 2388, *CRUX with Mathematical Mayhem*, 25, (1999) 445–445.

A–26. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Prove the following identity:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n}}{m^2n^2} = \left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\right\}^2.$$

Solution 1 by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. On the one hand, we see that $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ converges (and converges absolutely):

$$\left|\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\right| \le \sum_{n=1}^{\infty}\left|\frac{(-1)^n}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}.$$

On the other hand,

$$\sum_{m=1}^{r}\sum_{n=1}^{s}\frac{(-1)^{m+n}}{m^2n^2} = \sum_{m=1}^{r}\left(\frac{(-1)^m}{m^2}\sum_{n=1}^{s}\frac{(-1)^n}{n^2}\right)$$
$$= \left(\sum_{n=1}^{s}\frac{(-1)^n}{n^2}\right)\left(\sum_{m=1}^{r}\frac{(-1)^m}{m^2}\right).$$

This way, we see that the partial sums converge to $\left(\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\right)^2$, as we wanted to show.

Remark: Let $E = \sum_{n=1}^{\infty}\frac{1}{(2n)^2}$, $O = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$. Now, let $A = \sum_{n=1}^{\infty}\frac{1}{n^2}$, $S = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$. It is immediate to see that $E = \frac{A}{4}$, $A = E + O$, $S = E - O$. From the well known fact that $A = \sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}$, we get $E = \frac{\pi^2}{24}$, $O = \frac{3\pi^2}{24}$, $S = -\frac{\pi^2}{12}$.

Solution 2 by the proposer. The Fourier series expansions of the functions

$$f(x) = x^2, -\pi \le x \le \pi, f(x+2\pi) = f(x)$$

and

$$g(x,y) = x^2y^2, -\pi \le x, y \le \pi, g(x+2\pi, y+2\pi) = g(x,y)$$

are

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx, \quad -\pi \le x \le \pi \tag{5}$$

and

$$x^2y^2 = \frac{\pi^4}{9} + \frac{4\pi^2}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + \frac{4\pi^2}{3}\sum_{m=1}^{\infty}\frac{(-1)^m}{m^2}\cos my$$
$$+ 16\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^{m+n}}{m^2n^2}\cos nx\cos my, \quad -\pi \le x, y \le \pi, \tag{6}$$

respectively. Setting $x = y = 0$, from (5) we have

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}, \tag{7}$$

and (6) reduces to

$$\frac{\pi^4}{9} + \frac{8\pi^2}{3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} + 16\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n}}{m^2n^2} = 0. \tag{8}$$

Substituting (7) into (8) we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{m+n}}{m^2n^2} = \frac{\pi^4}{144} = \left(-\frac{\pi^2}{12}\right)^2 = \left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\right\}^2.$$

This completes the proof.

Also solved by Alberto Espuny-Díaz, BarcelonaTech, Barcelona, Spain, and Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany.

A–27. *Proposed by Alberto Espuny-Díaz, CFIS, BarcelonaTech, Barcelona, Spain.* Let $\{a_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers defined by

$$\frac{1}{a_n} = \frac{1}{n}\sum_{k=1}^{n-1} a_k$$

with a_1 a nonzero number. Determine its convergence and, if it is convergent, compute its limit depending on a_1.

Solution 1 by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin, Germany. Let us first assume that $a_1 > 0$. We define $s_n = \sum_{k=1}^{n} a_k$, and $t_n = a_n - (n+1)$. From the recurrence,

$$a_n = \frac{n}{s_{n-1}} = \frac{n}{t_{n-1}+n}. \tag{9}$$

We can use this to find a recurrence for the t_n,

$$t_n = t_{n-1} + a_n - 1 = t_{n-1} + \frac{n}{t_{n-1}+n} - 1 = t_{n-1}\left(1 - \frac{1}{t_{n-1}+n}\right).$$

Note that, if $t_{n-1} > -(n-1)$, then the sign of t_n is the same as the sign of t_{n-1}, and $|t_n| < |t_{n-1}|$. In particular, if $t_{k-1} > -(k-1)$ for

some k, then $t_{n-1} > -(n-1)$ for all $n > k$. But this is satisfied for $k = 2$, as $t_2 = a_1 + a_2 - 3 = a_1 + \frac{2}{a_1} - 3 \geq 2\sqrt{2} - 3 > -2$ (using AM-GM for $\{a_1, \frac{2}{a_1}\}$). We conclude that $|t_k| < |t_{k-1}|$ for $k > 2$, and $|t_k| < |t_2|$ for $k > 2$. The sequence $\{t_k\}$ is bounded.

Finally, we return to the equation (9) to find $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{n}{t_{n-1}+n} = 1$. If a_1 is negative, then the terms of the sequence are the same as the terms for the same recurrence and $b_1 = -a_1$ with the signs changed, so $\lim\limits_{n\to\infty} a_n = -1$.

Solution 2 by the proposer. Let us call $S_n = \sum\limits_{k=1}^{n} a_k$ and let us consider a new sequence, $T_n = S_n - (n+1)$. We can then write the sequence above in the following fashion:

$$a_n = \frac{n}{n + T_{n-1}}. \tag{10}$$

A different expression that will come in handy is

$$\begin{aligned} T_{n+1} &= S_{n+1} - (n+2) \\ &= S_n + a_{n+1} - (n+1) - 1 \\ &= T_n + \frac{n+1}{S_n} - 1. \end{aligned} \tag{11}$$

In order to solve the problem, let us first consider that $a_1 > 0$. We then have that all the elements of the sequence $\{a_n\}$ are positive, as are all the elements of $\{S_n\}$. However, the elements of $\{T_n\}$ can be either positive or negative. Let us study the different cases.

- Case 1. If $T_{k-1} = 0 \iff S_{k-1} = k$, then using (11) and (10) we get that $T_k = 0 \implies a_k = 1$. From this, it is easy to see that $a_n = 1 \ \forall\, n \geq k$. This would mean that $\{a_n\}$ is convergent to 1.
- Case 2. If $T_{n-1} > 0 \iff S_{n-1} > n$ we then have that $0 < \dfrac{n}{S_{n-1}} < 1$ and

$$-1 < T_n = T_{n-1} + \frac{n}{S_{n-1}} - 1 < T_{n-1}.$$

- Case 3. If $T_{n-1} < 0 \iff S_{n-1} < n$, we have that $\dfrac{n}{S_{n-1}} > 1$ and
$$T_n = T_{n-1} + \frac{n}{S_{n-1}} - 1 > T_{n-1}.$$

Now, we study the second and third cases.

If at a given point $T_{n-1} = c,\ c \in \mathbb{R}^+$, then $S_{n-1} = n + c$, which means that $T_n = c + \dfrac{n}{n+c} - 1 = c - \dfrac{c}{n+c}$. From this, we can obtain that $T_n = 0 \iff c = n - 1$ (since $c > 0$). Hence,

- Case 2.1. If $c > n - 1$, then $-1 < T_n < 0$.
- Case 2.2. If $c < n - 1$, then $T_{n-1} > T_n > 0$ and $\{T_n\}$ will thereafter be decreasing and bounded by zero, which means it is convergent.

If, at a given point, $T_{n-1} = -c,\ c \in \mathbb{R}^+$, then $S_{n-1} = n - c$. Similarly to the previous case, we get that $T_n = \dfrac{c}{n-c} - c$ and that $T_n = 0 \iff c = n - 1$. Then,

- Case 3.1. If $c > n - 1$, then $T_n > 0$ and we are back to the previous case.
- Case 3.2. If $c < n - 1$ (as would happen if $T_{n-2} > n - 2$), then $T_{n-1} < T_n < 0$ and $\{T_n\}$ will thereafter be increasing and bounded by zero, and hence, convergent.

All these cases mean that $\{T_n\}$ is convergent and bounded. Taking this into account, if we consider (10) and apply a limit, we have that
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n + T_{n-1}} = 1,$$
which obviously means that the sequence $\{a_n\}$ is convergent too.

If $a_1 < 0$, setting $T_n = S_n + n + 1$, all the signs of the previous development change and the limit for the sequence is -1.

Also solved by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain.

A–28. *Proposed by José Luis Díaz-Barrero, BarcelonaTech, Barcelona, Spain.* Let n be a positive integer. Prove that

$$\frac{1}{2n}\left[\prod_{k=1}^{n} F_k^{2/n} + \left(\frac{1}{n}\sum_{k=1}^{n} F_k^n\right)^{1/n}\right] \geq \frac{(F_1F_2\ldots F_n)^{4/n}}{F_nF_{n+1}},$$

where F_n represents the nth Fibonacci number defined by $F_1 = F_2 = 1$ and for $n \geq 3$, $F_n = F_{n-1} + F_{n-2}$.

Solution by Víctor Martín Chabrera, CFIS, BarcelonaTech, Barcelona, Spain. Let us define $M_k(n)$ as:

$$M_k(n) = \begin{cases} \left(\frac{F_1^k+F_2^k+\ldots F_n^k}{n}\right)^{\frac{1}{k}} & \text{if } k \neq 0, \\ \sqrt[n]{F_1F_2\cdots F_n} & \text{if } k = 0. \end{cases}$$

By power means inequality we know $k < m$ implies $M_k(n) \leq M_m(n)$. We can rewrite our inequality as:

$$\frac{1}{2n}\left[M_0^2(n) + M_n(n)\right] \geq \frac{M_0^4(n)}{F_nF_{n+1}}.$$

We can use the following well-known identity for Fibonacci numbers: $F_1^2 + F_2^2 + \ldots + F_n^2 = F_nF_{n+1}$ (which can be very easily proved by induction), to get $F_nF_{n+1} = nM_2^2(n)$, and so we can rewrite our inequality as

$$\frac{1}{2n}\left[M_0^2(n) + M_n(n)\right] \geq \frac{M_0^4(n)}{nM_2^2(n)},$$

and even simpler,

$$M_0^2(n)M_2^2(n) + M_n(n)M_2^2(n) \geq 2M_0^4(n).$$

Using the closed form of the Fibonacci sequence, $F_n = \frac{\varphi^n-(-\varphi)^{-n}}{\sqrt{5}}$

(where $\varphi = \frac{1+\sqrt{5}}{2}$), we have, on the one hand,

$$\begin{aligned}
M_2^2(n) &= \frac{F_n F_{n+1}}{n} = \frac{1}{5n}(\varphi^n - (-\varphi)^{-n})(\varphi^{n+1} - (-\varphi)^{-(n+1)}) \\
&= \frac{1}{5n}(\varphi^{2n+1} - (-1)^{-n-1}(-\varphi)^{-1} - (-1)^{-n}\varphi^1 + (-\varphi)^{-2n-1}) \\
&= \frac{1}{5n}(\varphi^{2n+1} - (-1)^{-n}(\varphi^1 - (-\varphi)^{-1}) + (-\varphi)^{-2n-1}) \\
&= \frac{1}{5n}(\varphi^{2n+1} - (-1)^{-n}(2\varphi - 1) + (-\varphi)^{-2n-1}) \geq \frac{\varphi^{2n+1}}{5n} - \frac{3}{5n}
\end{aligned}$$

We see that, if m is an even number, $F_m \leq \frac{\varphi^m}{\sqrt{5}}$ and $F_m F_{m+1} \leq \frac{\varphi^{m+(m+1)}}{5}$, and so, if n is even,

$$\begin{aligned}
M_0(n) &= \sqrt[n]{F_1 F_2 \cdots F_n} = \sqrt[n]{F_1(F_2F_3)\cdots(F_{n-2}F_{n-1})F_n} \\
&\leq \sqrt[n]{\frac{\varphi^{\frac{n(n+1)}{2}-1}}{\sqrt{5}^{n-1}}} = \sqrt[n]{\frac{\sqrt{5}}{\varphi}\frac{\varphi^{\frac{n(n+1)}{2}}}{\sqrt{5}^n}} = \frac{\varphi^{\frac{n+1}{2}}}{\sqrt{5}}\sqrt[n]{\frac{\sqrt{5}}{\varphi}}.
\end{aligned}$$

And if it is odd,

$$\begin{aligned}
M_0(n) &= \sqrt[n]{F_1 F_2 \cdots F_n} = \sqrt[n]{F_1(F_2F_3)\cdots(F_{n-1}F_n)} \\
&\leq \sqrt[n]{\frac{\varphi^{\frac{n(n+1)}{2}-1}}{\sqrt{5}^{n-1}}} = \sqrt[n]{\frac{\sqrt{5}}{\varphi}\frac{\varphi^{\frac{n(n+1)}{2}}}{\sqrt{5}^n}} = \frac{\varphi^{\frac{n+1}{2}}}{\sqrt{5}}\sqrt[n]{\frac{\sqrt{5}}{\varphi}}.
\end{aligned}$$

Thus, we have

$$\frac{M_2^2(n)}{2M_0^2(n)} \geq \frac{\frac{\varphi^{2n+1}}{5n} - \frac{2}{5n}}{\frac{2}{5}\varphi^{n+1}\sqrt[n]{\frac{5}{\varphi^2}}} = \frac{\varphi^n}{2n}\sqrt[n]{\frac{\varphi^2}{5}} - \frac{3}{2n\varphi^{n+1}\sqrt[n]{\frac{5}{\varphi^2}}}.$$

We can calculate that for $n = 6$ this is strictly larger than 1. We can see that, if this inequality holds for n, it holds for $n+1$. To prove that, we will check three things:

- If $n \geq 6$, $\frac{\varphi^{n+1}}{2(n+1)} > \frac{\varphi^n}{2n}$ holds:

$$\begin{aligned}
\frac{\frac{\varphi^{n+1}}{2(n+1)}}{\frac{\varphi^n}{2n}} > 1 &\iff \varphi\frac{n}{n+1} > 1 \iff \varphi n > n+1 \\
&\iff (\varphi - 1)n > 1 \iff n > \varphi.
\end{aligned}$$

- $f(x) = a^{1/x}$, defined over the positive real numbers, is monotonically increasing if $0 < a < 1$:

$$f'(x) = -\ln(a)\frac{1}{x^2}a^{1/x} > 0.$$

Having proved this, we are left with:

$$1 < \frac{\varphi^n}{2n}\sqrt[n]{\frac{\varphi^2}{5}} - \frac{3}{2n\varphi^{n+1}\sqrt[n]{\frac{5}{\varphi^2}}} = \sqrt[n]{\frac{\varphi^2}{5}}\left(\frac{\varphi^n}{2n} - \frac{3}{2n\varphi^{n+1}}\right)$$
$$< \sqrt[n+1]{\frac{\varphi^2}{5}}\left(\frac{\varphi^{n+1}}{2(n+1)} - \frac{3}{2(n+1)\varphi^{n+2}}\right) \leq \frac{M_2^2(n+1)}{2M_0^2(n+1)}.$$

For $n = 5$ we see that $2M_0^2(5) = 2(\sqrt[5]{1 \times 1 \times 2 \times 3 \times 5})^2 = 2(\sqrt[5]{30})^2 < 2 \times 2^2 = 8$ and $M_2^2(5) = \frac{F_5F_6}{5} = \frac{5\times 8}{5} = 8$. With this, we have seen that $M_2^2(n) > 2M_0^2$ holds for all $n \geq 5$, and so, if $n \geq 5$:

$$M_0^2(n)M_2^2(n) + M_n(n)M_2^2(n) > M_0^2(n)M_2^2(n) > 2M_0^4(n).$$

We are simply left with four cases. For $n = 1$, $n = 2$ we have $F_1 = F_2 = 1$ and so, $M_k(n) = 1$ for all k, and we get $M_0^2(n)M_2^2(n) + M_n(n)M_2^2(n) = 2 = 2M_0^4(n)$. Finally for $n = 3$, $n = 4$ it can be easily checked by hand that the inequality is satisfied, without reaching equality.

Also solved by the proposer.

Arhimede Mathematical Journal

Volume 3, No. 1 Spring 2016

Aim and Scope

The goal of *Arhimede Mathematical Journal* is to provide a means of publication of useful materials to train students for Mathematical Contests at all levels. Potential contributions include any work involving fresh ideas and techniques, problems and lessons helpful to train contestants, all written in a clear and elegant mathematical style. All areas of mathematics, including algebra, combinatorics, geometry, number theory and real and complex analysis, are considered appropriate for the journal.

Information for Authors

A detailed statement of author guidelines is available at the journal's website:

http://www.amj-math.com

ISSN 2462-537X