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Some discrete inequalities

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Abstract

In this short paper some ideas to derive new inequalities with constrains are presented. More precisely, by means of applying discrete classical Jensen's inequality to convex functions or integrating algebraic discrete inequalities, some elementary numerical inequalities involving positive real numbers are obtained. Moreover, an additive inequality is also given.

1 Introduction

The three great pillars of the theory of inequalities are positivity, monotonicity, and convexity. The notions of positivity and monotonicity combined with convexity are the source of a lot of nice inequalities and beautiful problems, such as the ones which appeared in [1, 2, 3, 4]. Our aim in this short paper is to use these concepts to obtain new constrained inequalities.

2 The Results

Hereafter, some general ideas to obtain new discrete inequalities are presented. Several examples applying them are also given. We begin with

Theorem 1 *Let $x_1, x_2, \ldots, x_n$, $n \geq 3$, and α be positive real numbers such that*

$$\sum_{cyclic} x_1^2 x_2^2 \ldots x_{n-1}^2 = n\alpha^{n-2} x_1 x_2 \ldots x_n$$

Then it is true that $\frac{1}{n}\sum_{j=1}^{n} f(a_j) \geq f\left(\alpha^{n-2}\right)$, *where f is a convex function and* $a_j = \frac{1}{x_j^2}\prod_{k=1}^{n} x_k$ *for all $1 \leq j \leq n$ (if f is concave then the inequality reverses).*

Proof. Applying Jensen's inequality to f with the a_j $(1 \leq j \leq n)$ given in the statement, we obtain

$$\frac{1}{n}\sum_{j=1}^{n} f(a_j) \geq f\left(\frac{1}{n}\sum_{j=1}^{n} a_j\right)$$

But

$$\frac{1}{n}\sum_{j=1}^{n} a_j = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{1}{x_j^2}\prod_{k=1}^{n} x_k\right)$$

$$= \frac{1}{n}\left(\frac{x_1 x_2 \ldots x_{n-1}}{x_n} + \frac{x_2 x_3 \ldots x_n}{x_1} + \ldots + \frac{x_n x_1 \ldots x_{n-2}}{x_{n-1}}\right) = \alpha^{n-2}$$

The statement immediately follows and the proof is complete. □

Corollary 2 *Let a, b, c be positive real numbers such that $a^2b^2 + b^2c^2 + c^2a^2 = 9abc$. Then the following holds:*

$$\sqrt[3]{\frac{a}{a+bc}} + \sqrt[3]{\frac{b}{b+ca}} + \sqrt[3]{\frac{c}{c+ab}} \geq 3\sqrt[3]{\frac{1}{4}}$$

Proof. Let $f : (0, +\infty) \to \mathbb{R}$ be the function defined by $f(t) = \frac{1}{\sqrt[3]{1+t}}$. Since $f'(t) = -\frac{1}{3(1+t)^{4/3}} < 0$ and $f''(t) = \frac{4}{9(1+t)^{7/3}} > 0$, f is convex. Applying Jensen's inequality to the function f with $t_1 = \frac{bc}{a}, t_2 = \frac{ca}{b}, t_3 = \frac{ab}{c}$, we have

$$\frac{1}{3}\sum_{k=1}^{3} f(t_k) \geq f\left(\frac{1}{3}\sum_{k=1}^{3} t_k\right)$$

or

$$\frac{1}{3}\left(\sqrt[3]{\frac{a}{a+bc}} + \sqrt[3]{\frac{b}{b+ca}} + \sqrt[3]{\frac{c}{c+ab}}\right) \geq f(3) = \sqrt[3]{\frac{1}{4}}$$

because from $a^2b^2 + b^2c^2 + c^2a^2 = 9abc$ it immediately follows that

$$\frac{t_1 + t_2 + t_3}{3} = \frac{1}{3}\left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c}\right) = 3$$

Equality holds when $a = b = c = 3$, and we are done. □

Combining convexity with mean inequalities, we have the following

Theorem 3 *Let $x_1, x_2, \ldots, x_n$ be $n \geq 2$ positive real numbers and let $f : (0, +\infty) \to \mathbb{R}$ be a convex function.*
(i) If $x_1 + x_2 + \ldots + x_n = 1$, then

$$\frac{1}{n}\sum_{j=1}^{n} f(x_j) \geq f\left(\frac{1}{n}\right)$$

(If f is concave then the inequality reverses).

(ii) If $x_1x_2\ldots x_n = 1$ *and* f *is increasing, then*

$$\frac{1}{n}\sum_{j=1}^{n} f(x_j) \geq f\,(1)$$

(iii) If $\dfrac{1}{x_1} + \dfrac{1}{x_2} + \ldots \dfrac{1}{x_n} = 1$ *and* f *is increasing, then*

$$\frac{1}{n}\sum_{j=1}^{n} f(x_j) \geq f\,(n)$$

Proof. (i) Applying Jensen's inequality to f with the x_j $(1 \leq j \leq n)$ given in the statement, we obtain

$$\frac{1}{n}\sum_{j=1}^{n} f(x_j) \geq f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) = f\left(\frac{1}{n}\right)$$

(ii) We have

$$\frac{1}{n}\sum_{j=1}^{n} f(x_j) \geq f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) \geq f\left(\left[\prod_{j=1}^{n} x_j\right]^{1/n}\right) = f(1)$$

on account of Jensen's and AM-GM inequality.

(iii) We have

$$\frac{1}{n}\sum_{j=1}^{n} f(x_j) \geq f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right) \geq f\left[\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)^{-1}\right] = f(n)$$

on account of Jensen's and AM-HM inequality. □

Corollary 4 *Let* n *be a positive integer. Then*

$$\frac{F_1}{F_1 + \sqrt{F_nF_{n+1}}} + \frac{F_2}{F_2 + \sqrt{F_nF_{n+1}}} + \ldots + \frac{F_n}{F_n + \sqrt{F_nF_{n+1}}} \leq \frac{n}{1+\sqrt{n}}$$

holds, where F_n *represents the* n^{th} *Fibonacci number defined by* $F_0 = 0, F_1 = 1$ *and* $F_n = F_{n-1} + F_{n-2}$ *for all* $n \geq 2$.

Proof. We consider the function $f : (0, +\infty) \to \mathbb{R}$ defined by $f(t) = \dfrac{\sqrt{t}}{1+\sqrt{t}}$. Since $f'(t) = \dfrac{1}{2\sqrt{t}(1+\sqrt{t})^2} > 0$ and $f''(t) = \dfrac{-(1+3\sqrt{t})}{4t^{3/2}(1+\sqrt{t})^3} < 0$, f is concave. Applying (i) to f with the numbers $x_i = \dfrac{F_i^2}{F_n F_{n+1}}$ for $1 \le i \le n$, whose sum is one – as can be easily proven – the statement immediately follows. Equality holds when $n = 1$, and the proof is complete. □

Corollary 5 *Let $a_1, a_2, \ldots, a_n$ be $n \ge 2$ positive real numbers. Then the following holds:*

$$\frac{a_1}{3a_2 + 5\sqrt[5]{a_1 a_2^4}} + \frac{a_2}{3a_3 + 5\sqrt[5]{a_2 a_3^4}} + \ldots + \frac{a_n}{3a_1 + 5\sqrt[5]{a_n a_1^4}} \ge \frac{n}{8}$$

Proof. Consider the function $f : (0, +\infty) \to \mathbb{R}$ defined by $f(t) = \dfrac{t^5}{3+5t}$. We have $f'(t) = \dfrac{5t^4(3+4t)}{(3+5t)^2} > 0$, so f is increasing, and $f''(t) = \dfrac{30t^3(6+15t+10t^2)}{(3+5t)^3} > 0$, so it is convex. Setting $t_k = \sqrt[5]{\dfrac{a_k}{a_{k+1}}}$ for $1 \le k \le n-1$ and $t_n = \sqrt[5]{\dfrac{a_n}{a_1}}$ into $f(t)$, we get

$$\frac{1}{n}\sum_{k=1}^{n} f(t_k) \ge f\left(\frac{1}{n}\sum_{k=1}^{n} t_k\right) \ge f\left(\left[\prod_{k=1}^{n} t_k\right]^{1/n}\right) = f(1) = \frac{1}{8}$$

on account of (ii). Using the preceding inequalities, we have

$$\frac{1}{n}\sum_{k=1}^{n} \frac{a_k/a_{k+1}}{3 + 5\sqrt[5]{a_k/a_{k+1}}} \ge \frac{1}{8},$$

where $a_{n+1} = a_1$, and the statement immediately follows. Equality holds when $a_1 = a_2 = \ldots = a_n$, and the proof is complete. □

Using monotonicity jointly with convexity and Chebyshev's inequality, we have the following

Theorem 6 *Let $a_1 \leq a_2 \leq \ldots \leq a_n$, $b_1 \leq b_2 \leq \ldots \leq b_n$ be two nondecreasing sequences of positive real numbers, each of them with sum up to one, and let $f : (0, +\infty) \to \mathbb{R}$ be an increasing and convex function. Then it is true that*

$$\frac{1}{n}\sum_{j=1}^{n} f(a_j b_j) \geq f\left(\frac{1}{n^2}\right)$$

Proof. Applying Jensen's inequality to f with the $a_j b_j$ $(1 \leq j \leq n)$ given in the statement yields

$$\frac{1}{n}\sum_{j=1}^{n} f(a_j b_j) \geq f\left(\frac{1}{n}\sum_{j=1}^{n} a_j b_j\right) \geq f\left[\left(\frac{1}{n}\sum_{j=1}^{n} a_j\right)\left(\frac{1}{n}\sum_{j=1}^{n} b_j\right)\right] = f\left(\frac{1}{n^2}\right)$$

on account of Chebyshev's inequality and the constraint. □

Next we give the following additive inequality.

Theorem 7 *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be sequences of positive reals, and let $c_1, c_2, \ldots, c_n$ and $d_1, d_2, \ldots, d_n$ be sequences of nonnegative reals, each with sum one. Then,*

$$\sum_{k=1}^{n} c_k a_k^3 + \sum_{k=1}^{n} d_k b_k^3 \geq \sum_{k=1}^{n} c_k a_k \sum_{k=1}^{n} d_k b_k^2 + \sum_{k=1}^{n} d_k b_k \sum_{k=1}^{n} c_k a_k^2$$

Proof. Using the elementary inequality $a^3 + b^3 \geq ab^2 + a^2 b$ valid for all positive real numbers a, b (with equality when $a = b$), we have

$$a_i^3 + b_j^3 \geq a_i b_j^2 + a_i^2 b_j, \quad 1 \leq i, j \leq n$$

Multiplying by $c_i d_j \geq 0$, for all $1 \leq i, j \leq n$, and adding up the resulting inequalities yields

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i d_j (a_i^3 + b_j^3) \geq \sum_{i=1}^{n}\sum_{j=1}^{n} c_i d_j a_i b_j^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} c_i d_j a_i^2 b_j$$

from which

$$\sum_{k=1}^{n} d_k \sum_{k=1}^{n} c_k a_k^3 + \sum_{k=1}^{n} c_k \sum_{k=1}^{n} d_k b_k^3 \geq \sum_{k=1}^{n} c_k a_k \sum_{k=1}^{n} d_k b_k^2 + \sum_{k=1}^{n} d_k b_k \sum_{k=1}^{n} c_k a_k^2$$

follows. Taking into account the constraint, we can deduce the statement. Equality holds when $a_i = b_j$, $1 \leq i, j \leq n$, and the proof is complete. □

Finally, integrating an algebraic inequality we have obtained the following result.

Theorem 8 *Let x, y, z be three nonnegative real numbers. Then, the following holds:*

$$\left(\sum_{cyc} \frac{3x+1}{(2x+1)(4x+1)}\right)\left(\sum_{cyc} \frac{(x+y+1)(2x+y+z+1)}{3x+2y+z+2}\right) \geq \frac{9}{2}$$

Proof. First, we write the claimed inequality in the form

$$\frac{1}{3}\sum_{cyc} \frac{3x+1}{(2x+1)(4x+1)} \geq \frac{3}{2}\left(\sum_{cyc} \frac{(x+y+1)(2x+y+z+1)}{3x+2y+z+2}\right)^{-1}$$

Now we insert the term $\frac{1}{6}\sum_{cyc} \frac{3x+2y+z+2}{(x+y+1)(2x+y+z+1)}$ and we get

$$\begin{aligned}\frac{1}{3}\sum_{cyc} \frac{3x+1}{(2x+1)(4x+1)} &\geq \frac{1}{6}\sum_{cyc} \frac{3x+2y+z+2}{(x+y+1)(2x+y+z+1)} \\ &\geq \frac{3}{2}\left(\sum_{cyc} \frac{(x+y+1)(2x+y+z+1)}{3x+2y+z+2}\right)^{-1}\end{aligned}$$

The last inequality is an immediate consequence of AM-HM inequality and the first is equivalent to

$$\sum_{cyc} \frac{3x+1}{(2x+1)(4x+1)} \geq \frac{1}{2}\sum_{cyc} \frac{3x+2y+z+2}{(x+y+1)(2x+y+z+1)}$$

that can be written as

$$\sum_{cyc} \frac{1}{4x+1} + \sum_{cyc} \frac{1}{2x+1} \geq \sum_{cyc} \frac{3x+2y+z+2}{(x+y+1)(2x+y+z+1)}$$

To prove the preceding, we observe that for any positive reals a, b, c it is true that

$$\sum_{cyc} a^4 + \sum_{cyc} a^2 \geq \sum_{cyc} ab(1+ca)$$

Indeed, we have

$$\sum_{cyc} ab(1+ca) = (a^2bc + ab^2c + abc^2) + (ab+bc+ca)$$
$$\leq (ab)^2 + (bc)^2 + (ca)^2 + (ab+bc+ca) \leq \sum_{cyc} a^4 + \sum_{cyc} a^2$$

where we have used the well-known inequality $a^2 + b^2 + c^2 \geq ab + bc + ca$. Equality holds when $a = b = c$.

Putting $a = t^x, b = t^y$, and $c = t^z (t > 0)$ in

$$\sum_{cyc} a^4 + \sum_{cyc} a^2 - \sum_{cyc} ab(1+ca) \geq 0$$

we obtain

$$f(t) = \sum_{cyc} t^{4x} + \sum_{cyc} t^{2x} - \sum_{cyc} (t^{x+y} + t^{2x+y+z}) \geq 0$$

Then,

$$\int_0^1 \left(\sum_{cyc} t^{4x} + \sum_{cyc} t^{2x} - \sum_{cyc} (t^{x+y} + t^{2x+y+z}) \right) \mathrm{d}t \geq 0$$

That is,

$$\left(\sum_{cyc} \frac{1}{4x+1} \right) + \left(\sum_{cyc} \frac{1}{2x+1} \right) - \sum_{cyc} \left(\frac{1}{x+y+1} + \frac{1}{2x+y+z+1} \right)$$
$$= \left(\sum_{cyc} \frac{1}{4x+1} \right) + \left(\sum_{cyc} \frac{1}{2x+1} \right) - \sum_{cyc} \frac{3x+2y+z+2}{(x+y+1)(2x+y+z+1)} \geq 0$$

from which the inequality we wanted to prove follows. Equality holds when $x = y = z$, and we are done.

□

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Problems

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical problems. Proposals are always welcome. Please observe the following guidelines when submitting proposals or solutions:

1. Proposals and solutions must be legible and should appear on separate sheets, each indicating the name and address of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editor has supplied a solution.

2. Send submittals to: **José Luis Díaz-Barrero**, Applied Mathematics III, UPC BARCELONA TECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain or by e-mail to: *jose.luis.diaz@upc.edu*.

The section is divided into four subsections: Elementary Problems, Easy-Medium High School Problems, Medium-Hard High School Problems, and Advanced Problems mainly for undergraduates. Proposals that appeared in Math Contests around the world and most appropriate for Math Olympiads training are always welcome. The source of these proposals will appear when the solutions are published.

Solutions to the problems stated in this issue should be posted before

September 30, 2014

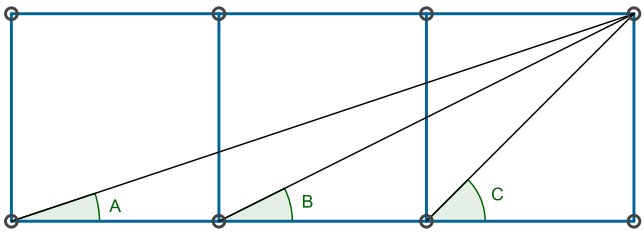
Elementary Problems

E–1. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Without the aid of a computer, calculate the following sum:

$$2014^2 - 2013^2 + 2012^2 - 2011^2 + \ldots + 4^2 - 3^2 + 2^2 - 1^2$$

E–2. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* On a table there are several notes of $5, 10, 20, 50, 100,$ 200 and 500 Euro. If the total number of notes is 211, then show that there are at least 31 with the same value.

E–3. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Suppose that in the following figure all the squares are of side length one. Compute the value of $A + B + C$.



Statement of the problem

E–4. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Find the value of

$$\frac{1}{2 + \sqrt{2}} + \frac{1}{3\sqrt{2} + 2\sqrt{3}} + \ldots + \frac{1}{2025\sqrt{2024} + 2024\sqrt{2025}}$$

E–5. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let I be the sum of the interior angles of a convex n-sided polygon and let E be the sum of its exterior angles taken counterclockwise. Compute

$$\frac{I-E}{I+E}$$

E–6. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Find the number of zeros at the end of

$$2014! = 1 \cdot 2 \cdot 3 \ldots 2012 \cdot 2013 \cdot 2014$$

E–7. *Proposed by Alberto Espuny-Díaz, CFIS, Barcelona Tech, Barcelona, Spain.* Compute the value of

$$\log \frac{1}{2} + \log \frac{2}{3} + \log \frac{3}{4} + \ldots + \log \frac{9999}{10000}$$

E–8. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Find the value of

$$\left(1 + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2014} + \frac{1}{2015}\right)\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2014}\right)$$
$$-\left(\frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2014} + \frac{1}{2015}\right)\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{2014}\right)$$

E–9. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* A set of 2016 points in general position (no three of them are collinear) are chosen in the plane. How many disjoint triangles can be built with them?

E–10. *Proposed by José Gibergnas-Báguena, Barcelona Tech, Barcelona, Spain.* Find the last four digits of the number

$$9 + 99 + 999 + 9999 + \ldots + \underbrace{999\ldots9}_{2014}$$

Easy–Medium Problems

EM–1. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* A triangle is drawn in a sheet of paper. A blind man draws a line that does not pass through any vertex of the triangle. Show that the line does not cut the three sides of the triangle.

EM–2. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let a, b, c be positive real numbers. Prove that

$$\left(a^{bc}\, b^{ca}\, c^{ab}\right)^{\frac{1}{ab+bc+ca}} \leq \frac{(a+b+c)^3}{9\,(ab+bc+ca)}$$

EM–3. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Show that there exist infinitely many triples (x, y, z) of positive integers such that $x < y < z$ and $2y^2 = x^2 + z^2$.

EM–4. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let a, b, c be positive real numbers. Prove that

$$\frac{a^4+10a^2+5}{b^3+b^2+b+1} + \frac{b^4+10b^2+5}{c^3+c^2+c+1} + \frac{c^4+10c^2+5}{a^3+a^2+a+1} \geq 12$$

EM–5. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* How many ways are there to weigh 31 grams with a balance if we have 7 weights of one gram, 5 of two grams, and 6 of five grams, respectively?

EM–6. *Proposed by Mihály Bencze, Braşov, Romania.* Let a, k be positive integers and let n be a nonnegative integer. Show that $(ka^2+1)^{2n+1}$ can be expressed as a sum of $k+1$ squares and $(ka^2+1)^{2n+2}$ can be expressed as a sum of $(k+1)^2$ squares.

EM–7. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* If $P(a, b, c)$ is a common point of the plane $x + y + z + 3 = 0$ and the sphere $x^2 + y^2 + z^2 = 9$ not lying on the cartesian axes, then show that

$$\frac{ab}{a+b} + \frac{bc}{b+c} + \frac{ca}{c+a}$$

does not depend on the position of point P.

EM–8. *Proposed by Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.* Find all pairs of non-negative integers (a, b) satisfying

$$a^2 2^a = 8b^3 3^b$$

EM–9. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let A, B, C, D be four points lying on the circle $\Gamma(O, R)$ such that the chords AC and BD are perpendicular and meet at point P. Prove that

$$AP^2 + BP^2 + CP^2 + DP^2 = 4R^2$$

EM–10. *Proposed by Nicolae Papacu, Slobozia, Romania.* Let $n \geq 1$ be a natural number and consider the equation

$$\left\lfloor \frac{x}{1 \cdot 3} \right\rfloor + \left\lfloor \frac{x}{3 \cdot 5} \right\rfloor + \left\lfloor \frac{x}{5 \cdot 7} \right\rfloor + \ldots + \left\lfloor \frac{x}{(2n-1)(2n+1)} \right\rfloor = n,$$

where x is a natural number. Find all the values of n for which the equation has n solutions for x ($\lfloor \cdot \rfloor$ denotes the integer part).

Medium–Hard Problems

MH–1. *Proposed by Diana Alexandrescu, Bucharest, Romania and José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let p, n be positive integers such that p is prime and $p < n$. If p divides $n+1$ and $\left(\left[\frac{n}{p}\right], (p-1)!\right) = 1$, then prove that $p \cdot \left[\frac{n}{p}\right]^2$ divides $\binom{n}{p} - \left[\frac{n}{p}\right]$ (here $[x]$ is the integer part of the real number x).

MH–2. *Proposed by Nicolae Papacu, Slobozia, Romania.* For all positive integer n we consider the number $a_n = 4^{6^n} + 1943$. Prove that a_n is divisible by 2013 for all $n \geq 1$, and find all values of n for which $a_n - 207$ is the cube of a positive integer.

MH–3. *Proposed by Radu Bairac, Chisinau, Republic of Moldova.* Let ABC be a triangle with $\angle ABC = 120°$ and angle bisectors AA_1, BB_1, CC_1, respectively. Let F be a point such that $B_1F \perp A_1C_1$, where $F \in A_1C_1$. If R, I and S are the incenters of the triangles C_1B_1F, $C_1B_1A_1$ and A_1B_1F, and $B_1S \cap A_1C_1 = \{Q\}$, then show that R, I, S and Q are concyclic.

MH–4. *Proposed by Iván Geffner Fuenmayor, Barcelona Tech, Barcelona, Spain.* Let Γ be the circumcircle of a triangle ABC and let E and F be the intersections of the bisectors of $\angle ABC$ and $\angle ACB$ with Γ. If EF is tangent to the incircle γ of $\triangle ABC$, then find the value of $\angle BAC$.

MH–5. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let x_i, $1 \leq i \leq 12$ be positive real numbers with product 1. Prove that

$$\sum_{cyclic} \frac{x_1^{13} + x_2^{11}}{x_2 + x_3 + \ldots + x_{12}} \geq \frac{24}{11}$$

MH–6. *Proposed by Sorin Radulescu, Bucharest, Romania.* Let p be an odd positive integer. Find all natural numbers $n \geq 2$ for which the following holds:

$$\sum_{i=1}^{n} \prod_{j \neq i} (x_i - x_j)^p \geq 0 \quad \forall\, x_1, \ldots, x_n \in \mathbb{R}$$

MH–7. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let u be the real root of the equation $x^3 - 3x^2 + 5x - 27 = 0$, and let v be the real root of the equation $x^3 - 3x^2 + 5x + 21 = 0$. Find $u + v$.

MH–8. *Proposed by Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.* We have $2n$ positive integers, the sum of which is a multiple of n. A step consists of choosing n numbers and adding them a positive integer (the same to all of them). Prove that we can make all $2n$ numbers equal in at most $2n - 1$ steps.

MH–9. *Proposed by Marius Dragan, Bucharest, Romania.* Let n be a positive integer. For $0 < x < 1$ prove that

$$\left\lfloor \sqrt{n - x} + \sqrt{n + x} + \sqrt{n} \right\rfloor = \left\lfloor \sqrt{9n - 2} \right\rfloor$$

MH–10. *Proposed by Ander Lamaison Vidarte, CFIS, Barcelona Tech, Barcelona, Spain.* Let $S = \{1, 2, \ldots, n\}$, $n \geq 2$, and let $f : S \longrightarrow S$ be a bijective function distinct from the identity. Let $u = \sum\limits_{k=1}^{n} |f(k) - k|$ and let v be the number of ordered pairs (a, b) of elements of S such that $a > b$ and $f(a) < f(b)$. Show that $v < u \leq 2v$, and that $u = 2v$ if and only if there do not exist positive integers $a > b > c$ such that $f(a) < f(b) < f(c)$.

Advanced Problems

A–1. *Proposed by Mihály Bencze, Braşov, Romania.* Find all real solutions of the system

$$\begin{aligned} \sin x + \sin y + \sin z &= 0, \\ \cos x + \cos y + \cos z &= 0, \\ \tan 3^k x + \tan 3^k y + \tan 3^k z &= 3(2-\sqrt{3}), \end{aligned}$$

where k is a positive integer.

A–2. *Proposed by Dan Popescu, Suceava, Romania.* Let p be a prime number. Show that for all $x \in \mathbb{R}$ the following is true:

$$\sum_{k=0}^{p-1} (-1)^k \cos^p \left(x + \frac{k\pi}{p} \right) = \frac{p \cos px}{2^{p-1}}$$

A–3. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain, and José Gibergans-Báguena, Barcelona Tech, Barcelona, Spain.* Calculate

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{1\le i<j\le n} \left(\frac{(n^2+i^2)(n^2+j^2)}{n^4+i^2+j^2} \right)^{1/2}$$

A–4. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let $\{x_n\}_{n\ge1}$ be a sequence of real numbers. Compute

$$\sum_{n=1}^{\infty} \left[\sin^2 x_n \prod_{k=1}^{n-1} \cos^2 x_k \right] + \prod_{n=1}^{\infty} \cos^2 x_n$$

A–5. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let $a < b$ be real numbers and let $f : [a,b] \to \mathbb{R}$ be an increasing (decreasing) function. Show that the set of discontinuities of f is countable.

A–6. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \ln\left(x + \sqrt{1+x^2}\right) + \left(1+x^2\right)^{-1/2} - x\,\left(1+x^2\right)^{-3/2}$$

and let $a < b$ be two real numbers such that $\ln\left(\dfrac{f(b)}{f(a)}\right) = b - a.$ Show that there exists $c \in (a, b)$ for which it is true that

$$2c^2 = 1 + (1+c^2)^{5/2} \ln(c + \sqrt{1+c^2})$$

A–7. *Proposed by Nicolae Papacu, Slobozia, Romania.* Let $\{I_n\}_{n\geq 1}$ be the sequence defined by

$$I_n = \int_0^n x^n \arctan x \, dx$$

Compute $\lim\limits_{n\to\infty} \dfrac{I_n}{n^n}$.

A–8. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let a, b be real numbers such that $-b \leq a \leq b.$ Compute the sum

$$\sum_{n=0}^{\infty} (-1)^n \frac{\sinh\left((2n+1)a\right)}{(2n+1)e^{(2n+1)b}}$$

A–9. *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Let $A(x) \in \mathbb{Z}[x]$ be a polynomial of degree n and let $a_0 < a_1 < \ldots < a_n$ be integers. Show that $|A(a_k)| \geq 2^{-n}\, n!$ for some $k \in \{0, 1, 2, \ldots, n\}$.

A–10. *Proposed by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain.* Compute

$$\int_0^{\infty} \frac{dx}{(1+x^2)(1+x^e)}$$

Mathlessons

This section of the Journal offers readers an opportunity to exchange interesting and elegant mathematical notes and lessons including material useful to solve mathematical problems.

Send submittals to: **José Luis Díaz-Barrero**, Applied Mathematics III, UPC BARCELONA TECH, Jordi Girona 1-3, C2, 08034 Barcelona, Spain or by e-mail to: *jose.luis.diaz@upc.edu*.

Thales, Ceva and Menelaus

J. L. Díaz-Barrero and F. Gispert-Sánchez

1 Introduction

In geometry, a *transversal* is a line that cuts two lines in the same plane at two distinct points. There is no doubt that one of the ancient and most famous results involving transversal lines is Thales theorem, who used it to determine the height of the Cheops pyramid, among other applications. Another important result in classical elementary geometry, often attributed to the Italian mathematician Giovanni Ceva, gives a necessary and sufficient condition of concurrence of segments in a triangle. Finally, a third pillar in the geometry of the triangle relating the lengths of the six resulting segments when the three sides of a triangle are cut by a transversal is Menelaus' theorem. The goal of these notes is to revisit these well-known results and to give some of their applications.

2 Three main results

In the following theorems, we are going to consider that line segments are oriented, that is to say, that they have a sign (i.e., $PQ = -QP$ for any two points P and Q). Thus, for any three points A, B, X that lie on the same line, $\frac{AX}{XB}$ is positive if X lies between A and B and is negative if X lies on the extension of the segment AB.

We begin this section stating and proving the **intercept theorem**, also known as Thales' theorem. It is a basic result in classical elementary geometry relating the ratios of several line segments that are obtained if two intersecting lines are intercepted by a pair of parallels. It is equivalent to the theorem about ratios in similar triangles and it must not be confused with another theorem with that name related to right triangles with hypothenuse equal to the diameter of its circumcircle. It is stated and proven in the following

Theorem 1 (Thales of Mileto, 600 AC) *If two concurrent lines are cut by two parallel lines, then the segments that they determine are proportional.*

Proof. We have to prove that if BD is parallel to CE, then

$$\frac{AB}{BC} = \frac{AD}{DE}$$

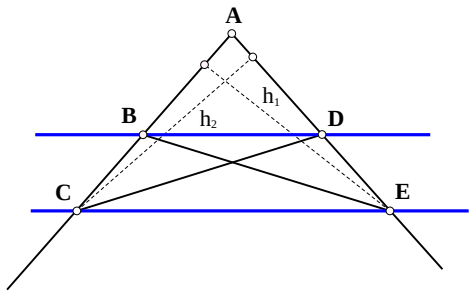
holds true. Since triangles BCE and CBE have equal area (both have the same base EC and the same altitude, the distance between BD and CE), then we have

$$\frac{1}{2}\, BC \cdot h_1 = \frac{1}{2}\, DE \cdot h_2$$

on account of the fact that h_1 is an altitude of $\triangle BCE$ and h_2 is an altitude of $\triangle CED$. Moreover, since triangles ABE and ADC have equal area, then it holds that $\frac{1}{2}\, AB \cdot h_1 = \frac{1}{2}\, AD \cdot h_2$. Dividing the preceding equalities yields

$$\frac{AB}{BC} = \frac{AD}{DE}$$

□



Scheme for Thales theorem

Remark. From the above expression, we also have

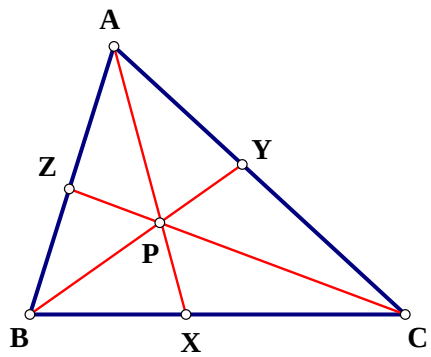
$$\frac{AB}{AD} = \frac{BC}{DE} = \frac{AB + BC}{AD + DE} = \frac{AC}{AE}$$

The line segment joining a vertex of a triangle to any given point on the opposite side (or its extension) is called a *cevian*, named after Giovanni Ceva (1648-1734), who proved a theorem in elementary geometry that unifies several other statements. It is stated and proven in the following

Theorem 2 (Giovanni Ceva, 1678) *Three cevians AX, BY, CZ, one trough each vertex of a triangle ABC, are concurrent if and only if*

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$$

Proof. The key idea to prove Ceva's theorem is that the areas of triangles with equal altitudes are proportional to the bases of the triangles.
Assume that the three cevians AX, BY and CZ are concurrent. Either (1) the points X, Y, Z lie all on the sides of the triangle or (2) one of them lies on a side of the triangle while the other two lie on the extensions of the other sides. Thus, we distinguish two cases.



Scheme (1) for Ceva's theorem

(1) From the figure, we have $\dfrac{BX}{XC} = \dfrac{\mathcal{A}(ABX)}{\mathcal{A}(AXC)}$ and also $\dfrac{BX}{XC} = \dfrac{\mathcal{A}(PBX)}{\mathcal{A}(PXC)}$. So,

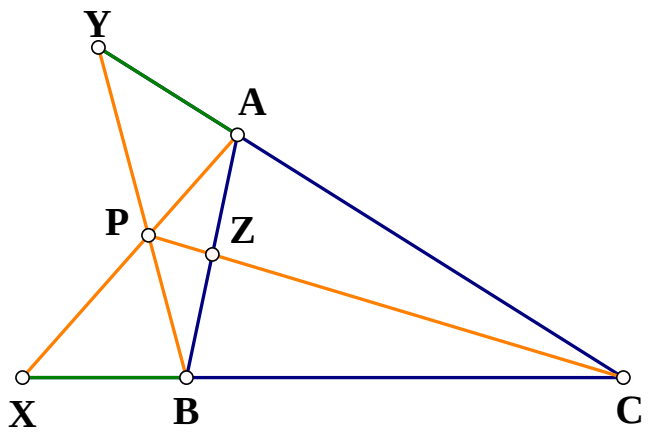
$$\frac{BX}{XC} = \frac{\mathcal{A}(ABX)}{\mathcal{A}(AXC)} = \frac{\mathcal{A}(PBX)}{\mathcal{A}(PXC)} = \frac{\mathcal{A}(ABX) - \mathcal{A}(PBX)}{\mathcal{A}(AXC) - \mathcal{A}(PXC)} = \frac{\mathcal{A}(ABP)}{\mathcal{A}(CAP)}$$

Likewise, $\dfrac{CY}{YA} = \dfrac{\mathcal{A}(BCP)}{\mathcal{A}(ABP)}$ and $\dfrac{AZ}{ZB} = \dfrac{\mathcal{A}(CAP)}{\mathcal{A}(BCP)}$. Multiplying the preceding expressions yields

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{\mathcal{A}(ABP)}{\mathcal{A}(CAP)} \frac{\mathcal{A}(BCP)}{\mathcal{A}(ABP)} \frac{\mathcal{A}(CAP)}{\mathcal{A}(BCP)} = 1$$

(2) From the figure, we have $\dfrac{BX}{XC} = -\dfrac{\mathcal{A}(ABX)}{\mathcal{A}(AXC)}$ and also $\dfrac{BX}{XC} = -\dfrac{\mathcal{A}(PBX)}{\mathcal{A}(PXC)}$. So,

$$\frac{BX}{XC} = -\frac{\mathcal{A}(ABX)}{\mathcal{A}(AXC)} = -\frac{\mathcal{A}(PBX)}{\mathcal{A}(PXC)} = -\frac{\mathcal{A}(ABX) - \mathcal{A}(PBX)}{\mathcal{A}(AXC) - \mathcal{A}(PXC)} = -\frac{\mathcal{A}(ABP)}{\mathcal{A}(CAP)}$$



Scheme (2) for Ceva's theorem

Likewise, $\dfrac{CY}{YA} = -\dfrac{\mathcal{A}(BCP)}{\mathcal{A}(ABP)}$ and $\dfrac{AZ}{ZB} = \dfrac{\mathcal{A}(CAP)}{\mathcal{A}(BCP)}$. Multiplying the preceding expressions yields

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{-\mathcal{A}(ABP)}{\mathcal{A}(CAP)} \frac{-\mathcal{A}(BCP)}{\mathcal{A}(ABP)} \frac{\mathcal{A}(CAP)}{\mathcal{A}(BCP)} = 1$$

To prove the converse, we have to see that if three cevians AX, BY, CZ satisfy $\dfrac{BX}{XC} \cdot \dfrac{CY}{YA} \cdot \dfrac{AZ}{ZB} = 1$, then they are concurrent. Indeed, suppose that the first two cevians meet at P, as before, and that the third cevian by P is CZ'. Then, by Ceva's theorem, $\dfrac{BX}{XC} \cdot \dfrac{CY}{YA} \cdot \dfrac{AZ'}{Z'B} = 1$. From the hypothesis $\dfrac{BX}{XC} \cdot \dfrac{CY}{YA} \cdot \dfrac{AZ}{ZB} = 1$, it immediately follows that $\dfrac{AZ}{ZB} = \dfrac{AZ'}{Z'B}$, and Z coincides with Z'. This completes the proof. $\square$

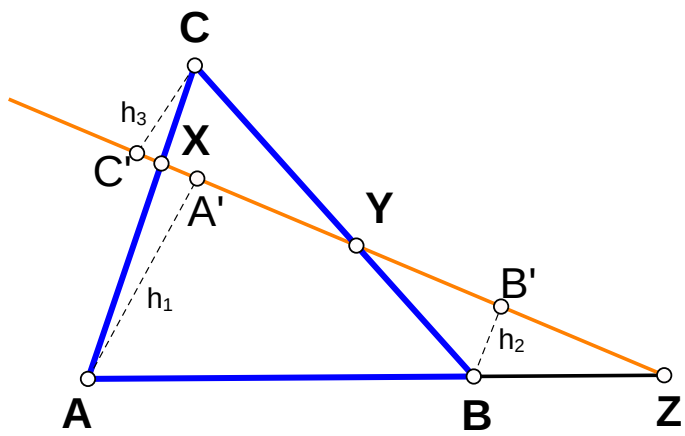
Another basic result about triangles in plane geometry relating a triangle and a transversal line is the following theorem, named after Menelaus of Alexandria.

Theorem 3 (Menelaus' Theorem) *Three points X, Y, Z on the sides CA, BC, AB, respectively, of $\triangle ABC$ (or their extensions) are collinear if and only if*

$$\frac{AZ}{ZB} \cdot \frac{BY}{YC} \cdot \frac{CX}{XA} = -1$$

Proof. Firstly, let us assume that X, Y, Z are collinear. We distinguish two cases: (1) the line through X, Y, Z intersects two sides of $\triangle ABC$; (2) the line through X, Y, Z intersects none of them.

(1) In the next figure we have $\triangle AA'Z \sim \triangle BB'Z$, from which it follows that $\frac{AZ}{ZB} = -\frac{h_1}{h_2}$. Likewise, from $\triangle BB'Y \sim \triangle CC'Y$

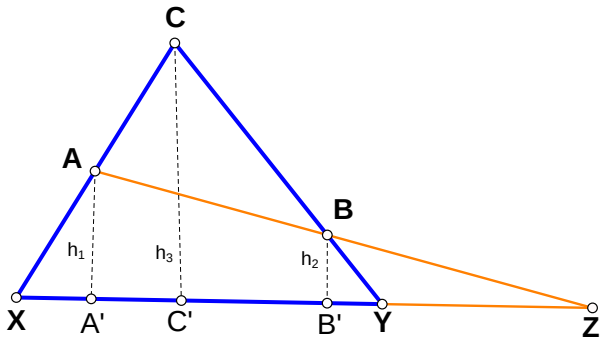


Scheme (1) for Menelaus theorem

we get $\frac{BY}{YC} = \frac{h_2}{h_3}$. Finally, from $\triangle CC'X \sim \triangle AA'X$ we obtain $\frac{CX}{XA} = \frac{h_3}{h_1}$. Multiplying the preceding expressions yields

$$\frac{AZ}{ZB} \cdot \frac{BY}{YC} \cdot \frac{CX}{XA} = -\frac{h_1 h_2 h_3}{h_1 h_2 h_3} = -1$$

(2) If the line trough X, Y, Z does not intersect any of the triangle sides, then, from the figure below, we have $\triangle AA'Z \sim \triangle BB'Z$,



Scheme (2) for Menelau's theorem

from which it follows that $\dfrac{AZ}{ZB} = -\dfrac{h_1}{h_2}$. Likewise, from $\triangle BB'Y \sim \triangle CC'Y$ we get $\dfrac{BY}{YC} = -\dfrac{h_2}{h_3}$. Finally, from $\triangle CC'X \sim \triangle AA'X$ we obtain $\dfrac{CX}{XA} = -\dfrac{h_3}{h_1}$. Multiplying the preceding expressions, the statement follows and the proof is complete.

To prove the converse, we suppose that X, Y, Z satisfy $\dfrac{AZ}{ZB} \cdot \dfrac{BY}{YC} \cdot \dfrac{CX}{XA} = -1$ and consider Z' the point where XY meets AB.
Note that this point Z' must exist, for if this were not the case, AB and XY would be parallel. Thus, from Thales' theorem, we would have $\dfrac{CX}{XA} = \dfrac{CY}{YB}$, from which $\dfrac{AZ}{ZB} = -1$ follows. And that is clearly impossible: if Z lies between A and B, then AZ and ZB will have the same sign; otherwise, AZ and ZB cannot have the same magnitude.
Then, by Menelaus' theorem, $\dfrac{AZ'}{Z'B} \cdot \dfrac{BY}{YC} \cdot \dfrac{CX}{XA} = -1$. Therefore, $\dfrac{AZ'}{Z'B} = \dfrac{AZ}{ZB}$ and Z coincides with Z'. $\square$

3 Applications

We close this section by presenting some theorems and problems that can be proven or solved by applying the theoretical results previously given.

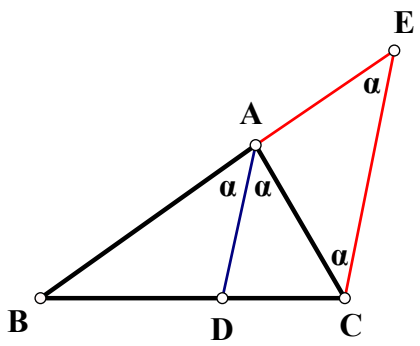
Theorem 4 (Bisector's Theorem) *Suppose that the angle bisector of angle A in a triangle ABC intersects the side BC at a point D. Then, the ratio of the length of the line segment BD to the length of the segment DC is equal to the ratio of the length of the side AB to the length of the side AC. That is,*

$$\frac{BD}{DC} = \frac{AB}{AC}$$

Proof. First, we draw a parallel to bisector AD by vertex C that meets the side AB at the point E. Since $\angle DAC = \angle ACE$ and $\angle AEC = \angle BAD$, then the triangle ACE is isosceles with $AE = AC$. Now, we have

$$\frac{BD}{DC} = \frac{AB}{AE} = \frac{AB}{AC}$$

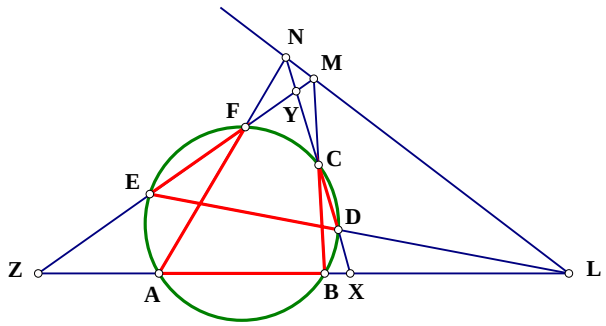
on account of Thales' theorem, and the proof is complete. □



Theorem 5 (Pascal's Theorem) *Let $ABCDEF$ be an arbitrary hexagon inscribed in a circle. Then, the three intersection points of the opposite sides or their extensions: $L = AB \cap DE$, $M = BC \cap EF$, and $N = CD \cap FA$, are collinear.*

Proof. This result known as Pascal's theorem for the circle is proven using several times Menelaus' and the secant or power point theorems. Indeed, applying Menelaus' theorem to the triangle XYZ and the line LDE yields

$$\frac{XD}{DY} \cdot \frac{YE}{EZ} \cdot \frac{ZL}{LX} = -1$$



Possible scheme for Pascal's Theorem

Applying again Menelaus' theorem to the triangle XYZ and the line MCB, we get

$$\frac{XC}{CY} \cdot \frac{YM}{MZ} \cdot \frac{ZB}{BX} = -1$$

and applying Menelaus' theorem to the triangle XYZ and the line NFA, we obtain

$$\frac{XN}{NY} \cdot \frac{YF}{FZ} \cdot \frac{ZA}{AX} = -1$$

Multiplying the preceding expressions, we get

$$\frac{XD}{DY} \cdot \frac{YE}{EZ} \cdot \frac{ZL}{LX} \cdot \frac{XC}{CY} \cdot \frac{YM}{MZ} \cdot \frac{ZB}{BX} \cdot \frac{XN}{NY} \cdot \frac{YF}{FZ} \cdot \frac{ZA}{AX} = -1$$

This equation can be simplified by applying the secant or power point theorem to the points X, Y, Z and the circle of the figure, from which it follows that:

$$XD \cdot XC = XB \cdot XA, \quad YD \cdot YC = YF \cdot YE, \quad ZF \cdot ZE = ZB \cdot ZA$$

Notice that $XA = -AX$, $XB = -BX$, and so on. After simplification, the above expression becomes

$$\frac{ZL}{LX} \cdot \frac{YM}{MZ} \cdot \frac{XN}{NY} = -1$$

By the converse of Menelaus' theorem, applied to the triangle XYZ, the points L, M, N are collinear and we are done. $\square$

Next, we present other practical applications in the following problems.

Problem 1. *Let X, Y, Z be three points on the sides BC, CA, AB respectively of a triangle ABC. If the segments AX, BY, CZ meet in a point P, show that*

$$\left(1+\frac{XC}{XB}+\frac{YA}{YC}\right)^{-1}+\left(1+\frac{YA}{YC}+\frac{ZB}{ZA}\right)^{-1}+\left(1+\frac{ZB}{ZA}+\frac{XC}{XB}\right)^{-1}\leq 1$$

Solution. Applying Ceva's theorem, we have

$$\frac{BX}{XC}\cdot\frac{CY}{YA}\cdot\frac{AZ}{ZB}=1\Leftrightarrow\frac{XC}{XB}\cdot\frac{YA}{YC}\cdot\frac{ZB}{ZA}=1$$

Let us denote by α, β and γ the ratios $\frac{XC}{XB}$, $\frac{YA}{YC}$ and $\frac{ZB}{ZA}$, respectively. We have to prove that

$$\frac{1}{1+\alpha+\beta}+\frac{1}{1+\beta+\gamma}+\frac{1}{1+\gamma+\alpha}\leq 1$$

when $\alpha\beta\gamma=1$. Indeed, putting $\alpha=x^3, \beta=y^3$ and $\gamma=z^3$, we have $xyz=1$ and

$$\alpha+\beta=x^3+y^3=(x+y)(x^2-xy+y^2)\geq(x+y)xy$$

from which it follows that

$$\frac{1}{1+\alpha+\beta}\leq\frac{1}{1+xy(x+y)}=\frac{z}{x+y+z}$$

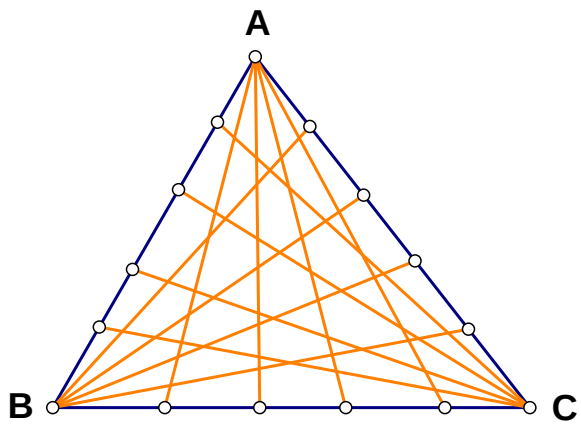
Likewise, $\frac{1}{1+\beta+\gamma}\leq\frac{x}{x+y+z}$ and $\frac{1}{1+\gamma+\alpha}\leq\frac{y}{x+y+z}$. Adding the preceding inequalities yields

$$\sum_{cyclic}\frac{1}{1+\alpha+\beta}\leq 1$$

Equality holds when $\alpha=\beta=\gamma$. That is, when the cevians are the medians of $\triangle ABC$, and we are done. $\square$

Problem 2. *Let $p \geq 3$ be a prime number. Each side of a triangle ABC is divided into p equal parts and each division point is joined to the opposite vertex. Calculate the maximum number of pairwise disjoint interior regions in which the triangle is divided.*

Solution. We claim that no three lines concur in the same point. Then, from the figure we observe that by drawing all the lines from A the triangle is divided into p triangles.



The lines from B divide each such triangle into p parts, so these lines determine p^2 regions. Every line from vertex C increases the number of regions by exactly the number of intersection points it contains. But these lines intersect $2(p-1)+1$ other lines, so the number of regions is

$$N = p^2 + \Big(2(p-1)+1\Big)(p-1) = p^2 + (2p-1)(p-1) = 3p^2 - 3p + 1$$

To prove that no three lines have a point in common, we will argue by contradiction. Suppose that three of the drawn lines (cevians) AX, BY and CZ have a common point. Then, on account of Ceva's theorem, we have

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$$

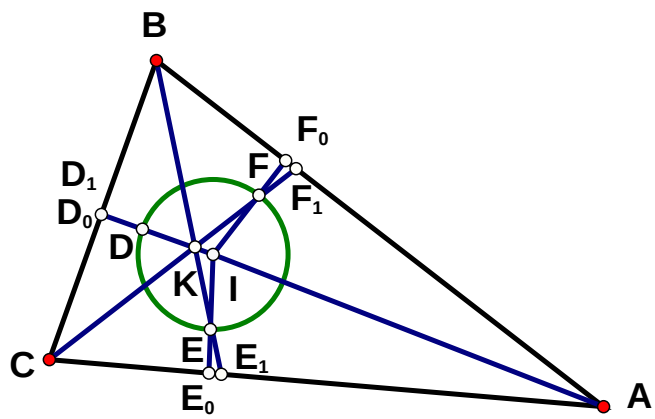
But, due to the construction of the division points, there exist integers k, l, m, $1 \leq k, l, m \leq p-1$ such that

$$\frac{BX}{XC} = \frac{k}{p-k}, \quad \frac{CY}{YA} = \frac{l}{p-l}, \quad \frac{AZ}{ZB} = \frac{m}{p-m}$$

Then, $klm = (p-k)(p-l)(p-m)$ or $2mkl = p\,(p^2 - pk - pl - pm + km + kl + lm)$, from which it follows that $p \mid 2kml$. Since k, l, m are smaller than p, then $p \mid 2$, which is impossible. Therefore, the claim is proven. □

Problem 3. *Let I be the incenter of an acute $\triangle ABC$. Let γ be a circle with center I that lies inside $\triangle ABC$. Let D, E, F be the intersection points of circle γ with the perpendicular rays from I to sides BC, CA, AB respectively. Prove that lines AD, BE, CF are concurrent.*

Solution. Let $D_0 = ID \cap BC$, $E_0 = IE \cap CA$, and $F_0 = IF \cap AB$. Since AI bisects $\angle CAB$, then IE_0 and IF_0 are symmetric respect



Scheme for Kariya's point

to AI. Now $IE = IF$ implies E and F are symmetric respect to AI. Hence, $d(E, AB) = d(F, AC)$. Then

$$\frac{[CFA]}{[AEB]} = \frac{CA \cdot d(F, AC)/2}{AB \cdot d(E, AB)/2} = \frac{CA}{AB}$$

Likewise, $\dfrac{[BEC]}{[CDA]} = \dfrac{BC}{CA}$ and $\dfrac{[ADB]}{[BFC]} = \dfrac{AB}{BC}$. Let $D_1 = AD \cap BC$, $E_1 = BE \cap CA$, and $F_1 = CF \cap AB$. Then, we have

$$\frac{AF_1}{F_1B} = \frac{[CF_1A]}{[BF_1C]} = \frac{d(A, CF)}{d(B, CF)} = \frac{[CFA]}{[BFC]}$$

Similarly, we get $\dfrac{BD_1}{D_1C} = \dfrac{[ADB]}{[CDA]}$ and $\dfrac{CE_1}{E_1A} = \dfrac{[BEC]}{[AEB]}$. Now, from the preceding, we obtain

$$\frac{AF_1}{F_1B} \cdot \frac{BD_1}{D_1C} \cdot \frac{CE_1}{E_1A} = \frac{CA}{BC} \cdot \frac{AB}{CA} \cdot \frac{BC}{AB} = 1$$

from which follows that lines AD, BE, and CF are concurrent on account of Ceva's theorem. □

Remark. Point K is known in the literature as Kariya's point.

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